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Differential Topology

Problem Set 1

Here are some problems related to the material of the course up to this point. They are neither equally difficult nor equally important. If you want to get feedback on your solution to a particular exercise, you may hand it in after any lecture, and I will try to comment within a week.

1. Prove that any topological manifold according to our definition (Hausdorff, second countable, locally homeomorphic to $\mathbb{R}^n$ for some fixed $n \geq 0$) is *paracompact*, meaning that every open covering has a locally finite refinement.
Recall: A refinement *of an open covering $\{U_\alpha\}_{\alpha \in A}$ is an open covering $\{W_\beta\}_{\beta \in B}$ such that for each $\beta \in B$ there exists some $\alpha \in A$ with $W_\beta \subseteq U_\alpha$.*
An open covering is locally finite *if every point has a neighborhood meeting only finitely many of the sets of the covering.*

2. Prove that the subset

$$Q := \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{j=1}^{n} z_j^2 = 1\} \subseteq \mathbb{C}^n$$

is diffeomorphic to the tangent bundle of S^{n-1}.

3. Let M be a smooth differentiable manifold, and let $\tau : M \to M$ be a smooth fixed point free involution, i.e. $\tau(p) \neq p$ for all $p \in M$ and $\tau \circ \tau = \mathrm{id}_M$.

- **a)** Prove that the quotient space M/τ which is obtained by identifying every point with its image under τ is a topological manifold, and it admits a unique smooth structure for which the projection map $\pi : M \to M/\tau$ is a local diffeomorphism.
- **b)** Give examples of this phenomenon.

4. Let $U \subset \mathbb{R}^n$ be a connected open subset, and let $p : U \to U$ be a smooth map such that $p \circ p = p$. Prove that the subset $F \subset U$ of fixed points of p forms a smooth submanifold of $\mathbb{R}^n$.

5. Prove that every connected compact manifold (without boundary) of dimension 1 is diffeomorphic to S^1. Similarly, prove that every connected compact manifold with nonempty boundary is diffeomorphic to $[0, 1]$.

6. Prove directly that every product of spheres of total dimension n can be embedded into $\mathbb{R}^{n+1}$.

7. Describe an immersion of the punctured torus $S^1 \times S^1 \setminus \{pt\}$ into $\mathbb{R}^2$. Can such an immersion be injective?

Bitte wenden!

8. Give a proof of the Whitney embedding theorem for noncompact manifolds: Every smooth manifold of dimension n has an embedding as a submanifold into $\mathbb{R}^{2n+1}$ such that the image is a closed subset.

Hint: You may find the following result useful: Every open cover of a topological manifold M admits a locally finite refinement $\{V_\alpha\}_{\alpha\in A}$ such that any point $x \in M$ meets at most $\dim M+1$ of the open sets V_α. (You do not need to prove this, although it might be fun to try.)

Possible strategy: Assuming the original cover to be by charts with bounded image in $\mathbb{R}^n$, the V_α will still be chart domains. Now partition the index set α into finitely many subsets such that the V_α for α in a given subset are disjoint, and use these to build an embedding into $\mathbb{R}^d$ for a suitable d, along the lines of the proof of Theorem 1. Use an arbitrary additional proper function (show existence!) to make the embedding closed, and then use the argument from the proof of Theorem 2 to conclude.

9. Prove the following assertion from the lecture: If $U \subseteq \mathbb{R}^n$ is open and $A \subseteq U$ has measure zero and if $f : U \to V \subseteq \mathbb{R}^n$ is a map of class C^1, then $f(A)$ has measure zero in V.

10. Let M be a connected smooth manifold of dimension $d \geq 2$. Prove that given two k-tupels $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ of distinct points in M, there exists a diffeomorphism $\varphi : M \to M$ satisfying $\varphi(x_j) = y_j$ for all $j = 1, \dots k$.

11. Prove that if M is a connected smooth manifold, then any two distinct points $x \neq y$ in M can be connected by a smooth embedded path, i.e. there exists a smooth embedding $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

12. Prove that every Lie group has trivial tangent bundle.

Hint: A Lie group is a group G which also admits the structure of a smooth manifold such that the structure maps, i.e. the group multiplication $\mu : G \times G \to G$ and the forming inverses $\iota : G \to G$ are smooth maps.

13. Construct a vector field on S^2 with exactly one zero.