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Differential Topology

Problem Set 2

Here is a second set of problems related to the material of the course up to this point. If you want to get feedback on your solution to a particular exercise, you may hand it in after any lecture, and I will try to comment within a week.

1. **a)** Let $f : M \to N$ and $g : N \to P$ be smooth maps between closed connected oriented manifolds of equal dimension. Prove that

$$\deg(f \circ g) = \deg(f) \cdot \deg(g).$$

b) Prove that any smooth map between closed connected oriented manifolds which is a homotopy equivalence has degree ± 1.

2. What is the degree of

a) the map $f_k : S^1 \to S^1$ given by $f(z) = z^k$ with $k \in \mathbb{Z}$? (Here we view $S^1 \subseteq \mathbb{C}$ as the unit circle in $\mathbb{C}$.)

b) the map $S^2 \to S^2$ defined by a rational function $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials of degrees n and m without common roots?
Remark: To interpret f as a self-map of S^2, we identify $S^2 \cong \mathbb{C} \cup \{\infty\}$.

3. Prove that every map $S^d \to S^d$ with degree different from $(-1)^{d+1}$ has a fixed point.

4. Prove that any map $S^d \to S^d$ of odd degree maps some pair of antipodal points onto a pair of antipodal points.

5. Prove that a manifold M is orientable if and only if the restriction of the tangent bundle to every closed curve in M is an orientable vector bundle.

6. Prove that for any vector bundle $p : E \to B$, the direct sum $E \oplus E \to B$ is orientable. Deduce as a consequence that the manifold TM is orientable for any smooth manifold M.

7. Let $p : E \to B$ be a vector bundle over a connected base space B and let $F : E \to E$ be a bundle morphism covering the identity on B and satisfying $F \circ F = F$. Prove that F has constant rank, and deduce that $\ker F$ and $\operatorname{Im} F$ are subbundles of E.

8. In this exercise, we denote by $\underline{\mathbb{R}}^k$ the trivial bundle of rank k over the given base.

Bitte wenden!

a) Prove that S^n admits a non-vanishing vector field if and only if n is odd.
Hint: Use such a vector field to construct a homotopy from the identity to the antipodal map.

b) Suppose M and N are manifolds of positive dimension such that $TM \oplus \underline{\mathbb{R}}^1$ and $TN \oplus \underline{\mathbb{R}}^1$ are trivial and assume that TM has a nonvanishing section. Prove that $T(M \times N)$ is a trivial bundle.

c) Deduce that a product of two or more spheres has trivial tangent bundle if and only if at least one of them has odd dimension.

d) Illustrate your proof by giving an explicit trivialization of $T(S^2 \times S^5)$.
(If you find this too hard, try $T(S^1 \times S^2)$ first.)

9. Let E_i be vector bundles over the same base B. A sequence of vector bundle morphisms, all covering the identity map on B,

$$\dots \xrightarrow{F_{i-2}} E_{i-1} \xrightarrow{F_{i-1}} E_i \xrightarrow{F_i} E_{i+1} \xrightarrow{F_{i+1}} \dots$$

is called exact, if for each $b \in B$ and each index i we have

$$\mathrm{image}(F_{i-1})_b = \ker(F_i)_b.$$

a) Prove that in every exact sequence of vector bundles all maps have constant rank over each connected component of B.

A short exact sequence is an exact sequence of the form

$$0 \to E_1 \xrightarrow{F_1} E_2 \xrightarrow{F_2} E_3 \to 0.$$

b) State explicitly which properties exactness of the sequence implies for each of the maps F_1 and F_2.

c) Prove that in a short exact sequence as above, E_2 is isomorphic to the direct sum $E_1 \oplus E_3$.

10. Let $f : M \to M'$ be a smooth map between manifolds transverse to the submanifold $Z' \subset M'$. We know that $Z := f^{-1}(Z') \subset M$ is a smooth submanifold of the same codimension as Z'. Prove that the normal bundle of Z in M is isomorphic to the pullback via f of the normal bundle of Z' in M'.

11. Let $f_1 : M_1 \to M'$ and $f_2 : M_2 \to M'$ be two smooth maps which are *transverse* in the sense that for every pair of points $(p_1, p_2) \in M_1 \times M_2)$ with $f_1(p_1) = f_2(p_2) =: q'$ one has

$$Df_1(T_{p_1}M_1) + Df_2(T_{p_2}M_2) = T_{q'}M'.$$

Prove that under these conditions the *fiber product of* M_1 *and* M_2 *over* M',

$$M_1 \times_{M'} M_2 := \{(p_1, p_2) \mid f_1(p_1) = f_2(p_2)\} \subseteq M_1 \times M_2,$$

is a smooth submanifold. What is its dimension? Can you find interesting examples of this construction?

Siehe nächstes Blatt!

12. Prove that if M is oriented and S_1 and S_2 are oriented closed submanifolds of complementary dimension, then the intersection number satisfies

$$S_2 \bullet S_1 = (-1)^{\dim S_1 \cdot \dim S_2} S_1 \bullet S_2.$$

Use the same argument that the Euler characteristic of an odd dimensional manifold vanishes.

13. Let M be a compact manifold and $f : M \to M$ a smooth map. The Lefschetz number $L(f)$ of f is defined to be the intersection number (defined in $\mathbb{Z}$ if M is oriented or in $\mathbb{Z}_2$ otherwise)

$$L(f) := \Delta \bullet \operatorname{graph} f$$

of the diagonal $\Delta \subset M \times M$ and $\operatorname{graph} f = \{(x, f(x)) \in M \times M \mid x \in M\}$.

a) Prove that if $L(f)$ is not zero, then f has a fixed point.

b) If f and g are homotopic maps, then $L(f) = L(g)$.

c) If f is homotopic to the identity, then $L(f) = \chi(M)$.

d) Show that $\operatorname{graph} f \pitchfork \Delta$ if and only if at every fixed point $x \in M$ for f the differential $f_{*,x} : T_xM \to T_xM$ does not have 1 as an eigenvalue.

e) Prove that in this case the contribution of the fixed point $x \in M$ of f to the Lefschetz number $L(f)$ is $\operatorname{sgn}\det(f_{*,x} - \mathbb{1})$, where $\mathbb{1} : T_xM \to T_xM$ is the identity map.