

The Forcing Idea

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- **The Goal:** We want to produce a model N for $ZFC + \neg CH$.
- **The Naive Idea:** Step outside our universe and create an ideal universe N with enough subsets of ω to make CH fail.

The Forcing Idea - A Motivating Example

- **The Goal:** We want a model $N \models ZFC + \neg CH$.
- **The Setup:** We start with a countable transitive model (ctm) M of ZFC . We will use a forcing poset $\mathbb{P} \in M$ to create a $\mathbb{P}$ -generic filter over M : G
- **The Strategy:** Using $\mathbb{P} = Fn(\kappa \times \omega, 2)$, the generic filter G will define a new function $f_G : \kappa \times \omega \rightarrow 2$ that lives in $N = M[G]$.
- **The Result:** Taking $\kappa = \omega_2^M$, people in N will see an ω_2 -sequence $\langle h_\alpha : \alpha < \omega_2 \rangle$ where $h_\alpha \in 2^\omega$, leading them to believe there are at least ω_2 distinct subset of 2^ω , that is, $2^{\aleph_0} \geq \aleph_2$, thus failing CH.

1.1: Constructing the New Universe $M[G]$

- Given a ground model M , a forcing poset $\mathbb{P} \in M$, and a filter $G \subseteq \mathbb{P}$
- $M[G]$ will be the set of all sets that can be built from G and elements of M using "simple set-theoretic processes".
- Each element of $M[G]$ will have a name in M that describes how it is built.

1.2: Defining $\mathbb{P}$ -Names

Definition IV.2.5

τ is a $\mathbb{P}$ -**name** iff τ is a relation and $\forall \langle \sigma, p \rangle \in \tau$ [σ is a $\mathbb{P}$ -name $\wedge p \in \mathbb{P}$].

- $V^{\mathbb{P}}$ is the class of all $\mathbb{P}$ -names.
- We relativize this to our ground model M (assuming M is a transitive model of $ZF - P$):

$$M^{\mathbb{P}} = V^{\mathbb{P}} \cap M = \{\tau \in M : (\tau \text{ is a } \mathbb{P}\text{-name})^M\}$$

- People in M can decide which sets are $\mathbb{P}$ -names and discuss their properties, even without access to the filter G .

1.3 $\mathbb{P}$ -Name is a Recursive Definition

- **The General Recursion Theorem:** Given a well-founded, set-like relation R on a class C and a rule H , there exists a unique function F such that:

$$F(x) = H(x, F \upharpoonright \{y : y R x\})$$

- **for $\mathbb{P}$ -names:**

- **The Relation (R):** $x R y \iff x \in \text{trcl}(y)$. Well-founded by Axiom of Foundation. Set-like by the fact that $\text{trcl}(y)$ is a set.
- **The Rule (H):** with $g = F \upharpoonright \text{trcl}(x)$, if we look at every element y of x , and find that $y = \langle z, p \rangle$ for some $p \in \mathbb{P}$ and $g(z) = 1$, we have $H(x, g) = 1$, otherwise $H(x, g) = 0$
- **The Result (F):** The theorem guarantees a unique, globally defined characteristic function $F : V \rightarrow \{0, 1\}$ corresponding to our definition.

1.4: Evaluating Names

Definition IV.2.7

If τ is a $\mathbb{P}$ -name and $G \subseteq \mathbb{P}$, define recursively:

$$val(\tau, G) = \tau_G = \{val(\sigma, G) : \exists p \in G[\langle \sigma, p \rangle \in \tau]\}$$

- The universe is then defined as $M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$.
- We can view this as collection "conditional on G "; that is, τ_G is some subset of named elements, and we can't tell which without knowing which conditions are in G .

1.5: Examples of Name Evaluation

- **The Empty Set:** $\emptyset$ is vacuously a valid $\mathbb{P}$ -name. Evaluating it gives $\emptyset_G = \{\} = \emptyset$.
- **Unconditional Collection:** Let $\tau = \{\langle \sigma^1, \mathbf{1} \rangle, \langle \sigma^2, \mathbf{1} \rangle, \langle \sigma^3, \mathbf{1} \rangle\}$.
- Since the largest element $\mathbf{1}$ is in any filter G , evaluating this always yields $\tau_G = \{\sigma_G^1, \sigma_G^2, \sigma_G^3\}$.
- **Conditional Collection:** Let $\pi = \{\langle \sigma^1, p^1 \rangle, \langle \sigma^2, p^2 \rangle, \langle \sigma^3, p^3 \rangle\}$.
- Here, π_G will be some subset of $\{\sigma_G^1, \sigma_G^2, \sigma_G^3\}$. We cannot tell exactly which subset this is without knowing which of the conditions p^1, p^2, p^3 belong to the generic filter G .

1.6: Special Canonical Names

Definition IV.2.9

For any set x , define $\check{x} = \{\langle \check{y}, \mathbf{1} \rangle : y \in x\}$.

- $\check{x}$ is read in English as “ x check”

Lemma IV.2.10

Assume that M transitive model of $ZF - P$ with $\mathbb{P} \in M$ and G a filter on $\mathbb{P}$, $\forall x \in M [\check{x} \in M^{\mathbb{P}} \wedge \text{val}(\check{x}, G) = x]$

- it follows that $M[G] \supseteq M$

1.7: Name for the Filter G

Definition IV.2.11

Given a forcing poset $\mathbb{P}$, define $\Gamma = \{\langle \check{p}, p \rangle : p \in \mathbb{P}\}$

- Since $\Gamma_G = \{\check{p}_G : p \in G\} = \{p : p \in G\} = G$, we have

Lemma IV.2.12

Assume that M transitive model of $ZF - P$ with $\mathbb{P} \in M$ and G a filter on $\mathbb{P}$, Γ is a $\mathbb{P}$ -name, and $\Gamma_G = G$. Hence, $G \in M[G]$.

1.8: Name for unordered and ordered pairs

Definition IV.2.13

$$up(\sigma, \tau) = \{\langle \sigma, \mathbf{1} \rangle, \langle \tau, \mathbf{1} \rangle\}$$

$$op(\sigma, \tau) = up(up(\sigma, \sigma), up(\sigma, \tau))$$

Lemma IV.2.14

Assume that M transitive model of $ZF - P$ with $\mathbb{P} \in M$ and G a filter on $\mathbb{P}$, if $\sigma, \tau \in M^{\mathbb{P}}$, then

$$up(\sigma, \tau), op(\sigma, \tau) \in M^{\mathbb{P}}$$

$$val(up(\sigma, \tau), G) = \{\sigma_G, \tau_G\}$$

$$val(op(\sigma, \tau), G) = \langle \sigma_G, \tau_G \rangle$$

1.9: A Concrete Example - Cohen Forcing

Cohen Forcing $\mathbb{P} = Fn(\omega, 2)$

- **Conditions** ($p \in \mathbb{P}$): Finite partial functions from ω to $2 = \{0, 1\}$.
- **Order** ($p \leq q$): $p \supseteq q$ (p extends q as a function).
- **Largest Element** (**1**): The empty function $\emptyset$.
- **Compatibility** ($p \not\perp q$): p and q are compatible if they do not disagree on any inputs in their shared domain. If they agree, $p \cup q$ is a common extension.
- **Incompatibility** ($p \perp q$): They disagree on some input n (e.g., $p(n) = 0$ and $q(n) = 1$).

What it does: It adds a new real (a new function $f : \omega \rightarrow 2$) to M .

1.9: Continue with the Example - Cohen Forcing

- $G \subseteq \mathbb{P}$ is a filter on $\mathbb{P}$ if and only if G is upward closed and directed, and $\mathbf{1} \in G$
- $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ if and only if $\forall p \in \mathbb{P} \exists q \in D \ q \leq p$
- **Dense Sets as Requirements:** In Cohen forcing, a dense set represents an "inevitable property" the final function must have.
- **Example:** Let $D_n = \{p \in Fn(\omega, 2) : n \in \text{dom}(p)\}$.
- D_n is dense: For any finite function q , if $n \notin \text{dom}(q)$, we can just extend q to a new condition p by defining $p(n) = 0$. Thus, we can always reach D_n .
- The filter G "stitches together" the consistent finite approximations

1.10: Generic Filter

Definition IV.2.2

For a forcing poset $\mathbb{P}$, G is a $\mathbb{P}$ -generic filter over M if and only if G is a filter on $\mathbb{P}$ and $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ such that $D \in M$

Lemma IV.2.3 (Generic Filter Existence Lemma)

Let M be a *ctm* for $ZF - P$ and let $\mathbb{P} \in M$ be a forcing poset. Then for every $p \in \mathbb{P}$, there exists a filter G on $\mathbb{P}$ such that $p \in G$ and G is a $\mathbb{P}$ -generic filter over M

Lemma IV.2.4

If $\mathbb{P} \in M$ is atomless (i.e. for every $p \in \mathbb{P}$ there exist $q_1, q_2 \leq p$, such that $q_1 \perp q_2$) and the filter G is $\mathbb{P}$ -generic over M , then $G \notin M$

1.10: Continue with the Example - Cohen Forcing

- Assuming M is a *ctm* of $ZF - P$, the Generic Filter Existence Lemma guarantees we have a $\mathbb{P}$ -generic filter over M , call it G .
- The Result:** Because G hits every D_n , the union of all conditions in G , $f_G = \bigcup G$, is a *total* function from ω to 2.
- $G \not\subseteq M$ since the poset $\mathbb{P} = Fn(\omega, 2)$ is atomless
- However, the names for G and $f_G = \bigcup G$ are in $M^{\mathbb{P}}$
- Name for f_G :**

$$\dot{f} = \{ \langle op(\check{i}, \check{j}), p \rangle : p \in \mathbb{P} \wedge \langle i, j \rangle \in p \}$$

- Evaluation:**

$$val(\dot{f}, G) = \{ \langle i, j \rangle : \exists p \in G \langle i, j \rangle \in p \} = \bigcup G$$

2.0: The Generic Extension $M[G]$

- With the construction of names and evaluation, people in M can already reason about the generic extension $M[G]$.
- For example, from the previous discussion, we know $M \subsetneq M[G]$ with atomless forcing.
- We can also prove that $M[G]$ is transitive, and is a model for Axioms of Extensionality, Foundation, Pairing, and Union

2.1: The Generic Extension $M[G]$

- Other things we know about $M[G]$ including:

Lemma IV.2.18

Assume M transitive model of $ZF - P$ with $\mathbb{P} \in M$ and G a filter on $\mathbb{P}$

- $rank(\tau_G) \leq rank(\tau)$ for all $\tau \in M^{\mathbb{P}}$
- $o(M[G]) = o(M)$
- $|M[G]| = |M|$

Lemma IV.2.19

Under the same assumption as above, let N be another transitive model of $ZF - P$ with $M \subseteq N$ and $G \in N$. Then $M[G] \subseteq N$.

2.1 Proof of Lemma IV.2.18 (a)

Lemma

For any $\mathbb{P}$ -name $\tau \in V^{\mathbb{P}}$ and filter G , $\text{rank}(\tau_G) \leq \text{rank}(\tau)$.

$$\tau_G = \{\sigma_G : \exists p \in G (\langle \sigma, p \rangle \in \tau)\}$$

$\Downarrow$ (Def. of Rank)

$$\text{rank}(\tau_G) = \sup\{\text{rank}(\sigma_G) + 1 : \langle \sigma, p \rangle \in \tau, p \in G\}$$

$\Downarrow$ (By IH: $\text{rank}(\sigma_G) \leq \text{rank}(\sigma)$)

$$\text{rank}(\tau_G) \leq \sup\{\text{rank}(\sigma) + 1 : \langle \sigma, p \rangle \in \tau, p \in G\}$$

$\Downarrow$ (Chain: $\sigma \in \{\sigma\} \in \langle \sigma, p \rangle \in \tau$)

$$\forall \sigma, \text{rank}(\sigma) < \text{rank}(\tau) \implies \text{rank}(\sigma) + 1 \leq \text{rank}(\tau)$$

$\Downarrow$ (Conclusion)

$$\text{rank}(\tau_G) \leq \text{rank}(\tau) \quad \square$$

2.1 Proof of Lemma IV.2.18 (b)

Lemma

$$o(M[G]) = o(M)$$

Dir 1: $o(M) \subseteq o(M[G])$

$$\begin{aligned} M &\subseteq M[G] \\ \Downarrow \\ \alpha &\in M \cap ON \\ \Downarrow \\ \alpha &\in M[G] \cap ON \\ \Downarrow \\ \alpha &\in o(M[G]) \end{aligned}$$

Dir 2: $o(M[G]) \subseteq o(M)$

$$\begin{aligned} \alpha \in o(M[G]) &\implies \alpha = \tau_G, \tau \in M^{\mathbb{P}} \\ &\Downarrow \text{ (By Lemma IV.2.18(a))} \\ \alpha = \text{rank}(\alpha) &= \text{rank}(\tau_G) \leq \text{rank}(\tau) \\ &\Downarrow (\tau \in M \implies \text{rank}(\tau) = \beta \in o(M)) \\ \alpha &\leq \beta, \text{ where } \beta \in o(M) \\ &\Downarrow \text{ (Ordinals are downward closed)} \\ \alpha &\in o(M) \quad \square \end{aligned}$$

2.1 Proof of Lemma IV.2.18 (c)

Lemma

$$|M[G]| = |M|$$

Definition IV.2.7: $M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$

$\Downarrow$ (There is a surjection $M^{\mathbb{P}} \rightarrow M[G]$ via $\tau \mapsto \tau_G$)

$$|M[G]| \leq |M^{\mathbb{P}}|$$

$\Downarrow$ (By inclusion: $M^{\mathbb{P}} \subseteq M$)

$$|M^{\mathbb{P}}| \leq |M|$$

$\Downarrow$ (By Lemma IV.2.10: $M \subseteq M[G]$)

$$|M| \leq |M[G]|$$

$\Downarrow$ (Cantor-Bernstein Theorem)

$$|M[G]| = |M| \quad \square$$

2.1 Proof of Lemma IV.2.19: Minimality of $M[G]$

Lemma IV.2.19

Under the same assumption as above, let N be another transitive model of $ZF - P$ with $M \subseteq N$ and $G \in N$. Then $M[G] \subseteq N$.

$$x \in M[G] \iff x = \text{val}(\tau, G) \text{ for some } \tau \in M^{\mathbb{P}}$$

$$\Downarrow (M \subseteq N)$$

$$\tau \in N \text{ and } G \in N$$

$$\Downarrow (\text{Definition IV.2.7 is recursive on a well-founded relation})$$

$$\text{val is absolute for transitive models of } ZF^-$$

$$\Downarrow (\tau, G \in N \text{ implies } \text{val}(\tau, G) \in N)$$

$$x \in N$$

$$\Downarrow$$

$$M[G] \subseteq N \quad \square$$

2.2 A motivation: Comprehension in $M[G]$

- **The Goal:** We want to prove $M[G] \models \text{Comprehension}$.
- **The Task:** Given $\sigma_G \in M[G]$ and a formula $\phi(x)$, we must show that the subset

$$A = \{x \in \sigma_G : M[G] \models \phi(x)\}$$

is also in $M[G]$.

- **The Obstacle:** To show $A \in M[G]$, we need to find a name $\mu \in M^{\mathbb{P}}$ such that $\mu_G = A$.
- **The Solution: The Forcing Relation ($\Vdash$)**
- We need a way to express truth in $M[G]$ **syntactically** within M :

The Core Idea

$p \Vdash \phi$ (" p forces ϕ ") means: Whenever G is a generic filter and $p \in G$, then $M[G] \models \phi$.

- Now M can define $\mu = \{\langle \tau, p \rangle \in \text{dom}(\sigma) \times \mathbb{P} : p \Vdash (\tau \in \sigma \wedge \phi(\tau))\}$.

2.3: The Forcing Language

- The $\mathbb{P}$ forcing language $\mathcal{FL}_{\mathbb{P}}$ is the class of logical formulas formed using the binary relation $\in$ and all the names in $V^{\mathbb{P}}$ as constant symbols.
- In discussing $M[G]$, we use only $\mathcal{FL}_{\mathbb{P}} \cap M$ which is the set of logical formulas formed using $\in$ and all the names in $M^{\mathbb{P}}$ as constant symbols.

2.4: Truth in the Extension ($M[G] \models \psi$)

- Let ψ be a sentence of the restricted forcing language $\mathcal{FL}_{\mathbb{P}} \cap M$.
- The statement $M[G] \models \psi$ has its standard model-theoretic meaning.
- **The Interpretation:**
 - The binary relation symbol $\in$ is interpreted as standard set membership.
 - Every constant symbol (which is a $\mathbb{P}$ -name $\tau \in M^{\mathbb{P}}$) is interpreted as its evaluated set τ_G .
- **Translating Syntax to Semantics:** A sentence like $\varphi(\check{n}, \sigma)$ written in M translates to a statement about the actual sets n and σ_G living in $M[G]$.

2.5: The (Semantic) Forcing Relation $\Vdash$

Let ψ be a sentence of $\mathcal{FL}_{\mathbb{P}} \cap M$.

Definition

$p \Vdash_{\mathbb{P}, M} \psi$ holds iff: $M[G] \models \psi$ for all filters G on $\mathbb{P}$ such that $p \in G$ and G is $\mathbb{P}$ -generic over M .

- " $p \Vdash \psi$ " is read " p forces ψ ".

2.6 Concrete Examples of the Forcing Relation

• Example 1: Upward Closure of Filters

- Let $\dot{G}$ be the name for the generic filter G .
- If $p \leq q$, then $p \Vdash \check{q} \in \dot{G}$.
- **Reason:** By definition of a filter, if $p \in G$ and $p \leq q$, then q *must* be in G . Thus, p “forces” q to be in the future filter.

• Example 2: Forcing the Axioms

- $1 \Vdash \psi$ iff $M[G] \models \psi$ for **all** generic filters G .
- Since we proved $M[G] \models$ Union Axiom, we have $1 \Vdash$ Union Axiom.

• Example 3: Forcing the Complement

- Let $\dot{C}$ be the name for the complement $\mathbb{P} \setminus G$.
- If $E \subseteq \mathbb{P}$ is not *centered* (i.e., its elements are incompatible), then:

$$1 \Vdash \exists x [x \in \dot{C} \wedge x \in \check{E}]$$

- **Reason:** A generic filter G cannot contain all of E . Thus, some $x \in E$ must fall into the complement $\dot{C}$.

2.7 Monotonicity of the Forcing Relation

Lemma IV.2.23

If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$.

Assume $q \in G$ where G is any $\mathbb{P}$ -generic filter over M

$\Downarrow$ ($q \leq p$ and filters are upward-closed)

$$p \in G$$

$\Downarrow$ (By the assumption $p \Vdash \varphi$)

$$M[G] \models \varphi$$

$\Downarrow$ (Since this holds for any generic G containing q)

$$\forall G [q \in G \implies M[G] \models \varphi]$$

$\Downarrow$ (By the intuitive definition of forcing)

$$q \Vdash \varphi \quad \square$$

2.8 Two Fundamental Challenges

In the previous slide, to show $M[G] \models \text{Comprehension}$, we "cooked up" a name for our subset A :

$$\mu = \{ \langle \tau, p \rangle \in \text{dom}(\sigma) \times \mathbb{P} : p \Vdash (\tau \in \sigma \wedge \phi(\tau)) \}$$

However, there are still two problems:

- ① **The Definability Problem:** Is the name μ actually in M ?
 - The definition of μ uses the relation $\Vdash$. If $\Vdash$ is not definable within M , then M cannot construct μ .
- ② **The Truth Problem:** Does the evaluation μ_G actually equal the set A ?
 - We need to be sure that $x \in A$ if and only if there is a condition $p \in G$ that *forces* this truth.

These are answered by the **Definability Lemma** and the **Truth Lemma**.

2.9 Lemma IV.2.25: The Definability Lemma

The Definability Lemma (Formal Statement)

Let M be a ctm for $ZF - P$. Let $\varphi(x_1, \dots, x_n)$ be a formula in $\mathcal{L} = \{\in\}$. Then the set:

$$\{(\mathbb{P}, p, \leq, \mathbf{1}, \vartheta_1, \dots, \vartheta_n) : \\ (\mathbb{P}, \leq, \mathbf{1}) \text{ is a forcing poset} \wedge p \in \mathbb{P} \wedge (\mathbb{P}, \leq, \mathbf{1}) \in M \\ \wedge \vartheta_1, \dots, \vartheta_n \in M^{\mathbb{P}} \wedge p \Vdash_{\mathbb{P}, M} \varphi(\vartheta_1, \dots, \vartheta_n)\}$$

is in $\mathcal{D}^-(M)$.

What does this mean?

- For any formula ϕ , the forcing relation $p \Vdash \phi(\tau_1, \dots, \tau_n)$ is **definable** in the ground model M .
- μ is actually in M

2.10 Lemma IV.2.24: The Truth Lemma

The Truth Lemma

Let M be a ctm for $ZF - P$ and G be $\mathbb{P}$ -generic over M . For any sentence ψ of the forcing language:

$$M[G] \models \psi \iff \exists p \in G (p \Vdash \psi)$$

- This lemma guarantees that our "syntactic" forcing relation in M perfectly matches the "semantic" truth in $M[G]$.
- It ensures that:

$$\begin{aligned} \tau_G \in \mu_G &\iff \exists p \in G [\langle \tau, p \rangle \in \mu] \iff \exists p \in G [p \Vdash \phi(\tau)] \\ &\iff M[G] \models \phi(\tau_G) \end{aligned}$$

- Thus, μ_G is exactly the set $A = \{x \in \sigma_G : M[G] \models \phi(x)\}$ we wanted to build.

Summary: The Basic Idea of Forcing

Ground Model $M \xrightarrow{\text{Poset } \mathbb{P}}$ Generic Extension $M[G]$

$\Downarrow$

Names (τ) allow us to denote elements of $M[G]$ within M

$\Downarrow$

Forcing Relation ($\Vdash$) allows us to predict truth in $M[G]$ within M

$\Downarrow$

Truth & Definability Lemmas ensure the construction is internally consistent

$\Downarrow$

$M[G] \models \text{ZFC} + \text{New properties (like } \neg CH)$