

$$M \models \varphi$$

$\hookrightarrow M$ is proper class, φ^M (relativization)

you cannot quantify over φ inside ZF.

$\hookrightarrow M$ is a set, φ^M or $M \models \varphi$

formal satisfaction relation

not definable for M proper classes.

Gödel's Constructible Universe:

How does $V = \bigcup_{\alpha \in ON} V_\alpha$

$$V_0 = \emptyset$$

:

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \rightarrow \text{You could say: } \mathcal{P}(V_\alpha) \text{ not really constructive}$$

:

$$V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$$

$$V = \bigcup_{\alpha \in ON} V_\alpha$$

Just look at $V_{\omega+1} = \mathcal{P}(V_\omega)$

$$\mathcal{P}(\omega) \subseteq V_{\omega+1}$$

$\uparrow$
contains many sets without definitions.

You cannot w.o. $\mathcal{P}(\omega)$ without AC.

Idea: instead of taking $\mathcal{P}(\dots)$ at succ. stages, take only the definable sets.

Def: $x \in A$ is definable over A if there is φ and param. $a_1 \dots a_n$ so that

$$X = \{x \in A \mid A \models \varphi(x, a_1 \dots a_n)\}$$

$$D(A) = \{x \in A \mid x \text{ definable over } A\}.$$

$$x \in D(A) \Leftrightarrow \underbrace{\exists \varphi}_{\text{formalized, } \exists \ulcorner \varphi \urcorner} \exists a_1 \dots a_n \in A \text{ so that } \forall x (x \in X \Leftrightarrow A \models \varphi(x, a_1 \dots a_n))$$

$\uparrow$
formalized relation.

Therefore $D(A)$ is a set.

Def:

$$L_0 = \emptyset$$

$$\vdots$$

$$L_{\alpha+1} = D(L_\alpha)$$

$$\vdots$$

$$L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha$$

$$\mathbb{L} = \bigcup_{\alpha \in \mathcal{ON}} L_\alpha$$

Def in ZF:

$$L_1 \text{ is a set}$$

$$\Rightarrow L_2 \text{ is a set}$$

$$\Rightarrow \vdots$$

$$L_\alpha \text{ is a set.}$$

Look at $\mathbb{L}$ and V :

$$L_0 = V_0$$

$\vdots$

$$L_n = V_n$$

because fin. sets are definable.

$\vdots$

$$L_\omega = V_\omega$$



$$L_{\omega+1} = \left\{ x \in L_\omega \mid x \text{ def'ble by } \ulcorner \varphi \urcorner \text{ and } a_1 \dots a_n \right\}$$

$$V_{\omega+1}$$

$$|V_{\omega+1}| = 2^{\aleph_0}$$

so $|L_{\omega+1}| = \text{ctbl}$

Lemma: for all infinite α , $|L_\alpha| = |\alpha|$.

Proof: induction as above.

So: V "grows" much faster than L , but still consistent: $V=L$.
(not: $V_\alpha = L_\alpha$!!)

Theorem: $(ZF) \models L \models ZF$.

Proof: Most axioms are easy; you need Reflection to prove
COMPREHENSION^L.. □

Theorem: $(ZF) \models L \models AC$.

Proof: In fact $L \models$ Global Choice, i.e. there is a
class-sized relation $<_L$, which well-orders
all of L .

For each L_α , wellorder it by $<_\alpha$, recursively.

• Suppose $(L_\alpha, <_\alpha)$ is w.o. $\Rightarrow <_\alpha^{lex}$ is also w.o.,

and actually $<_\alpha^{lex}$ on $\mathbb{N} \times L_\alpha^{<\omega}$
Gödel Numbers.

• Define $<_{\alpha+1}$ on $L_{\alpha+1}$: given $x, y \in L_{\alpha+1}$:

(1) $x, y \in L_\alpha \Rightarrow (x <_{\alpha+1} y \leftrightarrow x <_\alpha y)$ ($<_{\alpha+1}$ extends $<_\alpha$)

(2) $x \in L_\alpha$ and $y \in L_{\alpha+1} \setminus L_\alpha \Rightarrow x <_{\alpha+1} y$ (new after old)

(3) What if $x, y \in L_{\alpha+1} \setminus L_\alpha$?

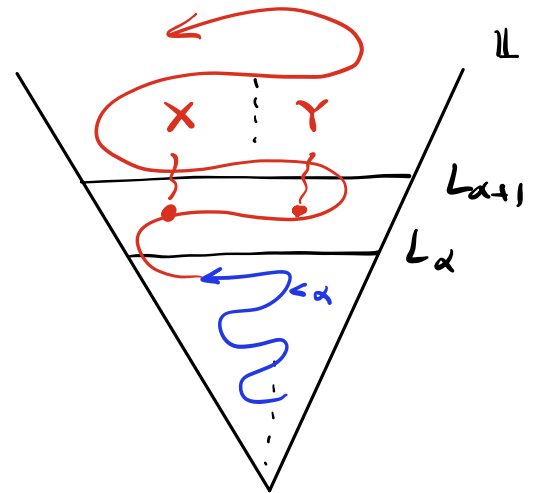
Then: $x <_{\alpha+1} y \iff$

The $<_{\alpha}^{\text{lex}}$ -least sequence
 $\langle \ulcorner \varphi \urcorner, a_1, \dots, a_n \rangle$ defining x

is $<_{\alpha}^{\text{lex}}$ -less than

the $<_{\alpha}^{\text{lex}}$ -least sequence

$\langle \ulcorner \varphi \urcorner, b_1, \dots, b_m \rangle$ defining y .



Need to check: $<_{\alpha+1}$ is still a well-order (exercise)

let's turn to CH & GCH.

Def: The Axiom of Constructibility is the statement

$$\forall x \exists \alpha \in \text{ON} (x \in L_\alpha)$$

Usually abbreviated by " $V=L$ "

Central Lemma: The definition of L_α is absolute for transitive models of ZF.

This really means: " $\alpha \mapsto L_\alpha$ " is absolute.

- if $\alpha \in M$ then $L_\alpha \in M$ ($L_\alpha^M = L_\alpha$)

Reason:

L_α = defined by recursion using $D(A)$.

and $D(A)$ is built from atomic statement,
boolean connectives and bounded quantifiers

$$A \models \forall x \exists y \dots$$

$$\Leftrightarrow \forall x \in A \exists y \in A \dots$$

$$\bullet \text{ if } A \in M, (A \models \varphi) \Leftrightarrow M \models (A \models \varphi)$$

Corollary 1: $L \models (V=L)$

Proof: $(V=L)^L \equiv \left(\forall x \exists \alpha \in ON (x \in L_\alpha) \right)^L$

$$\equiv \forall x \in L \exists \alpha \in ON \cap L (x \in L_\alpha)^L$$

$$\exists \alpha \in ON \quad x \in L_\alpha$$

$$\forall x \in L \exists \alpha (x \in L_\alpha) \quad \text{True by DEF. of } L.$$

So: to prove $L \models \Phi$, easier to prove (in ZF) $(V=L) \rightarrow \Phi$.

Corollary 2: Any transitive prop class $M \models ZF : L \subseteq M$

NB: M prop class $\Leftrightarrow ON \in M$.

Proof: $\forall \alpha : \alpha \in M \Rightarrow \forall \alpha L_\alpha \in M \Rightarrow L \subseteq M. \quad \square$

Corollary 3: Any trans. prop. class $M \models (ZF + V=L)$

Then $L = M$.

Proof: $M \models \forall x \exists \alpha (x \in L_\alpha)$, so $\forall x \in M \exists \alpha \in ON \cap M (x \in L_\alpha)^M$

This means: $\forall x \in M \exists \alpha (x \in L_\alpha)$

$$\forall x \in M \quad x \in \bigcup_{\alpha \in ON} L_\alpha = L.$$

$$\text{So } M \in L.$$

Q: What about set-sized models M ?

Def: If M is a set model of ZF^* notation: sufficiently large finite fragment of ZF .

The height of M is

$o(M) :=$ the least ordinal $\delta \notin M$.

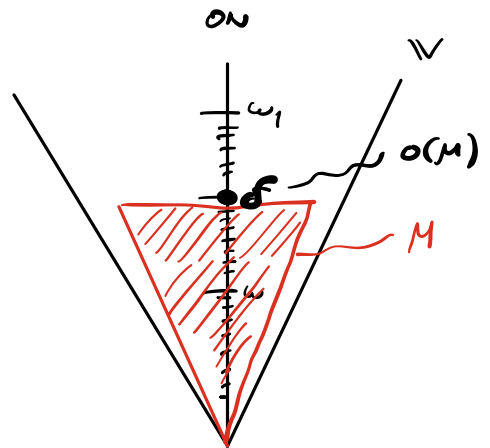
NB: $o(M) =$ always a limit ordinal (not cardinal) and

$$o(M) = ON \cap M.$$

eg: M is countable $\Rightarrow o(M) = \delta < \omega_1$

Corollary 4: M trans. set model of ZF^* with $o(M) = \delta$, then

$$L_\delta \subseteq M$$



Proof:

$$\forall \alpha < \delta : \alpha \in M$$

$$\forall \alpha < \delta : L_\alpha \in M \text{ so } L_\alpha \in M$$

$$\text{So } L_\delta = \bigcup_{\alpha < \delta} L_\alpha \subseteq M.$$

Corollary 5:

M trans. set model of $ZF^* + (V=L)$
with height $o(M) = \delta$. Then $M = L_\delta$.

Proof:

$$M \models (V=L)$$

$$M \models \forall x \exists \alpha (x \in L_\alpha)$$

$$\forall x \in M \underbrace{\exists \alpha \in ON \cap M}_{\exists \alpha < \delta} (x \in L_\alpha)^M$$

$$\forall x \in M \exists \alpha < \delta \text{ so that } x \in L_\alpha$$

$$\therefore M \in L_\delta.$$

□

Summary:

	$M \models ZF^*$	$M \models ZF^* + (V=L)$
$ON \in M$ (class model)	$L \subseteq M$	$L = M$
M with $o(M) = \delta$ (set model)	$L_\delta \subseteq M$	$L_\delta = M$ (*)

ZF^* = sufficient so that $ZF^* + " \alpha \mapsto L_\alpha "$ is absolute.

Theorem: If $V=L$ then GCH.

Proof: I'll prove it for CH.

Condensation Lemma: ($V=L$). If $x \in \omega$ then $\exists \delta < \omega_1$ so that $x \in L_\delta$.

Why is this enough? $\mathcal{P}(\omega) \subseteq \bigcup_{\delta < \omega_1} L_\delta = L_{\omega_1}$

$$\therefore |\mathcal{P}(\omega)| \leq |L_{\omega_1}| = |\omega_1| = \omega_1.$$

$\Rightarrow$ CH.

Proof of Condensation:

Let $x \in \omega$, let

$$a = \{x\} \cup \omega$$

(just to make it transitive). $a \in M$
 $x \in M$

By the Reflection Theorem, there is a model M , such that

- $|M| = \aleph_0$ $x \cup \text{trcl}(x) \in M$
- $a \in M$ and $\epsilon \in M$ $x \in M$
every $u \in M$
- $M \models ZF^* + V=L$

(Basically $M = \text{Skolem Hull of } a \text{ inside } V$)

$\hookrightarrow$ Reading for the project.

M may not be transitive, but there is a trans. $\bar{M}$

so that $\pi: M \rightarrow \bar{M}$ isomorphism, $\pi(a) = a$.

$\hookrightarrow$ Mostowski: Collapse (read. material)

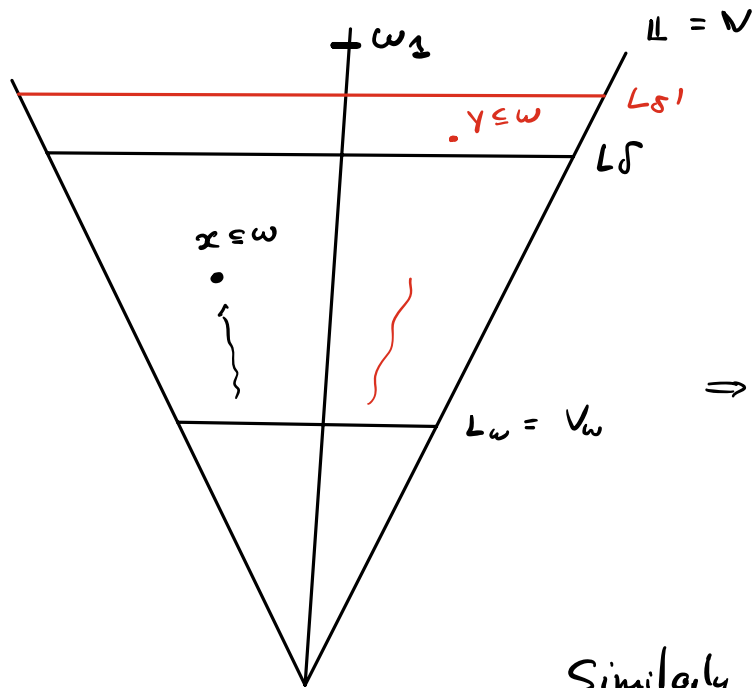
So we end up with $\bar{M}$ so that

- $a \in \bar{M}, x \in \bar{M}$
- $\bar{M} \models ZF^* + V=L$
- $\bar{M}$ is countable.

Now by (*) it follows that $\bar{M} = L_\delta$ for $\delta = o(\bar{M})$.

$\delta = \bar{M} \cap ON$ and $|\bar{M}| = \omega$ so $\delta < \omega_1$. So $x \in L_\delta$. $\square$

Visualize:



$p(\omega) \in L_{\omega_1}$
 $\Rightarrow CH.$

Similarly: $x \in K \rightarrow x \in L_{K^+}$

$\therefore |P(K)| \leq |L_{K^+}| = K^+$

$\Rightarrow GCH.$

Mostowski Collapse:

if (M, E) which is

$M = \text{class/set}$

(1) E is wellfounded

(2) E is set-like : $\{x \mid x E y\}$ is a

(3) E is extensional set for all y

$$\forall x \forall y (\forall z (z E x \leftrightarrow z E y) \rightarrow x = y)$$

Then there is trans. $\bar{M}$ and bij. $\pi: M \rightarrow \bar{M}$

$$\text{so that } x E y \Leftrightarrow \pi(x) \in \pi(y)$$

PROOF: By E -recursion:

$$\pi(x) := \{ \pi(y) \mid y E x \}$$

• If E is $\in$, $(M, \in)$ not necessarily transitive.

$$a \in M.$$

a is transitive set. $\text{trcl}(a) = a$

$$\text{Then } \pi(a) = a$$

e.g.: $x_7 \in M$, M cfl.

$$\pi[M] = \bar{M} \quad \bar{M} \text{ will contain } \pi(x_7)$$

~~$$x_7 \in \bar{M} \Rightarrow x_7 \in \bar{M}$$~~

$M \models ZF$

$\ulcorner \varphi \urcorner \in \mathbb{N}$

$$A \models \ulcorner \varphi \wedge \psi \urcorner \iff A \models \ulcorner \varphi \urcorner \text{ and } A \models \ulcorner \psi \urcorner$$

$$A \models \forall x \varphi \iff \forall x \in A (A \models \ulcorner \varphi(x) \urcorner)$$

If M contains $A, \ulcorner \varphi \urcorner, \ulcorner \varphi \urcorner, \dots$ then $M \models (A \models \ulcorner \varphi \urcorner)$

$$\iff A \models \ulcorner \varphi \urcorner$$