Step 3: We now use Newton iteration to get from the preglueing γ_ρ to an actual solution of the gradient flow equation.

Here is a model situation:

Let $f: \mathbb{R} \to \mathbb{R}$ be a function and write it as

$$f(x) = f(0) + f'(0) \cdot x + \underbrace{N(x)}_{\text{nonlinear part, } N(0)=0}$$

We assume that $f'(0) \neq 0$.
If $f(x_0) = 0$, we have

$$x_0 = -\frac{1}{f'(0)} \left(f(0) + N(x_0) \right),$$

so that such a zero is a fixpoint of the map

$$\varphi: \mathbb{R} \to \mathbb{R}$$
$$x \longmapsto -\frac{1}{f'(0)} \left(f(0) + N(x) \right)$$

In this situation we have

Lemma: Suppose $C > 0$ is such that

(i) $$\frac{1}{|f'(0)|} \cdot |N(x) - N(y)| \leq C \cdot (|x| + |y|) \cdot |x - y|$$

for all $x, y \in \overline{B(0, \frac{1}{5C})}$.

Then if also

(ii) $$\left| \frac{f(0)}{f'(0)} \right| \leq \frac{1}{10C},$$

there is a unique fixpoint x_0 of φ in the ball $\overline{B(0, \frac{1}{5C})}$ such that

$$|x_0| \leq 2 \left| \frac{f(0)}{f'(0)} \right|.$$

Pf: If $x \in \overline{B(0, \frac{1}{5C})}$, we have

$$|\varphi(x_0)| \leq |\varphi(0)| + |\varphi(x) - \varphi(0)|$$
$$\leq \frac{1}{10C} + C|x|^2$$
$$= \frac{1}{10C} + \frac{1}{25C} < \frac{1}{5C}.$$

Moreover, for $x, y \in B(0, \frac{1}{5C})$ we have

$$|\varphi(x) - \varphi(y)| \leq C \cdot (|x| + |y|)\, |x-y|$$
$$\leq C \cdot \frac{2}{5C}\, |x-y|$$
$$= \frac{2}{5}\, |x-y|$$

We have proven that φ maps $\overline{B(0, \frac{1}{5C})}$ to itself and is a contraction there.
So by the Banach fixpoint theorem, there is a unique fixpoint of φ in this ball.
Finally,

$$|x_0| = |x_0 - 0| \geq \frac{5}{2}\, |\varphi(x_0) - \varphi(0)| \geq 2\,(|x_0| - |\varphi(0)|)$$

so
$$|x_0| \leq 2\,|\varphi(0)| = 2\left|\frac{f(0)}{f'(0)}\right| .$$

□

<u>Remark:</u> Clearly, the above argument applies also to a smooth map $F : A \to B$, $F(x) = F(0) + DF_0(x) + N(x)$ between Banach spaces, if we interpret (i) as an estimate on

$$\| G N(x) - G N(y) \| \quad \text{where } G = (DF_0)^{-1}$$

We will apply this argument in our setting, which we now recall:

Given $(\gamma_1, \gamma_2) \in \tilde{\mathcal{F}}(p,r) \times \tilde{\mathcal{F}}(r,q)$ and $\rho \geq \rho_2$, we produced pregluings

$$\gamma_\rho = \gamma_1 \#_\rho^0 \gamma_2 \in C^\infty_{p,q} \subseteq \mathcal{P}^{1,2}_{p,q} .$$

We also found finite codimension subspaces

$$L_\rho^\perp \subseteq T_{\gamma_\rho} \mathcal{P}^{1,2}_{p,q}$$

such that the linearization

$$D_\rho := (DF)_{\gamma_\rho} \text{ maps } L_\rho^\perp \text{ onto } L^2_\rho ,$$

the fiber of $L^2(\mathcal{P}^{1,2\,*}_{p,q} TM)$ at γ_ρ

These linearizations have uniformly bounded inverses

$$G_\rho : L^2_\rho \to L^\perp_\rho$$

The spaces $L^\perp_\rho$ depend smoothly on ρ, because they are defined as the orthogonal complements of the glued kernels, and linear gluing is smooth in ρ. So we get a bundle

$$L^\perp \longrightarrow [\rho_2, \infty)$$

For each fixed $\rho \geq \rho_2$ we consider the (nonlinear!) map

$$F_\rho : L^\perp_\rho \longrightarrow L^2(\gamma_\rho^* TM) = L^2_\rho$$

obtained as

$$F_\rho(\xi) = F(\exp_{\gamma_\rho} \xi).$$

Now expand this nonlinear map of Banach spaces as

$$F_\rho(\xi) = F_\rho(0) + (DF_\rho)_0(\xi) + N_0(\xi)$$

Then

$$N_0(\xi) - N_0(\eta) = F_\rho(\xi) - F_\rho(\eta) - (DF_\rho)_0(\xi - \eta)$$

$F_\rho(\xi) = F_\rho(\eta) + (DF_\rho)_\eta(\xi-\eta) + N_\eta(\xi-\eta)$

$$= (DF_\rho)_\eta(\xi - \eta) + N_\eta(\xi - \eta) - (DF_\rho)_0(\xi - \eta)$$

Using this, one deduces the estimate

$$\| N_0(\xi) - N_0(\eta) \| \leq C_2 \left(\|\xi\| + \|\eta\| \right) \|\xi - \eta\|$$

Together with the estimate

$$\| G_\rho \xi \| \leq C_1 \|\xi\|$$

from last time this gives the estimate

$$\| G_\rho N_0(\xi) - G_\rho N_0(\eta) \| \leq C_1 C_2 \left(\|\xi\| + \|\eta\| \right) \|\xi - \eta\|$$

Finally, one achieves the analogue of (ii)

from

<u>Lemma:</u> There are constants $\alpha, \mu > 0$ such that

(Lemma 2.53 in Schwarz)

$$\| \mathcal{F}(\gamma_\rho) \|_{L^2_\rho} \le \alpha e^{-\mu\rho} \quad \text{for all } \rho \ge \rho_3 \ge \rho_2$$

This completes the verification of the requirements for the application of the Banach fixpoint argument, and in conclusion we get, for each $\rho \ge \rho_3$, a unique section

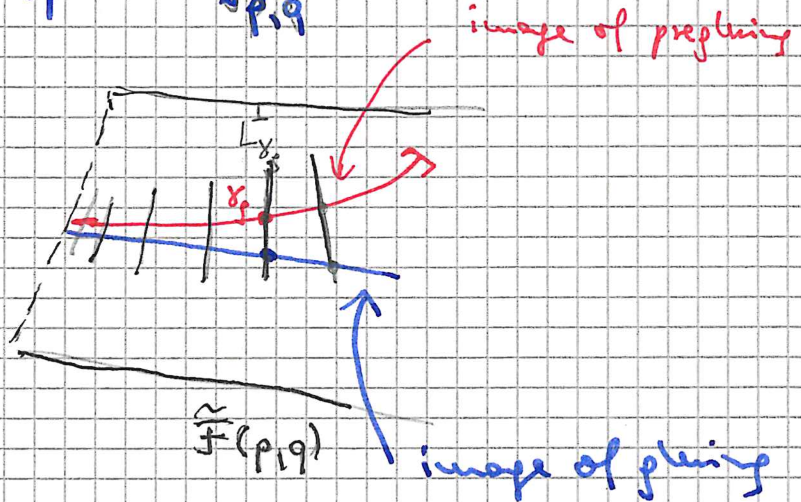
$$\xi_\rho \in L^\perp_{\gamma_\rho}$$

such that

$$\mathcal{F}(\exp_{\gamma_\rho} \xi_\rho) = 0$$

Here is a rough picture of the situation:

ambient space $\mathcal{P}^{1,2}_{p,q}$



Finally, one checks smoothness of the construction in the gluing parameter ρ, and argues the passage from the result for $\tilde{\mathcal{F}}$ to the result for $\mathcal{F}$.

This completes our very rough sketch of the proof of proposition 1.

The proofs of Propositions 2 and 3 follow the same pattern. One difference is that in the proof of compactness modulo breaking, one now starts from the "main" component centered near $0 \in \mathbb{R}$ (the domain) where the equation is time-dependent and extracts a C^∞_{loc}-convergent subsequence in this way.

If the asymptotic limits of the resulting limiting map

$$\gamma : \mathbb{R} \to M$$

do not agree with the original limits $p^\pm$, one then finds ~~rescaling~~ shift factors as before. As these time shifts necessarily satisfy $|t_n^i| \to \infty$, and the equations are gradient flow equations in these regions of the domain, all other possible components of a limiting object are gradient flow lines for (f^-, g^-) or (f^+, g^+).

As in the case of $\overline{\overline{\mathcal{F}}}(p,q)$, for index difference 1 we conclude that at most one break point can occur.

Together with the appropriate gluing theorem, this yields the description of the compactification of 1-dimensional spaces $\mathcal{M}(p^-, p^+)$ and $\mathcal{H}(p^-, p^+)$ as claimed in Prop. 2 and 3.

(Of course in the case of $\mathcal{H}(p^-, p^+)$ there is additional boundary arising from the boundary of the interval parametrising the homotopy).

What we did not do:

- give any details
- discuss orientations (needed to define theory over $\mathbb{Z}$)

One topic I want to briefly mention is the construction of operations on (co)homology using Morse theory.

Recall that on a smooth manifold M which is oriented one has an intersection product

$$H_k(M) \otimes H_\ell(M) \longrightarrow H_{k+\ell-n}(M)$$

which is dual to cup product in cohomology. Geometrically, if classes $a \in H_k(M)$ and $b \in H_\ell(M)$ are representable as images of the fundamental classes of smooth manifolds K of dimension k and L of dimension ℓ, respectively, under some smooth maps

$$f: K \to M \quad \text{and} \quad g: L \to M,$$

then one perturbs at least one of the maps to make them transverse to each other, and notes that in that case

$$S := \{(x,y) \in K \times L : f(x) = g(y)\} \subseteq K \times L$$

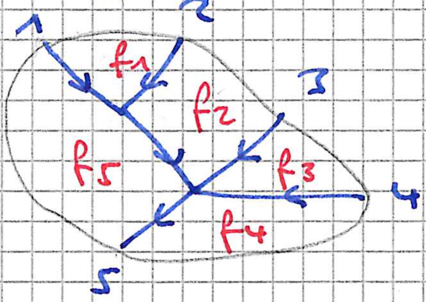
is a smooth submanifold of dimension $k+\ell-n$, which comes with an obvious map to M. Using the orientations of K, L and M one gets an orientation of S, and the pushforward of its fundamental class will ~~be~~ represent the intersection product of the original classes a and b.

In Morse theory, one can think of homology classes being represented by linear combinations of unstable manifolds of critical points. Ignoring the fact that their closures are not nice submanifolds, we have the more serious issue of failure of transversality: If we want to make sense of self-intersection of a class $x \in H_*(M)$, we would need to consider the intersection of $W^u(x)$ with itself which of course is never transverse unless x is a local maximum.

There are two approaches to overcome this:

1 (Fukaya)   Morse homotopy and its quantization

Consider metric trees T with k exterior vertices (with a fixed order) and all interior vertices at least trivalent, e.g.



Each interior edge has a length $l \in \mathbb{R}^+$.

The space of such trees has a natural topology which makes it homeomorphic to $\mathbb{R}^{k-3}$.
The ordering of the vertices determines a unique homotopy class of embeddings of any such tree T into $\mathbb{R}^2$. Label the complementary regions by functions $f_1, \dots, f_k$ as above, and orient all edges toward the last vertex.

Now ~~consider~~ given critical points $x_1, \dots, x_k$ of the functions $f_{left} - f_{right}$ for each exterior edge (assumed Morse) we can consider the space

$$\mathcal{T}^k(x_1, \dots, x_k)$$

of maps of such trees to M s.t. each edge is mapped to a flow line of $f_{left} - f_{right}$.
For a generic choice of these functions this space is a manifold of dimension

$$\sum_{j=1}^{k-1} \operatorname{ind}(x_j) - \operatorname{ind}(x_k) - (k-2)n + k - 3$$

One can define operations

$$\mu_{k-1}: CM_*(f_1 - f_k) \otimes CM_*(f_2 - f_3) \otimes \dots \otimes CM_*(f_{k-2} - f_{k-1}) \longrightarrow CM_*(f_{k-1} - f_k)$$

by counting elements of the spaces $\mathcal{T}^k(x_1, \dots, x_k)$

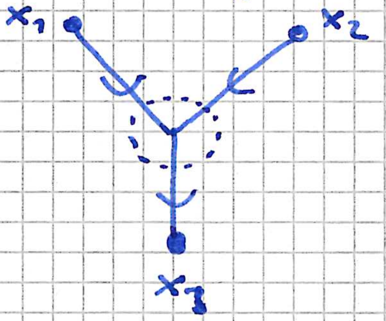
of expected dimension 0.

For $k=3$ we get an operation

$$CM_{r_1}(f_1-f_3)\otimes CM_{r_2}(f_2-f_1) \longrightarrow CM_{r_1+r_2-n}(f_2-f_3)$$

which on homology corresponds to the intersection product. The collection of operations give rise to what Fukaya calls a topological A_∞ category.

(2) If instead one wants to get operations on the Morse complex of a single Morse function, one has to perturb the gradient flow:



On each edges consider solutions of $\dot{\gamma}_i + \nabla f\circ\gamma_i + X_i(t) = 0$

$i=1,2,3$,

~~On outgoing~~ where $\gamma_1, \gamma_2 : (-\infty, 0] \to M$ and $\gamma_3 : [0,\infty) \to M$ and the vector fields $X_i(t)$ have support near $0\in\mathbb{R}$.

Geometrically, we consider the intersection of perturbations of $W^u(x_1)$, $W^u(x_2)$ and $W^s(x_3)$.

Again this gives rise to an operation

$$\mu_2 : CM_{r_1}(f)\times CM_{r_2}(f) \to CM_{r_1+r_2-n}(f)$$

which is the first in a hierarchy $\{\mu_k\}_{k\geq 2}$ giving CM_* the structure of an A_∞ algebra.

For details, see

S. Mescher "Perturbed gradient flow trees and A_∞-algebra structures in Morse cohomology"