Laver Trees in the Generalized Baire Space

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Abstract

We prove that any suitable generalization of Laver forcing to the space κ^κ, for uncountable regular κ, necessarily adds a Cohen κ-real. This is a contribution to the study of generalized Baire spaces and answers a question from [1]. We also study a dichotomy and an ideal naturally related to generalized Laver forcing, and use this to extend the first result to include a slightly wider class of forcing notions.

1 Introduction

In set theory of the reals, a basic questions is whether a forcing adds *Cohen reals* or *dominating reals*. It is well-known that Cohen forcing adds Cohen but not dominating reals while Laver forcing adds the opposite. In the language of cardinal characteristics of the continuum, this means that an appropriate iteration of Cohen forcing starting from CH yields $\mathfrak{b} < \mathrm{cov}(\mathcal{M})$, while an appropriate iteration of Laver forcing starting from CH yields $\mathrm{cov}(\mathcal{M}) < \mathfrak{b}$.

In recent years, the study of *generalized Baire spaces* has caught the attention of an increasing number of set theorists. Let κ be a regular, uncountable cardinal and consider elements of the spaces κ^κ or 2^κ as "κ-reals". The concepts *dominating κ-real* and *Cohen κ-real*, as well as the cardinal invariants $\mathfrak{b}_\kappa$ and $\mathrm{cov}(\mathcal{M}_\kappa)$, can be naturally generalized to this setting (see Section 2).

It is not hard to see that κ-Cohen forcing does not add dominating κ-reals, so an appropriate iteration of κ-Cohen forcing, starting from a model of GCH, yields a model in which $\mathfrak{b}_\kappa < \mathrm{cov}(\mathcal{M}_\kappa)$. A natural method for the converse direction, i.e., proving the consistency of $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{b}_\kappa$, would be to iterate a forcing which adds dominating κ-reals but not Cohen κ-reals. The authors of [1, p. 36] asked whether a forcing with such a property existed, and in particular, whether some generalization of Laver forcing had this property.

In this paper, we show that *any* generalization of Laver forcing necessarily adds a Cohen κ-real. Using a similar technique we can also show that if $\mathbb{P}$ is any forcing

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whose conditions are limit-closed trees on $\kappa^{<\kappa}$ and which adds a dominating κ-real obtained as the image of the generic under a continuous function in the ground model, also necessarily adds a Cohen κ-real. It is an open question whether there exists some other forcing adding dominating κ-reals but not adding Cohen κ-reals.[1]

We should note that a model for $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{b}_\kappa$ was recently constructed by Shelah (private communication). However, Shelah's method was to start from a model of $\mathrm{cov}(\mathcal{M}_\kappa) = \mathfrak{b}_\kappa = 2^\kappa > \kappa^+$ and add a witness to $\mathrm{cov}(\mathcal{M}_\kappa) = \kappa^+$ by a short forcing iteration. It is therefore still open whether an alternative proof exists by using a forcing iteration starting from a model of GCH which adds dominating κ-reals and no Cohen κ-reals.

The main result is proved in Section 3. Motivated by these methods, in the subsequent section we look at the ideal related to generalized Laver forcing and prove a somewhat surprising result concerning a generalization of the dichotomy for Laver forcing from [5]. This dichotomy is used in Section 5 to extend the results to include a wider range of forcing notions.

2 Preliminaries and definitions

We work in the setting where κ is an uncountable, regular cardinal, and consider the *generalized Baire space* κ^κ with the topology generated by basic open sets of the form $[\sigma] := \{x \in \kappa^\kappa : \sigma \subseteq x\}$ for $\sigma \in \kappa^{<\kappa}$, as well as the *generalized Cantor space* 2^κ, with the analogous topology.

A standard cardinal arithmetic assumption in this setting is $\kappa^{<\kappa} = \kappa$, which is sufficient to prove many pleasant properties of generalized Baire spaces, e.g., that it has a topology with base of size κ (without this assumption, the overall theory seems to be less coherent). In this paper, we will always work under this assumption. We refer the reader to [4] for a good introduction to generalized Baire spaces, and to [8] for an overview of the current state of the field and a list of open problems.

Definition 2.1. Let $f, g \in \kappa^\kappa$. We say that g *dominates* f, notation $f \leq^* g$, iff $\exists \alpha < \kappa \, \forall i > \alpha \, (f(i) \leq g(i))$.

Definition 2.2. A set $A \subseteq 2^\kappa$ is *nowhere-dense* if for every basic open $[\sigma]$ there exists a basic open $[\tau] \subseteq [\sigma]$ such that $[\tau] \cap A = \varnothing$. A set $A \subseteq 2^\kappa$ is *κ-meager* if it is contained in the union of κ-many nowhere-dense sets. The ideal of κ-meager sets is denoted by $\mathcal{M}_\kappa$. Analogous definitions work for κ^κ as well.

A *tree* in $\kappa^{<\kappa}$ or $2^{<\kappa}$ is a subset closed under initial segments. If T is a tree, we use $[T]$ to denote the set of branches (of length κ) through T, that is $[T] := \{x \in \kappa^\kappa : \forall \alpha \, (x \restriction \alpha \in T)\}$. For $\sigma \in T$ we use the notation $T{\uparrow}\sigma := \{\tau \in T : \sigma \subseteq \tau \vee \tau \subseteq \sigma\}$. A tree $T \subseteq \kappa^{<\kappa}$ is called *limit-closed*[2] if for any $\subseteq$-increasing sequence $\langle \sigma_i : i < \alpha \rangle$

[1] In an earlier version of this paper, we claimed that any $<\kappa$-closed forcing adding dominating κ-reals adds Cohen κ-reals, but this proof contained a mistake, so, to our knowledge, the question is still open, see Question 5.1.

[2] Other terminology used is "$<\kappa$-closed" and "sequentially closed".

from T of length $\alpha < \kappa$, the limit $\sigma := \bigcup_{\alpha<\kappa} \sigma_\alpha$ is in T. We call a set C *superclosed* if $C = [T]$ for a limit-closed tree T.

Every closed subset of κ^κ is the set of branches through a tree but not necessarily a limit-closed tree, so one could say that being superclosed is a topologically stronger property than being closes. We will also need to consider sets of branches of length shorter than κ. For any limit ordinal $\lambda < \kappa$ we use the notation $[T]_\lambda := \{x \in \kappa^\lambda : \forall \alpha < \lambda \ (x{\restriction}\alpha \in T)\}$. Thus T is limit-closed iff $[T]_\lambda \subseteq T$ for all limit ordinals $\lambda < \kappa$.

Definition 2.3. A *Laver tree* is a tree $T \subseteq \omega^{<\omega}$ with the property that such that for every $\sigma \in T$ extending stem(T), $|\mathrm{Succ}_T(\sigma)| = \omega$. *Laver forcing* $\mathbb{L}$ is the partial order of Laver trees ordered by inclusion.

Laver forcing adds dominating reals while satisfying the so-called *Laver property*, a well-known iterable property implying that no Cohen reals are added. There have been several attempts in the literature to generalize Laver forcing to κ^κ.

Definition 2.4. A *κ-Laver tree* is a tree $T \subseteq \kappa^{<\kappa}$ which is *limit-closed* and such that for every $\sigma \in T$ extending stem(T), $|\mathrm{Succ}_T(\sigma)| = \kappa$.

In itself, this partial order is not well-suited as a forcing on κ^κ—for example, it is not $<\kappa$-closed. But there have been several attempts to define subtler versions of Laver forcing, for example *club-Laver* $\mathbb{L}_\kappa^{club}$ (see [3]), where the requirement on the trees is strengthened to "$\mathrm{Succ}_T(\sigma)$ contains a club on κ". This is a $<\kappa$-closed forcing adding a dominating κ-real. However, it is also easy to see that it adds a Cohen real: let S be a stationary, co-stationary subset of κ and let $\varphi : \kappa^\kappa \to 2^\kappa$ be given by $\varphi(x)(\alpha) = 1 \iff x(\alpha) \in S$. If x_{gen} is the generic κ-real added by $\mathbb{L}_\kappa^{club}$, then $\varphi(x_{gen})$ is a Cohen κ-real.

Nevertheless, one could consider other ways of defining a forcing notion $\mathbb{P} \subseteq \mathbb{L}_\kappa$, by *carefully selecting* special types of trees, with the hope that this forcing would not add Cohen κ-reals. Our result, Theorem 3.5, says that such an approach cannot work. A slightly stronger result, Theorem 3.7, shows the same for a a wider class of trees. The results are further generalized in Section 5 where we prove that any "tree-like forcing" adding a dominating κ-real which is the image of the generic under a continuous function in the ground model, adds a Cohen κ-real.

3 The Supremum Game

The main ingredient of all the proofs is the following game.

Definition 3.1. Let $S \subseteq \kappa$. The *supremum game* $G^s(S)$ is played by two players, for ω moves, as follows:

$$\begin{array}{c||cccccc} \mathrm{I} & A_0 & & A_1 & & \dots & \\ \hline \mathrm{II} & & \beta_0 & & \beta_1 & & \dots \end{array}$$

where A_n are subsets of κ with $|A_n| = \kappa$ and $\beta_n \in A_n$. Player II wins the game iff $\sup_n \beta_n \in S$.

Lemma 3.2. *Let S be a stationary subset of $\mathrm{Cof}_\omega(\kappa) = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$. Then Player I does not have a winning strategy in $G^s(S)$.*

Proof. Suppose, towards contradiction, that I had a strategy σ in $G^s(S)$. Let θ be sufficiently large and let $M \prec \mathcal{H}_\theta$ be an elementary submodel such that $\sigma \in M$, $|M| < \kappa$, and $\delta := \sup(M \cap \kappa) \in S$. Note that we can always do that, because the set $\mathcal{M} := \{\sup(M \cap \kappa) : M \prec \mathcal{H}_\theta, \ \sigma \in M, \ |M| < \kappa\}$ contains a club.

Fix a sequence $\langle \gamma_n : n < \omega \rangle$ cofinal in δ, such that every $\gamma_n \in M$ (but the sequence itself is not). Inductively, Player II will construct a run of the game according to strategy σ.

At each step n, inductively assume A_k and β_k for $k < n$ have been fixed according to the rules of the game and the strategy σ, and assume they are all in M. Let $A_n := \sigma(A_0, \beta_0, \dots, A_{n-1}, \beta_{n-1})$. Since the finite sequence was in M and the strategy σ is in M, A_n is also in M. Furthermore, since $|A_n| = \kappa$, the following statement is true:

$$\exists \beta > \gamma_n \ (\beta \in A_n)$$

This statement holds in $\mathcal{H}_\theta$, so by elementarity, it also holds in M. Thus, there exists $\beta_n \in M$ with $\beta_n > \gamma_n$ and $\beta_n \in A_n$. This completes the construction.

We have produced a sequence $\langle \beta_n : n < \omega \rangle$ with $\beta_n \in M$ for all n. But clearly $\sup_n \beta_n = \sup_n \gamma_n = \delta \in S$, so Player II wins this game contrary to assumption. □

Definition 3.3. A *short κ-Laver tree* is a tree $L \subseteq \kappa^{<\omega}$ (i.e., depth ω), such that for all $\sigma \in L$ extending $\mathrm{stem}(L)$ we have $|\mathrm{Succ}_L(\sigma)| = \kappa$.

Corollary 3.4. *Let $S \subseteq \kappa$ be a stationary subset of $\mathrm{Cof}_\omega(\kappa)$. For every short κ-Laver tree L there exists a branch $\eta \in [L]_\omega$ such that $\sup_n \eta(n) \in S$.*

Proof. The short κ-Laver tree L induces a strategy σ_L for Player I in the supremum game:

$$\sigma_L(A_0, \beta_0, \dots, A_n, \beta_n) := \mathrm{Succ}_L(\mathrm{stem}(L)^\frown \langle \beta_0, \dots, \beta_n \rangle).$$

Whenever $\langle A_0, \beta_0, A_1, \beta_1, \dots \rangle$ is a run of the game according to σ_L, $\mathrm{stem}(L)^\frown \langle \beta_0, \beta_1, \dots \rangle$ is an element of $[L]_\omega$.

By Lemma 3.2, there exists a run of the game in which Player I follows σ_L but Player II wins. This yields a branch $\eta \in [L]_\omega$ such that $\sup_n \eta(n) \in S$. □

With this, we immediately obtain our main result.

Theorem 3.5. *Let $\mathbb{P} \subseteq \mathbb{L}_\kappa$ be any sub-forcing closed under the following condition: if $T \in \mathbb{P}$ and $\sigma \in T$, then $T{\uparrow}\sigma \in \mathbb{P}$. Then $\mathbb{P}$ adds a Cohen κ-real.*

Proof. We will use the following notation: if $T \in \kappa^{<\kappa}$ is a tree and $\sigma \in T$, then $T{\restriction}^\omega \sigma := \{\tau \in \kappa^{<\omega} : \sigma^\frown \tau \in T\}$. Note that if T is a κ-Laver tree, then for every $\sigma \in T$ extending $\mathrm{stem}(T)$, $T{\restriction}^\omega \sigma$ is a short κ-Laver tree, and moreover $[T{\restriction}^\omega \sigma]_\omega \subseteq T$.

Let $S_0 \cup S_1$ be a stationary/co-stationary partition of $\mathrm{Cof}_\omega(\kappa)$ and consider the mapping $\varphi : \kappa^\kappa \to 2^\kappa$ defined by

$$\varphi(x)(\alpha) = 1 \quad :\Leftrightarrow \quad \sup\{x(\omega\cdot\alpha + n) \ : \ n < \omega\} \in S_1.$$

In other words, partition x into κ-many blocks of length ω, and map each piece to 0 or 1 depending on whether its supremum lies in S_0 or S_1. We claim that if x_{gen} is $\mathbb{P}$-generic then $\varphi(x_{gen})$ is κ-Cohen-generic.

We use $\tilde{\varphi} : \kappa^{<\kappa} \to 2^{<\kappa}$ to denote the approximations of φ (defined as above). Let $T \in \mathbb{P}$ be given and let D be open dense in κ-Cohen forcing. Let $\sigma := \mathrm{stem}(T)$, w.l.o.g. $\mathrm{len}(\sigma)$ is a limit ordinal. Let $t \in D$ extend $\tilde{\varphi}(\sigma)$. Suppose $\tilde{\varphi}(\sigma)^\frown \langle 0\rangle \subseteq t$. By Corollary 3.4 there is $\eta \in [T{\restriction}^\omega \sigma]_\omega$ such that $\sup_n \eta(n) \in S_0$. If, instead, we have $\tilde{\varphi}(\sigma)^\frown \langle 1\rangle \subseteq t$, we can again apply Corollary 3.4 and find a branch $\mu \in [T{\restriction}^\omega\sigma]_\omega$ such that $\sup_n \mu(n) \in S_1$. Note that, since T is limit-closed, $\sigma^\frown\eta$ resp. $\sigma^\frown\mu$ are elements of T. Now proceed analogously until reaching τ, such that $\tilde{\varphi}(\tau) = t$. By assumption $T{\uparrow}\tau \in \mathbb{P}$, and now clearly $T{\uparrow}\tau \Vdash t \subseteq \varphi(\dot{x}_{gen})$. Thus $\varphi(x_{gen})$ is a Cohen κ-real. $\square$

By a slight modification of the above result, we can obtain a stronger theorem.

Definition 3.6. A tree $T \subseteq \kappa^{<\kappa}$ is called a *pseudo-κ-Laver tree* if it is limit-closed and has the following property: $\forall \sigma \in T\ \exists \tau \in T$ s.t. $\sigma \subseteq \tau$ and $T{\restriction}_\omega \tau$ is a short κ-Laver tree. We use $\mathrm{p}\mathbb{L}_\kappa$ to denote the partial order of pseudo-κ-Laver trees ordered by inclusion.

Theorem 3.7. *Let $\mathbb{P} \subseteq \mathrm{p}\mathbb{L}_\kappa$ be any sub-forcing closed under the following condition: if $T \in \mathbb{P}$ and $\sigma \in T$, then $T{\uparrow}\sigma \in \mathbb{P}$. Then $\mathbb{P}$ adds a Cohen κ-real.*

Proof. The method is similar to the above, however here, we let $\{S_t \ : \ t \in \kappa^{<\kappa}\}$ be a disjoint partition of $\mathrm{Cof}_\omega(\kappa)$ into stationary sets, indexed by $\kappa^{<\kappa}$, which is possible by the assumption $\kappa^{<\kappa} = \kappa$. Define the mapping $\pi : \kappa^\kappa \to 2^\kappa$ by $\pi(x) := t_0{}^\frown t_1{}^\frown t_2{}^\frown \dots$, where for all $\alpha < \kappa$, t_α is such that $\sup\{x(\alpha\cdot\omega + n) \ : \ n < \omega\} \in S_{t_\alpha}$. We also use $\tilde{\pi}$ to denote the same operation but from $\kappa^{<\kappa}$ to $2^{<\kappa}$.

Let x_{gen} be the $\mathbb{P}$-generic κ-real, and we show that $\pi(x_{gen})$ is κ-Cohen. Let D be κ-Cohen dense, and let $T \in \mathbb{P}$. Find $\sigma \in T$ such that $T{\restriction}^\omega \sigma$ is a short κ-Laver tree. Let $t \in D$ be such that $\tilde{\pi}(\sigma) \subseteq t$. Let u be such that $\tilde{\pi}(\sigma)^\frown u = t$. By Corollary 3.4 there is $\eta \in [T{\restriction}^\omega\sigma]_\omega$ such that $\sup_n \eta(n) \in S_u$. It follows that $\tilde{\pi}(\sigma^\frown\eta) = \tilde{\pi}(\sigma)^\frown u = t$. Therefore $T{\uparrow}(\sigma^\frown\eta) \Vdash t \subseteq \pi(\dot{x}_{gen})$. $\square$

4 The generalized Laver dichotomy

The supremum game and the arguments from Theorem 3.5 naturally lead us to consider a question in generalized descriptive set theory. The motivation for this is explained in Remark 4.6. In addition, in the following section we will use these results to draw some further consequences for forcings adding dominating κ-reals.

We need the following strengthening of the concept of a *dominating real*, which has been studied in the classical context in [5, 9, 2, 7].

Definition 4.1. For $f : \kappa^{<\kappa} \to \kappa$ and $x \in \kappa^\kappa$, we say that x *strongly dominates* f if $\exists \alpha_0 \, \forall \alpha > \alpha_0 \, (x(\alpha) > f(x \restriction \alpha))$. If M is a model of set theory with the same $\kappa^{<\kappa}$, then x is called *strongly dominating over* M if for all $f : \kappa^{<\kappa} \to \kappa$ with $f \in M$, x strongly dominates f.

Clearly, if x is strongly dominating, then it is also dominating. The converse is false in general, e.g., let d be dominating over M and let x be defined by $x(\alpha) := d(\alpha)$ for odd α and $x(\alpha) := d(\alpha+1)$ for even and limit α. Then d is dominating but not strongly dominating. However, the following is true:

Lemma 4.2. *Let M be a model of set theory such that $\kappa^{<\kappa} \cap M = \kappa^{<\kappa}$. Then, if there is a dominating real over M there is also a strongly dominating real over M.*

Proof. Let d be the dominating κ-real, and fix a bijection between $\kappa^{<\kappa}$ and κ in M. We can define a new dominating κ-real $d^* : \kappa^{<\kappa} \to \kappa$, i.e., such that for every $f : \kappa^{<\kappa} \to \kappa$ in M, $f(\sigma) < d^*(\sigma)$ holds for all but $<\kappa$-many $\sigma \in \kappa^{<\kappa}$. Now define inductively

$$e(\alpha) := d^*(e \restriction \alpha).$$

Then e is strongly dominating. □

Definition 4.3. A collection $X \subseteq \kappa^\kappa$ is a *strongly dominating family* if for every $f : \kappa^{<\kappa} \to \kappa$ there exists $x \in X$ which strongly dominates f. $\mathcal{D}_\kappa$ denotes the ideal of all $X \subseteq \kappa^\kappa$ which are *not* strongly dominating families.

For $\kappa = \omega$, the ideal $\mathcal{D}_\omega = \mathcal{D}$ is the well-known *non-strongly-dominating ideal*, introduced in [5] and independently in [12], and studied among others in [2]. The main interest in it stems from a perfect-set-like dichotomy theorem for Laver trees.

Theorem 4.4 (Goldstern et al [5])**.** *If T is a Laver tree then $[T] \notin \mathcal{D}$. Every analytic set $A \subseteq \omega^\omega$ is either in $\mathcal{D}$ or contains $[T]$ for some Laver tree T. In particular, there is a dense embedding from the order of Laver trees into the algebra of Borel subsets of ω^ω modulo $\mathcal{D}$.*

Dichotomies such as this one are common in classical descriptive set theory, the most notable example being the perfect set property and the closely related K_σ-dichomoty ([6]), all of which are false for arbitrary sets of reals but true for analytic sets. Interest in generalizing such dichotomies to the κ^κ-context was recently spurred by a result of Schlicht [11] showing that the generalized perfect set property for generalized projective sets is consistent, and Motto Ros-Lücke-Schlicht [10] showing that the generalized Hurewicz dichotomy for generalized projective sets is consistent. Thus, it might initially seem surprising that the generalized Laver dichotomy fails for closed sets, provably in ZFC.

Theorem 4.5. *There is a closed subset of κ^κ which is neither in $\mathcal{D}_\kappa$ nor contains the branches of a generalized Laver tree.*

Proof. Let φ be as in the proof of Theorem 3.5. Let z be the constant 0 function (or any other fixed element of 2^κ). We show that $C := \varphi^{-1}\{z\}$ is a counterexample to the dichotomy. Given any $T \in \mathbb{L}_\kappa$, we can easily find $x \in [T]$ such that $\varphi(x) \neq z$, therefore $[T] \nsubseteq C$. We claim that C is strongly dominating. Let $f : \kappa^{<\kappa} \to \kappa$ be given. Let

$$T_f := \{\sigma \in \kappa^{<\kappa} \ : \ \forall \beta < \mathrm{len}(\sigma)\ (\sigma(\beta) > f(\sigma \restriction \beta))\}.$$

It is not hard to see that T_f is a generalized Laver tree and $\mathrm{stem}(T_f) = \varnothing$. Therefore, we can find an $x \in [T]$ such that $\varphi(x) = z$. But then x strongly dominates f and $x \in C$, completing the argument. □

Remark 4.6. The relevance of this lemma is that it explains why Theorem 3.5 does not (as one might initially assume) yield a ZFC-proof of $\mathfrak{b}_\kappa \leq \mathrm{cov}(\mathcal{M}_\kappa)$. Indeed, it is not hard to verify that $\mathrm{cov}(\mathcal{D}_\kappa) = \mathfrak{b}_\kappa$ and that if $X \in \mathcal{M}_\kappa$ then $\varphi^{-1}[X]$ does not contain a κ-Laver tree. Thus, if the dichotomy would hold for generalized Borel (or just F_σ) sets then one could have concluded $\mathfrak{b}_\kappa = \mathrm{cov}(\mathcal{D}_\kappa) \leq \mathrm{cov}(\mathcal{M}_\kappa)$.

One could wonder whether there is *any* dichotomy for the ideal $\mathcal{D}_\kappa$, i.e., whether there is any collection $\mathbb{P}$ of limit-closed trees, such that for every $T \in \mathbb{P}$, $[T] \notin \mathcal{D}_\kappa$, and every analytic (or at least closed) set not in $\mathcal{D}_\kappa$ contains $[T]$ for some $T \in \mathbb{P}$. In fact, this is not the case either.

Lemma 4.7. *Let T be a tree such that $[T]$ is strongly dominating. Then there exists $s \in T$ such that $T \restriction^\omega s$ contains a short κ-Laver tree.*

Proof. Generalizing the game argument from [5], given $A \subseteq \kappa^\omega$ let $G^\star(A)$ be the game defined by:

I	α_0		α_1		$\dots$	
II		β_0		β_1		$\dots$

where $\alpha_n, \beta_n < \kappa$, $\alpha_n < \beta_n$ for all n, and Player II wins iff $\langle \beta_n \ : \ n < \omega \rangle \in A$.

It is easy to see that if Player II wins $G^\star(A)$ then there exists a short κ-Laver tree L such that $[L]_\omega \subseteq A$. Also it is clear that if A is closed (in the topology on κ^ω) then $G^\star(A)$ is determined.

Suppose, towards contradiction, that there is no $s \in T$ such that $T \restriction^\omega s$ contains a short κ-Laver tree. Then Player II does not have a winning strategy in $G^\star([T \restriction^\omega s]_\omega)$ for any $s \in T$, and therefore Player I has a winning strategy, call it σ_s. Define $f : \kappa^{<\kappa} \to \kappa$ as follows: for every $t \in T$, let $s \subseteq t$ be the maximal node of limit length, let u be such that $t = s^\frown u$, and define $f(t) := \sigma_s(u)$. Since $[T]$ is strongly dominating there is $x \in [T]$ and α such that $x(\beta) > f(x \restriction \beta)$ for all $\beta > \alpha$. In particular, there is $s \subseteq x$, of limit length, such that $x(|s|+n) > f(x \restriction (|s|+n))$ for all n. Letting $z \in \kappa^\omega$ be such that $s^\frown z = x \restriction (|s| + \omega)$, we see that $z(n) > f(s^\frown z \restriction n) = \sigma_s(z \restriction n)$, for every n. Therefore, $z \in [T \restriction^\omega s]_\omega$ satisfies the winning conditions for Player II in the game $G^\star([T \restriction^\omega s]_\omega)$, contradicting the assumption that σ_s was a winning strategy for Player I. □

Corollary 4.8. *There exists a closed strongly dominating set without a super-closed strongly dominating subset.*

Proof. Consider again the closed set $C := \varphi^{-1}\{z\}$ from Theorem 4.5. If C contains a strongly dominating $[T]$ for some limit-closed T, then by Lemma 4.7 there is $s \in T$ such that $T{\restriction}^{\omega}s$ contains a short κ-Laver tree L. By Corollary 3.4 there is $\eta \in [L]_\omega$ such that $\sup_n \eta(n) \in S_1$, and by limit-closure, there is $x \in [T]$ such that $s^\frown\eta \subseteq x$. But then $\varphi(x)$ contains a "1" and thus is not equal to z, the constant 0-function. $\square$

5 Interpretation trees and tree-like forcings

We would like to generalize the results from Section 3 about Laver trees to a wider class of forcing notions. Specifically, we can ask the following question:

Question 5.1. *Is it true that every $<\kappa$-closed forcing adding a dominating κ-real adds a Cohen κ-real?*

Notice that if a forcing is $<\kappa$-closed then it does not add new elements of $\kappa^{<\kappa}$. So in the context of generalized Baire spaces, it is natural to focus attention on $<\kappa$-closed forcings (after all, forcings on the ordinary Baire space does not changes $\omega^{<\omega}$). We start with the following:

Definition 5.2. Let $\mathbb{P}$ be any forcing notion, let $\dot{x}$ be a name, and let $p \in \mathbb{P}$ be such that $p \Vdash \dot{x} \in \kappa^\kappa$. Then the *interpretation tree* of $\dot{x}$ below p is defined by:

$$\mathfrak{T}_{\dot{x},p} = \{\sigma \in \kappa^{<\kappa} \ : \ \exists q \leq p \ (q \Vdash \sigma \subseteq \dot{x})\}$$

It is clear that $\mathfrak{T}_{\dot{x},p}$ is always a tree in the ground model, although in general it need not be a limit-closed tree. First we prove a general result about arbitrary $<\kappa$-closed forcings adding (strongly) dominating reals.

Lemma 5.3. *Suppose $\mathbb{P}$ is a $<\kappa$-closed forcing. If $p \Vdash$ "$\dot{d}$ is strongly dominating", then $[\mathfrak{T}_{\dot{d},p}]$ is strongly dominating.*

Proof. Fix $f : \kappa^{<\kappa} \to \kappa$ in the ground model and $p \Vdash \exists\beta\, \forall\alpha > \beta\ (\dot{d}(\alpha) > f(\dot{d}{\restriction}\alpha))$. By $<\kappa$-closure, there is a $q_0 \leq p$ which decides $\dot{d}{\restriction}\beta =: \sigma_0$. Then $\sigma_0 \in \mathfrak{T}_{\dot{d},p}$, and every condition q' stronger than q_0 also forces

$$\forall\alpha > \beta\ (\dot{d}(\alpha) > f(\dot{d}{\restriction}\alpha)) \qquad (*)$$

Inductively we find a decreasing sequence $\{q_i \ : \ i < \kappa\}$ below q_0, together with a strictly increasing sequence $\{\sigma_i \ : \ i < \kappa\}$ extending σ_0, such that $q_i \Vdash \sigma_i \subseteq \dot{d}$. At limit steps, we use the fact that $\mathbb{P}$ is $<\kappa$-closed. Now each $\sigma_i \in \mathfrak{T}_{\dot{d},p}$, therefore $x := \bigcup_{i<\kappa} \sigma_i \in [\mathfrak{T}_{\dot{d},p}]$. Moreover, since for every $\alpha > \beta$ there exists i such that $\alpha < |\sigma_i|$, by $(*)$ we know that $x(\alpha) > f(x{\restriction}\alpha)$. $\square$

Lemma 5.4. *Suppose $\mathbb{P}$ is a $<\kappa$-closed forcing. Let $\dot{d}$ be a name for a κ-real, and suppose further that every $\mathfrak{T}_{\dot{d},p}$ is limit-closed. If $p \Vdash$ "$\dot{d}$ is strongly dominating", then $p \Vdash$ "there is a Cohen κ-real".*

Proof. Let π be the function defined in Theorem 3.7. We will show that $p \Vdash$"$\pi(\dot{d})$ is κ-Cohen". Let D be κ-Cohen dense and $q \le p$ arbitrary. Since q forces that $\dot{d}$ is strongly dominating, by Lemma 5.3 we know that $[\mathfrak{T}_{\dot{d},q}]$ is a strongly dominating set. By Lemma 4.7 there is $s \in \mathfrak{T}_{\dot{d},q}$ such that $\mathfrak{T}_{\dot{d},q}\restriction^{\omega} s$ contains a short κ-Laver tree. Just as in the proof of Theorem 3.7, let $t \in D$ be such that $\tilde{\pi}(\sigma) \subseteq t$, u such that $\tilde{\pi}(\sigma)^\frown u = t$, and find $\eta \in [\mathfrak{T}_{\dot{d},q}\restriction^{\omega}\sigma]_\omega$ such that $\sup_n \eta(n) \in S_u$. Now, notice that by the assumption that $\mathfrak{T}_{\dot{d},q}$ is limit-closed, $\sigma^\frown\eta \in \mathfrak{T}_{\dot{d},q}$, hence there is $r \le q$ forcing $\sigma^\frown\eta \subseteq \dot{d}$. But then this r also forces $\tilde{\pi}(\sigma^\frown\eta) = \tilde{\pi}(\sigma)^\frown u = t \subseteq \pi(\dot{d})$, and therefore also $t \subseteq \pi(\dot{d})$. $\square$

So we see that, within the class of $<\kappa$-closed forcings, the decisive issue concerning Question 5.1 is whether interpretation trees are limit-closed. Notice that without this assumption, we would not be able to conclude $\sigma^\frown\eta \in \mathfrak{T}_{\dot{d},q}$ in the proof of Lemma 5.4 (actually, closure under ω-limits is sufficient).

Next, we want to focus specifically on forcings $\mathbb{P}$ whose conditions are limit-closed trees on $\kappa^{<\kappa}$. We will prove that if such a forcing adds a dominating κ-real which is the image of the generic under a continuous function in the ground model, then $\mathbb{P}$ adds a Cohen κ-real.

Definition 5.5. A forcing partial order $\mathbb{P}$ is called *tree-like* if its conditions are limit-closed trees $T \subseteq \kappa^{<\kappa}$, and for every $T \in \mathbb{P}$ and $\sigma \in T$, the restriction $T\uparrow\sigma \in \mathbb{P}$.

We now need to review some basic principles concerning continuous functions on κ^κ. Let us call a function $h : \kappa^{<\kappa} \to \kappa^{<\kappa}$ *pre-continuous* if:

1. $\sigma \subseteq \tau \;\Rightarrow\; h(\sigma) \subseteq h(\tau)$.
2. $\forall x \in \kappa^\kappa$, $\{\mathrm{len}(h(\sigma)) \,:\, \sigma \subseteq x\}$ is cofinal in κ.

If h is pre-continuous, let $f := \lim(h)$ be the function defined as $f(x) := \bigcup\{h(\sigma) : \sigma \subseteq x\}$. Just as in the classical situation, it is easy to check that if h is pre-continuous, then $\lim(h)$ is continuous, and for every continuous f there exists a pre-continuous h such that $f = \lim(h)$.[3]

[3]Proof of the above, just to check, later we will delete this footnote

Suppose h is pre-continuous. By 1, all sequences from the set $\{h(\sigma) : \sigma \subseteq x\}$ are mutually consistent, so by 2, $f(x) \in \kappa^\kappa$. Now suppose $\tau \subseteq f(x)$. Then by 2 there is $\sigma \subseteq x$ such that $\tau \subseteq h(\sigma)$. But then, for every y, if $\sigma \subseteq y$ then $h(\tau) \subseteq f(y)$, i.e., f"$[\sigma] \subseteq [\tau]$, showing that f is continuous.

Conversely, suppose f is continuous. For every σ, let $h(\sigma)$ be the longest τ such that f"$[\sigma] \subseteq [\tau]$, in other words

$$h(\sigma) \;:=\; \bigcup\{\tau : f\text{``}[\sigma] \subseteq [\tau]\} \;=\; \bigcup\{\tau : \forall y\, (\sigma \subseteq y \;\to\; \tau \subseteq f(y))\}.$$

Note that $\sigma \subseteq \sigma' \;\Rightarrow\; [\sigma'] \subseteq [\sigma] \;\Rightarrow\; f\text{``}[\sigma'] \subseteq f\text{``}[\sigma] \;\Rightarrow\; \{\tau : f\text{``}[\sigma] \subseteq [\tau]\} \subseteq \{\tau : f\text{``}[\sigma'] \subseteq [\tau]\} \Rightarrow\; h(\sigma) \subseteq h(\sigma')$. Moreover, by definition of continuity, for every x and every $\tau \subseteq f(x)$ there exists $\sigma \subseteq x$ such that f"$[\sigma] \subseteq [\tau]$, implying that $\tau \subseteq h(\sigma)$ by definition. Therefore $\{h(\sigma) : \sigma \subseteq x\}$ is cofinal in $f(x)$, implying both condition 2 of the definition of pre-continuity, and also that $f = \lim(h)$, as had to be shown.

For the proof, we will need to consider the image $h``T$ of a limit-closed tree $T$ under a pre-continuous function $h$. In general, such objects are not necessarily trees, so we will look at the tree generated by $h``T$:

$$\mathrm{tr}(h``T) := \{\tau : \exists \sigma \in T\ (\tau \subseteq h(\sigma))\}$$

Lemma 5.6. *Let $\mathbb{P}$ be a tree-like forcing, $\dot{x}$ a name for a κ-real, f a continuous function in the ground model with h the corresponding pre-continuous function, and T such that $T \Vdash \dot{x} = f(\dot{x}_{gen})$. Then $\mathfrak{T}_{\dot{x},T} = \mathrm{tr}(h``T)$.*

Proof. First suppose $\sigma \in T$. Then $T{\uparrow}\sigma \Vdash \sigma \subseteq \dot{x}_{gen}$, therefore $T{\uparrow}\sigma \Vdash h(\sigma) \subseteq f(\dot{x}_{gen}) = \dot{x}$. Therefore $h(\sigma) \in \mathfrak{T}_{\dot{x}_{gen},T}$.

Conversely, let $\tau \in \mathfrak{T}_{\dot{x}_{gen},T}$ be given. We want to find $\sigma \in T$ such that $\tau \subseteq h(\sigma)$. By definition there is $S \leq T$ such that $S \Vdash \tau \subseteq \dot{x}$. But since $S \Vdash \dot{x} = f(\dot{x}_{gen})$, we also have

$$S \Vdash \exists \sigma \subseteq \dot{x}_{gen}\ (\tau \subseteq h(\sigma)).$$

By $<\kappa$-closure, there exists $S' \leq S$ which decides σ, i.e., we may assume that σ is in the ground model and $S' \Vdash \sigma \subseteq \dot{x}_{gen} \wedge \tau \subseteq h(\sigma)$. Moreover, $\sigma \subseteq \mathrm{stem}(S')$, because otherwise there would be some incompatible $\sigma' \in S'$, and we would have $S' \Vdash \sigma' \subseteq \dot{x}_{gen}$, contradicting $S' \Vdash \sigma \subseteq \dot{x}_{gen}$. We conclude that $\sigma \in S' \subseteq S \subseteq T$ and $\tau \subseteq h(\sigma) = \tau$ as desired. $\square$

An immediate corollary is that if $\mathbb{P}$ is a tree-like forcing such that the generic *itself* is strongly dominating, then $\mathfrak{T}_{\dot{x}_{gen},T} = T$ is limit-closed, so by Lemma 5.4 $\mathbb{P}$ adds Cohen κ-reals. In general, however, we will need to consider $\mathrm{tr}(h``T)$ for pre-continuous functions $h$ other than the identity. In those cases, it is not a guarantee that $\mathrm{tr}(h``T)$ is limit-closed. However, what we will show is that every pre-continuous function can be *dominated* by a pre-continuous function which, in addition, preserves limit closure of the tree.

Definition 5.7. A pre-continuous h is *limit-closure-preserving* if for every limit-closed tree T, the tree $\mathrm{tr}(h``T)$ is also limit-closed. A continuous f is *limit-closure-preserving* if $f = \lim(h)$ for a limit-closure-preserving h.

Lemma 5.8. *For every continuous f there is a continuous and limit-closure-preserving g such that for all x and all α:*

$$f(x)(\alpha) < g(x)(\alpha).$$

Proof. Fix a function $R : \kappa^{<\kappa} \times \kappa^{<\kappa} \to \kappa^{<\kappa}$ such that

1. $R(\rho, \varnothing) = \varnothing$ for all ρ,
2. If $\sigma \neq \varnothing$, then
   - $\mathrm{len}(R(\rho, \sigma)) = \mathrm{len}(\sigma)$ for all ρ,
   - $\sigma(\alpha) < R(\rho, \sigma)(\alpha)$ for all $\alpha < \mathrm{len}(\sigma)$.

3. If $\rho \neq \rho'$, then for any $\sigma, \sigma' \neq \varnothing$, we have $R(\rho, \sigma)(0) \neq R(\rho', \sigma')(0)$

In words: R takes every non-empty sequence σ and shifts it coordinate-wise to a higher sequence of the same length depending on ρ; this happens in such a way that for different $\rho \neq \rho'$, the first coordinates of $R(\rho, \dots)$ and $R(\rho', \dots)$ are never the same. It is easy to see that such a function exists since $\kappa^{<\kappa} = \kappa$.

Let f be continuous and h the corresponding pre-continuous function. Define j inductively:

- If $j(\sigma)$ is defined, then for every β define $j(\sigma^\frown \langle\beta\rangle)$ as follows: let w be such that $h(\sigma)^\frown w = h(\sigma^\frown \langle\beta\rangle)$ ($w = \varnothing$ is also allowed). Then let

$$j(\sigma^\frown \langle\beta\rangle) := j(\sigma)^\frown R(\sigma^\frown \langle\beta\rangle, w)$$

- For σ of limit length, let w be such that $h(\sigma) = \bigcup_{\sigma' \subset \sigma} h(\sigma')^\frown w$. Note that this is always possible because $h(\sigma') \subseteq h(\sigma)$ for all $\sigma' \subset \sigma$ ($w = \varnothing$ is allowed, in fact this is usually the case). Then let

$$j(\sigma) := \left(\bigcup_{\sigma' \subset \sigma} j(\sigma') \right)^\frown R(\sigma, w)$$

Let $g := \lim(j)$. We claim that g has all the required properties.

Notice that, inductively, $\mathrm{len}(j(\sigma)) = \mathrm{len}(h(\sigma))$ for every σ. It is also clear, by construction, that $\sigma \subseteq \sigma'$ implies $j(\sigma) \subseteq j(\sigma')$. Therefore j is pre-continuous. Moreover, by construction we immediately see that $f(x)(\alpha) < g(x)(\alpha)$ holds for every x and α. It remains to prove that j is limit-closure-preserving.

Let T be an arbitrary limit-closed tree, and let $U := \mathrm{tr}(j``T)$. Let $\{u_i : i < \lambda\}$ be an increasing sequence in U. We need to show that this sequence has an extension in U. For each i, let $s_i \in T$ be $\subseteq$-*minimal* such that $u_i \subseteq j(s_i)$.[4]

Claim. $s_i \subseteq s_j$ for all $i < j$.

Proof. Suppose, towards contradiction, that $s_i \not\subseteq s_j$. First, $s_j \subset s_i$ (proper extension) is clearly not possible, since this would imply $u_i \subseteq u_j \subseteq j(s_j) \subseteq j(s_i)$, and thus we would have picked s_j instead of s_i. Therefore, s_i and s_j are incompatible. Let r be maximal such that $r \subseteq s_i$ and $r \subseteq s_j$.

Next, notice that $j(r) \subset u_i$: otherwise, we would have $u_i \subseteq j(r) \subseteq u_j$, so we would have picked r instead of s_i.

So we also know that $j(r) \subset j(s_i)$ and $j(r) \subset j(s_j)$. Let r_0 be minimal such that

$$r \subseteq r_0 \subseteq s_i \quad \text{and} \quad j(r) \subset j(r_0)$$

[4]The s_i's do not need to be distinct; e.g., they could be all equal to a unique s, or there could be $\mathrm{cf}(\lambda)$-many distinct s_i's, etc.

and let r_1 be minimal such that

$$r \subseteq r_1 \subseteq s_j \quad \text{and} \quad j(r) \subset j(r_1)$$

See Figure 1. First we consider r_0: there are two cases.

- Suppose r_0 is of successor length. Then there is r_{00} such that $r_0 = r_{00}{}^\frown \langle \beta \rangle$ and $j(r) = j(r_{00})$. Then there exists $w \neq \varnothing$ such that $h(r_{00}{}^\frown \langle \beta \rangle) = h(r_{00})^\frown w$, and by definition we have:

$$j(r_0) \;=\; j(r_{00})^\frown R(r_0, w) \;=\; j(r)^\frown R(r_0, w)$$

- Now suppose r_0 is of limit length. Then $j(r) = j(r')$ for all r' with $r \subseteq r' \subset r_0$, but $h(r_0) \supset \bigcup_{r' \subset r_0} h(r')$, hence there exists $w \neq \varnothing$ such that $h(r_0) = \bigcup_{r' \subset r_0} h(r')^\frown w$. Then, again, by definition we have

$$j(r_0) \;=\; \bigcup_{r' \subset r} j(r')^\frown R(r_0, w) \;=\; j(r)^\frown R(r_0, w)$$

Thus, in both cases we have $j(r_0) = j(r)^\frown R(r_0, w)$ for some non-empty w.

By exactly the same argument but looking at r_1, we see that $j(r_1) = j(r)^\frown R(r_1, v)$ for some non-empty v.

But $r_0 \neq r_1$, so by condition 3 of the definition of R, the first coordinates of $R(r_0, w)$ and of $R(r_1, v)$ are not the same. However, we also know $j(r)^\frown R(r_0, w) \subseteq j(s_i)$ while $j(r)^\frown R(r_1, v) \subseteq j(s_j)$. Together with the fact that $j(r) \subset u_i \subseteq j(s_i)$ and $j(r) \subset u_i \subseteq u_j \subseteq j(s_j)$, this gives us the desired contradiction (see Figure 1). We conclude that the only option is $s_i \subseteq s_j$. $\square$(Claim)

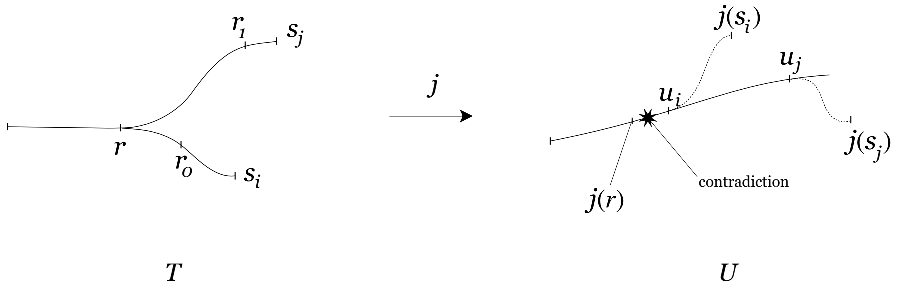

Figure 1: Contradiction assuming $s_i \perp s_j$

Thus, we have an increasing sequence $\{s_i : i < \lambda\}$ in T, and since T is limit-closed, there is $s_\lambda \in T$ with $s_i \subseteq s_\lambda$ for all i. Then $u_i \subseteq j(s_i) \subseteq j(s_\lambda)$ holds for all i. This completes the proof that U is limit-closed. $\square$

With this, we are ready to prove the final result.

Theorem 5.9. *Suppose $\mathbb{P}$ is tree-like, f a continuous function, and $T \Vdash$ "$f(\dot{x}_{gen})$ is a dominating κ-real". Then $T \Vdash$ "there is a Cohen κ-real".*

Proof. First, note that the function mapping a dominating to a strongly dominating κ-real, as in Lemma 4.2, may easily be chosen to be continuous. Thus, without loss of generality, we may assume that $T \Vdash$ "$f(\dot{x}_{gen})$ is a *strongly* dominating κ-real".

Let h be the pre-continuous function corresponding to f, let j be as in Lemma 5.8, and let $g(x) := \bigcup\{j(\sigma) : \sigma \subseteq x\}$. Then g is continuous and for all x and all α we have $f(x)(\alpha) < g(x)(\alpha)$. But then T also forces that $g(\dot{x}_{gen})$ is strongly dominating.

Let $\dot{d}$ be the name such that $T \Vdash g(\dot{x}_{gen}) = \dot{d}$. By Lemma 5.6, $\mathfrak{T}_{\dot{d},T} = tr(j``T)$, so, since j is limit-closure-preserving, $\mathfrak{T}_{\dot{d},T}$ is limit-closed. But this is all we need to apply Lemma 5.4, which shows that $T \Vdash$"there is a Cohen κ-real. $\square$

Unfortunately, none of the methods in this section seem to settle Question 5.1, which the authors consider very significant in the context of forcing over κ^κ: *"is it true that every $<\kappa$-closed forcing adding a dominating κ-real adds a Cohen κ-real?"*

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