Generic extensions preserve axioms of ZFC

Forcing & Independence Proofs Project

Quentin Gougeon

28 January 2021

Setup

We fix a countable model M of ZFC, a forcing poset $\mathbb{P} \in M$ and a $\mathbb{P}$-generic filter G over M.

Theorem

$M[G]$ is a model of ZFC

Axioms of ZFC

- Extensionality
- Pairing
- Union
- Power Set
- Comprehension
- Infinity
- Replacement
- Foundation
- Choice

Extensionality

Lemma

$M[G]$ is transitive.

Proof.

If $x \in \tau_G \in M[G]$ then x is of the form $x = \sigma_G$ with $(\sigma, p) \in \tau$ and $p \in G$. Since τ is a $\mathbb{P}$-name, so is σ, hence $\sigma_G \in M[G]$. $\square$

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Hence $M[G] \models$ Extensionality.

Pairing, Infinity, Foundation

- **Pairing:** Let $\sigma_G, \tau_G \in M[G]$. Remember that

$$\text{up}(\sigma, \tau) := \{(\sigma, \mathbf{1}), (\tau, \mathbf{1})\}$$

Then $\text{up}(\sigma, \tau)_G = \{\sigma_G, \tau_G\}$.

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- **Infinity:** $\omega \in M \subseteq M[G]$
- **Foundation:** $M[G]$ is a submodel of **V**, which satisfies Foundation, so $M[G] \vDash$ Foundation too.

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Let $\tau_G, \mu_{1G}, \ldots, \mu_{nG} \in M[G]$ and $\phi(x, y_1, \ldots, y_n)$ a formula. We want for a name for

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Idea:

$$\gamma := \{(\sigma, p) \in \tau \mid M[G] \vDash \phi(\sigma, \overline{\mu})\}$$

Lemma (Truth Lemma)

For any formula ψ,

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$$\gamma := \{(\sigma, q) \in \mathrm{Dom}(\tau) \times \mathbb{P} \mid q \Vdash^* \phi(\sigma, \overline{\mu}) \wedge (\sigma \in \tau)\}$$

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We then can check that in $M[G]$, $\gamma_G = \{\sigma_G \in \tau_G \mid \phi(\sigma_G, \overline{\mu}_G)\}$.

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Union

In $M[G]$:

$$\begin{aligned} \mu_G \in \bigcup \tau_G \quad &\iff \quad \exists \sigma_G, \mu_G \in \sigma_G \in \tau_G \\ &\Longleftarrow \quad \exists \sigma \in M^{\mathbb{P}}, p, q \in G, (\mu, q) \in \sigma \text{ and } (\sigma, p) \in \tau \end{aligned}$$

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So we define

$$\begin{array}{rcl} \gamma & := & \{(\mu, q) \in \sigma \mid (\sigma, p) \in \tau\} \\ & = & \{(\mu, q) \in \sigma \mid \sigma \in \mathrm{Dom}(\tau)\} \\ & = & \bigcup \mathrm{Dom}(\tau) \end{array}$$

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We have $\bigcup \tau_G \subseteq \gamma$ and we conclude by using comprehension.

In $M[G]$:

$$\begin{aligned} \mu_G \in \mathcal{P}(\tau_G) \quad & \iff \quad \forall \sigma_G \in \mu_G, \sigma_G \in \tau_G \\ & \Longleftarrow \quad \forall (\sigma, p) \in \mu, (p \in G \implies \exists q \in G, (\sigma, q) \in \tau) \end{aligned}$$

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$$\begin{aligned} \gamma \quad &:= \quad \{(\mu, \mathbf{1}) \mid \forall (\sigma, p) \in \mu, \exists q \in \mathbb{P}, (\sigma, q) \in \tau\} \\ &= \quad \{(\mu, \mathbf{1}) \mid \forall (\sigma, p) \in \mu, \sigma \in \mathrm{Dom}(\tau)\} \\ &= \quad \{(\mu, \mathbf{1}) \mid \mu \subseteq \mathrm{Dom}(\tau) \times \mathbb{P}\} \\ &= \quad \mathcal{P}(\mathrm{Dom}(\tau) \times \mathbb{P})^M \times \{\mathbf{1}\} \end{aligned}$$

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We check that $\mathcal{P}(\tau_G)^{M[G]} \subseteq \gamma_G$. Let $\mu_G \subseteq \tau_G$ in $M[G]$ and define

$$\theta := \{(\sigma, p) \mid \sigma \in \mathrm{Dom}(\tau) \text{ and } p \Vdash^* \sigma \in \mu\}$$

We have $(\theta, \mathbf{1}) \in \gamma$ and we check that $\mu_G = \theta_G$.

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- If $(\sigma, p) \in \theta$ with $p \in G$ then $\sigma_G \in \mu_G$ by the Truth Lemma.

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- If $(\sigma, p) \in \theta$ with $p \in G$ then $\sigma_G \in \mu_G$ by the Truth Lemma.
- If $\sigma_G \in \mu_G$ then by the Truth Lemma there is some $p \in G$ such that $p \Vdash^* \sigma \in \mu$. Also $\sigma_G \in \tau_G$ by assumption so $\sigma \in \text{Dom}(\tau)$, and $(\sigma, p) \in \theta$.

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Replacement

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$$M[G] \models \forall x \in \tau \exists y.\phi(x, y, \overline{\theta})$$

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If $\sigma_G \in \tau_G$ we have

$$\begin{aligned} \exists \mu_G, M[G] \vDash \phi(\sigma, \mu) \quad &\Longleftrightarrow \quad \exists \mu_G, p \in G, p \Vdash \phi(\sigma, \mu) \\ &\Longleftrightarrow \quad \exists \mu_G, p \in G, p \Vdash^* \phi(\sigma, \mu) \end{aligned}$$

So we need a name of the form

$$\{\mu \mid \exists \sigma \in \mathrm{Dom}(\tau), p \in \mathbb{P}, p \Vdash^* \phi(\sigma, \mu)\}$$

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Let $\sigma \in \mathrm{Dom}(\tau)$ and $p \in \mathbb{P}$. If there exists a $\mathbb{P}$-name μ such that $p \Vdash^* \phi(\sigma, \mu)$ then there is one of minimal rank α, and we set

$$F(\sigma, p) := \{\mu \in V_\alpha \mid p \Vdash^* \phi(\sigma, \mu)\}$$

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otherwise $F(\sigma, p) := \emptyset$.
By replacement we let $Q := F[\mathrm{Dom}(\tau) \times \mathbb{P}]$ in M, and

$$\gamma := (\bigcup Q) \times \{\mathbf{1}\}$$

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Choice

Let $\tau_G \in M[G]$. We want to define a well-order on τ_G.
Using the axiom of choice in M, there exists an ordinal α and a bijection $f : \alpha \to \mathrm{Dom}(\tau)$. Let us note $f(\xi) = \sigma^\xi$ for all $\xi < \alpha$.

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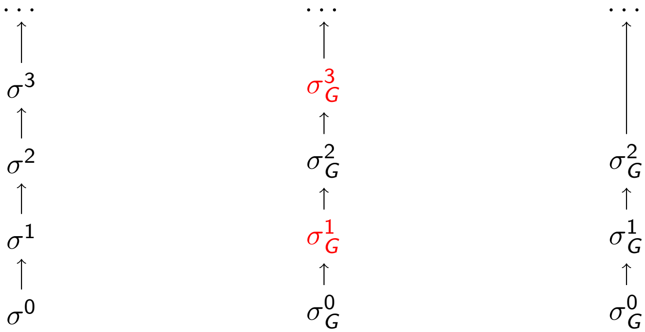

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We then define the order well-order $\leq$ on τ_G by setting

$$\sigma_G \leq \mu_G \iff \min g^{-1}(\sigma_G) \leq \min g^{-1}(\mu_G)$$

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