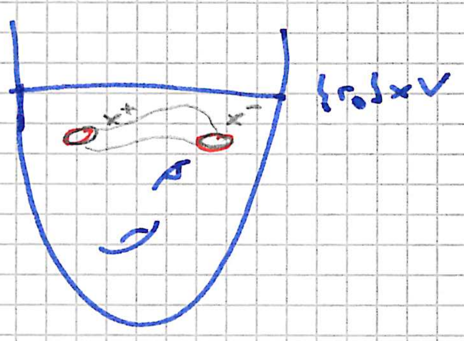
continuement

Compactness Lemma:

Let $\hat{W}$ be a Liouville manifold, $H: \hat{W} \to \mathbb{R}$ a Hamiltonian which is asymptotically linear and J an almost complex structure which is convex near some hypersurface $\{r_0\} \times V$ in the end. If $x^\pm$ are 1-periodic orbits of H contained in the region "below" $\{r_0\} \times V$, then every Floer cylinder connecting them ~~...~~ does not enter $[r_0, \infty) \times V$.



Pf: Assume by contradiction that there is a Floer cylinder u connecting x^+ and x^- whose image rises above $\{r_0\} \times V$.
Fix a regular value r_1 near r_0 of $\pi_{\mathbb{R}} \circ u$ and let $\Sigma \subseteq \mathbb{R} \times S^1$ denote the preimage

$$u^{-1}([r_1, \infty) \times V) \subseteq \mathbb{R} \times S^1$$

and set $v := u|_\Sigma$.

We know that

$$0 \leq E(v) = \frac{1}{2} \int_\Sigma \left(|\partial_s v|^2 + |\partial_t v - X_H|^2 \right) ds \wedge dt$$

$$= \int_\Sigma \omega(\partial_s v, J \partial_s v)\, ds \wedge dt$$

$$= \int_\Sigma \left(\omega(\partial_s v, \partial_t v) - \omega(\partial_s v, X_H \circ v) \right) ds \wedge dt$$

$H = br + c$ on the image of v

$$= \int_\Sigma v^* \omega - b\, d(r \circ v\, dt)$$

$$= \int_{\partial \Sigma} v^* \lambda - b\, (r \circ v)\, dt \qquad (*)$$

Now $$v^*\lambda = v^*(r\alpha) = -v^*dr\circ J$$
$$= -dr\circ J\circ dv$$
$$= -dr\circ dv\circ j + \underbrace{dr(X_H\circ v)\,ds}_{=0} - \underbrace{dr(JX_H\circ v)\,dt}_{=b(r\circ v)dt}$$
Floer equation

So
$$\int_{\partial\Sigma} v^*\lambda - b(r\circ v)\,dt = -\int_{\partial\Sigma} d(r\circ v)\circ j.$$

Now the boundary orientation on $\partial\Sigma$ is such that if Y is tangent to $\partial\Sigma$, then $j(Y)$ points inward.

But by construction $r\circ v$ reaches its minimum on $\partial\Sigma$, so we conclude that $d(r\circ v)\circ j$ is a positive form on $\partial\Sigma$, implying that

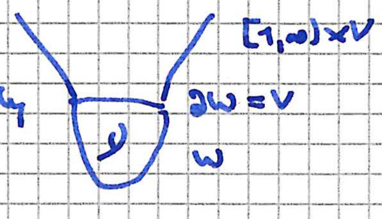
$$\int_{\partial\Sigma} v^*\lambda - b(r\circ v)\,dt < 0$$

This contradicts (*), proving that the original cylinder
$$u : \mathbb{R}\times S^1 \to \hat{W}$$
entering $[r_0,\infty)\times V$ does not exist. $\square$

Armed with this lemma, we can now proceed to define the Floer homology of an asymptotically linear Hamiltonian.

Recall our setup:

- $(\hat{W}, d\lambda)$ is a Liouville manifold, obtained from completing $(W, d\lambda)$.



- H is a (time dependent) asymptotically linear Hamiltonian with slope $b>0$, $b\notin \mathrm{Spec}(V,\alpha)$.
- J is convex, i.e.
$$dr\circ J = -r\alpha \quad \text{on } [1,\infty)\times V$$
(Note $r\alpha = \lambda$ on the end).

As in the closed case, given asymptotics $x^\pm$ the energy of a Floer cylinder connecting x^+ and x^- is given by

$$E(u) = \mathcal{A}_H(x^+) - \mathcal{A}_H(x^-).$$

As $\omega = d\lambda$ is exact, Stokes' theorem implies that $\omega_{|\pi_2} = 0$ (in fact the integral of ω over any closed surface vanishes).

By the lemma, we know that all Floer cylinders are contained in a compact subset of $\hat{W}$. The bubbling argument now works as in the closed case to exclude sphere bubbling and instead provide a priori gradient bounds. Now the Arzela-Ascoli argument shows that ~~any~~ moduli space $\mathcal{M}(x^+, x^-)$ of Floer cylinders is compact up to breaking of cylinders.

If we assume $c_1(\hat{W}) = 0$, then contractible 1-periodic orbits of X_H have a well defined integer grading, which in our conventions was

$$|x| = n + \mu_{CZ}(x).$$

Choosing a reference loop in each nontrivial free homotopy class of loops in W and trivializing TW over these loops allows one to assign integer gradings also to noncontractible 1-periodic orbits (contrary to the closed case, these will now contain interesting information).

For a generic choice of (time-dependent) J the moduli spaces of Floer cylinders

$$\hat{\mathcal{M}}(x^+, x^-; H, J)$$

will be transversally cut out for all pairs of periodic orbits $x^\pm$, just as in the compact case.

As before this implies that for $|x^+| - |x^-| = 1$, the quotient

$$\mathcal{M}(x^+, x^-; H, J) = \hat{\mathcal{M}}(x^+, x^-; H, J) / \mathbb{R}$$

is compact and 0-dimensional, i.e. finite.

Similarly, a gluing theorem shows that for

$$|x^+| - |x^-| = 2$$

the moduli space $\mathcal{M}(x^+, x^-; H, J)$ can be compactified to a 1-manifold with boundary

$$(*) \qquad \partial \overline{\mathcal{M}}(x^+, x^-; H, J) = \bigcup_{|x^+| > |x| > |x^-|} \mathcal{M}(x^+, x) \times \mathcal{M}(x, x^-) .$$

So as usual we set

$$CF_k(H, J) := \bigoplus_{|x| = k} \mathbb{Z} \cdot x$$

and define $\partial : CF_k(H, J) \to CF_{k-1}(H, J)$ by

$$\partial x^+ := \sum_{|x^-| = |x^+| - 1} \#_2 \mathcal{M}(x^+, x^-) \cdot x^- .$$

Then $\partial^2 = 0$ by $(*)$, and so the Floer homology

$$HF_*(H, J) := H_*(CF(H, J)_*, \partial)$$

is defined.

<u>This homology will generally depend on H.</u>

To compare the Floer homologies of different Hamiltonians $H_t^{\pm}$, we need to study continuation maps.

We start by picking a family of slopes $b : \mathbb{R} \to (0, \infty)$ with

$$b(s) = \begin{cases} b(H_t^-) & \text{for } s \ll 0 \quad (\text{e.g. for } s \leq -1) \\ b(H_t^+) & \text{for } s \gg 0 \quad (\text{e.g. for } s \geq 1) \end{cases}$$

Then we choose a family of Hamiltonians

$$H_{s,t} : W \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

(which are 1-periodic in t) s.t.

(a) $H_{s,t} = b(s) \cdot r - c(s)$ on a fixed cylindrical end $[r_0, \infty) \times V$

(b) $H_{s,t} = H_t^-$ for $s \ll 0$ and $H_{s,t} = H_t^+$ for $s \gg 0$.

Similarly, we fix a family $\{J_{s,t}\}$ of convex a.c. structures interpolating between given $J_t^{\pm}$.

The moduli space of continuation cylinders is defined as usual as

$$\mathcal{K}(x^+,y^-) := \Big\{ u : \mathbb{R}\times S^1 \to W \;:\; \partial_s u + J_{s,t}\left(\partial_t u - X_{H_{s,t}}\circ u\right) = 0,$$
$$\lim_{s\to\infty} u(s,t) = x^+(t),\ \lim_{s\to-\infty} u(s,t) = y^-(t) \Big\}$$

Here x^+ is a 1-periodic orbit of X_{H^+}, and y^- is a 1-periodic orbit of H^-.
The transversality theory works as before to show that for a generic choice of $\{J_{s,t}\}$, $\mathcal{K}(x^+,y^-)$ is a manifold of dimension
$$|x^+| - |y^-| .$$

The crucial question now is: can we understand its compactness properties?

Let us try to repeat the proof of our confinement lemma, which told us that Floer cylinders cannot enter the cylindrical region $[r_0,\infty)\times V$ where both H and J are standard:
We set $\Sigma := u^{-1}([r_0,\infty)\times V)$ and $v = u|_\Sigma$ and redo the computation of the energy:

$$0 < E(v) = \frac{1}{2}\int_\Sigma \| dv - dt\otimes X_{H_{s,t}} \|^2 dt$$

$$= \int_\Sigma \omega(\partial_s v, \partial_t v) - \omega(\partial_s v, X_{H_{s,t}}\circ v)\, ds\wedge dt$$

$$= \int_\Sigma v^*\omega - \int_\Sigma b(s)\, d(r\circ v)\, dt$$

Now the second integrand is no longer exact, so we rewrite this as

$$= \int_\Sigma v^*\omega - \int_\Sigma d\big(b(s)\cdot(r\circ v)\,dt\big) + \int_\Sigma b'(s)\,(r\circ v)\, ds\wedge dt$$

$$= \int_{\partial\Sigma} v^*\lambda - b(s)\,(r\circ v)\,dt + \int_\Sigma b'(s)\,(r\circ v)\, ds\wedge dt$$

In order to complete the argument as for Floer cylinders, we want to say that this is a lower bound for

$$\int_{\partial\Sigma} v^*\lambda - b(s)\,(r\circ v)\,dt .$$

This leads us to make the crucial additional assumption that

$$\boxed{b'(s) \leq 0} .$$

With this assumption, the proof proceeds and we find that continuation cylinders are indeed confined to a compact part of $\hat{W}$.

So we can define continuation maps

$$c : CF(H^+, J^+) \longrightarrow CF(H^-, J^-)$$

provided that $\quad b(H^+) \leq b(H^-)$.

The next steps again proceed in complete analogy with the case of closed manifolds:

1. Continuation maps are independent (up to chain homotopy) of the choice of interpolating families $\{H_{s,t}, J_{s,t}\}$ between the given "end points" (H_t^+, J_t^+) and (H_t^-, J_t^-), as long as the slope conditions are satisfied.
2. For $(H_t^+, J_t^+) = (H_t^-, J_t^-)$ one can choose the families constant in s, and in this case the continuation map only counts constant (in s) cylinders $u(s,t) = x(t)$ for $x \in \mathcal{P}(H^\pm)$. It follows that this particular continuation map is the identity on the Floer complex and hence on homology.
3. Given 3 pairs of Floer data (H^+, J^+), (H, J) and (H^-, J^-) satisfying the slope conditions

$$b(H^+) \leq b(H) \leq b(H^-)$$

so that continuation maps are defined, we have that

$$c^{H^+,H^-} \text{ is chain homotopic to } c^{H,H^-} \circ c^{H^+,H},$$

so that the induced maps on homology agree:

$$c_*^{H^+,H^-} = c_*^{H,H^-} \circ c_*^{H^+,H} .$$

As a consequence, we obtain

Prop: For each fixed slope $b \notin \mathrm{Spec}(V,\alpha)$, the Floer homologies of any two nondegenerate Hamiltonians with that slope at ∞ are canonically isomorphic.

So how do we get an invariant of our Liouville manifold $(\hat{W}, d\lambda)$?

We can partially order the set of asymptotically linear Hamiltonians by declaring

$$H_1 \leq H_2 \quad \text{if and only if} \quad b(H_1) \leq b(H_2) .$$

The above proposition says that Floer homology only depends on the slope, and we saw that for $b(H^+) \leq b(H^-)$ we have a continuation map

$$HF_*(H^+, J^+) \longrightarrow HF_*(H^-, J^-)$$

So the Floer homologies $HF_*(H,J)$ also form a directed system.

Def: For a Liouville ~~dom~~ manifold $(\hat{W}, d\lambda)$ we define

$$SH_*(W, d\lambda) := \varinjlim HF_*(H,J),$$

where the direct limit is taken over slopes of nondegenerate asymptotically linear Hamiltonians.