

FORCING PROJECT PRESENTATION MARTIN'S AXIOM

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In this talk, I will cover the following topics:

1. Delta-System Lemma;
2. Statement of Martin's Axiom;
3. Definition of the Finitary Function Poset;
4. $\text{MA}(\kappa)$ is false for all $\kappa \geq 2^\omega$;
5. $\text{MA}(\omega)$ is true; (and if time permits)
6. some consequences of MA

1. DELTA-SYSTEM LEMMA

Martin's Axiom is a generalization of many combinatorial results obtained by forcing. It can be understood without any knowledges of forcing but employs many infinitary combinatorial machineries. Among these, we introduce the delta-system lemma.

Definition 1. A family A of sets forms a *delta-system with root R* if

$$X \cap Y = R \text{ for all } X, Y \in A.$$

Theorem 2 (Delta-system lemma). *Let κ be an uncountable regular cardinal and A be a collection of finite sets with $|A| \geq \kappa$. Then, there exists $B \subseteq A$ so that $|B| = \kappa$ and B is forms a delta-system.*

Proof. Let $A_n = \{X \in A : |X| = n\}$. Then,

$$\theta_0 = |A_0|, \theta_1 = |A_0 \cup A_1|, \dots, \theta_n = \left| \bigcup_{i \leq n} A_i \right|, \dots$$

is a non-decreasing sequence such that $\lim_{n < \omega} \theta_n = \kappa$. As κ is regular, we can take $N \in \omega$ so that $\forall n \geq N \theta_n = \kappa$. Take N to be the smallest such then $|A_N| = \kappa$ and for each $X \in A_N$, $|X| = N$.

We can thus assume without loss of generality that A consists of finite sets of the same size; i.e.,

$$\exists n < \omega \forall X \in A \ |X| = n.$$

We now induct on n that there exists a delta-system $B \subseteq A$ with $|B| = \kappa$.

If $n = 1$, then A is a collection of distinct singletons. It is itself a delta-system with empty root.

For $n > 1$, we consider whether there exist some $p \in \cup A$ so that $A_p := \{X \in A : p \in X\}$ is of size κ .

- (1) If there exists some p so that $|A_p| = \kappa$, fix such p and take $A' := \{X \setminus \{p\} : X \in A_p\}$. By IH, we have some delta-system $B \subseteq A'$ with root R so that $|B| = \kappa$. Then, $\{X \cup \{p\} : X \in B\}$ is a subset of A and forms a delta-system with root $R \cup \{p\}$.
- (2) If for all $p \in \cup A$ $|A_p| < \kappa$, then given any set $S \subseteq \cup A$ with $|S| < \kappa$, $\{X \in A : X \cap S \neq \emptyset\} = \cup_{p \in S} A_p$ is a $< \kappa$ union of $< \kappa$ set. As κ is regular, it has size $< \kappa$. Hence, there exists $X \in A$ with $X \cap S = \emptyset$. For $\beta < \kappa$, choose recursively, X_β so that $X_\beta \cap \cup_{\alpha < \beta} X_\alpha = \emptyset$. Then, $\{X_\beta : \beta < \kappa\}$ is a delta-system with empty root of size κ .

□

Corollary 3. *Every uncountable collection of finite sets has an uncountable sub-collection that forms a delta-system.*

2. MARTIN'S AXIOM

Before stating Martin's Axiom, we need to define some terminologies.

Definition 4. A *forcing poset* is a triplet $(P, \leq, 1)$ so that $(P, \leq)$ forms a pre-order and $\forall p \in P$ $p \leq 1$.

Elements of the forcing poset P are called *conditions* and we say “ p extends q ” if $p \leq q$.

Note 5. While the relation on a forcing poset need only be a pre-order, i.e., reflexive and transitive, we mostly work with relations that are partial orders which additionally require that

$$\forall p, q \in P \ (p \leq q \wedge q \leq p \rightarrow p = q).$$

Definition 6. Let $(P, \leq, 1)$ be a forcing poset.

Given $p, q \in P$, we say p and q are *compatible* ($p \not\perp q$) if there is some $r \in P$ so that $r \leq p$ and $r \leq q$ and *incompatible* ($p \perp q$) if they are not compatible.

An *antichain* is a subset $A \subseteq P$ so that $\forall p, q \in A$ ($p \neq q \rightarrow p \perp q$).

We say P has the *countable chain condition* (c.c.c.) if every antichain in P is countable.

Example 7. Let X be a topological space and P be the collection of non-empty open sets in X with $\leq$ defined as $\subseteq$. Then, $p \perp q$ iff $p \cap q = \emptyset$.

We can easily check that P has c.c.c. iff there is no uncountable family of pairwise disjoint open subsets of X .

Definition 8. Let $(P, \leq, 1)$ be a forcing poset.

A subset $D \subseteq P$ is *dense* if $\forall p \in P \exists q \in D (q \leq p)$.

A subset G is a *filter* if

- i. $1 \in G$,
- ii. $\forall p, q \in G \exists r \in G (r \leq p \wedge r \leq q)$, and
- iii. $\forall q \in G \forall p \in P (p \geq q \rightarrow p \in G)$.

Example 9. Let X be any nonempty set and $P = \mathcal{P}(X) \setminus \{\emptyset\}$ is a forcing poset with $\leq$ being $\subseteq$ and $1 = X$. Then, $p \perp q$ iff $p \cap q = \emptyset$. $A \subseteq P$ is an antichain iff the elements of A are pairwise disjoint, so P has c.c.c. iff $|X| \leq \omega$.

A subset $F \subseteq P$ is a filter iff

- i. $X \in F$,
- ii. $\forall A, B \in F A \cap B \in F$, and
- iii. $\forall A \in F \forall B \in X (A \subseteq B \rightarrow B \in F)$.

We now state Martin's Axiom.

Definition 10. Let $(P, \leq, 1)$ be a forcing poset and κ be a cardinal.

$\text{MA}_P(\kappa)$ is the statement that if $\mathcal{D}$ is a family of dense subsets of P with $|\mathcal{D}| \leq \kappa$, then there exists a filter G so that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

$\text{MA}(\kappa)$ is the statement that $\text{MA}_P(\kappa)$ is true for all c.c.c. P .

MA is the statement that MA is true for all $\kappa < 2^\omega$.

3. FINITARY FUNCTION POSET

Some terminologies of the previous section may seem arbitrary, we justify these with the following example.

Example 11. Let I and J be sets. Define $\text{Fn}(I, J)$ to be the collection of finite partial functions from I to J ; i.e., $p \in \text{Fn}(I, J)$ iff $p : S \rightarrow J$ with $S = I$ and $|S| < \omega$.

Then, $\text{Fn}(I, J)$ is a forcing poset with $\leq$ being $\supseteq$ and $1 = \emptyset$. Note that $p \leq q$ iff p extends q as a function and p and q are compatible iff $\forall x \in \text{dom}(p) \cap \text{dom}(q) p(x) = q(x)$.

Theorem 12. $\text{Fn}(I, J)$ has c.c.c. iff $I = \emptyset$ or J is countable.

Proof. If I or J is empty, the lemma is trivial, we assume without loss of generality that $I \neq \emptyset$ and $J \neq \emptyset$.

($\implies$) Assume that J is uncountable. Fix $i \in I$, then the collection of singleton functions $\{(i, j) : j \in J\}$ is an uncountable antichain.

($\Leftarrow$) Assume to the contrary that J is countable and $A \subseteq \text{Fn}(I, J)$ is an uncountable anti-chain.

Claim: $\{\text{dom}(p) : p \in A\}$ is uncountable.

Proof of the claim. If otherwise, $A \subseteq \bigcup_{p \in A}^{\text{dom}(p)} J$ is also countable.

By the delta-system lemma, we have some uncountable $B \subseteq A$ and some finite $R \subseteq I$ so that $\text{dom}(p) \cap \text{dom}(q) = R$ whenever $p, q \in B$. As R is finite, ${}^R J$ is countable, so we have some $p, q \in B$ such that $p \upharpoonright_R = q \upharpoonright_R$. But as $\text{dom}(p) \cap \text{dom}(q) = R$, p and q are compatible, contradicting the assumption that A is an antichain. $\square$

Lemma 13. *Let I be infinite and J be non-empty. Then sets of the form $\{q \in \text{Fn}(I, J) : i \in \text{dom}(q)\}$ and $\{p \in \text{Fn}(I, J) : j \in \text{ran}(p)\}$ are dense whenever $i \in I$ and $j \in J$.*

4. BASIC PROPERTIES OF MA

We now show some basic properties of MA.

Lemma 14. *Let P be any forcing poset. Given cardinals $\kappa \geq \kappa'$, $\text{MA}_P(\kappa) \rightarrow \text{MA}_P(\kappa')$ and $\text{MA}(\kappa) \rightarrow \text{MA}(\kappa)$.*

This lemma is rather trivial, the proof of the following one illustrates the use of filters and dense sets in $\text{Fn}(I, J)$.

Theorem 15. *$\text{MA}(2^\omega)$ is false.*

Proof. Let $P = \text{Fn}(\omega, 2)$. By lemma 12, P is c.c.c.. If G be a filter on P , then elements in G are pairwise compatible, so the union of these functions $\bigcup G$ is itself a function, we denote it f_G .

MA states the existence of a filter that intersect some dense sets. By requiring G to intersect some dense sets, we can force f_G to have certain properties. In particular, we can make f_G total by requiring G to intersect all $D_n := \{q : n \in \text{dom}(q)\}$ for $n < \omega$. Similarly, we can make f_G onto by having G intersect all $E_i := \{p : i \in \text{ran}(p)\}$ for $i < 2$. Note that D_n and E_i are dense by Lemma 13.

The critical condition that we can force upon f_G to show the inconsistency of $\text{MA}(2^\omega)$ is that f_G is not equal to any specific function in ${}^\omega 2$. Let $h : \omega \rightarrow 2$ be any function, define $F_h := \{p : p \not\subseteq h\}$. Let $g \in P$, as $\text{dom}(g)$ is finite, we can take some n so that $n \notin \text{dom}(g)$, then $g \cup \{(n, 1 - h(n))\}$ extends g and is in F_h . Hence, F_h is dense. If G intersects F_h , $f_G \neq h$.

Note that $\mathcal{D} := \{D_n : n < \omega\} \cup \{E_0, E_1\} \cup \{F_h : h \in {}^\omega 2\}$ is a collection of dense sets with $|\mathcal{D}| = 2^\omega$ and it is not possible for any filter G to intersect all $D \in \mathcal{D}$. $\text{MA}(2^\omega)$ is false. $\square$

Corollary 16. $\text{MA}(\kappa) \rightarrow \kappa < 2^\omega$.

Theorem 17 (Generic Filter Existence Lemma). *Let P be any forcing posets and D be a countable family of dense subsets of P . Fix any $p \in P$, then there exists a filter G so that $p \in G$ and $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.*

Proof. List $D := \{D_n : n < \omega\}$. For each $n < \omega$, recursively choose r_n so that $r_0 = p$, $r_{n+1} \leq r_n$, and $r_{n+1} \in D_n$. As each D_n is dense, such r_{n+1} always exist. Then, $G := \{x : \exists n < \omega \ r_n \leq x\}$ is a filter that intersects each D_n with $p \in G$. $\square$

Corollary 18. $\text{MA}(\omega)$ is true. In particular, $\text{CH} \rightarrow \text{MA}$.

The above lemma establishes $\text{MA}_P(\omega)$ for each P . However, if we drop the c.c.c., $\text{MA}_P(\kappa)$ is false for any $\kappa > \omega$.

Example 19. Fix $\kappa > \omega$. Take $P = \text{Fn}(\omega, J)$. Let D_n and E_i be as in the proof of lemma 15. Then, any filter G intersecting all D_n and E_i produces a total onto function f_G . This is only possible iff P has c.c.c..

5. SOME CONSEQUENCES OF MA

Martin's Axiom is only interesting when CH is false, in which case, we can ask many combinatorial questions. We answer one of these here:

Theorem 20. *If $\kappa < 2^\omega$, then $\text{MA}(\kappa) \rightarrow 2^\kappa = 2^\omega$.*

Definition 21. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$. The *almost disjoint sets partial order*, $P_{\mathcal{A}}$ is

$$\{(s, F) : s \subseteq \omega \wedge |s| < \omega \wedge F \subseteq \mathcal{A} \wedge |F| < \omega\},$$

where $(s', F') \leq (s, F)$ iff

$$s \subseteq s' \wedge F \subseteq F' \wedge \forall x \in F (x \cap s' \subseteq s).$$

Lemma 22. *In $P_{\mathcal{A}}$, (s_1, F_1) and (s_2, F_2) are compatible iff*

$$\forall x \in F_1 (x \cap s_2 \subseteq s_1) \wedge \forall x \in F_2 (x \cap s_1 \subseteq s_2),$$

in which case $(s_1 \cup s_2, F_1 \cup F_2)$ is a common extension.

Definition 23. If G is a filter in $P_{\mathcal{A}}$, $d_G = \bigcup \{s : \exists F ((s, F) \in G)\}$

Lemma 24. *If G is a filter and $(s, F) \in G$, then $\forall x \in F (x \cap d_G \subseteq s)$.*

Definition 25. If $x \in \mathcal{A}$, then $D_x = \{(s, F) \in P_{\mathcal{A}} : x \in F\}$.

Lemma 26. *If G is a filter in $P_{\mathcal{A}}$ and $G \cap D_x \neq \emptyset$, then $|x \cap d_G| < \omega$*

Lemma 27. *If $x \in \mathcal{A}$, then D_x is dense in $P_{\mathcal{A}}$.*

Lemma 28. $P_{\mathcal{A}}$ has c.c.c..

Theorem 29. Assume $\text{MA}(\kappa)$. Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{P}(\omega)$ where $|\mathcal{A}| \leq \kappa$, $|\mathcal{C}| \leq \kappa$, and assume that for all $y \in \mathcal{C}$ and finite $F \subseteq \mathcal{A}$, $|y \setminus \bigcup F| = \omega$; then there is $d \subseteq \omega$ such that $\forall x \in \mathcal{A} (|d \cap x| < \omega)$ and $\forall y \in \mathcal{C} (|d \cap y| = \omega)$.

Proof. For $y \in \mathcal{C}$ and $n \in \omega$, let

$$E_n^y := \{(s, F) \in P_{\mathcal{A}} : s \cap y \not\subseteq n\}.$$

Apply MA on

$$\{D_x : x \in \mathcal{A}\} \cup \{E_n^y : y \in \mathcal{C} \wedge n \in \omega\}.$$

□

Corollary 30. Let $\mathcal{B} \subseteq \mathcal{P}(\omega)$ be an almost disjoint family of size κ where $\omega \leq \kappa < 2^\kappa$. Let $A \subseteq B$. If $\text{MA}(\kappa)$, then there is some $d \subseteq \omega$ s.t. $\forall x \in \mathcal{A} (|d \cap x| < \omega)$ and $\forall x \in \mathcal{B} \setminus \mathcal{A} (|d \cap x| = \omega)$.

Proof. Apply theorem 29 with $\mathcal{C} = \mathcal{B} \setminus \mathcal{A}$. □

Proof of Theorem 20. Fix any a.d. family $\mathcal{B}$ of size κ . Define $\Phi : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\mathcal{B})$ by $\Phi(d) = \{x \in \mathcal{B} : |d \cap x| < \omega\}$. Lemma 2.17 says Φ is onto, so $2^\kappa = |\mathcal{P}(\mathcal{B})| \leq |\mathcal{P}(\omega)| = 2^\omega$. □