Homework 7, due Monday 4 April

1. Prove the last line of theorem 50 (page 32 of the lecture notes): it is easy to check that "$\rightarrow$" of Definition 48 satisfies conditions 1–4. [6 pts]

2. (a) Show that, if $\vdash_{\mathbf{HA}} \neg\varphi \rightarrow \exists x \psi(x)$, then $\vdash_{\mathbf{HA}} \neg\varphi \rightarrow \psi(\bar{n})$ for some n (here φ has no free variables.) [3 pts]

   (b) Use (a) to prove that not for all $\alpha(x)$ that express primitive recursive formulas, $\vdash_{\mathbf{HA}} \neg\neg\exists x \alpha(x) \rightarrow \exists x \alpha(x)$ (Markov's principle).

   You can use the fact that for each n, $\vdash_{\mathbf{HA}} \alpha(\bar{n})$ or $\vdash_{\mathbf{HA}} \neg\alpha(\bar{n})$ holds. You can also assume that **HA** proves only true formulas, and you can use that it is not decidable whether a primitive recursive predicate is empty or not. [3 pts]

3. Show that a theory Γ has the disjunction property iff for all $\mathfrak{M}$ and $\mathfrak{M}'$ with $\mathfrak{M} \models \Gamma$ and $\mathfrak{M}' \models \Gamma$, there exists a rooted $\mathfrak{N} \models \Gamma$ such that $\mathfrak{M}$ and $\mathfrak{M}'$ are generated submodels of $\mathfrak{N}$.

   Hint: One direction is easy. For the other direction*: one cannot just put a new root under $\mathfrak{M}$ and $\mathfrak{M}'$; we need a third model above the new root, think for that of the canonical model. [6 pts]