Further dense properties of the space of circle diffeomorphisms with a Liouville rotation number

Philipp Kunde

Department of Mathematics, University of Hamburg, Hamburg, Germany

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Abstract

In continuation of Matsumoto's paper [Ma12] we show that various subspaces are C^∞-dense in the space of orientation-preserving C^∞-diffeomorphisms of the circle with rotation number α, where $\alpha \in \mathbb{S}^1$ is any prescribed Liouville number. In particular, for every odometer $\mathcal{O}$ of product type we prove the denseness of the subspace of diffeomorphisms which are orbit-equivalent to $\mathcal{O}$.

Key words: circle diffeomorphisms, orbit equivalence, rotation number, approximation by conjugation-method, odometer

AMS subject classification: 37E10 (primary), 37A20, 37C05, 37E45 (secondary).

Introduction

Let F be the group of orientation-preserving C^∞-diffeomorphisms of the circle. Furthermore, for $\alpha \in \mathbb{S}^1$ we consider the subspace F_α of F consisting of all the C^∞-diffeomorphisms of the circle with rotation number α. If α is irrational, for any $f \in F_\alpha$ there is a unique orientation-preserving homeomorphism H_f of the circle such that $f = H_f \circ R_\alpha \circ H_f^{-1}$ and $H_f(0) = 0$, where R_α denotes the rotation by α on $\mathbb{S}^1$. J.-C. Yoccoz proved that the subspace O_α of F_α of all the diffeomorphisms, for which H_f are C^∞-diffeomorphisms, is C^∞-dense in F_α ([Yo95]).
In the following, let α be a Liouville number. By [He79], chapter IV, section 6, the unique f-invariant probability measure μ_f is given by $\mu_f = (H_f)_* m$, where m is the Lebesgue measure on $\mathbb{S}^1$, and μ_f is either equivalent to m (then H_f maps any Lebesgue null set to a null set and H_f is called absolutely continuous) or singular to m (then H_f maps some Lebesgue null set to a conull set and H_f is called singular). In [Ma12] S. Matsumoto considered several subspaces of F_α according to the regularity of H_f:

- $G_{0,\text{sing}}$: H_f is singular and is not d-Hölder for any $d \in (0, 1)$.
- $G_{0,\text{ac}}$: H_f is absolutely continuous and is not d-Hölder for any $d \in (0, 1)$.
- For $\beta \in (0, 1)$ G_β: H_f is bi-β-Hölder, but is not d-Hölder for any $d \in (\beta, 1)$.
- $G_{1,\text{sing}}$: H_f is singular and is bi-d-Hölder for any $d \in (0, 1)$.
- $G_{1,\text{ac}}$: H_f is absolutely continuous and is d-Hölder for any $d \in (0, 1)$, but is not bi-Lipschitz.

- For $k \in \mathbb{N}$ G_k: H_f is a C^k-diffeomorphism, but is not a C^{k+1}-diffeomorphism.

Then Matsumoto proved that $G_{0,\text{sing}}$ is C^∞-dense in F_α in [Ma13]. In [Ma12], Theorem 1, the C^∞-denseness of all the other spaces is shown. In this paper we examine the subsequent subspaces of G_β:

Theorem 1. *For any Liouville number α and for any $\beta \in (0,1)$ the subspaces*

- $G_{\beta,sing}$*: H_f is singular, bi-β-Hölder, but is not d-Hölder for any $d \in (\beta, 1)$*
- $G_{\beta,ac}$*: H_f is absolutely continuous, bi-β-Hölder, but is not d-Hölder for any $d \in (\beta, 1)$*

are C^∞-dense in F_α.

This statement was conjectured in [Ma12], Remark 1.6, but not pursued.

In the second part of this paper we study odometers of product type.

Definition 1. Let $(d_k)_{k\in\mathbb{N}}$ be a sequence of positive integers. We consider the compact metrisable space $X = \prod_{k\in\mathbb{N}} \{0, ..., d_k - 1\}$ with Borel algebra $\mathcal{B}$ on it. For $x = (x_k)_{k\in\mathbb{N}} \in X$ we set $r(x) := \inf \{k : x_k < d_k - 1\}$. Hereby, we define the transformation $T : X \to X$, $Tx = (y_k)_{k\in\mathbb{N}}$, where

$$y_k = \begin{cases} 0 & \textit{for } k < r(x) \\ x_k + 1 & \textit{for } k = r(x) \\ x_k & \textit{for } k > r(x) \end{cases}$$

- Let μ be a continuous measure on $(X, \mathcal{B})$ which is ergodic and quasi-invariant with respect to T. Then $(X, \mathcal{B}, \mu, T)$ is called a measured odometer and we will denote it by $\mathcal{O}\left((d_k)_{k\in\mathbb{N}}, \mu\right)$.
- Let $\mathcal{O}\left((d_k)_{k\in\mathbb{N}}\right)$ be a measured odometer and assume that for every $k \in \mathbb{N}$ ν_k is a probability measure on $\{0, 1, ..., d_k - 1\}$ such that the probability of every digit is positive and the product measure $\nu = \prod_{k\in\mathbb{N}} \nu_k$ is non-atomic on $\mathcal{O}\left((d_k)_{k\in\mathbb{N}}\right)$. Then ν is ergodic and quasi-invariant under T. We call $\mathcal{O}\left((d_k)_{k\in\mathbb{N}}, \nu\right)$ an odometer of product type and denote it also by $\mathcal{O}\left((d_k)_{k\in\mathbb{N}}, \{\nu_k\}\right)$. We also use the notation $\nu_k^{(i)} = \nu_k(\{i-1\})$.

Moreover, we recall the notion of orbit equivalence (also referred to as "weak-equivalence" or "Dye-equivalence"): The non-singular systems $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ are orbit-equivalent if there is an isomorphism ψ of $(X_1, \mathcal{B}_1, \mu_1)$ onto $(X_2, \mathcal{B}_2, \mu_2)$ such that $\psi\left(\left\{T_1^i x\right\}_{i\in\mathbb{Z}}\right) = \left\{T_2^i \psi(x)\right\}_{i\in\mathbb{Z}}$ almost everywhere. Y. Katznelson proved that for every odometer of product type the set of C^∞-diffeomorphisms of the circle, which are orbit equivalent to this odometer, is C^∞-dense in the set of all C^∞-diffeomorphisms with irrational rotation number ([Ka79], Theorem 2.7). We obtain such a statement in the restricted space F_α for any Liouville number α:

Theorem 2. *Let $\alpha \in \mathbb{S}^1$ be a Liouville number. For every odometer of product type $\mathcal{O} = \mathcal{O}\left((d_k)_{k\in\mathbb{N}}, \{\nu_k\}\right)$ the set of C^∞-diffeomorphisms of the circle which are orbit-equivalent to $\mathcal{O}$ is C^∞-dense in F_α.*

We point out that it is still an open problem to find a smooth realization of an odometer (cf. [FK04], Problem 7.10).

1 Proof of Theorem 1

The proof bases upon the "approximation by conjugation"-method developed by D. Anosov and A. Katok ([AK70]): We construct the desired diffeomorphisms as limits of conjugates $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \in \mathbb{Q}$, $(p_{n+1}, q_{n+1}) = 1$ and $H_n = H_{n-1} \circ h_n$ with an orientation-preserving circle diffeomorphism h_n satisfying $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$.

1.1 Denseness of $G_{\beta,\mathbf{sing}}$

Fix $\beta \in (0,1)$. We use a C^∞-function $\psi : \mathbb{R} \to [0,1]$ satisfying $\psi((-\infty, 0]) = 0$, $\psi\left(\left[\frac{1}{4}, \infty\right)\right) = 1$ and ψ is strictly monotone increasing on $\left[0, \frac{1}{4}\right]$. For any $t \in (0,1)$ we define the orientation-preserving diffeomorphism $\hat{h}_t$ of the circle as follows

$$\hat{h}_t(x) = \begin{cases} \left(1 - \psi\left(t^{-1}x\right)\right) tx + \psi\left(t^{-1}x\right) t^{-1}x & \textit{if } x \in \left[0, \frac{t}{4}\right] \\ t^{-1}x & \textit{if } x \in \left[\frac{t}{4}, \frac{t}{1+t}\right] \\ \left(1 - \psi\left(t^{-1}x - \frac{1}{t+1}\right)\right) t^{-1}x + \psi\left(t^{-1}x - \frac{1}{t+1}\right) \cdot (t(x-1)+1) & \textit{if } x \in \left[\frac{t}{1+t}, \frac{t}{1+t} + \frac{t}{4}\right] \\ t \cdot (x-1) + 1 & \textit{if } x \in \left[\frac{t}{1+t} + \frac{t}{4}, 1\right] \end{cases}$$

See figure 1 for a visualisation of such a map. In particular, $\hat{h}_t$ is a smooth joining of the two affine functions $x \mapsto t^{-1}x$ and $x \mapsto t \cdot (x-1) + 1$, which coincide at $x = \frac{t}{1+t}$. Moreover, we observe for any $r \in \mathbb{N}$

$$|||\hat{h}_t|||_r \leq C_r \cdot t^{-m(r)} \tag{1}$$

with some constant $C(r) > 0$ and an integer $m(r) \geq r - 1$ which are independent of t. The notation $|||\cdot|||_r$ is the same as in [Ma12].
We present step n of the inductive process of our construction. Hence, we have already defined the orientation-preserving diffeomorphism $H_{n-1} = h_1 \circ ... \circ h_{n-1}$ as well as the numbers $\alpha_{n-1} = \frac{p_{n-1}}{q_{n-1}} \in \mathbb{Q}$, $t_{n-1} = \frac{c_{n-1}}{d_{n-1}} \in \mathbb{Q}$ and $Q_{n-1} \in \mathbb{N}$. We put

$$Q_n = 12 \cdot d_{n-1} \cdot (d_{n-1} + c_{n-1}) \cdot Q_{n-1} \cdot q_n \tag{2}$$

and

$$t_n = Q_n^{1-\beta_n^{-1}}, \tag{3}$$

where the numbers q_n and $\beta_n \in [\beta, \beta_{n-1}]$ will be determined later (see Lemma 1.5). In particular, part 4 of Lemma 1.5 shows $t_n \in \mathbb{Q}$. Note that

$$Q_n^{-1} = \left(t_n Q_n^{-1}\right)^{\beta_n}. \tag{4}$$

Let h_n be the lift of $\hat{h}_{t_n}$ by the cyclic Q_n-fold covering map π_{Q_n} such that $\mathrm{Fix}(h_n) \neq \emptyset$. In particular, we have $h_n \circ R_{\frac{1}{Q_n}} = R_{\frac{1}{Q_n}} \circ h_n$. Since Q_n is a multiple of q_n, this yields $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$ as well as

$$\left\|H_n^{-1} - H_{n-1}^{-1}\right\|_0 = \left\|(h_n^{-1} - \mathrm{id}) \circ H_{n-1}^{-1}\right\|_0 = \left\|h_n^{-1} - \mathrm{id}\right\|_0 \leq Q_n^{-1}.$$

Then we obtain

$$\left\|H_{n+k}^{-1} - H_{n-1}^{-1}\right\|_0 \leq \sum_{l=0}^{k} \left\|H_{n+l}^{-1} - H_{n+l-1}^{-1}\right\|_0 \leq \sum_{l=0}^{k} Q_{n+l}^{-1}.$$

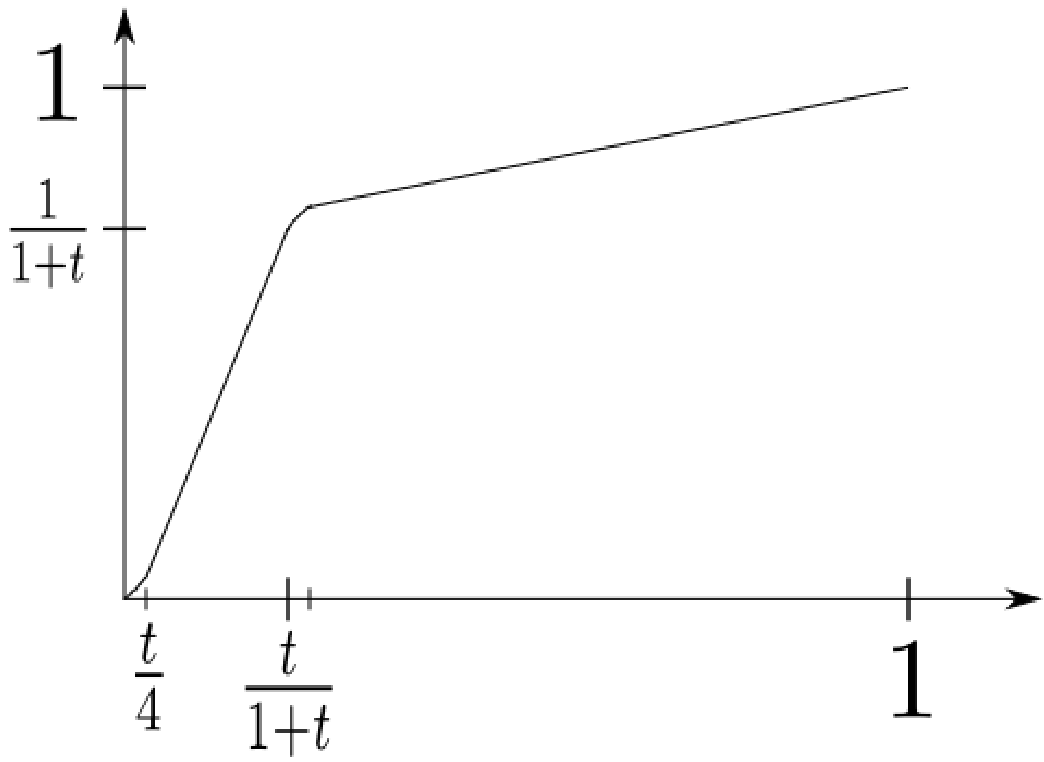

Figure 1: Qualitative shape of the function $\hat{h}_t$.

Since $\sum_{n=1}^{\infty} Q_n^{-1} < \infty$ by Lemma 1.5, $\left(H_n^{-1}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. This shows the uniform convergence of $\left(H_n^{-1}\right)_{n\in\mathbb{N}}$ to a continuous map $H^{-1}: \mathbb{S}^1 \to \mathbb{S}^1$. Additionally, H^{-1} is monotone as a uniform limit of homeomorphisms. Due to Lemma 1.6 the sequence of C^∞-diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges to a diffeomorphism f in the Diff$^\infty$-topology. Hereby, we conclude $H^{-1} \circ f = R_\alpha \circ H^{-1}$. Hence, H^{-1} is a homeomorphism.

Moreover, we introduce the intervals

$$\hat{I}_{t_n} = \left[\frac{1}{1+t_n} + \frac{t_n^2}{4}, 1\right], \ \hat{J}_{t_n} = \left[\frac{t_n}{1+t_n} + \frac{t_n}{4}, 1\right] \text{ and } \hat{K}_{t_n} = \left[\frac{t_n}{4}, \frac{7t_n}{12}\right].$$

By construction $\hat{h}_{t_n}|_{\hat{J}_{t_n}}$ is an affine transformation with slope t_n. Due to $t_n < 1$ we have $\hat{I}_{t_n} \subset \hat{J}_{t_n}$. Additionally, we observe $\hat{h}_{t_n}\left(\hat{J}_{t_n}\right) = \hat{I}_{t_n}$ and $\hat{K}_{t_n} \subset \left[\frac{t_n}{4}, \frac{t_n}{1+t_n}\right]$.
Finally, we define $I_n = \pi_{Q_n}^{-1}\left(\hat{I}_{t_n}\right)$, $J_n = \pi_{Q_n}^{-1}\left(\hat{J}_{t_n}\right)$ as well as $K_n = \pi_{Q_n}^{-1}\left(\hat{K}_{t_n}\right)$. By direct computation, $m\left(J_n\right) = \frac{1}{1+t_n} - \frac{t_n}{4} \geq 1 - 2t_n$, $m\left(I_n\right) = \frac{t_n}{1+t_n} - \frac{t_n^2}{4} \leq t_n$ (where m stands for the Lebesgue measure) and every component of K_n has length $\frac{t_n}{3Q_n}$.
Now we are able to prove the singularity of H by an approach similar to [Ma12], section 3:

Lemma 1.1. *H is singular.*

Proof. Let $C_n = \bigcap_{i=1}^n J_i$ and $C = \bigcap_{i=1}^\infty J_i$. By construction of the number Q_n any component of J_{n-1} is a $\left(Q_n^{-1}\mathbb{Z}\right)/\mathbb{Z}$-interval. Then

$$m\left(C_n\right) \geq \prod_{i=1}^n \left(1-2t_i\right).$$

Note that $\prod_{n=1}^\infty (1-a_n)$, $a_n \geq 0$, converges to a positive number if and only if $\sum_{n=1}^\infty a_n$ converges ([Kn64], section 28, Theorem 4). Since

$$\sum_{n=1}^\infty t_n = \sum_{n=1}^\infty Q_n^{1-\beta_n^{-1}} \leq \sum_{n=1}^\infty Q_n^{1-\beta_{n-1}^{-1}} < \infty$$

by part 3 of Lemma 1.5, we obtain $m(C) > 0$. This yields

$$\mu_f(H(C)) = (H_* m)(H(C)) = m(C) > 0.$$

By construction of h_j with the aid of the Q_j-fold covering map $\left(Q_{n+1}^{-1}\mathbb{Z}\right)/\mathbb{Z}$ is pointwise fixed under h_j, $j > n$. Since any component of J_n is a $\left(Q_{n+1}^{-1}\mathbb{Z}\right)/\mathbb{Z}$-interval, we get $h_j(J_n) = J_n$ for $j > n$. Then we obtain for any $j > n$:

$$H_j(J_n) = H_n(J_n) = H_{n-1}h_n(J_n) = H_{n-1}(I_n),$$

which yields $J_n = H_j^{-1}H_{n-1}(I_n)$, where $H_0 = \mathrm{id}$. This shows that the uniform limit H^{-1} of H_j^{-1} satisfies $J_n = H^{-1}\left(H_{n-1}(I_n)\right)$ for any $n \in \mathbb{N}$. Hereby, we observe

$$H\left(C_n\right) = H\left(\bigcap_{i=1}^n J_i\right) = \bigcap_{i=1}^n H\left(J_i\right) = \bigcap_{i=1}^n H_{i-1}\left(I_i\right).$$

In order to have $H_{i-1}\left(A\right) \subset H_{i-2}\left(I_{i-1}\right)$ a set $A \subset \mathbb{S}^1$ has to satisfy $h_{i-1}\left(A\right) \subset I_{i-1}$ which implies the condition $A \subset J_{i-1}$. Since $m(I_i) \leq t_i$ and the slope of $h_{i-1}|_{J_{i-1}}$ is equal to t_{i-1}, this yields $m\left(H\left(C_n\right)\right) \leq \prod_{i=1}^{n-1} t_i$ which converges to 0 as $n \to \infty$. Therefore, $m(H(C)) = 0$.
Finally, we note that μ_f is not equivalent to m because $\mu_f(H(C)) > 0$ and $m(H(C)) = 0$. Hence, H is a singular map. □

In the next steps, we examine the Hölder continuity of H:

Lemma 1.2. *H is not d-Hölder for any $d \in (\beta, 1)$.*

Proof. For any component $\tilde{K}_i$ of K_i there is a component $\tilde{K}_{i+1}$ of K_{i+1} such that $\tilde{K}_{i+1} \subset \tilde{K}_i$. This proves the existence of a component $\tilde{K}_n$ which is contained in $\bigcap_{i=1}^n K_i$. By construction, $H_n|_{\tilde{K}_n}$ is an affine transformation of slope $t_1^{-1} \cdot \dots \cdot t_n^{-1}$. In the following, we denote $\tilde{K}_n = [x', y']$. In particular, we have $|y' - x'| = 3^{-1} t_n Q_n^{-1}$.
Moreover, we define $x, y \in \mathbb{S}^1$ by $H^{(n+1)}(x) = x'$ as well as $H^{(n+1)}(y) = y'$ using the notation $H^{(n+1)} = H_n^{-1}H$. Its inverse $\left(H^{(n+1)}\right)^{-1}$ is the uniform limit of $h_{n+m}^{-1} \circ \dots \circ h_{n+2}^{-1} \circ h_{n+1}^{-1}$ as $m \to \infty$. Since $\sum_{i=n+1}^\infty Q_i^{-1} \leq 3^{-1} t_n Q_n^{-1}$ for any $n \in \mathbb{N}$ by Lemma 1.5 and the maps h_i are

Q_i^{-1}-cyclic, we have

$$\begin{aligned}
|x-y| &= \left|\left(H^{(n+1)}\right)^{-1}(x') - \left(H^{(n+1)}\right)^{-1}(y')\right| \\
&\leq \left|\left(H^{(n+1)}\right)^{-1}(x') - x'\right| + |x'-y'| + \left|y' - \left(H^{(n+1)}\right)^{-1}(y')\right| \\
&\leq \sum_{i=n+1}^{\infty} Q_i^{-1} + |x'-y'| + \sum_{i=n+1}^{\infty} Q_i^{-1} \\
&\leq 3^{-1} t_n Q_n^{-1} + |x'-y'| + 3^{-1} t_n Q_n^{-1} \\
&= 3\,|x'-y'|\,.
\end{aligned}$$

With the aid of $|y'-x'| = 3^{-1} t_n Q_n^{-1}$ we estimate

$$\begin{aligned}
|H(x)-H(y)| &= |H_n(x') - H_n(y')| = t_1^{-1}\cdot\dots\cdot t_n^{-1}\cdot|x'-y'| = t_1^{-1}\cdot\dots\cdot t_n^{-1}\cdot 3^{-1} t_n Q_n^{-1} \\
&= 3^{-1} t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot Q_n^{-1}
\end{aligned}$$

as well as

$$|x-y|^d \leq 3^d\,|x'-y'|^d = 3^d\left(3^{-1} t_n Q_n^{-1}\right)^d = t_n^d Q_n^{-d}.$$

Using equation 3 both estimates together yield in case of $\beta_n < d$ (which is fulfilled for sufficiently large $n\in\mathbb{N}$ due to $\beta_n \to \beta < d$):

$$\begin{aligned}
\frac{|H(x)-H(y)|}{|x-y|^d} &\geq 3^{-1} t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot Q_n^{d-1}\cdot t_n^{-d} \\
&= 3^{-1} t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot Q_n^{d-1}\cdot Q_n^{\beta_n^{-1}d-d} \\
&= 3^{-1} t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot Q_n^{\beta_n^{-1}d-1} \\
&\geq 3^{-1} t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}.
\end{aligned}$$

Since this expression can be arbitrarily large, we conclude that H cannot be d-Hölder for any $d\in(\beta,1)$. □

Lemma 1.3. *H^{-1} is β-Hölder.*

Proof. For any pair of $x,y\in\mathbb{S}^1$, $x\neq y$, there is $n\in\mathbb{N}$ such that $t_{n+1}Q_{n+1}^{-1} \leq |x-y| \leq t_n Q_n^{-1}$. Since the Lipschitz constant of h_i^{-1} is t_i^{-1}, we have

$$\begin{aligned}
\left|H_n^{-1}(x) - H_n^{-1}(y)\right| &\leq t_1^{-1}\cdot\dots\cdot t_n^{-1}\cdot|x-y| = t_1^{-1}\cdot\dots\cdot t_n^{-1}\cdot|x-y|^{1-\beta}\cdot|x-y|^{\beta} \\
&\leq t_1^{-1}\cdot\dots\cdot t_n^{-1}\cdot\left(t_n Q_n^{-1}\right)^{1-\beta}\cdot|x-y|^{\beta} \\
&= t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot\left(Q_n^{1-\beta_n^{-1}}\right)^{-\beta}\cdot Q_n^{\beta-1}\cdot|x-y|^{\beta} \\
&= t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot Q_n^{-1+\beta\beta_n^{-1}}\cdot|x-y|^{\beta}
\end{aligned}$$

By Lemma 1.5 this shows

$$\left|H_n^{-1}(x) - H_n^{-1}(y)\right| \leq |x-y|^{\beta}\,. \tag{5}$$

There are two possible cases

- Case 1: $\left|H_n^{-1}(x) - H_n^{-1}(y)\right| \geq Q_{n+1}^{-1}$
  Since h_{n+1}^{-1} is Q_{n+1}^{-1}-cyclic, we get in this case $\left|H_{n+1}^{-1}(x) - H_{n+1}^{-1}(y)\right| \leq 2\left|H_n^{-1}(x) - H_n^{-1}(y)\right|$. With the aid of Lemma 1.5 we see that the numbers Q_i, $i > n+1$, grow fast enough such that
  $$\left|H^{-1}(x) - H^{-1}(y)\right| \leq 3\left|H_n^{-1}(x) - H_n^{-1}(y)\right| \leq 3\,|x-y|^\beta$$
  using equation (5) in the last step.

- Case 2: $\left|H_n^{-1}(x) - H_n^{-1}(y)\right| < Q_{n+1}^{-1}$
  Once again, we exploit the fact that h_{n+1}^{-1} is Q_{n+1}^{-1}-cyclic. In the case under consideration, this yields $\left|H_{n+1}^{-1}(x) - H_{n+1}^{-1}(y)\right| \leq Q_{n+1}^{-1}$. With the aid of equation 4 we get
  $$\left|H^{-1}(x) - H^{-1}(y)\right| \leq 2Q_{n+1}^{-1} \leq 2\left(t_{n+1}Q_{n+1}^{-1}\right)^\beta \leq 2\,|x-y|^\beta$$

Hence, H^{-1} is β-Hölder. □

Lemma 1.4. *H is β-Hölder.*

Proof. As above, for any pair of $x, y \in \mathbb{S}^1$, $x \neq y$, there is $n \in \mathbb{N}$ such that $t_{n+1}Q_{n+1}^{-1} \leq |x-y| \leq t_n Q_n^{-1}$. Let $x' = H^{(n+2)}(x)$ and $y' = H^{(n+2)}(y)$. Recall $H^{(n+2)} = H_{n+1}^{-1}H$. Since H is the uniform limit of H_n and the maps h_i are Q_i^{-1}-cyclic and the numbers Q_i, $i > n+1$, are sufficiently large due to Lemma 1.5, we have $|x'-y'| \leq 2\,|x-y|$. Once again, we have to examine two cases:

- Case 1: $|x'-y'| \geq Q_{n+1}^{-1}$
  Since h_{n+1} is Q_{n+1}^{-1}-cyclic, we get in this case $|x''-y''| \leq 2\,|x'-y'|$ for $x'' = h_{n+1}(x')$ and $y'' = h_{n+1}(y')$. Since the Lipschitz constant of h_i is t_i^{-1}, we obtain by the same calculations as in the first case of the previous Lemma
  $$\begin{aligned}|H(x) - H(y)| = |H_n(x'') - H_n(y'')| &\leq t_1^{-1} \cdot \dots \cdot t_n^{-1} \cdot |x''-y''| \\ &\leq t_1^{-1} \cdot \dots \cdot t_n^{-1} \cdot 2\,|x'-y'| \\ &\leq t_1^{-1} \cdot \dots \cdot t_n^{-1} \cdot 4\,|x-y| \\ &\leq 4\,|x-y|^\beta .\end{aligned}$$

- Case 2: $|x'-y'| < Q_{n+1}^{-1}$
  Since h_{n+1} is Q_{n+1}^{-1}-cyclic, we have
  $$|x''-y''| = |h_{n+1}(x') - h_{n+1}(y')| \leq Q_{n+1}^{-1} = \left(t_{n+1}Q_{n+1}^{-1}\right)^{\beta_{n+1}} \leq |x-y|^{\beta_{n+1}}$$
  Hereby, we conclude
  $$\begin{aligned}|H(x) - H(y)| &= |H_n(x'') - H_n(y'')| \\ &\leq t_1^{-1} \cdot \dots \cdot t_n^{-1} \cdot |x''-y''|^{1-\beta\beta_{n+1}^{-1}} \cdot |x''-y''|^{\beta\beta_{n+1}^{-1}} \\ &\leq t_1^{-1} \cdot \dots \cdot t_n^{-1} \cdot \left(Q_{n+1}^{-1}\right)^{1-\beta\beta_{n+1}^{-1}} \cdot |x''-y''|^{\beta\beta_{n+1}^{-1}} \\ &\leq |x-y|^\beta\end{aligned}$$
  using Lemma 1.5, part 4, in the last step.

Hence, H is β-Hölder. □

Finally, we want to prove convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Diff}^\infty\left(\mathbb{S}^1\right)$. For this purpose, we deduce the subsequent statement.

Lemma 1.5. *Let $(l_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with $\sum_{n=1}^{\infty}\frac{1}{l_n}<\infty$ and C_{l_n} be the constants from [Ma12], Lemma 2.4. For any Liouvillean number α there are sequences $\alpha_n=\frac{p_n}{q_n}$ of rational numbers and $(\beta_n)_{n\in\mathbb{N}}$ of real numbers, such that $\beta_n \searrow \beta$ and the following conditions are satisfied:*

1. *For every $n\in\mathbb{N}$:*
$$|\alpha-\alpha_n|<\frac{1}{2^{n+1}\cdot l_n\cdot C_{l_n}\cdot |||H_n|||_{l_n+1}^{l_n+1}}.$$
2. *For every $n\in\mathbb{N}$:*
$$\sum_{i=n+1}^{\infty}Q_i^{-1}\leq 3^{-1}t_nQ_n^{-1}$$
3. *For every $n\in\mathbb{N}$:*
$$t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot Q_n^{-1+\beta\beta_n^{-1}}\leq 1 \quad \text{and} \quad Q_n^{1-\beta_{n-1}^{-1}}\leq\frac{1}{n^2}.$$
4. *$t_n\in\mathbb{Q}$ for every $n\in\mathbb{N}$.*

Proof. Since the numbers t_i, $i<n$, are independent of q_n and $1>\beta_{n-1}>\beta$, we can demand the number q_n to be sufficiently large such that

$$t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot q_n^{\frac{1}{2}\left(1-\frac{\beta_{n-1}}{\beta}\right)}\leq 1 \tag{6}$$

as well as
$$q_n^{1-\beta_{n-1}^{-1}}\leq\frac{1}{n^2}.$$

Additionally, we can satisfy the second property of the Lemma by choosing the numbers q_i sufficiently large in each step.
By equations (1) and (3) we have

$$\begin{aligned}
|||h_n|||_r &\leq C_r\cdot Q_n^{r-1}\cdot\left(Q_n^{\beta_n^{-1}-1}\right)^{m(r)}\leq C_r\cdot Q_n^{\beta^{-1}\cdot m(r)}\\
&= C_r\cdot\left(12\cdot d_{n-1}\cdot(d_{n-1}+c_{n-1})\cdot Q_{n-1}\right)^{\beta^{-1}m(r)}\cdot q_n^{\beta^{-1}m(r)}\\
&\leq C_{r,n-1}\cdot q_n^{\beta^{-1}m(r)}
\end{aligned}$$

for any $r\in\mathbb{N}$ due to the condition $\beta_n\geq\beta$. Using [Ma12], Lemma 2.3, we obtain

$$|||H_n|||_r=|||H_{n-1}\circ h_n|||_r\leq\tilde{C}_r\cdot|||H_{n-1}|||_r^r\cdot|||h_n|||_r^r\leq\hat{C}_{r,n-1}\cdot q_n^{\beta^{-1}m(r)\cdot r},$$

where $\hat{C}_{r,n-1}$ is a constant independent of q_n. In particular, we can demand $q_n\geq\hat{C}_{l_n+1,n-1}$. Then we get
$$|||H_n|||_{l_n+1}\leq q_n^{\beta^{-1}m(l_n+1)\cdot(l_n+1)+1}\leq q_n^{A_n}$$

using the notation $A_n = \lceil \beta^{-1} m(l_n+1)\cdot(l_n+1)+1\rceil$.
Since α is a Liouvillean number, we find a rational number $\alpha_n = \frac{p_n}{q_n}$, p_n, q_n relatively prime, satisfying the above restrictions and

$$|\alpha-\alpha_n| = \left|\alpha - \frac{p_n}{q_n}\right| < \frac{|\alpha-\alpha_{n-1}|}{2^{n+1}\cdot l_n \cdot C_{l_n}\cdot q_n^{A_n\cdot(l_n+1)}} \leq \frac{1}{2^{n+1}\cdot l_n\cdot C_{l_n}\cdot |||H_n|||^{l_n+1}_{l_n+1}}.$$

After the number q_n is determined with respect to these restrictions we can choose a number $\beta_n \in \left[\beta + \frac{\beta_{n-1}-\beta}{2}, \beta_{n-1}\right)$ such that $t_n = Q_n^{1-\beta_n^{-1}} \in \mathbb{Q}$. Then we have

$$\begin{aligned} -1+\beta\beta_n^{-1} &= \beta_n^{-1}\cdot(-\beta_n+\beta) \leq \beta^{-1}\cdot\left(-\left(\beta+\frac{\beta_{n-1}-\beta}{2}\right)+\beta\right) \\ &= \beta^{-1}\frac{\beta-\beta_{n-1}}{2} = \frac{1}{2}\left(1-\frac{\beta_{n-1}}{\beta}\right) < 0. \end{aligned}$$

By condition 6 this yields

$$t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot Q_n^{-1+\beta\beta_n^{-1}} < t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot q_n^{-1+\beta\beta_n^{-1}} \leq t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot q_n^{\frac{1}{2}\left(1-\frac{\beta_{n-1}}{\beta}\right)} \leq 1.$$

□

The previous Lemma shows that the requirements of the following convergence result deduced in [Ku16], Lemma 5.8, are fulfilled.

Lemma 1.6. *Let $\varepsilon > 0$ be arbitrary and $(l_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers satisfying $\sum_{n=1}^{\infty}\frac{1}{l_n} < \varepsilon$. Furthermore, we assume that in our constructions the following conditions are fulfilled:*

$$|\alpha-\alpha_1| < \varepsilon \qquad \textit{and} \qquad |\alpha-\alpha_n| \leq \frac{1}{2\cdot l_n\cdot C_{l_n}\cdot |||H_n|||^{l_n+1}_{l_n+1}} \textit{ for every } n\in\mathbb{N},$$

where C_{l_n} are the constants from [Ma12], Lemma 2.4. Then the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges in the Diff$^\infty$-topology to a smooth diffeomorphism f, for which $d_\infty(f, R_\alpha) < 3\cdot\varepsilon$ holds.

Hence, the rotation R_α is contained in the C^∞-closure of $G_{\beta,\text{sing}}$. Since this closure is invariant under conjugation by any $h \in F$ and O_α is C^∞-dense in F_α ([Yo95]), we obtain the C^∞-denseness of $G_{\beta,\text{sing}}$ in F_α by the same reasoning as in [Ma12], section 2.1.

1.2 Denseness of $G_{\beta,\mathbf{ac}}$

Fix $\beta\in(0,1)$. We slightly modify the construction of the map $\hat{h}_t$ from the previous chapter. Once again, we use a C^∞-map $\psi:\mathbb{R}\to[0,1]$ satisfying $\psi((-\infty,0]) = 0$, $\psi\left(\left[\frac{1}{4},\infty\right)\right) = 1$ and ψ is strictly monotone increasing on $\left[0,\frac{1}{4}\right]$. Then, for any $t\in(0,1)$ we define the orientation-preserving diffeomorphism $\breve{h}_t : [0,1]\to[0,1]$ as follows

$$\breve{h}_t(x) = \begin{cases} \left(1-\psi\left(t^{-1}x\right)\right)x + \psi\left(t^{-1}x\right)t^{-1}x & \textit{if } x\in\left[0,\frac{t}{4}\right] \\ t^{-1}x & \textit{if } x\in\left[\frac{t}{4},\frac{t}{1+t}\right] \\ \left(1-\psi\left(t^{-1}x-\frac{1}{t+1}\right)\right)t^{-1}x+\psi\left(t^{-1}x-\frac{1}{t+1}\right)\cdot(t(x-1)+1) & \textit{if } x\in\left[\frac{t}{1+t},\frac{t}{1+t}+\frac{t}{4}\right] \\ t\cdot(x-1)+1 & \textit{if } x\in\left[\frac{t}{1+t}+\frac{t}{4},1-\frac{t}{4}\right] \\ \left(1-\psi\left(t^{-1}\left(x-1+\frac{t}{4}\right)\right)\right)\cdot(t(x-1)+1)+\psi\left(t^{-1}\left(x-1+\frac{t}{4}\right)\right)x & \textit{if } x\in\left[1-\frac{t}{4},1\right] \end{cases}$$

Note that $\breve{h}_t$ coincides with the identity in a neighbourhood of the boundary. Using the maps $C_n : \left[0, \frac{1}{2^{n+1}}\right] \to [0,1]$, $C_n(x) = 2^{n+1} \cdot x$, we construct the orientation-preserving circle diffeomorphism h_{t_n} as follows:

$$h_{t_n}(x) = \begin{cases} C_n^{-1} \circ \breve{h}_{t_n} \circ C_n(x) & \textit{if } x \in \left[0, \frac{1}{2^{n+1}}\right] \\ x & \textit{if } x \in \left[\frac{1}{2^{n+1}}, 1\right] \end{cases},$$

where we define the numbers $Q_n = 2^n \cdot 12 \cdot d_{n-1} \cdot (d_{n-1} + c_{n-1}) \cdot Q_{n-1} \cdot q_n$ and t_n as in the previous section. This time, we demand the additional requirement

$$t_n \leq 2^{-(n+2)} \tag{7}$$

which can be satisfied by choosing q_n sufficiently large in the proof of Lemma 1.5.
Let h_n be the lift of h_{t_n} by the cyclic Q_n-fold covering map π_{Q_n} such that $\mathrm{Fix}(h_n) \neq \emptyset$. By the same reasoning as above, the sequence $\left(H_n^{-1}\right)_{n\in\mathbb{N}}$ converges to a homeomorphism H^{-1}.
First of all, we prove the absolute continuity of H by the same method as in [Ma12], section 4:

Lemma 1.7. *H is absolutely continuous.*

Proof. We introduce the sets

$$\hat{L}_n = \left[2^{-(n+1)}, 1\right] \quad \text{and} \quad L_n = \pi_{Q_n}^{-1}\left(\hat{L}_n\right).$$

According to our construction h_n is the identity on L_n. Let $X = \bigcap_{n=1}^{\infty} L_n$. Then we have

$$m(X) \geq 1 - \sum_{n=1}^{\infty} m\left(\mathbb{S}^1 \setminus L_n\right) = 1 - \sum_{n=1}^{\infty} 2^{-(n+1)} = \frac{1}{2}.$$

Since H is the identity on the positive measure set X, we have for any Borel set B $\mu_f(B \cap X) = m(B \cap X)$ and $\mu_f(X) = m(X) > 0$.
Assume that μ_f is not equivalent to m. Then μ_f is singular to m and there is a Borel set $B \subset \mathbb{S}^1$ such that $m(B) = 1$ and $\mu_f(B) = 0$. But then we obtain the contradiction $m(B \cap X) = m(X) > 0$ but $\mu_f(B \cap X) \leq \mu_f(B) = 0$.
Hence, H is absolutely continuous. $\square$

We start to examine the Hölder-continuity of H.

Lemma 1.8. *H is not d-Hölder for any $d \in (\beta, 1)$.*

Proof. Let

$$K_{t_n} = \left[\frac{t_n}{4 \cdot 2^{n+1}}, \frac{7t_n}{12 \cdot 2^{n+1}}\right] \quad \text{and} \quad K_n = \pi_{Q_n}^{-1}\left(K_{t_n}\right).$$

For any component $\tilde{K}_i$ of K_i there is a component $\tilde{K}_{i+1}$ of K_{i+1} such that $\tilde{K}_{i+1} \subset \tilde{K}_i$. This proves the existence of a component $\tilde{K}_n$ which is contained in $\bigcap_{i=1}^{n} K_i$. By construction, $H_n|_{\tilde{K}_n}$ is an affine transformation of slope $t_1^{-1} \cdot \dots \cdot t_n^{-1}$. In the following, we denote $\tilde{K}_n = [x', y']$. In particular, we have $|y' - x'| = 3^{-1} t_n Q_n^{-1} \cdot 2^{-(n+1)}$.
According to this we modify the second requirement of Lemma 1.5 as follows $\sum_{i=n+1}^{\infty} Q_i^{-1} \leq 3^{-1} t_n Q_n^{-1} \cdot 2^{-(n+1)}$. Then we get for $x, y \in \mathbb{S}^1$ defined by $H^{(n+1)}(x) = x'$ as well as $H^{(n+1)}(y) = y'$: $|x - y| \leq 3\,|x' - y'|$. With the aid of $|y' - x'| = 3^{-1} t_n Q_n^{-1} \cdot 2^{-(n+1)}$ we estimate

$$|H(x) - H(y)| = |H_n(x') - H_n(y')| = t_1^{-1} \cdot \dots \cdot t_n^{-1} \cdot |x' - y'| = t_1^{-1} \cdot \dots \cdot t_{n-1}^{-1} \cdot 3^{-1} Q_n^{-1} \cdot 2^{-(n+1)}$$

as well as

$$|x-y|^d \leq 3^d\,|x'-y'|^d = 3^d\left(3^{-1}t_nQ_n^{-1}2^{-(n+1)}\right)^d = t_n^dQ_n^{-d}\cdot 2^{-(n+1)d}.$$

Using equation 3 both estimates together yield in case of $\beta_n < d$ (which is fulfilled for sufficiently large $n\in\mathbb{N}$ due to $\beta_n\to\beta<d$):

$$\begin{aligned}\frac{|H(x)-H(y)|}{|x-y|^d} &\geq 3^{-1}t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot Q_n^{d-1}\cdot t_n^{-d}\cdot 2^{(d-1)\cdot(n+1)}\\ &> 3^{-1}t_1^{-1}\cdot\dots\cdot t_{n-1}^{-1}\cdot Q_n^{\beta_n^{-1}d-1}\cdot 2^{-(n+1)}\\ &\geq 3^{-1}t_1^{-1}\cdot\dots\cdot t_{n-2}^{-1}.\end{aligned}$$

(where we used the additional requirement $t_{n-1}\leq 2^{-(n+1)}$ from equation 7 in the last step). Since this expression can be arbitrarily large, we conclude that H cannot be d-Hölder for any $d\in(\beta,1)$. □

Lemma 1.9. *H and H^{-1} are β-Hölder.*

Proof. Since the Lipschitz constants of h_i and h_i^{-1} are equal to t_i^{-1}, we can copy the proofs of Lemma 1.3 and Lemma 1.4. □

Then we conclude the C^∞-denseness of $G_{\beta,\mathrm{ac}}$ by the same reasoning as in the previous section.

2 Proof of Theorem 2

For any $n\in\mathbb{N}$ we use a C^∞-map $\psi_n:\mathbb{R}\to[0,1]$ satisfying $\psi_n\left((-\infty,0]\right)=0$, $\psi_n\left(\left[\frac{1}{4\cdot 2^n},\infty\right)\right)=1$ and ψ is strictly monotone increasing on $\left[0,\frac{1}{4\cdot 2^n}\right]$. We define the orientation-preserving diffeomorphism $\tilde{h}_n$ of the circle as follows

- If $x\in\left[0,\frac{1}{4\cdot 2^n\cdot d_n}\right]$

$$\tilde{h}_n(x) = \left(1-\psi_n\left(d_nx\right)\right)d_n\nu_n^{(d_n)}x + \psi_n\left(d_nx\right)d_n\nu_n^{(1)}x$$

- For $i=0,\dots,d_n-1$ and $x\in\left[\frac{i}{d_n}+\frac{1}{4\cdot 2^n\cdot d_n},\frac{i+1}{d_n}\right]$:

$$\tilde{h}_n(x) = d_n\nu_n^{(i+1)}\cdot\left(x-\frac{i}{d_n}\right)+\sum_{k=0}^{i}\nu_n^{(k)}$$

- For $i=1,\dots,d_n-1$ and $x\in\left[\frac{i}{d_n},\frac{i}{d_n}+\frac{1}{4\cdot 2^n\cdot d_n}\right]$:

$$\begin{aligned}\tilde{h}_n(x) = &\left(1-\psi_n\left(d_n\cdot\left(x-\frac{i}{d_n}\right)\right)\right)\cdot\left(d_n\nu_n^{(i)}\left(x-\frac{i}{d_n}\right)+\sum_{k=0}^{i-1}\nu_n^{(k)}\right)\\ &+\psi_n\left(d_n\cdot\left(x-\frac{i}{d_n}\right)\right)\cdot\left(d_n\nu_n^{(i+1)}\left(x-\frac{i}{d_n}\right)+\sum_{k=0}^{i}\nu_n^{(k)}\right)\end{aligned}$$

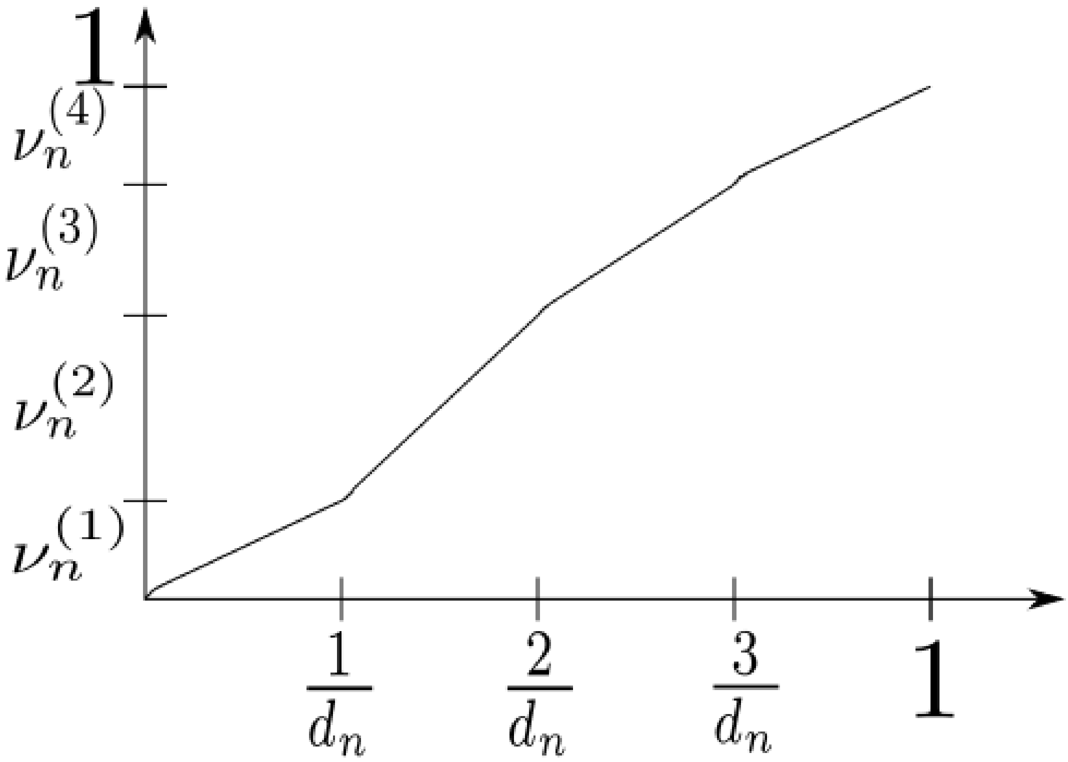

Figure 2: Qualitative shape of the function $\tilde{h}_n$.

using the notation $\nu_n^{(0)} = 0$. See figure 2 for a visualisation of such a map.

We present step n of the inductive process of our construction. Hence, we have already defined the orientation-preserving diffeomorphism $H_{n-1} = h_1 \circ \cdots \circ h_{n-1}$ and the numbers $\alpha_{n-1} = \frac{p_{n-1}}{q_{n-1}} \in \mathbb{Q}$.
Let h_n be the lift of $\tilde{h}_n$ by the cyclic q_n-fold covering map π_{q_n} such that $\mathrm{Fix}(h_n) \neq \emptyset$ where the number q_n will be determined later (see Lemma 2.1). In particular, we have $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$ and for any $l \in \mathbb{N}$

$$|||h_n|||_l \leq C_{n,l} \cdot q_n^{l-1}, \tag{8}$$

where the constant $C_{n,l}$ is independent of q_n. Then we can prove a statement analogous to Lemma 1.5:

Lemma 2.1. *Let $(l_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with $\sum_{n=1}^{\infty} \frac{1}{l_n} < \infty$ and C_{l_n} be the constants from [Ma12], Lemma 2.4. For any Liouvillean number α there are sequences $\alpha_n = \frac{p_n}{q_n}$ of rational numbers, such that the following conditions are satisfied:*

1. *For every* $n \in \mathbb{N}$:
$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} \cdot l_n \cdot C_{l_n} \cdot |||H_n|||_{l_n+1}^{l_n+1}}.$$

2. *For every* $n \in \mathbb{N}$:
$$|\alpha - \alpha_n| < \frac{1}{4 \cdot 2^{n+1} d_n q_n^2}.$$

As in the previous chapter we prove the denseness of constructed C^∞-diffeomorphisms and observe that the sequence $\left(H_n^{-1}\right)_{n\in\mathbb{N}}$ converges uniformly to a homeomorphism $H^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$. In the following, we will consider systems induced on a set A:

Definition 2. Let $(X, \mathcal{B}, \mu, T)$ be a non-singular system and $A \in \mathcal{B}$ be a set of positive measure. Then T induces a mapping T^A on A by $T^A(x) = T^{n(x)}(x)$ where $n(x)$ is the smallest positive integer n for which $T^n(x) \in A$. Then $(A, \mathcal{B}^A, \mu^A, T^A)$ is called the induced system where $\mathcal{B}^A$ is the algebra of $\mathcal{B}$-subsets of A and μ^A is the normalized restriction of μ to $\mathcal{B}^A$.

On each interval $\Delta_1^{(i)} = \left[\frac{i}{d_1 q_1} + \frac{1}{2 \cdot 2 d_1 q_1}, \frac{i+1}{d_1 q_1} - \frac{1}{4 \cdot 2 d_1 q_1}\right] \subset \mathbb{S}^1$, $i = 0, \dots, d_1 q_1 - 1$, we choose $\lfloor \frac{q_2 \cdot \left(1 - \frac{3}{8}\right)}{d_1 q_1} \rfloor$ intervals of type $\left[\frac{t}{q_2}, \frac{t+1}{q_2}\right]$ contained in $\Delta_1^{(i)}$ completely. In case $\alpha_2 < \alpha_1$ the rightmost of these contained in $\Delta_1^{(0)}$ is labelled by $B_1 = \left[\frac{t_1}{q_2}, \frac{t_1+1}{q_2}\right]$ (in case $\alpha_2 > \alpha_1$ the leftmost of these contained in $\Delta_1^{(0)}$ is labelled by B_1 analogously). The union of these chosen intervals $\left[\frac{t}{q_2}, \frac{t+1}{q_2}\right]$ will be denoted by F_1 and they will be numbered serially by the dynamical order determined by the induced map $f_1^{h_1(F_1)}$, the first one being $h_1(B_1)$. We note that there are $N_1 := \lfloor \frac{q_2 \cdot \left(1 - \frac{3}{8}\right)}{d_1 q_1} \rfloor \cdot d_1 q_1$ numbers on the first level.

We proceed by an inductive process describing the constructions on the level $n \geq 2$ under the induction assumptions that $B_{n-1} = \left[\frac{t_{n-1}}{q_n}, \frac{t_{n-1}+1}{q_n}\right]$ is the base level of level $n-1$ and that there are
$$N_{n-1} = q_1 \cdot \prod_{i=1}^{n-1} \lfloor \frac{q_{i+1} \cdot \left(1 - \frac{3}{4 \cdot 2^i}\right)}{d_i q_i} \rfloor \cdot d_i$$
chosen intervals $\left(f_{n-1}^{H_{n-1}(F_{n-1})}\right)^l \left(H_{n-1}\left(B_{n-1}\right)\right)$ in $H_{n-1}(F_{n-1})$.

We consider $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ with $\alpha_{n+1} > \alpha_n$ (the proof in the case $\alpha_{n+1} < \alpha_n$ is similar). Let $\Delta_n^{(t_{n-1},i)} = \left[\frac{t_{n-1}}{q_n} + \frac{i}{d_n q_n} + \frac{1}{2 \cdot 2^n \cdot d_n q_n}, \frac{t_{n-1}}{q_n} + \frac{i+1}{d_n q_n} - \frac{1}{4 \cdot 2^n \cdot d_n q_n}\right] \subset B_{n-1}$ for $i = 0, \dots, d_n - 1$. In each of these we choose $\lfloor \frac{q_{n+1} \cdot \left(1 - \frac{3}{4 \cdot 2^n}\right)}{d_n q_n} \rfloor$ intervals $\left[\frac{s}{q_{n+1}}, \frac{s+1}{q_{n+1}}\right]$ contained in it and denote their union by $F_n^{(1,i)}$. The collective union of these chosen $\frac{1}{q_{n+1}}$-intervals is called $F_n^{(1)}$ and the leftmost of all of these is labelled by $B_n = \left[\frac{t_n}{q_{n+1}}, \frac{t_n+1}{q_{n+1}}\right]$.
On each of the other chosen intervals $\left[\frac{t}{q_n}, \frac{t+1}{q_n}\right] \subset F_{n-1} \setminus B_{n-1}$ we consider the intervals $\Delta_n^{(t,i)} = \left[\frac{t}{q_n} + \frac{i}{d_n q_n} + \frac{1}{4 \cdot 2^n \cdot d_n q_n}, \frac{t}{q_n} + \frac{i+1}{d_n q_n}\right]$ of size slightly bigger than $\Delta_n^{(t_{n-1},i)}$ and on each of these we chose all possible intervals $\left[\frac{s}{q_{n+1}}, \frac{s+1}{q_{n+1}}\right]$ contained in it. Let $F_n^{(2)}$ denote their union. Moreover,

we put $\tilde{F_n} = F_n^{(1)} \cup F_n^{(2)}$. Then we consider the iterates of $H_n(B_n)$ under $f_n^{H_n(\tilde{F_n})}$. Put

$$l_0 = 0,\ l_m = \min\left\{ l \geq l_{m-1} + N_{n-1} \;\middle|\; \left(f_n^{H_n(\tilde{F_n})}\left(H_n(B_n)\right)\right)^l \subset H_n\left(F_n^{(1)}\right)\right\} \text{ for } m \geq 1.$$

All the iterates $\left(f_n^{H_n(\tilde{F_n})}\right)^i (H_n\left(B_n\right))$, $i = l_m, \dots, l_m + N_{n-1} - 1$ are numbered with "m" on the n-th level and their union is denoted by $B_m^{(n)}$. Note that $R^l_{\alpha_{n+1}}(B_n) \subset \Delta_n^{(t_{n-1},i)}$ implies $R^{l+k}_{\alpha_{n+1}}\left(B_n\right) \subset F_n^{(2)} \cap \bigcup_{t\in\mathbb{Z}} \Delta_n^{(t,i)}$ for every $k = 1, \dots, N_{n-1} - 1$ because

$$N_{n-1} \cdot |\alpha_{n+1} - \alpha_n| \leq q_n \cdot 2\,|\alpha - \alpha_n| \leq \frac{1}{4 \cdot 2^n d_n q_n}$$

due to Lemma 2.1. Hence, there are $\lfloor \frac{q_{n+1}\cdot\left(1-\frac{3}{4\cdot 2^n}\right)}{d_n q_n} \rfloor \cdot d_n$ different numbers of the form l_m and we can choose

$$\lfloor \frac{q_{n+1}\cdot\left(1-\frac{3}{4\cdot 2^n}\right)}{d_n q_n} \rfloor \cdot d_n \cdot N_{n-1}$$

iterates of B_n from the family $\tilde{F_n}$. The complete union of chosen $\frac{1}{q_{n+1}}$ intervals is called F_n. More precisely, we introduce the clusters

$$\tilde{A}_i^{(n)} = \left\{ R^l_{\alpha_{n+1}}(I) \;\middle|\; I \in F_n^{(1,i)}.\ l = 0, \dots, N_{n-1} - 1\right\} \text{ and } A_i^{(n)} = H_n\left(\tilde{A}_i^{(n)}\right)$$

for $i = 0, \dots, d_n - 1$. Furthermore, let $\bar{A}_i^{(n)}$ denote the set of the corresponding numberings in F_n.

Lemma 2.2. *We have*

$$m\left(A_{i_n}^{(n)}\right) = q_1 \cdot \frac{\nu_n^{(i_n+1)}}{q_{n+1}} \cdot \prod_{i=1}^{n} \lfloor \frac{q_{i+1}\cdot\left(1-\frac{3}{4\cdot 2^i}\right)}{d_i \cdot q_i} \rfloor \cdot d_i.$$

Proof. In order to compute $m\left(A_i^{(n)}\right)$ we point out that

$$h_n\left(\left[\frac{t}{q_n} + \frac{i}{d_n q_n} + \frac{1}{2\cdot 2^n \cdot d_n q_n}, \frac{t}{q_n} + \frac{i+1}{d_n q_n} - \frac{1}{4\cdot 2^n \cdot d_n q_n}\right]\right) \subset \left[\frac{t}{q_n}, \frac{t+1}{q_n}\right]$$

because h_n is a q_n-cyclic covering. For every $i = 0, \dots, d_1 - 1$ there are q_1 intervals of type $\left[\frac{t}{q_1} + \frac{i}{d_1 q_1}, \frac{t}{q_1} + \frac{i+1}{d_1 q_1}\right]$ and each domain $\left[\frac{t}{q_k} + \frac{i}{d_k q_k}, \frac{t}{q_k} + \frac{i+1}{d_k q_k}\right] \subset F_{k-1}$ contains $\lfloor \frac{q_{k+1}\cdot\left(1-\frac{3}{4\cdot 2^k}\right)}{d_k q_k} \rfloor$ many chosen intervals $\left[\frac{s}{q_{k+1}}, \frac{s+1}{q_{k+1}}\right]$. Everyone of these contains one domain of type

$$\left[\frac{s}{q_{k+1}} + \frac{i}{d_{k+1}q_{k+1}}, \frac{s}{q_{k+1}} + \frac{i+1}{d_{k+1}q_{k+1}}\right]$$

for each $i = 0, \dots, d_{k+1} - 1$. Since for any fixed i the iterates $\left(f_n^{H_n(\tilde{F_n})}\right)^k (H_n\left(I\right))$, $k = l_m, \dots, l_m + N_{n-1} - 1$ of $I \in F_n^{(1,i)}$ meet every occurring domain $\Delta_n^{(t,i)}$ exactly once, we get for $B_m^{(n)}$ in the situation of $R^{l_m}_{\alpha_{n+1}}|_{\tilde{F}_n}(B_n) \subset F_n^{(1,i_n)}$:

$$\begin{aligned} & m\left(B_m^{(n)}\right) \\ =& \sum_{i_1=1}^{d_1} q_1 \sum_{i_2=1}^{d_2} \lfloor \frac{q_2 \cdot \left(1-\frac{3}{4\cdot 2}\right)}{d_1 q_1} \rfloor \cdots \sum_{i_{n-1}=1}^{d_{n-1}} \lfloor \frac{q_n \cdot \left(1-\frac{3}{4\cdot 2^{n-1}}\right)}{d_{n-1}q_{n-1}} \rfloor \nu_1^{(i_1)} d_1 \cdot \nu_2^{(i_2)} d_2 \cdot \dots \cdot \nu_n^{(i_n+1)} d_n \cdot \frac{1}{q_{n+1}} \\ =& q_1 \lfloor \frac{q_2 \cdot \left(1-\frac{3}{4\cdot 2}\right)}{d_1 q_1} \rfloor \dots \lfloor \frac{q_n \cdot \left(1-\frac{3}{4\cdot 2^{n-1}}\right)}{d_{n-1}q_{n-1}} \rfloor d_1 \cdot d_2 \cdot \dots \cdot d_n \cdot \frac{\nu_n^{(i_n+1)}}{q_{n+1}} \end{aligned}$$

Then we have

$$m\left(A_{i_n}^{(n)}\right) = q_1 \lfloor \frac{q_2 \cdot \left(1-\frac{3}{4\cdot 2}\right)}{d_1 q_1} \rfloor \dots \lfloor \frac{q_n \cdot \left(1-\frac{3}{4\cdot 2^{n-1}}\right)}{d_{n-1}q_{n-1}} \rfloor d_1 \cdot d_2 \cdot \dots \cdot d_n \cdot \frac{\nu_n^{(i_n+1)}}{q_{n+1}} \cdot \lfloor \frac{q_{n+1} \cdot \left(1-\frac{3}{4\cdot 2^{n}}\right)}{d_n q_n} \rfloor$$

recalling that $\lfloor \frac{q_{n+1} \cdot \left(1-\frac{3}{4\cdot 2^{n}}\right)}{d_n q_n} \rfloor$ numbers belong to this cluster. □

By this Lemma we get for the set $E_n = H_n(F_n)$

$$\begin{aligned} m(E_n) &= \sum_{i_n=0}^{d_n-1} m\left(A_{i_n}^{(n)}\right) \\ &= q_1 \lfloor \frac{q_2 \cdot \left(1-\frac{3}{4\cdot 2}\right)}{d_1 q_1} \rfloor \dots \lfloor \frac{q_n \cdot \left(1-\frac{3}{4\cdot 2^{n-1}}\right)}{d_{n-1}q_{n-1}} \rfloor d_1 \cdot d_2 \cdot \dots \cdot d_n \cdot \frac{1}{q_{n+1}} \cdot \lfloor \frac{q_{n+1} \cdot \left(1-\frac{3}{4\cdot 2^{n}}\right)}{d_n q_n} \rfloor \end{aligned}$$

In particular, we observe

$$m(E_n) \geq \prod_{i=1}^{n} \left(1 - \frac{1}{2^i}\right).$$

Moreover, we introduce the set

$$E = \bigcap_{n \in \mathbb{N}} E_n.$$

$F_n \subset F_{n-1}$ for every $n \in \mathbb{N}$ implies $E_n \subset E_{n-1}$ and we conclude

$$m(E) \geq \prod_{i=1}^{\infty} \left(1 - \frac{1}{2^i}\right) > 0.$$

By Dye's Theorem $(E, \mathcal{L}^E, m^E, f^E)$ is orbit equivalent to $(\mathbb{S}^1, \mathcal{L}, m, f)$ ([Ka79], Theorem 1.5), where $\mathcal{L}$ is the σ-algebra of the Lebesgue measurable subsets of $\mathbb{S}^1$. Hence, it is sufficient to examine the induced system:

Lemma 2.3. *The induced system $(E, \mathcal{L}^E, m^E, f^E)$ is orbit equivalent to the odometer of product type $\mathcal{O}\left((M_k d_k)_{k\in\mathbb{N}}, \{\nu_k^*\}\right)$, where $M_k = \lfloor \frac{q_{k+1}\cdot\left(1-\frac{3}{4\cdot 2^k}\right)}{d_k q_k} \rfloor$ and $\nu_k^*(t) = M_k^{-1}\nu_k(i)$ for $t \in \bar{A}_i^{(k)}$, $i = 0, \dots, d_k - 1$.*

Proof. The element $(t_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} \{0, \dots, M_k d_k - 1\}$ corresponds to

$$\left\{ x \in E \,\middle|\, x \in \bigcap_{k\in\mathbb{N}} H_k\left(B_{t_k}^{(k)}\right) \right\}.$$

The domains were introduced and labelled by numbers in such a way that f_n imitates the behaviour of the odometer up to level n and for the first q_{n+1} iterates. By the same calculations as above we get for $B_m^{(n)}$ in the situation of $R^{l_m}_{\alpha_{n+1}}|_{\tilde{F}_n}(B_n) \subset F_n^{(1,i_n)}$:

$$\frac{m\left(H_n\left(B_m^{(n)}\right)\cap E\right)}{m(E)} = \frac{\nu_n^{(i_n+1)}}{\left\lfloor \frac{q_{n+1}\cdot\left(1-\frac{3}{4\cdot 2^n}\right)}{d_n q_n}\right\rfloor}$$

and

$$\frac{m\left(A_{i_n}^{(n)}\cap E\right)}{m(E)} = \nu_n^{(i_n+1)} = \nu_n\left(\{i_n\}\right).$$

Moreover, we have for $B_{m_k}^{(k)}$ in the situation of $R^{l_{m_k}}_{\alpha_{k+1}}|_{\tilde{F}_k}(B_k) \subset F_k^{(1,i_k)}$:

$$\frac{m\left(\bigcap_{k=1}^{N} H_k\left(B_{m_k}^{(n)}\right)\cap E\right)}{m(E)} = \prod_{k=1}^{N}\frac{\nu_k^{(i_k+1)}}{\left\lfloor \frac{q_{k+1}\cdot\left(1-\frac{3}{4\cdot 2^k}\right)}{d_k q_k}\right\rfloor}.$$

□

Finally, we use the subsequent result on orbit equivalence of odometers stated in [Ka79], Theorem 1.8:

Lemma 2.4. *Let $\mathcal{O}\left((d_k)_{k\in\mathbb{N}}, \{\nu_k\}_{k\in\mathbb{N}}\right)$ be an odometer of product type. For every $k \in \mathbb{N}$ let M_k be a positive integer and ν_k^* be a probability measure on $A = \{0,1,\dots,M_k d_k - 1\}$ such that there exists a partition $A = \bigcup_j A_j$ where $\#A_j = M_k$ and $\nu_k^*(n) = M_k^{-1}\nu_k(j)$ for $n \in A_j$, $j = 0,\dots,d_k-1$. Then $\mathcal{O}\left((M_k d_k)_{k\in\mathbb{N}}, \{\nu_k^*\}_{k\in\mathbb{N}}\right)$ is orbit equivalent to $\mathcal{O}\left((d_k)_{k\in\mathbb{N}}, \{\nu_k\}_{k\in\mathbb{N}}\right)$*

Hereby, we conclude that $(\mathbb{S}^1, \mathcal{L}, m, f)$ is orbit equivalent to $\mathcal{O}\left((d_k)_{k\in\mathbb{N}}, \{\nu_k\}\right)$.

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