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Questions on generalised Baire spaces

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We provide a list of open problems in the research area of *generalised Baire spaces*, compiled with the help of the participants of two workshops held in Amsterdam (2014) and Hamburg (2015).

1 Introduction

When studying questions about real numbers, it is common practice in set theory to investigate the closely related Baire space ω^ω and Cantor space 2^ω. These spaces have been extensively studied by set theorists from various points of view, e.g., questions about *cardinal characteristics of the continuum*, *descriptive set theory* and other combinatorial questions. Furthermore, the investigation of 2^ω and ω^ω has played a role in *model theory*, since countable structures can be coded as elements in these spaces (e.g., Scott's and Lopez-Escobar's theorems). Various motivations from the above areas have led to an interest among set theorists to study the uncountable analogues 2^κ and κ^κ. In recent years, this subject has developed in its own right, with internally motivated open questions and a rich overarching theory. Moreover, unexpected applications to other areas of set theory and mathematics have been discovered (e.g., connections to large cardinals and forcing axioms).

This survey paper is the output of two workshops on *generalised Baire spaces*, the first (*Amsterdam Workshop on Set Theory 2014*) held in Amsterdam in 2014 (3 & 4 November 2014), and the second (*Hamburg Workshop on Set Theory 2015*) in Hamburg in 2015 (20 & 21 September 2015). During both meetings, a group of set theorists met and presented some of the recent developments in this area. This compilation is based on the open questions raised in the talks and the discussions during these two workshops, and it aims to provide a structured overview of the current state of this field.

2 Background and preliminary notions

2.1 Basic definitions

Let κ be an uncountable regular cardinal. We consider the spaces κ^κ and 2^κ with the following topological structure:

Definition 2.1 The *bounded topology on* κ^κ is the one generated by basic open sets of the form

$$[s] = \{f \in \kappa^\kappa \mid f \restriction |s| = s\}$$

with $s \in \kappa^{<\kappa}$. The bounded topology on 2^κ is defined analogously. We call κ^κ and 2^κ with this topology the *generalised Baire space* and *generalised Cantor space*, respectively.

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Remark 2.2 For the study of the generalised Baire space from a topological or descriptive set theoretic point of view, the additional assumption $\kappa^{<\kappa} = \kappa$ is nearly always necessary, since otherwise the elementary structure of the space is very unclear; cf. [22, § II.2.1] for a discussion of this point.

Throughout this paper, any topological notion will refer to the bounded topology, except for the sections dealing with universally Baire sets in 2^κ (due to Ikegami and Viale). Also, for the majority of the questions, κ will be assumed to be regular satisfying $\kappa^{<\kappa} = \kappa$. The only exceptions are the questions about cardinal characteristics (§ 3.1.1). In case of doubt, this will be specified.

Definition 2.3 The family of *Borel sets* is defined as the smallest collection containing the open sets and closed under complements and unions of size $\leq\kappa$. The *projective hierarchy* is defined in the same way as for the classical Baire or Cantor space.

$$A \in \mathbf{\Sigma}^1_0 \textit{ iff } A \text{ is open},$$

$$A \in \mathbf{\Pi}^1_0 \textit{ iff } A \text{ is closed},$$

$$A \in \mathbf{\Sigma}^1_{n+1} \textit{ iff } A \text{ is the projection of a } \mathbf{\Pi}^1_n \text{ set } B \subseteq (\kappa^\kappa)^2,$$

$$A \in \mathbf{\Pi}^1_{n+1} \textit{ iff } A \text{ is the complement of a } \mathbf{\Sigma}^1_{n+1} \text{ set},$$

$$A \in \mathbf{\Delta}^1_n \textit{ iff } A \in \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n.$$

$$\textsc{Proj} := \bigcup_{n\in\omega} \mathbf{\Sigma}^1_n = \bigcup_{n\in\omega} \mathbf{\Pi}^1_n.$$

We note that (assuming $\kappa^{<\kappa} = \kappa$), in contrast to the classical setting, the class of generalised Borel sets does not coincide with the class of generalised $\mathbf{\Delta}^1_1$ sets, cf., e.g., [22, Theorem 18]. In view of this, Halko defined the *Borel** sets as a generalisation of the concept of Borel codes as well-founded trees to non-well-founded trees; cf. [32] for a definition. It is known that Borel $\subsetneq \mathbf{\Delta}^1_1 \subseteq$ Borel* $\subseteq \mathbf{\Sigma}^1_1$. Moreover, both Borel* $= \mathbf{\Sigma}^1_1$ and Borel* $\neq \mathbf{\Sigma}^1_1$ are consistent, and $\mathbf{\Delta}^1_1 \neq$ Borel* is also consistent; cf. [22, 37]. The consistency of $\mathbf{\Delta}^1_1 =$ Borel* is an open problem; cf. Question 3.21.

Definition 2.4 *A tree* is a partially ordered set $(T, <)$ such that for all $t \in T$ the set $\{s \in T \mid s < t\}$ is well-ordered. We are using standard terminology for trees such as *rank* and *height*.

Remark 2.5 As in classical descriptive set theory, it is often useful to consider "descriptive set theoretic" trees as subsets of $\kappa^{<\kappa}$ or $2^{<\kappa}$ closed under initial segments. For such trees, $[T]$ denotes the set of branches through T (i.e., $x \in \kappa^\kappa$ or 2^κ such that for every $\alpha < \kappa$, $x{\restriction}\alpha \in T$). It is clear that $[T]$ is a closed set and every closed set has the form $[T]$ for some tree T in the above sense.

Definition 2.6 A κ*-Kurepa tree* is a tree T such that

1. $\text{height}(T) = \kappa$,
2. T has strictly more than κ branches,
3. for each $\alpha < \kappa$, $|\{t \in T \mid \text{height}(t) = \alpha\}| \leq |\alpha| + \aleph_0$.

Definition 2.7 Let X be some set of cardinality κ. An ideal I on $\wp(X)$ is κ*-complete* if any $<\kappa$-union of elements of I is in I. We put $I^+ := \wp(X)\setminus I$.

In the following definitions, we always refer to κ-complete ideals.

Definition 2.8

1. An ideal $I \subseteq \wp(\kappa)$ is called a *weak P-point* iff for all $A \in I^+$ and $f \in \kappa^A$ with $\{f^{-1}(\{\alpha\}) \mid \alpha < \kappa\} \subseteq I$, there exists a $B \in I^+ \cap \wp(A)$ such that f is $(<\kappa)$-to-one on B.
2. An ideal $I \subseteq \wp(\kappa)$ is called a *local Q-point* iff for every $g \in \kappa^\kappa$ there exists a $B \in I^+$ such that for every $(\alpha, \beta) \in B \times B$ with $\alpha < \beta$, we have that $g(\alpha) < \beta$. The ideal I is a *weak Q-point* iff $I{\restriction}A$ is a local Q-point for every $A \in I^+$.

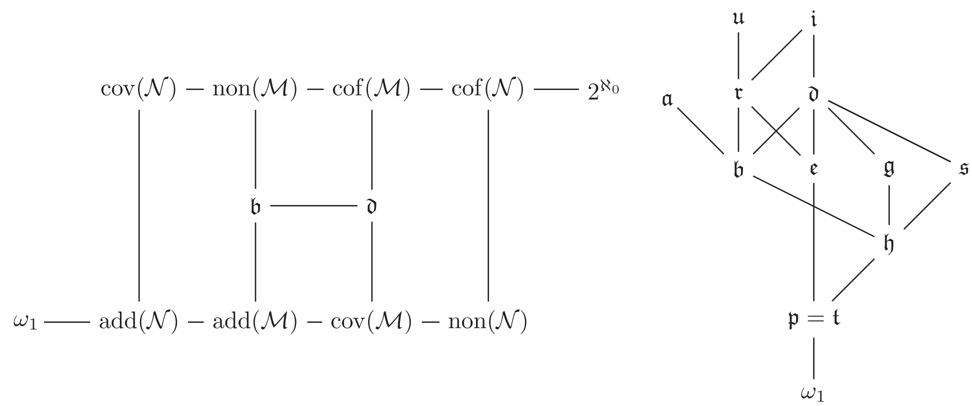

Fig. 1 Cichoń's diagram and the diagram of combinatorial cardinal characteristics.

Definition 2.9 Let $\kappa < \lambda$ be cardinals. An ideal $I \subseteq \wp([\lambda]^{<\kappa})$ is called a *weak* χ*-point* iff given $A \in I^+$ and $g \in ([\lambda]^{<\kappa})^\kappa$, there exists $B \in I^+ \cap \wp(A)$ such that $g(\bigcup(a \cap \kappa)) \subseteq b$, for all $a, b \in B$ such that $\bigcup(a \cap \kappa) < \bigcup(b \cap \kappa)$.

Definition 2.10 Let $\mathbb{P}$ be a forcing partial order and λ any cardinal. We say that $\mathbb{P}$ has the λ*-c.c.* iff every antichain has size $< \lambda$. We say that $\mathbb{P}$ is $<\lambda$*-closed* iff for every $\gamma < \lambda$ and every decreasing sequence $\langle p_\beta \mid \beta < \gamma \rangle$ there is $p \in \mathbb{P}$ with $p \leq p_\beta$ for all $\beta < \gamma$. We say that $\mathbb{P}$ is κ^κ*-bounding* iff for every condition $p \in \mathbb{P}$ and every $\mathbb{P}$-name $\dot{f}$ for an element of κ^κ in the generic extension, there is $q \leq p$ and a ground model $g \in \kappa^\kappa$ such that $q \Vdash \dot{f}(\alpha) \leq g(\alpha)$ for all α.

2.2 Cardinal characteristics

Cardinal characteristics of the continuum have been studied extensively in recent decades. Questions about the consistency of cardinal characteristic inequalities have largely motivated the development of sophisticated forcing iteration and preservation techniques.

We refer the reader to the expositions in [4] and [6] for a detailed overview. The former focuses mainly on the cardinal characteristics occurring in *Cichoń's diagram*, i.e., those associated with the null and meager ideals, as well as the bounding number $\mathfrak{b}$ and the dominating number $\mathfrak{d}$. [6] presents many cardinal characteristics associated with combinatorial aspects of ω^ω, such as the splitting number $\mathfrak{s}$ [6, Definition 3.1], the reaping number $\mathfrak{r}$ [6, Definition 3.6], the ultrafilter number $\mathfrak{u}$ [6, Definition 9.6], the tower number $\mathfrak{t}$ and the pseudointersection number $\mathfrak{p}$ [6, Definition 6.2], which are equal by [52], the distributivity number $\mathfrak{h}$ [6, Definition 6.5], the groupwise density number $\mathfrak{g}$ [6, Definition 6.26], the almost disjointness number $\mathfrak{a}$ [6, Definition 8.3], the independence number $\mathfrak{i}$ [6, Definition 8.11] and the evasion number $\mathfrak{e}$ [6, Definition 10.1].

The traditional way of representing the ZFC-provable relations between various cardinal characteristics is Cichoń's diagram and the diagram of combinatorial cardinal characteristics (Figure 1).[1]

Definition 2.11 Let κ be an uncountable cardinal. A set A in κ^κ or 2^κ is *nowhere dense* if the interior of its closure is empty, and a set A *is* κ*-meager* if it is a $\leq\kappa$-sized union of nowhere dense sets. The κ-ideal of κ-meager sets is denoted by $\mathcal{M}_\kappa$. This allows us to define $\mathrm{add}(\mathcal{M}_\kappa)$, $\mathrm{cov}(\mathcal{M}_\kappa)$, $\mathrm{non}(\mathcal{M}_\kappa)$ and $\mathrm{cof}(\mathcal{M}_\kappa)$ in the standard way.

Usually the above definition is only applied to cardinals satisfying $\kappa^{<\kappa} = \kappa$. In the absence of this assumption, $\mathrm{add}(\mathcal{M}_\kappa) = \mathrm{cov}(\mathcal{M}_\kappa) = \kappa^+$ by [45, Lemma 1.3 (d)], and $\mathrm{non}(\mathcal{M}_\kappa) \geq \kappa^{<\kappa}$ by [7, Proposition 4.15]. The exact values of $\mathrm{non}(\mathcal{M}_\kappa)$ and $\mathrm{cof}(\mathcal{M}_\kappa)$ may still be non-trivial.

[1] In Cichoń's diagram, the lines from left to right and from bottom to top represent $\leq$, provable in ZFC. Additionally the equalities $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$ and $\mathrm{cof}(\mathcal{M}) = \max(\mathrm{non}(\mathcal{M}), \mathfrak{d})$ hold. In the combinatorial diagram, lines from bottom to top represent $\leq$. The first of these two diagrams is *complete* in the sense that any implications missing from it are consistently false. The second diagram is almost complete in this sense, but there are still some open questions (e.g., the consistency of $\mathfrak{i} < \mathfrak{a}$).

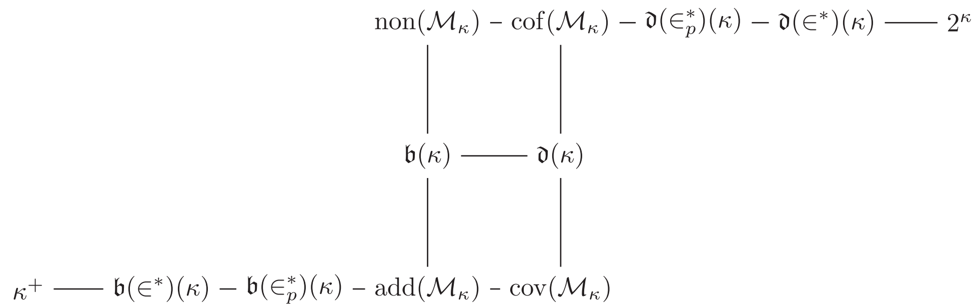

Fig. 2 Cichoń's diagram for strongly inaccessible κ.

Definition 2.12 Let κ be regular and let $f, g \in \kappa^\kappa$. Then we define $f \leq^\kappa g$ iff $\exists \alpha < \kappa \ \ \forall \beta > \alpha : f(\beta) \leq g(\beta)$ and say that g dominates f. A family $\mathcal{B} \subseteq \kappa^\kappa$ is called *unbounded* if for all $g \in \kappa^\kappa$ there is $f \in \mathcal{B}$ such that $f \not\leq^\kappa g$. A family $\mathcal{D} \subseteq \kappa^\kappa$ is called *dominating* if for all $g \in \kappa^\kappa$ there is $f \in \mathcal{D}$ such that $g \leq^\kappa f$. The cardinals

$$\mathfrak{b}(\kappa) = \min\{|\mathcal{B}| \ \mid \ \mathcal{B} \subseteq \kappa^\kappa \text{ is an unbounded family}\} \text{ and}$$

$$\mathfrak{d}(\kappa) = \min\{|\mathcal{D}| \ \mid \ \mathcal{D} \subseteq \kappa^\kappa \text{ is a dominating family}\}$$

are called the *bounding number* and the *dominating number*, respectively. We also define $\overline{\mathfrak{d}(\kappa)}$ as the least size of a family $\mathcal{D} \subseteq \kappa^\kappa$ such that for every $g \in \kappa^\kappa$ there is $X \in [\mathcal{D}]^{<\kappa}$ such that for all $\alpha < \kappa$, $g(\alpha) \in \bigcup_{f \in X} f(\alpha)$.

There is currently no agreement on the right generalisation of the *Lebesgue null* ideal to the generalised Baire space: indeed, the search for a suitable generalisation is an important open problem; cf. Question 3.19.[2] Nevertheless, for strongly inaccessible κ, one can consider generalisations of certain combinatorial characteristics which are equivalent to add($\mathcal{N}$) and cof($\mathcal{N}$) in the classical setting.

Definition 2.13 (Brendle, Brooke-Taylor, Friedman, Montoya; [11]) Let κ be strongly inaccessible.

1. A *slalom* is a function $F : \kappa \to [\kappa]^{<\kappa}$ such that $F(\alpha) \in [\kappa]^{|\alpha|}$ for all $\alpha < \kappa$.
2. A *partial slalom* is a partial function $F : \mathrm{dom}(F) \subseteq \kappa \to [\kappa]^{<\kappa}$ such that $F(\alpha) \in [\kappa]^{|\alpha|}$ for all $\alpha \in \mathrm{dom}(F)$.
3. If $f \in \kappa^\kappa$ and F is a slalom, then $f \in^* F$ iff $\exists \beta \, \forall \alpha > \beta \, (f(\alpha) \in F(\alpha))$. If F is a partial slalom, then $f \in^*_p F$ iff $\exists \beta \, \forall \alpha > \beta, \alpha \in \mathrm{dom}(F) \, (f(\alpha) \in F(\alpha))$.
4. Then we can define

$$\mathfrak{b}(\in^*)(\kappa) := \min\{|\mathcal{F}| \mid \text{for all slaloms } F \text{ there is an } f \in \mathcal{F} \text{ such that } f \notin^* F\} \text{ and}$$

$$\mathfrak{d}(\in^*)(\kappa) := \min\{|\mathcal{G}| \mid \mathcal{G} \text{ is a family of slaloms such that } \forall f \in \kappa^\kappa \, \exists F \in \mathcal{G} \, (f \in^* F)\},$$

and analogously $\mathfrak{b}(\in^*_p)(\kappa)$ and $\mathfrak{d}(\in^*_p)(\kappa)$.

Theorem 2.14 *(Brendle, Brooke-Taylor, Friedman, Montoya;* [11]*) If κ is strongly inaccessible then all the implications in Figure 2 hold.*

In [11], several models are produced to separate cardinal invariants in this diagram (e.g., κ-Cohen forcing increases cov($\mathcal{M}_\kappa$) but leaves non($\mathcal{M}_\kappa$) small) although many questions remain; cf. Question 3.1.

The combinatorial cardinal characteristics are, in general, easy to generalise. In particular, the following numbers are defined by a direct replacement of ω by κ and "finite" by "$<\kappa$": $\mathfrak{a}(\kappa)$, $\mathfrak{e}(\kappa)$, $\mathfrak{g}(\kappa)$, $\mathfrak{i}(\kappa)$, $\mathfrak{r}(\kappa)$, $\mathfrak{s}(\kappa)$, and $\mathfrak{u}(\kappa)$.

Some care needs to be taken concerning $\mathfrak{p}$ and $\mathfrak{t}$, since a straightforward generalisation would yield a cardinal number which is always equal to ω. The "correct" definition of $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$ is to require that the family in

[2] We should like to mention that Galeotti developed the basic theory of an generalised analogue $\mathbb{R}_\kappa$ of the real numbers in [27, 28] on the basis of Conway's surreal numbers [16]. The space $\mathbb{R}_\kappa$ allows us to define appropriate notions of κ-metric and κ-Polish spaces and gives hope for a generalisation of measure theory that might shed some light on the question of the generalisation of random forcing.

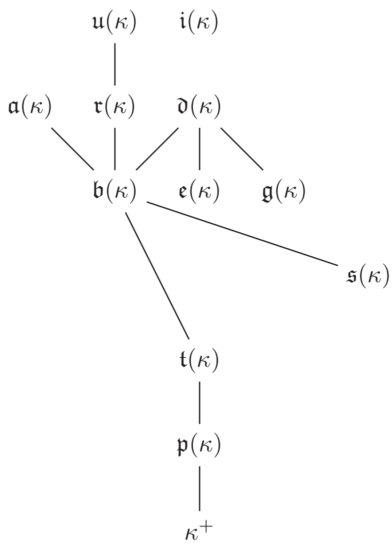

Fig. 3 Diagram of the generalised combinatorial cardinal characteristics (for a fixed κ).

question have size at least κ. In [64] it was shown that under this assumption, it follows that $\mathfrak{t}(\kappa) \geq \kappa^+$, and a similar argument works for $\mathfrak{p}(\kappa)$.

A similar point can be made for $\mathfrak{h}(\kappa)$—by [3], a straightforward generalisation would yield a cardinal number which is always ω if κ has uncountable cofinality and ω_1 if κ is singular of countable cofinality. One can adjust the definition as above, but we do not know whether this would imply $\mathfrak{h}(\kappa) \geq \kappa^+$. Lacking an agreed-upon definition for $\mathfrak{h}(\kappa)$, we leave it out of the discussion and the diagram.

We also introduce a new characteristic, which is equal to $\mathrm{cov}(\mathcal{M})$ in the classical case but may be more complicated in the generalised case. It was first isolated and studied by Landver in [45], who, among other things, proved that it is equal to $\mathrm{cov}(\mathcal{M}_\kappa)$ for strongly inaccessible κ; cf. Question 3.8.

Definition 2.15 Let κ be regular. $f, g \in \kappa^\kappa$. We say that f, g are *eventually different* iff there is $\beta < \kappa$ such that for all $\alpha \geq \beta$, $f(\alpha) \neq g(\alpha)$. A family $\mathcal{E} \subseteq \kappa^\kappa$ is called *eventually different* iff for every $g \in \kappa^\kappa$ there is $f \in \mathcal{E}$ which is eventually different from g. The cardinal

$$\mathfrak{in}(\kappa) = \min\{\, |\mathcal{E}| \;\mid\; \mathcal{E} \subseteq \kappa^\kappa \text{ is an eventually different family}\}$$

is called the *inequality number*.

The currently known relations between these cardinal characteristics (for a fixed κ) are summarised in Figure 3; cf. Question 3.2. As in Figure 1, lines from bottom to top signify $\leq$, however, this diagram is far from complete, in the sense that for many implications, it is not known whether they are consistent or not. Moreover, we should note that, unlike the classical situation, the consistency of such relations can have substantial large cardinal strength. Most notably, the following holds: for all regular κ, the statement "$\mathfrak{s}(\kappa) > \kappa^+$" is equiconsistent with $\mathsf{ZFC} + o(\kappa) \geq \kappa^{++}$ (where $o(\kappa)$ refers to the Mitchell order). The lower bound was proved by Zapletal in [67], and the equiconsistency in a recent result of Ben-Neria and Gitik [5].

A different (and little explored) line of inquiry is what happens when one examines several (or even all) cardinals κ at the same time. This was done for $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ in [18]. A yet further question could be how far large cardinal properties of κ play a role.

Cardinal characteristics for uncountable κ have been studied by various researchers. Relevant contributions include [5, 14, 15, 18, 29, 30, 57, 63, 64, 67].

2.3 Regularity properties

The three classical properties that have played a crucial role in the development of descriptive set theory are the *Baire property*, *Lebesgue measurability* and the *perfect set property*. All analytic sets satisfy these properties,

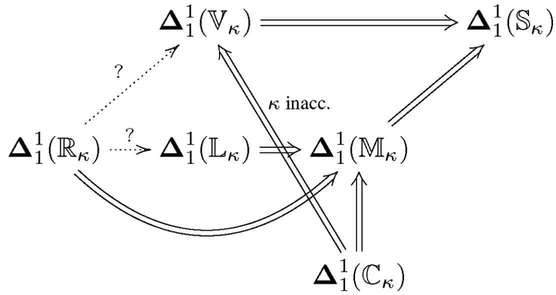

Fig. 4 Diagram of implications for $\mathbb{P}$-measurability of $\mathbf{\Delta}^1_1$ sets.

the Axiom of Choice allows us to construct counterexamples, and in the *Solovay model*[3] all sets satisfy these properties. Moreover, the statement "all $\mathbf{\Pi}^1_1$ sets have the perfect set property" is equivalent to $\omega_1^{\mathbf{L}[a]} < \omega_1$ for all $a \in \omega^\omega$. The Baire property and Lebesgue measurability for $\mathbf{\Sigma}^1_2$-sets is false if $\mathbf{V}=\mathbf{L}$ but holds in suitable generic extensions of $\mathbf{L}$. By [60], an inaccessible is necessary to prove the consistency of "all projective sets are Lebesgue measurable", whereas the strength of "all projective sets have the property of Baire" is just ZFC. Several people have studied various generalisations of these properties, e.g., the *Ramsey property*, K_σ*-regularity* (or *Hurewicz dichotomy*), and various properties naturally related to definable forcing notions.

In the generalised setting, we do not have an adequate notion of Lebesgue measurability, but we do have natural definitions for the other properties.

Definition 2.16 A set $A \subseteq 2^\kappa$ has the *κ-Baire property* iff $A \triangle O$ is κ-meager for some open set $O \subseteq 2^\kappa$ (here "open" refers to the bounded topology; cf. Definition 2.1).

All generalised Borel sets have the *κ-Baire property*, but by results of Halko and Shelah [33, Theorem 4.2], the *club filter* on κ is a generalised-$\mathbf{\Sigma}^1_1$ set without the Baire property, in stark contrast to the classical setting. On the other hand, it is independent whether all generalised $\mathbf{\Delta}^1_1$ sets satisfy the Baire property; cf. [22, § IV.3] (recall that in the generalised setting, the class of Borel sets is smaller than the class of $\mathbf{\Delta}^1_1$ sets).

The Baire property can also be generalised to measurability properties generated by tree-like forcing notions, in a way similar to the classical results [12, 13, 39, 43].

Definition 2.17 Let $\mathbb{P}$ be a forcing notion whose conditions are trees on $\kappa^{<\kappa}$ or $2^{<\kappa}$, ordered by inclusion. Let $\mathcal{N}_\mathbb{P}$ be the ideal of subsets A such that for every $T \in \mathbb{P}$ there is $S \leq T$ with $[S] \cap A = \varnothing$. Let $\mathcal{I}_\mathbb{P}$ be the κ^+-ideal generated by $\mathcal{N}_\mathbb{P}$. A subset A of 2^κ or κ^κ is called $\mathbb{P}$-measurable if for every $T \in \mathbb{P}$ there is $S \leq T$ such that $[S] \subseteq^* A$ or $[S] \cap A =^* \varnothing$, where $\subseteq^*$ and $=^*$ refers to "modulo $\mathcal{I}_\mathbb{P}$".

In this setting, the Baire property is the same as $\mathbb{P}$-measurability for $\mathbb{P}$ being the κ-Cohen forcing on 2^κ. A first systematic study of such regularity properties, where $\mathbb{P}$ was a suitably generalised version of Cohen, Sacks, Miller, Laver, Mathias and Silver forcing, was conducted in [24], where it was established that (1) all Borel sets satisfy $\mathbb{P}$-measurability for all $\mathbb{P}$; (2) $\mathbf{\Sigma}^1_1$ sets do not satisfy $\mathbb{P}$-measurability for any $\mathbb{P}$, and (3) $\mathbb{P}$-measurability for $\mathbf{\Delta}^1_1$ sets is independent, and the implications between statements of the form "all $\mathbf{\Delta}^1_1$ sets are $\mathbb{P}$-measurable" follows the pattern shown in Figure 4,[4] in parallel to the situation on the $\mathbf{\Delta}^1_2$ level in the classical setting.

Another classical property that has interesting generalisations is the perfect set property and the related Hurewicz dichotomy.

Definition 2.18 A set A satisfies the *κ-perfect set property* if either $|A| \leq \kappa$ or A contains a closed homeomorphic copy of 2^κ (alternatively, a perfect subset). A set A satisfies the *Hurewicz dichotomy* if A is either a κ-union of κ-compact sets,[5] or A contains a closed homeomorphic copy of κ^κ.

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[3] A model of ZF without Choice, obtained by collapsing an inaccessible to ω_1 using the Lévy collapse and then taking the $\mathbf{L}(\mathbb{R})$ of the generic extension.

[4] Here $\mathbb{C}_\kappa$, $\mathbb{S}_\kappa$, $\mathbb{M}_\kappa$, $\mathbb{L}_\kappa$, $\mathbb{R}_\kappa$ and $\mathbb{V}_\kappa$ stand for Cohen, Sacks, Miller, Laver, Mathias and Silver forcing, respectively, and $\mathbf{\Delta}^1_1(\mathbb{P})$ abbreviates "all $\mathbf{\Delta}^1_1$ sets are $\mathbb{P}$-measurable".

[5] A set A is called κ-compact if every open cover of A has an open subcover of size $\leq \kappa$.

Here, the situation diverges even more from the classical setting: if there exists a κ-Kurepa tree T, then $[T]$ cannot have the perfect set property, so it is consistent for the perfect set property to fail even for closed sets. In [59], Schlicht constructed a model where all projective sets satisfy the generalised perfect set property. The related Hurewicz dichotomy was first studied in [48]. It consistently fails for closed sets and consistently holds for $\mathbf{\Sigma}^1_1$ sets. Moreover, it is well-known that 2^κ and κ^κ are homeomorphic if and only if κ is not a weakly compact cardinal (cf. [36, Theorem 1] and [48, Corollary 2.3]). Therefore, if κ is not weakly compact, the perfect set property implies the Hurewicz dichotomy, and hence it consistently holds for all projective sets. It is an open question whether for weakly compact κ such an implication holds and whether the Hurewicz dichotomy for projective sets is consistent (cf. Questions 3.37 and 3.39).

Relevant contributions to these and related questions include [22, 24, 44, 48, 55, 59].

2.4 Model theory and Borel equivalence relations

Perhaps the strongest motivation for studying generalised Baire spaces comes from connections between the model theory of uncountable structures and the generalised theory of Borel equivalence relations.

Classically, a central topic in descriptive set theory has been the study of definable equivalence relations on Polish spaces. Given an equivalence relation E on a Polish space X and an equivalence relation F on a Polish space Y, one says that E is *Borel reducible* to F ($E \leq_{\mathrm{B}} F$) iff there exists a Borel function $\varphi : X \to Y$ such that $x E y$ if and only if $\varphi(x) F \varphi(y)$ (analogously, one can define *continuous reducibility* $\leq_{\mathrm{c}}$, by replacing "Borel" with "continuous").

Two essential results in this area are the *Silver dichotomy* and the *Harrington-Kechris-Louveau* (or *Glimm-Effros*) *dichotomy*. The former is the statement: if $E \subseteq 2^\omega \times 2^\omega$ is a $\mathbf{\Pi}^1_1$ equivalence relation, then either it has countably many equivalence classes or there are perfectly many E-inequivalent points (equivalently: either $E \leq_{\mathrm{B}} \mathrm{id}_\omega$ or $\mathrm{id}_{2^\omega} \leq_{\mathrm{B}} E$). The latter is the statement: if $E \subseteq 2^\omega \times 2^\omega$ is a Borel equivalence relation, then either $E \leq_{\mathrm{B}} \mathrm{id}_{2^\omega}$ or $\mathbf{E}_0 \leq_{\mathrm{c}} E$, where $x \mathbf{E}_0 y$ if and only if for all but finitely many n, we have $x(n) = y(n)$.

Analytic equivalence relations of particular interest include the *isomorphism relation* and the *(bi-)embeddability relation* on countable structures. Fix a canonical encoding of all countable structures by reals, writing M_x to refer to the model coded by x. For a theory T, define

$$\begin{aligned} \mathrm{Mod}^\omega_T &:= \{x \in 2^\omega \mid M_x \models T\}, \\ \cong^\omega_T &:= \{(x, y) \mid M_x \models T \text{ and } M_y \models T \text{ and } M_x \cong M_y\}, \text{ and} \\ \sqsubseteq^\omega_T &:= \{(x, y) \mid M_x \models T \text{ and } M_y \models T \text{ and } M_x \sqsubseteq M_y\}. \end{aligned}$$

Here $M_x \cong M_y$ means that M_x and M_y are isomorphic and $M_x \sqsubseteq M_y$ means that there is an injection $f : M_x \to M_y$ such that $M_x \cong M_y \restriction \mathrm{ran}(f)$. For two theories T, T', one defines $T \leq^\omega T'$ iff $(\cong^\omega_T) \leq_{\mathrm{B}} (\cong^\omega_{T'})$. We refer the reader to [42, § 16.C] for a good introduction, and to [34] for a more extensive survey. Cf. also [26, 47] for results about the (bi-)embeddability relation on countable structures.

A natural question is whether $T \leq^\omega T'$ gives us information about the relationship between T and T', such as whether T is in some sense a simpler theory than T'. It turns out that the answer is negative (e.g., let T be the theory of the rationals and T' the theory of vector spaces over the rationals; cf. [22, p. 5]). However, if ω is replaced by an uncountable κ, and the definitions of Mod^κ_T, $\cong^\kappa_T$, $\sqsubseteq^\kappa_T$ and $T \leq^\kappa T'$ are adequately generalised, then there is a strict relationship between $\leq^\kappa$ and the classification in stability theory. This provides strong incentive to study these relationships in the generalised, rather than the classical, Baire and Cantor spaces.

We note that in this connection it is important to consider theories in the *infinitary languages* $\mathcal{L}_{\lambda,\kappa}$, as well as the *infinitely deep* languages $M_{\lambda,\kappa}$. Cf. [22, 37] or the original source [41] for more on such languages; cf. [22] for an in-depth exposition of model theory in this generalised setting, as well as the generalisations of the Silver and the Harrington-Kechris-Louveau dichotomy.

2.5 Universally Baire sets

Finally we include a recent topic due to Ikegami and Viale [40]. Recall that a set $A \subseteq 2^\omega$ is called *universally Baire* if for every complete Boolean algebra B and every continuous function $f : \mathrm{St}(B) \to \omega^\omega$ (where $\mathrm{St}(B)$ is the Stone space of B), the pre-image $f^{-1}[A]$ has the Baire property in $\mathrm{St}(B)$. This notion is due to Feng, Magidor and Woodin [19] and plays a crucial role in bridging diverse areas of set theory: descriptive set theory, large cardinals, determinacy and inner model theory.

Definition 2.19 (Ikegami & Viale) Let 2^κ be endowed with the *product topology*. Let A be a subset of 2^κ and let Γ be any class of complete Boolean algebras. We say that A is *universally Baire in* 2^κ *with respect to* Γ ($\mathrm{uB}^\Gamma_\kappa$) if for any $B \in \Gamma$ and any continuous function $f : \mathrm{St}(B) \to 2^\kappa$, the set $f^{-1}[A]$ has the κ-Baire property in the Stone space of B.

For $\kappa = \omega$ and Γ being the class of all complete Boolean algebras, the above is an equivalent characterisation of the classical notion of universally Baire sets of reals. In this definition, it is essential to consider the *product topology* rather than the *bounded topology* on 2^κ as in the other sections. This is necessary to achieve the correspondence between names for elements of 2^κ and continuous functions from the corresponding Stone space to 2^κ.

Ikegami and Viale study to what extent this concept resembles the classical one under suitable large cardinal hypotheses and forcing axioms. Their main observation is that they can lift to the generalised framework the standard characterisations of universally Baire sets of reals as the sets $A \subseteq 2^\omega$ such that

1. A is obtained by the projection on 2^ω of one of a pair of trees (T, U) on $(2 \times V)^{<\omega}$ which project to complements, and
2. for any forcing Boolean algebra B, A admits a B-name $\dot{A}$, such that for club many countable models $M \prec H_{|B|^+}$, and for all G, M-generic filters for B, we have that $\{\bar{\tau}_{\bar{G}} : [\![\tau \in \dot{A}]\!]_B \in G\} = A \cap \bar{M}[\bar{G}]$, where $\bar{\tau}_{\bar{G}}$ is the evaluation of the $\bar{B}$-name $\bar{\tau}$ by the $\bar{M}$-generic filter $\bar{G}$, and $\bar{a}$ denotes the image of $a \cap M$ under the transitive collapse of M, for all objects a.

Many other nice properties of universally Baire sets of reals can be naturally formulated in this general framework. Nevertheless, many basic questions remain open, cf. [40] and § 3.3.2.

3 The list of open questions

The list of open questions is organised according to the three categories below.

3.1. **Set theory and combinatorics of the generalised reals:** This section deals with questions on generalisations of results concerning cardinal characteristics, filters and ideals, tree combinatorics, properties of forcings, and related questions. Many of these questions can be viewed as attempts at generalising the theory summarised in [4] and [6].
3.2. **Generalised descriptive set theory:** This includes questions about 2^κ and κ^κ from the topological perspective, as well as questions concerning definable subsets of these spaces.
3.3. **Borel reducibility, model theory and other topics:** This includes questions on the application of descriptive set theoretic methods to model theory (e.g., complexity of embeddability relations) and applications to stability theory. In the last part of this section we also consider questions concerning universally Baire sets and connections with forcing axioms.

Each category is further subdivided into sub-categories (although there will invariably be some overlap between the subcategories). We have attempted to list closely related questions next to one another, and in some cases the questions are preceded by a short introduction. Some questions are followed up by "*Further background*": this is intended to provide background material for a better understanding or motivation of the question at hand (e.g., references to undefined notions, explanation of what is already known, potential applications etc.) We have tried to reference the authors who posed the questions whenever this is known, and we give exact references if the corresponding question has appeared in written form.

3.1 Set theory and combinatorics of the generalised reals

3.1.1 Cardinal characteristics on κ

The most general question is to what extent the classical Cichoń's diagram and the diagram of combinatorial characteristics (cf. Figure 1) lift to higher κ. In fact, the next two questions can be understood as a summary of a vast array of questions concerning a better understanding of cardinal characteristics for uncountable κ.

Question 3.1 To what extent does Cichoń's diagram generalise? Specifically, are the implications shown in Figure 2 the only possible ones or are there any additional implications? What can be proved for several cardinals simultaneously? How do large cardinal properties of κ affect the results?

Question 3.2 To what extent does the diagram of combinatorial cardinal characteristics generalise? Are the known implications in Figure 3 the only ones or are there additional implications? Which consistency statements do generalise? If they do, can they be proved in ZFC or do they have large cardinal strength? Which assumptions on κ are necessary and how do they affect the results? What can be proved for several cardinals simultaneously? What about large cardinal properties of κ? Does $\mathfrak{h}(\kappa)$ have a canonical generalisation? If so, how does it relate to other combinatorial cardinal characteristics?

Question 3.3 (Jech, Veličković; Blass, Hyttinen, Zhang; [7]. Brendle, Brooke-Taylor; [9,10]) Let κ be regular. Is $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$ consistent?

Question 3.4 (Kojman, Kubis, Shelah; Brendle; [9])

1. Is $\mathfrak{a}(\aleph_\omega) = \aleph_\omega$ consistent?
2. Is $\mathfrak{b} \neq \mathfrak{a}(\aleph_\omega)$ consistent?
3. Is it consistent to have $\mathfrak{a}(\kappa) < \mathfrak{a}(\mathrm{cf}(\kappa))$ for some singular κ of uncountable cofinality?

Further background. In Question 3.4.3., we require $\mathrm{cf}(\kappa) > \omega$ because, in a recent unpublished work, $\mathfrak{a}(\kappa) < \mathfrak{a}(\omega)$ was proved to be consistent for κ of countable cofinality by Brendle.

One of the obstacles on the way to solving the above questions concerns Canjar ultrafilters on κ. A *Canjar ultrafilter U* on κ is an ultrafilter on κ for which the generalised ultrafilter-Mathias forcing does not add dominating reals.

Question 3.5 (Brooke-Taylor) Is there a κ-complete Canjar ultrafilter on κ, for measurable κ? Do Canjar ultrafilters have a characterisation using P-points?

Further background. By an observation of Friedman and Montoya, a Canjar ultrafilter on a measurable κ cannot be normal.

Question 3.6 (Brooke-Taylor) What is the consistency strength of $\mathfrak{u}(\kappa) < 2^\kappa$?

In the definition of $\mathfrak{d}(\kappa)$ one could replace the ordering $\leq^\kappa$ with $\leq^{\mathrm{club}}$, i.e., $f \leq^{\mathrm{club}} g$ iff $\{\alpha < \kappa \mid f(\alpha) < g(\alpha)\}$ contains a club. Let $\mathfrak{d}_{\mathrm{club}}(\kappa)$ be the dominating number for this relation. Cummings and Shelah [18] proved some analogies between these two different versions, but the following remained open:

Question 3.7 (Cummings, Shelah; [18]) Is it consistent that $\mathfrak{d}_{\mathrm{club}}(\kappa) < \mathfrak{d}(\kappa)$?

Recall the cardinal invariant $\mathfrak{in}(\kappa)$ from Definition 2.15. In the classical setting, this is always equal to $\mathrm{cov}(\mathcal{M})$, and the same holds for strongly inaccessible κ by [45]. It is also known that if κ is successor and $\kappa^{<\kappa} = \kappa$, then $\mathfrak{in}(\kappa) = \mathfrak{d}(\kappa)$. But the following remains open:

Question 3.8 (Matet, Shelah; [54]) If κ is successor but $\kappa^{<\kappa} \neq \kappa$, is $\mathfrak{in}(\kappa) < \mathfrak{d}(\kappa)$ consistent?

Further background. Another related question is the possibility of $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{in}(\kappa)$ for limit cardinals κ, which was recently proved to be consistent in an argument to appear in [11].

The last question refers to κ-complete ideals.

Question 3.9 (Matet, Shelah; [54]) Let $I \subseteq \wp(\kappa)$ be a κ-complete ideal on κ. Let $\overline{\mathrm{cof}}(I)$ be the least size of $X \subseteq I$ such that for every $A \in I$ there is $\mathcal{B} \in [X]^{<\kappa}$ such that $A \subseteq \bigcup \mathcal{B}$. Let P be a property on I. We define non(P) (respectively, $\overline{\mathrm{non}}(P)$) as the least cardinal λ for which there exists an ideal $I \subseteq \wp(\kappa)$ such that $\mathrm{cof}(I) = \lambda$ (respectively, $\overline{\mathrm{cof}}(I) = \lambda$) and I does not satisfy P.

1. Is $\overline{\mathfrak{d}(\kappa)} > \overline{\mathrm{non}}$(weak P-point) consistent?
2. Is $\kappa^+ < \overline{\mathrm{non}}$(weak Q-point) consistent, for κ limit cardinal with $2^{<\kappa} > \kappa$?

3.1.2 Borel conjecture, filters and ideals

Halko and Shelah gave a definition of the concept of *strong measure zero* subsets of 2^κ, see [32] and [33, § 2]. The definition uses straightforward combinatorics which does not presuppose the existence of a measure on 2^κ. The *Borel conjecture on* κ, abbreviated by $\mathsf{BC}(\kappa)$, is the statement "every strong measure zero set in 2^κ has size $\leq \kappa$". In [33] it is proved that $\mathsf{BC}(\kappa)$ is false if κ is a successor satisfying $\kappa = \kappa^{<\kappa}$.

Question 3.10 (Halko, Shelah; [33]) Is it consistent that $\mathsf{BC}(\kappa)$ holds for inaccessible κ?

Question 3.11 (Matet, Shelah; [54]) Given cardinals $\kappa < \lambda$, let $I_{\kappa,\lambda}$ be the ideal of all subsets of $\wp([\lambda]^{<\kappa})$ which are not cofinal in $([\lambda]^{<\kappa}, \subseteq)$. Is it consistent that $\kappa < \kappa^{<\kappa}$ and I_{κ,κ^+} is a weak χ-point?

Shelah has introduced the concepts *reasonable ultrafilter*, *very reasonable ultrafilter* and *super-reasonable filter* in an attempt to generalise the notion of P-points on ω. The technical definitions can be found in [61, Definition 2.5 (4)–(5)] and [58, Definition 1.11 (4)].

Question 3.12 (Shelah; [61]) Let κ be regular. Is it provable in ZFC that there exist reasonable ultrafilters? Is it provable in ZFC that there exist very reasonable ultrafilters?

Question 3.13 (Shelah; [61]) Let $D \subseteq \wp(\kappa)$ be a filter on κ and $f \in \kappa^\kappa$. Let $D/f := \{A \subseteq \kappa \mid f^{-1}[A] \in D\}$.

1. Is it consistent that there exists a very reasonable ultrafilter D on κ such that for every very reasonable ultrafilter D' on κ there exists a non-decreasing and unbounded $f \in \kappa^\kappa$ such that $D/f = D'/f$?
2. Is it consistent that for every ultrafilter D on κ there is a non-decreasing unbounded function $f \in \kappa^\kappa$ such that either D/f is normal or D/f is reasonable (or even very reasonable)?

Question 3.14 (Shelah; [58]) Is it consistent that there is no super-reasonable filter?

3.1.3 Trees and tree forcings

Every closed, non-empty subset of ω^ω is a continuous image of the Baire space ω^ω (in fact it is even a retract of the whole space) [42, Proposition 2.8]. Every closed subset of κ^κ can be written as $[T]$ for some tree $T \subseteq \kappa^{<\kappa}$, however, by results from [49, Theorem 1.5] there is always a tree T such that $[T]$ is not a continuous image of κ^κ. Therefore it is interesting to ask whether the closed sets induced by trees with certain special properties (e.g. Kurepa trees) can be continuous images of κ^κ.

Question 3.15 (Holy, Lücke, Schlicht) Suppose that $\kappa \geq \omega_2$. Is it consistent that there are κ-Kurepa trees, and for every κ-Kurepa tree T, $[T]$ is a continuous image of ${}^\kappa\kappa$?

Further background. It is known to be consistent [51] that

1. there are κ-Kurepa trees, but for every κ-Kurepa tree T, $[T]$ is not the continuous image of κ^κ, and
2. there are κ-Kurepa trees, and for some κ-Kurepa tree T, $[T]$ is a continuous image of κ^κ while for some other κ-Kurepa tree S, $[S]$ is a not a continuous image of κ^κ.

Also, (1) holds when $\kappa = \omega_1$.

The class of countably infinite trees without infinite branches has a universal family of size ω_1, i.e., there is a family $\mathcal{U}$ of size ω_1 of countably infinite trees without infinite branches, such that every such tree can be mapped into some $T \in \mathcal{U}$ by an order-preserving mapping. It is natural to ask how small a universal family can be when considering the analogue of such trees on ω_1.

Let $\mathcal{T}$ be the class of all trees of size ω_1 without branches of length $\geq \omega_1$. We can order these trees by the relation $T \leq T'$ if and only if there is a strict order preserving map from T to T'. We say that $T \in \mathcal{T}$ is a *largest tree* if it is the largest element in $\mathcal{T}$ with respect to $\leq$.

Question 3.16 (Väänänen) Is it consistent that there is a largest tree in $\mathcal{T}$?

Further background. If CH holds, then there is no such tree by [55].

The following question refers to a generalisation of the following well-known classical fact: if there is a non-constructible real, then for every perfect set P there is a non-constructible real $x \in P$.

Question 3.17 (Woodin. Groszek, Slaman; [31]) Assume that there is a non-constructible subset of ω_1. Does every countably closed perfect tree on ω_1 have a non-constructible branch?

In [8], Jörg Brendle considered Marczewski-style ideals (cf. the $\mathcal{N}_{\mathbb{P}}$ from Definition 2.17) associated to various combinatorial tree forcing notions (Sacks, Mathias, Miller, Laver, Silver and others) and determined the inclusion and orthogonality relations between these ideals.

Question 3.18 Investigate the relations between ideals generated by tree forcing notions in the generalised context, similarly to [8].

Since there is no adequate generalisation of the concepts *Lebesgue measure* and *Lebesgue null*, there is also no adequate generalisation of *random forcing* (cf. § 2.2). One approach is to try to find a forcing which at least satisfies some of the properties of random forcing.

Question 3.19 (Shelah; [63]; Friedman, Laguzzi; [25]) Is there a (non-trivial) tree forcing notion $\mathbb{P}$ satisfying the following properties?

(1) $\mathbb{P}$ is κ^+-c.c.,
(2) $\mathbb{P}$ is $<\kappa$-closed,
(3) $\mathbb{P}$ is κ^κ-bounding,
(4) $\mathbb{P}$ does not have the generalised Sacks property.

Is there a non-trivial tree forcing notion $\mathbb{P}$ fulfilling conditions (1)–(3) for a successor cardinal κ?

Further background. Here, the generalised Sacks property is the following statement: for every name $\dot{f}$ and $T \Vdash \dot{f} \in \kappa^\kappa$, there is a slalom F in the ground model (cf. Definition 2.13) and an $S \leq T$ such that $S \Vdash \forall\alpha\ (\dot{f}(\alpha) \in \check{F}(\alpha))$. A tree forcing satisfying conditions (1)–(3) for weakly compact κ was constructed by Shelah in [62], and for inaccessible κ assuming $\diamondsuit_{\kappa^+}(S^{\kappa^+}_\kappa)$ (where $S^{\kappa^+}_\kappa$ is the set of κ-cofinal ordinals below κ^+) by Friedman and Laguzzi in [25].

An "amoeba" for a forcing poset is a forcing adding a specific tree of generic branches. Amoeba forcings play a central role in increasing additivity numbers and other properties of the ideals.

Question 3.20 Investigate the status of *amoebas* for tree forcings in the generalised context. In particular:

1. Is there an *amoeba* for κ-Sacks, κ-Miller and κ-Laver forcing which does not add κ-Cohen reals?
2. Can we prove that any reasonable *amoeba* for κ-Silver forcing necessarily adds κ-Cohen reals?

3.2 Generalised descriptive set theory

3.2.1 Topology and Silver dichotomy

Recall that in the generalised context we have the Borel and Borel* sets, and we have Borel $\subsetneq \mathbf{\Delta}^1_1 \subseteq$ Borel* $\subseteq \mathbf{\Sigma}^1_1$. It is known that both Borel* $= \mathbf{\Sigma}^1_1$ and Borel* $\neq \mathbf{\Sigma}^1_1$ are consistent, and that $\mathbf{\Delta}^1_1 \neq$ Borel* is consistent, however, the consistency of $\mathbf{\Delta}^1_1 =$ Borel* is still open; cf. [22, 32, 37] for more detail.

Question 3.21 (Friedman, Hyttinen, Kulikov; [22, 37]) Is it consistent that $\mathbf{\Delta}^1_1 =$ Borel*?

Questions 3.22 to 3.27 questions refer to [49], where various sub-classes of $\mathbf{\Sigma}^1_1$-sets are examined.

Question 3.22 (Holy, Lücke, Schlicht; [49]) Is it consistent that the club filter on κ is an injective continuous image of κ^κ?

Further background. It is consistent that the club filter on κ is not a continuous injective image of any closed subset of κ^κ (Lücke, Schlicht; [49]).

Question 3.23 (Lücke, Schlicht; [49]) Let C^{κ,κ^+} be the class of all continuous images of $(\kappa^+)^\kappa$. Is it consistent that every set in C^{κ,κ^+} is a continuous injective image of a closed subset of κ^κ?

Question 3.24 (Holy, Lücke, Schlicht) Is it consistent that every closed relation on κ^κ has a definable uniformisation and there is no definable wellorder of κ^κ?

Further background. It is known to be consistent that there is a closed relation with no definable uniformisation [50].

Question 3.25 (Holy, Lücke, Schlicht) Is it consistent that $\wp(\kappa) \nsubseteq \mathbf{L}$ and there is a wellorder of $\mathbf{H}_{\kappa^+}$ definable over $\mathbf{H}_{\kappa^+}$ by a Σ_1-formula without parameters?

Further background. Note that such a wellorder exists if **V**=**L**. Also, recall that in the classical setting, the existence of such a wellorder of the reals implies that the reals are constructible by a classical result of Mansfield.

Question 3.26 (Holy, Lücke, Schlicht) Is it consistent that $\kappa > \omega_1$, there is a supercompact cardinal $\lambda > \kappa$, and there is a $\Sigma_1(\kappa)$ wellorder of $\mathbf{H}_{\kappa^+}$?

Further background. Results of Woodin on the Π_2-maximality of the $\mathbb{P}_{\max}$-extension of $\mathbf{L}(\mathbb{R})$ directly imply that this is not possible for $\kappa = \omega_1$ (cf. [35, Proposition 1.8]). Aspero and Friedman have shown that the above is possible (also for $\kappa = \omega_1$) for well-orders of higher logical complexity (cf. [1, 2]) and it follows, e.g., from the results of [21] that the above is possible (again also for $\kappa = \omega_1$) if one allows for a subset of κ rather than κ as a parameter.

Question 3.27 (Holy, Lücke, Schlicht; [49]) Let $S_1^{\mathbf{L}}$ be the class of subsets $A := \{x \in \kappa^\kappa \mid \mathbf{L}[x, y] \models \varphi(x, y)\}$, for some $y \in \kappa^\kappa$ and some Σ_1-formula φ. If $\mathbf{\Sigma}_1^1 = S_1^{\mathbf{L}}$, is there an $x \subseteq \kappa$ such that $\kappa^\kappa \subseteq \mathbf{L}[x]$?

Further background. This question is motivated by [49, Proposition 1.13], which shows that $\mathbf{\Sigma}_1^1 = S_1^{\mathbf{L}}$ in models of the form $\mathbf{L}[x]$, with $x \subseteq \kappa$.

Question 3.28 (Coskey, Schlicht; [17]) Suppose that X is a regular strong κ-Choquet space of size $> \kappa$ and weight $\leq \kappa$. Is there a closed nonempty subset of κ^κ which is not a continuous image of κ^κ?

Further background. For a definition of strong κ-Choquet spaces one can check [17, Definition 2.1]. Roughly speaking, it is obtained by considering the Choquet game of length κ instead of ω.

Question 3.29 (Coskey, Schlicht; [17]) Is there a universal space for regular strong κ-Choquet spaces of weight $\leq \kappa$?

Question 3.30 (Friedman; [20]) The Silver dichotomy at κ is the statement that if a Borel equivalence relation E on κ^κ has more than κ classes, then there is a continuous reduction of the identity relation on 2^κ to E. Is the Silver dichotomy for κ consistent without assuming the consistency of $0^\#$?

Further background. It is known that at least the consistency of an inaccessible is needed [22, Corollary 43]. It is shown in [20, Theorem 7] that the general Silver dichotomy for Borel equivalence relations is consistent assuming the consistency of $0^\#$.

Question 3.31 (Holy, Lücke, Schlicht) Does the Silver dichotomy for closed sets imply the Silver dichotomy for Borel sets in κ^κ?

3.2.2 Regularity properties

Recall the regularity properties generalising the Baire property from Definition 2.17.

Question 3.32 (Friedman, Khomskii, Kulikov; [24]) Complete the diagram of implications (Figure 4) for regularity properties related to forcing notions at the $\mathbf{\Delta}_1^1$ level.

Since the club filter is a counterexample to all the properties considered above, a natural question is the following:

Question 3.33 (Friedman, Khomskii, Kulikov; [24]) Are there adequate generalisations of classical regularity properties related to forcing notions for which the club filter is regular?

Further background. Note that the notions of "κ-Miller measurability", "κ-Sacks measurability" and "κ-Silver measurability" considered in [48] and [44] are potential candidates for such properties; however, they are not generated by $<\kappa$-closed forcing notions on κ^κ.

Question 3.34 (Friedman, Laguzzi) Is there a version of *generalised Silver forcing* which is $<\kappa$-closed and such that for the corresponding regularity property, one can force that all generalised projective sets are regular?

Recall the perfect set property and Hurewicz dichotomy from Definition 2.18.

Question 3.35 (Holy, Lücke, Schlicht) Does the perfect set property for closed subsets of κ^κ imply the perfect set property for $\mathbf{\Delta}^1_1$ subsets of κ^κ?

Question 3.36 (Holy, Lücke, Schlicht) Is a $\mathbf{\Sigma}^1_1$ wellorder of κ^κ compatible with the perfect set property for $\mathbf{\Sigma}^1_1$ subsets of κ^κ?

Let $\lambda > \kappa$ be inaccessible and let $\mathrm{Coll}(\kappa, <\lambda)$ be the Lévy forcing collapsing λ to κ^+. In [59, Theorem 1.2] it was shown that in $V^{\mathrm{Coll}(\kappa,<\lambda)}$ (the Silver model) all sets definable from ordinals and subsets of κ, and therefore all generalised projective sets, satisfy the perfect set property. If κ is not weakly compact, then the same sets satisfy the Hurewicz dichotomy (cf. § 2.3). But it is not clear what happens in the weakly compact case.

Question 3.37 (Lücke, Motto Ros, Schlicht; [48]) Let κ be weakly compact.

1. Does $\mathrm{Coll}(\kappa, <\lambda)$ force that all sets definable from ordinals and subsets of κ satisfy the Hurewicz dichotomy?
2. If the Hurewicz dichotomy holds for κ-coanalytic sets, is there an inner model with an inaccessible cardinal?

In the next two questions, "κ-Miller measurability" and "κ-Silver measurability" refer to weaker notions than those from Question 3.32.

Question 3.38 (Lücke, Motto Ros, Schlicht; [48]) Can we force κ-Miller measurability for all sets definable from ordinals and subsets of κ, without assuming an inaccessible above κ? Can we do the same for κ-Silver measurability, for κ successor?

Further background. κ-Miller measurability can be forced to hold for all sets definable from ordinals and subsets of κ in the Silver model (which requires an inaccessible $\lambda > \kappa$); cf. [44, Lemma 5.4]. If κ is inaccessible, then κ-Silver measurability for all sets definable from ordinals and subsets of κ holds in the κ-Cohen model; cf. [44, Lemma 4.2].

Question 3.39 (Lücke, Motto Ros, Schlicht; [48])

1. Is it consistent that for a weakly compact κ, all κ-analytic sets have the κ-perfect set property but there is a closed set not satisfying the Hurewicz dichotomy?
2. Is it consistent that all κ-analytic sets are κ-Miller measurable but there is a κ-analytic (closed?) set that does not satisfy the Hurewicz dichotomy? Can such a κ be weakly compact?
3. Can we separate the κ-Miller measurability from the κ-perfect set property in the non-weakly compact case?

3.3 Borel reducibility, model theory and other topics

3.3.1 Isomorphism and embeddability relations

This subsection refers to connections between Borel reducibility of isomorphism/embeddability relations of structures, and the model theory for uncountable models; cf. § 2.4.

For the purpose of this subsection, if L is some (possibly infinitary) logic, a subset $A \subseteq \kappa^\kappa$ is said to be *definable in* L if $A = \{x \mid M_x \models \varphi\}$ for some L-formula φ (where M_x is the model with domain κ coded by x). A set A is *closed under isomorphisms* if whenever $M_x \cong M_y$ then $x \in A$ if and only if $y \in A$. By a result of Vaught [65] (cf. also [22, Theorem 24]), if $A \subseteq \kappa^\kappa$ is closed under isomorphisms, then it is Borel iff it is definable in $\mathcal{L}_{\kappa^+\kappa}$. Since the difference between Borel and Borel* sets in κ^κ has close parallels to the difference between $\mathcal{L}_{\kappa^+\kappa}$ and $M_{\kappa^+\kappa}$, one might expect that a similar characterisation holds for Borel* and $M_{\kappa^+\kappa}$.

In this subsection $\kappa^{<\kappa} = \kappa$ is assumed unless stated otherwise.

Question 3.40 (Hyttinen, Kulikov; [37]) Is it consistent that the sets $B \subseteq \kappa^\kappa$ definable in $M_{\kappa^+\kappa}$ are precisely the Borel* sets closed under isomorphism?

Further background. It is consistently false; cf. [37, § 2].

Question 3.41 Is there a $\varphi \in M_{\kappa^+\kappa}$ such that for all $\psi \in M_{\kappa^+\kappa}$, for some model M of size κ, $M \not\models (\neg\varphi \leftrightarrow \psi)$?

Question 3.42 (Friedman, Hyttinen, Kulikov; [22]) Is it consistent that $\cong^\kappa_T$ is $\mathbf{\Delta}^1_1$ for some complete first-order non-classifiable theory T?

Question 3.43 (Friedman, Hyttinen, Kulikov; [22]) Under which assumptions on T and κ does it hold that if the number of equivalence classes of $\cong^\kappa_T$ is greater than κ, then $\mathrm{Id} \leq_{\mathrm{B}} (\cong^\kappa_T)$?

Further background. By [22], this holds if κ is strongly inaccessible.

Question 3.44 (Friedman, Hyttinen, Kulikov; [22]) How much can we do without the assumption $\kappa^{<\kappa} = \kappa$? In particular, can we prove in ZFC that if $\kappa^{<\kappa} \neq \kappa$, then there are Borel sets closed under isomorphisms which are not definable in $\mathcal{L}_{\kappa^+\kappa}$, or, vice versa, that there are $\mathcal{L}_{\kappa^+\kappa}$-definable sets which are not Borel?

Further background. A consequence of [56, Theorem 4.4] is that consistently, there are even open (and closed) subsets of κ^κ which are not $\mathcal{L}_{\kappa^+\kappa}$-definable. Conversely, it is consistent that there are $\mathcal{L}_{\kappa^+\kappa}$-definable sets which are not Borel; cf. [22, Remark 25].

Question 3.45 (Friedman, Hyttinen, Kulikov; [22]) Suppose T is a classifiable theory, and T' a non-classifiable theory. Is it true that $(\cong^\kappa_T) \leq_{\mathrm{B}} (\cong^\kappa_{T'})$? What about other relations between isomorphisms of theories?

Further background. In a very recent result [38], it was shown to be consistently true, but it is still open whether it is true in ZFC.

Question 3.46 (Friedman, Hyttinen, Kulikov; [23]) Let E^λ_μ for $\lambda \in \{2, \kappa\}$ and $\mu < \kappa$ regular be the equivalence relation on λ^κ where $(\eta, \xi) \in E^\lambda_\mu$ iff the set $\{\alpha \mid \eta(\alpha) = \xi(\alpha)\}$ contains a μ-club, i.e., an unbounded set which contains all the limits of its increasing μ-long sequences. Is $E^\kappa_\mu \leq_{\mathrm{B}} E^2_\mu$?

Further background. If the answer is "yes", then a partial answer to Question 3.45 is obtained (cf. [23]): if T_1 is classifiable and shallow, T_2 is non-classifiable and $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$, then $(\cong^\kappa_{T_1}) \leq_{\mathrm{B}} (\cong^\kappa_{T_2})$.

Question 3.47 (Friedman, Hyttinen, Kulikov; [22]) Assuming $\kappa = \omega_2$ and using the notation of Question 3.46, is it consistent that $E^2_{\omega_1}$ Borel reduces to E^2_ω?

In [56], Motto Ros analysed the embeddability relation for the theories Tree and Graph, which axiomatise trees and graphs of size κ, respectively. Thus the relations $\sqsubseteq^\kappa_{\mathsf{Tree}}$ and $\sqsubseteq^\kappa_{\mathsf{Graph}}$ (as defined in § 2.4) are analytic quasi-orders on 2^κ, which represent the embeddability relations among trees and graphs of size κ, respectively. Motto Ros shows that if κ is weakly compact, then $\sqsubseteq^\kappa_{\mathsf{Tree}}$ and $\sqsubseteq^\kappa_{\mathsf{Graph}}$ are both complete for analytic quasi-orders on 2^κ, in the sense that for each such quasi-order R, we have Borel reductions $R \leq_{\mathrm{B}} (\sqsubseteq^\kappa_{\mathsf{Tree}})$ and $R \leq_{\mathrm{B}} (\sqsubseteq^\kappa_{\mathsf{Graph}})$. The first question is about relaxing the conditions on κ:

Question 3.48 (Motto Ros; [56]) Suppose κ is uncountable with $\kappa^{<\kappa} = \kappa$. Are the relations $\sqsubseteq^{\kappa}_{\mathsf{Tree}}$ and/or $\sqsubseteq^{\kappa}_{\mathsf{Graph}}$ complete for analytic quasi-orders? What if we drop the assumption $\kappa^{<\kappa} = \kappa$? What about $\sqsubseteq^{\kappa}_{\mathsf{Graph}}$ in that case?

Further background. Recent unpublished work of Mildenberger and Motto Ros strongly suggests that $\sqsubseteq^{\kappa}_{\mathsf{Tree}}$ and $\sqsubseteq^{\kappa}_{\mathsf{Graph}}$ are complete for arbitrary κ satisfying $\kappa^{<\kappa} = \kappa$. Without this assumption, it seems likely that the answer is negative, and $\sqsubseteq^{\kappa}_{\mathsf{Graph}}$ would be the most natural counterexample.

A more basic structure than trees or graphs are linear orders, with $\sqsubseteq^{\kappa}_{\mathsf{LO}}$ denoting the corresponding embeddability relation.

Question 3.49 (Motto Ros; [56]) Given a weakly compact cardinal κ, is $\sqsubseteq^{\kappa}_{\mathsf{LO}}$ complete for analytic quasi-orders? What about arbitrary regular κ?

A possible approach to solve this problem is to first answer the following question.

Question 3.50 (Motto Ros; [56]) For $X, Y \in [\kappa]^{\kappa}$, we write $X \subseteq^{\mathsf{NS}} Y$ if $X \setminus Y$ is nonstationary. If κ is weakly compact, is $\subseteq^{\mathsf{NS}}$ complete for analytic quasi-orders (on $[\kappa]^{\kappa}$)?

In fact the completeness of $\sqsubseteq^{\kappa}_{\mathsf{Tree}}$ and $\sqsubseteq^{\kappa}_{\mathsf{Graph}}$ follows from a stronger result, namely that both relations are *(strongly) invariantly universal* (cf. [56, Definitions 6.5 and 6.7]).

Question 3.51 (Motto Ros; [56]) For which uncountable cardinals κ satisfying $\kappa^{<\kappa} = \kappa$ are $\sqsubseteq^{\kappa}_{\mathsf{Tree}}$ and $\sqsubseteq^{\kappa}_{\mathsf{Graph}}$ (strongly) invariantly universal? What if we drop the assumption $\kappa^{<\kappa} = \kappa$? Is $\sqsubseteq^{\kappa}_{\mathsf{Graph}}$ a counterexample in that case?

Another interesting open problem concerns the possibility of distinguishing the notions of completeness, invariant universality, and strong invariant universality with suitable embeddability relations.

Question 3.52 (Motto Ros; [56]) Is it consistent that there is an infinite cardinal κ, a countable relational language $\mathcal{L}$, and two $\mathcal{L}_{\kappa^+\kappa}$-sentences φ_0 and φ_1, such that $\sqsubseteq^{\kappa}_{\varphi_0}$ is complete but not invariantly universal, and $\sqsubseteq^{\kappa}_{\varphi_1}$ is invariantly universal but not strongly invariantly universal?

Further background. Note that this question is also open for $\kappa = \omega$.

3.3.2 Universally Baire sets

Here we consider the setting from § 2.5. All the questions in this subsection refer to [40], and the topology on 2^{κ} is assumed to be the *product topology*. Recall Definition 2.19 of sets *universally Baire in* 2^{κ} *with respect to* Γ ($\mathrm{uB}^{\Gamma}_{\kappa}$), where Γ denotes an arbitrary class of Boolean algebras.

Let B be a Boolean algebra. For $b \in B$, let $B \restriction b := \{c \in B \mid c \leq b\}$. Let $\mathsf{FA}_{\kappa}(B)$ be the statement "for any κ-sequence of dense subsets of B, there is a filter on B meeting all these dense sets".

The first question concerns Wadge reducibility. Let A, A' be subsets of 2^{κ}. We say that A is *Wadge reducible to* A' ($A \leq_{\mathrm{W}} A'$) if there is a continuous function $f : 2^{\kappa} \to 2^{\kappa}$ such that $A = f^{-1}[A']$.

Question 3.53 (Ikegami, Viale) Assume that suitable large cardinals exist, e.g., a proper class of Woodin cardinals. Let Γ be the class of complete Boolean algebras B such that for all b in B, $\mathsf{FA}_{\kappa}(B \restriction b)$ holds. Let A, A' be $\mathrm{uB}^{\Gamma}_{\kappa}$ subsets of 2^{κ}. Can one prove that either $A \leq_{\mathrm{W}} A'$ or $(2^{\kappa} \setminus A') \leq_{\mathrm{W}} A$? Can one prove that the order $\leq_{\mathrm{W}}$ on $\mathrm{uB}^{\Gamma}_{\kappa}$ sets is well-founded?

Further background. Note that for universally Baire sets in the classical setting ($\kappa = \omega$), the answer to both questions is positive by universally Baire determinacy assuming large cardinals.

The second question is more vague and concerns the possibility of using $\mathrm{uB}^{\Gamma}_{\kappa}$ sets to measure the complexity of first-order theories.

Question 3.54 (Ikegami, Viale) Let Γ be the class of complete Boolean algebras B such that for all b in B, $\mathsf{FA}_{\kappa}(B \restriction b)$ holds. Can the theory of $\mathrm{uB}^{\Gamma}_{\kappa}$ sets provide tools to measure the model theoretic complexity of mathematical theories? In particular, is the notion of Borel reducibility from generalised descriptive set theory meaningful to compare the complexity of $\mathrm{uB}^{\Gamma}_{\kappa}$ equivalence relations?

It is known that universally Baire sets of reals are exactly the "∞-homogeneously Suslin sets of reals" under suitable large cardinals (cf. [46, 53] for more on ∞-homogeneously Suslin sets). The third question concerns a characterisation of $\mathrm{uB}_\kappa^\Gamma$ sets in terms of homogeneously Suslin sets.

Question 3.55 (Ikegami, Viale) Suppose there is a proper class of Woodin cardinals. Let Γ be the class of complete Boolean algebras B such that for all b in B, $\mathsf{FA}_\kappa(B\restriction b)$ holds. Can one formulate the definition of a *homogeneously Suslin subset of* 2^κ, which generalises the classical definition, and prove that homogeneously Suslin subsets of 2^κ are exactly the $\mathrm{uB}_\kappa^\Gamma$ sets?

It is known that assuming large cardinals, universally Baire sets of reals are exactly the "generically invariant sets of reals" (for the precise statement see [46, Theorem 3.3.7]). One can ask whether a similar equivalence could be established when $\kappa = \omega_1$. In this case, the natural class of forcings to look at are the *stationary set preserving* (SSP) forcings, because forcings which are *not* SSP do not even preserve the Σ_1-theory of projective subsets of 2^{ω_1}.

In [66], Viale established a generic absoluteness result for statements about subsets of 2^{ω_1} under the forcing axiom MM^{+++}, which is a natural strengthening of MM^{++} and of Martin's Maximum. A forcing axiom of this sort is required, if one aims to obtain the equivalence between $\mathrm{uB}_{\omega_1}^{\mathsf{SSP}}$ sets and SSP-generically invariant sets. For the definitions and basics on MM^{+++}, super almost huge cardinals, totally rigid partial orders, and category forcings U_δ, we refer the reader to [66].

Question 3.56 (Ikegami, Viale) Suppose MM^{+++} holds and there is a proper class of super almost huge cardinals. Let Γ be the class of SSP-complete Boolean algebras B which are totally rigid, and force MM^{+++}. Is the family of $\mathrm{uB}_{\omega_1}^\Gamma$ sets the same as those subsets of 2^κ which are generically invariant with respect to forcings in Γ (i.e., those $A \subseteq 2^\kappa$ defined by a formula φ which is absolute among V, V^B for $B \in \Gamma$, and generic ultrapowers M obtained by the category forcings $\mathsf{U}_\delta\restriction B$ where $\delta > |B|$ is a super almost huge cardinal)?

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