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Set Theory

Kenneth Kunen

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Set Theory
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Preface

This book is a total rewrite of the author's *Set Theory: An Introduction to Independence Proofs*, first published in 1980. Both books are intended for readers who have studied axiomatic set theory and who want to learn the basics of the independence proofs, starting with the independence of the Continuum Hypothesis (CH). There are two reasons why this rewrite was needed.

First, there has been a lot of set theory discovered over the past thirty years. Not all of this impacts on these two books, which just cover the basics of the subject, but some of the discoveries need to be reflected in the text. For example, the first book could essentially culminate with the consistency of Martin's Axiom (MA) with $\neg CH$, but now even an elementary text in set theory must give some discussion of results independent of $MA + \neg CH$, and at least mention the Proper Forcing Axiom.

Second, model-theoretic methods have become increasingly prevalent in mathematical arguments. Working in an axiomatic set theory (such as ZFC), one can formalize the syntax of logic and then prove theorems about model theory, and in particular about models of set theory. This idea was well-known to logicians going back to the 1940s, but even in the 1980s, there were still many researchers in set theory and its applications who did not know much logic. So, pains were taken to avoid model-theoretic methods in the first book. But now, anyone doing research in the more set-theoretic areas of topology and analysis has learned basic model theory and knows how to apply model-theoretic techniques to set-theoretic problems. So, the present book describes these techniques in Chapter I, as part of a review of basic logic, and then in later chapters applies these techniques to prove mathematical theorems and independence results.

One spinoff of this rewrite is that Gödel's proof of the consistency of the Generalized Continuum Hypothesis (GCH) has been downgraded from a whole chapter in the original text to one section in this text (part of Chapter II, "Easy Consistency Proofs"). The reason is that once one is familiar with the subject of models of set theory (which historically grew out of Gödel's proof), the construction of his model for GCH seems like almost the obvious thing to do.

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Chapter 0

Introduction

0.1 Why Read This Book?

Conventional mathematics is based on *ZFC* (the Zermelo-Fraenkel axioms, including the Axiom of Choice). Working within *ZFC*, one develops:

1. Abstract properties of the ordinals $0, 1, 2, \ldots, \omega \ldots$ and the infinite cardinal numbers $\aleph_0, \aleph_1, \ldots, \aleph_\omega, \ldots$.
2. All the mathematics found in basic texts on analysis, topology, algebra, etc.

However, some fundamental questions are not settled by *ZFC*. Most well-known is the Continuum Hypothesis, *CH*. This is the statement that $2^{\aleph_0} = \aleph_1$; equivalently, every uncountable subset of the set $\mathbb{R}$ of real numbers has the same size as $\mathbb{R}$.

CH is independent of *ZFC*. This means that it is consistent to have *ZFC* + *CH* and it is also consistent to have *ZFC* + $\neg$*CH*. The proofs of these consistency facts proceed by building models. We can construct one model $M \models ZFC + CH$ (Gödel, 1938) and another model $N \models ZFC + \neg CH$ (Cohen, 1963). In fact, there are models in which the continuum $2^{\aleph_0}$ is anything not contradicted by Cantor's theorem that $2^{\aleph_0} > \aleph_0$ or by the somewhat later result of König (1905) that $2^{\aleph_0} \neq \aleph_\gamma$ if γ is a limit ordinal of countable cofinality, such as ω, $\omega^3 + \omega^2$, $\omega_1 + \omega$, etc. But, $2^{\aleph_0}$ can be $\aleph_2$ or $\aleph_{17}$ or $\aleph_{\omega+1}$ or $\aleph_{\omega_1}$.

Most mathematicians are aware of these independence results, usually without knowing the proofs. The main purpose of this book is to present the proofs, along with proofs of many related independence results. So, the question is: Why should you spend your time studying these proofs?

Likewise, most mathematicians know that the Four Color Theorem was proved in 1976, but they will usually not know the details of the proof unless it is related to their area of research.

Set-theoretic independence proofs are worth studying because the independence of *CH* is just the beginning of the story, and in many areas of mathematics, questions come up that depend not only on cardinal arithmetic (the values of $2^{\aleph_0}, 2^{\aleph_1}, 2^{\aleph_2}, \ldots$), but also on other statements that are independent of cardinal arithmetic. We give two examples; one from elementary topology and one from elementary measure theory.

In topology, the Category Theorem (Baire, 1899) implies that the real line $\mathbb{R}$ is not the union of $\aleph_0$ closed nowhere dense (*cnwd*) sets. Here, $K \subset \mathbb{R}$ is cnwd iff K is closed and has empty interior; simple examples are Cantor sets and finite sets.

In measure theory, the Lebesgue theory of integration (1904) assigns a measure $\mu(X)$ to many sets $X \subseteq \mathbb{R}$. This generalizes the notion of length, since $\mu([2,5]) = 3$ and $\mu((1,2) \cup [3,4)) = 2$. Finite and countable sets are Lebesgue *null* sets — that is, they have measure 0. Also, the standard middle-third Cantor set is a null set. A countable union of null sets is null, so $\mathbb{R}$ is not the union of $\aleph_0$ null sets.

It is now natural to ask:

Q1. Is $\mathbb{R}$ the union of $\aleph_1$ cnwd sets?
Q2. Is $\mathbb{R}$ the union of $\aleph_1$ null sets?

Of course, if *CH* is true, then the answers are "yes" and "yes", since you may take your $\aleph_1$ sets to be singletons. But if *CH* is false, then the questions are non-trivial, and in fact the two questions are independent of $\neg CH$ and of each other. That is, there are models of $ZFC + \neg CH$ plus each of the $2 \cdot 2 = 4$ possible answers to Q1,Q2; see Proposition V.4.32.

These results on cnwd and null sets are from the 1960s, due to Solovay and others. They are by now well-known, and proofs are in standard texts, such as this one. But similar set-theoretic question keep coming up in analysis and in general topology, and in some parts of algebra. So, if you are working on problems in these areas, you need to know the methods for proving independence results, so that you can produce your own models.

With the introduction of axiomatic set theory in the early 1900s, it appeared that mathematics was the study of consequences of *ZFC*. But, by now it is clear that modern mathematics is the study of *models* of *ZFC*, which is the subject of this book.

0.2 Prerequisites

If the previous section made sense to you, then you probably know enough to read this book. But, more specifically:

It is assumed that you know basic set theory. This includes a knowledge of *ZFC* and the development, within *ZFC*, of basic properties of the ordinals and the infinite cardinals. We also assume some rudimentary knowledge of model

theory and recursion theory. In particular, you should have seen formal logic and the terminology $\Gamma \vdash \varphi$ used to mean that there is a formal proof of φ from Γ; here, Γ is a set of logical axioms (such as *ZFC*) and φ is a logical sentence. Logic texts vary as to the precise definition of "logical sentence" and "formal proof", but you should be aware that these notions are decidable; that is, there is an algorithm that will input a finite symbolic object and tell you whether it is a logical sentence, or an axiom of *ZFC*, or a formal proof. You should also know that $\Gamma \vdash \varphi$ is equivalent to the semantic notion $\Gamma \models \varphi$ (φ is true in all models of Γ) by the Soundness and Completeness Theorems.

The prerequisite material is usually covered in a beginning graduate level course in logic. Everything needed can be found in the author's [57]. However, much of [57] is not relevant here, and many readers will have learned the equivalent material from other texts, such as [28, 29, 53, 69, 71, 89]. Chapter I contains a summary of what is important. This chapter stresses basic logic issues, such as properties of models of set theory, since these are sometimes skimmed over in beginning courses, while it omits proofs of technical results such as König's Theorem I.13.12 ($\mathrm{cf}(\kappa^\lambda) > \lambda$), which such courses usually do cover.

We also assume some knowledge of elementary topology and analysis. This is not strictly necessary for understanding the proof that *CH* is independent of *ZFC*, but, as indicated in the previous section, the main reason for being interested in independence results is their relevance to these subjects, so we try to integrate examples from topology and analysis into the text.

0.3 How to Read This Book

The chapters are arranged by topic, and there may be more on a given topic than you care to know. So, reading this book from cover to cover, from the beginning straight through to the end, may be a mistake.

Chapter I contains background material. It is assumed that you know 70% of this, but you might scan it to pick up the other 30%, and also to see the notation that we are using here. All the material in this chapter *could* be contained in a beginning graduate logic course, but some of it often isn't, since the motivation for studying some of the topics isn't clear until you go on to more advanced material. In particular, we pay a lot of attention to some of the sub-theories of *ZFC*. This isn't very interesting if you just want to understand how to develop mathematics in *ZFC*, but it becomes important later because advanced work in set theory uses models for these sub-theories. Also, we assume that the reader has already seen recursively defined functions on the ordinals, such as the function $\alpha \mapsto \aleph_\alpha$, but Section I.9 discusses recursions on arbitrary well-founded relations, which isn't always covered in elementary works.

Chapter II covers "Easy Consistency Proofs". Sections II.1 through II.6

probably should be read in detail; they present the logical background behind all the independence results and explain Gödel's model L for $ZFC + GCH$. Cohen's method of *forcing*, which is the subject of Chapters IV and V, uses the same logical background. Section II.7 gives a very brief statement of the idea behind forcing. Sections II.8 through II.10 cover some issues of a more technical nature and are only occasionally referred to in the rest of this book.

Chapter III, on infinitary combinatorics, is a collection of results that mostly can be proved without any knowledge of independence proofs. Most of these get used eventually in the later chapters. But, it is possible to skip much of this chapter and go on to Chapter IV, referring back later to Chapter III as needed. The beginning of Chapter III says what parts of that chapter are really essential to understand the basic Cohen construction of a model for $ZFC + \neg CH$.

But also, most of the "classical" set theory in Chapter III can be read without any knowledge of independence results or Chapter II.

Most of Chapter IV (Forcing) is needed to understand Chapter V (Iterated Forcing). The main exception is Section IV.8 on the independence of the Axiom of Choice, which is rarely referred to again. Also, Section IV.7 is a rather long collection of assorted results proved by non-iterated forcing; the section says in the beginning how much of this is needed for the basics of iterated forcing.

Exercises are sprinkled throughout the various sections. Some are collected at the end of a section, but some are inserted in the middle of a section to provide examples in context. Some exercises will get used in later sections, so they are really lemmas whose proof is left to the reader.

Since the main reason for studying independence proofs is their application to mathematical problems, some exercises will involve some technical knowledge of analysis or topology going beyond the normal prerequisites for reading this book. Obviously, those exercises won't be used as lemmas, except possibly for other exercises of the same nature.

Chapter I

Background Material

Here is a summary of what you need to know about basic logic and set theory to understand the independence proofs. We shall often refer to logic texts for further details and some of the proofs. But, we shall occasionally include proofs of very elementary facts when these proofs are relevant to the logical foundations of set theory.

I.1 The Formalist Philosophy of Mathematics

Although not strictly necessary, some knowledge of philosophy is useful because it provides an explanation for the way we discuss logic and the foundations of set theory. For a deeper discussion of this topic, see [9, 12, 38, 73, 104].

Discussions of the philosophy of mathematics go back over two thousand years, but we focus on the controversy engendered by Cantor's development of the theory of infinite and uncountable sets in the late 1800s. Prior to that, most of mathematics was not too far removed from physical reality. But with Cantor's discussion of various sizes of infinity, $\aleph_0, \aleph_1, \aleph_2, \cdots, \aleph_\omega, \aleph_{\omega+1}, \cdots\cdots$, mathematics entered a different realm. We assume that the reader has seen statements such as:

Theorem I.1.1 $2^{\aleph_0} \geq \aleph_1$ *and* $2^{\aleph_0} \neq \aleph_\omega$.

Since this theorem has nothing to do with any physics that we can currently conceive of, there are various philosophical views, such as *Finitism*, *Platonism*, and *Formalism*, regarding what this theorem really means.

Finitism says that it means nothing at all. The notion of an infinite set is a fiction, and has no real meaning. A finitist will accept the use of *specific* infinite sets as abbreviations, if it is clear that you could express what you mean without using such sets. For example, the statement, "P and $\mathbb{N}\backslash P$ are both infinite, where $\mathbb{N}$ is the set of natural numbers and P is the set of primes", would be accepted by a finitist. The sets $\mathbb{N}$ and P don't really exist, but our

statement could be expressed more verbosely by saying that whenever $n > 1$ is a natural number, there is a non-prime larger than n (such as $2n$) and a prime larger than n (somewhere between $n+1$ and $n!+1$). However, it is meaningless to quantify over arbitrary subsets of $\mathbb{N}$ (e.g., $\forall X \subseteq \mathbb{N}(\cdots)$), and even more meaningless to claim that you can collect all these subsets into the power set $\mathcal{P}(\mathbb{N})$ and discuss its cardinality, $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$.

Plato [73] advanced the idea that abstract mathematical statements have an absolute meaning in some ideal universe, and that their truth is independent of us or what we think about them. For example, $2^{2^4} + 1 = 65537$ was always prime and $2^{2^5} + 1$ was never prime; and these are absolute facts that have nothing at all to do with people such as Fermat and Euler talking about them. This view is consistent with finitism; Plato did not say that his ideal universe contains any infinite sets, so you can share his philosophy without also believing that it makes sense to talk about $2^{\aleph_0}$. However, in modern times, *Platonism* has come to mean the philosophy that these infinitistic concepts really do have some meaning, and that Theorem I.1.1 expresses a fact about this abstract mathematical universe, and was always true, even before it was published by Cantor and by König. Modern platonists do not necessarily believe that *ZFC* is precisely the "right" set of axioms, but they usually believe that something approximately like *ZFC* is true in this ideal mathematical universe.

In this book, our *official* philosophy is *Formalism* — mainly because it helps to clarify the logical structure of the independence proofs. In this point of view, all we assert as being "really true" is elementary finitistic reasoning; this is called the *metatheory*. Using this reasoning, we can explain what a logical formula is and what a formal proof is and which logical formulas are axioms of *ZFC*. Then, all discussions about infinite and uncountable sets take place within *ZFC*, which is called the *formal theory*. Normally, in modern mathematical writing, we would just announce Theorem I.1.1 as being "true", since it is usually understood that our formal theory is *ZFC*. However, if we are stressing some point about formal logic, we might be more pedantic and announce that $ZFC \vdash 2^{\aleph_0} \geq \aleph_1$ (Cantor) and $ZFC \vdash 2^{\aleph_0} \neq \aleph_\omega$ (König). Finitists would agree that Theorem I.1.1 is a formal theorem of *ZFC*, since the formal proof is finite and is something that they can believe in. So, the difference between finitists and formalists is really one of attitude; a finitist would say that theorems of *ZFC* are completely uninteresting. Formalists remain agnostic towards questions about what kind of "absolute truth" is really expressed by set-theoretic axioms, so they maintain some interest in studying the consequences of these axioms.

We remark that it is okay for finitists and formalists to talk about infinite "sets" of axioms, such as *ZFC*, just as it is okay to talk about $\mathbb{N}$, as long as it is clear how to express what you mean without referring to such "sets".

Our finitistic reasoning in the metatheory is never analyzed formally. In fact, it cannot be, since you could not understand the formal analysis if you did not already understand some basic finitistic reasoning. There is a formal

axiomatic theory, called *Primitive Recursive Arithmetic* (see [91]), which is an approximation to finitistic reasoning, but to understand this formal system, you have to understand what a logical formula is and what a formal proof is, which involves some basic finitistic reasoning.

Whatever your philosophy, you should distinguish between *theorems of ZFC* and *schemes in the metatheory*. This is sometimes confusing because these are usually presented as statements in English, not in formal logic. For example, consider induction on the natural numbers and on the ordinals (see Section I.9 for more on this topic). Induction on $\mathbb{N}$ is the assertion that every non-empty set of natural numbers contains a least element. Let ON denote the class of all (von Neumann) ordinals; then induction on ON is the assertion that every non-empty class of ordinals contains a least element.

Working in ZFC, we can define the set $\mathbb{N} = \omega$; then induction on $\mathbb{N}$ is expressed by one sentence,

$$\forall S \subseteq \mathbb{N}\,[S \neq \emptyset \rightarrow \exists x \in S \forall z \in \mathbb{N}\,[z < x \rightarrow z \notin S]] \ ,$$

which is provable from ZFC; of course, it takes some work to unwind this sentence to an official sentence in the language of set theory (just using $\in, =$), but no one (not even a finitist) doubts that this can be done and that one can then come up with a formal proof of it from ZFC.

Working in ZFC, we can define the property "x is an ordinal" (Definition I.7.3) and then prove (Theorem I.7.12) that there is no set containing all the ordinals. Since ZFC has only sets, the class ON doesn't really exist. It is permissible to use the symbol ON by way of an abbreviation, if it is clear that you could express what you mean without using it, in the same way that a finitist will allow you to talk about $\mathbb{N}$; e.g., "$x \in ON$" is a convenient abbreviation of the assertion that x satisfies Definition I.7.3. However, you cannot express quantification over arbitrary subclasses of ON, so you cannot express induction by saying $\forall S \subseteq ON\,[S \neq \emptyset \rightarrow \cdots\cdots]$. You could talk about all subsets of ON, but this does not convey the true strength of induction; for example, once we have proved that there exists an uncountable ordinal, we want to apply induction with the "class" (*not* a set) S of all uncountable ordinals to claim that there is a least one (called ω_1). So, induction on ON is really a scheme in the metatheory: For each formula $\varphi(x, y_1, \ldots y_n)$ in the language of set theory, we may think informally about the class $\{x \in ON : \varphi(x, \vec{y})\}$ (which depends on $\vec{y}$), and we are asserting that $ZFC \vdash I_\varphi$, where I_φ is the sentence (abbreviated by):

$$\forall \vec{y}\,[\exists x \in ON\, \varphi(x, \vec{y}) \ \rightarrow \ \exists x \in ON\,[\varphi(x, \vec{y}) \wedge \forall z \in ON\,[z < x \rightarrow \neg\varphi(z, \vec{y})]]] \ ;$$

When we assert that "every non-empty class of ordinals contains a least element", we are really giving a finitistic explanation in the metatheory of how, given any φ, we may form I_φ and then write a formal proof of I_φ from ZFC. The fact that it was a bit tedious to write all this out in detail explains why peo-

ple use the convenient shorthand, "every non-empty class of ordinals contains a least element".

Now, the independence of the Continuum Hypothesis is another finitistic statement about formal proofs that is proved in the metatheory. People say, informally, that Gödel showed that "*CH* is relatively consistent with *ZFC*". What he really showed is that if φ is a statement of elementary number theory and $ZFC + CH \vdash \varphi$, then $ZFC \vdash \varphi$. In Section II.6 (Theorem II.6.24), we shall show explicitly, by a finitistic argument, how to construct a proof of φ from *ZFC* if you are given a proof of φ from $ZFC + CH$. φ could be $0 = 1$, so "*CH* is relatively consistent with *ZFC*" (*ZFC* + *CH* can't be inconsistent unless *ZFC* is inconsistent). But φ could also be the Twin Prime Conjecture; so, if you could prove the Twin Prime Conjecture using *CH*, then you can prove it without using *CH*. Likewise, Cohen showed that "$\neg CH$ is also relatively consistent with *ZFC*", with the same formal interpretation, as we shall see in Chapter IV. We say that *CH* is *independent* of *ZFC* because we have proved that both *CH* and $\neg CH$ are relatively consistent with *ZFC*.

Our official formalist point of view doesn't stop you from being a platonist, but it will help you to keep track of what is done in the formal theory and what is done in the metatheory. If you are a platonist reading this book, then you will believe that *CH* must be either true or false in your ideal mathematical universe, and you may even have an opinion as to which one it is, but the independence results are still of interest, since they show that you will never decide whether *CH* is true by using only principles formalizable in *ZFC*.

If you are a finitist, then you should still be able to follow all the arguments in this book and become convinced that *CH* is independent of *ZFC*, in the formal sense described above. You should also feel that reading this book is a waste of your time, given that most of the topics discussed, such as *CH*, have a meaning only within a formal axiomatic system that is completely devoid of any interest.

Regardless of your philosophy, you should be aware of Gödel's Second Incompleteness Theorem. Roughly, this says that if Γ is some formal axiomatization for the foundations of mathematics, such as *ZFC*, and $\Gamma \vdash \mathrm{Con}(\Gamma)$, then Γ is inconsistent. Here, $\mathrm{Con}(\Gamma)$ is a formal statement in the language of Γ asserting that Γ is consistent (that is, cannot prove any $\varphi \wedge \neg\varphi$). The theorem assumes that membership in Γ is decidable and that Γ is strong enough to develop at least finitistic reasoning. See [29, 53, 57, 69, 89] for a more formal discussion and a proof. The Incompleteness Theorem is never used in this book, except for some points of philosophy and motivation that we bring up now.

Philosophy. If you are a platonist, then you must believe that no formal system Γ can axiomatize all that is true. If you believe that all the axioms of Γ are really true, then Γ must be consistent, and then $\mathrm{Con}(\Gamma)$ is a true statement not provable from Γ.

Motivation. The independence of *CH* is usually understood informally as

the assertion, already made on page 1, that we can produce models of $ZFC+CH$ and of $ZFC+\neg CH$. But where are these models being produced? If we could work in ZFC and produce such models, then we would in particular be producing models of ZFC, so $ZFC \vdash \mathrm{Con}(ZFC)$, and hence ZFC is inconsistent. So, you should expect that our actual proof of independence will involve a somewhat more complex use of logic.

I.1.1 On doing things twice

Most people learn arithmetic in elementary school, before learning axiomatic set theory. So, you learn facts such as

1. $x+y=y+x$ for natural numbers x, y.
2. $2+8=10$.
3. Every natural number >1 is a product of prime numbers.

These statements about natural numbers may be viewed as part of the finitistic metatheory. Then, when you learn about ZFC, you realize that these, and thousands of other such facts, can be expressed as formal theorems of ZFC. For most of these facts, you don't ever write out a detailed proof. You are essentially using

Thesis I.1.2 *Every finitistic fact can be formalized and proved within ZFC.*

This is similar in spirit to the Church–Turing Thesis I.17.6 in recursion theory, that every function that is computable in the informal sense can be proved to satisfy your favorite mathematical definition of "computable". Neither thesis is a precise mathematical statement, and so can never be given a precise mathematical proof, but no one doubts the correctness of these theses in the specific instances where they are used, such as Thesis I.1.2 applied to (1)(2)(3) above. But, to be completely honest, finitistic mathematics gets done *twice*, once in the metatheory and once formalized in ZFC, whereas theorems about infinite sets are viewed *only* as theorems formalized within ZFC. Actually, ZFC is much stronger than what you need for Thesis I.1.2; we shall rephrase it later using a weaker but less well-known theory of sets (see Thesis I.3.2).

To understand even how to *say* (1)(2)(3) within axiomatic set theory, you do need some discussion, as is provided by elementary set theory texts. These texts explain that a natural number is really a finite von Neumann ordinal (so $2=\{0,1\}=\{\emptyset,\{\emptyset\}\}$), and they often include a rigorous proof of (1) from the axioms, since the commutative law fails for infinite ordinals. You will probably not find detailed proofs of (2)(3) in set theory texts, although it might be mentioned that proofs can be found for statements like (2) using a recursive definition: $x+0=x$; $x+S(y)=S(x+y)$, since $8=S(S(S(S(S(S(S(S(0)))))))$. For (3), no one doubts that the inductive proof that most people learn before studying set theory can be replicated within ZFC.

Thesis I.1.2 is often used without thinking when discussing, within *ZFC*, various theorems about infinite sets. For example, some non-trivial argument is needed to develop the theory of $\mathbb{R}$ within *ZFC*, but once this is done, it is clear that *ZFC* proves that $\forall x[x^2 \cdot x^8 = x^{10}]$ holds in the (uncountable) field of real numbers. Of course, this proof uses, among other things, the fact that $ZFC \vdash 2 + 8 = 10$.

Most people don't learn much about formal logic in elementary school, but the basics of logical syntax and formal proofs, as outlined in Section I.2, are at the same philosophical level as elementary arithmetic; they are both equally finitistic, so that Thesis I.1.2 applies equally to them. The difference is mainly historical. It was obvious from the beginning of set theory that ancient facts about natural numbers may get used in arguments about infinite sets. But logic was originally viewed as being external to the formal theory (*ZFC* here); logic just provided a means of saying precisely what is a correct argument. The first use of logic within a formal theory was probably in Gödel's Incompleteness Theorems (1931), since before you even *ask* whether $ZFC \vdash \mathrm{Con}(ZFC)$, you have to realize that the discussion in the metatheory of formal proofs and axioms must be formalized within *ZFC*, so that one can write $\mathrm{Con}(ZFC)$ in the language of set theory.

A related issue of philosophy and logic: *ZFC* is a set of sentences in the language of set theory. The language and the axioms must be specified at the outset (Section I.3), before we start to discuss whether the axioms can prove theorems about numbers, predicate logic, etc. The symbols of this language are $\in$ and $=$, together with variables, quantifiers, and propositional connectives, and we string these symbols together to form logical sentences. So how can we even *ask* whether $ZFC \vdash 2 + 8 = 10$, given that $+, 8, \cdots$ are not even symbols in the language of set theory?

There are actually two issues here: The philosophical issue, "Does the von Neumann $\{\emptyset, \{\emptyset\}\}$ really capture the true essence of 2?", is beyond the scope of this book. But the logic issue is: given that we agree, in the metatheory, to represent 2 as $\{\emptyset, \{\emptyset\}\}$, how exactly do we talk about this and other such objects in our formal logic system? To address this issue, we need to describe some mechanism for introducing defined notions into the formal theory; for further discussion, see Section I.2.

It is obvious that *ZFC* must be able to formalize theorems about numbers and arithmetic, or it would be completely useless. But also, the independence results require formalizing logic within *ZFC*, since these rely on the structure of infinite models for *ZFC* and various subtheories thereof. In addition, logic and model theory, formalized within *ZFC*, has become a useful tool for proving abstract theorems in set theory and general topology which seem at first to have nothing at all to do with logic; for more on this, see Section III.8.

I.2 Formal Logic

A *formal language* is a language with precisely defined rules of formation, unlike *natural languages* such as English. Working in the metatheory, people have defined many formal languages. Many of these are designed to interact with computers, such as programming languages (Java, Python, etc.), text processing languages (TEX, Postscript, etc.), and languages for web page design, such as html.

Predicate logic was designed for developing formal axiomatic theories. We assume that the reader has seen some development of formal logic, and we just make a few basic points here, without dwelling on fine points of syntax.

Regarding the basic symbols, one must distinguish between the *logical symbols* and *nonlogical symbols*. The *logical symbols* are fixed, and they include the equality symbol $=$, propositional connectives, such as $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$, quantifiers $\forall, \exists$, and variables $x, y, z, \ldots$. Many definitions of syntax require parentheses as well, in which case these also are logical symbols. Parentheses are used to distinguish between $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$; but if you write things in Polish (or prefix) notation, then you do not use parentheses, since you will write these formulas as $\vee\wedge pqr$ and $\wedge p \vee qr$.

The *nonlogical symbols* vary with context. Each application of logic will specify a set $\mathcal{L}$ of nonlogical symbols. Working in the finitistic metatheory, we assume that $\mathcal{L}$ is finite. Each symbol in $\mathcal{L}$ has a specified *arity*, which is some natural number, and a specified *type*, which is either "predicate symbol" or "function symbol". If we are discussing *ZFC*, then $\mathcal{L} = \{\in\}$, where $\in$ is a predicate symbol of arity 2 (more briefly, a 2–ary or binary predicate symbol). If we are axiomatizing the theory of ordered rings with unity, then we might take $\mathcal{L} = \{+, \cdot, <, 0, 1\}$, where $+$ and $\cdot$ are binary function symbols, $<$ is a binary predicate symbol, and $0, 1$ are 0–ary function symbols (usually called *constant symbols*). 0–ary predicate symbols are called *proposition letters*; examples are the p, q, r used above in $(p \wedge q) \vee r$.

The set $\mathcal{L}$, together with the assignment of types and arities to its symbols, is called the *lexicon* or *signature*. Given $\mathcal{L}$, we build the *formal language* consisting of the *formulas* of $\mathcal{L}$. *Sentences* are formulas with no free variables.

Since $=$ is a logical symbol, $\mathcal{L}$ could be empty, and we will still have logical formulas, such as the sentence $\exists x, y\,[x \neq y]$, which is true in a model iff that model contains more than one element. We follow the usual convention in logic that models are non-empty by definition, so that the sentence $\exists x[x = x]$, asserting "something exists" is true in every model.

Axioms of a theory are always sentences; that is, they have no free variables. In expositions, an axiom may be presented with free variables, but it is understood that the actual axiom is the *universal closure*, obtained by universally quantifying each free variable. For example, the multiplicative inverse law for fields might be written as the formula $x \neq 0 \rightarrow x \cdot x^{-1} = 1$. But it is understood that we mean *for all* x. It is also understood that x^{-1} is a way

of writing the multiplicative inverse which more formally should be something like $i(x)$, so our axiom becomes the sentence $\forall x\,[x \neq 0 \rightarrow x \cdot i(x) = 1]$. Of course, for this to be a sentence, $\mathcal{L}$ must contain at least the binary function symbol $\cdot$ and the unary function symbol i and the constant symbols 0 and 1.

Logic books usually contain some precise definition of syntax, and the definition is usually very simple, so that it can be presented quickly and so that theorems about the formal logic can be proved quickly. Such simple definitions can never encompass the wide variety of formalisms in every-day mathematical usage, so that what you really see most of the time, in those books and in this book, are *abbreviations* of logical formulas.

For example, in [57], the official definition of syntax uses Polish notation, so that the multiplicative inverse law becomes $\forall x \rightarrow \neg = x0 = \cdot xix1$. On the other hand, the text of Chang and Keisler [13] uses some parentheses, and uses infix for logical symbols but prefix for non-logical symbols, so the inverse law becomes $(\forall x)(\ \neg(\ (\neg x \equiv 0) \wedge (\neg \cdot (x, i(x)) \equiv 1)\ )\ )$; in [13], $\rightarrow$ is not a symbol of the language, and the only propositional connectives are $\wedge$ and $\neg$.

To understand set theory and model theory, it is never necessary to memorize any formal definition of syntax. It is important only to know that there *is* such a precise definition, and that the basic notions of logical syntax are decidable (see Section I.17). So, it is important to know that there is an algorithm that will read a string of symbols and decide whether it is a sentence in the language of set theory, and decide whether that sentence is an axiom of *ZFC*, as described in Section I.3. Then, once one has developed *ZFC*, it should be clear, as we mentioned in Subsection I.1.1, that within *ZFC*, one may develop the theory of formal logic and define the set of axioms of *ZFC*. One can also, working within *ZFC*, write a sentence Con(*ZFC*) asserting that *ZFC* is consistent:

Definition I.2.1 *If* Γ *is a set of sentences of* $\mathcal{L}$*, then* Incon(Γ) *is the assertion that there is a formal proof of a contradiction from* Γ. Con(Γ) *is* $\neg$Incon(Γ).

When $\mathcal{L}$ is finite and Γ is something like *ZFC*, which is described, as in Section I.3, by a finitistically acceptable definition, then Definition I.2.1 makes sense in the finitistic metatheory. But also, it makes sense as formalized within *ZFC*, in which case, $\mathcal{L}$ can have arbitrary cardinality. Also, within *ZFC*, one can talk about infinite models and prove the Completeness Theorem, which shows that Con(Γ) is equivalent to the statement that Γ has a model. Section I.15 will continue our discussion of formal logic and model theory within *ZFC*.

Although we said that $\mathcal{L} = \{\in\}$ for *ZFC*, the development of mathematics within *ZFC* would be virtually impossible if we never introduced any new symbolism. For example, very early in the development of *ZFC*, we say that $x \subseteq y$ means $\forall z(z \in x \rightarrow z \in y)$. What exactly does "means" mean? In an informal development of *ZFC*, one assumes that the axioms are "true" and one then proceeds by "ordinary mathematical reasoning", where such definitions are very common; one can let the logicians sort out the formal analysis behind

them. But here, we shall be a little more pedantic, since we also want to discuss the meaning of these defined notions in various models for set theory. We follow roughly the treatment in [57] §II.15, to which we refer the reader for proofs of the following two theorems; see also Shoenfield [89] §4.6. Our view is that any mathematical terminology beyond $\in$ and $=$ defined anywhere is an extension by definitions over *ZFC*.

Definition I.2.2 *Assume that $\mathcal{L} \subseteq \mathcal{L}'$ and Γ is a set of sentences of $\mathcal{L}$.*

If $p \in \mathcal{L}'\backslash\mathcal{L}$ is an n-ary predicate symbol, then a definition of p over $\mathcal{L}, \Gamma$ *is a sentence of the form $\forall x_1, \ldots, x_n\, [p(x_1, \ldots, x_n) \leftrightarrow \theta(x_1, \ldots, x_n)]$, where θ is a formula of $\mathcal{L}$.*

If $f \in \mathcal{L}'\backslash\mathcal{L}$ is an n-ary function symbol, then a definition of f over $\mathcal{L}, \Gamma$ *is a sentence of the form $\forall x_1, \ldots, x_n\, [\theta(x_1, \ldots, x_n, f(x_1, \ldots, x_n))]$, where θ is a formula of $\mathcal{L}$ and*

$$\Gamma \vdash \forall x_1, \ldots, x_n \exists! y\, \theta(x_1, \ldots, x_n, y) \quad . \qquad (*)$$

A set of sentences Γ' of $\mathcal{L}'$ is an extension by definitions *of Γ iff $\Gamma' = \Gamma \cup \Delta$, where $\Delta = \{\delta_s : s \in \mathcal{L}'\backslash\mathcal{L}\}$ and each δ_s is a definition of s over $\mathcal{L}, \Gamma$.*

The idea that extensions by definitions add nothing essentially new is made precise by:

Theorem I.2.3 *Assume that $\mathcal{L} \subseteq \mathcal{L}'$, Γ is a set of sentences of $\mathcal{L}$, and Γ' in $\mathcal{L}'$ is an extension by definitions of Γ in the sense of Definition I.2.2. Then*

1. *If φ is any sentence of $\mathcal{L}$, then $\Gamma \vdash \varphi$ iff $\Gamma' \vdash \varphi$.*
2. *If φ is any formula of $\mathcal{L}'$, then there is a formula $\widehat{\varphi}$ of $\mathcal{L}$ with the same free variables such that Γ proves the universal closure of $[\widehat{\varphi} \leftrightarrow \varphi]$.*

For an example of a defined predicate, consider the subset relation. Informally, with $\mathcal{L} = \{\in\}$ and Γ any axioms about $\in$, it is always permissible to "define" the relation $x_1 \subseteq x_2$ to mean $\forall z(z \in x_1 \rightarrow z \in x_2)$. More formally, we are now arguing from Γ' in $\mathcal{L}' = \{\in, \subseteq\}$, where Γ' adds the new axiom $\forall x_1, x_2\, [x_1 \subseteq x_2 \leftrightarrow \forall z(z \in x_1 \rightarrow z \in x_2)]$. Item (1) says that Γ' is a *conservative extension* of Γ; that is, any statement that just talks about $\in, =$ and that is provable from Γ' is already provable from Γ. An informal view of item (2) is that we may consider any formula φ of $\mathcal{L}'$ to be an abbreviation for a formula $\widehat{\varphi}$ of $\mathcal{L}$; more formally, φ and $\widehat{\varphi}$ are equivalent with respect to Γ'. We get $\widehat{\varphi}$ by replacing all occurrences of $\subseteq$ in φ by the definition of $\subseteq$.

Item (2) is used implicitly when developing *ZFC*. We frequently apply the Comprehension Axiom to form a set $\{x \in w : \varphi(x)\}$; see the discussion in Sections I.3 and I.4. Often, φ will involve $\subseteq$ or other defined notions. However, the axiom is stated in the beginning, before we have defined anything, so the φ in the Comprehension Axiom is in $\mathcal{L} = \{\in\}$. To justify an application of

Comprehension where φ uses defined notions, we replace φ by some equivalent $\widehat{\varphi}$ of $\mathcal{L}$ and form $\{x \in w : \widehat{\varphi}(x)\}$.

As an example of a defined function, consider the intersection of two sets. Informally, with $\mathcal{L} = \{\in\}$ and Γ any axioms about $\in$, we may "define" $x_1 \cap x_2$ to be the y such that $\text{int}(x_1, x_2, y)$, where $\text{int}(x_1, x_2, y)$ abbreviates the formula $\forall z(z \in y \leftrightarrow z \in x_1 \wedge z \in x_2)$. More formally, $\mathcal{L}' = \{\in, \cap\}$, and Γ' is Γ together with the axiom $\delta_\cap : \forall x_1, x_2\, \text{int}(x_1, x_2, x_1 \cap x_2)$. To illustrate (1): Γ' proves the $\mathcal{L}$–sentence $\forall x_1, x_2 \exists y\, \text{int}(x_1, x_2, y)$ (because it follows from $\delta_\cap$); but this sentence is provable from Γ (as claimed by item (1)) because of our proviso $(*)$ in Definition I.2.2. This proviso holds as long as Γ contains the Comprehension and Extensionality Axioms of *ZFC*; see Section I.4.

When we develop set theory, starting in Section I.4, we shall not be so pedantic; but shall point out why the existence and uniqueness of y is provable from the axioms we are considering when we introduce $\cap$ in Definition I.4.3.

As an example of (2), the formula $a \cap b = c \cap d$ is equivalent, using $\delta_\cap$, to $\exists s, t\, [\text{int}(a, b, s) \wedge \text{int}(c, d, t) \wedge s = t]$. The proof will use $\forall x_1, x_2 \exists! y\, \text{int}(x_1, x_2, y)$.

The $n = 0$ case of Definition I.2.2 occurs frequently; here the $x_1, \ldots, x_n$ is missing. A 0–ary function symbol is a constant symbol. For example, in developing set theory with $\mathcal{L} = \{\in\}$, we shall introduce (in Section I.4) the constant $\emptyset$ as the unique y satisfying the formula $\text{emp}(y) : \forall z(z \notin y)$. Then $\delta_\emptyset$ is just $\text{emp}(\emptyset)$, and the required $\exists! y\, \text{emp}(y)$ is provable from Γ as long as Γ contains the Comprehension and Extensionality Axioms. A 0–ary predicate symbol is a proposition letter. An example of this in set theory is *CH*; so δ_{CH} is just $CH \leftrightarrow \theta_{CH}$, where θ_{CH} is a sentence just using $\in$ and $=$ that expresses *CH*. Of course, no one ever writes such a sentence. Rather, we think of *CH* as arising from a long chain of extensions by definitions, applying:

Theorem I.2.4 *Assume that $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{L}''$, Γ is a set of sentences of $\mathcal{L}$, Γ' is a set of sentences of $\mathcal{L}'$, and Γ'' is a set of sentences of $\mathcal{L}''$. Assume that Γ' is an extension by definitions of Γ and Γ'' is an extension by definitions of Γ'. Then Γ'' is equivalent to an extension by definitions of Γ.*

In practice, we get $\{\in\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{1000} \subset \cdots$ by repeatedly iterating this theorem when developing *ZFC*. Then the proposition letter *CH* appears in some $\mathcal{L}_{\mathbf{k}+1}$, and θ_{CH} is $2^{\aleph_0} = \aleph_1$, where $\mathcal{L}_{\mathbf{k}}$ already contains the constants $0, 1, 2$, the unary aleph function, and the binary cardinal exponentiation function. We shall never be so pedantic as to say exactly what $\mathbf{k}$ is, but this general development will be important when we discuss defined notions in models. If we are trying to decide whether *CH* is true in a given model, we don't rewrite *CH* into an equivalent sentence in $\mathcal{L}_0 = \{\in\}$. Rather, retracing our development of set theory, we first decide what the model thinks about $0, 1, 2$ and the aleph function; see Section I.15 for a more formal discussion.

Actually, you could not say exactly what $\mathbf{k}$ is based on this book, because our presentation is not 100% formal. Even in the above discussion, when we were trying to be extremely pedantic, this matter is not entirely clear. Does

the symbol $\emptyset$ occur in $\mathcal{L}_1$, defined using the formula $\forall z(z \notin y)$, or does it occur in $\mathcal{L}_2$, where $\mathcal{L}_1 = \{\in, \text{emp}\}$ and emp is a defined 1–ary predicate?

The aleph function brings up two other points in formal logic. First, $\aleph_x$ means $\aleph(x)$; for historical reasons, some functions are commonly written using subscripts. Second, what is $\aleph_x$ when x isn't an ordinal? Informally, it's "undefined", but most treatments of formal logic and model theory don't allow for partially defined functions, since that would only complicate the semantics. Instead, we let it be some default value, such as $\emptyset$.

We return to the issue mentioned on page 10, at the end of Subsection I.1.1: How exactly do we say that $ZFC \vdash 2 + 8 = 10$ and $ZFC \vdash 2 \mid 8$, given that $\mathcal{L} = \{\in\}$, so that $+, |, 8, \cdots$ are not even symbols in our language? By now, the above discussion should make it clear that we are regarding $+$ as a defined binary function symbol, $|$ (divides) as a defined binary relation symbol, and $2, 8, 10$ as defined constant symbols.

A *quoting convention* is sometimes useful when discussing constants that also have a meaning in the finitistic metatheory. The sentence $|\mathbb{R}| = |\mathbb{C}|$ is meaningless in the metatheory, so this could only denote a sentence in the formal theory, where $\mathbb{R}$ and $\mathbb{C}$ are defined constants. The precise definition of these constants is different in different texts; one possible definition is given in Section I.14; but all definitions will satisfy $ZFC \vdash |\mathbb{R}| = |\mathbb{C}|$.

But, $2 + 8 = 10$ does make sense in the metatheory. If we want to be hyperpedantic, we shall write $ZFC \vdash \ulcorner 2 \urcorner + \ulcorner 8 \urcorner = \ulcorner 10 \urcorner$. Here, "$2 + 8 = 10$" refers to the fact in the metatheory that everyone knows — even people who don't think of 2 as $\{\emptyset, \{\emptyset\}\}$. But "$\ulcorner 2 \urcorner + \ulcorner 8 \urcorner = \ulcorner 10 \urcorner$" refers to the statement in the formal theory. Most of the time we shall dispense with the $\ulcorner\ \urcorner$, but formally there is a distinction between the concepts 2, 8, 10 known in the metatheory and the defined constant symbols $\ulcorner 2 \urcorner, \ulcorner 8 \urcorner, \ulcorner 10 \urcorner$.

Using this terminology, we might assert in the metatheory, as an example of Thesis I.1.2, that for all natural numbers n, m, k: If $n + m = k$ then $ZFC \vdash \ulcorner n \urcorner + \ulcorner m \urcorner = \ulcorner k \urcorner$. We might also assert that whenever a formula φ has n free variables, then $ZFC \vdash \psi(\ulcorner \varphi \urcorner, \ulcorner n \urcorner)$, where $\psi(x, y)$ is a formula in set theory expressing the notion that x is a logical formula, $y \in \mathbb{N}$, and x has y free variables. So, we then have $ZFC \vdash \psi(\ulcorner \psi \urcorner, \ulcorner 2 \urcorner)$.

Since there are many such objects in the metatheory that one might wish to quote, it is sometimes useful to have a uniform way of doing so. We take the view in the metatheory that we are representing all finite objects as *hereditarily finite sets*. These are sets, such as $\{0, 2\} = \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$, that can be displayed explicitly as finite expressions using $\emptyset$ and brackets(i.e., $\{, \}$). For a precise definition of "hereditarily finite" in set theory, see Definition I.10.19. For more on representing logical formulas as hereditarily finite sets, see page 82. Now, in the metatheory, we can define $\ulcorner b \urcorner$ by recursion on b. We let the constant $\ulcorner b \urcorner$ be introduced by the sentence $\theta_b(\ulcorner b \urcorner)$. Then $\theta_\emptyset(y)$ is just $\text{emp}(y)$, as described above. If $b = \{a_1, \ldots, a_n\}$, then $\theta_b(y)$ is the formula $\forall x\, [x \in y \leftrightarrow x = \ulcorner a_1 \urcorner \vee \cdots \vee x = \ulcorner a_n \urcorner]$. The required $\exists! y\, \theta_b(y)$ is provable from Γ as long

as Γ contains the Extensionality, Comprehension, Pairing, and Union Axioms; see Section I.4.

I.3 The Axioms of Set Theory

We assume that the reader knows what *ZFC* is, but books differ as to exactly how they state the axioms, so for reference, we list them all here. We shall also give names to some sub-theories of *ZFC*, such as $ZF-P$, which is *ZFC* with the Axioms of Choice and Power Set deleted. In most of mathematics, these sub-theories are not of much interest; you don't usually keep track of exactly which proofs use the Power Set Axiom. However, for discussing models and independence proofs, it is important to have some knowledge of what axioms get used where in the elementary development of set theory.

There is no natural order in which to list the axioms. We try to list the most "basic" axioms first, but which one is most "basic" depends on the application. In some sense, Extensionality, Comprehension, Pairing, Union are the most basic, since they let us develop some basic facts about pairing, finite unions, relations, functions, and the natural numbers $0, 1, 2, 3, 4, \ldots$. They are not enough to prove that cartesian products $A \times B$ always exist. This can be done either with the Power Set Axiom or the Replacement Axiom, and it is not clear which of these to list first. Also, Infinity is pretty basic, since without that, you can't get the set ω of all natural numbers. Many of the axioms say that some sets that "obviously should exist" (e.g., $\omega, \mathcal{P}(A)$, etc.) really do exist. We list the Axiom of Choice last because it has a different flavor, since there is no natural definition of a choice function. The Foundation Axiom, which says that $\in$ is well-founded, has a rather strange position among the axioms. It is irrelevant to the development of ordinary mathematics within *ZFC*; in fact, the proof of its relative consistency shows that *in principle* Foundation is irrelevant; see Remark II.3.2. However, by simplifying the structure of the $\in$ relation, Foundation greatly simplifies the discussion of models of set theory (see Section II.4), and almost all the models that we consider here will satisfy Foundation anyway, so we list it second, right after Extensionality.

We work in predicate logic with $\mathcal{L} = \{\in\}$. As we said in Section I.2, $=$ is a logical symbol and the statement that the universe is non-empty (i.e., $\exists x(x = x)$) is a logical fact, so we do not list this as an axiom. Axioms stated with free variables are understood to be universally quantified.

Axiom 1. Extensionality.

$$\forall z(z \in x \leftrightarrow z \in y) \quad \rightarrow \quad x = y$$

Axiom 2. Foundation.

$$\exists y(y \in x) \quad \rightarrow \quad \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y))$$

Axiom 3. Comprehension Scheme. For each formula, φ, without y free,

$$\exists y \forall x (x \in y \quad \leftrightarrow \quad x \in v \wedge \varphi(x))$$

Axiom 4. Pairing.

$$\exists z (x \in z \wedge y \in z)$$

Axiom 5. Union.

$$\exists A \forall Y \forall x (x \in Y \wedge Y \in \mathcal{F} \quad \rightarrow \quad x \in A)$$

Axiom 6. Replacement Scheme. For each formula, φ, without B free,

$$\forall x \in A \, \exists! y \, \varphi(x, y) \quad \rightarrow \quad \exists B \, \forall x \in A \, \exists y \in B \, \varphi(x, y)$$

The final three axioms are a little easier to state using some defined notions. On the basis of Axioms 1,3,4,5, define $\subseteq$ (subset), $\emptyset$ (or 0; empty set), S (ordinal successor function), $\cap$ (intersection), and SING(x) (x is a singleton) by:

$$\begin{array}{rcl}
x \subseteq y & \iff & \forall z (z \in x \rightarrow z \in y) \\
x = \emptyset & \iff & \forall z (z \notin x) \\
y = S(x) & \iff & \forall z (z \in y \leftrightarrow z \in x \vee z = x) \\
y = v \cap w & \iff & \forall x (x \in y \leftrightarrow x \in v \wedge x \in w) \\
\mathrm{SING}(x) & \iff & \exists y \in x \, \forall z \in x (z = y)
\end{array}$$

Section I.2 explains more precisely what defined notions mean in a formal logical system. One could, as per Theorem I.2.3(2), rewrite the final three axioms into longer sentences not using any defined notions.

Axiom 7. Infinity.

$$\exists x \big(\emptyset \in x \wedge \forall y \in x (S(y) \in x) \big)$$

Axiom 8. Power Set.

$$\exists y \forall z (z \subseteq x \rightarrow z \in y)$$

Axiom 9. Choice, or AC.

$$\emptyset \notin F \wedge \forall x \in F \, \forall y \in F (x \neq y \rightarrow x \cap y = \emptyset) \quad \rightarrow \quad \exists C \, \forall x \in F (\mathrm{SING}(C \cap x))$$

- *ZFC* = Axioms 1–9. *ZF* = Axioms 1–8.
- *ZC* and *Z* are *ZFC* and *ZF*, respectively, with the Replacement Scheme deleted.
- X^- denotes X with the Foundation Axiom deleted.
- $X-P$ denotes X with the Power Set Axiom deleted.
- $X-Inf$ denotes X with the Axiom of Infinity deleted.

ZFC is an infinite set of axioms because Axioms 3 and 6 are infinite *schemes* (one axiom for each φ). *ZFC* is not finitely axiomatizable; that is, there is no finite set of axioms equivalent to *ZFC*; see Theorem II.5.9.

In this book, we are mainly interested in the various *extensions* of *ZFC*, such as $ZFC + 2^{\aleph_0} = \aleph_5$. But it is useful to consider a few of the *subtheories* of *ZFC* for the following two reasons:

First, it aids in the understanding of the axioms to see how they are used in mathematics. Foundation is never used, so all standard mathematics takes place in ZFC^-. Basic properties of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ do not require the Axiom of Choice, so these are developed in ZF^-. The Power Set Axiom is needed to build uncountable sets, but $ZF^- - P$ proves the basic facts about $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$.

Second, working in *ZFC*, we can define some some natural models for some of the subtheories of *ZFC*. Most important in applications are the models $H(\kappa)$ (Definition I.13.27) and $R(\gamma)$ (Definition I.9.28). We shall see in Section II.2 that $H(\kappa) \models ZFC-P$ whenever κ is a regular uncountable cardinal and $R(\gamma) \models ZC$ whenever $\gamma > \omega$ is a limit ordinal. Understanding what part of mathematics can be developed in $ZFC-P$ and *ZC* give us some idea of what is true in these models.

The *hereditarily finite sets*, $HF = H(\aleph_0) = R(\omega)$, is of particular importance; see Definition I.10.19. Informally, these are the sets that can be displayed explicitly as finite expressions using $\emptyset$ and brackets (i.e., $\{, \}$). These sets include all the natural numbers and rational numbers, all finite sequences of these numbers, and all logical formulas (when $\mathcal{L}$ is finite). *HF* may be considered to be the home of finite mathematics. $HF \models ZFC-Inf$ (see Theorem II.2.14), so it satisfies all mathematical theorems that may be proved without the Axiom of Infinity.

We now bring up two additional points about axiomatics that are relevant when discussing models of theories weaker than *ZF*. First,

Definition I.3.1 *BST ("Basic Set Theory") denotes the axioms of Extensionality, Foundation, Comprehension, Pairing and Union,* plus *the* disjunction*: the Power Set Axiom holds* or *the Replacement Axiom holds. Then BST^- denotes these same axioms without Foundation.*

Since Power Set is a single sentence while Replacement is an infinite scheme, the disjunction is the set of all PSA $\vee \psi$ such that ψ is an instance of Replacement and PSA denotes the Power Set Axiom. We shall see in Section I.6 that many of the basic facts about functions and relations have dual proofs; one using Power Set and one using Replacement; these facts include the existence of $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ and the existence of Cartesian product $A \times B$. Also, many of the basic facts about ordinals proved in Section I.7 require only BST^-. So, all these facts are theorems of BST^-.

Assuming a disjunction in this way seems highly inelegant, and we are not advocating *BST* as a reasonable foundation for mathematics. We mention it here because the models $R(\gamma)$ (for γ a limit) and $H(\kappa)$ (for κ a regular cardinal)

all satisfy BST, but "most" $R(\gamma)$ do not satisfy Replacement and "most" $H(\kappa)$ do not satisfy Power Set. So, by tracking the consequences of BST, we shall know that both the $R(\gamma)$ and the $H(\kappa)$ satisfy some of the basic facts about functions, relations, and ordinals. We may restate Thesis I.1.2 as

Thesis I.3.2 *Every finitistic fact can be formalized and proved within* BST^-.

The second point involves the Axiom of Choice:

Definition I.3.3 AC^+ *is the statement that every set can be well-ordered.*

This definition will not be "official" until we have defined "well-order" (Definition I.6.21). Working in ZF^-, all the standard forms of the Axiom of Choice (Axiom 9, AC^+, Zorn's Lemma, etc.) are equivalent (see Section I.12). So, it is somewhat more elegant to use Axiom 9 (following Zermelo [105]) for our official AC, since it requires very few defined notions to state. But, the proof that Axiom 9 implies AC^+ seems to require the Power Set Axiom (see Lemma I.12.4), whereas all standard forms of Choice follow from AC^+ without Power Set. In applications of Choice in mathematics, it is often AC^+ or Zorn's Lemma or minor variants thereof that are used. Usually, when we verify that a model, such as $H(\kappa)$, satisfies $ZFC-P$, we shall in fact verify the stronger AC^+, so it will be useful to have a name for this axiom.

I.4 Extensionality, Comprehension, Pairing, Union

In this section, we assume only these four axioms, and we point out that they are sufficient to develop some basic facts about pairs, finite unions and relations and functions, and to define the numbers $0, 1, 2, 3, 4$.

Axiom 1 is Extensionality:

$$\forall x, y\, [\forall z(z \in x \leftrightarrow z \in y) \quad \rightarrow \quad x = y] \quad .$$

We have added "$\forall x, y$" to the axiom listed Section I.3 because, as we said there, all free variables are implicitly universally quantified.

Informally, the truth of this axiom is clear if we recall that the intended universe of set theory consists of all *hereditary sets*. That is, if x is in our universe, then x is a set and all members of x are also in our universe (and hence also sets). Then, $\forall z(z \in x \leftrightarrow z \in y)$ says that x, y are sets with exactly the same members, so they must be the same set.

Axiom 3 is the Comprehension *scheme*; for each logical formula φ without y free, we are asserting the universal closure of

$$\exists y \forall x(x \in y \ \leftrightarrow \ x \in v \wedge \varphi(x)) \quad .$$

So, the v gets universally quantified, along with all other variables other than x that happen to be free in φ.

A simple example of the use of Axioms 1,3 is the definition of the intersection of two sets, $v \cap w$, as indicated on page 17. Since this is our first such example, we shall be extremely pedantic:

Definition I.4.1 $\text{int}(v, w, y) \leftrightarrow \forall x(x \in y \leftrightarrow x \in v \wedge x \in w)$.

Introducing a defined relation such as $\text{int}(v, w, y)$ requires no justification at all. Informally, $\text{int}(v, w, y)$ says that $y = v \cap w$, although defining $v \cap w$ to be *the* y such that $\text{int}(v, w, y)$ *does* require a justification, namely:

Lemma I.4.2 $\forall v, w \, \exists! y \, \text{int}(v, w, y)$.

Proof. To prove $\exists y \, \text{int}(v, w, y)$, use Comprehension, with φ the formula $x \in w$:

$$\forall v, w \, [\exists y \forall x(x \in y \;\leftrightarrow\; x \in v \wedge x \in w)] \quad .$$

To prove that y is unique, observe, from the definition of $\text{int}(v, w, y)$, that

$$\text{int}(v, w, y_1) \wedge \text{int}(v, w, y_2) \rightarrow \forall x(x \in y_1 \leftrightarrow x \in y_2) \quad ,$$

so that $y_1 = y_2$ by Extensionality. □

This justifies, as we indicated in Section I.2, page 14:

Definition I.4.3 $v \cap w$ *is the unique* y *such that* $\text{int}(v, w, y)$.

As usual, before giving a name to an object satisfying some property, we must prove that that property really is held by a unique object; see Section I.2 for a more formal discussion of such definitions.

To avoid having to go through such a long justification for similar definitions in the future, we introduce some notation:

Notation I.4.4 *For any formula* $\varphi(x)$*:*

- ☞ $\{x : \varphi(x)\}$ *is,* informally, *called a class.*
- ☞ *If there is a set* A *such that* $\forall x[x \in A \leftrightarrow \varphi(x)]$*, then* A *is unique by Extensionality, and we denote this set by* $\{x : \varphi(x)\}$*, and we say that* $\{x : \varphi(x)\}$ exists.
- ☞ *If there is no such set, then we say that* $\{x : \varphi(x)\}$ doesn't exist, *or* forms a proper class.
- ☞ $\{x \in v : \varphi(x)\}$ *abbreviates* $\{x : x \in v \wedge \varphi(x)\}$.

Comprehension asserts that sets of the form $\{x : x \in v \wedge \varphi(x)\}$ always exist. As an illustration, we restate the above definition of $v \cap w$, together with the definition of set difference:

Definition I.4.5 *Given* v, w*:*

☞ $v \cap w := \{x \in v : x \in w\}$.
☞ $v \setminus w := \{x \in v : x \notin w\}$.

Here, φ is, respectively, $x \in w$ and $x \notin w$. To form $v \cup w$, which can be bigger than v and w, we need the Union Axiom, discussed below. First, a few more examples. For $\emptyset$:

Definition I.4.6 $\mathrm{emp}(y)$ *iff* $\forall x(x \notin y)$.

Definition I.4.7 $\emptyset$ *denotes the (unique)* y *such that* $\mathrm{emp}(y)$.

Justification. Fix any v, and form $y = \{x \in v : x \neq x\}$, which is empty, so $\exists y\, \mathrm{emp}(y)$. Then, y is unique by Extensionality. As remarked on page 16, the statement that there really exists some v (our universe is non-empty, or $\exists v(v = v)$) is a logical fact, and was not listed as an axiom of set theory. □

One example of a set that doesn't exist, or forms a proper class, is our universe:

Notation I.4.8 *The universe,* $V := \{x : x = x\}$.

In view of Notation I.4.4, the statement "V doesn't exist" is just shorthand for $\neg\exists A\, \forall x\, [x \in A \leftrightarrow x = x]$, or equivalently $\neg\exists A\, \forall x\, [x \in A]$. To prove that V doesn't exist, we use the Russell Paradox, which applies Cantor's diagonal argument (Theorem I.10.4) to V:

Lemma I.4.9 V *doesn't exist, and neither does* $R := \{x : x \notin x\}$.

Proof. For R, we need no axioms at all; if we had an R such that $\forall x\, [x \in R \leftrightarrow x \notin x]$, then $R \in R \leftrightarrow R \notin R$, a contradiction. For V, we use Comprehension; if we had a V with $\forall x\, [x \in V]$, then we could form R as $\{x \in V : x \notin x\}$. □

Informally, it is sometimes useful to introduce letters to represent proper classes, such as V, since it will simplify our notation. Note that we called I.4.8 "Notation", not "Definition". For example, we shall later (Section I.7) introduce ON as the class of (von Neumann) ordinals; this also doesn't exist, but this notation will be useful; e.g. we can then write $x \subset ON$ as a convenient shorthand for saying that all members of the set x are ordinals. When using this sort of notation,

Caution I.4.10 *Make sure that anything you say with classes is really an abbreviation for something that you can say without classes.*

This was also discussed on page 7. As another example, using the R from Lemma I.4.9, "$R = V$" is shorthand for $\forall x\,[x \notin x]$. Is $R = V$ true? In fact, $R = V$ is easily proved if we add the Foundation Axiom to the ones already discussed; see Exercise I.4.18. However, it is consistent with ZFC^- (all the axioms except Foundation) to have $R \neq V$; in fact, $\{x : x = \{x\}\}$ can be a proper class (see Exercise II.9.5).

We can also define subset, $x \subseteq y$, as on page 17. Then Extensionality is equivalent to the assertion $\forall x, y\,[x \subseteq y \wedge y \subseteq x \to x = y]$.

So far, $\emptyset$ is the only set whose existence we have actually demonstrated, and in fact, Extensionality plus Comprehension is consistent with $V = \{\emptyset\}$ (equivalently, $\forall x, y\,[x \notin y]$).

We can construct non-empty sets using Pairing (Axiom 4):

$$\forall x, y\,\exists z\,(x \in z \wedge y \in z) \quad .$$

This gives us $\{x, y\}$. The axiom allows z to contain other elements as well, but given z, we can form $u = \{w \in z : w = x \vee w = y\}$. This justifies the following definition of unordered and ordered pairs:

Definition I.4.11

- $\{x, y\} = \{w : w = x \vee w = y\}$.
- $\{x\} = \{x, x\}$.
- $\langle x, y\rangle = (x, y) = \{\{x\}, \{x, y\}\}$.

The key fact about ordered pairs is:

Lemma I.4.12

$$\langle x, y\rangle = \langle x', y'\rangle \to x = x' \wedge y = y' \quad .$$

There are many other definitions of ordered pair that satisfy this lemma, although our definition, from Kuratowski [60], is the one commonly used now. In most of mathematics, it does not matter which definition is used — it is only important that x and y are determined uniquely from their ordered pair. But sometimes it will be useful to refer to this specific Kuratowski definition when we are discussing foundational issues or properties of models for set theory (see, e.g., Definitions I.6.4 and I.6.5 and Lemma II.4.3).

It is common in mathematics to use the "(x, y)" notation for ordered n–tuples in general; e.g., $s := (x, y, z, t)$ is an ordered 4–tuple. In set theory, a *sequence* is a function whose domain is an ordinal, so s is the function whose domain is $4 = \{0, 1, 2, 3\}$, with $s(0) = x$, $s(1) = y$, $s(2) = z$, $s(3) = t$. When you study calculus, you never ask whether (x, y) denotes the Kuratowski pair $\{\{x\}, \{x, y\}\}$, or a function with domain $2 = \{0, 1\}$, namely $\{\langle 0, x\rangle, \langle 1, y\rangle\}$.

Of course, we need to define $0, 1, 2, 3, 4$:

Definition I.4.13

$$0 = \emptyset \qquad 1 = \{0\} = \{\emptyset\} \qquad 2 = \{0,1\} = \{\emptyset, \{\emptyset\}\} \ .$$

Following von Neumann, we are regarding each natural number to be the set of smaller natural numbers. Note that we stopped at 2. The axioms so far don't let us construct any sets any with more than two elements. To do that we use the Union Axiom; then we can get $3 = 2 \cup \{2\} = \{0,1,2\}$. The usual statement of the Union Axiom can be used to justify infinite unions as well (once we are able to build infinite sets):

$$\forall \mathcal{F}\, \exists A\, \forall Y\, \forall x\, [x \in Y \wedge Y \in \mathcal{F} \ \rightarrow \ x \in A] \ .$$

That is, for any $\mathcal{F}$ (in our universe), view $\mathcal{F}$ as a family of sets. This axiom gives us a set A that contains all the members of members of $\mathcal{F}$. We can now take the union of the sets in this family:

Definition I.4.14

$$\bigcup \mathcal{F} = \bigcup_{Y \in \mathcal{F}} Y = \{x : \exists Y \in \mathcal{F}(x \in Y)\}$$

That is, the members of $\bigcup \mathcal{F}$ are the members of members of $\mathcal{F}$. As usual when writing $\{x : \cdots\cdots\}$, we must justify the existence of this set.

Justification. Let A be as in the Union Axiom, and apply Comprehension to form $B = \{x \in A : \exists Y \in \mathcal{F}(x \in Y)\}$. Then $x \in B \leftrightarrow \exists Y \in \mathcal{F}(x \in Y)$. □

Definition I.4.15 $u \cup v = \bigcup\{u, v\}$, $\{x, y, z\} = \{x, y\} \cup \{z\}$, *and* $\{x, y, z, t\} = \{x, y\} \cup \{z, t\}$.

To prove facts such as $u \cup v = v \cup u$, prove $\forall x\, [x \in u \cup v \leftrightarrow x \in v \cup u]$ from the definition of union, and then use Extensionality. Likewise, one may derive from the axioms the other basic properties of $\cup$ and $\cap$ that are known from elementary mathematics, such as the associative laws $u \cap (v \cap w) = (u \cap v) \cap w$ and $u \cup (v \cup w) = (u \cup v) \cup w$, and the distributive laws $u \cap (v \cup w) = (u \cap v) \cup (u \cap w)$ and $u \cup (v \cap w) = (u \cup v) \cap (u \cup w)$.

We can also define the intersection of a family of sets; as with pairwise intersection, this only requires the Comprehension and Extensionality Axioms:

Definition I.4.16 *When* $\mathcal{F} \neq \emptyset$,

$$\bigcap \mathcal{F} = \bigcap_{Y \in \mathcal{F}} Y = \{x : \forall Y \in \mathcal{F}(x \in Y)\}$$

Justification. Fix $E \in \mathcal{F}$, and form $\{x \in E : \forall Y \in \mathcal{F}(x \in Y)\}$. □

Note that $\bigcup \emptyset = \emptyset$, while $\bigcap \emptyset$ would be the universal class, V, which doesn't exist.

We can now get 3 and 4:

Definition I.4.17 *The* ordinal successor function, $S(x)$, *is* $x \cup \{x\}$. *Then:*

$$\begin{array}{lllll} 3 & = & S(2) & = & \{0,1,2\} \\ 4 & = & S(3) & = & \{0,1,2,3\} \\ 5 & = & S(4) & = & \{0,1,2,3,4\} \end{array} \qquad \begin{array}{lllll} 6 & = & S(5) & = & \{0,1,2,3,4,5\} \\ 7 & = & S(6) & = & \{0,1,2,3,4,5,6\} \\ 8 & = & S(7) & = & \{0,1,2,3,4,5,6,7\} \end{array}$$

Of course, we could also define 9. But, to collect all the natural numbers in the infinite set $\mathbb{N} = \omega$, we need the Axiom of Infinity (see Section I.5). The natural numbers are also called the *finite ordinals.* If we keep counting past all the natural numbers, we hit the first infinite ordinal. Since each ordinal is the set of all smaller ordinals, this first infinite ordinal is $\omega = \mathbb{N}$. The next two ordinals are $S(\omega) = \omega \cup \{\omega\} = \{0,1,2,\ldots,\omega\}$ and $S(S(\omega)) = \{0,1,2,\ldots,\omega,S(\omega)\}$. We then have to explain how to add and multiply ordinals. Not surprisingly, $S(S(\omega)) = \omega + 2$ and $\omega + \omega = \omega \cdot 2$ and $\omega \cdot \omega = \omega^2$, so the ordinals start out:

$$0,1,2,\ldots,\omega,\omega+1,\omega+2,\ldots,\omega\cdot 2,\omega\cdot 2+1,\ldots,\omega\cdot 3,\ldots,\omega\cdot\omega,\ldots,\omega^3,\ldots\omega^\omega,\ldots$$

These are all countable ordinals, but after we count for $\aleph_1$ years, we shall come to the first uncountable ordinal, called ω_1 or $\aleph_1$. To develop all this rigorously in *ZFC*, we first have to develop the theory of well-orders, which we shall do in Section I.6.

Exercise I.4.18 *Derive* $\forall y\,[y \notin y]$ *from the Axioms of Comprehension and Foundation.* Don't *use the Pairing or Extensionality Axioms. Then find a two element model for Foundation, Extensionality, Pairing, and Union, plus* $\exists y \forall x\,[x \in y]$ *(so, of course,* $y \in y$*).*

Hint. If $y \in y$, use Comprehension to form $x := \{y\}$; the fact that there may be more than one such x doesn't affect the proof. □

I.5 Infinity, Replacement, and Power Set

These three axioms are discussed separately and very briefly here, just to review how they are used. They all imply that some natural collections $\{x : \varphi(x)\}$ that "should" exist really do exist. The remaining axioms, Foundation and Choice, have a completely different flavor.

In this section, we assume throughout the axioms of Extensionality, Comprehension, Pairing, and Union; the use of other axioms will be noted explicitly.

The *Axiom of Infinity* lets us prove that the collection of natural numbers exists.

Definition I.5.1 *A set* I *is* inductive *iff* $\emptyset \in I$ *and* $\forall y \in I(S(y) \in I)$.

The Axiom of Infinity says that there exists an inductive set. Every limit ordinal is an inductive set, but we need the Axiom of Infinity to prove that limit ordinals exist. One definition of $\mathbb{N}$ is "the smallest inductive set":

Definition I.5.2 *Assuming the Axiom of Infinity:*

$$\mathbb{N} = \{y : \forall I\,[I \text{ is inductive } \rightarrow y \in I]\} \ .$$

Justification. We may form $\mathbb{N}$ as $\{y \in J : \forall I\,[I \text{ is inductive } \rightarrow y \in I]\}$, where J is any inductive set; there is one by the Axiom of Infinity. □

Immediately from the definition of $\mathbb{N}$, we see that $\mathbb{N}$ is inductive and:

Theorem I.5.3 (Principle of Ordinary Induction) *Assuming the Axiom of Infinity: For any set $X \subseteq \mathbb{N}$, if X is inductive then $X = \mathbb{N}$.*

Induction is commonly used as a proof method. To prove $\forall y \in \mathbb{N}\,\varphi(y)$, it is sufficient to verify that $X := \{y \in \mathbb{N} : \varphi(y)\}$ is inductive; that is, we verify $\varphi(0)$ (the *basis*) and $\varphi(y) \rightarrow \varphi(S(y))$ (the *induction step*).

Definition I.5.2 is sometimes used in undergraduate texts (see Moschovakis [71]), since one may now prove theorems about the natural numbers without studying the general theory of ordinals. However, we really do need ordinals in set theory, and we shall develop them soon (Section I.7); we shall then see that $\mathbb{N}$ is the same as ω, the smallest limit ordinal (Theorem I.7.21).

The *Replacement Axiom* is, like Comprehension, an infinite *scheme*. For each formula, φ, without B free, it asserts

$$\forall x \in A\,\exists! y\,\varphi(x,y) \quad \rightarrow \quad \exists B\,\forall x \in A\,\exists y \in B\,\varphi(x,y) \ .$$

That is, suppose that for each $x \in A$, we associate a unique object y_x (the unique y such that $\varphi(x,y)$). Then we "should" be able to form the set $C = \{y_x : x \in A\}$ by *replacing* each x by y_x in A. Using Replacement plus Comprehension, we form C as: $C = \{y \in B : \exists x \in A\ \varphi(x,y)\}$.

For example, say we have a set A. Let $A^* = \{\{x\} : x \in A\}$; this notation is shorthand for $\{y : \exists x\,[x \in A \wedge y = \{x\}]\}$. Informally, this set "should" exist, and it "should" have the same size as A. Using Replacement, A^* does exist; it is the C in the above discussion, where $\varphi(x,y)$ says that $y = \{x\}$ (that is, $x \in y \wedge \forall z \in y\,[x = z]$).

The *Power Set Axiom* says that for all x, there is a y such that $\forall z(z \subseteq x \rightarrow z \in y)$. Thus, applying Comprehension, $\{z : z \subseteq x\}$ exists, justifying:

Definition I.5.4 *Assuming the Power Set Axiom: The* power set of x *is* $\mathcal{P}(x) = \{z : z \subseteq x\}$.

The Power Set Axiom can be used instead of Replacement to prove that the above A^* exists, since $A^* = \{y \in \mathcal{P}(A) : \exists x\,[x \in A \wedge y = \{x\}]\}$. This proof is a bit inelegant, since it needs the existence of the uncountable set $\mathcal{P}(\mathbb{N})$ to construct the countable set $\mathbb{N}^*$. A similar example from Section I.6: Cartesian products $A \times B$ can be proved to exist using either Power Set or Replacement. These dual proofs suggest the theory *BST*; see the discussion following Definition I.3.1.

Neither $\mathbb{N}^*$ nor $\mathbb{N} \times \mathbb{N}$ can be proved to exist in $ZC-P$ (deleting both Power Set and Replacement; see Exercise II.4.22).

I.6 Discrete Mathematics; Relations and Functions

In this section, we work in the theory BST^-, for the reasons explained after Definition I.3.1; the use of axioms beyond BST^- in definitions or lemmas will be mentioned explicitly. Historically, much of this discrete mathematics was developed within Zermelo set theory before the Replacement Axiom was stated, whereas now, as we shall indicate below, the proofs using Replacement seem more elegant and natural than the proofs using Power Set.

Definition I.6.1 *R is a (binary)* relation *iff R is a set of ordered pairs — that is,*

$$\forall u \in R \, \exists x, y \, [u = \langle x, y \rangle] \quad .$$

xRy abbreviates $\langle x, y \rangle \in R$ and $x \not R y$ abbreviates $\langle x, y \rangle \notin R$.

The following are some commonly used properties of relations:

Definition I.6.2

- *R is* transitive *on A iff* $\forall xyz \in A \, [xRy \wedge yRz \to xRz]$.
- *R is* irreflexive *on A iff* $\forall x \in A \, [x \not R x]$
- *R is* reflexive *on A iff* $\forall x \in A \, [xRx]$
- *R satisfies* trichotomy *on A iff* $\forall xy \in A \, [xRy \vee yRx \vee x = y]$.
- *R is* symmetric *on A iff* $\forall xy \in A \, [xRy \leftrightarrow yRx]$.
- *R* partially orders *A* strictly *iff R is transitive and irreflexive on A.*
- *R* totally orders *A* strictly *iff R is transitive and irreflexive on A and satisfies trichotomy on A.*
- *R is an* equivalence relation *on A iff R is reflexive, symmetric, and transitive on A.*
- *R is a* pre-order *on A iff R is reflexive and transitive on A.*
- *R* partially orders *A* non-strictly *iff R is a pre-order on A and satisfies* $\neg \exists xy \in A \, [xRy \wedge yRx \wedge x \neq y]$.

Definition I.6.2 does not assume that $R \subseteq A \times A$, although this *is* implied in some contexts. For example, we may say, "there are exactly 6 strict total orders R on the set $A = \{e, \pi, 1\}$, one of which is the usual $<$." Obviously, to get the "6", we are only counting $\{R \subseteq A \times A : \cdots\}$, whereas "$<$" denotes the ordering on all the real numbers.

In the literature, the word "strict" is not often used when discussing partial orders; rather, $<$ is used for the strict partial order and $\leq$ for the corresponding non-strict partial order $x \leq y \leftrightarrow x < y \vee x = y$ and $x < y \leftrightarrow x \leq y \wedge x \neq y$. Both $<$ and $\leq$ are referred to as "partial orders".

If R is a pre-order on A, then the relation $x \approx y$ iff $xRy \wedge yRx$ is an equivalence relation on A. A simple example: If A is the set of sentence of $\mathcal{L}$ and $\varphi R \psi$ iff $\varphi \vdash \psi$, then R is a pre-order and $\approx$ is logical equivalence.

Definition I.6.3 *R is a* function *iff R is a relation and for every x, there is at most one y such that $(x, y) \in R$. If $\exists y\,[xRy]$, then $R(x)$ denotes that unique y.*

Most of these definitions should be familiar from elementary mathematics, along with the rest of the definitions in this section. We list them to specify our notation and to make it clear that they can all be developed within our axiomatic set theory.

Definition I.6.4 *For any set A, $id_A = \{(x, x) : x \in A\}$ is the identity function on A.*

Justification. Assuming the Power Set Axiom: Recall (Definition I.4.11) that $(x, y) = \{\{x\}, \{x, y\}\}$. If $x, y \in A$ then $\{x\}, \{x, y\} \in \mathcal{P}(A)$, so that $(x, y) \in \mathcal{P}(\mathcal{P}(A))$. Thus, we can form id_A using Comprehension as $\{z \in \mathcal{P}(\mathcal{P}(A)) : \exists x\,[z = (x, x)]\}$.

Assuming the Replacement Axiom: This is exactly like the justification for A^* discussed in Section I.5. □

The justification using Replacement is more elegant, in that it doesn't refer to the specific Kuratowski definition of ordered pair, and would work equally well with any definition. Also, once we have $\mathbb{N}$ (using the Axiom of Infinity), it seem a bit strange that to justify the existence of $id_{\mathbb{N}}$, which is a countable object, we need to construct $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, which is uncountable, of size $2^{2^{\aleph_0}}$.

Definition I.6.5 $A \times B = \{(x, y) : x \in A \ \wedge \ y \in B\}$.

Justification. Assuming the Power Set Axiom: As above, use the Kuratowski definition of pair, so $A \times B = \{(x, y) \in \mathcal{P}(\mathcal{P}(A \cup B)) : x \in A \ \wedge \ y \in B\}$.

Assuming the Replacement Axiom: Use Replacement twice. Use it once to prove that for each y, we may form $A \times \{y\} = \{(x, y) : x \in A\}$. Then, use it again to show that we may form $S := \{A \times \{y\} : y \in B\}$. Then get $A \times B = \bigcup S$ by the Union Axiom. For more details, see [57]. □

Definition I.6.6 *For any set R, define:*

$$\mathrm{dom}(R) = \{x : \exists y[(x, y) \in R]\} \qquad \mathrm{ran}(R) = \{y : \exists x[(x, y) \in R]\} \ .$$

Justification. Using the Kuratowski definition, $\{\{x\}, \{x, y\}\} \in R$ implies that $\{x\}, \{x, y\} \in \bigcup R$ and hence $x, y \in \bigcup\bigcup R$, so we get $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ using the Union Axiom plus Comprehension. Neither Power Set nor Replacement is needed.

Using Replacement: This is exactly like the justification for A^* discussed in Section I.5, and works equally well with any definition of ordered pair. □

Definition I.6.7 $R \restriction A = \{(x, y) \in R : x \in A\}$.

Here, $R \restriction A$ is a defined subset of R, so it exists by Comprehension. This $\restriction$ (restriction) is most often used for functions.

We can also make the usual definitions of "injection", "surjection", and "bijection":

Definition I.6.8

- ☞ $F : A \to B$ *means that* F *is a function,* $\mathrm{dom}(F) = A$, *and* $\mathrm{ran}(F) \subseteq B$.
- ☞ $F : A \xrightarrow[\text{onto}]{} B$ *or* $F : A \twoheadrightarrow B$ *means that* $F : A \to B$ *and* $\mathrm{ran}(F) = B$ (F *is a* surjection *or maps* A onto B).
- ☞ $F : A \xrightarrow{1-1} B$ *or* $F : A \hookrightarrow B$ *means that* $F : A \to B$ *and* $\forall x, x' \in A[F(x) = F(x') \to x = x']$ (F *is an* injection *or maps* A 1-1 into B).
- ☞ $F : A \xrightarrow[\text{onto}]{1-1} B$ *or* $F : A \rightleftarrows B$ *means that both* $F : A \xrightarrow{1-1} B$ *and* $F : A \xrightarrow[\text{onto}]{} B$ (F *is a* bijection *from* A *onto* B).

Replacement justifies the following common way of defining functions. Unlike most of other results in this section, there is no way to do this using Power Set instead, since the following lemma is easily seen to be equivalent to Replacement:

Lemma I.6.9 *Assume that* $\forall x \in A \exists! y \varphi(x, y)$, *and assume the Axiom of Replacement. Then there is a function* f *with* $\mathrm{dom}(f) = A$ *such that for each* $x \in A$, $f(x)$ *is the unique* y *such that* $\varphi(x, y)$.

Proof. Let $\psi(x, z)$ say $\exists y [\varphi(x, y) \wedge z = (x, y)]$, and use Replacement to form f as $\{z : \exists x \in A \, \psi(x, z)\}$. □

Definition I.6.10 *Assuming the Power Set Axiom, define* $B^A = {}^A B$ *to be the set of all* F *such that* $F : A \to B$.

Justification. If $F : A \to B$, then $F \subseteq A \times B$, so that B^A is a defined subset of $\mathcal{P}(A \times B)$. □

Here, we really need the Power Set Axiom, and cannot get by with the Replacement Axiom. For example, ${}^{\omega}\omega$ is an uncountable set of size $2^{\aleph_0}$, and the model HC satisfies $ZFC-P$, plus the statement that there are no uncountable sets; see Lemma II.4.4. In the case that A is a natural number, we can avoid Power Set and induct on $n \in \mathbb{N}$ to construct ${}^n B$ and ${}^{<\omega} B$:

Definition I.6.11 *Assume the Axiom of Infinity. For* $n \in \mathbb{N}$, *define* $B^n = {}^n B$ *to be the set of all* F *such that* $F : n \to B$. *Let* $B^{<\omega} = {}^{<\omega} B = \bigcup \{B^n : n \in \mathbb{N}\}$. *For* $F \in B^{<\omega}$, $\mathrm{lh}(F) = \mathrm{dom}(F)$, *the* length *of* F.

Justification. Assuming the Power Set Axiom: Exactly like the justification for Definition I.6.10.

Assuming the Replacement Axiom: Prove that B^n forms a set by ordinary induction (Theorem I.5.3) on n. For the basis, $B^0 = \{\emptyset\}$. For the induction step, assume that we have the set B^n. We then have the product $B^n \times B$. Then, by Lemma I.6.9, we have a function (set of ordered pairs) f such that $\mathrm{dom}(f) = B^n \times B$ and $f(s,x) = s \cup \{\langle n, x\rangle\}$ for each $\langle s,x\rangle \in B^n \times B$. Then $\mathrm{ran}(f) = B^{S(n)}$.

Now that we have B^n for each $n \in \mathbb{N}$, we may use Replacement again to form $\{B^n : n \in \mathbb{N}\}$, and we then obtain $B^{<\omega}$ by the Union Axiom. □

When we are thinking of $F \in B^{<\omega}$ as a sequence of objects, we shall use $\mathrm{lh}(F)$ rather than $\mathrm{dom}(F)$. A logical formula is a sequence of symbols. The set of formulas of $\mathcal{L}$ is a subset of $B^{<\omega}$ where B is the union of $\mathcal{L}$ with the set of logical symbols. Since $ZF^- - P$ is sufficient to form this set, it is also strong enough to develop the basic concepts of model theory; see Section I.15.

Definition I.6.12 $F(A) = F\text{“}A = \mathrm{ran}(F\restriction A)$.

In most applications, F will be a function with $A \subseteq \mathrm{dom}(F)$. We shall usually use $F(A)$, except where A might also be a member of $\mathrm{dom}(F)$. For example, if $F = \{(0,2),(1,1),(2,0)\}$, then $F : 3 \underset{\text{onto}}{\overset{1\text{–}1}{\to}} 3$ and $F(2) = 0$, while $F\text{“}2 = \{F(0), F(1)\} = \{2,1\}$.

Definition I.6.13 $R^{-1} = \{(y,x) : (x,y) \in R\}$.

Justification. This is a defined subset of $\mathrm{ran}(R) \times \mathrm{dom}(R)$. □

Definition I.6.14 $G \circ F =$
$\{(x,z) \in \mathrm{dom}(F) \times \mathrm{ran}(G) : \exists y[(x,y) \in F \wedge (y,z) \in G\}]$.

Usually, F, G will be functions with $\mathrm{ran}(F) \subseteq \mathrm{dom}(G)$, in which case we have $\mathrm{dom}(G \circ F) = \mathrm{dom}(F)$ and $(G \circ F)(x) = G(F(x))$.

If S and T are ordered sets, then we can order their cartesian product $S \times T$ *lexicographically* (i.e., as in the dictionary).

Definition I.6.15 *If $<$ and $\sqsubset$ are relations and S and T are sets, then their* lexicographic product *on $S \times T$ is the relation $\lhd$ on $S \times T$ defined by:*

$$\lhd = \left\{\big((s,t),\,(s',t')\big) \in (S \times T) \times (S \times T) :\ s < s' \vee [s = s' \wedge t \sqsubset t']\right\}\ .$$

It is easily seen that the dictionary is totally ordered:

Lemma I.6.16 *if $<$ and $\sqsubset$ are strict total orders of S, T, respectively, then their lexicographic product on $S \times T$ is a strict total order of $S \times T$.*

Definition I.6.17 *F is an* isomorphism from $(A;<)$ onto $(B;\sqsubset)$ *iff F is a bijection $(F : A \underset{\text{onto}}{\overset{1\text{–}1}{\to}} B)$ and $\forall x, y \in A\,[x < y \leftrightarrow F(x) \sqsubset F(y)]$. Then, $(A;<)$ and $(B;\sqsubset)$ are* isomorphic *(in symbols, $(A;<) \cong (B;\sqsubset)$) iff there exists an isomorphism from $(A;<)$ onto $(B;\sqsubset)$.*

The notion of isomorphism applies to general mathematical structures, where one has sets A, B with some relations and/or functions on them; for example, one speaks of isomorphic groups, isomorphic rings, etc.; see Definition I.15.18. A special case is just pure sets A, B with *no* relations or functions on them, where we use injections and bijections to compare the sizes of sets:

Definition I.6.18

1. $X \preccurlyeq Y$ *iff there is a function* $f : X \xrightarrow{1-1} Y$.
2. $X \approx Y$ *iff there is a function* $f : X \xrightarrow[\text{onto}]{1-1} Y$.

So $X \approx Y$ says that X, Y have the *same size* in the sense that they can be put into 1-1 correspondence by some function, whereas $X \preccurlyeq Y$ means that there is a function that embeds X 1-1 into Y, so the size of X is equal to or less than that of Y.

Lemma I.6.19

1. $\preccurlyeq$ *is transitive and reflexive.*
2. $X \subseteq Y \rightarrow X \preccurlyeq Y$.
3. $\approx$ *is an equivalence relation.*

Proof. For (2), and in the proof of reflexivity in (1) and (3), use id_X. To prove transitivity in (1) and (3), compose functions. □

Further results on these concepts are discussed in Section I.10.

The quotient of a structure by an equivalence relation (see Definition I.6.2) is commonly used in algebra and topology; for example, in forming the quotient of a group by a normal subgroup.

Definition I.6.20 *Let R be an equivalence relation on a set A. For $x \in A$, let $[x] = \{y \in A : yRx\}$; $[x]$ is called the* equivalence class *of x. Let $A/R = \{[x] : x \in A\}$.*

Justification. Forming each equivalence class $[x]$ just requires the Comprehension Axiom, but to justify forming A/R, the set of all equivalence classes:

Assuming the Power Set Axiom: note that $A/R \subseteq \mathcal{P}(A)$.

Assuming the Replacement Axiom: This is exactly like the justification for A^* discussed in Section I.5. □

We now turn to well-founded relations and well-order relations.

Definition I.6.21 *Let R be a relation. $y \in X$ is* R–minimal *in X iff*

$$\neg\exists z(z \in X \wedge zRy)) \quad ,$$

and R–maximal *in X iff*

$$\neg\exists z(z \in X \wedge yRz)) \quad .$$

R is well-founded *on A iff for all non-empty $X \subseteq A$, there is a $y \in X$ that is R–minimal in X. R* well-orders *A iff R totally orders A strictly and R is well-founded on A.*

A simple lemma, to illustrate the concepts:

Lemma I.6.22 *Assume that the relation R is well-founded on A. Then R is irreflexive on A (i.e., $\neg\exists x \in A\,[xRx]$), and $\neg\exists x, y \in A\,[xRy \wedge yRx]$ and $\neg\exists x, y, z \in A\,[xRy \wedge yRz \wedge zRx]$.*

Proof. If $xRy \wedge yRx$ then $X := \{x, y\}$ would have no R–minimal element. The other two statements are similar. □

Of course, this generalizes to the fact that a well-founded R is *acyclic* on A, that is, one cannot have $x_0 R x_1 R \cdots R x_n R x_0$ for any $n \in \mathbb{N}$, but to state and prove this formally, one should first develop the theory of $\mathbb{N}$; see Definition I.9.7 and Exercise I.9.8. Informally (for now), the order $<$ on $\mathbb{N}$ is a well-order, but $>$ on $\mathbb{N}$ is acyclic but not a well-order, since infinite subsets of $\mathbb{N}$ do not contain a $>$–minimal (that is, $<$–maximal) element. In this spirit:

Exercise I.6.23 *Define the relation R on $\mathbb{N} \times \mathbb{N}$ by saying: $(x_0, x_1) R (y_0, y_1)$ iff $x_0 \leq y_0$ and $x_1 \leq y_1$ and $(x_0, x_1) \neq (y_0, y_1)$. Show that R is well-founded on $\mathbb{N}$ but not a total order. Show that every non-empty $X \subseteq \mathbb{N} \times \mathbb{N}$ contains only finitely many R–minimal elements, and, for each $k \in \mathbb{N}$, there is such an X with exactly k R–minimal elements.*

This R shouldn't be confused with the lexicographic product order, which is a well-order of order type ω^2. More generally,

Lemma I.6.24 *If $<$ and $\sqsubset$ are well-orders of S, T, respectively, then their lexicographic product on $S \times T$ is a well-order of $S \times T$.*

Lemma I.6.25 *If R well-orders A and $X \subseteq A$, then R well-orders X.*

Informally, the usual order (which is $\in$) is a well-order on ω and every other ordinal. Formally, well-order is part of the definition of ordinal (see Section I.7). In any case, it is easy to check directly from the definition (I.4.17) of $3 = \{0, 1, 2\}$ that

Exercise I.6.26 *Show that $\in$ is a well-order of* 3.

Clearly, $\in$ is a not a well-order of the universe, V, since it is not transitive: $\emptyset \in \{\emptyset\}$ and $\{\emptyset\} \in \{\{\emptyset\}\}$ but $\emptyset \notin \{\{\emptyset\}\}$; also, trichotomy fails. But $\in$ might be well-founded on V; that's equivalent to the Axiom of Foundation. Before discussing this further, we make a basic point of logic. Up to now, our definitions only talked about *sets*. That is, a relation R is a *set* of ordered pairs, and we talked about it having various properties, such as transitivity, on various

sets A. But V is not a set and neither is $\in$. $\in$ is one of the symbols in the language of set theory, although we may think of it as a class $E := \{(x, y) : x \in y\}$. E cannot exist, since $\mathrm{dom}(E) = V$, which doesn't exist. Nevertheless, it does make sense to take the various properties of relations (sets of ordered pairs) R on sets A and plug *specific* proper classes into them. For example, if we take the statement of trichotomy and plug in $\in$ as the "relation" and V as the "set", we get $\forall xy\,[x \in y \vee y \in x \vee x = y]$, which makes a meaningful (but false) assertion in set theory, whereas the trichotomy of $\in$ on 3, $\forall xy \in 3\,[x \in y \vee y \in x \vee x = y]$ is true. Likewise, it makes sense to say that $\subseteq$ is transitive on V, and that $\approx$ is an equivalence relation on V and $\preccurlyeq$ is a pre-order on V (see Lemma I.6.19).

A "defined relation" may be thought of as a defined class which happens only to contain ordered pairs. Pretending that classes really exist gives us some more efficient notation, but be aware of Caution I.4.10.

Generalizing our conventions:

Notation I.6.27 *Let A be a defined class and R a defined relation. To express the truth of a defined first-order property (which talks only about* elements *of A, such as the ones in Definition I.6.2), we write the defining formula* relativized *to A — that is, interpreting the quantified variables to range over A.*

We say that R is well-founded *on A iff every non-empty sub***set** *of A contains an R-minimal element.*

Since a function is a just a special type of relation, we may also plug defined functions into our definitions. For example, we may say "$\bigcup : V \xrightarrow{\text{onto}} V$" as shorthand for $\forall y \exists x\,[\bigcup x = y]$ (let $x = \{y\}$). The statement $\bigcup : V \xrightarrow{1\text{–}1} V$ is also meaningful, but false (e.g., $\bigcup\{2, 3\} = \bigcup\{1, 2, 3\} = 3$), whereas, assuming the Power Set Axiom, $\mathcal{P} : V \xrightarrow{1\text{–}1} V$ is true.

As an example of Caution I.4.10, one must exercise care when a property of a relation on a set A is second order – that is, involves quantification over all subsets of A. We may talk about a *specific* proper class as a convenient abbreviation, but it's meaningless to quantify over all proper classes. So, we interpret such a property on a proper class A as quantifying over all sub*sets* of A, as we just did with well-foundedness in Notation I.6.27. To illustrate this:

Lemma I.6.28 *The Axiom of Foundation is equivalent to the statement that the $\in$ relation is well-founded on V.*

Proof. "$\in$ is well-founded on V" means that for every non-empty sub*set* $x \subseteq V$, there is a $y \in x$ that is $\in$–minimal in x. But $x \subseteq V$ is trivially true and minimality of y in x means $\neg\exists z(z \in x \wedge z \in y)$, so we are asserting

$$\forall x\,[\exists y(y \in x) \;\rightarrow\; \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y))]\ ,$$

which is the statement of Foundation in Section I.3. □

Since the Axiom of Foundation implies that $\in$ is well-founded on V, it also implies that $\in$ is irreflexive on V; that is $a \notin a$ for all a; but this proof, quoting Lemmas I.6.28 and I.6.22, just restates Exercise I.4.18 in different words.

Also, $\in$ is a well-order of the class ON of all ordinals; see Theorem I.7.11. The precise statement of this theorem is obtained using our Notation I.6.27. The assertion of trichotomy becomes $\forall xy \in ON\,[x \in y \vee y \in x \vee x = y]$. But well-foundedness becomes the assertion that every non-empty sub*set* of ON has a least element, since there is no way to quantify over proper classes. Because of the definition (I.7.3) of ordinal, the Axiom of Foundation will not be needed to prove well-foundedness.

Exercise I.6.29 *Show that the Axiom of Foundation implies that the successor function S is 1-1 on V.*

As noted above, this is just shorthand for the assertion that one cannot have both $x \cup \{x\} = y \cup \{y\}$ and $x \neq y$. It is consistent with ZFC^- to have $x \neq y$ and $x = \{y\}$ and $y = \{x\}$ (see Exercise II.9.5); then $S(x) = S(y) = \{x, y\}$.

One final remark about classes and functions on classes:

Lemma I.6.30 *If A, B, C are classes and $A \approx B$ and $B \approx C$, then $A \approx C$.*

Proof. Mathematically, this is completely trivial (just compose bijections), so we state the lemma only to illustrate the logic involved. Unlike Exercise I.6.29, which is shorthand for one sentence in the language of set theory, this lemma is shorthand for an assertion in the metatheory that given five formulas $\alpha, \beta, \gamma, \varphi, \psi$, we can write a sixth formula χ and prove something about it.

That is, say $A = \{x : \alpha(x)\}$ and $B = \{y : \beta(y)\}$ and $C = \{z : \gamma(z)\}$. Then we can write $\chi(x, z)$ which provides a bijection from A to C, assuming that $\varphi(x, y)$ provides a bijection from A to B and $\psi(y, z)$ provides a bijection from B to C. So, $\chi(x, z)$ is $\exists y\,[\varphi(x, y) \wedge \psi(y, z) \wedge \beta(y)]$, and we can prove

$$\begin{array}{l}[\,\forall x \in A\,\exists! y \in B\,\varphi(x, y) \;\wedge\; \forall y \in B\,\exists! x \in A\,\varphi(x, y) \;\wedge \\ \;\forall y \in B\,\exists! z \in C\,\psi(y, z) \;\wedge\; \forall z \in C\,\exists! y \in B\,\psi(y, z)\,] \;\rightarrow \\ \;\forall x \in A\,\exists! z \in C\,\chi(x, z) \;\wedge\; \forall z \in C\,\exists! x \in A\,\chi(x, z)\end{array}$$

Of course, you have to rewrite this into a sentence in the language of set theory by replacing the A, B, C by α, β, γ. Although we are thinking here about set theory and classes, this sentence is actually provable in pure logic, without using any set theory axioms.

Actually, the sentence that needs to be proved is really the universal closure of the displayed formula, since the formulas $\alpha, \beta, \gamma, \varphi, \psi, \chi$ could all depend on some tuple $\vec{t}$ of variables, which are viewed as "fixed" parameters. For example, we may assert that for every set t, $A_t \approx B_t$, where $A_t = \{x : t \notin x\}$ and $B_t = \{x : t \in x\}$ (use the bijection $x \mapsto \{t\} \cup x$). $\square$

Theorem I.9.11, justifying transfinite recursion in set theory, will have the same logical form: "given some formulas, we can write another one", but the mathematics is not quite so trivial.

This lemma also suggests the mathematical question: For which classes A, B can we prove $A \approx B$? Assuming Gödel's Axiom of Constructibility, $A \approx ON$ for all proper classes A (including V); this follows easily from the fact that there is a defined relation that well-orders V in type ON (Theorem II.6.20). However, in some models of ZFC constructed by forcing, $V \approx ON$ is false (see Exercise V.2.17).

I.7 Ordinals I

We split the theory of the von Neumann ordinals into two sections. In this section, we continue working in BST^- and develop some elementary facts, so that we shall know that these facts hold in any model we plan to consider. In the next section, we shall add the Axiom of Infinity, which gives us ω, and the Replacement Axiom, which implies that ordinal arithmetic can be defined.

Definition I.7.1 *z is a* transitive set *iff* $\forall y \in z[y \subseteq z]$.

For example, $3 = \{0, 1, 2\}$, is a transitive set, since $0, 1, 2 \subseteq 3$. Also, 4 is transitive, as are $\{0, 1, \{1\}\}$ and $\{0, 1, 2, \{1\}\}$, but $\{0, 1, \{2\}\}$ is not transitive.

Exercise I.7.2 *Assuming the Axiom of Foundation, show that every non-empty transitive set contains* 0 *and show that every non-singleton transitive set contains* 1. *Then show that* 1 *is the only one-element transitive set, and* 2 *is the only two-element transitive set.*

Hint. If t is transitive and $x \in t$ and $x \cap t = \emptyset$, then $x = \emptyset$. □

Definition I.7.3 *z is a (von Neumann) ordinal iff z is a transitive set and z is well-ordered by $\in$.*

So, 3 is an ordinal because it is a transitive set and is well-ordered by $\in$. The set $\{1, 2, 3\}$ is not an ordinal; it is well-ordered by $\in$ but it is not transitive. The sets $\{0, 1, \{1\}\}$ and $\{0, 1, 2, \{1\}\}$ are transitive but not ordinals because they are not well-ordered by $\in$; in fact trichotomy fails because $0 \notin \{1\}$ and $\{1\} \notin 0$. Also, 0 is an ordinal because the definition of "well-ordered" is trivially true for the empty set.

Lemma I.7.4 *Assuming the Axiom of Foundation, z is an ordinal iff z is a transitive set and z is totally ordered by $\in$.*

Proof. By Foundation and Lemma I.6.28, $\in$ is well-founded on V, and hence on z. □

This lemma is important in set theory because it shows that the property of being an ordinal is absolute for transitive models; see Lemma II.4.2. A well-known improvement on this lemma, that in fact only trichotomy is necessary, is primarily a curiosity; see Exercise I.7.25.

All the basic properties of ordinals can be developed without Foundation because ordinals are well-ordered by definition. For example, Foundation excludes "pathological" sets x such that $x \in x$, but without this,

Lemma I.7.5 *If x is an ordinal, then $x \notin x$.*

Proof. If $x \in x$, then $\in$ would not be irreflexive on x. □

Lemma I.7.6 *If x is an ordinal, then $S(x)$ is an ordinal. Hence, the sets $0, 1, 2, 3, 4, 5, 6, 7, 8$ are all ordinals.*

Proof. To prove that $S(x)$ is well-ordered by $\in$, use Lemma I.7.5 to conclude that the order on $x \cup \{x\}$ consists of the object x sitting on top of all the elements of x. □

Notation I.7.7 *When discussing ordinals, Greek letters (especially $\alpha, \beta, \gamma, \delta, \zeta, \eta, \xi, \mu$) "range over the ordinals"; that is, $\forall \alpha \varphi(\alpha)$ abbreviates*

$$\forall x[x \text{ is an ordinal } \rightarrow \varphi(x)] \ .$$

Also, $\alpha < \beta$ means (by definition) $\alpha \in \beta$, and $\alpha \leq \beta$ means $\alpha \in \beta \vee \alpha = \beta$.

ON denotes the class of all ordinals.

In fact, ON is a proper class – that is it doesn't really exist (Theorem I.7.12). However, the informal concept of ON yields some useful terminology; for example "$x \cap ON$" abbreviates "$\{y \in x : y \text{ is an ordinal}\}$".

Our use of the symbol $<$ for $\in$ on ON indicates that we plan to prove that $\in$ really is an order relation on ON; see Theorem I.7.11.

Lemma I.7.8 *ON is a transitive class. That is, if $\alpha \in ON$ and $z \in \alpha$, then $z \in ON$.*

Proof. α is a transitive set, so $z \subseteq \alpha$. Since $\in$ well-orders α, it well-orders every subset of α, and it is easy to check that z is a transitive set. □

Similarly, one easily checks, from the definition of ON:

Lemma I.7.9 *For all $\alpha, \beta \in ON$, $\alpha \cap \beta \in ON$.*

On the ordinals, $<$ is $\in$ by definition. But also, $\leq$ is $\subseteq$; this is "obvious" from the fact that $\in$ totally orders ON, but we haven't proved that fact yet.

Lemma I.7.10 *For all* $\alpha, \beta \in ON$: $\alpha \subseteq \beta \leftrightarrow \alpha \in \beta \vee \alpha = \beta$.

Proof. For $\rightarrow$: Assume $\alpha \subseteq \beta$ and $\alpha \neq \beta$. We show that $\alpha \in \beta$. Let $X = \beta \backslash \alpha$. Then $X \neq \emptyset$, so let ξ be the $\in$-least member of X. Then prove that $\xi = \alpha$. For details, see [57]. □

The next theorem says that ON is well-ordered by $\in$. Since ON doesn't exist, we use our remarks at the end of Section I.6 on "relations" on proper classes to express this precisely as a theorem about ordinals:

Theorem I.7.11 *ON is well-ordered by* $\in$. *That is*:

1. $\in$ *is transitive on the ordinals*: $\forall \alpha\beta\gamma[\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma]$.
2. $\in$ *is irreflexive on the ordinals*: $\forall \alpha[\alpha \not< \alpha]$.
3. $\in$ *satisfies trichotomy on the ordinals*: $\forall \alpha\beta[\alpha < \beta \vee \beta < \alpha \vee \alpha = \beta]$.
4. $\in$ *is well-founded on the ordinals: every non-empty* set *of ordinals has an* $\in$*-least member.*

Proof. (1) follows from the fact that γ is a transitive set, and (2) is Lemma I.7.5. For (3), let $\delta = \alpha \cap \beta$. Then δ is an ordinal by Lemma I.7.9, and $\delta \subseteq \alpha$ and $\delta \subseteq \beta$, so $\delta \in \alpha$ or $\delta = \alpha$, and $\delta \in \beta$ or $\delta = \beta$ by Lemma I.7.10. If either $\delta = \alpha$ or $\delta = \beta$, we are done. If not, then $\delta \in \alpha$ and $\delta \in \beta$, so $\delta \in \delta$, contradicting (2).

For (4), let X be any non-empty set of ordinals, and fix an $\alpha \in X$. If α is least, we're done. Otherwise, $\alpha \cap X = \{\xi \in X : \xi < \alpha\} \neq \emptyset$, and $\alpha \cap X$ has a least element, ξ, since α is well-ordered by $\in$. Then ξ is also the least element of X. □

Theorem I.7.12 *ON is a proper class.*

Proof. If ON were a set, then by Theorem I.7.11, ON would be an ordinal, so $ON \in ON$, contradicting Theorem I.7.11. □

Lemma I.7.13 *If* X *is a subset of* ON *and* X *is transitive, then* $X \in ON$.

Proof. Theorem I.7.11 implies that X is well-ordered by $\in$. □

Note that when $X \subset ON$, X is transitive iff X is an initial segment of ON; that is $\forall \beta \in X \forall \alpha < \beta[\alpha \in X]$, since $<$ is $\in$. Then $X \in ON$ is the least ordinal not in X. So, for example, once we verify (in Section I.11) that one can form the set X of all countable ordinals, then X is automatically the first uncountable ordinal (called ω_1).

Lemma I.7.14 *If* α, β *are ordinals, then* $\alpha \cup \beta$ *and* $\alpha \cap \beta$ *are ordinals, with* $\alpha \cup \beta = \max(\alpha, \beta)$ *and* $\alpha \cap \beta = \min(\alpha, \beta)$. *If* X *is a non-empty set of ordinals, then* $\bigcap X$ *and* $\bigcup X$ *are ordinals, with* $\bigcap X = \min(X)$ *and* $\bigcup X = \sup(X)$.

Lemma I.7.15 *If α is any ordinal, then $S(\alpha)$ is an ordinal, $\alpha \in S(\alpha)$, and for all ordinals γ: $\gamma < S(\alpha)$ iff $\gamma \leq \alpha$.*

Proof. $S(\alpha)$ is an ordinal by Lemma I.7.6; the rest is immediate when you replace "$<$" by "$\in$" and "$\leq$" by "$\in$ or $=$". □

We partition $ON\backslash\{0\}$ into successors and limits.

Definition I.7.16 *An ordinal β is*

- *a* successor ordinal *iff $\beta = S(\alpha)$ for some α.*
- *a* limit ordinal *iff $\beta \neq 0$ and β is not a successor ordinal.*
- *a* natural number *iff every $\alpha \leq \beta$ is either 0 or a successor.*

So, 8 is a natural number because each of $8, 7, 6, 5, 4, 3, 2, 1, 0$ is either 0 or the successor of some ordinal. The natural numbers form an initial segment of the ordinals:

Lemma I.7.17 *If n is a natural number, then $S(n)$ is a natural number and every element of n is a natural number.*

Definition I.7.18 $\omega = \{n : n \text{ is a natural number }\}$.

Theorem I.7.19 (Principle of Ordinary Induction) *For any class X: If $\emptyset \in X$ and $\forall y \in X(S(y) \in X)$, then X contains all natural numbers.*

Proof. Suppose that n is a natural number and $n \notin X$. Then $n \in S(n)\backslash X$, so $S(n)\backslash X \neq \emptyset$. $S(n)$ is well-ordered, so let m be the least element of $S(n)\backslash X$. Then $m \neq 0$ (since $0 \in X$), and $m \neq S(y)$ for any y (since then $y \in X$ and $S(y) \notin X$), so m is not a natural number, contradicting Lemma I.7.17. □

Without assuming the Axiom of Infinity, we cannot claim that ω is a set (maybe it is a proper class). But the proof of Theorem I.7.19 is still valid even though the X may also be a proper class. A simple example of its use:

Lemma I.7.20 *For all $y \in \omega$, $S(y) \not\preccurlyeq y$ (see Definition I.6.18).*

Proof. Let $X = \{y \in \omega : S(y) \not\preccurlyeq y\}$. Clearly $\emptyset \in X$, so it is sufficient to show that $y \in X \to S(y) \in X$; equivalently, $S(S(y)) \preccurlyeq S(y) \to S(y) \preccurlyeq y$. So, suppose that $f : S(S(y)) \xrightarrow{1-1} S(y) = \{y\} \cup y$.

If $f(S(y)) = y$ or $y \notin \text{ran}(f)$, then $f\restriction S(y) : S(y) \xrightarrow{1-1} y$, so $S(y) \preccurlyeq y$.

Now, suppose that $f(S(y)) \neq y$ and $y \in \text{ran}(f)$. Then $f(i) = y$ for some $i \leq y$, and $j := f(S(y)) \in y$. Let $h = ((f\restriction S(y)) \setminus \{(i, y)\}) \cup \{(i, j)\}$. Then $h : S(y) \xrightarrow{1-1} y$, so $S(y) \preccurlyeq y$. □

Note that we already discussed Ordinary Induction in Theorem I.5.3, but that used a different notion of "natural number" and had a different proof. However, both theorems used exactly the same hypothesis — that X is *inductive*, as in Definition I.5.1. Putting these together, we have, in BST^-, a number of equivalent ways of thinking about Infinity:

Theorem I.7.21 *The following are equivalent:*

1. *The Axiom of Infinity.*
2. *ω exists (is a set).*
3. *Some limit ordinal exists.*

Assuming the Axiom of Infinity, $\omega = \mathbb{N}$ and ω is the smallest limit ordinal.

Proof. (1) $\rightarrow$ (2): If X is an inductive set, then $\omega \subseteq X$ by Theorem I.7.19, so ω exists by Comprehension.

(2) $\rightarrow$ (3): Applying Lemma I.7.13, ω is a limit ordinal.

(3) $\rightarrow$ (1): Every limit ordinal is an inductive set.

Now, assuming Infinity, we have just noted that ω is a limit ordinal, and it is the least one because all ordinals $<$ (i.e, $\in$) ω are not limit ordinals. To prove that $\omega = \mathbb{N}$, let $X = \mathbb{N} \cap \omega$. Then $X = \mathbb{N}$ by Ordinary Induction (Theorem I.5.3) and $X = \omega$ by Ordinary Induction (Theorem I.7.19). □

Without assuming Infinity, we can use Definition I.6.18 of $\preccurlyeq$ to define "finite" and prove that the finite ordinals are the natural numbers:

Definition I.7.22 *A set A is* finite *iff $A \preccurlyeq n$ for some natural number n. "*infinite*" means "not finite".*

Lemma I.7.23 *For any ordinal α: α is finite iff α is a natural number.*

Proof. For $\leftarrow$: use $\alpha \preccurlyeq \alpha$ and the definition of "finite". For $\rightarrow$: Assume that α is not a natural number. Since $\in$ totally orders ON, Lemma I.7.17 implies that α is larger than all natural numbers; i.e., $\omega \subseteq \alpha$. If α is finite, let $f : \alpha \xrightarrow{1-1} n \in \omega$. Then $f \restriction S(n) : S(n) \xrightarrow{1-1} n$, contradicting Lemma I.7.20. □

Using ordinals, we extend our terminology for sequences and lengths to include transfinite sequences as well:

Definition I.7.24 *A* sequence *is a function F whose domain is some ordinal. Then* $\mathrm{lh}(F) = \mathrm{dom}(F)$*, the* length *of F. If F is a function with* $\mathrm{dom}(F) = \xi$*, then $F^\frown b = F \cup \{(\xi, b)\}$, which is a function whose domain is $\xi + 1$.*

For example, $(3,1,4,1)^\frown 6 = (3,1,4,1,6)$. This illustrates the standard way of displaying explicit sequences; $F := (3,1,4,1)$ means that $\mathrm{dom}(F) = 4$ and $F(0) = 3$ and $F(1) = F(3) = 1$ and $F(2) = 4$.

Exercise I.7.25 *Assuming the Axiom of Foundation, z is an ordinal iff z is a transitive set and $\in$ satisfies trichotomy on z.*

Hint. $z \cap ON$ is an ordinal by Lemma I.7.13. Then use trichotomy to derive a contradiction from an $\in$–minimal element of $z \setminus ON$. □

Exercise I.7.26 *Assuming the Axiom of Foundation, z is an ordinal iff z is a transitive set and all elements of z are transitive sets.*

The following exercise shows that using induction, one can construct power sets of finite sets in BST^-:

Exercise I.7.27 *Show, in BST^-, that if A, B are finite, then $A \cup B$ and $A \times B$ are finite, and $\mathcal{P}(A)$ exists and is finite.*

I.8 Ordinals II

This section proceeds in $ZF^- - P$; equivalently, we add Infinity and Replacement to BST^-. We have just seen that Infinity gives us ω, and we can then form even larger ordinals, such as $S(\omega) = \omega + 1, S(S(\omega)) = \omega + 2, \ldots$. We would like to form $\omega + \omega = \omega \cdot 2 = \sup_{n<\omega}(\omega + n)$, but this requires Replacement. Our official definition of $\omega \cdot 2$ will be the ordinal isomorphic to the set $2 \times \omega$ well-ordered by the lexicographic product order (see Definition I.6.15). Of course, we first must prove that every well-ordered set is isomorphic to an ordinal; see Theorem I.8.2. The proof of this theorem does not require the Axiom of Infinity, but does require the Replacement Axiom. We first verify:

Lemma I.8.1 *If $f : \alpha \xrightarrow[\text{onto}]{1-1} \beta$ is an isomorphism from $(\alpha; <)$ to $(\beta; <)$ then f is the identity map and hence $\alpha = \beta$.*

Proof. If f is not the identity map, then the least ordinal in $\{\xi \in \alpha : f(\xi) \neq \xi\}$ leads to a contradiction. For details, see [57]. □

So, different ordinals are not isomorphic. Also, the lemma applied with $\alpha = \beta$ implies that the ordering $(\alpha; <)$ has no automorphisms other than id_α.

Now, say R well-orders a set A and we want to produce an ordinal α such that $(A; R) \cong (\alpha; \in)$. This α must be unique by Lemma I.8.1. Say $f : A \xrightarrow{\text{onto}} \alpha$ is the desired isomorphism; f must also be unique because if g is another isomorphism, then $f \circ g^{-1}$ is an automorphism of α, so $f \circ g^{-1} = id_\alpha$. Say that R has A ordered as $a, b, c, \ldots, x \ldots$. It is natural to compute f recursively as $f(a) = 0$, $f(b) = 1$, $f(c) = 2 = \{0, 1\}$, etc. In general $f(x)$ is the first ordinal not in $\{f(y) : yRx\}$, which, as we remarked after Lemma I.7.13, is the same as saying $f(x) = \{f(y) : yRx\}$. To make this into a rigorous proof, one could prove a general theorem justifying such recursions, which we shall do later (see Theorem I.9.11). However, it is easy to prove that the isomorphism exists without a general discussion of recursion, and this proof may be considered to be a warmup for the proof of Theorem I.9.11. Roughly, we show that $f(x)$ exists because the R–least x for which it doesn't exist would yield a contradiction:

Theorem I.8.2 *If R well-orders A, then there is a unique $\alpha \in ON$ such that $(A; R) \cong (\alpha; \in)$.*

Table I.1: Ordinal Arithmetic

Associative Laws

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$; $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$

¬ Commutative Laws

$1 + \omega = \omega < \omega + 1$; $2 \cdot \omega = \omega < \omega \cdot 2$

Left Distributive Law: $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

¬ Right Distributive Law: $(1 + 1) \cdot \omega = \omega < \omega + \omega = 1 \cdot \omega + 1 \cdot \omega$

0 and 1

$S(\alpha) = \alpha + 1$

$\alpha + 0 = 0 + \alpha = \alpha$; $\alpha \cdot 0 = 0 \cdot \alpha = 0$; $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$

$\alpha^0 = 1$; $\alpha^1 = \alpha$; $1^\alpha = 1$

$\alpha > 0 \rightarrow 0^\alpha = 0$

Left Cancellation

$\alpha + \beta = \alpha + \gamma \rightarrow \beta = \gamma$

$\alpha \cdot \beta = \alpha \cdot \gamma \wedge \alpha \neq 0 \rightarrow \beta = \gamma$

¬ Right Cancellation

$1 + \omega = 2 + w = 1 \cdot w = 2 \cdot w = \omega$

Subtraction

$\alpha \leq \beta \rightarrow \exists! \gamma \, (\alpha + \gamma = \beta)$

Division

$\alpha > 0 \rightarrow \exists! \gamma \delta \, (\alpha \cdot \gamma + \delta = \beta \wedge \delta < \alpha)$

Exponentiation

$\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$; $a^{\beta \cdot \gamma} = (\alpha^\beta)^\gamma$

Logarithms

$\alpha > 1 \wedge \beta > 0 \rightarrow \exists! \gamma \delta \xi \, (\beta = \alpha^\delta \cdot \xi + \gamma \wedge \xi < \alpha \wedge \gamma < \alpha^\delta \wedge \xi > 0)$

For finite ordinals, $\delta = \lfloor \log_\alpha \beta \rfloor$

Example: $873 = 10^2 \cdot 8 + 73 \wedge 8 < 10 \wedge 73 < 10^2$

Order

$\beta < \gamma \rightarrow \alpha + \beta < \alpha + \gamma \wedge \beta + \alpha \leq \gamma + \alpha$

$\beta < \gamma \wedge \alpha > 0 \rightarrow \alpha \cdot \beta < \alpha \cdot \gamma \wedge \beta \cdot \alpha \leq \gamma \cdot \alpha$

$\beta < \gamma \wedge \alpha > 1 \rightarrow \alpha^\beta < \alpha^\gamma \wedge \beta^\alpha \leq \gamma^\alpha$

$2 < 3$; $2 + \omega = 3 + \omega = 2 \cdot \omega = 3 \cdot \omega = 2^\omega = 3^\omega = \omega$

Recursive Computation

$\alpha + 0 = \alpha$; $\alpha + S(\beta) = S(\alpha + \beta)$; $\alpha + \gamma = \sup_{\beta<\gamma}(\alpha + \beta)$ (for γ a limit)

$\alpha \cdot 0 = 0$; $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$; $\alpha \cdot \gamma = \sup_{\beta<\gamma}(\alpha \cdot \beta)$ (for γ a limit)

$\alpha^0 = 1$; $\alpha^{S(\beta)} = \alpha^\beta \cdot \alpha$; $\alpha^\gamma = \sup_{\beta<\gamma}(\alpha^\beta)$ (for γ a limit)

Continuity

The functions $\alpha + \beta$, $\alpha \cdot \beta$, α^β are continuous in β, but not in α.

Proof. Uniqueness is immediate from Lemma I.8.1.

If $a \in A$, let $a{\downarrow} = \{x \in A : xRa\}$. Then $a{\downarrow}$ is well-ordered by R as well. Call $a \in A$ *good* iff the theorem holds for $a{\downarrow}$ — that is, $(a{\downarrow}; R) \cong (\xi; \in)$ for some ordinal ξ. Note that this ξ is then unique by Lemma I.8.1. Let G be the set of good elements of A. Let f be the function with domain G such that $f(a)$ is the (unique) ξ such that $(a{\downarrow}; R) \cong (\xi; \in)$; this definition is justified by the Replacement Axiom (see Lemma I.6.9).

Let $\alpha = \text{ran}(f)$. Then α is an initial segment of ON, and hence is an ordinal by Lemma I.7.13.

If $G = A$, then f is an isomorphism from $(A; R)$ onto $(\alpha; \in)$.

If $G \neq A$, let e be the least element of $A \backslash G$. Then $e{\downarrow} = G$, and f is an isomorphism from $(e{\downarrow}; R)$ onto $(\alpha; \in)$, so $e \in G$, which is a contradiction. □

Definition I.8.3 *If R well-orders A, then* $\text{type}(A; R)$ *is the unique $\alpha \in ON$ such that $(A; R) \cong (\alpha; \in)$. We also write* $\text{type}(A)$ *(when R is clear from context), or* $\text{type}(R)$ *(when A is clear from context).*

Now, we define ordinal sum and product by:

Definition I.8.4 $\alpha \cdot \beta = \text{type}(\beta \times \alpha)$. $\alpha + \beta = \text{type}(\{0\} \times \alpha \cup \{1\} \times \beta)$. *Here, we're using lexicographic order to compare ordered pairs of ordinals.*

For example, $\omega + \omega = \omega \cdot 2 = \text{type}(\{0, 1\} \times \omega)$. This well-ordering consists of a copy of ω followed by a second copy stacked on top of it.

We remark that the Replacement Axiom (via the proof of Theorem I.8.2) is needed to prove that the ordinal $\omega + \omega$ exists. The model $R(\omega + \omega)$ satisfies ZC (all of ZFC except Replacement), and satisfies the statement that $\omega + \omega$ doesn't exist; see Section II.4. In ZC, one can produce the set $2 \times \omega$ and the lexicographic order on it; one can also do most of "ordinary" mathematics, but one does not have the convenience of using von Neumann ordinals.

Table I.1 lists some facts about ordinal arithmetic. The remarks there on recursive computation and exponentiation will be explained in Section I.9.

Lemma I.8.5 *If R well-orders A and $X \subseteq A$, then R well-orders X and* $\text{type}(X; R) \leq \text{type}(A; R)$.

Proof. Applying Theorem I.8.2, WLOG A is an ordinal, α, and R is $<$, the usual (membership) relation on the ordinals. Let f be the isomorphism from $(X; <)$ onto some ordinal, $\delta = \text{type}(X; <)$, and prove that $f(\xi) \leq \xi$ by transfinite induction on ξ; that is, the least ξ such that $f(\xi) > \xi$ would be contradictory. Then $\delta = \{f(\xi) : \xi \in X\} \subseteq \alpha$, so $\delta \leq \alpha$. □

I.9 Induction and Recursion and Foundation

This section is developed in $ZF^- - P$; that is, we assume all the axioms of set theory except Foundation, Power Set, and Choice; any use of these three axioms (except in informal examples) will be mentioned explicitly. We shall discuss induction and recursion on well-founded relations, and we shall elaborate on the significance of the Axiom of Foundation, which says that $\in$ is well-founded.

As applied on $\mathbb{N} = \omega$, induction and recursion are well-known in elementary mathematics. Induction is a standard method of proof. Recursion is a method of defining functions. A simple example of recursion is the factorial function:

$$f(0) = 1 \quad ; \quad f(n) = n \cdot f(n-1) \text{ when } n > 0 \ .$$

Here, $f : \omega \to \omega$ is defined by recursion on $n \in \omega$. This is a recursion because the value of $f(n)$ is defined from the values of f on numbers smaller than n; that is, $f(n)$ is defined using n and $f\restriction n$ (since $n = \{0, 1, \ldots, n-1\}$). We may write this abstractly in the form

$$f(n) = G(n, f\restriction n) \ .$$

Here, $G(0, \emptyset) = 1$ and $G(n, s) = n \cdot s(n-1)$ whenever $n \in \omega \backslash \{0\}$ and $s \in {}^n\omega$; the value of $G(x, y)$ for other x, y is completely irrelevant.

A related example is the $\sum$ notation. If $h : \omega \to \omega$, one may define $f(n) = \sum_{j<n} h(j)$. With the understanding that the empty sum is 0, our recursion is:

$$f(0) = 0 \quad ; \quad f(n) = h(n-1) + f(n-1) \text{ when } n > 0 \ .$$

Again, $f(n)$ is defined from $f\restriction n$, together with h, which may be viewed as a fixed parameter in this definition.

In the last two examples, we only needed $f(n-1)$, not all of $f\restriction n$, to compute $f(n)$. An example where all of $f\restriction n$ is used is in the definition:

$$f(n) = n + \sum_{j<n} f(j) \ .$$

Here, there is no special case for $n = 0$. Informally, the definition is justified because it tells us how to compute successively the values $f(0), f(1), f(2), \ldots$; for example, we compute $f(4) = 4+0+1+3+7 = 15$ after we have computed $f(0), f(1), f(2), f(3)$ as $0, 1, 3, 7$. Formally, we need to prove a theorem (I.9.11) that such a *recursion* really does define a function. In this example, we might now conjecture that $f(n) = 2^n - 1$, and this is easily proved by *induction.*

Another simple example:

Definition I.9.1 *If $f : A \to A$, define $f^n : A \to A$ by iterating f n times; so f^0 is the identity function and $f^n = f \circ f^{n-1}$ when $n > 0$.*

This definition will be justified trivially once we prove Theorem I.9.11, but we must realize that the recursion is on $n \in \omega$, with A, f as a fixed parameters. So, we are really defining a function I_f on ω, and we should say:

$$I_f(0) = id_A \quad ; \quad I_f(n) = f \circ (I_f(n-1)) \text{ when } n > 0 \ .$$

As with the factorial function, $\mathrm{dom}(I_f) = \omega$ and $I_f(n)$ is defined from $I_f \restriction n$. Then, f^n is just shorthand for $I_f(n)$.

Now, before presenting Theorem I.9.11, we discuss a few generalizations of induction and recursion.

In elementary mathematics, there are two basic types of induction used to prove a theorem of the form $\forall \xi \in \omega \ \varphi(\xi)$. *Ordinary Induction* (Theorems I.5.3 and I.7.19) is specific to ω. *Transfinite Induction*, or the *Least Number Principle*, looks at the least ξ such that $\neg\varphi(\xi)$, and derives a contradiction. This is justified by the fact that ω is well-ordered, and is thus equally valid if you replace ω by any ordinal α, as we already did in the proofs of Lemmas I.8.1 and I.8.5. Actually, once ordinals are understood, ordinary induction is reduced to the observation that there are no limit ordinals below ω. A proof of $\forall \xi \in \omega \ \varphi(\xi)$ by ordinary induction could be replaced by a transfinite induction, with the contradiction from the least ξ such that $\neg\varphi(\xi)$ being split into two cases: the basis, when $\xi = 0$, and the successor step, when $\xi = y + 1$. So, in the general discussion of induction, we need only consider generalizations of *transfinite* induction.

Likewise, the general form of defining a function f on ω by recursion is: $f(\xi) = G(\xi, f \restriction \xi)$, where G is some given way of computing a value from ξ and $f \restriction \xi$. In some cases, as with the factorial function, $G(\xi, f \restriction \xi)$ only uses the value $f(\xi - 1)$, not all of $f \restriction \xi$, but we need not mention this in our general theorems about recursion. If we replace ω by an arbitrary ordinal α, the general form of defining f by $f(\xi) = G(\xi, f \restriction \xi)$ is still valid.

Note that transfinite induction on α requires no proof; we are simply applying the definition of well-order, and α is well-ordered by definition of "ordinal". But transfinite recursion, stating that such a function f *exists*, does require a proof.

Next, we point out that these inductions and recursions generalize further in two important ways:

1. We may replace the ordinal α by any set A and replace the well-order of α by any relation R that is well-founded on A.
2. A in (1) may even be a proper class.

Regarding (1): Note that R need not be a total order. Again, induction requires no justification: if we are trying to prove that $\forall y \in A \, \varphi(y)$, and we let $X = \{y \in A : \neg\varphi(y)\}$, then *if* we can derive a contradiction from an R–minimal element of X, then $X = \emptyset$ is obvious from the definition (I.6.21) of well-foundedness. Recursion requires a proof, even when A is finite.

Regarding (2): This is familiar from elementary set theory in the case that $A = ON$, the class of all ordinals, using the natural well-ordering $<$, which is $\in$; for example, we prove $|\alpha \times \alpha| = |\alpha|$ for $\alpha \geq \omega$ by induction on α (see Theorem I.11.6), and we define $\aleph_\xi$ by recursion on ξ (see Definition I.11.4). But also, the Axiom of Foundation implies that $\in$ is well-founded on V, so assuming Foundation, one also can do inductions and recursions on all of V.

When A is a proper class, even induction is non-trivial. As pointed out in Section I.6 (see Notation I.6.27), well-foundedness can only mean that every sub*set* of A has an R–minimal element, so if $X = \{y \in A : \neg\varphi(y)\}$ happens to be a proper class, then well-foundedness says nothing about X.

In this entire section, we shall prove a number of results about relations R on classes A. These results will have a dual interpretation: A could be a set, and R a set of ordered pairs, or A could be a proper class (defined by a formula), and R a relation defined by another formula. In the case of sets, each lemma or theorem is an abbreviation for one sentence in the language of set theory that is provable from our axioms. In the case of proper classes, the lemma or theorem is really a scheme in the metatheory for producing theorems and formal proofs; see page 7 and Lemma I.6.30 for more on such schemes.

When A is a proper class, an additional assumption needs to be added for induction and recursion to work:

Definition I.9.2 *Let R be a relation on a class A. If $y \in A$, let $y{\downarrow} = \mathrm{pred}_R(y) = \mathrm{pred}_{A,R}(y) = \{x \in A : xRy\}$. Then R is* set-like *on A iff $y{\downarrow}$ is a set for all $y \in A$.*

We shall usually use $y{\downarrow}$, although $\mathrm{pred}_R(y)$ or $\mathrm{pred}_{A,R}(y)$ will be used when R and/or A are not clear from context.

For a set A, "set-like" is trivial. Some examples when A is a proper class: If $A = V$ and R is membership, then $y{\downarrow} = y$, so $\in$ is obviously set-like. If $A = V$ and R is $\subseteq$, then $y{\downarrow} = \mathcal{P}(y)$, so $\subseteq$ is set-like iff the Power Set Axiom holds. The $\in$ relation is set-like on any A; in particular $\in$ (or $<$) is set-like on ON. Now consider $ON \times ON$, and let $<$ denote lexicographic order: $(\alpha, \beta) < (\gamma, \delta)$ iff $\alpha < \gamma$ or $\alpha = \gamma \wedge \beta < \delta$. This is a well-order in the sense of Theorem I.7.11; it is easy to check that every non-empty subset of $ON \times ON$ has a least element. This $<$ is not set-like; $(0, \beta){\downarrow}$ is the set $\{0\} \times \beta$, but $(\alpha, \beta){\downarrow} \supseteq \{0\} \times ON$ is a proper class whenever $\alpha > 0$.

Note that Definition I.9.2 is really a *definition scheme* in the metatheory. That is, for each formula defining a class A and each formula defining a binary relation, we have a third formula expressing the assertion "R is set-like on A". Likewise, the next theorem is a theorem scheme, and involves three classes (formulas) A, R, X. Recall that the assertion that R is well-founded on A is the assertion that every non-empty sub*set* of A has an R–minimal element, and there is no way to quantify over sub*classes*. However, given a specific sub-class (formula) X, we can assert that X has a minimal element:

Theorem I.9.3 (Transfinite Induction on Well-founded Relations) *Assume that R is well-founded and set-like on A, and that X is a non-empty sub-class of A. Then X has an R-minimal element.*

This theorem is often used to justify inductive proofs on ON. To prove $\forall \xi \varphi(\xi)$, we form $X = \{\xi : \neg\varphi(\xi)\}$. If X is non-empty (and possibly a proper class), we derive a contradiction from the least ξ such that $\xi \in X$.

This is a step up in logical complexity from Notation I.6.27. The theorem is really an abbreviation for a scheme in the metatheory. It says that given any formulas $\rho(x, y, \vec{z})$ (defining R) and $\alpha(x, \vec{z})$ (defining A) and $\chi(x, \vec{z})$ (defining X), we can prove from our axioms the sentence that asserts that for all $\vec{z}$,

if $\{(x, y) : \rho(x, y, \vec{z})\}$ is well-founded and set-like on $\{x : \alpha(x, \vec{z})\}$
and $\{x : \chi(x, \vec{z}) \wedge \alpha(x, \vec{z})\} \neq \emptyset$
then $\exists x\, [\chi(x, \vec{z}) \wedge \alpha(x, \vec{z}) \wedge \neg\exists u\, [\chi(u, \vec{z}) \wedge \alpha(u, \vec{z}) \wedge \rho(u, x, \vec{z})]$.

This formula looks really ugly and hard to understand. As explained when we discussed Notation I.6.27, it simplifies our terminology to pretend that these classes A, R, X really exist, so that we are not writing a bunch of ugly formulas, but, in line with Caution I.4.10, we do have to make sure that anything we say with these classes is really an abbreviation for something that we *can* say without mentioning classes if we have to.

The $\vec{z}$ in this formalism is a tuple of 0 or more parameters that are "fixed" during the induction.

The proof of Theorem I.9.3 for ON in elementary texts (e.g., [57]) is very easy, using the order properties of $<$ (fix $\xi \in X$; then the least element of the *set* $X \cap (\xi + 1)$ is the least element of X). But we plan to apply the theorem when R is not even transitive; assuming the Axiom of Foundation, R could be $\in$ and A could be V. Then, our proof uses the *transitive closure* of R:

Definition I.9.4 *For a relation R and a class A:*

1. *s is a* path *(or, R-path) of n* steps *in A iff $n \in \omega$, $n \geq 1$, s is a function,* $\mathrm{dom}(s) = n + 1$, $\mathrm{ran}(s) \subseteq A$, *and* $\forall j < n\, [s(j)\, R\, s(j + 1)]$.
2. *The s in* (1) *is called a path* from $s(0)$ to $s(n)$.
3. *The* transitive closure *of R on A is the relation $R^* = R^*_A$ on A defined by xR^*y iff there exists a path in A from x to y.*

In graph theory, we would say that n is the "length" of the path s, but in set theory, $\mathrm{lh}(s) = \mathrm{dom}(s) = n + 1$, as per Definitions I.6.11 and I.7.24.

Note that we are not assuming that $R \subseteq A \times A$. For example, R could be the $\in$ relation, and we shall frequently use induction or recursion on $\in$. In (3), the subscript A is a bit pedantic, and we shall usually omit it, but we sometimes need to specify A, since the paths lie in A. For example, if $a = 2$ and $b = \{\{\{2\}\}\}$, and $A = \{a, b, \{2\}\}$, then $a \in^*_V b$ is true, since $a \in \{2\} \in \{\{2\}\} \in b$ but $a \in^*_A b$ is false, since there is no path in A. Note that $R \cap (A \times A) \subseteq R^*_A \subseteq A \times A$ always holds.

Lemma I.9.5 *For a relation R and a class A:*

1. *R^* is transitive on A.*
2. *If R is set-like on A, then R^* is set-like on A.*

Proof. (1) is easily proved by combining paths.

For (2): Fix $a \in A$, and let $D_n(a)$ be the set of all $x \in A$ such that there exists a path in A from x to a of exactly n steps. Then $D_0(a) = \emptyset$ (since paths have at least one step), and $D_1(a) = \text{pred}_R(a)$, which is a set because R is set-like. For $n \geq 1$, if we assume that $D_n(a)$ is a set, then $D_{n+1} = \bigcup E$, where $E = \{\text{pred}_R(y) : y \in D_n(a)\}$; E is a set by the Replacement Axiom and then D_{n+1} is a set by the Union Axiom. Thus, by induction on ω, each $D_n(a)$ is a set. By the Replacement Axiom again, we can form the set $F = \{D_n(a) : n \in \omega\}$, and then $\text{pred}_{R^*}(a) = \bigcup F$ is a set. Thus, R^* is set-like. □

Proof of Theorem I.9.3. Fix any $a \in X$. Let b be an R–minimal element of the *set* $\{a\} \cup (\text{pred}_{R^*}(a) \cap X)$. Then b is an R–minimal element of X, since $yRb \to y \in \text{pred}_{R^*}(a)$. □

Exercise I.9.6 *Derive the Axioms of Infinity and Replacement from (2) of Lemma I.9.5.*

Hint. For Infinity, let A be the (possibly proper) class of all natural numbers, and let xRy iff $x = y + 1$. □

Definition I.9.7 *A relation R is* acyclic *on A iff R^*_A is irreflexive on A.*

Here, "acyclic" means "no cycles". If aR^*a, then we have a path s from a to a, and the sequence $a = s(0), s(1), \ldots, s(n) = a$ is called a *cycle.*

Exercise I.9.8 *If R is well-founded on A, then R is acyclic on A; hence, R^* is a strict partial order on A. Conversely, if A is finite and R is acyclic on A, then R is well-founded on A.*

Note that a total order is always acyclic, but is not well-founded unless it is a well-order.

The transitive closure of a relation is well-known in discrete mathematics. For example, if A is the set of all humans who have lived until now, and xRy iff x is a child of y, then xR^*y iff x is a descendent of y, and $D_2(a)$ from the proof of Lemma I.9.5 is the set of grandchildren of a. Now, let $\Phi(x)$ be the age of x in years, rounded off to the nearest year. It is obvious from human biology that $xRy \to \Phi(x) < \Phi(y)$. It follows that R is well-founded. This argument generalizes, using the fact that the ordinals are well-ordered:

Lemma I.9.9 *Let R be a relation on a class A, and assume that we have defined a function $\Phi : A \to ON$ such that $xRy \to \Phi(x) < \Phi(y)$ for all $x, y \in A$. Then R is well-founded.*

Proof. If $X \subseteq A$, then any $a \in X$ with $\Phi(a) = \min\{\Phi(x) : x \in X\}$ is R–minimal in X. □

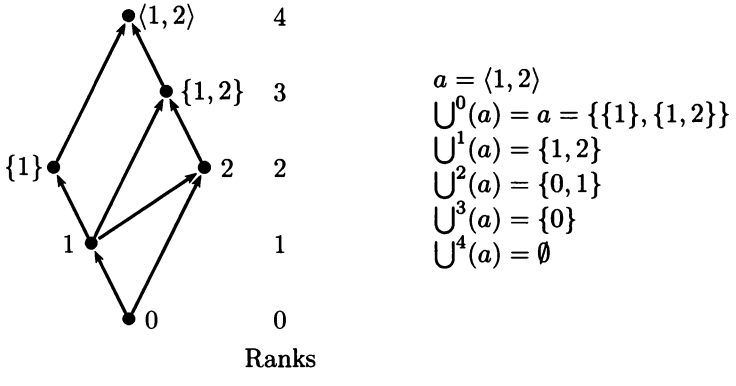
The Φ such that $xRy \to \Phi(x) < \Phi(y)$ is not uniquely defined. In our example with humans, we could have just as well used the age of x in days, rounded off to the nearest day. For well-founded relations, there is an optimum Φ, called the *rank function*, and defined by

$$\text{rank}(y) = \sup\{\text{rank}(x) + 1 : xRy\} \quad . \qquad (*)$$

Of course, we need our result on recursive definitions (Theorem I.9.11 below) to justify this definition. Once that is done, it will be clear that $xRy \to \text{rank}(x) < \text{rank}(y)$. In our example with humans, $\text{rank}(y) = 0$ iff y has no children, $\text{rank}(y) = 1$ iff y has children but no grandchildren, and $\text{rank}(y) = 2$ iff y has children and grandchildren but no great-grandchildren. In the case that R is a well-order on a set A, the rank function is the isomorphism from $(A; R)$ onto an ordinal; see Theorem I.8.2.

In set theory, R^* is related to the transitive closure of a set. Consider $\in$ on V. If $D_n(a)$ is as in the proof of Lemma I.9.5, then $D_1(a) = a\downarrow = a$, $D_2(a) = \bigcup a$, $D_3(a) = \bigcup\bigcup a$, etc. The set $\text{pred}_{V,\in^*}(a)$ occurs so often that is given a name, a (sometimes confusing) re-use of the word "transitive closure".

Figure I.1: Transitive Closures and Ranks



Definition I.9.10 *For any set a, the* transitive closure *of a,* $\text{trcl}(a)$, *is* $\{x : x \in^* a\}$.

For $x \in \mathrm{trcl}(a)$, there may be multiple paths from x to a of different numbers of steps, as we see in Figure I.1 with $a = \langle 1, 2\rangle$. There are four paths from 0 to a, three with 3 steps and one with 4 steps. The six nodes shown in this picture actually constitute $\mathrm{trcl}(\{a\})$, with a at the top. No other set has this picture (see Lemma I.9.38); assuming the Foundation Axiom, every set a is uniquely determined by the isomorphism type of $\in$ on $\mathrm{trcl}(\{a\})$. The ranks for elements of $\mathrm{trcl}(\{a\})$ under the $\in$ relation, shown on the right of the picture, are computed using (❋). For example, $\mathrm{rank}(0) = \sup\{\mathrm{rank}(x) + 1 : x \in \emptyset\} = \sup(\emptyset) = 0$, and $\mathrm{rank}(a) = \sup\{\mathrm{rank}(x) + 1 : x \in a\} = \sup\{2 + 1, 3 + 1\} = 4$.

Informally, $\mathrm{trcl}(a)$ contains all sets used in building a. The Axiom of Foundation says, informally, that every set can be built up from $0 = \emptyset$. Elementary set theory texts [57, 71] often define $\mathrm{trcl}(y) = \bigcup\{\bigcup^n y : n \in \omega\}$, where $\bigcup^n y$ is defined recursively by $\bigcup^0 y = y$ and $\bigcup^{n+1} y = \bigcup \bigcup^n y$. This is easily seen to be equivalent to our definition, once we have justified such recursive definitions. $x \in \bigcup^n y$ iff there is a path from x to y with $n + 1$ steps. In the picture, $1 \in \bigcup^1 a \cap \bigcup^2 a$ because there are 2–step and 3–step paths from 1 to a.

Theorem I.9.11 (Transfinite Recursion on Well-founded Relations) *Assume that R is well-founded and set-like on A and $\forall x, s\, \exists! y\, \varphi(x, s, y)$. Define $G(x, s)$ to be the unique y such that $\varphi(x, s, y)$. Then we can write a formula ψ for which the following are provable:*

1. $\forall x \exists! y \psi(x, y)$, *so ψ defines a function F, where $F(x)$ is the y such that $\psi(x, y)$.*
2. $\forall a \in A\, [F(a) = G(a, F\restriction(a\downarrow))]$.

Remarks. In our formal statement of the theorem, $G(x, s)$ is defined for all sets s, even though the computation of $F(a)$ only uses $G(x, s)$ when s is relevant to the computation of a function on A (i.e., $x \in A$ and s is a function whose domain is $x\downarrow$). This is no problem, since we can always let $G(x, s)$ take some default value, such as $\emptyset$, for the "uninteresting" inputs. Likewise, we are really "thinking" of F as defined only on A, but (1) says $\forall x \exists! y \psi(x, y)$, so $F(x)$ defined for all x; this is no problem, since we can let $F(x) = \emptyset$ when $x \notin A$.

In the case that $A = ON$ and R is the usual order, $\alpha\downarrow = \alpha$, and the recursion scheme is often written $F(\alpha) = G(F\restriction\alpha)$; that is, G is a function of one variable, not two. It is not necessary to mention α, since $\alpha = \mathrm{dom}(F\restriction\alpha)$. In the general case, x need not be uniquely determined from $x\downarrow$.

Formally, Theorem I.9.11 is a scheme in the metatheory, just as is Theorem I.9.3, but it is yet another step up in logical complexity. In both theorems, the classes involved can depend on 0 or more parameters $\vec{z}$ that are "fixed" during the induction/recursion. Theorem I.9.3 asserted that given formulas defining R, A, X, a certain sentence was provable. Here we are asserting that given formulas defining R, A, G, you can write *another* formula defining F such that (1) and (2) are provable. This is the same logical form as Lemma I.6.30, where the mathematics was much simpler.

An easy induction (using Theorem I.9.3) shows that the F of Theorem I.9.11 is unique:

Lemma I.9.12 *In Theorem I.9.11, suppose that F and F' both satisfy* (2). *Then* $\forall a \in A\,[F(a) = F'(a)]$.

Proof. If not, and a is R–minimal in $\{x \in A : F(x) \neq F'(x)\}$, then $F(a) = G(a, F\restriction(a\downarrow)) = G(a, F'\restriction(a\downarrow)) = F'(a)$, a contradiction. □

The existence of F also uses induction; roughly, we induct on a to prove that "$F(a)$ makes sense". Of course, we need to make this precise. The proof of Theorem I.8.2, which produced an isomorphism from a well-order R of a set A onto an ordinal, used the same ideas in a much simpler setting.

Proof of Theorem I.9.11. For sets d, h, let $\mathrm{App}(d,h)$ say that h is a function, $\mathrm{dom}(h) = d \subseteq A$, $\forall y \in d\,[y\downarrow \subseteq d]$, and $\forall y \in d\,[h(y) = G(y, h\restriction(y\downarrow))]$; so, we're saying that h is an *approximation* to F, defined on some set d. Here, $y\downarrow$ abbreviates $\mathrm{pred}_{A,R}(y)$ as before. Note that the property $\forall y \in d\,[y\downarrow \subseteq d]$ implies also $\forall y \in d\,[\mathrm{pred}_{A,R^*}(y) \subseteq d]$. An important set with this property is:

$$d_x := \{x\} \cup \mathrm{pred}_{A,R^*}(x)$$

for any $x \in A$; this is a set because R^* is set-like by Lemma I.9.5.

Assuming that the theorem is true, $\mathrm{App}(d,h)$ implies that $h = F\restriction d$, since $h(y) = F(y)$ for all $y \in d$ is easily proved by transfinite induction. Since we don't know yet that the theorem is true, we shall prove the theorem by using this $\mathrm{App}(d,h)$ to write down the following definition of ψ:

$$\begin{aligned}\psi(x,y) \iff & \\ & [x \notin A \ \wedge\ y = \emptyset] \ \vee \\ & [x \in A \ \wedge\ \exists d, h\,[\mathrm{App}(d,h) \wedge x \in d \wedge h(x) = y]] \quad .\end{aligned}$$

We now need to check that this definition works, which will be easy once we have verified the *existence* and *uniqueness* of these approximations.

Uniqueness means that all the approximations agree wherever they are defined:

$$\mathrm{App}(d,h) \wedge \mathrm{App}(d',h') \to \mathrm{App}(d \cap d', h \cap h') \quad . \tag{U}$$

To verify this, note first that $\forall y \in (d \cap d')\,[y\downarrow \subseteq (d \cap d')]$. Then, note that $h(y) = h'(y)$ for all $y \in d \cap d'$, since an R–minimal element of $\{y \in d \cap d' : h(y) \neq h'(y)\}$ would be contradictory, using $h(y) = G(y, h\restriction(y\downarrow))$. So, the intersection $h \cap h'$ is really a function with domain $d \cap d'$ that takes $y \in d \cap d'$ to $h(y) = h'(y)$. Then $\mathrm{App}(d \cap d', h \cap h')$ is clear.

By (U), we know that for all x, there is at most one y such that $\psi(x,y)$. To prove that such a y always exists, use:

$$\forall x \in A\,\exists d, h\,[\mathrm{App}(d,h) \wedge x \in d] \quad . \tag{E}$$

To prove (E), we apply transfinite induction on R (using Theorem I.9.3). First observe that $\text{App}(d,h) \wedge x \in d \to \text{App}(d_x, h_x)$, where $h_x = h\restriction d_x$. Assuming that (E) is false, let $X = \{x \in A : \neg\exists d,h\,[\text{App}(d,h) \wedge x \in d]\} \neq \emptyset$. Observe that for $x \notin X$, we have an h_x such that $\text{App}(d_x, h_x)$, and this h_x is unique by (U). Let $a \in X$ be R–minimal in X. Let $\tilde{d} = \text{pred}_{A,R^*}(a) = \bigcup\{d_x : xRa\}$. By minimality, $xRa \to x \notin X$, so by the Replacement Axiom, we may define the set $\tilde{h} = \bigcup\{h_x : xRa\}$, which is a function by (U), and it is then easily verified that $\text{App}(\tilde{d}, \tilde{h})$. Now $a \notin \tilde{d}$, but $a{\downarrow} \subseteq \tilde{d}$. Informally $F\restriction\tilde{d}$ "should be" $\tilde{h}$, so $F(a)$ "should be" $G(a, \tilde{h}\restriction(a{\downarrow}))$. Formally, let $d = \tilde{d} \cup \{a\}$ and let $h = \tilde{h} \cup \{(a,\, G(a, \tilde{h}\restriction(a{\downarrow}))\,)\}$. Then $\text{App}(d,h)$ and $a \in d$, contradicting $a \in X$.

Combining (U) and (E), we know that $\forall x \in A\,\exists! y \psi(x,y)$, so ψ defines a function F, as in (1). Then (2) follows from the definition of $\text{App}(d,h)$. □

The use of Replacement is essential here. For example, on ON, we may define $F(\xi+1) = \{F(\xi)\}$ and $F(\alpha) = \{F(\xi) : \xi < \alpha\}$ when α is a limit or 0. Then $F(\omega)$ is the set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots\}$ but one cannot prove that there exists such a set in ZC (see Exercise II.4.21).

Next, we shall discuss in detail a very important example of recursion — namely, the definition of the rank function, formalizing (❋) on page 47.

Definition I.9.13 *Assume that R is well-founded and set-like on A. Define, recursively, for $y \in A$,* $\text{rank}(y) = \text{rank}_{A,R}(y) = \bigcup\{S(\text{rank}(x)) : x \in y{\downarrow}\}$. *Let* $\text{rank}(y) = \emptyset$ *for $y \notin A$.*

Justification. Use Theorem I.9.11. Let $G(x,s) = \bigcup\{S(t) : t \in \text{ran}(s)\}$; note that this doesn't depend on x, and is defined for all sets x, s. Then $F(a) = G(a, F\restriction(a{\downarrow}))$ translates to $F(a) = \bigcup\{S(F(c)) : c \in A \wedge cRa\}$. □

In our notation for rank, we are suppressing mention of A, R when these are clear from context. Note that $S(u)$ is the "ordinal successor" $u \cup \{u\}$, but it makes sense for any set u. But, the rank is always an ordinal, as we shall prove in the next lemma. Also, for sets of ordinals, it is more common to write sup for $\bigcup$ (see Lemma I.7.14) and $\alpha+1$ for $S(\alpha)$, so we have:

Lemma I.9.14 *If R is well-founded and set-like on A, then* $\text{rank}(y)$ *is an ordinal for all $y \in A$, so* $\text{rank}(y) = \sup\{\text{rank}(x)+1 : x \in A \wedge xRy\}$. *Furthermore,* $xRy \to \text{rank}(x) < \text{rank}(y)$ *for all $x, y \in A$.*

Proof. Prove $\text{rank}(y) \in ON$ by transfinite induction; an R–minimal y such that $\text{rank}(y) \notin ON$ would be contradictory because the union of a set of ordinals is an ordinal. Then the "Furthermore" is clear, since we are taking a sup, so $xRy \to \text{rank}(x) + 1 \leq \text{rank}(y)$. □

See Figure I.1, page 47, for an example of the rank function.

Since the existence of a rank function implies well-foundedness by Lemma I.9.9, Theorem I.9.11 is *false* whenever R is not well-founded, with the rank function providing an explicit counter-example; see Exercise I.9.50.

When A is a proper class, F is not a set, but each restriction $F\restriction(a\downarrow) = \{(x, F(x)) : xRa\}$ is a set by the Replacement Axiom (see Lemma I.6.9), since each $a\downarrow$ is a set. Note that when R is not set-like, Theorem I.9.11 is *meaningless*, not false — that is, we can't ask for a counter-example because we can't even state the theorem. Recall that only sets exist, and (2) is really an abbreviation for $\forall a \in A\,[\varphi(a, F\restriction(a\downarrow), F(a))]$. Of course, we can replace the "$a \in A$" by the formula defining A and we can replace "$F(a)$" by the formula defining F, but there is no way to plug the non-set $F\restriction(a\downarrow)$ into a formula.

A simple use of the rank function, continuing Lemma I.9.5:

Lemma I.9.15 *For a relation R and a class A: If R is set-like and well-founded on A, then R^* is well-founded on A.*

Proof. Observe that $xR^*y \to \text{rank}(x) < \text{rank}(y)$ for all $x, y \in A$, and apply Lemma I.9.9. □

We proceed to develop some further facts about the rank function, using, as an illustration, Figure I.1, which shows the rank function on $A = \{0, 1, 2, \{1\}, \{1, 2\}, \langle 1, 2\rangle\}$, where R is $\in$. It is reasonable to use $\in$ for our example, since by Exercise I.9.49, every well-founded relation on a set is isomorphic to $\in$ on some other set. Note that the rank is just the level in the picture, with nodes of rank 0 having no children, nodes of rank 1 having a child but no grandchildren, etc.

In Figure I.1, $\{\text{rank}(x) : x \in A\} = \{0, 1, 2, 3, 4\} = 5$. More generally, the range of the rank function doesn't skip any ordinals:

Lemma I.9.16 *Assume that R is well-founded and set-like on A. Fix $b \in A$, and fix $\alpha < \text{rank}(b)$. Then $\alpha = \text{rank}(a)$ for some $a \in A$ such that aR_A^*b.*

In view of Lemma I.7.13, this says that $\{\text{rank}(x) : x \in A\}$ is either a member of ON or equal to ON. Informally, the "aR_A^*b" occurs because in computing $\text{rank}(b)$, only the descendents of b are used.

Proof. Facts about recursively defined functions are often proved by induction. Fix α, and let b be a minimal counter-example. More precisely, let

$$X = \{b \in A : \text{rank}(b) > \alpha \wedge \neg\exists a \in A\,[\text{rank}(a) = \alpha \wedge aR^*b]\}\ .$$

If the lemma fails, then $X \neq \emptyset$, so fix some $b \in X$ that is R–minimal in X and let $\beta = \text{rank}(b)$. Now $\alpha < \beta = \text{rank}(b) = \sup\{\text{rank}(t) + 1 : t \in A \wedge tRb\}$, so fix $t \in A$ such that tRb and $\text{rank}(t) + 1 > \alpha$, so $\text{rank}(t) \geq \alpha$.

If $\text{rank}(t) = \alpha$: Then $b \notin X$ (since tR^*b), a contradiction.

If $\text{rank}(t) > \alpha$: Since $t \notin X$ (by tRb and minimality of b), fix $a \in A$ such that $\text{rank}(a) = \alpha$ and aR^*t. But then aR^*b, contradicting $b \in X$. □

The Axiom of Foundation says that $\in$ is well-founded on V (see Lemma I.6.28). So, it is natural to consider $\text{rank}_{V,\in}(x)$. For example, $\text{rank}_{V,\in}(0) = \sup(\emptyset) = 0$ because 0 has no members. Then $1 = \{0\}$, so then $\text{rank}_{V,\in}(1) =$

$\sup\{0+1\} = 1$ and then $2 = \{0, 1\}$, so $\text{rank}_{V,\in}(2) = \sup\{0+1, 1+1\} = 2$. Also, $\text{rank}_{V,\in}(\{1\}) = \sup\{1+1\} = 2$. Note that these values for $\text{rank}_{V,\in}$ agree with $\text{rank}_{A,\in}$, where A is as in Figure I.1. This holds in general for *transitive* A (see Lemma I.9.21). It can fail for non-transitive sets. For example, if $B = \{1, 2, 3\}$, then $\text{rank}_{B,\in}(1) = \sup(\emptyset) = 0$ and $\text{rank}_{B,\in}(2) = \sup\{0+1\} = 1$.

The fact that $\text{rank}_{V,\in}(2) = 2$ generalizes to:

Lemma I.9.17 *For $\alpha \in ON$: $\text{rank}_{ON,\in}(\alpha) = \alpha$; also, assuming the Axiom of Foundation, $\text{rank}_{V,\in}(\alpha) = \alpha$.*

Proof. First, observe that $\in$ is a well-order of ON, so we do not need Foundation to define $\text{rank}_{ON,\in}$. To prove $\text{rank}_{ON,\in}(\alpha) = \alpha$, assume this fails and let α be $\in$–minimal in $\{\xi \in ON : \text{rank}_{ON,\in} \neq \alpha\}$. Applying minimality, $\text{rank}_{ON,\in}(\alpha) = \sup\{\xi + 1 : \xi < \alpha\} = \alpha$, a contradiction.

Exactly the same proof works for $\text{rank}_{V,\in}$, assuming Foundation so that $\text{rank}_{V,\in}$ is defined. □

The fact that $\text{rank}_{ON,\in}(\alpha) = \text{rank}_{V,\in}(\alpha)$ generalizes:

Lemma I.9.18 *Suppose that $A \subseteq B$, and R is well-founded and set-like on B. Let R^* denote R^*_B. If $b \in A$ then $\text{rank}_{A,R}(b) \leq \text{rank}_{B,R}(b)$. If $b \in A$ and $\text{pred}_{B,R^*}(b) \subseteq A$ then $\text{rank}_{A,R}(b) = \text{rank}_{B,R}(b)$.*

Proof. First prove $\text{rank}_{A,R}(b) \leq \text{rank}_{B,R}(b)$ by induction: If b is R–minimal in $\{x \in A : \text{rank}_{A,R}(x) > \text{rank}_{B,R}(x)\}$, then we get a contradiction from

$$\begin{aligned}\text{rank}_{A,R}(b) = &\sup\{\text{rank}_{A,R}(x) + 1 : x \in A \wedge xRb\} \leq \\ &\sup\{\text{rank}_{B,R}(x) + 1 : x \in B \wedge xRb\} = \text{rank}_{B,R}(b) \ .\end{aligned}$$

The "$\leq$" uses minimality of b.

Next, prove $\text{rank}_{A,R}(b) = \text{rank}_{B,R}(b)$ when $\text{pred}_{B,R^*}(b) \subseteq A$ by induction: If b is R–minimal in $\{x \in A : \text{rank}_{A,R}(x) \neq \text{rank}_{B,R}(x) \wedge \text{pred}_{B,R^*}(x) \subseteq A\}$ we get a contradiction from

$$\begin{aligned}\text{rank}_{A,R}(b) = &\sup\{\text{rank}_{A,R}(x) + 1 : x \in A \wedge xRb\} = \\ &\sup\{\text{rank}_{B,R}(x) + 1 : x \in B \wedge xRb\} = \text{rank}_{B,R}(b) \ .\end{aligned}$$

The second "=" uses minimality of b, plus the fact that from $\text{pred}_{B,R^*}(b) \subseteq A$ we conclude that xRb implies both $x \in A$ and $\text{pred}_{B,R^*}(x) \subseteq A$. □

If we fix B and $b \in B$, then the smallest set A that works in this lemma is $\{b\} \cup \text{pred}_{B,R^*}(b)$. In the case that $B = V$ and R is $\in$, this reduces to $\{b\} \cup \text{trcl}(b)$ (see Definition I.9.10). Without assuming the Axiom of Foundation, we shall define b to be a well-founded set iff the computation of $\text{rank}(b)$ "works":

Lemma I.9.19 *For any set b: $\in$ is well-founded on $\text{trcl}(b)$ iff $\in$ is well-founded on $\{b\} \cup \text{trcl}(b)$.*

Proof. This is obvious if $b \in \mathrm{trcl}(b)$, so assume that $b \notin \mathrm{trcl}(b)$; equivalently $\neg(b \in^* b)$. The $\leftarrow$ direction is still clear from the fact that $\mathrm{trcl}(b) \subseteq \{b\} \cup \mathrm{trcl}(b)$.

For the $\rightarrow$ direction: Assume that $\in$ is well-founded on $\mathrm{trcl}(b)$, and let X be a non-empty subset of $\{b\} \cup \mathrm{trcl}(b)$; we need to produce an $\in$–minimal element a of X. This is obvious if $b \notin X$ or if $X = \{b\}$. But if $\{b\} \subsetneq X$, then any $\in$–minimal element $a \in X \backslash \{b\}$ is $\in$–minimal in X, since $b \in a$ would imply that $b \in^* b$. □

We remark that if $b \in^* b$, then $\in$ is not well-founded on $\mathrm{trcl}(b)$ because a well-founded relation is acyclic (see Exercise I.9.8).

Definition I.9.20 *The set b is a* well-founded set *iff $\in$ is well-founded on* $\mathrm{trcl}(b)$, *in which case* $\mathrm{rank}(b)$ *denotes* $\mathrm{rank}_{\{b\} \cup \mathrm{trcl}(b), \in}(b)$. *WF denotes the class of all well-founded sets.*

Lemma I.9.21 *If T is a transitive class and $\in$ is well-founded on T, then $T \subseteq WF$ and* $\mathrm{rank}(b) = \mathrm{rank}_{T, \in}(b)$ *for all $b \in T$.*

Proof. Use Lemma I.9.18, since $b \in T \rightarrow \mathrm{pred}_{T, \in^*}(b) = \mathrm{trcl}(b) \subseteq T$. □

Setting $T = ON$:

Lemma I.9.22 *$ON \subseteq WF$, so WF is a proper class, and* $\mathrm{rank}(\alpha) = \alpha$ *for $\alpha \in ON$.*

Lemma I.9.23 *The Axiom of Foundation is equivalent to $V = WF$.*

Proof. For $\rightarrow$: $\in$ is well-founded on every set.

For $\leftarrow$: If x is a non-empty set with no $\in$–minimal elements, then $\in$ is not well-founded on $\mathrm{trcl}(x)$, so $x \notin WF$. □

Lemma I.9.24

1. $x \in b \in WF \rightarrow x \in WF \wedge \mathrm{rank}(x) < \mathrm{rank}(b)$.
2. *$\in$ is well-founded on WF.*
3. *For all* sets *b, $b \in WF$ iff $b \subseteq WF$, so that WF is a transitive class.*
4. *For $b \in WF$,* $\mathrm{rank}(b) = \sup\{\mathrm{rank}(x) + 1 : x \in b\}$.
5. $\mathrm{rank}(b) = \mathrm{rank}_{WF, \in}(b)$ *for all $b \in WF$.*

Proof. For (1): $x \in WF$ because $\mathrm{trcl}(x) \subseteq \mathrm{trcl}(b)$, and then, using Lemma I.9.21 with $T = \{b\} \cup \mathrm{trcl}(b)$: $\mathrm{rank}(x) = \mathrm{rank}_{T, \in}(x) < \mathrm{rank}_{T, \in}(b) = \mathrm{rank}(b)$; the "$<$" is by Lemma I.9.14. Now, (2) follows by Lemma I.9.9. Note that $x \in b \rightarrow \mathrm{trcl}(x) \subseteq \mathrm{trcl}(b)$ is clear from the definition (I.9.10) of trcl using $\in^*$.

For (3): The $\rightarrow$ direction is clear from (1). For $\leftarrow$: If $b \subseteq WF$ then $\mathrm{trcl}(b) \subseteq WF$ (since WF is transitive), so $\in$ is well-founded on $\mathrm{trcl}(b)$ by (2).

(5) is immediate by Lemma I.9.21 with $T = WF$. Then, (4) follows using the definition (I.9.13) of rank_{WF}. □

Assuming the Axiom of Foundation, Lemma I.9.24 implies that $\text{rank}(b) = \text{rank}_{V,\in}(b)$ for all b. In set theory, $\text{rank}(b)$ always means $\text{rank}_{WF,\in}(b)$ (which equals $\text{rank}_{V,\in}(b)$, assuming Foundation), unless it is being made clear from context that we are discussing some other $\text{rank}_{A,R}$.

Lemma I.9.24 can be used to show that all "reasonable" sets (including anything we have already constructed) lie in WF, and it is now easy to compute the rank of a given set. We already have the ordinals by Lemma I.9.22. Then, for $\alpha \in ON$, we know that $\{\alpha\} \in WF$ by (3), and then $\text{rank}(\{\alpha\}) = \alpha+1$ by (4). Then $\{\{8\},\{2\}\} \in WF$ by (3) and $\text{rank}(\{\{8\},\{2\}\}) = \max(9+1, 3+1) = 10$.

The next two lemmas say a bit more about computation of ranks; they are easily derived from Lemma I.9.24:

Lemma I.9.25 *If $z \subseteq y \in WF$ then $z \in WF$ and $\text{rank}(z) \leq \text{rank}(y)$.*

Lemma I.9.26 *Suppose that $x, y \in WF$. Then:*

1. $\{x,y\} \in WF$ *and* $\text{rank}(\{x,y\}) = \max(\text{rank}(x), \text{rank}(y)) + 1$.
2. $\langle x,y\rangle \in WF$ *and* $\text{rank}(\langle x,y\rangle) = \max(\text{rank}(x), \text{rank}(y)) + 2$.
3. *If $\mathcal{P}(x)$ exists, then $\mathcal{P}(x) \in WF$ and* $\text{rank}(\mathcal{P}(x)) = \text{rank}(x) + 1$.
4. $\bigcup x \in WF$ *and* $\text{rank}(\bigcup x) \leq \text{rank}(x)$.
5. $x \cup y \in WF$ *and* $\text{rank}(x \cup y) = \max(\text{rank}(x), \text{rank}(y))$.
6. $\text{trcl}(x) \in WF$ *and* $\text{rank}(\text{trcl}(x)) = \text{rank}(x)$.

Roughly, these lemmas imply that WF is a model of set theory, including Foundation (since $\in$ is well-founded on WF). The model WF will be used in von Neumann's proof that $\text{Con}(ZF^-) \rightarrow \text{Con}(ZF)$; see Chapter II.

These lemmas also imply that WF contains all mathematical objects. We have already seen that WF contains all ordinals, and hence all sets of ordinals, pairs of ordinals, etc. Also, WF contains $\mathbb{R}$, $\mathbb{C}$, and Lebesgue measure on $\mathbb{R}$, assuming that these objects exist (which uses the Power Set Axiom) and are defined in any reasonable way; see Section I.14.

Exercise I.9.27 *Assume that each element of $\mathbb{R}$ has rank ω; this will be true using our Definition I.14.2. Let μ be Lebesgue measure on $\mathbb{R}$. Prove that $\mathbb{R}, \mu \in WF$, with $\text{rank}(\mathbb{R}) = \omega + 1$ and $\text{rank}(\mu) = \omega + 4$.*

Hint. All you need to know about μ is that it is a function with $\text{ran}(\mu) \subseteq \mathbb{R}$ and $\{\emptyset\} \subsetneqq \text{dom}(\mu) \subseteq \mathcal{P}(\mathbb{R})$. □

Modern set theory always assumes Foundation (equivalently, $V = WF$) as a basic axiom. This does not imply any philosophical belief that Foundation must be "really true" (whatever that means). Rather, we're just saying that WF contains everything of mathematical interest anyway, and Foundation simplifies the discussion of models of set theory (see Section II.4), so we might as well consider WF to be our set-theoretic universe.

Definition I.9.28 $R(\alpha) = \{x \in WF : \text{rank}(x) < \alpha\}$.

V_α is also commonly used as a synonym of $R(\alpha)$.

It is apparent from Exercise I.9.27 that $R(\omega + \omega)$ contains everything constructed in an elementary analysis text. In fact, $R(\omega + \omega)$ is a model of ZC (ZFC without Replacement) (see Theorem II.2.2), and Replacement is never used in the development of elementary analysis.

Lemma I.9.29 *For all sets b and ordinals α: $b \in R(\alpha + 1)$ iff $b \subseteq R(\alpha)$.*

Proof. For $\rightarrow$: We have $b \in WF$ and $\text{rank}(b) < \alpha + 1$. Then by Lemma I.9.24, $x \in b$ implies that $x \in WF$ and $\text{rank}(x) < \alpha$, so $b \subseteq R(\alpha)$.

For $\leftarrow$: If $b \subseteq R(\alpha)$ then, by Lemma I.9.24 and the definition of $R(\alpha)$: $b \in WF$ and $\text{rank}(b) = \sup\{\text{rank}(x) + 1 : x \in b\} \leq \alpha$, so $b \in R(\alpha + 1)$. □

It is not immediately clear whether $R(\alpha)$ is a set or a proper class. $\omega \subseteq R(\omega)$ by Lemma I.9.22, so $\mathcal{P}(\omega) \subseteq R(\omega + 1)$ by Lemma I.9.29. Since $ZFC-P$ does not prove that $\mathcal{P}(\omega)$ is a set (Lemma II.4.4), we really need the Power Set Axiom to prove that $R(\omega + 1)$ is a set.

Lemma I.9.30 *Assume the Power Set Axiom. Then $R(\delta)$ is a set for each $\delta \in ON$, and furthermore:*

1. $R(0) = \emptyset$.
2. $R(\alpha + 1) = \mathcal{P}(R(\alpha))$.
3. $R(\gamma) = \bigcup_{\alpha < \gamma} R(\alpha)$ *for limit ordinals* γ.

Proof. We prove that $R(\delta)$ is a set by induction on $\delta \in ON$; that is, assume that δ is least such that $R(\delta)$ is not a set, and derive a contradiction by proving that $R(\delta)$ is a set. At the same time, we establish (1)(2)(3).

$\delta \neq 0$, since $R(0) = \emptyset$ is obvious from the definition of $R(0)$.

If $\delta = \alpha + 1$, then $R(\delta) = \mathcal{P}(R(\alpha))$ by Lemma I.9.29, and this is a set by the Power Set Axiom.

Finally, if δ is a limit ordinal, then $R(\delta) = \bigcup_{\alpha < \delta} R(\alpha)$ is clear from the definition of $R(\delta)$, and this is a set by the Replacement Axiom. □

Note that Replacement is essential here. $R(\omega + \omega)$ is a model for ZC plus the statement that $R(\omega + \omega)$ doesn't exist (see Exercise II.4.23).

Lemma I.9.30 may be viewed as saying that the well-founded sets are those sets that can be obtained from nothing, $\emptyset$, by iterating *collection*; that is, taking a bunch of sets and putting them between brackets, $\{\cdots\cdots\}$. If at stage α we have "constructed" all the elements of $R(\alpha)$, then putting the various collections of these between brackets forms the various elements of $\mathcal{P}(R(\alpha)) = R(\alpha + 1)$.

Most set theory texts [28, 49, 54, 57] take (1)(2)(3) of Lemma I.9.30 as the definition of the $R(\alpha)$, and then define WF to be the class $\bigcup\{R(\alpha) : \alpha \in ON\}$;

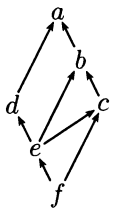
rank(x) is defined to be the α such that $x \in R(\alpha+1)\backslash R(\alpha)$. This approach has the advantage of giving a somewhat simpler definition of WF, requiring only recursion on ON, which is easier to describe then the general case of recursion. It has the disadvantage of making it appear that the notion of rank depends on the Power Set Axiom, which it doesn't. Using our presentation here, it will be easy to see that rank is defined and absolute for transitive models of $ZF-P$ (see Corollary II.4.16).

We turn now to another simple but important example of recursion, which is related to, but different than, the rank function (Definition I.9.13).

Definition I.9.31 *Assume that R is well-founded and set-like on A. Define, recursively, for $y \in A$,* $\text{mos}(y) = \text{mos}_{A,R}(y) = \{\text{mos}(x) : x \in y{\downarrow}\}$. *This is called the* Mostowski collapsing function. *Here,* $y{\downarrow} = \text{pred}_{A,R}(y)$.

Justification. Use Theorem I.9.11. Let $G(x,s) = \text{ran}(s)$; note that this doesn't depend on x, and is defined for all sets x, s. Then $F(y) = G(y, F\restriction(y{\downarrow}))$ translates to $F(y) = \{F(x) : x \in A \wedge zRy\} = \{F(x) : x \in y{\downarrow}\}$. □

We could write the recursion more compactly as $\text{mos}(y) = \text{mos}``(y{\downarrow})$ (see Definition I.6.12). As an illustration, suppose that you were just given the abstract directed graph shown in Figure I.1 on page 47, without the labeling by sets. You could easily compute the labels, working from the bottom up:



$$\begin{aligned}
&\text{mos}(f) = \text{mos}``\emptyset = \emptyset = 0\\
&\text{mos}(e) = \text{mos}``\{f\} = \{0\} = 1\\
&\text{mos}(d) = \text{mos}``\{e\} = \{1\}\\
&\text{mos}(c) = \text{mos}``\{f,e\} = \{0,1\} = 2\\
&\text{mos}(b) = \text{mos}``\{e,c\} = \{1,2\}\\
&\text{mos}(a) = \text{mos}``\{d,b\} = \langle 1,2\rangle
\end{aligned}$$

This is the only way to label the nodes by elements of a *transitive* set so that the relation is $\in$. There's infinitely many such labelings by non-transitive sets. Perhaps $A = \{f,e,d,c,b,a\}$, where $f = 0$, $e = 1$, $c = 2$, $d = \{1,3\}$, $b = \{1,2,8\}$, $a = \{d,b,11\}$. Then $\in$ on A has the same directed graph. The Mostowski *collapsing* function is the identity on the "transitive part" $\{0,1,2\}$ of A and collapses d, b, a down to $\text{mos}(d) = \{1\}$, $\text{mos}(b) = \{1,2\}$, and $\text{mos}(a) = \{\text{mos}(d), \text{mos}(b)\}$. In general, mos for $(A,\in)$ is the identity on the transitive part of A; see Lemma I.9.36.

The Mostowski function on A need not be 1-1, even when R is $\in$. For example, if $\in$ is empty on A (e.g., $A = \{\{1\},\{2\},\{3\}\}$, then $\text{mos}_{A,\in}(y) = \emptyset$ for all $y \in A$. But, if (A,R) is a model for the Axiom of Extensionality, then mos is 1-1 and provides an isomorphism from (A,R) onto the membership relation on some transitive set; see Lemma I.9.35 below. First we note:

Lemma I.9.32 *Assume that R is well-founded and set-like on A. Then* mos$``A$ *is transitive.*

Proof. By definition, every $\mathrm{mos}(y) = \{\mathrm{mos}(x) : x \in y{\downarrow}\}$ is a subset of $\mathrm{mos}\text{“}A$. □

Definition I.9.33 *The relation R is* extensional *on A iff (A, R) satisfies the Axiom of Extensionality; equivalently $\forall x, y \in A\,[x{\downarrow} = y{\downarrow} \to x = y]$.*

Since $x{\downarrow} = x$ when R is $\in$ and A is transitive,

Lemma I.9.34 *The $\in$ relation is extensional on A whenever A is transitive.*

Lemma I.9.35 *Assume that R is well-founded and set-like on A. Then the function $\mathrm{mos}_{A,R}$ is 1-1 iff R is extensional on A, in which case* mos *provides an isomorphism from (A, R) onto $(\mathrm{mos}\text{“}A, \in)$.*

Proof. If R is not extensional on A, then there are $a \neq b$ in A with $a{\downarrow} = b{\downarrow}$, which implies that $\mathrm{mos}(a) = \mathrm{mos}(b)$; thus, $\mathrm{mos}_{A,R}$ is not 1-1.

Conversely, assume that R is extensional on A. We shall prove that mos is 1-1. We wish to show $a \neq b \to \mathrm{mos}(a) \neq \mathrm{mos}(b)$ by "induction on (a, b)", but rather than introducing a concept of "double induction", we quantify out the b, and let $X = \{a \in A : \exists y \in A\,[a \neq y \wedge \mathrm{mos}(a) = \mathrm{mos}(y)]$. If $X = \emptyset$, then mos is 1-1, so assume that $X \neq \emptyset$. Applying induction (Theorem I.9.3), let $a \in X$ be R–minimal in X. Then fix any $b \in X$ such that $a \neq b$ and $\mathrm{mos}(a) = \mathrm{mos}(b)$. Since R is extensional, $a{\downarrow} \neq b{\downarrow}$. there are now two cases:

Case I. There is a c with cRa and $c \not R\, b$. Since $\mathrm{mos}(c) \in \mathrm{mos}(a) = \mathrm{mos}(b) = \{\mathrm{mos}(z) : z \in b{\downarrow}\}$, there is a d with dRb and $\mathrm{mos}(d) = \mathrm{mos}(c)$, and note that $c \neq d$ because $c \not R\, b$. But then $c \in X$ and cRa, contradicting minimality of a.

Case II. There is a d with dRb and $d \not R\, a$. Exactly as above, we find a c with cRa and $c \neq d$ and $\mathrm{mos}(d) = \mathrm{mos}(c)$. Again, $c \in X$ and cRa, contradicting minimality of a.

Now that we know that mos is 1-1, the fact that mos is an isomorphism (that is, $\forall x, y \in A\,[xRy \leftrightarrow \mathrm{mos}(x) \in \mathrm{mos}(y)]$), follows immediately from the definition of $\mathrm{mos}(y)$ as $\{\mathrm{mos}(x) : x \in y{\downarrow}\}$. □

A special case occurs when R is the $\in$ relation. If A is transitive, then $\in$ is trivially extensional on A (since then $x{\downarrow} = x$), but then mos is the identity function on A. More generally,

Lemma I.9.36 *Assume that $\in$ is well-founded and extensional on A. Let $T \subseteq A$ be transitive. Then $\mathrm{mos}_{A,\in}(y) = y$ for all $y \in T$.*

Proof. If a is $\in$–minimal in $\{y \in T : \mathrm{mos}(a) \neq a\}$, then $\mathrm{mos}(a) = \{\mathrm{mos}(y) : y \in A \wedge y \in a\} = \{y : y \in a\} = a$, a contradiction. The second $=$ used minimality of a and the fact that $y \in a \to y \in T \subseteq A$. □

A special case is where T is all of A, which gives us:

Lemma I.9.37 *Let A, B be transitive sets, with $A \in WF$, and let $f : A \xrightarrow[\text{onto}]{1\text{-}1} B$ be an isomorphism from $(A; \in)$ onto $(B; \in)$. Then $f = id_A$ and hence $A = B$.*

Proof. This generalizes Lemma I.8.1, which is the special case where $A, B \in ON$. Say $f(a) = b$. Since f is an isomorphism, we have

$$\forall y \in B\,[y \in b \leftrightarrow \exists x \in A\,[x \in a \wedge f(x) = y]] \quad .$$

Since A, B are transitive, the "$y \in B$" and "$x \in A$" are redundant, so we get $\forall y\,[y \in b \leftrightarrow \exists x \in a\,[f(x) = y]]$; so, $f(a) = \{f(x) : x \in a\}$. But, since A is well-founded, this means that f is just the Mostowski isomorphism function mos on $(A; \in)$ from Definition I.9.31, so $f = id_A$ by Lemma I.9.36. □

Of course, then $B \in WF$ also, since $B = A$. We needed to assume that one of $A, B \in WF$, since it is consistent with ZFC^- to have $A \neq B$ but $A = \{A\}$ and $B = \{B\}$ (see Exercise II.9.5). Then A, B are transitive and $(A; \in) \cong (B; \in)$ but $A \neq B$.

Lemma I.9.37 says that a transitive well-founded set A is determined by the isomorphism type of $\in$ on $A = \mathrm{trcl}(A)$. This is false for non-transitive sets. For example, if $A = \{1, 2\}$ and $B = \{0, 2\}$, then $A \neq B$ but $(A; \in) \cong (B; \in)$ and $(\mathrm{trcl}(A); \in) \cong (\mathrm{trcl}(B); \in)$ (since $\mathrm{trcl}(A) = \mathrm{trcl}(B) = 3$). But A is determined by the type of $\in$ on $\mathrm{trcl}(\{A\}) = \mathrm{trcl}(A) \cup \{A\}$:

Lemma I.9.38 *If $A \in WF$ and $(\mathrm{trcl}(A) \cup \{A\}; \in) \cong (\mathrm{trcl}(B) \cup \{B\}; \in)$, then $A = B$.*

Proof. We have $\mathrm{trcl}(A) \cup \{A\} = \mathrm{trcl}(B) \cup \{B\}$ by Lemma I.9.37. Then $A = B$, since, for $y \in WF$, y is the unique $\in$–maximal element of $\mathrm{trcl}(y) \cup \{y\}$. □

We may measure the "complexity" of a set $x \in WF$ by how many sets are needed in the construction of x from $\emptyset$ — that is, by the cardinality of $\mathrm{trcl}(x)$. The models $H(\kappa) = \{x \in WF : |\mathrm{trcl}(x)| < \kappa\}$ are used very frequently in set theory. We shall make this an official definition and discuss them further after we have discussed cardinalities; see Definition I.13.27.

A simple example of recursion is ordinal arithmetic; see Table I.1 on page 40. In particular, the table gives the standard definition of the ordinal exponent α^β using recursion on β:

$$\alpha^0 = 1 \quad ; \quad \alpha^{S(\beta)} = \alpha^\beta \cdot \alpha \quad ; \quad \alpha^\gamma = \sup_{\beta<\gamma}(\alpha^\beta) \text{ (for } \gamma \text{ a limit)}$$

The table also gives a recursion for $\alpha + \beta$ and $\alpha \cdot \beta$. Since we had already defined sum and product in another way (Definition I.8.4), these lines in the table are lemmas for us, but other texts (e.g., [71]) take them as definitions.

We conclude this section with a few exercises.

Lemma I.9.35 gave the conditions for the Mostowski function to be 1-1. The conditions for when the rank function is 1-1 are less important in practice, but it makes a good exercise:

Exercise I.9.39 *Assume that R is well-founded and set-like on A. Then* $\text{rank}_{A,R}$ *is 1-1 iff R^* is a total order of A.*

Recall that R^* must be well-founded, so it is a total order iff it is a well-order.

Exercise I.9.40 *If $R_1 \subseteq R_2$ are both well-founded and set-like on A, then* $\text{rank}_{A,R_1}(y) \leq \text{rank}_{A,R_2}(y)$ *for all $y \in A$. Also,* $\text{rank}_{A,R_1}(y) = \text{rank}_{A,R_2}(y)$ *if $R_1 \subseteq R_2 \subseteq (R_1)^*$.*

Exercise I.9.41 *Assume that R is well-founded and set-like on A, and that R is a transitive relation on A. Then* $\text{mos}_{A,R}(a) = \text{rank}_{A,R}(a)$ *for all $a \in A$.*

Exercise I.9.42 *Assume that R well-orders the set A. Then the three functions* $\text{mos}_{A,R}$, $\text{rank}_{A,R}$, *and the isomorphism of A onto an ordinal described in Theorem I.8.2, are all the same.*

Lemma I.9.24 shows that for all sets b: $b \in WF$ iff $b \subseteq WF$. The $\rightarrow$ direction just says that WF is a transitive class, but the $\leftarrow$ direction is a very strong closure property:

Exercise I.9.43 *Work in $ZF^- - P$. Let K be any class and assume that every subset of K is a member of K. Prove that $WF \subseteq K$.*

Hint. Consider an $\in$–minimal element of $WF \backslash K$. □

In particular, the Axiom of Foundation implies that $K = V$.

Our discussion of WF is a special case of a general notion of *well-founded initial segment*, which makes sense for an arbitrary set-like relation R on a class A. This occurs frequently when discussing total orders. The idea is that even if R fails to be well-founded on A, it may be well-founded below (in the sense of R^*) some elements of A, and these make up the well-founded initial segment.

Exercise I.9.44 *For any relation R on a class A, and $a \in A$: R is well-founded on* $\text{pred}_{A,R^*}(a)$ *iff R is well-founded on* $\{a\} \cup \text{pred}_{A,R^*}(a)$.

Definition I.9.45 *Let R be set-like on A. Define the* well-founded initial segment $WF_{A,R}$ *to be the set of all $a \in R$ such that R is well-founded on* $\text{pred}_{A,R^*}(a)$ *(equivalently, on* $\{a\} \cup \text{pred}_{A,R^*}(a)$*). For $a \in WF_{A,R}$, define* $\text{rank}(a) = \text{rank}_{A,R}(a) = \text{rank}_{\{a\} \cup \text{pred}_{A,R^*}(a)}(a)$.

Remark. If R is also well-founded, then we have already defined $\text{rank}_{A,R}(a)$, but this is the same as $\text{rank}_{\{a\} \cup \text{pred}_{A,R^*}(a)}(a)$ by Lemma I.9.18.

For example, let $A = \{1 - 2^{-n} : n \in \omega\} \cup [3,4] \subset \mathbb{R}$, ordered in the usual way. Then A is not well-ordered, but $WF_A = \{0, 1/2, 3/4, 7/8, \ldots 3\}$, which is well-ordered in type $\omega + 1$, after which the interval $(3,4]$ has no least element. For another example, let $A = \mathcal{P}(S)$, where S is an infinite set; partially order A by $\subsetneq$. Then WF_A is the set of all the finite subsets of S, with $\text{rank}(x) = |x| \in \omega$.

Exercise I.9.46 *Let R be set-like on A, and let $W = WF_{A,R}$ Then:*

1. *$A \backslash W$ has no R-minimal elements.*
2. *R is well-founded on W.*
3. *For $a \in W$, $\text{rank}_{A,R}(a) = \text{rank}_{W,R}(a)$.*
4. *For $a \in A$: $a \in W$ iff $\text{pred}_{A,R^*}(a) \subseteq W$ iff $\text{pred}_{A,R}(a) \subseteq W$.*

Exercise I.9.47 *Call a relation R on a set A* anti-transitive *iff it satisfies $\forall xyz \in A\,[xRy \wedge yRz \to x\not R z]$. Prove that if R is any well-founded relation on a set A, and all elements of A have finite rank, then there is a unique anti-transitive $H \subseteq R$ such that $H^* = R^*$. Then give a counter-example to this where R is a well-order of a countable set.*

Note that if R is also transitive (and hence a strict partial order), then $H^* = R$, and a picture of H (when A is finite) is called a *Hasse diagram* of R; it's a common way of picturing a partial order without drawing more arrows than are necessary. The prerequisite relation in a college catalogue is frequently anti-transitive. If trig is a prerequisite for calculus and calculus is a prerequisite for ODE, they don't need to tell you that trig is needed for ODE.

Exercise I.9.48 *The requirements for a Ph.D. in math at $\aleph U$ are only the courses $\boxed{\sigma}$ for each $\sigma \in \omega^{<\omega}$ (see Definition I.6.11). The prerequisites for $\boxed{\sigma}$ are all $\boxed{\tau}$ such that τ is obtained from σ by replacing one term n from τ by a finite (possibly empty) sequence of numbers less than n. For example, $\boxed{2,1,7,1}$ teaches you all about the sequence $(2,1,7,1)$, and has as prerequisites $\boxed{1,7,1}$, $\boxed{0,1,1,0,1,7,1}$, $\boxed{2,1,5,5,4,0,1,1}$, etc. $\boxed{\ }$ teaches you all about the empty sequence $\emptyset$, and has no prerequisite. No number is less than 0, so the only prerequisite for $\boxed{0}$ is $\boxed{\ }$.*

a. Prove that the prerequisite relation is well-founded; so, you can graduate in some ordinal number of semesters.

b. Compute the rank function; note that here, $\text{rank}(x)$ is the first semester in which you can possibly take course x.

Hint. Since you can prove well-foundedness by producing a rank function, you may wish to do (*b*) before doing (*a*). The rank function can be expressed using a sum of finite powers of ω; see also Exercise I.9.54. Note that the rank of a sequence is the same as the rank of any permutation of that sequence. □

When we use the Mostowski function, we need to assume that R is extensional on the set A to get (A, R) isomorphic to $\in$ on a *transitive* set M. But, if we don't care about M being transitive, then we don't need R to be extensional.

Exercise I.9.49 *Let R be well-founded and set-like on A. Prove that there is a class M and an isomorphism from (A, R) onto $(M, \in)$.*

Hint. Define the isomorphism F by recursion; then M is just $\mathrm{ran}(F)$. One way is: $F(y) = \{F(x) : x \in y\downarrow\} \cup \{\langle 0, y\rangle, \langle 1, y\rangle, \langle 2, y\rangle\}$; this is like the Mostowski function, but we throw in some extra junk to make sure that F is 1-1.

First prove that F is 1-1; then prove that F is an isomorphism. □

Exercise I.9.50 *Assume that R is set-like on A and not well-founded. Define a $G(x, s)$ that is an explicit counter-example to Theorem I.9.11.*

Hint. The rank function $F(y) = \bigcup\{S(F(x)) : x \in y\downarrow\}$ doesn't quite work, without assuming Foundation. For example, say R is the usual total order on $\mathbb{Z}$. It is consistent with ZFC^- that there is a transitive "fake ordinal" t that is totally ordered by $\in$ and is isomorphic to $\mathbb{Z}$ (see Exercise II.9.4); then F could be the isomorphism. Instead, consider $\widehat{S}(y)$, which is $y+1$ when y is an ordinal and 0 otherwise. □

The next exercise is a warmup for the Reflection Theorem (see Section II.5). It shows that using Foundation, you can drop the "set-like" in Theorem I.9.3:

Exercise I.9.51 *Work in ZF. Assume that R is well-founded on A, and that X is a non-empty sub-class of A. Prove that X has an R-minimal element.*

Hint. We know that every non-empty sub-*set* of A has a minimal element. Assume that the class X does not have a minimal element. Use $V = WF$ to prove that this situation *reflects* to some set $X \cap R(\gamma)$.

For each $x \in X$, let $f(x) = \min\{\mathrm{rank}(z) : z \in X \wedge zRx\}$. Then get γ such that $X \cap R(\gamma) \neq \emptyset$ and $\forall x \in X \cap R(\gamma)\,[f(x) < \gamma]$. □

Now that recursion has given us a rigorous definition of ordinal exponentiation, we list a few exercises on exponentiation:

Exercise I.9.52 *Verify the statements made about exponentiation in Table I.1 on page 40.*

Exercise I.9.53 *Let γ be a limit ordinal. Show that the following are equivalent:*

a. $\forall \alpha, \beta < \gamma\,[\alpha + \beta < \gamma]$.
b. $\forall \alpha < \gamma\,[\alpha + \gamma = \gamma]$.
c. $\forall X \subseteq \gamma\,[\mathrm{type}(X) = \gamma \vee \mathrm{type}(\gamma \backslash X) = \gamma]$.
d. $\exists \delta\,[\gamma = \omega^\delta]$.

Such γ are called *indecomposable*. The δ in (d) might equal γ. The least γ such that $\gamma = \omega^\gamma$ is called $\epsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$

Exercise I.9.54 *Prove the* Cantor Normal Form Theorem*: Each ordinal $\alpha > 0$ can be represented uniquely in the form:*

$$\alpha = \omega^{\beta_l} \cdot n_1 + \cdots + \omega^{\beta_k} \cdot n_k \quad ,$$

where $k, n_1, \ldots, n_k \in \omega \backslash \{0\}$ and $\alpha \geq \beta_1 > \cdots > \beta_k$.

If $\alpha = \beta_1$ then $\alpha = \omega^\alpha$ and $k = n_1 = 1$.

The following exercise yields a combinatorial definition of ordinal exponentiation that does not use recursion:

Exercise I.9.55 *Let $F(\alpha, \beta)$ be the set of all functions f from β to α such that $\{\xi < \beta : f(\xi) \neq 0\}$ is finite. For $f, g \in F(\alpha, \beta)$ with $f \neq g$, define $f \lhd g$ iff $f(\xi) < g(\xi)$, where ξ is the* largest *ordinal such that $f(\xi) \neq g(\xi)$. Prove that $\lhd$ well-orders $F(\alpha, \beta)$ in type α^β.*

Although our "official" definition of ordinal sum and product did not use recursion, some of the facts about them (such as the one labeled "Division" in Table I.1 on page 40) are most easily proved by induction. Also, using induction, one can prove the transfinite version of the Euclidean algorithm:

Exercise I.9.56 *If α, β are non-zero ordinals, then there is a largest ξ such that $\xi \mid \alpha$ and $\xi \mid \beta$. Here, $\xi \mid \alpha$ iff $\alpha = \xi \cdot \mu$ for some μ.*

Hint. Induct on $\max(\alpha, \beta)$. □

I.10 Cardinalities

This section is developed in $ZF^- - P$; that is, we assume all the axioms of set theory except Foundation, Power Set, and Choice; any use of these three axioms (except in informal examples) will be mentioned explicitly. Lacking the Power Set, we cannot actually produce any cardinalities larger than $\aleph_0$, and lacking the Axiom of Choice, we cannot in general define the size $|X|$ of an infinite set X as von Neumann cardinal (some $\aleph_\alpha$). However, we can develop some basic facts about cardinalities, including properties that the uncountable cardinals must have if they do exist.

As we pointed out in Section I.6 (Definition I.6.18 and Lemma I.6.19), we use injections and bijections to compare the sizes of sets. A basic result about this concept is called the Schröder–Bernstein Theorem (see below), although the next lemma, which contains the meat of the theorem, was proved earlier by Dedekind; see [57] Lemma I.11.4 or [16] for more details:

Lemma I.10.1 *If $B \subseteq A$ and $f : A \xrightarrow{1-1} B$ then $A \approx B$.*

Proof. We have $f^n : A \xrightarrow{1-1} A$ for $n \in \omega$ (see Definition I.9.1). Then

$$f^0(A) = A \supseteq f^0(B) = B \supseteq f^1(A) \supseteq f^1(B) \supseteq f^2(A) \supseteq f^2(B) \supseteq f^3(A) \supseteq \cdots$$

Let $H_n = f^n(A) \backslash f^n(B)$, let $K_n = f^n(B) \backslash f^{n+1}(A)$. Then for each n, f maps H_n bijectively onto H_{n+1} and f maps K_n bijectively onto K_{n+1}. Let $P = \bigcap_{n<\omega} f^n(A) = \bigcap_{n<\omega} f^n(B)$. Then

$$\begin{array}{rcccccc} A = & P & \cup & H_0 \cup H_1 \cup H_2 \cdots & \cup & K_0 \cup K_1 \cup K_2 \cdots \\ B = & P & \cup & H_1 \cup H_2 \cup H_3 \cdots & \cup & K_0 \cup K_1 \cup K_2 \cdots \end{array}$$

and the sets P, H_m, K_n are all pairwise disjoint. Let $g(x) = f(x)$ for $x \in \bigcup_n H(n)$, and $g(x) = x$ for $x \in P \cup \bigcup_n K(n)$. Then $g : A \xrightarrow[\text{onto}]{1\text{-}1} B$. □

Theorem I.10.2 (Schröder–Bernstein Theorem) *$A \approx B$ iff $A \preccurlyeq B$ and $B \preccurlyeq A$.*

Proof. Given $f : A \xrightarrow[\text{onto}]{1\text{-}1} B$, we have $A \preccurlyeq B$ using f, and $B \preccurlyeq A$ using f^{-1}. For the nontrivial direction, we have $f : A \xrightarrow{1\text{-}1} B$ and $h : B \xrightarrow{1\text{-}1} A$, and we need to prove that $A \approx B$. Let $\widehat{B} = h(B) \subseteq A$; then $h : B \xrightarrow[\text{onto}]{1\text{-}1} \widehat{B}$, so $\widehat{B} \approx B$. Also, $h \circ f : A \xrightarrow{1\text{-}1} \widehat{B}$, so $A \approx \widehat{B}$ follows by Lemma I.10.1. □

Definition I.10.3 *$X \prec Y$ iff $X \preccurlyeq Y$ and $Y \not\preccurlyeq X$.*

By the Schröder-Bernstein Theorem, $X \prec Y$ iff $X \preccurlyeq Y$ and $X \not\approx Y$. To state Cantor's theorem $A \prec \mathcal{P}(A)$ without using the Power Set Axiom, we say:

Theorem I.10.4 (Cantor) *If $\mathcal{P}(A)$ exists, then $A \prec \mathcal{P}(A)$.*

Proof. $A \preccurlyeq \mathcal{P}(A)$ because the map $x \mapsto \{x\}$ defines an injection from A to $\mathcal{P}(A)$. $A \not\approx \mathcal{P}(A)$ because if $f : A \xrightarrow[\text{onto}]{1\text{-}1} \mathcal{P}(A)$, then Cantor's diagonal argument would produce a contradiction: Let $D = \{x \in A : x \notin f(x)\}$, and then fix $c \in A$ with $f(c) = D$; then $c \in D \leftrightarrow c \notin D$. □

Definition I.10.5 *A is* countable *iff $A \preccurlyeq \omega$. "*uncountable*" means "not countable". A is* countably infinite *iff A is countable and infinite.*

So, $\mathcal{P}(\omega)$ is uncountable if it exists.

The following continuation of Lemma I.8.5 is useful when dealing with sizes of sets:

Lemma I.10.6

1. $B \subseteq \alpha \to \text{type}(B; \in) \leq \alpha$.
2. *If $B \preccurlyeq \alpha$ then $B \approx \delta$ for some $\delta \leq \alpha$.*
3. *If $\alpha \leq \beta \leq \gamma$ and $\alpha \approx \gamma$ then $\alpha \approx \beta \approx \gamma$.*

Note that (3) is clear from Lemma I.10.1 (set $A = \gamma$ and $B = \beta$). By (3), the ordinals come in blocks of the same *size* or *cardinality*. The first ordinal in its block is called a *cardinal* (or a cardinal number).

Definition I.10.7 *A (von Neumann)* cardinal *is an ordinal α such that $\xi \prec \alpha$ for all $\xi < \alpha$.*

Since $\xi < \alpha \to \xi \subseteq \alpha \to \xi \preccurlyeq \alpha$, an ordinal α fails to be a cardinal iff there is some $\xi < \alpha$ with $\xi \approx \alpha$. It is then easy to check the following; for (2), one can use Lemma I.7.20.

Theorem I.10.8

1. *Every cardinal* $\geq \omega$ *is a limit ordinal.*
2. *Every natural number is a cardinal.*
3. *If* A *is a set of cardinals, then* $\sup(A)$ *is a cardinal.*
4. ω *is a cardinal.*

Now, we would like to define $|A|$ to be the cardinal κ such that $A \approx \kappa$. However, to prove that there *is* such a κ requires well-ordering A, which requires the Axiom of Choice (AC) (see Section I.12).

Definition I.10.9 *A set* A *is* well-orderable *iff there is some relation* R *that well-orders* A.

Lemma I.10.10 *A can be well-ordered in type* α *iff there is a bijection* $f : A \xrightarrow[\text{onto}]{1\text{-}1} \alpha$. *Hence,* A *is well-orderable iff* $A \approx \alpha$ *for some ordinal* α.

Definition I.10.11 *If* A *is well-orderable, then* $|A|$ *is the least ordinal* α *such that* $A \approx \alpha$.

Clearly, every set of ordinals is well-orderable. In particular, $|\alpha|$ is defined for all α. It is easy to see directly that $|\omega + \omega| = \omega$; more generally, ordinal sum, product, and exponentiation applied to *infinite* ordinals does not raise the cardinality (see [57], Exercise I.11.34).

Lemma I.10.12 *If* κ *is a cardinal and* B *is a non-empty set, then* $B \preccurlyeq \kappa$ *iff there is a function* $f : \kappa \xrightarrow{\text{onto}} B$.

In particular, with $\kappa = \omega$, a non-empty B is countable by Definition I.10.5 (i.e., $B \preccurlyeq \omega$) iff there is a function $f : \omega \xrightarrow{\text{onto}} B$; that is, you can count B using the natural numbers.

Lemma I.10.13 *Assume that* A, B *are well-orderable. Then:*

- ☞ $|A|$ *is a cardinal.*
- ☞ $A \approx B$ *iff* $|A| = |B|$.
- ☞ $A \preccurlyeq B$ *iff* $|A| \leq |B|$.
- ☞ $A \prec B$ *iff* $|A| < |B|$.

In view of this, the notations $A \preccurlyeq B$, $A \approx B$, $A \prec B$ are used mainly when working in set theory without the Axiom of Choice. When using AC, the terminology $|A| \leq |B|$, $|A| = |B|$, $|A| < |B|$ is more standard.

We have already defined "finite" (Definition I.7.22). An equivalent definition, in terms of cardinalities, is given by:

Lemma I.10.14 *A is finite iff* A *is well-orderable and* $|A| < \omega$.

In the case of *finite* ordinals, ordinal sum, product, and exponentiation are important in computing cardinalities.

Lemma I.10.15 *Assume that A and B are finite, with $m = |A|$ and $n = |B|$. Then $A \cup B$ is finite, with $|A \cup B| \leq m + n$ (equality holding iff $A \cap B = \emptyset$), and $A \times B$ is finite, with $|A \times B| = m \cdot n$. Also, ${}^B A$ exists and is finite and $|{}^B A| = m^n$, and $\mathcal{P}(B)$ exists and is finite and $|\mathcal{P}(B)| = 2^n$.*

Proof. Induct on $|B|$, using the recursive computations in Table I.1. □

Note that we are not assuming the Power Set Axiom, but Lemma I.10.15 shows that we do not need Power Set to prove that ${}^B A$ and $\mathcal{P}(B)$ exist for finite sets (see also Exercise I.7.27). So, the proof, using Power Set, that all $R(\delta)$ are sets (Lemma I.9.30) works for $\delta \leq \omega$:

Lemma I.10.16 *$R(\delta)$ is a set for all $\delta \leq \omega$ and $R(\delta)$ is finite for all $\delta < \omega$.*

It is easily seen now that $R(\omega)$ is countably infinite; i.e., it is well-orderable and $|R(\omega)| = \omega$. In fact, the following elegant computable bijection from $R(\omega)$ onto ω is due to Ackermann in 1937; for more details, see [3] or [57].

Lemma I.10.17 *Define $E \subseteq \omega \times \omega$ by: $nEm \leftrightarrow 2 \nmid \lfloor m2^{-n} \rfloor$ (equivalently, there is a 1 in place n in the binary representation of m (counting from the right; so $43 = 101011_b$ has a 1 in places 0,1,3,5)). Then $(R(\omega); \in) \cong (\omega; E)$, with the isomorphism Γ defined recursively by $\Gamma(y) = \sum\{2^{\Gamma(x)} : x \in y\}$. For example,*

$\Gamma(\emptyset) = \sum \emptyset = 0$
$\Gamma(1) = \Gamma(\{\emptyset\}) = 2^0 = 1$
$\Gamma(\{1\}) = 2^1 = 2$
$\Gamma(2) = 2^0 + 2^1 = 3$
$\Gamma(\{\{1\}, 0\}) = 2^2 + 2^0 = 5$
$\Gamma(\{\{\{1\}, 0\}, 2, 1, 0\}) = 2^5 + 2^3 + 2^1 + 2^0 = 43$.

Γ^{-1} is the Mostowski function $\mathrm{mos}_{\omega,E}$.

Another view of the set $R(\omega)$ is given by:

Lemma I.10.18 *$x \in R(\omega)$ iff $x \in WF$ and $\mathrm{trcl}(x)$ is finite.*

Proof. For $\rightarrow$: If $x \in R(\omega)$, then $x \in R(n)$ for some finite n. Since $R(n)$ is transitive, we have $\mathrm{trcl}(x) \subseteq R(n)$, and each $R(n)$ is finite.

For $\leftarrow$: Let $n = \{\mathrm{rank}(z) : z \in \mathrm{trcl}(x)\}$. Then n is finite, and $n \in ON$ by Lemma I.9.16, so $n \in \omega$. Since $\mathrm{rank}(z) < n$ for all $z \in x \subseteq \mathrm{trcl}(x)$, $x \subseteq R(n)$ and hence $x \in R(n+1)$. □

In the $\leftarrow$, we cannot drop the assumption that $x \in WF$ because it is consistent with ZFC^- to have a set x such that $x = \{x\}$ (see Exercise II.9.5); then $x = \mathrm{trcl}(x)$, which is finite, but $x \notin WF$.

Definition I.10.19 *$HF = R(\omega)$ is called the set of* hereditarily finite sets.

If you think of $\in$ as a directed graph and draw a picture of x together with $\mathrm{trcl}(x)$ (see Figure I.1 on page 47), then x is the set of children of x, while $\mathrm{trcl}(x)$ is the set of all descendents of x; hence the name "hereditarily finite".

Informally, all of finite mathematics lives in HF. It is clear that any set that can be displayed explicitly as a finite expression using $\emptyset$ and brackets (i.e., $\{,\}$) has finite transitive closure, and hence is in HF. This includes the finite ordinals, plus, e.g., ${}^m n$ whenever $m, n \in \omega$, plus anything else used in finite combinatorics. For example, every finite group is isomorphic to one of the form $\langle n, \cdot \rangle$ where $n \in \omega$ and $\cdot \subseteq (n \times n) \times n$ (since the product, $\cdot$, is a function from $n \times n \to n$). The rank of this $\langle n, \cdot \rangle$ is finite (in fact, $n + 6$).

For generalizations of HF to larger cardinalities, see Definition I.13.27 below; for example, HC is the set of all x whose transitive closure is countable, and $|HC| = 2^{\aleph_0}$. We shall return to this subject after we have reviewed some basic facts about computing with infinite cardinalities.

I.11 Uncountable Cardinalities

This section works in ZF^-; that is, we assume all the axioms of set theory except Foundation and Choice. We shall use the Power Set Axiom to produce uncountable (von Neumann) cardinals. We have already produced uncountable sets, such as $\mathcal{P}(\omega)$, by Cantor's Theorem I.10.4. Assuming the Axiom of Choice, every set is well-orderable (see Section I.12), so that $|\mathcal{P}(\omega)|$ is an uncountable cardinal. However, by the following argument of Hartogs, uncountable cardinals exist without using AC.

Theorem I.11.1 (Hartogs, 1915) *For every set A, there is a cardinal κ such that $\kappa \not\preccurlyeq A$.*

Proof. Let W be the set of pairs $(X, R) \in \mathcal{P}(A) \times \mathcal{P}(A \times A)$ such that $R \subseteq X \times X$ and R well-orders X. So, W is the set of all well-orderings of all subsets of A. Note that $\alpha \preccurlyeq X$ iff $\alpha = \mathrm{type}(X; R)$ for some $(X, R) \in W$. Applying the Replacement Axiom, let $\kappa = \sup\{\mathrm{type}(X, R) + 1 : (X, R) \in W\}$. Then κ is a cardinal and $\kappa > \alpha$ whenever $\alpha \preccurlyeq A$, so $\kappa \not\preccurlyeq A$. □

Definition I.11.2 *For each ordinal α: α^+ is the least cardinal greater than α.*

Note that α^+ is the same as the κ produced in the proof of Theorem I.11.1 with $A = \alpha$. By Theorem I.10.8, we have:

Lemma I.11.3 *For ordinals, α: $\alpha^+ = \alpha + 1$ when $\alpha < \omega$, while $\alpha^+ > \alpha + 1$. when $\alpha \geq \omega$.*

Applying transfinite recursion:

Definition I.11.4 *The $\aleph_\xi = \omega_\xi$ are defined by recursion on ξ by:*

- $\aleph_0 = \omega_0 = \omega$.
- $\aleph_{\xi+1} = \omega_{\xi+1} = (\aleph_\xi)^+$.
- $\aleph_\eta = \omega_\eta = \sup\{\aleph_\xi : \xi < \eta\}$ *when η is a limit ordinal.*

Frequently, "$\aleph_\xi$" is used when talking about cardinalities and "ω_ξ" is used when talking about order types.

Lemma I.11.5 $\xi < \zeta \to \aleph_\xi < \aleph_\zeta$. *$\kappa$ is an infinite cardinal iff $\kappa = \aleph_\xi$ for some ξ.*

We already know that $\alpha \times \alpha$ can be well-ordered lexicographically, so that $|\alpha \times \alpha|$ is defined. The lexicographic order type of $\alpha \times \alpha$ is $\alpha \cdot \alpha$, which is greater than α whenever $\alpha \geq 2$. However, the cardinality is the same as that of α whenever α is infinite. We prove this, without using the Axiom of Choice, by defining a slightly more complicated well-order of pairs of ordinals:

Theorem I.11.6 *If $\alpha \geq \omega$ then $|\alpha \times \alpha| = |\alpha|$. Hence, if $\kappa \geq \omega$ is a cardinal, then $|\kappa \times \kappa| = \kappa$.*

Proof. Define $(\xi_1, \xi_2) \lhd (\eta_1, \eta_2)$ to hold iff either $\max(\xi_1, \xi_2) < \max(\eta_1, \eta_2)$ or we have both $\max(\xi_1, \xi_2) = \max(\eta_1, \eta_2)$ and (ξ_1, ξ_2) precedes (η_1, η_2) lexicographically. It is easy to verify that $\lhd$ well-orders $ON \times ON$ (in the same sense that $\in$ well-orders ON; see Theorem I.7.11). If S is a subset of $ON \times ON$, then type(S) denotes type$(S, \lhd)$. For example,

$$(0,0) \lhd (0,1) \lhd (1,0) \lhd (1,1) \lhd (0,2) \lhd (1,2) \lhd \cdots$$
$$(0,\omega) \lhd (1,\omega) \lhd \cdots (\omega,0) \lhd (\omega,1) \lhd \cdots (\omega,\omega) \lhd \cdots \quad ,$$

so type$(\omega \times \omega) = \omega$ and type$((\omega + 1) \times (\omega + 1)) = \omega \cdot 3 + 1$.

In general, type$(\kappa \times \kappa) = \kappa$ whenever κ is an infinite cardinal; this is easily proved by transfinite induction (Theorem I.9.3); for details, see [57]. It follows that $|\kappa \times \kappa| = \kappa$ whenever κ is an infinite cardinal.

When α is any infinite ordinal, we let $\kappa = |\alpha|$, and then $\alpha \times \alpha \approx \kappa \times \kappa \approx \kappa \approx \alpha$. □

For all ordinals, type$(\alpha \times \alpha, \lhd) \geq \alpha$, since $\alpha \times \alpha$ contains the subset $\{0\} \times \alpha$ ordered in type α. As an exercise in ordinal arithmetic:

Exercise I.11.7 *For $\alpha \geq \omega$, show that* type$(\alpha \times \alpha, \lhd) = \alpha$ *iff $\alpha = \omega^\mu$, where μ is either* 1 *or an infinite indecomposable ordinal (see Exercise I.9.53).*

Exercise I.11.8 *Applying transfinite recursion, define δ_ξ by:*

- $\delta_0 = 0$.
- $\delta_{\xi+1} = \aleph_{\delta_\xi}$.
- $\delta_\eta = \sup\{\delta_\xi : \xi < \eta\}$ *when η is a limit ordinal.*

Show that $\aleph_{\delta_\eta} = \delta_\eta$ for all limit η.

It is consistent with *ZFC* to have the continuum $\mathfrak{c}$ be δ_{ω_1}; then $\mathfrak{c} = \aleph_{\mathfrak{c}}$ (see Corollary IV.3.14). One cannot have $\mathfrak{c} = \delta_\omega$ by König's Theorem I.13.12.

Lemma I.11.9 ${}^A2 \approx \mathcal{P}(A)$.

Proof. Associate a subset of A with its characteristic function. □

It is also easy to see:

Lemma I.11.10 *If* $A \preccurlyeq B$ *and* $\emptyset \neq C \preccurlyeq D$ *then* ${}^AC \preccurlyeq {}^BD$. *If* $2 \preccurlyeq C$ *then* $A \prec \mathcal{P}(A) \preccurlyeq {}^AC$.

Lemma I.11.11

1. ${}^C({}^BA) \approx {}^{C\times B}A$
2. ${}^{(B\cup C)}A \approx {}^BA \times {}^CA$ *if* B, C *are disjoint.*

I.12 The Axiom of Choice (AC)

This section works in $ZF^- - P$. In fact, the Power Set Axiom will be important for some of our results, but we shall mention it explicitly when we use it.

There are many known equivalents of *AC*: see Howard and Rubin [47]. We mention only the seven forms listed in [57]:

Theorem I.12.1 *The following all follow from* (3), *and they are all equivalent, assuming the Power Set Axiom:*

1. *The Axiom of Choice (as in Section I.3).*
2. *Every family of non-empty sets has a choice function.*
3. AC^+ *(every set can be well-ordered).*
4. $\forall xy(x \preccurlyeq y \vee y \preccurlyeq x)$.
5. *Tukey's Lemma.*
6. *The Hausdorff Maximal Principle.*
7. *Zorn's Lemma.*

The meaning of items (3) and (4) should be clear from Definitions I.6.21 and I.6.18 respectively. Regarding (1) (2):

Definition I.12.2 *Let* F *be a family of non-empty sets. A* choice function *for* F *is a function* g *with* $\mathrm{dom}(g) = F$ *such that* $g(x) \in x$ *for all* $x \in F$. *A* choice set *for* F *is a set* C *such that* $C \cap x$ *is a singleton set for all* $x \in F$.

The version of *AC* in Section I.3 (Axiom 9) says that whenever F is a *disjoint* family of non-empty sets, there is a choice set C for F. In practice, it is often more convenient to use a choice function, since one may need to choose elements from sets that are not disjoint.

We shall state the definitions of (5), (6), and (7), and outline the proof of the theorem, referring to [57] for more details. Our outline will be contained in a sequence of lemmas, pointing out where the Power Set Axiom is used. This is of interest because, working in *ZFC*, we frequently consider models M for $ZF-P$. What does $M \models AC$ mean? Although it is often easy to verify $M \models (1)$, this is not the form commonly used in mathematical arguments, and it would be nice to know that all seven forms are true in M. In fact, we shall usually verify $M \models AC^+$, and the proofs outlined below show, without using the Power Set Axiom, that the other forms listed follow from (3). Although we used (1) as our "official" version because one can state it with very few defined notions, it is really version (3) that is the "basic" version of AC in work on independence proofs.

Lemma I.12.3 (1) $\leftrightarrow$ (2), (3) $\rightarrow$ (2), *and* (3) $\rightarrow$ (4).

Proof. For (2) $\rightarrow$ (1): Given a *disjoint* family F of non-empty sets: If g is a choice function for F, then $C = \{g(x) : x \in F\}$ is a choice set for F.

For (1) $\rightarrow$ (2): Given *any* family F of non-empty sets, let $F^* = \{\{x\} \times x : x \in F\}$. Then F^* is a disjoint family. If C is any choice set for F^*, then C is a set of ordered pairs, and is a function (as per Definition I.6.3), and $\mathrm{dom}(C) = F$, and C is a choice function for F.

For (3) $\rightarrow$ (2): If R well-orders $\bigcup F$, we can define a choice function g by letting $g(x)$ be the R-least element of x.

For (3) $\rightarrow$ (4): If x and y can be well-ordered, then $|x|$ and $|y|$ are von Neumann cardinals, and either $|x| \leq |y|$ (and hence $x \preccurlyeq y$) or $|y| \leq |x|$ (and hence $y \preccurlyeq x$). $\square$

Lemma I.12.4 (1) $\leftrightarrow$ (2) $\leftrightarrow$ (3) $\leftrightarrow$ (4), assuming *the Power Set Axiom.*

Proof. For (4) $\rightarrow$ (3): By Theorem I.11.1, fix κ so that $\kappa \not\preccurlyeq A$, so (4) implies that $A \preccurlyeq \kappa$, so A can be well-ordered; if $h : A \xrightarrow{1-1} \kappa$ define $a < b$ iff $h(a) < h(b)$.

For (2) $\rightarrow$ (3): Informally, given any set A, we well-order A by listing A in a transfinite sequence, $a_0, a_1, a_2, a_3, \ldots, a_\alpha, \ldots\ldots$, using a choice function g to choose $a_\alpha \in A \setminus \{a_\xi : \xi < \alpha\}$. But, since we are not given the listing in advance, how do we know what $\mathrm{dom}(g)$ should be? Using the Power Set Axiom, we let g be a choice function for all of $\mathcal{P}(A)\setminus\{\emptyset\}$. Fix any $S \notin A$. Think of S as signifying "Stop". Fix κ so that $\kappa \not\preccurlyeq A$. By transfinite recursion (Theorem I.9.11), fix $f : \kappa \rightarrow A \cup \{S\}$ so that for all $\alpha \in ON$: $f(\alpha) = g(A \setminus \{f(\xi) : \xi < \alpha\})$ if $A \setminus \{f(\xi) : \xi < \alpha\}$ is non-empty; otherwise, $f(\alpha) = S$.

Observe that $f(\xi) \neq f(\alpha)$ whenever $\xi < \alpha$ and $f(\alpha) \neq S$. If $f(\alpha) \neq S$ for all α, then $f : \kappa \xrightarrow{1-1} A$, which is impossible, since $\kappa \not\preccurlyeq A$. Now, let α be least such that $f(\alpha) = S$, and note that $f \restriction \alpha : \alpha \xrightarrow[\text{onto}]{1-1} A$; thus, A can be well-ordered in type α as above, using $h = f^{-1}$. $\square$

The last three principles are maximal principles. The following exercise will illustrate the plusses and minuses of using them:

Exercise I.12.5 *Assume AC^+, and* do not *assume the Power Set Axiom. Let A be a vector space over some field K. Prove that A has a basis.*

Hint. List $A\backslash\{0\}$ as $\{a_\alpha : \alpha < \kappa\}$, and let B be the set of all "vectors" a_α such that a_α is not in the linear span of $\{a_\xi : \xi < \alpha\}$. □

The more "standard" method of doing this exercise, common in algebra texts, is to let $\mathcal{F} \subseteq \mathcal{P}(A)$ be the family of all linearly independent subsets of A, and then apply Zorn's Lemma or Tukey's Lemma to get a maximal element $B \in \mathcal{F}$, which is then a basis. This proof can be understood without any knowledge of ordinals and recursion. There are two drawbacks to this method. Foundationally, you need the Power Set Axiom to form $\mathcal{F}$. Pedagogically, it is not clear why Zorn's Lemma or Tukey's Lemma should be true, unless you are familiar with ordinals and recursion.

We remark that Blass [10] has proved a converse to Exercise I.12.5; that is, working in ZF plus the statement that every vector space has a basis, one can prove AC. His proof really uses all of ZF, including Foundation. The appearance of Foundation in such a result may seem a bit strange; for a much simpler example requiring Foundation, AC follows from the statement that the power set of every well-ordered set can be well-ordered; see Exercise I.12.17.

Notation I.12.6 *When we don't assume the Power Set Axiom, if $\mathcal{F}, A$ are any sets, then $\mathcal{F} \subseteq \mathcal{P}(A)$ means $\forall X \in \mathcal{F}\,[X \subseteq A]$.*

This notation is consistent with our earlier notation for dealing with classes such as ON; here, $\mathcal{P}(A)$ is possibly a proper class.

Definition I.12.7 *If $\mathcal{F} \subseteq \mathcal{P}(A)$, then $X \in \mathcal{F}$ is* maximal *in $\mathcal{F}$ iff it is maximal with respect to the relation $\subsetneq$ (see Definition I.6.21); that is, X is not a proper subset of any set in $\mathcal{F}$.*

Definition I.12.8 *$\mathcal{F} \subseteq \mathcal{P}(A)$ is of* finite character *iff for all $X \subseteq A$: $X \in \mathcal{F}$ iff every finite subset of X is in $\mathcal{F}$.*

For example, if A is a vector space, the family of all linearly independent subsets is of finite character.

Definition I.12.9 Tukey's Lemma *is the assertion that whenever the set $\mathcal{F} \subseteq \mathcal{P}(A)$ is of* finite character *and $X \in \mathcal{F}$, there is a maximal $Y \in \mathcal{F}$ such that $X \subseteq Y$.*

This is often called the Teichmüller–Tukey Lemma. It was discovered independently by John Tukey and Oswald Teichmüller in the 1930s.

Definition I.12.10 *Let $<$ be a strict partial order of a set A. Then $C \subseteq A$ is a* chain *iff C is totally ordered by $<$; C is a* maximal chain *iff in addition, there are no chains $X \supsetneq C$.*

Definition I.12.11 The Hausdorff Maximal Principle *asserts that whenever* $<$ *is a strict partial order of a set* A*, there is a maximal chain* $C \subseteq A$.

Definition I.12.12 Zorn's Lemma *is the assertion that whenever* $<$ *is a strict partial order of a set* A *satisfying*

(☠) *For all chains* $C \subseteq A$ *there is some* $b \in A$ *such that* $x \leq b$ *for all* $x \in C$,

then for all $a \in A$*, there is a maximal* (*see Definition I.6.21*) $b \in A$ *with* $b \geq a$.

Lemma I.12.13 (3) $\rightarrow$ (6) $\rightarrow$ (7) $\rightarrow$ (5).

Proof. For (3) $\rightarrow$ (6): Let $\lhd$ be a strict partial order A. Applying (3), list A as $\{a_\alpha : \alpha < \kappa\}$. Recursively define $C \subseteq A$ so that $a_\alpha \in C$ iff $\{a_\alpha\} \cup \{a_\xi : \xi < \alpha\}$ is a chain. It is easily checked that C is a maximal chain in $\lhd$.

To formalize this recursive definition of a *set* C: Define $f : \kappa \rightarrow 2$ by transfinite recursion so that $f(\alpha) = 1$ iff $\{a_\alpha\} \cup \{a_\xi : \xi < \alpha \wedge f(\xi) = 1\}$ is a chain. This defines $f(\alpha)$ from $f\restriction\alpha$. Then $C = \{a_\xi : \xi < \kappa \wedge f(\xi) = 1\}$.

For (6) $\rightarrow$ (7): Fix $a \in A$, and let $A' = \{x \in A : x \geq a\}$. Let $C \subseteq A'$ be a maximal chain. Fix $b \in A'$ such that $x \leq b$ for all $x \in C$. Then b is a maximal element of A and $b \geq a$.

For (7) $\rightarrow$ (5): Fix $X \in \mathcal{F}$. Apply Zorn's Lemma to the set $\mathcal{F}' = \{Y \in \mathcal{F} : X \subseteq Y\}$, partially ordered by $\subsetneq$. A maximal $Y \in \mathcal{F}'$ is also maximal in $\mathcal{F}$. □

Proof of Theorem I.12.1. It is sufficient to assume the Power Set Axiom and prove that (5) $\rightarrow$ (3). So, fix a set A, and then, by Theorem I.11.1, fix κ so that $\kappa \not\preccurlyeq A$. Let $\mathcal{F}$ be the set of all $f \in \mathcal{P}(\kappa \times A)$ such that f is (the graph of) a 1-1 function from a subset of κ into A. Then $\mathcal{F}$ is of finite character, so by Tukey's Lemma, let $f \in \mathcal{F}$ be maximal.

If $\mathrm{ran}(f) = A$, then we well-order A by defining $f(\xi) < f(\eta)$ iff $\xi < \eta$.

If $\mathrm{ran}(f) \neq A$, fix $a \in A \backslash \mathrm{ran}(f)$ and fix $\alpha \in \kappa \backslash \mathrm{dom}(f)$ (which is non-empty because $\kappa \not\preccurlyeq A$. Then $f \cup \{(\alpha, a)\}$ contradicts maximality of f. □

Some additional applications of the Axiom of Choice:

By AC, a countable union of countable sets is countable (Theorem I.12.14 below). It is consistent with ZF that $\mathbb{R}$ is a countable union of countable sets; it is also consistent that ω_1 is a union (= supremum) of a countable set of countable ordinals. See [48] for more on consistency results involving $\neg AC$.

Theorem I.12.14 (ZFC^-) *Let* κ *be an infinite cardinal. If* $\mathcal{F}$ *is a family of sets with* $|\mathcal{F}| \leq \kappa$ *and* $|X| \leq \kappa$ *for all* $X \in \mathcal{F}$*, then* $|\bigcup \mathcal{F}| \leq \kappa$.

Proof. Assume $\mathcal{F} \neq \emptyset$ (otherwise the result is trivial) and $\emptyset \notin \mathcal{F}$ (since removing $\emptyset$ from $\mathcal{F}$ does not change the union). Then (see Lemma I.10.12), fix $f : \kappa \overset{\text{onto}}{\rightarrow} \mathcal{F}$. Likewise, for each $B \in \mathcal{F}$, there are functions $g : \kappa \overset{\text{onto}}{\rightarrow} B$. By AC, choose $g_\alpha : \kappa \overset{\text{onto}}{\rightarrow} f(\alpha)$ for all $\alpha < \kappa$ (to do this, well-order $\mathcal{S} := {}^\kappa(\bigcup \mathcal{F})$, and

let g_α be the least $g \in \mathcal{S}$ with $\mathrm{ran}(g) = f(\alpha)$). This defines $h : \kappa \times \kappa \xrightarrow{\text{onto}} \bigcup \mathcal{F}$, where $h(\alpha, \beta) = g_\alpha(\beta)$. Since $|\kappa \times \kappa| = \kappa$ (see Theorem I.11.6), we can map κ onto $\bigcup \mathcal{F}$, so $|\bigcup \mathcal{F}| \leq \kappa$ (see Lemma I.10.12). □

Another elementary application of AC is that one can use surjections instead of injections to compare the sizes of sets (i.e., to define $B \preccurlyeq A$):

Lemma I.12.15 ($ZFC^- - P$) *For any sets A, B with $B \neq \emptyset$: There is a function $g : B \xrightarrow{1-1} A$ iff there is a function $f : A \xrightarrow{\text{onto}} B$.*

Proof. For $\leftarrow$: Given f, use a choice set for $\{f^{-1}\{b\} : b \in B\}$ to get g with $f \circ g$ the identity function on B. For $\rightarrow$: Given g, fix $b \in B$ and let $f = g^{-1} \cup ((A \backslash \mathrm{ran}(g)) \times \{b\})$. □

Note that only the $\leftarrow$ direction used AC. As an example of how it could fail, ZF does not prove that there is an injection from ω_1 into $\mathcal{P}(\omega)$, but ZF does prove that one can map $\mathcal{P}(\omega)$ onto ω_1.

Exercise I.12.16 (ZF^-) *Prove that the Tychonov Theorem is equivalent to AC.*

Hint. Of course, $\leftarrow$ is the famous theorem of Tychonov (1935). For $\rightarrow$ (Kelley, 1950), let F be a family of non-empty sets. Fix an object $\infty \notin \bigcup F$. Let P be the product, $\prod_{X \in F}(X \cup \{\infty\})$. So, the elements of P are the functions g with $\mathrm{dom}(g) = F$ such that $g(X) \in X \cup \{\infty\}$ for all $X \in F$. Then any $g \in P$ such that $g(X) \neq \infty$ for all X is a choice function for F. So, topologize the $X \cup \{\infty\}$ to make each one compact, with X a closed subset. □

In this exercise, it is easy to make each $X \cup \{\infty\}$ a T_1 space, but not a T_2 (Hausdorff) space. Working in ZF, the Tychonov Theorem for compact Hausdorff spaces does not imply AC, although it is equivalent to several famous consequences of AC, such as the Completeness Theorem I.15.13 in model theory and the Boolean Prime Ideal Theorem (every Boolean algebra has a prime (= maximal) ideal). For more on this, see Chapter 2 of Jech [48].

For the equivalents of AC proved so far, the Axiom of Foundation was totally irrelevant. But there are some implications that really need Foundation, since they use $V = WF$ to prove AC by showing that all $R(\alpha)$ (see Definition I.9.28) can be well-ordered.

Exercise I.12.17 *Prove, in ZF, that AC is equivalent to the statement that $\mathcal{P}(\delta)$ can be well-ordered for all $\delta \in ON$.*

Hint. For the non-trivial direction: Note that the obvious inductive proof of "$R(\alpha)$ can be well-ordered" fails. There is no problem at successors, since $R(\alpha) \approx \delta \rightarrow R(\alpha + 1) = \mathcal{P}(R(\alpha)) \approx \mathcal{P}(\delta)$, but what do you do at a limit γ? Naively, apply induction and *choose*, for $\alpha < \gamma$, a well-order $\lhd_\alpha$ of $R(\alpha)$; then define $x \lhd_\gamma y$ iff

$$[\mathrm{rank}(x) < \mathrm{rank}(y)] \vee [\mathrm{rank}(x) = \mathrm{rank}(y) \wedge x \lhd_{\mathrm{rank}(x)+1} y] \ ,$$

but this *choosing* requires AC, which you're trying to prove. Instead, first fix γ, and then fix κ such that $\kappa \not\leq R(\gamma)$, and then fix a well-order $\sqsubset$ of $\mathcal{P}(\kappa)$. Then, from $\sqsubset$, recursively *define* $\lhd_\alpha$ on $R(\alpha)$ for $\alpha \leq \gamma$. □

It is consistent with ZF^- that AC is false, but AC is true in WF, so that all $\mathcal{P}(\delta)$ can be well-ordered; see Exercise II.9.10.

I.13 Cardinal Arithmetic

This section is done in ZFC^-. Using AC, every set can be well-ordered, so $|x|$ is defined for all x. We define cardinal addition, multiplication, and exponentiation by:

Definition I.13.1 *If κ, λ are cardinals, then*

- ☞ $\boxed{\kappa + \lambda} = |\{0\} \times \kappa \cup \{1\} \times \lambda|$
- ☞ $\boxed{\kappa \cdot \lambda} = |\kappa \times \lambda|$
- ☞ $\boxed{\kappa^\lambda} = |{}^\lambda\kappa|$

The boxes are omitted when it is clear from context that cardinal arithmetic is intended. In particular, if a cardinal is named as "$\aleph_\alpha$", then cardinal arithmetic is intended.

Usually, we avoid using boxes by making our intentions clear, but boxes are occasionally needed when we are using both ordinal and cardinal arithmetic in the same expression. The rest of this section summarizes some properties of cardinal arithmetic; see, e.g., [57] for some of the missing proofs.

Lemma I.13.2 *If $\kappa, \lambda, \kappa', \lambda'$ are cardinals and $\kappa \leq \kappa'$ and $\lambda \leq \lambda'$, then $\kappa + \lambda \leq \kappa' + \lambda'$, $\kappa \cdot \lambda \leq \kappa' \cdot \lambda'$, and $\kappa^\lambda \leq (\kappa')^{\lambda'}$ (unless $\kappa = \kappa' = \lambda = 0$), where* cardinal *arithmetic is meant throughout.*

Lemma I.13.3 *If κ, λ, θ are cardinals, then using cardinal arithmetic:*

1. $\kappa + \lambda = \lambda + \kappa$.
2. $\kappa \cdot \lambda = \lambda \cdot \kappa$.
3. $(\kappa + \lambda) \cdot \theta = \kappa \cdot \theta + \lambda \cdot \theta$.
4. $\kappa^{(\lambda \cdot \theta)} = (\kappa^\lambda)^\theta$.
5. $\kappa^{(\lambda + \theta)} = \kappa^\lambda \cdot \kappa^\theta$.

Lemma I.13.4 *For any ordinals α, β: $|\alpha + \beta| = \boxed{|\alpha| + |\beta|}$ and $|\alpha \cdot \beta| = \boxed{|\alpha| \cdot |\beta|}$.*

Lemma I.13.5 *If κ, λ are finite cardinals, then $\boxed{\kappa + \lambda} = \kappa + \lambda$, $\boxed{\kappa \cdot \lambda} = \kappa \cdot \lambda$, and $\boxed{\kappa^\lambda} = \kappa^\lambda$.*

Lemma I.13.6 *If κ, λ are cardinals and at least one of them is infinite, then $\boxed{\kappa + \lambda} = \max(\kappa, \lambda)$. Also, if neither of them are 0 then $\boxed{\kappa \cdot \lambda} = \max(\kappa, \lambda)$.*

Lemma I.13.7 *$\kappa^\lambda = 2^\lambda = |\mathcal{P}(\lambda)|$ whenever κ, λ are cardinals with λ infinite and $2 \leq \kappa \leq 2^\lambda$. Also, $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$ for every ordinal α.*

Definition I.13.8 *CH is the statement $2^{\aleph_0} = \aleph_1$. The* Generalized Continuum Hypothesis, *GCH, is the statement $\forall\alpha[2^{\aleph_\alpha} = \aleph_{\alpha+1}]$.*

Lemma I.13.7 indicates that κ^λ should be trivial to compute under *GCH*, but there are further complexities about κ^λ when $\kappa > \lambda^+$. Assuming *GCH*, Theorem I.13.13 will yield $(\aleph_7)^{\aleph_0} = \aleph_7$ and $(\aleph_{\omega_1})^{\aleph_0} = \aleph_{\omega_1}$, but $(\aleph_\omega)^{\aleph_0} = \aleph_{\omega+1}$. The key feature about $\aleph_\omega$ here is that it has countable cofinality:

Definition I.13.9 *If γ is any limit ordinal, then the* cofinality *of γ is*

$$\mathrm{cf}(\gamma) = \min\{\mathrm{type}(X) : X \subseteq \gamma \wedge \sup(X) = \gamma\} \quad .$$

γ is regular *iff $\mathrm{cf}(\gamma) = \gamma$ and* singular *iff $\mathrm{cf}(\gamma) < \gamma$.*

Note that $\mathrm{cf}(\gamma) \leq \gamma$ (taking $X = \gamma$) and $\mathrm{cf}(\gamma) \geq \omega$ (since finite subsets of γ have a largest element). Two simple examples: $\mathrm{cf}(\omega^2) = \omega$ (using $X = \{\omega \cdot n : n \in \omega\}$) and $\mathrm{cf}(\aleph_\omega) = \omega$ (using $X = \{\aleph_n : n \in \omega\}$). As an aid to computing cofinalities in more complicated situations:

Lemma I.13.10 *For any limit ordinal γ:*

1. *If $A \subseteq \gamma$ and $\sup(A) = \gamma$ then $\mathrm{cf}(\gamma) = \mathrm{cf}(\mathrm{type}(A))$.*
2. *$\mathrm{cf}(\mathrm{cf}(\gamma)) = \mathrm{cf}(\gamma)$, so $\mathrm{cf}(\gamma)$ is regular.*
3. *$\omega \leq \mathrm{cf}(\gamma) \leq |\gamma| \leq \gamma$*
4. *If γ is regular then γ is a cardinal.*
5. *If $\gamma = \aleph_\alpha$ where α is either 0 or a successor, then γ is regular.*
6. *If $\gamma = \aleph_\alpha$ where α is a limit ordinal, then $\mathrm{cf}(\gamma) = \mathrm{cf}(\alpha)$.*

Actually, Definition I.13.9 and all of Lemma I.13.10 *except* (5) do not require *AC*. But *AC* is used in the proof of (5): If $\theta = \aleph_{\beta+1} = \kappa^+$ then $\mathrm{cf}(\theta) = \theta$ because if $\mathcal{F} \subseteq \theta$ with $\mathrm{type}(\mathcal{F}) < \theta$ then $\mathcal{F}$ is a family of $\leq \kappa$ sets (ordinals), each of size $\leq \kappa$, so by Theorem I.12.14 (which used *AC*), $\sup(\mathcal{F}) = \bigcup\mathcal{F}$ has size $\leq \kappa$, and hence $\sup(\mathcal{F}) \neq \theta$.

Part (1) if the next theorem generalizes Theorem I.12.14, which is the special case when $\theta = \kappa^+$.

Theorem I.13.11 *Let θ be any infinite cardinal.*

1. *If θ is regular, and $\mathcal{F}$ is a family of sets with $|\mathcal{F}| < \theta$, and $|S| < \theta$ for all $S \in \mathcal{F}$, then $|\bigcup\mathcal{F}| < \theta$.*
2. *If $\mathrm{cf}(\theta) = \lambda < \theta$, then there is a family $\mathcal{F}$ of subsets of θ with $|\mathcal{F}| = \lambda$ and $\bigcup\mathcal{F} = \theta$, such that $|S| < \theta$ for all $S \in \mathcal{F}$.*

By generalizing Cantor's diagonal argument for proving $2^\lambda > \lambda$, we get:

Theorem I.13.12 (König, 1905) *If $\kappa \geq 2$ and λ is infinite, then* $\mathrm{cf}(\kappa^\lambda) > \lambda$.

For example, $\mathrm{cf}(2^{\aleph_0}) > \omega$, so that $2^{\aleph_0}$ cannot be $\aleph_\omega$. Roughly, Cohen showed that it is consistent for $2^{\aleph_0}$ to be any cardinal of uncountable cofinality, such as $\aleph_1$ (*CH*), or $\aleph_7$, or $\aleph_{\omega+1}$, or $\aleph_{\omega_1}$; see Section IV.3. More generally, by a result of Easton (see Section V.2), the function $\kappa \mapsto 2^\kappa$ for regular κ can be anything not contradicted by König's Theorem plus the obvious monotonicity restriction, $\kappa < \lambda \to 2^\kappa \leq 2^\lambda$. For singular κ, the situation is more complicated. For example, if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$; see Exercise III.6.18.

The following lemma tells us how to compute κ^λ under *GCH* when at least one of κ, λ is infinite. It shows that κ^λ is the smallest possible value, given König's Theorem.

Theorem I.13.13 *Assume GCH, and let κ, λ be cardinals with* $\max(\kappa, \lambda)$ *infinite.*

1. *If $2 \leq \kappa \leq \lambda^+$, then $\kappa^\lambda = \lambda^+$.*
2. *If $1 \leq \lambda \leq \kappa$, then κ^λ is κ if $\lambda < \mathrm{cf}(\kappa)$ and κ^+ if $\lambda \geq \mathrm{cf}(\kappa)$.*

Definition I.13.14 *The infinite cardinal λ is a* limit cardinal *iff $\lambda \neq \kappa^+$ for any κ; equivalently, $\lambda = \aleph_\gamma$, where γ is either 0 or a limit ordinal.*

We now consider some set-theoretic constructs and compute their cardinality.

Definition I.13.15 $B^{<\alpha} = {}^{<\alpha}B = \bigcup\{{}^\xi B : \xi < \alpha\}$.

This extends Definition I.6.11 (for the case $\alpha = \omega$). So, ${}^{<\alpha}B$ is the set of well-ordered sequences from B of length less than α. When discussing cardinal arithmetic, we use ${}^{<\lambda}\kappa$ for the set of sequences and $\kappa^{<\lambda}$ for the cardinal:

Definition I.13.16 *For cardinals κ, λ:* $\kappa^{<\lambda} = |{}^{<\lambda}\kappa|$.

This is most often used when λ is a limit cardinal.

Lemma I.13.17 $\kappa^{<\lambda} = \sup\{\kappa^\theta : \theta < \lambda \wedge \theta = |\theta|\}$ *whenever κ, λ are cardinals with λ infinite and $\kappa \geq 2$.*

Here, "$\theta = |\theta|$" is just a short way of saying "θ is a cardinal" in a formula. Note that we're taking the sup just over cardinals, not ordinals, although ${}^{<\lambda}\kappa$ contains all ordinal sequences. In particular, if $\lambda = \theta^+$, then $\kappa^{<\lambda} = \kappa^\theta$.

Proof. Let $\rho = \sup\{\kappa^\theta : \theta < \lambda \wedge \theta = |\theta|\}$. Then $\kappa^{<\lambda} \geq \rho$ follows from the fact that ${}^\theta\kappa \subseteq {}^{<\lambda}\kappa$ whenever $\theta < \lambda$.

Also, $\rho \geq \lambda$ follows from the fact that $\kappa^\theta \geq 2^\theta \geq \theta^+$. So, to prove that $\kappa^{<\lambda} \leq \rho$, it is sufficient to prove that $|{}^\alpha\kappa| \leq \rho$ for all $\alpha < \lambda$, since the union of $\leq \rho$ sets of size $\leq \rho$ has size $\leq \rho$ (Theorem I.12.14). But if $\alpha < \lambda$ and $\theta = |\alpha|$, then $|{}^\alpha\kappa| = |{}^\theta\kappa| = \kappa^\theta \leq \rho$. □

In particular, $\kappa^{<\omega} = \kappa$ whenever κ is infinite; so, when we construct a formal language using κ basic symbols (as we do in model theory; see Section I.15), our language has only κ expressions.

Definition I.13.18 $[B]^\theta = \{x \subseteq B : |x| = \theta\}$ *and* $[B]^{<\theta} = \{x \subseteq B : |x| < \theta\}$.

Lemma I.13.19 *If B is an infinite set then $|[B]^n| = |[B]^{<\omega}| = |B|$ for each finite n. Also, $|[B]^\theta| = |B|^\theta$ and $|[B]^{<\theta}| = |B|^{<\theta}$ whenever $\aleph_0 \leq \theta \leq |B|$.*

If S is an infinite subset of a group, then the subgroup generated by S has size only $|S|$. This is a special case of the Lemma I.13.22 below, which may be viewed as the set-theoretic part of the Downward Löwenheim–Skolem–Tarski Theorem (I.15.10) in model theory:

Definition I.13.20 *Let B be any non-empty set:*

1. *An* α–ary function *on B is a function $f : {}^\alpha B \to B$.*
2. *A* $<\theta$–ary function *on B is an f such that $f : {}^\alpha B \to B$ for some $\alpha < \theta$.*
3. *A* finitary function *is a $<\omega$–ary function.*
4. *If $f : {}^\alpha B \to B$ then $A \subseteq B$ is* closed under *f iff $f({}^\alpha A) \subseteq A$.*
5. *If $\mathcal{F}$ is a family of $<\theta$–ary functions on B, then $A \subseteq B$ is* closed under *$\mathcal{F}$ iff A is closed under every $f \in \mathcal{F}$, and the* closure *of $S \subseteq B$ under $\mathcal{F}$ is the least set A such that $S \subseteq A \subseteq B$ and A is closed under $\mathcal{F}$.*

In (5), the closure, A, of S under $\mathcal{F}$, can be obtained trivially by:

$$A = \bigcap \{X : S \subseteq X \subseteq B \ \wedge \ X \text{ is closed under } \mathcal{F}\} \ .$$

But, if you want to compute $|A|$, you need to "construct" A recursively, using

Lemma I.13.21 *Let θ be a regular infinite cardinal, and let $\mathcal{F}$ be a family of $<\theta$–ary functions on B. Fix $S \subseteq B$, and define S_ξ recursively by:*

1. *$S_0 = S$.*
2. *$S_\eta = \bigcup_{\xi<\eta} S_\xi$ when η a limit ordinal.*
3. *$S_{\xi+1} = S_\xi \cup \{f(\vec{x}) : f \in \mathcal{F} \ \wedge \ f$ is α–ary $\ \wedge \ \vec{x} \in {}^\alpha(S_\xi)\}$.*

Then S_θ is the closure of S under $\mathcal{F}$.

Applying this with $\theta = \omega$, it is easily seen that:

Lemma I.13.22 *If $\mathcal{F}$ is a family of finitary functions on B, κ is an infinite cardinal, $S \subseteq B$, $|S| \leq \kappa$, and $|\mathcal{F}| \leq \kappa$, then the closure of S under $\mathcal{F}$ has size no more than κ.*

In particular, as mentioned above, if B is a group and $S \subseteq B$ with $|S| = \kappa$, then the subgroup A of B generated by S has size κ also. Here, A is the closure of S under the family $\mathcal{F}$ consisting of the 2–ary product function, the 1–ary inverse function, and the 0–ary identity element.

We remark on a notational quirk regarding 0–ary functions. ${}^0B = \{\emptyset\}$, so a 0–ary function is an $f : \{\emptyset\} \to B$, and A is closed under f iff $f(\emptyset) \in A$. But we usually identify f with the element $f(\emptyset) \in B$; e.g., we talk about the identity element e of a group B, instead of the 0–ary function $\emptyset \mapsto e$, and we express "closure" by saying simply that $e \in A$. Likewise, in predicate logic (Sections I.2 and I.15), 0–ary function symbols are called constant symbols, and denote elements of a structure, not functions from $\{\emptyset\}$ into the structure.

If we apply Lemma I.13.21 with $\theta > \omega$, we can compute the sizes of closures under various infinitary functions:

Lemma I.13.23 *Let θ be a regular infinite cardinal, and let $\mathcal{F}$ be a family of $<\theta$–ary functions on B. Fix $S \subseteq B$, and let A be the closure of S under $\mathcal{F}$. Let κ be an infinite cardinal, and assume that $|S| \leq \kappa$ and $|\mathcal{F}| \leq \kappa$. Then $|A| \leq \kappa^{<\theta}$.*

As an example with $\theta = \aleph_1$ and $\kappa = 2^{\aleph_0}$ (so $\kappa^{<\theta} = 2^{\aleph_0}$): If X is a topological space (such as $\mathbb{R}$) with $2^{\aleph_0}$ open sets, then X has only $2^{\aleph_0}$ Borel sets. Here, $B = \mathcal{P}(X)$, S is the family of open sets, and the family A of Borel sets is the closure of S under the family $\mathcal{F}$ consisting of the 1–ary complement operation, $Y \mapsto X \backslash Y$, and the ω–ary union operation, $\langle Y_n : n \in \omega \rangle \mapsto \bigcup_n Y_n$.

Analogously to Definition I.11.4, we define the beth function:

Definition I.13.24 *The $\beth_\xi$ are defined by recursion on ξ by:*

- $\beth_0 = \aleph_0 = \omega$.
- $\beth_{\xi+1} = 2^{\beth_\xi}$.
- $\beth_\eta = \sup\{\beth_\xi : \xi < \eta\}$ *when η is a limit ordinal.*

So, *CH* is equivalent to the statement that $\beth_1 = \aleph_1$, and *GCH* is equivalent to the statement that $\beth_\xi = \aleph_\xi$ for all ξ.

Lemma I.13.25 *$\xi \leq \aleph_\xi \leq \beth_\xi$ for all ξ, so that $\beth_\xi = \xi \to \aleph_\xi = \xi$.*

To produce a ξ such that $\beth_\xi = \xi$, replace $\aleph$ by $\beth$ in Exercise I.11.8.

The beth function is not used much in elementary mathematics, where one is mostly discussing sets such as $\mathbb{R}$ and $\mathbb{C}$ of size $\beth_1 = 2^{\aleph_0}$, or occasionally $\mathcal{P}(\mathbb{R})$ of size $\beth_2$ — one could just as well write $2^{2^{\aleph_0}}$. The $\aleph$ function comes up more naturally, since, as we shall see (Corollary IV.3.14), $2^{\aleph_0}$ could be $\aleph_\xi$ for some large ξ; one can even have $\aleph_\xi = \xi$. There are some constructions in elementary algebra and analysis that lead to the cardinal $\beth_\omega$ (see Exercises I.13.34 and I.13.35), but elementary textbooks don't dwell on these.

However, the beth function is useful when discussing the well-founded sets because, as is easily seen by induction,

Lemma I.13.26 *$|R(\omega+\alpha)| = \beth_\alpha$ for all α, so that $|R(\alpha)| = \beth_\alpha$ for all $\alpha \geq \omega^2$.*

We shall see in Section II.2 that $R(\gamma)$ for limit $\gamma > \omega$ is a natural model of *ZC* (all of *ZFC* except Replacement). The smallest of these is $R(\omega + \omega)$, of size $\beth_\omega$. To get a natural model of *ZFC−P* (all of *ZFC* except Power Set), we shall use the $H(\kappa)$ where κ is regular and uncountable:

Definition I.13.27 *For any cardinal κ, $H(\kappa) = \{x \in WF : |\mathrm{trcl}(x)| < \kappa\}$. $HC = H(\aleph_1)$ is called the set of* hereditarily countable sets.

We have already defined $HF = H(\aleph_0)$, the set of the hereditarily finite sets, and $HF = R(\omega)$; see Lemma I.10.18 and Definition I.10.19. Of course, we need to prove that $H(\kappa)$ is a set for any κ:

Lemma I.13.28 *For any infinite cardinal κ, $H(\kappa)$ is a set of size $2^{<\kappa}$, and $H(\kappa) \subseteq R(\kappa)$.*

Proof. If $\alpha = \mathrm{rank}(x)$ and $\xi < \alpha$, then $\xi = \mathrm{rank}(z)$ for some $z \in \mathrm{trcl}(x)$ (by Lemma I.9.16); hence, $\{\mathrm{rank}(z) : z \in \mathrm{trcl}(x)\} = \alpha$. So, if $x \in H(\kappa)$, then $\alpha < \kappa$ because $|\mathrm{trcl}(x)| < \kappa$; hence $H(\kappa) \subseteq R(\kappa)$.

$|H(\kappa)| \geq 2^{<\kappa}$ because $\mathcal{P}(\lambda) \subseteq H(\kappa)$ for all $\lambda < \kappa$, so that $|H(\kappa)| \geq 2^\lambda$.

To see that $|H(\kappa)| \leq 2^{<\kappa}$, get $F : H(\kappa) \xrightarrow{1-1} \bigcup\{\mathcal{P}(\lambda \times \lambda) : \lambda < \kappa\}$ as follows: If $x \in H(\kappa)$, let $\lambda = |\mathrm{trcl}(x) \cup \{x\}| < \kappa$, and, by *AC*, choose a relation $F(x) \subseteq \lambda \times \lambda$ such that $(\lambda; F(x)) \cong (\mathrm{trcl}(x) \cup \{x\}; \in)$; F is 1-1 because x is determined by the isomorphism type of $\in$ on $\mathrm{trcl}(x) \cup \{x\}$ (Lemma I.9.38). □

For cardinals $\kappa > \omega$, $|R(\kappa)| = \beth_\kappa$, while $|H(\kappa)|$ is "only" $2^{<\kappa}$, so "usually", $H(\kappa)$ is much smaller than $R(\kappa)$; for example, $|HC| = 2^{\aleph_0} = \beth_1$ while $|R(\omega_1)| = \beth_{\omega_1}$.

Lemma I.13.29 *For cardinals $\kappa > \omega$, $H(\kappa) = R(\kappa)$ iff $\kappa = \beth_\kappa$.*

Proof. Assume that $H(\kappa) = R(\kappa)$. Whenever $\omega^2 \leq \alpha < \kappa$, $R(\alpha)$ is a transitive set of size $\beth_\alpha$ by Lemma I.13.26, and $R(\alpha) \in R(\kappa) = H(\kappa)$, so $\beth_\alpha < \kappa$. Thus, $\beth_\kappa \leq \kappa$; and then $\beth_\kappa = \kappa$, since $\beth_\xi \geq \xi$ for all ξ.

Assume that $\kappa = \beth_\kappa$. If $x \in R(\kappa)$, then $x \in R(\alpha)$ for some α with $\omega^2 \leq \alpha < \kappa$, and then $|\mathrm{trcl}(x)| \leq |R(\alpha)| = \beth_\alpha < \beth_\kappa = \kappa$, so $x \in H(\kappa)$. Thus, $R(\kappa) \subseteq H(\kappa)$, and then $H(\kappa) = R(\kappa)$ by Lemma I.13.28. □

Definition I.13.30 *A cardinal κ is* weakly inaccessible *iff $\kappa > \omega$, κ is regular, and $\kappa > \lambda^+$ for all $\lambda < \kappa$. κ is* strongly inaccessible *iff $\kappa > \omega$, κ is regular, and $\kappa > 2^\lambda$ for all $\lambda < \kappa$.*

Lemma I.13.31 *If κ is weakly inaccessible then $\kappa = \aleph_\kappa$. If κ is strongly inaccessible then $\kappa = \aleph_\kappa = \beth_\kappa$.*

It is clear from the definitions that all strong inaccessibles are weak inaccessibles, and that the two notions are equivalent under *GCH*. *ZFC* alone does not prove the existence of weak inaccessibles, although as we have just noted above, $ZFC \vdash \exists \kappa\, [\kappa = \aleph_\kappa = \beth_\kappa]$. The continuum $2^{\aleph_0}$ is obviously not strongly inaccessible, but it could be weakly inaccessible; see Lemma IV.3.17 and the discussion following it.

We shall see later (Section II.2) that $H(\kappa) \models ZFC-P$ whenever $\kappa > \omega$ is regular. The proof that $H(\kappa) \models$ Replacement uses the following lemma, which is easily proved from the definitions:

Lemma I.13.32 *Assume that κ is regular. If $y \subseteq H(\kappa)$ and $|y| < \kappa$, then $y \in H(\kappa)$. If $f : z \to H(\kappa)$ and $z \in H(\kappa)$, then $f \in H(\kappa)$ and $\mathrm{ran}(f) \in H(\kappa)$.*

Exercise I.13.33 *Prove that $(\beth_\omega)^{\aleph_0} = \prod_{n\in\omega} \beth_n = \beth_{\omega+1}$. Here $\prod_{n\in\omega} \beth_n$ is defined to be $|\mathcal{F}|$, where $\mathcal{F} = \{f \in {}^\omega(\beth_\omega) : \forall n\, f(n) \in \beth_n\}$.*

Hint. Every subset of $\beth_\omega$ can be coded by a function from ω into $\beth_\omega$. □

The following two exercises show that sets of size $\beth_\omega$ could (but usually don't) arise naturally in beginning graduate courses in algebra and analysis:

Exercise I.13.34 *Let W be a vector space over some field F, and let $W^* = \mathrm{Hom}(W, F)$ be the dual vector space. Consider W to be a subspace of W^{**} in the usual way (identify $x \in W$ with $\varphi \mapsto \varphi(x)$ in W^{**}). Let $W_0 = W$ and $W_{n+1} = (W_n)^{**}$, so that $W_0 \subseteq W_1 \subseteq \cdots$. Let $W_\omega = \bigcup_n W_n$. Now assume that $|F| < \beth_\omega$ and $\aleph_0 \leq \dim(W) < \beth_\omega$. Prove that $|W_\omega| = \dim(W_\omega) = \beth_\omega$.*

Exercise I.13.35 *Let B be a Banach space over $\mathbb{R}$ or $\mathbb{C}$, and let B^* be the dual Banach space. Consider B to be a subspace of B^{**} in the usual way. Let $B_0 = B$ and $B_{n+1} = (B_n)^{**}$. Let B_ω be the completion of $\bigcup_n B_n$.*

a. Give an example of a separable B such that $|B_\omega| = \beth_{\omega+1}$.
b. Prove that if B is a Banach space and $|B| \geq \beth_\omega$, then $|B| \geq \beth_{\omega+1}$.

Hint. For (b), see Exercise I.13.33. For (a), you can use ℓ^1. □

Exercise I.13.36 *Assume CH, but don't assume GCH. Show that $(\aleph_n)^{\aleph_0} = \aleph_n$ whenever $1 \leq n < \omega$.*

Hint. Induct on n. □

Exercise I.13.37 *Let κ be an infinite cardinal and let $\lhd$ be any well-order of κ. Show that there is an $X \subseteq \kappa$ with $|X| = \kappa$ such that $\lhd$ and $<$ agree on X.*

Hint. For regular κ, it's easy to choose X by recursion in κ steps. □

Exercise I.13.38 *Prove the Milner-Rado "Paradox" (1965): If κ is an infinite cardinal and $\kappa \leq \alpha < \kappa^+$, then there are $X_n \subseteq \alpha$ for $n < \omega$ such that $\bigcup_{n<\omega} X_n = \alpha$ and each* $\mathrm{type}(X_n) \leq \kappa^n$.

Hint. Induct on α. □

This may seem paradoxical because the types are bounded for finite unions; see Exercise I.9.53, or

Exercise I.13.39 *If κ is an infinite cardinal and $\alpha = \bigcup_{n<c} X_n$, where $c < \omega$ and each* $\mathrm{type}(X_n) < \kappa^\omega$, *then* $\alpha < \kappa^\omega$.

I.14 The Identity of Mathematical Objects

All mathematicians know that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$. They may not know, or even care, what each real and complex number is as a set. They probably are aware that most (or all, depending on the exposition) of these $\subseteq$s aren't really $\subseteq$s after all, but are shorthands for embeddings.

For example, we all know that $3 + 4i \in \mathbb{C} \setminus \mathbb{R}$ and that $|3 + 4i| = 5 \in \mathbb{N}$. But what is the true identity of $3 + 4i$? What is its rank? This is usually irrelevant. It is important at some point to prove rigorously that it makes sense to compute with complex numbers. The simplest way of doing this is to *define* the complex plane $\mathbb{C}$ to be the set $\mathbb{R} \times \mathbb{R}$, so i denotes $(0, 1)$ and $3 + 4i$ denotes $(3, 4)$. Then $\mathbb{R}$ isn't a subset of $\mathbb{C}$ after all, but there is a natural embedding $x \mapsto (x, 0)$ from $\mathbb{R}$ into $\mathbb{C}$. Of course, you also have to define how to add and multiply elements of $\mathbb{C}$; e.g., $(x, y) \cdot (x', y') = (xx' - yy', xy' + yx')$.

After learning this once, and verifying that the fields of real and complex numbers behave the way "you think they should", it is usually safe to forget the details. But for the purpose of doing independence proofs, it is important to recall some of this now. For example, if we are proving that a model M for *ZFC* satisfies some statement about $\mathbb{R}$, we may need to identify $\mathbb{R}^M$, which is the object that M thinks is the set of real numbers.

There are many different expositions of these various number systems. Following [57], we choose one that is set-theoretically simple, and not necessarily the most elegant mathematically. Also, we try to make all "essentially finite" objects be members of $HF = R(\omega)$, which is where all finite mathematics should live. These "essentially finite" objects include things such as natural numbers, rational numbers, and logical formulas. Also, *every* object that we define explicitly will be in *WF*, as we would expect from the discussion in Section I.9 (see Exercise I.9.27).

We already have $\mathbb{N} = \omega$, and we know that $\omega \subset HF$. Now,

Definition I.14.1 $\mathbb{Q}$ *is the union of ω with the set of all $\langle \ell, \langle m, n \rangle \rangle \in \omega \times (\omega \times \omega)$ such that:*

1. $m, n \geq 1$
2. $\ell \in \{0, 1\}$
3. $\gcd(m, n) = 1$
4. *If* $\ell = 0$ *then* $n \geq 2$

$\mathbb{Z} = \omega \cup \{\langle 1, \langle m, 1\rangle\rangle : 0 < m < \omega\}$.

With this definition, $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$ is literally true, and $\mathbb{Q} \subseteq HF$. The ℓ in $\langle \ell, \langle m, n\rangle\rangle$ is a sign bit, with 0 signifying + and 1 signifying −. So, 3/5 is really $\langle 0, \langle 3, 5\rangle\rangle$, and $-4/10$ is really $\langle 1, \langle 2, 5\rangle\rangle$. Of course, one must define, in the obvious way, +, ·, and < on $\mathbb{Q}$; later (Lemma II.4.14), we shall see that these definitions are absolute, in that they have the same meaning in all transitive models. This absoluteness will follow immediately from the fact that these notions are defined by quantifying only over HF.

There are several well-known constructions of $\mathbb{R}$ from $\mathbb{Q}$. Again, we choose one that is set-theoretically simple. Since $\mathbb{R}$ is uncountable, it cannot be a subset of $R(\omega)$, so we make it a subset of $R(\omega + 1)$. Informally, each $x \in \mathbb{R}$ defines a lower Dedekind cut $C_x = \{q \in \mathbb{Q} : q < x\} \subseteq \mathbb{Q}$. So, we define $\mathbb{R}$ officially so that each $x \in \mathbb{R}$ *is* C_x. Then, $\mathbb{R} \subseteq \mathcal{P}(\mathbb{Q}) \subseteq \mathcal{P}(R(\omega)) = R(\omega + 1)$ and each element of $\mathbb{R}$ will have rank ω, so that $\mathrm{rank}(\mathbb{R}) = \omega + 1$. So,

Definition I.14.2 $\mathbb{R}$ *is the set of all* $x \in \mathcal{P}(\mathbb{Q})$ *such that* $x \neq \emptyset$, $x \neq \mathbb{Q}$, x *has no largest element, and*

$$\forall p, q \in \mathbb{Q}[p < q \in x \to p \in x] \quad . \tag{$*$}$$

Then, $\mathbb{Q}$ is not a subset of $\mathbb{R}$, but there is a natural embedding, $q \mapsto C_q$, from $\mathbb{Q}$ into $\mathbb{R}$. Finally, as indicated above,

Definition I.14.3 $\mathbb{C} = \mathbb{R} \times \mathbb{R}$.

It is easily seen from these definitions that $\mathbb{C}, \mathbb{R} \in WF$, $\mathrm{rank}(x) = \omega$ for each $x \in \mathbb{R}$, $\mathrm{rank}(z) = \omega + 2$ for each $z \in \mathbb{C}$, $\mathrm{rank}(\mathbb{R}) = \omega + 1$, and $\mathrm{rank}(\mathbb{C}) = \omega + 3$.

The set-theoretic identity of symbolic objects, such as logical formulas, is discussed in Section I.15.

I.15 Model Theory

This section continues the discussion in Sections I.1 and I.2. Working in ZFC, we develop all of conventional mathematics, including the subject of model theory. The core of this section takes place in $ZF^- - P$, which is sufficient to develop the basic model theory notions. The Axioms of Power Set and Choice are important in model theory, but we shall point out explicitly where they are used when we state lemmas and theorems. Foundation is never used in model theory itself, and becomes important only when discussing models of set theory.

Now, model theory is a very big subject, and is described in many specialized texts, such as [13, 65]. However, what we need for the independence proofs

is really just the elements of the subject, which are contained in elementary logic books, such as [29, 57, 69]. These elements amount to:

1. The syntax of formal logic.
2. The notion of model, $\mathfrak{A} \models \varphi$, and of elementary submodel, $\mathfrak{A} \prec \mathfrak{B}$.
3. The downward Löwenheim–Skolem–Tarski Theorem.
4. The notion of formal provability, $\Gamma \vdash \varphi$.
5. The Soundness Theorem.

We summarize these items briefly here. In addition, it is good to be aware of the Completeness Theorem and the Second Incompleteness Theorem; these are not actually needed for the independence proofs in set theory, but they provide useful motivation. The Incompleteness Theorem was already discussed in Section I.1. The Completeness Theorem I.15.13 is discussed briefly below.

Regarding (1): We begin with a *lexicon* or *signature* $\mathcal{L}$, and we use this to build formulas and sentences. This is exactly as in Section I.2, which discussed doing this in the finitistic metatheory, except that now, working in set theory, we do not need to assume that $\mathcal{L}$ is finite. As in Section I.2, we do not dwell on the fine details of syntax.

In line with our desire, as in Section I.14, that all "essentially finite" objects be members of $HF = R(\omega)$, we shall always assume that all logical symbols are members of HF. Also, we assume that membership in the (infinite) set of variables is decidable. Furthermore, when $\mathcal{L}$ is finite, we shall assume that $\mathcal{L} \in HF$. Formulas are always finite sequences of symbols. Since a finite sequence is just a function whose domain is a natural number, all formulas will be in HF when $\mathcal{L}$ is finite. Then for finite $\mathcal{L}$, with any reasonable definition of syntax, basic syntactic notions, such as "formula", "sentence", and "free variable" will be decidable.

Even if $\mathcal{L}$ is infinite, $ZF^- - P$ is sufficient to form the set of symbols B (the union of $\mathcal{L}$ with the logical symbols), and then the set of formulas, which is a subset of $B^{<\omega}$; see Definition I.6.11.

Regarding (2): Given $\mathcal{L}$, a *structure* $\mathfrak{A}$ for $\mathcal{L}$ consists of a non-empty set A together with an *assignment* of *appropriate* semantic entities for the elements (symbols) in $\mathcal{L}$. For example, if we are discussing models of set theory, then $\mathcal{L} = \{\in\}$ and $\mathfrak{A} = (A, E)$, where $A \neq \emptyset$ and $E \subseteq A^2$. *Every* such pair $\mathfrak{A}$ forms a structure for $\mathcal{L}$, regardless of whether $\mathfrak{A}$ satisfies any of the ZFC axioms. Most of the time in this book, E will be the membership relation on A, which will often (but not always) be a transitive set. To avoid excess verbiage,

Definition I.15.1 *An* $\in$*–model is any structure* $\mathfrak{A} = (A, E)$ *for* $\mathcal{L} = \{\in\}$ *such that* $E = \{(a, b) \in A \times A : a \in b\}$. *For these, we often use* A *for* $\mathfrak{A}$. *A* transitive model *is any* $\in$*–model* A *such that* A *is transitive.*

So, if we say "$A \models ZFC$", without specifying a binary relation on A, then it is understood that we are using the ordinary membership relation on A.

This sort of abuse of notation should be familiar from algebra. Continuing in the spirit of Section I.2, say we are axiomatizing the theory of rings with unity (1), using $\mathcal{L} = \{0, 1, +, \cdot, -\}$, where "$-$" denotes the unary additive inverse function. Of course, $0, 1$ are constant symbols and $+, \cdot$ are binary function symbols. Speaking in the abstract, we might specify structures for $\mathcal{L}$ as $\mathfrak{A} = (A; 0_{\mathfrak{A}}, 1_{\mathfrak{A}}, +_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, -_{\mathfrak{A}})$. The "assignment of *appropriate* semantic entities" means that $0_{\mathfrak{A}}, 1_{\mathfrak{A}}$ must be elements of A, and $+_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, -_{\mathfrak{A}}$ must be functions, with $+_{\mathfrak{A}}, \cdot_{\mathfrak{A}} : A^2 \to A$ and $-_{\mathfrak{A}} : A \to A$. In the abstract, such pedantry is needed because there are many different ring structures on the same set; we might have a different $\mathfrak{B}$ with $A = B$, but $0_{\mathfrak{B}}, 1_{\mathfrak{B}}, +_{\mathfrak{B}}, \cdot_{\mathfrak{B}}, -_{\mathfrak{B}}$ not the same as $0_{\mathfrak{A}}, 1_{\mathfrak{A}}, +_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, -_{\mathfrak{A}}$. However, if A is $\mathbb{R}$ or $\mathbb{Q}$ and we are using the expected ring operations, then we shall simply write $\mathbb{R}$ or $\mathbb{Q}$ for the structure, and say, e.g., the sentence $\forall x \exists y\, [x = y \cdot (y \cdot y)]$ is true in $\mathbb{R}$ and false in $\mathbb{Q}$.

To be even more pedantic, which is useful only to see how the whole subject is formalized in *ZF*, the "*assignment* of appropriate semantic entities" means that there is a function $\mathcal{I}$ that does the assigning. So,

Definition I.15.2 *Given a lexicon $\mathcal{L}$ for predicate logic, a* structure *for $\mathcal{L}$ is a pair $\mathfrak{A} = (A, \mathcal{I})$ such that A is a non-empty set (called the* universe *of $\mathfrak{A}$) and $\mathcal{I}$ is a function with domain $\mathcal{L}$ with the $\mathcal{I}(s)$ satisfying (writing $s_{\mathfrak{A}}$ for $\mathcal{I}(s)$):*

1. *If f is an n–ary function symbol with $n > 0$, then $f_{\mathfrak{A}} : A^n \to A$.*
2. *If p is an n–ary predicate symbol with $n > 0$, then $p_{\mathfrak{A}} \subseteq A^n$.*
3. *If c is a 0–ary function symbol then $c_{\mathfrak{A}} \in A$.*
4. *If p is a 0–ary predicate symbol then $p_{\mathfrak{A}} \in 2 = \{0, 1\} = \{F, T\}$.*

Actually, the special case (4) for 0–ary predicate symbols (proposition letters) is not required, since $A^0 = \{\emptyset\}$, whose only subsets are 0 and 1. For 0–ary function symbols (constant symbols), see the remark following Lemma I.13.22.

We now say more about the meaning of "φ is true in $\mathfrak{A}$" (or, "$\mathfrak{A}$ is a model for φ", or, in symbols, "$\mathfrak{A} \models \varphi$"); this is defined whenever φ is a sentence of $\mathcal{L}$ and $\mathfrak{A}$ is a structure for $\mathcal{L}$. Most mathematicians would say that the meaning of this is obvious by *relativization*, as in Notation I.6.27. If φ is the sentence $\forall x \exists y\, [x = y \cdot (y \cdot y)]$, and you want to see whether it is true in a structure $\mathfrak{A}$, you simply rewrite it, *relativizing* the variables x, y to A and interpreting $\cdot$ the way $\mathfrak{A}$ says that you should. So, it is *trivial* to understand what "φ is true in $\mathbb{R}$" and "φ is false in $\mathbb{Q}$" *mean*, although it takes a little knowledge of elementary analysis and algebra to prove these assertions. This "φ is true in A" even makes sense when A is a proper class. For example, if we give $\cdot$ its standard meaning for ordinals, then the statement "φ is true in ON" means, relativizing the quantified variables to ON, that $\forall x \in ON\, \exists y \in ON\, [x = y \cdot (y \cdot y)]$, which is clearly false. But, if we interpret the $\cdot$ as $\cap$, then we would get "φ is true in $(ON, \cap)$" as shorthand for the (trivial) fact $\forall x \in ON\, \exists y \in ON\, [x = y \cap (y \cap y)]$.

The study of logic was greatly advanced by two famous insights of Tarski in the 1930s:

Tarski's "*definability of truth*" shows that working in set theory, one can use recursion to define rigorously the notion "$\mathfrak{A} \models \varphi$" as a property of two sets, $\mathfrak{A}$ and φ; the property is false unless $\mathfrak{A}$ is a structure for $\mathcal{L}$ and φ is a sentence for the same $\mathcal{L}$. The study of this $\models$ led to the subject of model theory. It now makes sense to define $\mathfrak{A} \models \Lambda$ (or, $\mathfrak{A}$ is a model for Λ), where Λ is an infinite set of sentences, such as *ZFC*, to mean $\forall \varphi \in \Lambda\ \mathfrak{A} \models \varphi$.

Tarski's "*undefinability of truth*" shows that one cannot do this with proper class models. Naively, working in *ZFC*, it is trivial that each axiom φ of *ZFC* is true in V, so that $V \models ZFC$, so $\mathrm{Con}(ZFC)$. So, $ZFC \vdash \mathrm{Con}(ZFC)$. By Gödel's Second Incompleteness Theorem, there must be a flaw somewhere in this argument, but Tarski showed that one can't even define "φ is true in V" as a property of the set φ, so that you can't even *say* "$V \models ZFC$". This is different from *set* models, where you can use Tarski's definition of truth to write the statement $\exists \mathfrak{A}\ \mathfrak{A} \models ZFC$ in set theory; it just isn't provable in *ZFC* by the Incompleteness Theorem.

Still, for each *specific* sentence φ in the metatheory, it makes sense to say that φ is true in a class A, and this is the basis for many of the relative consistency proofs; see Section II.2. Tarski's undefinability theorem is discussed in logic books, such as [29, 57]. We assume that the reader has seen the notion $\mathfrak{A} \models \varphi$ before, but we shall outline the definition here, focusing on the reason why the definition is justified when A is a set and not when A is a proper class.

Roughly, the recursion is on φ, but note that although our goal might be to discuss the truth of *sentences*, our recursion must discuss arbitrary *formulas*, with free variables. For example, if φ is $\forall x \exists y\, [x = y \cdot (y \cdot y)]$, and we want to explain why it is false in $\mathbb{Q}$ (or, $\mathbb{Q} \not\models \varphi$), we would say that if $\psi(x)$ is the formula $\exists y\, [x = y \cdot (y \cdot y)]$, then we can find an $a \in \mathbb{Q}$ (such as $a = 2$) for which $\mathbb{Q} \not\models \psi[a]$; the fact that we can find other a (such as $a = 8$) for which $\mathbb{Q} \models \psi[a]$ is irrelevant here.

To handle formulas that may have arbitrarily many free variables, we say that σ is an *assignment* for φ in A iff σ is a function with $\mathrm{ran}(\sigma) \subseteq A$ and $\mathrm{dom}(\sigma)$ a finite set of variables that include at least all the free variables in φ. Now, we are really defining the notion "$\mathfrak{A} \models \varphi[\sigma]$", which depends on the pair (φ, σ). To apply our Theorem I.9.11 on recursive definitions, we view defining a *notion* as really defining a *function* $\mathrm{val}_{\mathfrak{A}}$ into $\{F, T\} = \{0, 1\}$, where $\mathrm{val}_{\mathfrak{A}}(\varphi)[\sigma] = T$ iff $\mathfrak{A} \models \varphi[\sigma]$. But also, before even discussing the truth values of formulas, we have to discuss the values of *terms*; that is, define $\mathrm{val}_{\mathfrak{A}}(\tau)[\sigma] \in A$ whenever τ is a term of $\mathcal{L}$ and σ is an assignment for τ in A (that is, $\mathrm{dom}(\sigma)$ contains all the variables in τ). For example, if τ is the term $y \cdot (y \cdot y)$, then $\mathrm{val}_{\mathbb{Q}}(\tau)[1.26] = 2.000376$. We need to know that $\mathrm{val}_{\mathbb{Q}}(\tau)[b]$ is never exactly 2 to conclude that $\mathbb{Q} \not\models \psi[2]$. Formally, the recursion is:

Definition I.15.3 *If $\mathfrak{A}$ is a structure for $\mathcal{L}$, then we define $\mathrm{val}_{\mathfrak{A}}(\tau)[\sigma] \in A$ whenever τ is a term of $\mathcal{L}$ and σ is an assignment for τ in A as follows:*

1. $\mathrm{val}_{\mathfrak{A}}(x)[\sigma] = \sigma(x)$ *when* $x \in \mathrm{dom}(\sigma)$.

2. $\mathrm{val}_{\mathfrak{A}}(c)[\sigma] = c_{\mathfrak{A}}$ *when* c *is a constant symbol.*
3. $\mathrm{val}_{\mathfrak{A}}(f(\tau_1, \ldots, \tau_n))[\sigma] = f_{\mathfrak{A}}(\mathrm{val}_{\mathfrak{A}}(\tau_1)[\sigma], \ldots, \mathrm{val}_{\mathfrak{A}}(\tau_n)[\sigma])$ *when* f *is an* n*-ary function symbol with* $n > 0$.

So, for each fixed $\mathfrak{A}$: If J is the collection of all pairs (τ, σ) such that τ is a term of $\mathcal{L}$ and σ is an assignment for τ in A, then we are defining a function $\mathrm{val}_{\mathfrak{A}}$ from J into A. This function $\mathrm{val}_{\mathfrak{A}}$ exists by Theorem I.9.11, using recursion on the relation R, where $(\tau', \sigma')R(\tau, \sigma)$ iff τ' is a *proper* subterm of τ and $\sigma' = \sigma$. This R is both well-founded (the Φ in Lemma I.9.9 can be the length of τ) and set-like (since τ has only finitely many subterms).

Note that R is well-founded and set-like even when A is a proper class, so our definition is still justified. Likewise, the following definition of truth for *atomic* formulas (ones with no quantifiers or propositional connectives) is justified even for proper class models; this definition is *not* a recursion:

Definition I.15.4 *If* $\mathfrak{A}$ *is a structure for* $\mathcal{L}$*, then we define* $\mathrm{val}_{\mathfrak{A}}(\varphi)[\sigma] \in \{0, 1\} = \{F, T\}$ *whenever* φ *is an atomic formula of* $\mathcal{L}$ *and* σ *is an assignment for* φ *in* A *as follows:*

1. $\mathrm{val}_{\mathfrak{A}}(p)[\sigma] = p_{\mathfrak{A}}$ *when* p *is a* 0*-ary predicate symbol.*
2. $\mathrm{val}_{\mathfrak{A}}(p(\tau_1, \cdots, \tau_n))[\sigma] = T$ *iff* $(\mathrm{val}_{\mathfrak{A}}(\tau_1)[\sigma], \ldots, \mathrm{val}_{\mathfrak{A}}(\tau_n)[\sigma]) \in p_{\mathfrak{A}}$ *when* p *is an* n*-ary predicate symbol with* $n > 0$.
3. $\mathrm{val}_{\mathfrak{A}}(\tau_1 = \tau_2)[\sigma] = T$ *iff* $\mathrm{val}_{\mathfrak{A}}(\tau_1)[\sigma] = \mathrm{val}_{\mathfrak{A}}(\tau_2)[\sigma]$.

The problem in defining the truth of formulas in V lies with the quantified variables. To handle quantified variables, one further preliminary: We define $\sigma + (y/a) = \sigma \restriction (\mathrm{dom}(\sigma) \setminus \{y\}) \cup \{(y, a)\}$. This $\sigma + (y/a)$ assigns y value a, and discards (in the case that $y \in \mathrm{dom}(\sigma)$) the value that σ gave to y.

Definition I.15.5 *If* $\mathfrak{A}$ *is a structure for* $\mathcal{L}$*, then we define* $\mathrm{val}_{\mathfrak{A}}(\varphi)[\sigma] \in \{0, 1\} = \{F, T\}$ *whenever* φ *is a formula of* $\mathcal{L}$ *and* σ *is an assignment for* φ *in* A *as follows:*

1. $\mathrm{val}_{\mathfrak{A}}(\neg\varphi)[\sigma] = 1 - \mathrm{val}_{\mathfrak{A}}(\varphi)[\sigma]$.
2. $\mathrm{val}_{\mathfrak{A}}(\varphi \wedge \psi)[\sigma]$, $\mathrm{val}_{\mathfrak{A}}(\varphi \vee \psi)[\sigma]$, $\mathrm{val}_{\mathfrak{A}}(\varphi \to \psi)[\sigma]$, *and* $\mathrm{val}_{\mathfrak{A}}(\varphi \leftrightarrow \psi)[\sigma]$, *are obtained from* $\mathrm{val}_{\mathfrak{A}}(\varphi)[\sigma]$ *and* $\mathrm{val}_{\mathfrak{A}}(\psi)[\sigma]$ *using the standard truth tables for* $\wedge, \vee, \to, \leftrightarrow$.
3. $\mathrm{val}_{\mathfrak{A}}(\exists y\varphi)[\sigma] = T$ *iff* $\mathrm{val}_{\mathfrak{A}}(\varphi)[\sigma + (y/a)] = T$ *for some* $a \in A$.
4. $\mathrm{val}_{\mathfrak{A}}(\forall y\varphi)[\sigma] = T$ *iff* $\mathrm{val}_{\mathfrak{A}}(\varphi)[\sigma + (y/a)] = T$ *for all* $a \in A$.

$\mathfrak{A} \models \varphi[\sigma]$ *means* $\mathrm{val}_{\mathfrak{A}}(\varphi)[\sigma] = T$. *If* φ *is a sentence, then* $\mathrm{val}_{\mathfrak{A}}(\varphi)$ *abbreviates* $\mathrm{val}_{\mathfrak{A}}(\varphi)[\emptyset]$, *and* $\mathfrak{A} \models \varphi$ *means* $\mathrm{val}_{\mathfrak{A}}(\varphi) = T$. *If* Λ *is a set of sentences, then* $\mathfrak{A} \models \Lambda$ *means* $\forall \varphi \in \Lambda\ \mathfrak{A} \models \varphi$.

So, for each fixed $\mathfrak{A}$: If K is the collection of all pairs (φ, σ) such that φ is a formula of $\mathcal{L}$ and σ is an assignment for φ in A, then we are defining a function $\text{val}_{\mathfrak{A}}$ from K into $\{0,1\}$. The function $\text{val}_{\mathfrak{A}}$ exists by Theorem I.9.11, using recursion on R, where $(\varphi', \sigma')R(\varphi, \sigma)$ iff φ' is a proper subformula of φ. This is well founded as before (see the discussion of Definition I.15.3), and is trivially set-like when A is a set.

R is not set-like when A is a proper class, since there is a proper class of possible σ', so our definition is not justified. One might try to modify R somewhat, since to evaluate the truth of $\forall y \varphi(y)$ one need not actually look at *all* possible σ', but there is no way in general to avoid having to consider all possible values for y, leading to a proper class of possible $\sigma' = \sigma + (y/a)$. It is true that for some very special proper class models, one can find a trick to limit the values of of y considered, thereby defining "$\mathfrak{A} \models \varphi$" (see Exercise I.15.22), but by Tarski's theorem, no such trick will work for $(V, \in)$.

Thus, our fake proof (outlined above) of $ZFC \vdash \text{Con}(ZFC)$, via showing that $V \models \text{Con}(ZFC)$, is doomed to failure. Now, a platonist who believes in the ZFC universe will also believe this proof of $\text{Con}(ZFC)$, in the sense that ZFC must be consistent because all its axioms are true in the universe of set theory, but this platonist will then view this proof as an example of a valid argument that cannot be formalized in the system ZFC. As we pointed out on page 8, to a platonist, no formal system can axiomatize all of truth.

The notation $\mathfrak{A} \models \varphi[\sigma]$ is sometimes a bit awkward, and we often use the following alternatives:

Notation I.15.6 *If $\mathfrak{A}$ is a structure for $\mathcal{L}$ and φ is a formula of $\mathcal{L}$ with free variables among $x_1, \ldots, x_n$, and $a_1, \ldots a_n \in A$, then $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$ and $A \models \varphi[a_1, \ldots, a_n]$ and $(\varphi(a_1, \ldots, a_n))^{\mathfrak{A}}$ and $(\varphi(a_1, \ldots, a_n))^{A}$ all denote $\mathfrak{A} \models \varphi[\sigma]$, where $\sigma = \{(x_1, a_1), \ldots, (x_n, a_n)\}$.*

To use notations such as $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$ with $n > 1$, some ordering on the variables has to be made clear from context. For example with $\mathcal{L} = \{0, 1, +, \cdot, -\}$, if we let $\varphi(x, y)$ be $\exists v\,[y = x + v \cdot v]$, and we ask, "does $\mathbb{R} \models \varphi[e, \pi]$?", our "let $\varphi(x, y)$ be $\cdots$" indicates that we're intending σ to be $\{(x, e), (y, \pi)\}$, not $\{(x, \pi), (y, e)\}$, and we would answer "yes", knowing that $e < \pi$. As always, the use of the set $\mathbb{R}$ alone as a model indicates that we are interpreting the symbols of $\mathcal{L}$ in the standard way. Sometimes we don't want to bother giving a name to our formula, in which case the superscripted notation is useful. For example, we might say simply that $(\exists v\,[\pi = e + v \cdot v])^{\mathbb{R}}$ holds, as does $(\exists v\,[e = \pi + v \cdot v])^{\mathbb{C}}$, whereas $(\exists v\,[e = \pi + v \cdot v])^{\mathbb{R}}$ is false. This notation will be clear to any mathematician, even one ignorant of logic, if we just explain that the superscript $\mathbb{R}$ means that we relativize the quantifiers to $\mathbb{R}$ — that is, the variables "range" only over $\mathbb{R}$.

This superscripted notation is very common in set theory when discussing $\in$–models, as we shall frequently do in this book. For example, Definition I.4.6 says that $\text{emp}(y)$ means $\forall x(x \notin y)$ ("y is empty"), so that $(\text{emp}(3))^M$ or

$\text{emp}^M(3)$ means $\forall x \in M(x \notin 3)$; this notion is defined only when $3 \in M$. We shall return specific facts about $\in$–models in Section I.16, but as a preview, we make two simple remarks here. First, if M is transitive, then $3 \in M \to 0, 1, 2 \in M$, so $(\text{emp}(3))^M$ is false; this is a special case of a general fact that $\text{emp}(y)$ and other "simple" set-theoretic formulas (called Δ_0) have their expected meaning in all transitive models. Second, for a specific given φ, it makes sense to talk about φ^M when M is a proper class; for example, $(\text{emp}(3))^{ON}$ is false but $(\text{emp}(3))^{ON \setminus \{0,1,2\}}$ is true. This is what we have already called relativization.

We next consider the definition of *substructure* and *elementary substructure*. The notion of substructure generalizes the notions of subgroup, subring, etc., from algebra:

Definition I.15.7 *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are structures for $\mathcal{L}$. Then $\mathfrak{A} \subseteq \mathfrak{B}$ means that $A \subseteq B$ and the functions and predicates of $\mathfrak{A}$ are the restrictions of the corresponding functions and predicates of $\mathfrak{B}$. Specifically:*

- ☞ *If f is an n–ary function symbol with $n > 0$, then $f_{\mathfrak{A}} = f_{\mathfrak{B}} \restriction A^n$.*
- ☞ *If p is an n–ary predicate symbol with $n > 0$, then $p_{\mathfrak{A}} = p_{\mathfrak{B}} \cap A^n$.*
- ☞ *If c is a constant symbol, then $c_{\mathfrak{A}} = c_{\mathfrak{B}}$.*
- ☞ *If p is a proposition letter, then $p_{\mathfrak{A}} = p_{\mathfrak{B}} \in 2 = \{0, 1\} = \{F, T\}$.*

$\mathfrak{A}$ is called a substructure (*or* submodel) *of $\mathfrak{B}$, and $\mathfrak{B}$ is called an* extension *of $\mathfrak{A}$.*

In the case of constants and functions, observe that $c_{\mathfrak{A}}$ must be an element of A and $f_{\mathfrak{A}}$ must map into A. So, if we start with $\mathfrak{B}$ and an arbitrary non-empty $A \subseteq B$, it is not true in general that A can be made into a substructure of $\mathfrak{B}$. For example, suppose $\mathfrak{B} = (B; \cdot, i, 1\}$ is a group, where we are taking the language of group theory to be $\mathcal{L} = \{\cdot, i, 1\}$ (where $i(x)$ means x^{-1}). If $A \subseteq B$, then it cannot be made into a substructure of $\mathfrak{B}$ unless it is closed under product and inverse and contains 1, that is, unless it is a subgroup.

Definition I.15.8 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathcal{L}$ with $\mathfrak{A} \subseteq \mathfrak{B}$. If φ is a formula of $\mathcal{L}$, then $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$ means that $\mathfrak{A} \models \varphi[\sigma]$ iff $\mathfrak{B} \models \varphi[\sigma]$ for all assignments σ for φ in A. $\mathfrak{A} \preccurlyeq \mathfrak{B}$* (elementary substructure or elementary submodel) *means that $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$ for all formulas φ of $\mathcal{L}$.*

Lemma I.15.9 *If $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$ whenever φ is quantifier-free, and $\text{val}_{\mathfrak{A}}(\tau)[\sigma] = \text{val}_{\mathfrak{B}}(\tau)[\sigma]$ whenever τ is a term of $\mathcal{L}$ and σ is an assignment for τ in A.*

For example, let $\mathcal{L} = \{<\}$, and consider intervals in $\mathbb{R}$, which are understood to have their usual order. Then $[0, 1] \subseteq [0, 2]$, meaning, with $\mathcal{L} = \{<\}$, that the elements of the set $[0, 1]$ are ordered the same way in $[0, 1]$ as in $[0, 2]$. But $[0, 1] \not\preccurlyeq [0, 2]$ because $[0, 1] \not\preccurlyeq_\varphi [0, 2]$ for some φ. Here, our φ cannot be quantifier-free (by Lemma I.15.9), and our φ cannot be a sentence (since

$[0,1] \cong [0,2]$, so they satisfy the same sentences). However, if φ is $\exists y\,(x < y)$, then $[0,1] \not\preccurlyeq_\varphi [0,2]$ because $[0,2] \models \varphi[1]$ but $[0,1] \not\models \varphi[1]$. It is true that $(0,1) \preccurlyeq (0,2)$ and $\mathbb{Q} \preccurlyeq \mathbb{R}$, although it is not completely trivial to prove these facts (see [57], Exercise II.16.13, or a model theory text [13, 65]).

As before, for a specific given φ, the notion $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$ makes sense even for proper class models, but $\mathfrak{A} \preccurlyeq \mathfrak{B}$ does not, since one must quantify over all φ.

When $\mathcal{L} = \{\in\}$ and we consider natural models for set theory, such as HC, it becomes hard to give non-trivial *explicit* examples of $\mathfrak{A} \preccurlyeq \mathfrak{B}$, although many such examples can be proved to exist by the *Downward Löwenheim–Skolem–Tarski Theorem*. This is an important tool in set theory as well as in model theory. It says, for example, that if $\mathcal{L}$ is countable, then there is always a countable $\mathfrak{A}$ with $\mathfrak{A} \preccurlyeq \mathfrak{B}$:

Theorem I.15.10 (Downward Löwenheim–Skolem–Tarski Theorem) *Work in ZFC^-. Let $\mathfrak{B}$ be any structure for $\mathcal{L}$. Fix κ such that $\max(|\mathcal{L}|, \aleph_0) \leq \kappa \leq |B|$, and fix $S \subseteq B$ with $|S| \leq \kappa$. Then there is an $\mathfrak{A} \preccurlyeq \mathfrak{B}$ such that $S \subseteq A$ and $|A| = \kappa$.*

Working in ZFC^-, we develop cardinal arithmetic (Section I.13), and we can then state and prove this theorem. As examples of its use in set theory: We shall see in Section II.2 that there are natural sets B such that $(B, \in) \models \Lambda$, where Λ contains many of the ZFC axioms; such as $ZFC-P$ (let $B = HC$) or ZC (let $B = R(\omega + \omega)$). We can even have Λ be all of ZFC if we assume that there is a strongly inaccessible cardinal κ and let $B = R(\kappa) = H(\kappa)$. All these natural B are uncountable. Applying Theorem I.15.10, we shall get a countable $A \preccurlyeq B$, so that we shall get countable $A \models \Lambda$. Then, the technique of *forcing* described in Chapter IV is a method of extending *countable* models of set theory to obtain models where, for example, CH fails in various ways. We do not know a simple way of describing explicitly a countable A such that $A \preccurlyeq HC$ or $A \preccurlyeq R(\omega + \omega)$, but such A exist by Theorem I.15.10.

Theorem I.15.10 is proved in model theory texts, such as [13, 65], and also in some general logic books, such as [57, 89]. For some applications in set theory, such as the proof of the Reflection Theorem (see Section II.5), one needs to know the proof of the theorem, not just its statement, so we shall outline the proof briefly. To obtain our $\mathfrak{A} \preccurlyeq \mathfrak{B}$, we use the following, which expresses $\mathfrak{A} \preccurlyeq \mathfrak{B}$ as a closure property of $\mathfrak{A}$.

Lemma I.15.11 (Tarski–Vaught criterion) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathcal{L}$ with $\mathfrak{A} \subseteq \mathfrak{B}$. Then the following are equivalent:*

1. $\mathfrak{A} \preccurlyeq \mathfrak{B}$.
2. *For all* existential *formulas $\varphi(\vec{x})$ of $\mathcal{L}$ (so $\varphi(\vec{x})$ is of the form $\exists y \psi(\vec{x}, y)$), and all $\vec{a} \in A$: if $\mathfrak{B} \models \varphi[\vec{a}]$, then there is some $b \in A$ such that $\mathfrak{B} \models \psi[\vec{a}, b]$.*

Proof of Theorem I.15.10. Following the terminology in Lemma I.15.11, for each existential formula φ of $\mathcal{L}$, choose a *Skolem function* f_φ as follows: Say φ has n free variables, which we list as $\vec{x} = (x_1, \ldots, x_n)$ and write φ as $\varphi(\vec{x})$. Then $\varphi(\vec{x})$ is of the form $\exists y \psi(\vec{x}, y)$. Let $f_\varphi : B^n \to B$ be such that for $\vec{a} \in B$, if $\mathfrak{B} \models \varphi[\vec{a}]$, then $\mathfrak{B} \models \psi[\vec{a}, f_\varphi(\vec{a})]$. So, if there is a $b \in B$ such that $\mathfrak{B} \models \psi[\vec{a}, b]$, then $f_\varphi(\vec{a})$ chooses one such b; if not, then $f_\varphi(\vec{a})$ can be chosen arbitrarily. Since $n = n_\varphi$ depends on φ, we really have $f_\varphi : B^{n_\varphi} \to B$. Such Skolem functions can be defined from a well-order of B.

Let $\mathcal{F}$ be the family of all these f_φ. Note that $|\mathcal{F}| \leq \kappa$ because $|\mathcal{L}| \leq \kappa$. Since $|B| \geq \kappa$, fix S' such that $S \subseteq S' \subseteq B$ and $|S'| = \kappa$. Let A be the closure of S' under $\mathcal{F}$. Then $|A| \leq \kappa$ by Lemma I.13.22, and hence $|A| = \kappa$.

Note that A is closed under all the functions of $\mathcal{L}$. That is, if $g \in \mathcal{L}$ is an n–ary function symbol, so that $g_{\mathfrak{B}} : B^n \to B$, then $g_{\mathfrak{B}}(A^n) \subseteq A$. To prove this, let $\varphi(\vec{x})$ be $\exists y[g(\vec{x}) = y]$, and note that the Skolem function f_φ must be the function $g_{\mathfrak{B}}$. In the $n = 0$ case, A contains $c_{\mathfrak{B}}$ for each constant symbol c. Now φ is the sentence $\exists y[c = y]$, and the Skolem function f_φ maps $B^0 = \{\emptyset\}$ into B, and $f_\varphi(\emptyset)$ must be $c_{\mathfrak{B}}$, so $c_{\mathfrak{B}} \in A$ (see the remark on 0–ary functions following Lemma I.13.22).

A thus defines a structure $\mathfrak{A} \subseteq \mathfrak{B}$, and then $\mathfrak{A} \preccurlyeq \mathfrak{B}$ is immediate from Lemma I.15.11. □

We turn now to formal provability and the notion $\Gamma \vdash \varphi$. Here, Γ is a set of sentences of $\mathcal{L}$ and φ is a sentence of $\mathcal{L}$. We assume that the reader has seen some precise definition of $\vdash$; the definition is different in different texts, but it always satisfies the Completeness and Soundness Theorems (see below), which relate the proof theory notion $\vdash$ to the model theory notion $\models$.

Using $\vdash$, we can make the definition of Incon(Γ) and Con(Γ) (see Definition I.2.1) more precise: Incon(Γ) means that $\Gamma \vdash \varphi \wedge \neg\varphi$ for some (equivalently for every) sentence φ of $\mathcal{L}$, and Con(Γ) means $\neg$Incon(Γ).

Theorem I.15.12 (Soundness) *Let Γ be a set of sentences of $\mathcal{L}$. Then*

1. *If $\mathfrak{A} \models \Gamma$ for some $\mathfrak{A}$, then* Con(Γ).
2. *For every sentence ψ of $\mathcal{L}$: If $\Gamma \vdash \psi$ then ψ is true in every $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$.*

Proof. We give a brief outline, since we plan to re-use the method later (see Lemma I.16.13). It is sufficient to prove (2). Then (1) will follow, since if Incon(Γ), so that $\Gamma \vdash \varphi \wedge \neg\varphi$, then $\varphi \wedge \neg\varphi$ must be true in every model of Γ, which implies that there are no such models.

The details of the proof of (2) depend on exactly which definition of $\vdash$ you are using. But, consider a Hilbert-style system such as described in [57]. Here, a formal proof of ψ consists of a finite sequence of sentences, $\varphi_0, \varphi_1, \ldots, \varphi_n$, where φ_n is ψ, and each φ_i is either in Γ, or is a logical axiom, or follows from

some φ_j, φ_k by Modus Ponens, where $j, k < i$. So, it is sufficient to assume that $\mathfrak{A} \models \Gamma$ and prove by induction on i that $\mathfrak{A} \models \varphi_i$.

The case $\varphi_i \in \Gamma$ is obvious, and the Modus Ponens case is handled by the observation that if $\mathfrak{A} \models \pi \to \theta$ and $\mathfrak{A} \models \pi$, then $\mathfrak{A} \models \theta$, which is obvious from the truth table for $\to$. Finally, one must check that $\mathfrak{A} \models \pi$ for any $\mathfrak{A}$ whenever π is a logical axiom. □

Theorem I.15.13 (Completeness) *Assume AC^+. Let Γ be a set of sentences of $\mathcal{L}$. Then*

1. *Con(Γ) iff there is some $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$.*
2. *For every sentence ψ of $\mathcal{L}$: $\Gamma \vdash \psi$ iff ψ is true in every $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$.*

One direction of these "iff"s is given by Soundness. The other (and *harder*!) direction is proved in most logic texts. The Completeness Theorem is not used much in set theory. We do often establish Con(Γ) by producing a model for Γ, but this only requires Soundness. But, as a matter of motivation, Completeness shows that this way of proving Con(Γ) is natural, since Con(Γ) cannot hold unless there is a model for Γ.

It is known that the Completeness Theorem is not provable in *ZF* (see the remarks after Exercise I.12.16). However, the Completeness Theorem is provable in $ZF^- - P$ in the case that $\mathcal{L}$ can be well-ordered (in particular, when $\mathcal{L}$ is countable).

We next consider reducts and expansions:

Definition I.15.14 *Assume that $\mathcal{L} \subseteq \mathcal{L}'$, $\mathfrak{A}$ is a structure for $\mathcal{L}$ and $\mathfrak{A}'$ is a structure for $\mathcal{L}'$. Then $\mathfrak{A}$ is a* reduct *of $\mathfrak{A}'$ and $\mathfrak{A}'$ is an* expansion *of $\mathfrak{A}$ iff $\mathfrak{A}$ and $\mathfrak{A}'$ have the same universe ($A = A'$), and the symbols of $\mathcal{L}$ are interpreted the same way in both structures.*

Let $\mathcal{L} = \{0, 1, +, -\}$, $\mathcal{L}' = \{0, 1, +, \cdot, -\}$, and $\mathcal{L}'' = \{0, 1, +, \cdot, -, <\}$. Let $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$ be, respectively, the *group* of real numbers, the *field* of real numbers, and the *ordered field* of real numbers, viewed in the obvious way as structures for $\mathcal{L}, \mathcal{L}', \mathcal{L}''$, respectively; so $A = A' = A'' = \mathbb{R}$. Then, $\mathfrak{A}'$ is an expansion of $\mathfrak{A}$ and $\mathfrak{A}''$ is an expansion of $\mathfrak{A}'$. $\mathfrak{A}$ and $\mathfrak{A}'$ have different model-theoretic properties. For example, $\mathfrak{A}$ has many (actually, $\beth_2$) automorphisms, whereas $id_{\mathbb{R}}$ is the only automorphism of $\mathfrak{A}'$ or $\mathfrak{A}''$.

But, $\mathfrak{A}'$ and $\mathfrak{A}''$ are "essentially the same" because $<$ is actually *definable* in $\mathfrak{A}'$: $x < y$ iff $x \neq y \wedge \exists z\, [y = x + z \cdot z]$. To state this in terms of the notion of extensions by definitions discussed in Section I.2:

Definition I.15.15 *If $\mathfrak{A}$ is a structure for $\mathcal{L}$, then the* theory of $\mathfrak{A}$, Th($\mathfrak{A}$), *is the set of all $\mathcal{L}$-sentences φ such that $\mathfrak{A} \models \varphi$.*

So, in the above example, $\mathrm{Th}(\mathfrak{A}'')$ is an extension by definitions of $\mathrm{Th}(\mathfrak{A}')$ in the sense of Definition I.2.2. Expansions arise naturally from any extension by definitions because of the following easy lemma:

Lemma I.15.16 *Assume that $\mathcal{L} \subseteq \mathcal{L}'$ and Γ is a set of sentences of $\mathcal{L}$, and assume that the set of sentences Γ' of $\mathcal{L}'$ is an extension by definitions of Γ. Then every model $\mathfrak{A} \models \Gamma$ has a unique expansion to a model $\mathfrak{A}' \models \Gamma'$.*

Using the Completeness Theorem, part (1) of Theorem I.2.3 is obvious from this; if φ is any sentence of $\mathcal{L}$, and $\Gamma' \vdash \varphi$ then $\Gamma \vdash \varphi$ because φ is true in every model $\mathfrak{A}$ for Γ (otherwise the expansion $\mathfrak{A}'$ would be a model for Γ' in which φ is false).

In our work in set theory, most of the expansions we consider will be definitional expansions, as in Lemma I.15.16. As a general convention:

Notation I.15.17 *Assume that $\mathcal{L} \subseteq \mathcal{L}^+$, where the symbols in $\mathcal{L}^+ \backslash \mathcal{L}$ have conventional definitions in the models we are considering. Let $\mathfrak{A}, \mathfrak{B}$ be structures for $\mathcal{L}$. If φ is a formula of $\mathcal{L}^+$ and if Γ is a set of sentences of $\mathcal{L}^+$, then the meaning of $\mathfrak{A} \models \Gamma$ and $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$ is obtained by* expanding $\mathfrak{A}, \mathfrak{B}$ *to structures for $\mathcal{L}^+$, where the symbols in $\mathcal{L}^+ \backslash \mathcal{L}$ are interpreted so that these conventional definitions are satisfied.*

There is obviously a lack of rigor here, since it may not be clear which of several "conventional definitions" is intended.

First, an example where there is no ambiguity. Let $\mathcal{L} = \{\in\}$ and consider the $\in$-model HF. We might say that $HF \models \forall x, y\, [x \cap y = y \cap x]$. Here, it is understood that we are using the "conventional" definition of $v \cap w$ as the unique y satisfying $\mathrm{int}(v, w, y)$, as we already discussed in Section I.3 and after Definition I.4.3. A different text might phrase the definition somewhat differently, but its clear that any reasonable definition of $v \cap w$ will be equivalent to our definition in any model that we consider.

Ambiguities may arise in set theory when a definition follows a long chain of earlier definitions (see the discussion of Theorem I.2.4). Even when discussing basic notions, where all mathematicians agree on their "conventional meaning" in ZFC, there are different ways of explaining the definitions that may not be equivalent in models of sub-theories of ZFC. For example, if we know that $M \models ZF$, what does "$M \models CH$" mean? CH is a defined proposition letter that has the same meaning to everyone working in ZFC, but what if you just work in ZF? From reading this book (Definition I.13.8), you might guess that CH means that $\mathcal{P}(\omega)$ can be well-ordered in type ω_1, but people who specialize in set theory without choice often mean the weaker statement that every uncountable subset of $\mathcal{P}(\omega)$ has a bijection onto $\mathcal{P}(\omega)$.

In this book, we shall try to be more explicit when it matters when talking about models for sub-theories of ZFC, without becoming too pedantic and worrying about the thousands of mathematical definitions for which there is no ambiguity.

The notions of isomorphism and elementary equivalence are often useful:

Definition I.15.18 *Suppose that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are structures for* $\mathcal{L}$. Φ *is an* isomorphism *from* $\mathfrak{A}$ *onto* $\mathfrak{B}$ *iff* $\Phi : A \xrightarrow[\text{onto}]{1-1} B$ *and* Φ *preserves the structure. Specifically:*

- ☞ *If* f *is an* n*-ary function symbol with* $n > 0$*, then* $f_{\mathfrak{B}}(\Phi(a_1), \ldots, \Phi(a_n)) = \Phi(f_{\mathfrak{A}}(a_1, \ldots, a_n))$ *for all* $a_1, \ldots, a_n \in A$.
- ☞ *If* p *is an* n*-ary predicate symbol with* $n > 0$*, then* $(\Phi(a_1), \ldots, \Phi(a_n)) \in p_{\mathfrak{B}}$ *iff* $(a_1, \ldots, a_n) \in p_{\mathfrak{A}}$ *for all* $a_1, \ldots, a_n \in A$.
- ☞ *If* c *is a* 0*-ary function symbol, then* $c_{\mathfrak{B}} = \Phi(c_{\mathfrak{A}})$.
- ☞ *If* p *is a* 0*-ary predicate symbol, then* $p_{\mathfrak{B}} = p_{\mathfrak{A}}$.

$\mathfrak{A}$,$\mathfrak{B}$ *are* isomorphic ($\mathfrak{A} \cong \mathfrak{B}$) *iff there exists an isomorphism from* $\mathfrak{A}$ *onto* $\mathfrak{B}$.

This generalizes the algebraic notion of isomorphism of groups, rings, etc., as well as our earlier Definition I.6.17 for binary relations.

Definition I.15.19 *If* $\mathfrak{A}$, $\mathfrak{B}$ *are structures for* $\mathcal{L}$*, then* $\mathfrak{A} \equiv \mathfrak{B}$ ($\mathfrak{A}, \mathfrak{B}$ *are* elementarily equivalent) *iff for all* $\mathcal{L}$*-sentences* φ*:* $\mathfrak{A} \models \varphi$ *iff* $\mathfrak{B} \models \varphi$.

So, $\mathfrak{A} \equiv \mathfrak{B}$ iff $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$ (see Definition I.15.15).

Lemma I.15.20 *If* $\mathfrak{A} \preccurlyeq \mathfrak{B}$ *or* $\mathfrak{A} \cong \mathfrak{B}$*, then* $\mathfrak{A} \equiv \mathfrak{B}$.

We shall frequently study proper class models and relativization in the case $\mathcal{L} = \{\in\}$, since this is a standard tool in obtaining independence proofs; see Sections I.16 and II.2. But, the next exercise illustrates that such relative interpretations make sense with other $\mathcal{L}$.

Exercise I.15.21 *Fix a prime* p*. Define a proper class* A *and a function* $+$ *such that* $(A, +)$ *is an abelian group in which every element other than* 0 *has order* p.

Hint. When $p = 2$, you can let $A = V$ and let $+ : V \to V$ be symmetric difference; $x + y = (x \backslash y) \cup (y \backslash x)$. □

Note that although V doesn't really exist, so you can't really form the pair $(V, +)$, it makes perfect sense to assert (in the metatheory) that each of the axioms of our group theory, when relativized to $(V, +)$, becomes a sentence in the language of set theory that is provable; e.g., with symmetric difference, $ZF^- - P \vdash \forall x, y, z\,[(x + y) + z = x + (y + z)]$ and $ZF^- - P \vdash \forall x\,[x + x = 0]$.

Exercise I.15.22 *Let* $\mathcal{L} = \{+\}$*, and consider symmetric difference* $+$ *on* V*. Show how to define the notion* $(V, +) \models \varphi[\sigma]$*, as in Definition I.15.5.*

Hint. Define $(V,+) \models \varphi[\sigma]$ iff $(A,+) \models \varphi[\sigma]$, where A is some (any) infinite *set* such that A is closed under $+$ and $A \supset \text{ran}(\sigma)$. To show that it doesn't matter which A you choose, use the fact that the theory of infinite abelian groups of exponent p is model-complete — that is, every submodel is an elementary submodel (see [57], Exercise II.16.10, or any model theory text). □

We next consider the notion of definability over a structure.

Definition I.15.23 *Let $\mathfrak{A}$ be a structure for $\mathcal{L}$ with $P \subseteq A$. Fix $k > 0$.*

1. *$S \subseteq A^k$ is* definable over $\mathfrak{A}$ with parameters in P *iff for some $n \geq 0$, there is a formula $\varphi(x_1, \ldots x_k, y_1, \ldots y_n)$ of $\mathcal{L}$ with $k+n$ free variables such that for some $b_1, \ldots, b_n \in P$, $S = \{\vec{a} \in A^k : \mathfrak{A} \models \varphi[a_1, \ldots, a_k, b_1, \ldots, b_n]\}$.*
2. *$S \subseteq A^k$ is definable over $\mathfrak{A}$* with parameters *iff S is definable over $\mathfrak{A}$ with parameters in A, and S is definable over $\mathfrak{A}$* without parameters *iff S is definable over $\mathfrak{A}$ with parameters in $\emptyset$.*
3. *For $a \in A$, we say the element a is* definable with parameters in P *or* definable without parameters *iff, respectively, the set $\{a\}$ is definable with parameters in P or definable without parameters.*

Note that if $P = \emptyset$, then n must be 0, so φ is just $\varphi(x_1, \ldots x_k)$. As usual, when $k = 1$, we identify S with a subset of A and talk about a definable *set*, rather than a 1-ary relation; we used this terminology already in (3).

Some examples from the field $\mathbb{R}$ of real numbers, with $\mathcal{L} = \{+, \cdot, <, 0, 1\}$: Both $\pm\sqrt{b}$ are definable with parameters in $\{b\}$ whenever $b \geq 0$, and $\sqrt{2}$ is definable without parameters. It is clear from cardinal arithmetic that $\mathbb{R}$ has $2^{2^{\aleph_0}}$ subsets, of which only $2^{\aleph_0}$ are definable with parameters, and only $\aleph_0$ are definable without parameters (since $\mathcal{L}$ is countable). Some more specific facts about definability require non-trivial model theory to verify. In particular, $S \subseteq \mathbb{R}$ is definable with parameters iff S is a finite union of intervals with endpoints in $\mathbb{R} \cup \{\pm\infty\}$; the intervals may be of the form (a, b) , $[a, \infty)$, $(-\infty, b)$, $[a, a] = \{a\}$, etc. When S is definable without parameters, the endpoints of the intervals must be $\pm\infty$ or algebraic numbers. See, e.g., §3.3 of Marker [65].

It is easy to see:

Lemma I.15.24 *Assume that every $a \in A$ is a definable (without parameters) element of $\mathfrak{A}$. Then every subset of A that is definable with parameters is also definable without parameters.*

In set theory, we consider these notions mainly for $\in$-models $(A, \in)$, in which case we shall, as per our usual convention, say "definable over A", rather than "definable over $(A, \in)$". Every $a \in HF$ is a definable element of HF; see our discussion of quoting conventions on page 15. Thus, by Lemma I.15.24 "definable without parameters" and "definable with parameters" are equivalent notions for subsets of HF, so we shall just use the term "definable".

In recursion theory, definable subsets of HF are called *arithmetical*. These include the *decidable* sets, which are definable by "very nice" formulas; see Section I.17.

Definition I.15.25 *For any set A and $P \subseteq A$: $\mathcal{D}(A,P)$ is the set of all subsets of A that are definable over $(A,\in)$ with parameters in P. Then $\mathcal{D}^+(A) = \mathcal{D}(A,A)$ and $\mathcal{D}^-(A) = \mathcal{D}(A,\emptyset)$. Whenever $\mathcal{D}^+(A) = \mathcal{D}^-(A)$, we denote them by $\mathcal{D}(A)$. $\mathcal{D}(\emptyset) = \mathcal{D}^+(\emptyset) = \mathcal{D}^-(\emptyset) = \{\emptyset\}$.*

We restated the obvious $\mathcal{D}(\emptyset) = \{\emptyset\}$ only because in model theory, structures are usually assumed to be non-empty. We shall use $\mathcal{D}^+$ in Section II.6, where we build Gödel's model L for $ZFC + GCH$ by replacing "$\mathcal{P}$" by "$\mathcal{D}^+$" in the recursive construction of WF described in Lemma I.9.30.

Note that the set $\mathcal{D}(A,P)$ can be defined in $ZF^- - P$, using the Replacement Axiom, not the Power Set Axiom.

A simple illustration of Definition I.15.25:

Lemma I.15.26 *Every finite subset of A is in $\mathcal{D}^+(A)$.*

Proof. To prove that $\{b_1,\dots,b_n\} \in \mathcal{D}^+(A)$, let $\varphi(x,y_1,\dots,y_n)$ from Definition I.15.23 be the formula $x = y_1 \vee \cdots \vee x = y_n$. □

This lemma is (an abbreviation of) a single logical sentence that is provable from $ZF^- - P$. The next lemma is really a scheme in the metatheory:

Lemma I.15.27 *For each set theory formula $\varphi(x,y_1,\dots y_n)$,*

$$ZF^- - P \vdash \forall A \forall b_1,\dots,b_n \in A \left[\{x \in A : (\varphi(x,b_1,\dots,b_n))^A\} \in \mathcal{D}^+(A)\right] .$$

This is essentially obvious from the definitions, although it does involve the correspondence between objects in the metatheory and objects in the formal theory discussed in Subsection I.1.1. Here the objects are formulas. We are really using our quoting convention described on page 15 to form the object $\ulcorner\varphi\urcorner$ and then Thesis I.3.2 to assert that one can prove in $ZF^- - P$ the assertion that $\ulcorner\varphi\urcorner$ is a logical formula with the correct syntactic properties; this formula is then plugged into Definitions I.15.25 and I.15.23.

The following exercise, which is easy to do using the Tarski–Vaught criterion (Lemma I.15.11), provides a slightly different proof of the Downward Löwenheim–Skolem–Tarski Theorem:

Exercise I.15.28 *Fix $P \subseteq A$, and assume that there is a binary relation R on A that well-orders A and that is definable over $\mathfrak{A}$ with parameters in P. Let $\mathfrak{H}(\mathfrak{A},P)$ be the set of elements of A that are definable with parameters in P. Then $\mathfrak{H}(\mathfrak{A},P)$ is an elementary submodel of $\mathfrak{A}$.*

Note that $P \subseteq \mathfrak{H}(\mathfrak{A}, P)$; $\mathfrak{H}(\mathfrak{A}, P)$ is sometimes called the *Skolem hull* of P. Now, say $\mathfrak{A}$ is arbitrary and $\mathcal{L}$ and P are countable, and we wish to find a countable $\mathfrak{C} \preccurlyeq \mathfrak{A}$ with $P \subseteq C$. $\mathfrak{H}(\mathfrak{A}, P)$ is certainly countable, but if $\mathfrak{A}$ lacks a definable well-order, we cannot claim that $\mathfrak{H}(\mathfrak{A}, P) \preccurlyeq \mathfrak{A}$ (for example, $\mathfrak{H}((\mathbb{R}, <), P)$ is just P for all $P \subseteq \mathbb{R}$). However, using AC, we can expand $\mathfrak{A}$ to $\mathfrak{A}'$ (Definition I.15.14) by adding a binary relation symbol to $\mathcal{L}$, interpreted as some well-order of A; then $\mathfrak{H}(\mathfrak{A}', P) \preccurlyeq \mathfrak{A}'$.

I.16 Models of Set Theory

The independence proofs in set theory use model-theoretic concepts applied with $\mathcal{L} = \{\in\}$ to study models that satisfy some (or all) of the *ZFC* axioms. In this section, we consider some of the notions of Section I.15 in this context. Most of the time, our models will be $\in$–models (see Definition I.15.1). A detailed discussion of which models satisfy which axioms will take place in Chapter II. Here, we focus on a few points of model theory. As in Section I.15, we shall work primarily in $ZF^- - P$.

As mentioned in Section I.15, it is hard to find explicit non-trivial examples of $A \preccurlyeq B$ where A, B are $\in$–models of some interesting set theory. But, if one considers $A \preccurlyeq_\varphi B$ for specific φ, then there are non-trivial interesting examples. The most well-known of these is when φ is a Δ_0 formula. With $\mathcal{L} = \{\in\}$, these are formulas in which all quantifiers are bounded; that is, occur as $\forall x \in y \cdots$ or $\exists x \in y \cdots$. Generalizing slightly:

Definition I.16.1 *Assume that $\mathcal{L}$ contains the symbol $\in$ plus possibly other predicate and function symbols. Then the* Δ_0 formulas *of $\mathcal{L}$ are those formulas constructed by the rules:*

a. *All atomic formulas are Δ_0 formulas.*
b. *If φ is a Δ_0 formula, y is a variable, and τ is a term that does not contain y, then $\forall y \in \tau\ \varphi$ and $\exists y \in \tau\ \varphi$ are Δ_0 formulas.*
c. *If φ is a Δ_0 formula, then so is $\neg\varphi$.*
d. *If φ and ψ are Δ_0 formulas, then so are $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \to \psi$, and $\varphi \leftrightarrow \psi$.*

As usual, $\forall y \in \tau\ \varphi$ and $\exists y \in \tau\ \varphi$ are just abbreviations for $\forall y\,[y \in \tau \to \varphi]$ and $\exists y\,[y \in \tau \wedge \varphi]$. Then

Lemma I.16.2 *Let $\mathcal{L}$ be as in Definition I.16.1, and assume that $\mathfrak{A} \subseteq \mathfrak{B}$. Also, assume that A is a transitive set and $\in_A = \{(a, b) \in A \times A : a \in b\}$ and $\in_B = \{(a, b) \in B \times B : a \in b\}$. Then $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$ for all Δ_0 formulas φ of $\mathcal{L}$.*

Proof. Induct on φ. The basis, when φ is atomic, is just Lemma I.15.9. The induction steps for propositional connectives are trivial. For the induction step for $\exists$, assume that $\varphi(\vec{x}, z)$ is $\exists y\,[y \in \tau(\vec{x}, z) \wedge \psi(\vec{x}, y, z)]$, where ψ is Δ_0, and assume (inductively) that $A \preccurlyeq_\psi B$. For any tuple $\vec{a}$ from A and $c \in A$, let $\tau_{\vec{a},c}$

abbreviate $\text{val}_{\mathfrak{A}}\tau[\vec{a}, c]$, which is the same as $\text{val}_{\mathfrak{B}}\tau[\vec{a}, c]$ because $\mathfrak{A} \subseteq \mathfrak{B}$. Then, for such $\vec{a}, c$, the definition of $\models$ yields:

$$A \models \varphi[\vec{a}, c] \leftrightarrow \exists b \in A \left\{ b \in \tau_{\vec{a},c} \wedge A \models \psi\,[\vec{a}, b, c] \right\} \leftrightarrow$$
$$\exists b \in B \left\{ b \in \tau_{\vec{a},c} \wedge B \models \psi\,[\vec{a}, b, c] \right\} \leftrightarrow B \models \varphi[\vec{a}, c]$$

The second $\leftrightarrow$ uses $A \preccurlyeq_\psi B$ along with the fact that A is transitive, so that $\tau_{\vec{a},c} \subseteq A \subseteq B$, so the "$\exists b \in A$" and "$\exists b \in B$" could both be replaced by "$\exists b$".

The induction step for $\forall$ is similar. □

The proviso in (b) of Definition I.16.1 that τ not contain y, was used implicitly in this proof, when we displayed the variables $\vec{x}, z$ that τ depends on. This proviso is necessary. For example, suppose that $S \in \mathcal{L}$, denoting $S(y) = y \cup \{y\}$. Then $\exists y\, \varphi$ is equivalent to $\exists y \in S(y)\, \varphi$, so if we deleted this proviso from (b), then every formula would be equivalent to a Δ_0 formula.

When $\mathcal{L} = \{\in\}$, the interest in this lemma is that many (but not all!) of the simpler properties of sets are expressed by Δ_0 formulas, so these have the same meaning in all transitive models. For now, we content ourselves with the following five examples; further examples will be covered in Chapter II. These five examples are the formulas we used in Section I.3 to define the notions $\subseteq$, $\emptyset$, S, $\cap$, $\text{SING}(x)$, which were used to express Axioms 7,8,9. Because of the way we customarily write formulas, they are not all actually Δ_0, but they are all logically equivalent to Δ_0 formulas:

Example I.16.3 *The following formulas in $\mathcal{L} = \{\in\}$ are logically equivalent to Δ_0 formulas:*

$\forall z(z \in x \to z \in y)$	$\forall z \in x(z \in y)$	$x \subseteq y$
$\forall z(z \notin x)$	$\forall z \in x(z \neq z)$	$\text{emp}(x)$ or $x = \emptyset$
$\forall z(z \in y \leftrightarrow z \in x \vee z = x)$	$x \in y \wedge x \subseteq y \wedge$ $\forall z \in y(z = x \vee z \in x)$	$y = S(x)$
$\forall x(x \in y \leftrightarrow x \in v \wedge x \in w)$	$y \subseteq v \wedge y \subseteq w \wedge$ $\forall x \in v(\forall x \in w(x \in y))$	$\text{int}(v, w, y)$ or $y = v \cap w$
$\exists y \in x \forall z \in x(z = y)$	$\exists y \in x \forall z \in x(z = y)$	$\text{SING}(x)$

The first column is the formula as in Section I.3, and the third column shows the notion defined by that formula. The second column is a logically equivalent Δ_0 formula. In the third and fourth row, one must replace the "$\subseteq$" by a Δ_0 formula that defines $\subseteq$ to obtain the actual Δ_0 formula for column two.

In set theory, one often phrases $A \preccurlyeq_\varphi B$ in terms of *absoluteness*:

Definition I.16.4 *φ is* absolute *for A, B iff $A \preccurlyeq_\varphi B$.*

So, using Lemma I.16.2 and Example I.16.3, we say "$\subseteq$ is absolute for transitive models", with $A \preccurlyeq_{x \subseteq y} B$ being an example of our Notation I.15.17. It should be clear from our discussion of that notation in Section I.15 that our terminology presumes that we are defining $\subseteq$ in its "natural" Δ_0 way.

We next note that the definition of "absolute" makes sense even when dealing with proper class models, so we can define

Definition I.16.5 *φ is* absolute *for A iff $A \preccurlyeq_\varphi V$.*

As we have noted in earlier examples, statements about set models that are single statements expressed in the language of set theory become schemes in the metatheory when applied to class models. For example, Lemma I.16.2, when talking about set models, is an abbreviation for one sentence in the language of set theory, $\forall \varphi, \mathfrak{A}, \mathfrak{B}\, [\cdots\cdots \to \mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}]$, and this is provable from $ZF^- - P$; we are using Tarski's Definition I.15.5 of truth to express $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$. When talking about classes $A = \{x : \alpha(x)\}$ and $B = \{x : \beta(x)\}$, where α, β are formulas, then statements about A and B are really expressed using the formulas α and β. So, Lemma I.16.2 is really a scheme in the metatheory that given any such α, β, and a Δ_0 formula φ, if we can prove that $A \subseteq B$ (i.e., $\forall x(\alpha(x) \to \beta(x))$), and we can prove that A is transitive, then we can also prove $A \preccurlyeq_\varphi B$, meaning, using our superscripted Notation I.15.6: $\forall x_1, \ldots, x_n \in A[\varphi^A(\vec{x}) \leftrightarrow \varphi^B(\vec{x})]$. In particular B could be V, in which case $A \subseteq B$ is trivial, and φ^V (relativize all the quantifiers to V) is equivalent to φ, so saying that φ is absolute for A means $\forall x_1, \ldots, x_n \in A[\varphi^A(\vec{x}) \leftrightarrow \varphi(\vec{x})]$; that is, φ and φ^A have the same meaning for members of A.

As another example, we can write the statement $\exists \gamma, \delta\, [\gamma < \delta \wedge R(\gamma) \preccurlyeq R(\delta)]$ as one sentence, although this sentence is not provable in ZFC, since it implies that $R(\gamma) \models ZFC$ (see Exercise I.16.8). But, we can't even *say* $\exists \gamma\, [R(\gamma) \preccurlyeq V]$. We might assert that $R(\omega + \omega) \not\preccurlyeq V$, conveying that we have an explicit φ in mind such that $R(\omega + \omega) \not\preccurlyeq_\varphi V$; φ could be a sentence that asserts that the ordinal $\omega + \omega$ exists; then φ is true but $\varphi^{R(\omega+\omega)}$ is false. Since φ is a sentence, we could also say that $R(\omega + \omega) \not\equiv V$.

Note that Lemma I.16.2 allowed $\mathcal{L}$ to contain symbols other than $\in$, although we haven't used this yet in any examples. This is important in set theory because it often gives us $A \preccurlyeq_\varphi B$ for formulas that are not Δ_0 in the usual sense. Again, we refer to Notation I.15.17. For many examples, see Section II.4, but as a preview, let $\varphi(x, y)$ be $x \approx y$, saying that there is a bijection from x onto y. Our definition (I.6.18) of this could be written as $\exists f \in \mathcal{P}(x \times y)\, [f : x \underset{\text{onto}}{1\text{–}1} y]$. This is Δ_0 if $\mathcal{L}$ contains the unary function $\mathcal{P}$, the binary function $\times$, and the ternary predicate bij (expressing $f : x \underset{\text{onto}}{1\text{–}1} y$), all of which are defined in models of ZC. Now $\times$ and bij are absolute for transitive models of ZC (see Lemma II.4.3), but $\mathcal{P}$ is not, and there are transitive $A \subseteq B$ such that $A \not\preccurlyeq_\varphi B$; this happens when there are $a, b \in A$, with an $f \in B$ such that $f : a \underset{\text{onto}}{1\text{–}1} b$, but no such bijection lies in A (see Exercise I.16.7 below). However, if γ, δ are limit ordinals with $\omega < \gamma < \delta$, then $R(\gamma) \preccurlyeq_\varphi R(\delta)$ because

this φ, along with most other formulas of mathematical interest, can be expressed by quantification bounded by appropriate power sets. The power set function is absolute for the $R(\gamma)$:

Exercise I.16.6 (ZF^-) *Let* $\mathrm{pow}(x,y)$ *be* $\forall z\,[z \in y \leftrightarrow z \subseteq x]$, *asserting that* y *is the power set of* x. *Let* γ *be a limit ordinal, with* $a, b \in R(\gamma)$. *Prove that* $R(\gamma) \models \mathrm{pow}[a,b]$ *iff* $b = \mathcal{P}(a)$; *that is,* $R(\gamma) \preccurlyeq_{\mathrm{pow}} V$.

Note also that $a \in R(\gamma) \to \mathcal{P}(a) \in R(\gamma)$, since $\mathrm{rank}(\mathcal{P}(a)) = \mathrm{rank}(a) + 1$. To deduce that $R(\gamma) \preccurlyeq_\varphi R(\delta)$ and $R(\gamma) \preccurlyeq_\varphi V$ with the above φ, we apply Lemma I.16.2, where $\mathcal{L} = \{\in, \mathcal{P}, \times, \mathrm{bij}\}$ and $\mathfrak{A}$ is the natural structure for $\mathcal{L}$ with universe $R(\gamma)$.

Exercise I.16.7 (ZFC^-) *Let* $\gamma > \omega_1$ *be a limit ordinal. Prove that there is a countable transitive* M *and ordinals* $\alpha, \beta \in M$ *such that* $M \equiv R(\gamma)$ *and* $(\alpha \approx \beta)^M$ *is false but* $(\alpha \approx \beta)^{R(\gamma)}$ *is true.*

Hint. By the Downward Löwenheim–Skolem–Tarski Theorem, get a countable A with $\omega, \omega_1 \in A \preccurlyeq R(\gamma)$. Then, let M be the Mostowski collapse of A; let $\alpha = \mathrm{mos}(\omega) = \omega$ and $\beta = \mathrm{mos}(\omega_1)$. Then β will be a countable ordinal that M "thinks" is uncountable. □

If γ is strongly inaccessible, then our models $R(\gamma), A, M$ will satisfy all of ZFC (see Corollary II.2.3), but even without strong inaccessibles, one can get them to satisfy any given finite subset of ZFC (see Corollary II.5.4); in particular, $R(\gamma)$ can satisfy enough set theory to prove all of known mathematics. This exercise is related to the so-called *Skolem Paradox*: We have a countable transitive model M satisfying enough set theory to produce an uncountable ordinal $\beta \in M$, but $\beta \in M$ implies that β is countable, yielding a "contradiction". Such "paradoxes" in set theory are really just errors in thinking — here, the error is in not realizing that "countable" is not absolute.

Exercise I.16.8 (ZFC^-) *Assume that* $0 < \gamma < \delta$ *and* $R(\gamma) \preccurlyeq R(\delta)$. *Prove that* $R(\gamma) \models ZFC$ *and hence* $R(\delta) \models ZFC$. *You may use the fact, to be proved in Section II.2, that* $R(\gamma) \models ZC$ *for any limit* $\gamma > \omega$.

Hint. γ is a limit, since if $\gamma = S(\alpha)$, then "$\preccurlyeq$" would fail with the formula "$S(\alpha)$ exists". Then show that $\gamma > \omega$. For the Replacement Axiom in $R(\gamma)$, if $A \in R(\gamma)$ and if $\forall x \in A\, \exists! y\, \varphi(x,y)$ holds, then $\exists B\, \forall x \in A\, \exists y \in B\, \varphi(x,y)$ must hold in $R(\delta)$. □

By the Incompleteness Theorem, ZFC cannot prove that there are such γ, δ. Note that ZFC does prove that there are γ, δ with $0 < \gamma < \delta$ and $R(\gamma) \equiv R(\delta)$, since there are only $2^{\aleph_0}$ possibilities for $\mathrm{Th}(R(\gamma))$.

We remark that one can see that $ZFC \nvdash \exists \gamma\, [R(\gamma) \models ZFC]$ by using a proof that is much simpler than that of the Incompleteness Theorem:

Exercise I.16.9 *Assume that* $ZFC \vdash \exists \gamma\, [R(\gamma) \models ZFC]$. *Show directly that ZFC is inconsistent.*

Hint. Let δ be least such that $R(\delta) \models ZFC$. Then $\exists \gamma \in R(\delta)\, [R(\gamma) \models ZFC]$, leading to a contradiction. To make this proof 100% rigorous, you need to use some of the absoluteness results discussed in Section II.4. □

Exercise I.16.10 *Show how to modify Definition I.15.5 to give a correct definition of* $(V, \in) \models \varphi[\sigma]$ *in the case of* Δ_0 *formulas.*

Hint. This is similar in spirit to Exercise I.15.22. Just define $(V, \in) \models \varphi[\sigma]$ iff $(A, \in) \models \varphi[\sigma]$, where A is some (any) transitive *set* such that $A \supset \text{ran}(\sigma)$. To show that it doesn't matter which A you choose, use the fact that Δ_0 formulas are absolute for transitive models. □

The use of proper classes as models is related to *relative interpretations*. This is a general notion in formal logic. We discuss it in this book because it is used in the relative consistency proofs in set theory, such as $\text{Con}(ZF^-) \rightarrow \text{Con}(ZF)$ (von Neumann; see Section II.2) and $\text{Con}(ZF) \rightarrow \text{Con}(ZFC + GCH)$ (Gödel; see Section II.6). More generally, relative interpretations yield proofs of $\text{Con}(\Lambda) \rightarrow \text{Con}(\Gamma)$ for various axiomatic theories which need have nothing at all to do with set theory; see, e.g., §2.7 of Enderton [29]. Here, Λ is written in some finite lexicon, $\mathcal{L}_0$, and Γ is in some possibly different finite lexicon, $\mathcal{L}$. Since we are focused on set theory applications, $\mathcal{L}_0$ will always be $\{\in\}$. $\mathcal{L}$ will often be $\{\in\}$, but it doesn't have to be. So, Λ will be some axioms for set theory and Γ will be some other axioms, often (but not always) for set theory. For example, if Λ is ZF^- and Γ is ZF, then $\mathcal{L} = \{\in\}$ and we shall prove that $\text{Con}(ZF^-) \rightarrow \text{Con}(ZF)$. To do this, we we work in ZF^- and define the class $M = WF$, and show that all axioms of ZF hold in M. Of course, the formal logic doesn't mention the "class"; it just uses the formula $\mu(x)$ saying that x is well-founded (Definition I.9.20). Details for WF are in Section II.2. In this section, we discuss the basic logic behind such proofs.

As a very simple illustration where $\mathcal{L} \neq \{\in\}$, let $\mathcal{L} = \{+, \cdot, 0\}$. Let Γ be the axioms for rings, written in this language. We may consider the whole universe V as a model for Γ if we interpret $+$ as symmetric difference (as in the hint to Exercise I.15.21), $\cdot$ as $\cap$, and 0 as $\emptyset$. This yields a relative interpretation of ring theory into set theory. We shall explain why even though V is a proper class, this relativization yields a relative consistency proof of $\text{Con}(ZF) \rightarrow \text{Con}(\Gamma)$.

In concept, the notion of relativization is exactly the same as the model theory notion of structure $\mathfrak{A}$ (Definition I.15.2), but formally, since A could be a proper class, the definition takes place in the metatheory.

Definition I.16.11 *If* Λ *is some axioms in* $\{\in\}$ *for set theory, and* $\mathcal{L}$ *is a finite lexicon, a* relative interpretation of $\mathcal{L}$ in Λ *consists of a non-empty class* A *together with an assignment of a suitable semantic entity* $s^{\mathfrak{A}}$ *to each symbol of* $\mathcal{L}$. *Specifically,*

1. *If f is an n-ary function symbol with $n > 0$, then $f^{\mathfrak{A}} : A^n \to A$.*
2. *If p is an n-ary predicate symbol with $n > 0$, then $p^{\mathfrak{A}} \subseteq A^n$.*
3. *If c is a 0-ary function symbol then $c^{\mathfrak{A}} \in A$.*
4. *If p is a 0-ary predicate symbol then $p^{\mathfrak{A}} \in 2 = \{0, 1\} = \{F, T\}$.*

Again, since A could be a proper class, we are really working in the metatheory and defining formulas. For A, we really have a formula $\alpha(x)$; the statement that A is non-empty really means $\Lambda \vdash \exists x \alpha(x)$. In items (2)(4), $p^{\mathfrak{A}}$ is really a formula with n free variables. In item (3), $c^{\mathfrak{A}}$ is really a defined constant, which means that we have a formula $\varphi(y)$ in $\{\in\}$ for which $\Lambda \vdash \exists! y \in A \varphi(y)$. In item (1), we have a formula $\varphi(x_1, \ldots, x_n, y)$ in $\{\in\}$ for which $\Lambda \vdash \forall x_1, \ldots, x_n \in A \; \exists! y \in A \; \varphi(\vec{x}, y)$. Note that φ need not quantify just over A; it is just a definition in set theory that specifies what $f^{\mathfrak{A}}$ is. For example, we may talk about the "structure" $(ON, <, \cdot)$. So, $<$ is just membership, but $\cdot$ is ordinal multiplication, so $\varphi(x_1, x_2, y)$ says that y is isomorphic to $x_2 \times x_1$; the isomorphism is a function, not an ordinal. Here Λ needs to have enough of ZF to be able to prove basic facts about functions and ordinals.

As with set models, we frequently use A for $\mathfrak{A}$ if it is clear from context what the interpretations for the symbols of $\mathcal{L}$ are.

We now restate the model theory notions of values of terms and truth values of formulas (Definitions I.15.3, I.15.4, and I.15.5), using our relativization notation. Again, this is exactly the same as these notions for set models, except that formally, the recursion is in the metatheory. That is, as discussed on page 83, we are using the obvious naive relativization notation, and *not* using Tarski's definition of truth (which wouldn't make sense in general for class models).

Definition I.16.12 *Given a relative interpretation $\mathfrak{A}$ as in Definition I.16.11, if τ is a term of $\mathcal{L}$ and φ is a formula, then $\tau^{\mathfrak{A}}$ is the term and $\varphi^{\mathfrak{A}}$ is the formula obtained by replacing the various symbols (f, p, c) of $\mathcal{L}$ by their relativized forms $(f^{\mathfrak{A}}, p^{\mathfrak{A}}, c^{\mathfrak{A}})$, and relativizing all quantifiers to A, replacing $\forall x \cdots$ by $\forall x \in A \cdots$ and $\exists x \cdots$ by $\exists x \in A \cdots$.*

We continue our example with $\mathcal{L} = \{+, \cdot, 0\}$, where $+^{\mathfrak{A}}$ denotes symmetric difference and $\cdot^{\mathfrak{A}}$ denotes intersection. A can be V, or, any other class or set containing $\emptyset$ and closed under symmetric difference and intersection. Then the commutativity of $\cdot$ holds: $(\forall x, y \, [x \cdot y = y \cdot x])^{\mathfrak{A}}$, since $x \cdot^{\mathfrak{A}} y = y \cdot^{\mathfrak{A}} x$ reduces to $x \cap y = y \cap x$. It is easy to verify also the axioms of ring theory, such as the distributive law, $(\forall x, y, z \, [(x + y) \cdot z = y \cdot z + y \cdot z])^{\mathfrak{A}}$. Also, we have $(\forall x \, [x \cdot x = x])^{\mathfrak{A}}$, so $\mathfrak{A}$ is a *boolean* ring.

So far, we are just elaborating a bit further on our old Notation I.6.27, but we now need to point out that such relativizations yield relative consistency proofs. To do that, we use the following lemma. This lemma conceptually is exactly the same as the Soundness Theorem I.15.12, although formally here the induction is in the metatheory:

Lemma I.16.13 *Let $\mathfrak{A}$ be a relative interpretation of $\mathcal{L}$ in Λ, in the sense of Definition I.16.11. Let ψ and $\varphi_1, \ldots, \varphi_k$ be sentences of $\mathcal{L}$ and assume that $\{\theta_1, \ldots, \theta_k\} \vdash \psi$. Then $\Lambda \vdash (\theta_1^{\mathfrak{A}} \wedge \cdots \wedge \theta_k^{\mathfrak{A}}) \to \psi^{\mathfrak{A}}$.*

Proof. As with the proof of Theorem I.15.12, we leave the details to the reader, since the details depend on exactly which definition of $\vdash$ you are using. In a Hilbert-style system, you have a formal proof of ψ consisting of a finite sequence of sentences, $\varphi_0, \varphi_1, \ldots, \varphi_n$, where φ_n is ψ, and each φ_i is either in $\{\theta_1, \ldots, \theta_k\}$, or is a logical axiom, or follows from some φ_j, φ_k by Modus Ponens, where $j, k < i$. So, it is sufficient to assume that $\mathfrak{A} \models \Gamma$ and prove by induction on i that $\Lambda \vdash (\theta_1^{\mathfrak{A}} \wedge \cdots \wedge \theta_k^{\mathfrak{A}}) \to \psi^{\mathfrak{A}}$.

The case $\varphi_i \in \{\theta_1, \ldots, \theta_k\}$ is obvious, and the Modus Ponens case is easy. Finally, one must check that $\Lambda \vdash \pi^{\mathfrak{A}}$ whenever π is a logical axiom.

In most of this argument, Λ is not used. It is used occasionally to prove that $A \neq \emptyset$, for the occasional logical axiom whose validity relies on this. Also, the set theory axioms in Λ may be needed to verify that the defined functions in $\mathfrak{A}$ really make sense. □

In many cases, this lemma is used implicitly in mathematics, and is considered to be essentially obvious. To continue our example with $\mathcal{L} = \{+, \cdot, 0\}$, using V as the "model", once you verify that the axioms for boolean rings are satisfied by $+$ and $\cdot$ and 0, you are justified in using any conclusion that you can prove from these axioms; for example, cancellation holds for symmetric difference: $\forall xyz[x + y = x + z \to y = z]$. Or, once one has verified that ON is totally ordered by $<$ (i.e., $\in$), one freely uses other facts about ON that are obvious consequences of the total order axioms, such as $\neg\exists\alpha, \beta, \gamma\,[\alpha < \beta \wedge \beta < \gamma \wedge \gamma < \alpha]$.

The following corollary to Lemma I.16.13 is the basis of many of the relative consistency proofs in set theory:

Corollary I.16.14 *Let $\mathfrak{A}$ be a relative interpretation of $\mathcal{L}$ in Λ, in the sense of Definition I.16.11. Let Γ be some sentences of $\mathcal{L}$, and assume that we can verify that $\Lambda \vdash \theta^{\mathfrak{A}}$ for each axiom θ of Γ. Then in the metatheory, we can conclude of* $\mathrm{Con}(\Lambda) \to \mathrm{Con}(\Gamma)$.

Proof. Arguing in the metatheory, suppose that we had a contradiction in Γ, so $\{\theta_1, \ldots, \theta_k\} \vdash (\psi \wedge \neg\psi)$, where $\theta_1, \ldots, \theta_k$ are axioms of Γ. Applying Lemma I.16.13, $\Lambda \vdash (\psi \wedge \neg\psi)^{\mathfrak{A}}$, which is $\psi^{\mathfrak{A}} \wedge \neg\psi^{\mathfrak{A}}$, which yields a contradiction from Λ. □

Using our example with $\mathcal{L} = \{+, \cdot, 0\}$ and V as the "model", where Γ is the axioms for boolean rings and Λ is ZF^-, we have proved $\mathrm{Con}(ZF^-) \to \mathrm{Con}(\Gamma)$. But, this is completely uninteresting, since $\mathrm{Con}(\Gamma)$ can be established easily by a finitistic argument, since Γ has a finite model (see also Exercise I.16.20). But, the same basic logic will give us a proof of $\mathrm{Con}(ZF^-) \to \mathrm{Con}(ZF)$; here, $\mathcal{L} = \{\in\}$ and we work in ZF^- and use WF as the model; see Section II.2.

While we are discussing the basic logic, consider the meaning of Notation I.15.17 when we add defined predicate and function symbols. In general, a formal axiomatic system is developed by going through a sequence of definitions of predicates and functions; see, e.g., Section I.2. Regarding the relativization of defined predicates:

Definition I.16.15 *If p is an n-ary predicate symbol introduced by the definition $\forall x_1, \ldots, x_n\, [p(x_1, \ldots, x_n) \leftrightarrow \theta(x_1, \ldots, x_n)]$, then $p^{M,E}(x_1, \ldots, x_n)$ is the formula $\theta^{M,E}(x_1, \ldots, x_n)$, making the obvious changes in the bound variables of θ. Now, the relativization $\varphi^{\mathfrak{A}}$ is defined, using Definition I.16.12, for all formulas using p. Also, the notion "φ is absolute for M" is now defined for all formulas using p.*

To illustrate the "obvious changes in the bound variables": we said in Section I.3 that $x \subseteq y$ was defined as $\forall z(z \in x \to z \in y)$. But it would be absurd to replace $z \subseteq x$ by $\forall z(z \in z \to z \in x)$; rather, we change the bound z to a v, obtaining $\forall v(v \in z \to v \in x)$.

Since $\subseteq$ is defined by a Δ_0 formula in $\in$, Lemma I.16.2 implies that $\subseteq$ is absolute for all transitive M; but then, Lemma I.16.2 with $\mathcal{L} = \{\in, \subseteq\}$ implies that every Δ_0 formula in $\in, \subseteq$ is also absolute for all transitive M.

In Definition I.16.15, an equivalent way to obtain the relativization $\varphi^{\mathfrak{A}}$ is first to translate φ in the standard way (see Theorem I.2.3) to a formula $\widehat{\varphi}$ that does not use p, and then relativize that formula.

Defined functions are handled similarly, but are slightly trickier:

Definition I.16.16 *Suppose that f is an n-ary function symbol introduced by the definition $\forall x_1, \ldots, x_n\, [\theta(x_1, \ldots, x_n, f(x_1, \ldots, x_n))]$, and suppose that we have verified $[\forall x_1, \ldots, x_n \exists! y\, \theta(x_1, \ldots, x_n, y)]^{M,E}$. Then, for $x_1, \ldots, x_n \in M$, $f^{M,E}(\vec{x})$ denotes the unique $y \in M$ such that $\theta(x_1, \ldots, x_n, y)^{M,E}$. To obtain the relativization $\varphi^{M,E}$ for formulas using f,* either *first translate φ in the standard way to a formula $\widehat{\varphi}$ that does not use f, and then apply Definition I.16.12,* or *apply Definition I.16.12 to the expanded language with f added. Also, the notion "φ is absolute for M" is now defined for all formulas using f. We say that f is absolute for M iff $f^{M,E}(\vec{x}) = f(\vec{x})$ for all $x_1, \ldots, x_n \in M$.*

For example, consider the intersection $v \cap w$, where we actually spelled out the defining formula $\mathrm{int}(v, w, y)$ as $\forall x(x \in y \leftrightarrow x \in v \wedge x \in w)$ in Definition I.4.1 (and see also Example I.16.3). Although int^M is defined for every M, $\cap^M$ is defined only if for all $v, w \in M$, there is a unique $y \in M$ such that $\forall x \in M\, [x \in y \leftrightarrow x \in v \wedge x \in w]$, in which case $v \cap^M w$ is that unique y. Then, $\cap$ is *absolute* for M iff $\cap^M$ is defined and also $v \cap^M w = v \cap w$ for all $v, w \in M$.

Since (Example I.16.3) int is equivalent to a Δ_0 formula, it is absolute for every transitive M. So for transitive M, *if* $\cap^M$ is defined, then $\cap$ is absolute for M. But also,

Exercise I.16.17 *Show that:*

1. *If* $M = ON \cup \{\{\alpha, \beta\} : \alpha < \beta \in ON\}$, *then* M *is transitive and* $\cap^M$ *is not defined.*
2. *Describe a two-element non-transitive* M *that is isomorphic to* $\{0, 1\}$, *such that* $\cap^M$ *is defined, but* $\cap$ *is not absolute for* M, *and such that* $\subseteq$ *is not absolute for* M.

The $\widehat{\varphi}$ referred to in Definition I.16.16 may involve the introduction of additional quantifiers. For example, if $\varphi(a, b, c, d)$ is $((a \cap b) \cap c) \in d$ then $\widehat{\varphi}(a, b, c, d)$ can be $\exists y \exists z\, [\mathrm{int}(a, b, y) \wedge \mathrm{int}(y, c, z) \wedge z \in d]$. Different logic texts will have slightly different methods of obtaining $\widehat{\varphi}$ from φ.

In practice, we usually verify the $[\forall x_1, \ldots, x_n \exists! y\, \theta(x_1, \ldots, x_n, y)]^M$ of Definition I.16.16 by verifying that it is a theorem of set theory and verifying that the axioms used in the proof hold in (M, E). For example, $\forall v, w\, \exists! y\, \mathrm{int}(v, w, y)$ is provable from Axioms 1,3 (see Lemma I.4.2), so working in $ZF^- - P$, we may assert that $\cap^M$ is defined for any model M for these two axioms. A similar discussion holds for the 0–ary function $\emptyset$ and the unary function S, which were discussed in Section I.4, using just the Extensionality, Comprehension, Pairing, and Union axioms. So, we have:

Lemma I.16.18 *If* M *is any model for the Axioms of Extensionality, Comprehension, Pairing, and Union, then* $\emptyset^M, S^M, \cap^M$ *are defined. Furthermore, if* M *is also transitive, then these are absolute for* M.

Of course, we can iterate the uses of Definitions I.16.15 and I.16.16 introducing a sequence of defined symbols (predicates and functions), $s_0, s_1, \ldots$, where the definition of s_{k+1} may mention $s_0, s_1, \ldots, s_k$. One such s_k may be the defined proposition letter (0–ary predicate) CH. Then, we can talk about CH^M by successively understanding $s_0^M, s_1^M, \ldots, s_k^M$. This will be *much* easier than trying to rewrite CH into a sentence of $\mathcal{L} = \{\in\}$; especially because many (but not all) of the s_i will be seen to be absolute, so that these s_i^M will have the same meaning as s_i. See also Theorem I.2.4 and the following discussion.

Exercise I.16.19 *Give a purely finitistic proof that the theory DTO of dense total orders without first or last elements is consistent. Here,* $\mathcal{L} = \{<\}$.

Hint. Give a purely finitistic definition of the notion $\mathbb{Q} \models \varphi[s]$, and then prove that $\mathbb{Q} \models \varphi$ whenever φ is a sentence that is provable from DTO. For example, if $\varphi(x, y)$ is $\exists z\, \psi(x, y, z)$, then your definition will say $\mathbb{Q} \models \varphi[1/3, 1/2]$ iff $\mathbb{Q} \models \psi[1/3, 1/2, 1/3]$ or $\mathbb{Q} \models \psi[1/3, 1/2, 1/2]$ or $\mathbb{Q} \models \psi[1/3, 1/2, -2/3]$ or $\mathbb{Q} \models \psi[1/3, 1/2, 3/2]$ or $\mathbb{Q} \models \psi[1/3, 1/2, 5/12]$.

You need not believe in the set $\mathbb{Q}$, but you can still define a function $\mathrm{val}(\varphi, s)$ by primitive recursion, and then prove that it "works". □

This exercise is the finitist's analog of Exercises I.16.10 and I.15.22, which show that even though V doesn't exist, one can define the notion of truth in V for *some* restricted classes of formulas.

The fact that one can find a finitistic proof of the consistency of a theory with only infinite models gives some plausibility to Hilbert's Program in the 1920s (later refuted by the Incompleteness Theorem) of looking for a finitistic proof of something like Con(ZFC).

Exercise I.16.20 *Let Γ be the axioms for* infinite *boolean rings. Give a purely finitistic proof that Γ is consistent.*

Hint. Repeat the argument of Exercise I.16.19, replacing $\mathbb{Q}$ by the countable atomless boolean algebra. □

I.17 Recursion Theory

Here, we just remark on the very small part of this very big subject that is relevant to set theory.

We have already observed that the objects of finite mathematics live in HF. These include natural numbers, rational numbers, and logical formulas (when $\mathcal{L}$ is finite). Informally, some relations among these objects are *decidable* (such as "the variable x is free in the formula φ") and some functions on these objects are *computable* (such as the number of free variables in φ).

In fact, "decidable" and "computable" may be given precise mathematical definitions. Using these definitions, we can prove that decidable relations and computable functions are absolute for transitive models of $ZF-P$; see Section II.4. We can then prove that many set theory notions that utilize computable functions are absolute. For example, we shall show the notion $\mathfrak{A} \models \varphi$ and related concepts, such as the functions $\mathcal{D}^+(A)$ and $\mathcal{D}^-(A)$ (Definition I.15.25), are absolute. Roughly, absoluteness holds because our notions use quantification bounded by A (which may be infinite), together with decidable properties of the logical formulas $\varphi \in HF$, such as which variables are free and which are bound. In this book, we have never spelled out a specific definition of logical syntax, but under any reasonable definition, the basic syntactic notions are all decidable, so we shall appeal to a general statement that all decidable notions are absolute. Actually, this absoluteness will apply to all *arithmetical* notions (see Lemma II.4.14) because the constant HF is absolute.

Definition I.17.1 *$R \subseteq HF^n$ is* arithmetical *iff $R \in \mathcal{D}(HF)$. A function $f : HF^n \to HF$ is* arithmetical *iff $f \in \mathcal{D}(HF)$.*

Note that R is a set of n–tuples and we identify a function with its graph, so R, f are naturally subsets of HF. We observe (below) that all decidable notions are arithmetical, as are many classical examples of undecidable notions, such as the set of logically valid sentences of a finite $\mathcal{L}$, and the set of inputs at which a given computer program halts.

Now, there are many syntactical notions (free variables, subformulas, etc.), and most readers will not have seen detailed proofs that each of these notions is decidable. Rather, once one understands the basics of recursion theory, one sees that they are decidable by the Church–Turing Thesis. But, since all arithmetical notions are absolute, we shall only need:

Thesis I.17.2 (Weakened Church–Turing Thesis) *If $R \subseteq HF^n$ is* decidable in the informal sense *then R is arithmetical. If a function $f : HF^n \to HF$ is* computable in the informal sense *then f is arithmetical.*

We shall now define the notions of "computable" and "decidable" and state the classical Church–Turing Thesis.

Definition I.17.3 *A formula φ is Σ_1 iff φ is of the form $\exists y_1 \exists y_2 \cdots \exists y_k \psi$, where $k \geq 0$ and ψ is Δ_0. A formula φ is Π_1 iff φ is of the form $\forall y_1 \forall y_2 \cdots \forall y_k \psi$, where $k \geq 0$ and ψ is Δ_0.*

Definition I.17.4 *Fix $S \subseteq HF$.*

1. *S is Σ_1 over HF iff there is a Σ_1 formula $\varphi(x)$ of $\mathcal{L} = \{\in\}$ with one free variable such that $S = \{a \in HF : HF \models \varphi[a]\}$.*
2. *S is Π_1 over HF iff there is a Π_1 formula $\varphi(x)$ of $\mathcal{L} = \{\in\}$ with one free variable such that $S = \{a \in HF : HF \models \varphi[a]\}$.*
3. *S is Δ_1 over HF iff S is both Σ_1 and Π_1 over HF.*

Note that there is no notion of a "Δ_1 formula". In (3), we are saying that S is defined by some Σ_1 formula and also by some Π_1 formula.

Definition I.17.5 *$R \subseteq HF^n$ is* decidable *iff R is Δ_1 over HF. A function $f : HF^n \to HF$ is* computable *iff f is Δ_1 over HF.*

It is obvious from Definitions I.17.5 and I.17.1 that all decidable relations and computable functions are arithmetical.

Thesis I.17.6 (Church and Turing) *$R \subseteq HF^n$ is* decidable in the informal sense *iff R is decidable in the sense of Definition I.17.5. A function $f : HF^n \to HF$ is* computable in the informal sense *iff f is computable in the sense of Definition I.17.5.*

We take the informal sense of "decidable" or "computable" to imply the existence of a finite set of instructions that may be followed by rote to compute a result in finite time, allowing, say, one second for each basic step. Not all philosophers believe that Thesis I.17.6 is necessarily true, although there have been no explicit suggestions for a counter-example, and no one doubts the truth of this thesis as applied to concrete examples arising in mathematics, such as the syntax of formal logic.

There have been some suggestions that if one weakens the notion of "decidable" to allow arbitrary physical experiments, then a "decidable" relation may fail to be Δ_1. For example, Etesi and Németi [32] suggest utilizing the time dilation created by a black hole to decide Σ_1 sets; the computation device does ω steps, taking ω years in its time frame, but the observer sees this happening in a finite time. We know of no suggested counter-examples, utilizing any physics currently considered to be plausible, to our weakened Thesis I.17.2.

There are many (equivalent) precise definitions of "computable" in the literature. Our definition of "computable" as "Δ_1 over HF" is well-known in logic, and is sometimes taken as a basis for an introduction to recursion theory (see, e.g. [33, 57]). The more common definitions involve specifying some simple abstract computing machine. The reader who prefers one of those approaches may easily verify directly that all computable functions are arithmetical because, as with all finite mathematics, the description of the progress of the machine over time takes place completely within HF.

The arithmetical sets are well-known in recursion theory; see, e.g., Chapter IV of Soare [92]. Usually, the focus is on subsets of ω, but this is essentially the same as studying subsets of HF, since there is a computable bijection between ω and HF (see Lemma I.10.17).

The main subject of [92] is the Σ_1 sets, which are of central importance in recursion theory. These are usually called c.e. (computably enumerable) or r.e. (recursively enumerable) because a non-empty $S \subseteq HF$ is Σ_1 iff S is the range of a computable $f : HF \to HF$.

Chapter II

Easy Consistency Proofs

There are two difficulties in understanding the proofs of consistency results such as $\mathrm{Con}(ZF) \rightarrow \mathrm{Con}(ZFC + \neg CH)$: understanding the basic logic, and understanding the technical details of constructing a model of $ZFC + \neg CH$.

The construction of models for various extensions of ZFC is taken up in Chapters IV and V, as well as in a number of more advanced texts [49, 84], and is the subject of many current research papers.

The basic logic needs only to be discussed once. It involves some general facts about models of set theory, and everything needed for the independence proofs is covered entirely in this chapter and the end of Chapter I. We shall also use this logic to provide some "easy" consistency proofs — that is, ones for which the models were already given in Chapter I. This includes the relative consistency of the Axiom of Foundation (using the model WF), and the facts that the Power Set Axiom is not provable in $ZFC-P$ (using the model HC) and that the Replacement Axiom is not provable in ZC (using the model $R(\omega+\omega)$).

We shall also give the proof of Gödel that $\mathrm{Con}(ZF) \rightarrow \mathrm{Con}(ZFC + GCH)$. Although this proof was certainly not considered to be "easy" in its day (1938), it has become easy given the development of models of set theory, which grew out of that proof.

II.1 Informal Remarks on Consistency Proofs

The definitions and lemmas in this section are *informal*, and are intended to give an overview of the types of consistency results you will find in this book and in other works on set theory. The reader who is familiar with proof theory will see how to make them precise and formal. These remarks take place in the metatheory, using just finitistic reasoning. Even if you are a platonist, it helps to keep the formal logic straight if you think of the metatheory as being purely finitistic.

We consider axiomatic theories Γ. For us, Γ will always be some system of set theory extending BST^-, such as ZFC^- or Z or $ZF-P$. In particular, Γ will always be strong enough to develop at least finite combinatorics, so that finitistic arguments can be formalized within Γ. Also, the axioms for Γ will always be a decidable set of sentences. Then, we consider two kinds of results that one might prove regarding the consistency of such theories:

Definition II.1.1 $\Gamma \lhd \Lambda$ *iff* $\Lambda \vdash \mathrm{Con}(\Gamma)$. *We say that* Λ *is* strictly stronger proof-theoretically *than* Λ.

We always take $\mathrm{Con}(\Gamma)$ to be the assertion that there is no formal proof of contradiction from Γ, as in Definition I.2.1; this is natural because the theory Λ might not be strong enough to produce any infinite models. But, most of the time, our Λ will be strong enough to prove the completeness and soundness of first-order logic, and our demonstration of $\Gamma \lhd \Lambda$ will involve working in Λ and producing a model for Γ. For example, $ZFC-P \lhd ZFC$, since working in ZFC, we can produce the model HC for $ZFC-P$ (see Section II.2).

The relation $\lhd$ is not reflexive, since by the Second Incompleteness Theorem, $\Gamma \lhd \Gamma$ holds iff Γ is inconsistent. A notion weaker than $\lhd$, which is reflexive, is given by:

Definition II.1.2 *Define* $\Gamma \leq \Lambda$ *iff we have a finitistic proof that* $\mathrm{Con}(\Lambda) \rightarrow \mathrm{Con}(\Gamma)$*; such a proof is called a* relative consistency proof. *Define* $\Gamma \sim \Lambda$ *iff* $\Gamma \leq \Lambda$ *and* $\Lambda \leq \Gamma$*; we say that* Γ *and* Λ *are* proof-theoretically equivalent.

This $\leq$ is clearly reflexive and transitive, so that $\sim$ is an equivalence relation. Also, since all theories under consideration are at least as strong as finitistic reasoning, if $\Gamma \leq \Lambda$ and Θ is one of our theories, then $\Theta \vdash \mathrm{Con}(\Lambda) \rightarrow \mathrm{Con}(\Gamma)$.

Our proofs of $\Gamma \leq \Lambda$ will often use Corollary I.16.14; that is, we shall work in Λ and define some class model for Γ.

Lemma II.1.3

1. $\Gamma \lhd \Lambda \rightarrow \Gamma \leq \Lambda$
2. $\Gamma \leq \Lambda \wedge \Lambda \lhd \Theta \rightarrow \Gamma \lhd \Theta$.
3. *The relation* $\lhd$ *is transitive.*
4. $\Gamma \leq \Lambda \wedge \Lambda \lhd \Gamma \rightarrow \neg\mathrm{Con}(\Gamma) \wedge \neg\mathrm{Con}(\Lambda)$

Proof. For (1): Assuming $\Lambda \vdash \mathrm{Con}(\Gamma)$, we may argue finitistically that if we had a formal proof of contradiction from Γ, then $\Lambda \vdash \neg\mathrm{Con}(\Gamma)$, which would give us a contradiction in Λ. So, $\mathrm{Con}(\Lambda) \rightarrow \mathrm{Con}(\Gamma)$.

For (2): Assuming $\Gamma \leq \Lambda \wedge \Lambda \lhd \Theta$, then working in Θ, we can prove $\mathrm{Con}(\Lambda)$ and $\mathrm{Con}(\Lambda) \rightarrow \mathrm{Con}(\Gamma)$, and hence $\mathrm{Con}(\Gamma)$.

(3) is immediate from (1)(2). For (4): We have $\Gamma \lhd \Gamma$ by (2), and hence $\neg\mathrm{Con}(\Gamma)$, and hence $\neg\mathrm{Con}(\Lambda)$ by $\Gamma \leq \Lambda$. $\square$

$ZFC^- \leq ZFC$ is obvious. But also $ZFC \leq ZFC^-$ by von Neumann's Theorem II.2.13. so $ZFC^- \sim ZFC$; the same theorem shows that $ZF^- \sim ZF$. But also, $ZFC + GCH \leq ZF$ by Gödel's Theorem II.6.24, so we shall have $ZF^- \sim ZF \sim ZFC^- \sim ZFC \sim ZFC + GCH$. Likewise, we shall see in Chapters IV and V that ZFC is proof-theoretically equivalent to ZFC plus various assertions of mathematical interest, including failures of the GCH, properties of category and Lebesgue measure on $\mathbb{R}$, etc.

Some illustrations of Lemma II.1.3(4): Section II.2 shows that $\Lambda \lhd ZFC$ when Λ is ZFC with one of the axioms Power Set, Replacement, or Infinity deleted. So, for these, $ZFC \leq \Lambda$ is false by Lemma II.1.3(4) unless ZFC is inconsistent; for example, there is no finitistic proof of $\mathrm{Con}(ZFC-P) \rightarrow \mathrm{Con}(ZFC)$. But Foundation is different; the fact that $ZFC \leq ZFC^-$ implies that $ZFC^- \lhd ZFC$ is false unless ZFC^- is inconsistent; that is, we cannot work in ZFC and produce a set that is a model for ZFC^-.

There are some well-known additions to ZFC that are strictly stronger proof-theoretically than ZFC. Most well-known are various large cardinal assumptions. For example, if Γ is ZFC plus the assertion that there is a strongly inaccessible cardinal κ, then $ZFC \lhd \Gamma$; this follows from Corollary II.2.3, which shows that $R(\kappa) \models ZFC$. Likewise, ZFC plus the assumption of 76 strong inaccessibles is strictly stronger proof-theoretically than ZFC plus the assumption of 75 strong inaccessibles; see Exercise II.4.25.

Less obvious examples involve failures of the GCH at singular cardinals. For example, let Γ be ZFC plus $\forall n < \omega\, [2^{\aleph_n} = \aleph_{n+1}]$ plus $2^{\aleph_\omega} \geq \aleph_{\omega+2}$. This does not mention large cardinals, but in fact Γ is strictly stronger proof-theoretically than ZFC plus the assumption that the strongly inaccessible cardinals are cofinal in ON. The results about this Γ are beyond the scope of this book; see Chapter V of [20]. However, we shall show (Exercises III.6.18 and III.6.19) that ZFC plus $\forall \xi < \omega_1\, [2^{\aleph_\xi} = \aleph_{\xi+1}]$ plus $2^{\aleph_{\omega_1}} \geq \aleph_{\omega_1+2}$ is simply inconsistent.

Remarks on Proof Theory. Definition II.1.2 could be made formal by defining "finitistic" as "formalizable in Primitive Recursive Arithmetic (PRA)"; this PRA (see [91]) is a formal system that is intended to capture finitistic reasoning. We shall not be that formal here; for the examples discussed in this book, we shall just present the proofs and explain why they are finitistically valid. Also, working in the finitistic metatheory, Γ and Λ are not really sets, but rather definitions of sets; for example, ZFC really denotes the standard definition (as in Section I.3) that tells you which sentences are axioms of ZFC. Finitism aside, there is no way to plug an infinite *set* Γ into a formal theory Λ to write $\mathrm{Con}(\Gamma)$; you are really expressing, in Λ, the *definition* of Γ.

Actually, the basic proof theory notion is $\mathrm{Incon}(\Gamma)$, as in Definition I.2.1. Then $\mathrm{Con}(\Gamma)$ is $\neg\mathrm{Incon}(\Gamma)$, so $\neg\mathrm{Con}(\Gamma)$ is $\neg\neg\mathrm{Incon}(\Gamma)$, which, to finitists or intuitionists, is weaker than $\mathrm{Incon}(\Gamma)$, since they reject the law of the excluded middle $\varphi \vee \neg\varphi$ in their metatheory; the fact that $\varphi \vee \neg\varphi$ is part of the formal system of set theory is totally irrelevant here. In this book, whenever we show $\Gamma \leq \Lambda$, our proof will really establish $\mathrm{Incon}(\Gamma) \rightarrow \mathrm{Incon}(\Lambda)$, which is stronger

than $\mathrm{Con}(\Lambda) \to \mathrm{Con}(\Gamma)$; the proof will actually provide a primitive recursive function that will convert every inconsistency in Γ to an inconsistency in Λ.

II.2 The Consistency of Foundation

We shall prove here the relative consistency of the Foundation Axiom, due to von Neumann (1929). That is, $ZFC \leq ZFC^-$ and $ZF \leq ZF^-$, and hence $ZFC \sim ZFC^-$ and $ZF \sim ZF^-$. Later, Gödel (1938) showed that $ZFC \leq ZF$, so that $ZFC \sim ZFC^- \sim ZF \sim ZF^-$; see Section II.6.

Naively, $ZFC \leq ZFC^-$ because, working in ZFC^-, we can define the class WF, and prove that $WF \models ZFC$. Now, verifying $WF \models ZFC$ is easy using the closure properties of WF proved in Section I.9, *once* we straighten out the logic. In particular, the statement $WF \models ZFC$ (that is $\forall \varphi \in ZFC\,[WF \models \varphi]$) is nonsense, since WF is a proper class and there is no way even to define this statement in set theory.

Formally, the statement "$WF \models ZFC$" must be viewed as a scheme in the metatheory. Using our notion of relativization discussed in Section I.16, we can, for each specific $\varphi \in ZFC$, write the sentence φ^{WF} as an abbreviation for the sentence we get from φ by restricting all the bound variables to "range" over WF (as in Notation I.6.27 and Definition I.16.12). Working in the metatheory, we can show that φ^{WF} is a theorem of ZFC^- whenever φ is any axiom of ZFC. Then $ZFC \leq ZFC^-$ will follow by Corollary I.16.14. Essentially the same argument will produce the following results about the sets $H(\kappa)$ and $R(\gamma)$, which may be viewed as "small approximations" to WF:

Theorem II.2.1 (ZFC^-) *$H(\kappa) \models ZFC-P$ whenever κ is a regular uncountable cardinal.*

Theorem II.2.2 (ZF^-) *$R(\gamma) \models Z$ whenever $\gamma > \omega$ and γ is a limit ordinal. The Axiom of Choice implies that $R(\gamma) \models ZC$.*

It then follows immediately, using Lemmas I.13.29 and I.13.31:

Corollary II.2.3 (ZFC^-) *$H(\kappa) = R(\kappa) \models ZFC$ whenever κ is strongly inaccessible.*

Informally, we may view ON as a strongly inaccessible cardinal and conclude also, working in ZFC^-, that $WF = H(ON) = R(ON) \models ZFC$. Formally, this is incorrect, as we just pointed out, since you can't even *say* $WF \models ZFC$; this is just shorthand for an infinite scheme. But, $H(\kappa)$ and $R(\gamma)$ are sets, and the two theorems and the corollary are (abbreviations of) single statements in the language of set theory that are provable in ZFC^- or ZF^-. But fortunately, the mathematics is very similar for all these models; you just have to keep the logic straight. For example, the proof that the Pairing Axiom holds in M is essentially the same for $M = WF$, $M = H(\kappa)$, and

$M = R(\gamma)$. To avoid having to do the same proof three times, we shall prove some lemmas giving useful sufficient conditions for models to satisfy the various axioms. These lemmas will get re-used in Chapter IV when we consider other models. For the first six axioms listed in Section I.3, we use:

Lemma II.2.4 ($ZF^- - P$) *For any class M:*

1. *If M is transitive, then the Axiom of Extensionality holds in M.*
2. *If $M \subseteq WF$, then the Foundation Axiom holds in M.*
3. *If $\forall z \in M\, \forall y \subseteq z\, [y \in M]$, then the Comprehension Axiom holds in M.*
4. *If $\forall x, y \in M\, [\{x, y\} \in M]$, then the Pairing Axiom holds in M.*
5. *If $\forall \mathcal{F} \in M\, [\bigcup \mathcal{F} \in M]$, then the Union Axiom holds in M.*
6. *Assume that M is transitive and for all functions f: If $\mathrm{dom}(f) \in M$ and $\mathrm{ran}(f) \subseteq M$, then $\mathrm{ran}(f) \in M$. Then the Replacement Axiom holds in M.*

Proof. As we just discussed, there is a dual view of this lemma. For set models, each of $(1-6)$ abbreviates one sentence, which we can prove in $ZF^- - P$; for example, (3) is $\forall M[\forall z \in M\, \forall y \subseteq z\, [y \in M] \rightarrow M \models$ Comprehension Axiom]. But for classes, (3) is a scheme. For each class M (defined by some formula $\mu(x)$) if we can verify $\forall z \in M\, \forall y \subseteq z\, [y \in M]$, then each instance of Comprehension relativized to M is provable in $ZF^- - P$. A similar duality is found in the discussion of induction and recursion in Section I.9; see page 44.

For ease of notation, we use the relativization terminology (as in Notation I.15.6), rather than the model-theoretic $\models$, for set models as well as class models.

For (4): the Pairing Axiom is $\forall x, y\, \exists z\, [x \in z \wedge y \in z]$, so relativized to M we get the sentence $\forall x, y \in M\, \exists z \in M\, [x \in z \wedge y \in z]$, so let $z = \{x, y\} \in M$.

For (1): The Axiom of Extensionality relativized to M is just the statement that the $\in$ relation is extensional on M, which is true whenever M is transitive; see Lemma I.9.34.

For (2): The Foundation Axiom relativized to M is

$$\forall x \in M\, [\exists y \in M(y \in x) \quad \rightarrow \quad \exists y \in M(y \in x \wedge \neg \exists z \in M(z \in x \wedge z \in y))] \ .$$

To prove this, use the fact that $\in$ is well-founded on WF (Lemma I.9.24) and let y be $\in$-minimal in $x \cap M$.

Axiom (5) is left as an exercise. Axiom (3) could also be left as an exercise, but we spell out the details to illustrate some points of logic. Fix a formula φ without y free. φ may have x, z free, along with possibly other free variables $v_0, \ldots, v_{n-1}$, so write it as $\varphi(x, z, v_0, \ldots, v_{n-1})$. Then we must verify

$$\forall z, v_0, \ldots, v_{n-1} \in M\, \exists y \in M\, \forall x \in M\, [x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi^M(x, z, \vec{v})]$$

This holds, since we may let $y = \{x \in z : \varphi^M(x, z, \vec{v})\}$; then $y \in M$ because $y \subseteq z$. Note that φ may have quantified variables, and φ^M relativizes all

of these to M. The formulas φ and φ^M need not be equivalent, but that is irrelevant here. We are using the Comprehension Axiom (which is part of $ZF^- - P$), applied with the formula φ^M (*not* φ), to assert that the set y exists, and then we are using the hypothesis of (3) to assert that $y \in M$.

(6) is similar to (3), so we shall be briefer. Assume that $A \in M$ and that $\forall x \in M\,[x \in A \rightarrow \exists! y \in M\, \varphi^M(x,y)]$. We need to produce a $B \in M$ such that $\left(\forall x \in A\, \exists y \in B\, \varphi(x,y)\right)^M$. So, let f be the function with $\mathrm{dom}(f) = A$ such that $f(x)$ is the (unique) $y \in M$ such that $\varphi^M(x,y)$; the set f exists by the Replacement Axiom (which is part of $ZF^- - P$) applied with the formula $\varphi^M(x,y) \wedge y \in M$ (*not* $\varphi(x,y)$). Then, let $B = \mathrm{ran}(f)$. □

In each of $(1-6)$, the given condition is not necessary; e.g., for (4), the Pairing Axiom alone does not imply that $\{x,y\}$ exists. For (1)(2)(4)(5), the given conditions will suffice for all the consistency results discussed in this book. For (3)(6), the given conditions will suffice in this chapter, but better methods will be needed later.

Corollary II.2.5 ($ZF^- - P$) *Axioms 1,2,3,4,5,6 hold in WF.*

Proof. The conditions of Lemma II.2.4 are easily verified using the results of Section I.9. Those results, along with Lemma II.2.4, were derived from $ZF^- - P$. □

When we add Axiom 7 (Infinity) to the list, we shall have $\mathrm{Con}(ZF^- - P) \rightarrow \mathrm{Con}(ZF - P)$ (Theorem II.2.13 below). When we prove (Corollary II.2.9) that Axiom 8 (Power Set) holds in WF, we shall work in ZF^-, which includes Axiom 8. Actually, most of Lemma II.2.4 used much less than $ZF^- - P$, but we don't bother listing exactly what each item needed, since we don't plan to prove results of the form $\mathrm{Con}(\Lambda) \rightarrow \mathrm{Con}(\Gamma)$ for any Λ weaker than $ZF^- - P$.

Corollary II.2.6 (ZF^-) $R(\gamma) \models$ *Axioms 1,2,3,4,5 whenever* γ *is a limit ordinal.* $HF = R(\omega) = H(\omega) \models$ *Axiom 6.*

Again, this is easy using the results of Section I.9. Note that $R(\alpha+1) \models \exists x \forall y\, x \notin y$, so that the Pairing Axiom fails, so the $R(\alpha+1)$ are not of much interest as models of set theory. Regarding Axiom 6 for limit $\gamma > \omega$, "most" $R(\gamma)$ do not satisfy Replacement; see Exercise II.4.24.

Corollary II.2.7 (ZFC^-) $H(\kappa) \models$ *Axioms 1,2,3,4,5 whenever* κ *is an infinite cardinal. If* κ *is also regular, then* $H(\kappa) \models$ *Axiom 6.*

Here, the required properties of $H(\kappa)$ are easy from the material in Section I.13. For Axiom 6, use Lemma I.13.32.

We shall now consider the remaining three Axioms, 7,8,9. In Section I.3, the first six axioms were displayed directly as sentence of $\mathcal{L} = \{\in\}$, but the last three were stated using the defined notions $\subseteq$, $\emptyset$, S, $\cap$, and $\mathrm{SING}(x)$ (x is

a singleton). For example, Axiom 8 (Power Set) is stated using the expression $z \subseteq x$. What exactly is (Axiom 8)^M? One approach is simply to replace the $z \subseteq x$ by its definition, $\forall v(v \in z \rightarrow v \in x)$, obtaining a sentence of $\mathcal{L} = \{\in\}$, which we may then relativize to M. This approach would work for Axioms 7,8,9, but is not feasible for more complicated statements. For example, say we wish to discuss $(2^{\aleph_1} = \aleph_5)^M$. It would be not be practical to rewrite $2^{\aleph_1} = \aleph_5$ as a sentence of $\mathcal{L} = \{\in\}$, since the aleph function $\aleph_\alpha$ and the cardinal exponentiation function κ^λ were obtained by a very long chain of definitions. It is more efficient to use the general discussion of relativization of defined notions from Section I.16. In particular, we saw there that $\subseteq$ was defined by a Δ_0 formula, and is thus absolute for transitive models. We can now handle Axiom 8 for most of the models we shall consider:

Lemma II.2.8 (ZF^-) *Let M be a transitive class. Then*

8. $\forall x \in M((\mathcal{P}(x) \cap M) \in M) \rightarrow$ the Power Set Axiom holds in M.

Also, the $\leftarrow$ direction holds if M satisfies the Comprehension Axiom.

Proof. By the absoluteness of $\subseteq$, the Power Set Axiom holds in M iff

$$\forall x \in M \exists y \in M \forall z \in M(z \subseteq x \rightarrow z \in y) \ ,$$

For the $\rightarrow$, let $y = \mathcal{P}(x) \cap M$. For the $\leftarrow$, fix $x \in M$. Then (Power Set)^M gives us a $y \in M$ such that $\mathcal{P}(x) \cap M \subseteq y$. But then Comprehension in M plus the absoluteness of $\subseteq$ implies that $\mathcal{P}(x) \cap M = \{z \in y : (z \subseteq x)^M\} \in M$. □

Corollary II.2.9 (ZF^-) *The Power Set Axiom holds in WF, and in $R(\gamma)$ for any limit γ.*

Proof. If $x \in WF$, then applying Lemmas I.9.25 and I.9.26, $\mathcal{P}(x) \subseteq WF$ and $\mathcal{P}(x) \cap WF = \mathcal{P}(x) \in WF$. Likewise for the $R(\gamma)$. □

Corollary II.2.10 (ZFC^-) *If κ is a regular uncountable cardinal, and κ is not strongly inaccessible, then the Power Set Axiom is false in $H(\kappa)$.*

Proof. Fix $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Then $\lambda \in H(\kappa)$ but $\mathcal{P}(\lambda) \cap H(\kappa) = \mathcal{P}(\lambda) \notin H(\kappa)$. □

Note that if κ is ω or strongly inaccessible, then $H(\kappa) = R(\kappa) \models$ Power Set. For most of the models M considered in Chapters IV and V, we won't have $\mathcal{P}(x) \subseteq M$, but we will have $\mathcal{P}(x) \cap M \in M$.

Axiom 7 and 9 were stated using the 0–ary, 1–ary, and 2–ary function symbols $\emptyset$, S, and $\cap$, and, by Lemma I.16.18, these are defined and absolute for transitive models of Extensionality, Comprehension, Pairing, and Union. This gives us:

Lemma II.2.11 ($ZF^- - P$) *Let M be a transitive class, and assume that the Axioms of Extensionality, Comprehension, Pairing, and Union hold in M.*

7. *The Axiom of Infinity holds in M if $\omega \in M$.*
9. *Axiom 9 holds in M iff every disjoint family of non-empty sets in M has a choice set in M.*

Proof. For Axiom 9: The form of AC stated in Section I.3 is logically equivalent to $\forall F\, \exists C\, [\mathrm{df}(F) \to \mathrm{cs}(C, F)]$, where $\mathrm{df}(F)$ is

$$\emptyset \notin F \wedge \forall x \in F\, \forall y \in F (x \neq y \to x \cap y = \emptyset) \ ,$$

asserting that F is a disjoint family of non-empty sets and $\mathrm{cs}(C, F)$ is

$$\forall x \in F(\mathrm{SING}(C \cap x)) \ ,$$

asserting that C is a choice set for F. Both df and cs are Δ_0 in the notions $\emptyset$, $\cap$, and SING, which we have verified in Section I.16 are absolute for M, so $(AC)^M$ is equivalent to $\forall F \in M\, \exists C \in M\, [\mathrm{df}(F) \to \mathrm{cs}(C, F)]$.

For Axiom 7: The Axiom of Infinity holds in M iff

$$\exists x \in M(\varphi(x)^M) \ , \quad \text{where} \quad \varphi(x) \ \text{ is } \ \emptyset \in x \wedge \forall y \in x (S(y) \in x)$$

But φ is Δ_0 in the notions $\emptyset$ and S, which we have verified in Section I.16 are absolute for M, so we can replace the $\varphi(x)^M$ by $\varphi(x)$, which holds if $x = \omega$. □

The $\to$ direction in (7) is provable in ZF (using Foundation; see Exercise II.4.26) but not in ZF^- (see Exercise II.9.5).

Corollary II.2.12 ($ZF^- - P$) *Axiom 7 holds in WF. Furthermore, AC implies that Axiom 9 holds in WF.*

Proof. $\omega \in WF$ by Lemma I.9.22. If $F \in WF$ is a disjoint family of non-empty sets, then F has a choice set C (assuming AC). $C \cap \bigcup F$ is also a choice set for F, and $(C \cap \bigcup F) \in WF$ by Lemmas I.9.25 and I.9.26. □

Theorem II.2.13 *Let Γ be one of the theories $ZF-P$, $ZFC-P$, ZF, ZFC. Let Γ^- be Γ with the Axiom of Foundation deleted. Then $\Gamma \leq \Gamma^-$; that is, there is a finitistic proof of* $\mathrm{Con}(\Gamma^-) \to \mathrm{Con}(\Gamma)$.

Proof. Applying Corollaries II.2.5, II.2.9, and II.2.12, we may work in Γ^- and prove each axiom of Γ relativized to WF. □

Of course, this is a theorem in the metatheory about formal proofs, whereas Theorems II.2.1 and II.2.2 are theorems about set models, proved within ZFC^- and ZF^- respectively.

Proof of Theorem II.2.2. Exactly as for WF. $\omega \in R(\gamma)$ because $\gamma > \omega$. For Axiom 9, observe that if C is a choice set for F, then $\text{rank}(C \cap \bigcup F) \leq \text{rank}(F)$. □

Proof of Theorem II.2.1. Again, $\omega \in H(\kappa)$ because $\kappa > \omega$. For Axiom 9, observe that if C is a choice set for F, then $\text{trcl}(C \cap \bigcup F) \subseteq \text{trcl}(F)$, so $F \in H(\kappa)$ implies that $C \cap \bigcup F$ in $H(\kappa)$. □

The model $HF = H(\omega) = R(\omega)$ is a special case:

Theorem II.2.14 (ZF^-) *$HF \models ZFC - Inf$, and the Axiom of Infinity is false in HF.*

Proof. Infinity fails because $\neg\exists x \in HF\ \varphi(x)$, where φ is as in the proof of Lemma II.2.11. For Choice, note that we don't have to work in ZFC^-. Without assuming AC, HF can be well-ordered (see Lemma I.10.17), so every disjoint family in HF has a choice set in HF. □

For these various set models, it is of some interest that whenever we verify AC in the model, we can also verify AC^+ (see Definition I.3.3). Recall from Section I.12 that in ZF^-, all the standard forms of AC are equivalent; but for $H(\kappa)$ and $R(\gamma)$, which are models for somewhat less than full ZF, it is worth pointing out that they satisfy the stronger form AC^+. Since AC^+ involves concepts such as "relation" and "well-order", we defer the proof until Section II.4, when we discuss the absoluteness of these notions; see Exercise II.4.9.

We always assumed AC in the discussions of the $H(\kappa)$ (working in ZFC^-, not ZF^-) because the "$|\text{trcl}(x)| < \kappa$" in Definition I.13.27 seemed to require AC for $|\text{trcl}(x)|$ to be defined. However, if we take "$|\text{trcl}(x)| < \kappa$" to be shorthand for "$\text{trcl}(x)$ can be well-ordered and $|\text{trcl}(x)| < \kappa$", then $H(\kappa)$ is defined in ZF^-. Curiously, AC is not needed to prove $H(\kappa) \models AC$ (this is like Exercise II.2.16), but AC was used to show that $H(\kappa)$ satisfies Replacement, via Lemma I.13.32.

Remark on Parameters. As we have emphasized throughout, the "class" M is really defined by some formula, $\mu(x)$; for example, "x is well-founded" in the case of the class WF. As usual, this $\mu(x)$ could also have some tuple $\vec{t}$ of free variables, which are considered parameters in this construction. In that case, the above lemmas can be rephrased to mention the $\vec{t}$ explicitly, using, e.g., $M_{\vec{t}}$. So, for example, Lemma II.2.8 is shorthand for the following lemma scheme in the metatheory: For each formula $\mu(x, \vec{t})$, the following sentence is provable from ZF^-:

$$\forall \vec{t}\,[M_{\vec{t}} \text{ is transitive } \wedge \forall x \in M_{\vec{t}}((\mathcal{P}(x) \cap M_{\vec{t}}) \in M_{\vec{t}}) \rightarrow (\text{Power Set Axiom})^{M_{\vec{t}}}]\ .$$

Of course, there are numerous abbreviations here. $M_{\vec{t}}$ "denotes" the class $\{x : \mu(x, \vec{t})\}$, which means that each use of "$M_{\vec{t}}$" is really an abbreviation. For example "$M_{\vec{t}}$ is transitive" abbreviates $\forall x, y\,[x \in y \wedge \mu(y, \vec{t}) \rightarrow \mu(x, \vec{t})]$.

Remark on Set Models. We never said that M or $M_{\tilde{t}}$ had to be a *proper* class, so all our results go through if $M_{\tilde{t}}$ happens to be a set. For example, working in ZFC^-, if we did not know the proof (Lemma I.13.28) that $H(\kappa)$ is a set, we could have stated Theorem II.2.1 by saying that for each axiom φ of $ZFC-P$, we can prove that $\varphi^{H(\kappa)}$ holds for all regular uncountable κ. In particular, consider $HC = H(\aleph_1)$, and let Γ be $ZFC-P$ plus the negation of the Power Set Axiom. Without knowing that HC is a set, we could apply II.2.8 and use the class HC as a model to prove that $\Gamma \leq ZFC^-$. But in fact, HC *is* a set, and we have just proved in ZFC^- that $H(\kappa) \models \Gamma$, so we have the stronger result that $ZFC^- \vdash \mathrm{Con}(\Gamma)$ — that is $\Gamma \lhd ZFC^-$.

We do not know of any mathematically interesting example of a Γ for which we can demonstrate a relative interpretation into a *set* M, but for which we cannot prove the stronger statement $M \models \Gamma$. However, the following "artificial" example indicates that such an example might be conceivable.

Exercise II.2.15 *We shall see (Corollary II.5.4) that* $ZFC \vdash \mathrm{Con}(\Gamma)$ *whenever* Γ *is a finite subset of ZFC. Use this to define* explicitly, *in ZFC, a binary relation* E *on* ω *such that* $ZFC \vdash \varphi^{\omega,E}$ *for each axiom* φ *of ZFC.*

Hint. List the axioms of ZFC in some computable way, as $\{\zeta_i : i < \omega\}$; let $ZFC_n = \{\zeta_i : i < n\}$. Working in ZFC, we can define Γ to be ZFC if $\mathrm{Con}(ZFC)$; if $\neg\mathrm{Con}(ZFC)$, let $\Gamma = ZFC_n$ where n is largest such that $\mathrm{Con}(ZFC_n)$. Then clearly $\mathrm{Con}(\Gamma)$, and observe that the proof of the Completeness Theorem yields an explicit E such that $(\omega; E) \models \Gamma$. □

Note that the statement "$(\omega; E) \models ZFC$" is expressible in ZFC, but we cannot prove it in ZFC unless ZFC is inconsistent.

The following relative consistency proof for AC, using the "hereditarily well-orderable sets", is much simpler than the proof of Gödel in Section II.6.

Exercise II.2.16 *Let HWO be the class of all* $x \in WF$ *such that* $\mathrm{trcl}(x)$ *can be well-ordered; it is clearly a proper class, since* $ON \subseteq HWO$. *Verify, in ZF, that Axioms 1,2,3,4,5,7,9 of Section I.3 hold in HWO; Axiom 9 is AC.*

Unfortunately, the fact that we cannot guarantee Axioms 6 and 8 (Replacement and Power Set) in this model makes this consistency result mathematically uninteresting. Likewise, the relative consistency of $ZF^- + \neg AC$ was well-known long before Cohen's work (see Section II.9), but the non-well-orderable set was also not well-founded, and was not anything mathematically interesting (such as $\mathcal{P}(\omega)$). Thus, everyone gives credit to Gödel and Cohen for proving the relative consistency of AC and of $\neg AC$ with a mathematically interesting theory (such as ZF).

Note that $\mathcal{P}(\omega) \subseteq HWO$, so that the Power Set Axiom is false in HWO whenever $\mathcal{P}(\omega)$ cannot be well-ordered, which happens in Cohen's model of $\neg AC$.

II.3 The Last Word on Foundation

Henceforth in this book, we shall always assume Foundation as a basic axiom (equivalently, we shall live in WF to do our mathematics), *except* in Section II.9, where we shall discuss the independence of Foundation. That is, our basic theory will be ZFC or ZF, or, occasionally $ZF-P$, but Foundation will always be included. There are two justifications for this: formal and philosophical.

As a formal matter, for relative consistency proofs: We already know that $\mathrm{Con}(ZF^-) \to \mathrm{Con}(ZF)$, so to prove that $\mathrm{Con}(ZF^-) \to \mathrm{Con}(ZFC+GCH)$, it is sufficient to prove that $\mathrm{Con}(ZF) \to \mathrm{Con}(ZFC + GCH)$ (this will be Theorem II.6.24), which means that we work in ZF and produce a model for $ZFC + GCH$. Assuming the Foundation Axiom will simplify our discussion of models of set theory, since the fact that $\in$ is always well-founded greatly expands the collection of concepts that we can prove to be absolute; see Section II.4.

Similar remarks apply to other statements that we prove consistent relative to ZF^-. Say we wish to prove that $\mathrm{Con}(ZF^-) \to \mathrm{Con}(ZFC + 2^{\aleph_0} = \aleph_2)$, as we shall in Chapter IV. But in fact we shall prove $\mathrm{Con}(ZFC + GCH) \to \mathrm{Con}(ZFC + 2^{\aleph_0} = \aleph_2)$, which is sufficient by Theorem II.6.24; and our work will be made much easier by starting from a model of $AC + GCH$.

Philosophically, our use of WF to prove the relative consistency of Foundation (Theorem II.2.13) is unique among the relative consistency results in set theory, because we are making the informal claim that all mathematics lives in WF anyway, so we might as well also live in WF, where the Foundation Axiom is true. We have already seen that specific mathematical objects, such as $\mathbb{R}$ and $\mathbb{C}$, lie in WF, as do things constructed from them, such as functions from $\mathbb{R}$ to $\mathbb{R}$, or Lebesgue measure (which maps a subset of $\mathcal{P}(\mathbb{R})$ to $\mathbb{R}$). As to abstract objects, such as groups and topological spaces,

Exercise II.3.1 (ZFC^-) *Show that every group is isomorphic to a group in WF and every topological space is homeomorphic to a space in WF. Also show that the cardinal exponentiation map $\kappa, \lambda \mapsto \kappa^\lambda$ is absolute for WF.*

Hint. By AC, we need only consider groups or spaces whose base set is an ordinal. For absoluteness, first check directly that $\mathcal{P}(x)$ is absolute for WF. Then, look at Section II.4, which uses Foundation to prove absoluteness results, but most of these (and much more) can be proved absolute for WF using the absoluteness of $\mathcal{P}$ (see also page 97). For example x is an ordinal iff x is a transitive set (which is Δ_0) and $\forall y \in \mathcal{P}(x)\,[y \neq \emptyset \to y$ has an $\in$-minimal element] (which is Δ_0 in $\mathcal{P}$). And, κ is a cardinal iff $\neg\exists f \in \mathcal{P}(\kappa \times \kappa)\,\exists \alpha \in \kappa$ $[f : \kappa \xrightarrow{1-1} \alpha]$. □

In view of this exercise and hint,

Remark II.3.2 The Axiom of Foundation is irrelevant to mathematics because $\varphi^{WF} \leftrightarrow \varphi$ holds for any statement φ of mathematical interest. So, if $ZFC \vdash \varphi$, then $ZFC^- \vdash \varphi^{WF}$ and hence $ZFC^- \vdash \varphi$.

Of course, this remark is informal (philosophical), since "mathematical interest" is not precisely defined. For example, we are declaring $\exists x\, x \in x$ to be not of mathematical interest.

When we provide models later in this book for various extensions of ZF, we shall never claim that our models contain all of mathematics. For example, Gödel defined the class L (see Section II.6) and showed that it is a model for $ZFC + GCH$, but he never said that all mathematics lives in L. In fact, he seemed to believe [41] that CH is not a natural assumption, so that his L should be viewed as just a clever trick for proving the relative consistency of $AC + GCH$. Even in the case of AC, which Gödel and many other mathematicians have felt should be included as a basic axiom, we shall prove $\mathrm{Con}(ZF) \to \mathrm{Con}(ZFC)$ by using the model L, but we shall build many other models in which AC is true, none of which is claimed by anyone to contain all of mathematics.

II.4 More on Absoluteness

Now that we are assuming Foundation, we can expand our library of absolute functions and relations. We shall work in ZF, but we shall consider absoluteness of concepts for transitive models of the weaker theory, BST (see Definition I.3.1), so they apply to all $H(\kappa)$ for regular κ (assuming AC) and to all $R(\gamma)$ for limit γ by Theorems II.2.1, II.2.2, and II.2.14.

Recall (Definition I.16.5) that φ is absolute for M iff $M \preccurlyeq_\varphi V$. The following lemma is obvious from this definition, but we state it because it is used so often, usually implicitly.

Lemma II.4.1 *If $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are two formulas in $\mathcal{L} = \{\in\}$ and $\forall \vec{x}\, [\varphi(\vec{x}) \leftrightarrow \psi(\vec{x})]$ holds in V and in M, then φ is absolute for M iff ψ is absolute for M.*

In most of our applications, ψ is the "official" definition of a concept and φ is a formula that is known to be absolute by some general principle (e.g., when φ is Δ_0 and M is transitive). Usually, the equivalence $\forall \vec{x}\, [\varphi(\vec{x}) \leftrightarrow \psi(\vec{x})]$ is something that we have proved from some axioms $\Lambda \subseteq ZF$ that we have already verified to hold in M; so, working in ZF, we may claim that the equivalence holds both in V and in M. In this section, Λ is usually at least as strong as BST. In Section I.16, where we discussed the absoluteness of the concepts in Example I.16.3, $\Lambda = \emptyset$; for example, the statement $y = v \cap w$ is logically equivalent to a Δ_0 formula.

The importance of Foundation is that now, the notion of "ordinal" and many related concepts are absolute for transitive models:

Lemma II.4.2 *The following set-theoretic notions are defined by formulas that BST proves are equivalent to Δ_0 formulas. Hence, they are absolute for every transitive model M for BST.*

1. *x is a transitive set.*
2. *x is an ordinal.*
3. *x is a successor ordinal.*
4. $x = 0$.
5. *x is a limit ordinal.*
6. *x is a natural number.*
7. $x \subseteq \omega$.
8. $x = \omega$.

Proof. For (1), x is transitive iff $\forall y \in x(\forall z \in y(z \in x))$; this is clearly equivalent to Definition I.7.1.

For (2): Transitivity is Δ_0 by (1), and the statement that x is totally ordered by $\in$ is easily expressed by quantifying over x. For x to be an ordinal, Definition I.7.3 required also that x be *well*-ordered by $\in$, but this follows by Foundation (see Lemma I.7.4).

$(3-8)$ are easy, using (2). For example, x is a successor ordinal iff x is an ordinal and $\exists y \in x \forall z \in x[z = y \vee z \in y]$; and x is a natural number iff x is an ordinal and x and all its elements are either successor ordinals or 0; see Definition I.7.16. Of course (4), $x = \emptyset$, or $\mathrm{emp}(x)$, was already discussed in Example I.16.3. □

We are implicitly using Lemma II.4.1 here, since the "official" definitions of most of these concepts are not Δ_0, but *BST* is sufficient to prove the equivalence. Note that *BST* does not contain the Axiom of Infinity, but the properties of *being* a limit ordinal and of *being* ω (that is, a limit ordinal all of whose elements are natural numbers) are Δ_0, even if one cannot prove in *BST* that any such ordinals exists.

Of course, "official" means the definition given in this book. Other authors may phrase the definitions differently, but for *most* of these concepts, the definitions are provably equivalent to ours from *BST*. For example, some texts assume Foundation from the start and define "ordinal" as a transitive set totally ordered by $\in$, or even something weaker, as in Exercise I.7.25. Then their definition is not *logically equivalent* to ours, but *BST* proves that the two definitions are equivalent.

There are a few notable exceptions. A common definition of "x is a natural number" is that x is in every inductive set, as in our Definition I.5.2. Working in *ZF*, this is equivalent to our "official" Definition I.7.16, so it doesn't matter which definition is used, but when studying the model *HF*, one should use Definition I.7.16. In this model, the Axiom of Infinity is false (see Theorem II.2.14), so there are no inductive sets, so *every* x is in every inductive set.

When we prove more complicated properties to be absolute, it is not practical to reduce them to formulas in $\mathcal{L} = \{\in\}$. Rather, we express them using some defined notions, following the discussion in Section I.16 of absoluteness for formulas involving defined notions. We shall frequently make use of Definition I.16.1 and Lemma I.16.2, which say that to prove absoluteness of a concept, it is sufficient to prove that it is equivalent to a Δ_0 formula that uses some defined notions, provided that those notions are absolute. This will allow us to prove the absoluteness of properties that are not expressible by Δ_0 formulas in $\mathcal{L} = \{\in\}$ alone.

Lemma II.4.3 *The following are absolute for transitive models of BST:*

1. *The* 0*-ary function* $\emptyset$.
2. *The* 1*-ary successor function* S.
3. *The* 2*-ary intersection function* $\cap$.
4. *The* 2*-ary union function* $\cup$, *and the* 1*-ary union and intersection functions* $\bigcup$ *and* $\bigcap$, *where we define* $\bigcap \emptyset = \emptyset$.
5. *The ternary relation:* $\{x, y\} = z$.
6. *The* 2*-ary unordered pairing function* $\{x, y\}$, *and the* 1*-ary singleton function* $\{x\}$, *and the* 2*-ary ordered pairing function* $\langle x, y\rangle$.
7. *The properties:* z *is an ordered pair, and* x *is a relation.*
8. $\mathrm{dom}(x)$ *and* $\mathrm{ran}(x)$.
9. *The properties:* f *is a function,* f *is an injection,* f *is a surjection, and* f *is a bijection.*
10. *The binary function* $f(x)$, *defined to be* $\emptyset$ *unless* f *is a function and* $x \in \mathrm{dom}(f)$.
11. *The binary function* $x \times y$.
12. *All relational properties of* R *and* A *defined in Definition I.6.2.*

Proof. (1)(2)(3) were discussed earlier; see Section I.16. For example, the formula $\mathrm{int}(v, w, y)$ (defining $y = v \cap w$) was defined (Definition I.4.1) by a formula logically equivalent to a Δ_0 formula (see Example I.16.3). Since *BST* is sufficient to prove $\forall v, w\, \exists! y\, \mathrm{int}(v, w, y)$ (Lemma I.4.2), $\cap^M$ is defined for every model M for *BST*, and is also absolute for M if M is transitive.

(4) is similar to (3), since *BST* is sufficient to prove the existence of these functions and it is easy to express them by Δ_0 formulas (see Section I.4). For example, $y = \bigcup x$ iff $\forall v \in x\, v \subseteq y$ and $\forall w \in y\, \exists v \in x\, w \in v$. Recall that $\bigcup \emptyset = \emptyset$, but $\bigcap \emptyset$ would be V, which would be meaningless, if we did not insert a special case such as $\bigcap \emptyset = \emptyset$.

(5)(6) are similar. Any reasonable definition of the notion $\{x, y\} = z$ is equivalent in *BST* to the Δ_0 formula

$$\mathrm{up}(x, y, z): \qquad x \in z \wedge y \in z \wedge \forall u \in z\, (u = x \vee u = y) \ ,$$

but also, $BST \vdash \forall x, y\, \exists! z\, \mathrm{up}(x, y, z)$, so that the binary unordered pairing function is absolute for transitive models for *BST*. Formally, we are introducing a binary function symbol here for the pair, say $p(x, y)$, but everyone writes $p(x, y)$ as $\{x, y\}$. Then, the unary singleton function $\{x\} = \{x, x\}$ and the ordered pairing function $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ are compositions of the unordered pairing function, so they are quantifier-free in a lexicon including unordered pairing, and hence absolute for transitive models for *BST*.

For (7), we can say z is an ordered pair (a set of the form $\{\{x\}, \{x, y\}\}$) by $\exists v \in z\, \exists x, y \in v\, [z = \langle x, y\rangle]$, which is Δ_0 in the ordered pairing function. Then x is a relation iff $\forall z \in x\, [z$ is an ordered pair$]$.

For (8), these functions were proved to exist in *BST* (see Section I.6), and the relations $y = \mathrm{dom}(x)$ and $y = \mathrm{ran}(x)$ are Δ_0 in absolute concepts. For domain, note that $y = \mathrm{dom}(x)$ iff $\forall v \in y\, \exists w \in \bigcup\bigcup x\, [\langle v, w\rangle \in x]$ and $\forall v, w \in \bigcup\bigcup x\, [\langle v, w\rangle \in x \to v \in y]$. We are using the fact (see Definition I.6.6) that $\mathrm{dom}(x)$ and $\mathrm{ran}(x)$ are subsets of $\bigcup\bigcup x$.

For (9), the notion of "function" and the various types theoreof are easily expressed by first saying that f is a relation, and then using some quantifiers bounded by $\mathrm{dom}(f)$ or $\mathrm{ran}(f)$, so that they are Δ_0 in absolute notions. For example, we say f is a function as $\forall x \in \mathrm{dom}(f)\, \forall y, z \in \mathrm{ran}(f)\, [\langle x, y\rangle \in f \wedge \langle x, z\rangle \in f \to y = z]$. (10) is similar, once we realize that function application is really a defined binary function $\mathrm{apply}(f, x)$, which is then always written $f(x)$.

For (11), the cartesian product was proved to exist from *BST* (see Definition I.6.5), and the relation $x \times y = z$ is easily seen to be Δ_0 in the defined $\bigcup$ and ordered pair functions.

For (12), for example "R totally orders A strictly" is easily expressed by quantifying over A and R. □

A simple illustration of absoluteness is in

Lemma II.4.4 *(ZFC) HC is a model for ZFC−P plus "all sets are countable" plus "$\mathcal{P}(\omega)$ doesn't exist".*

Proof. For example, $\forall x \in HC\, \exists f \in HC\, [f : x \xrightarrow{1-1} \omega]$ is clear from the definition of $H(\aleph_1)$, so use the fact that "$f : x \xrightarrow{1-1} \omega$" is absolute. □

All of the concepts in Lemma II.4.3 are, in fact, expressible by Δ_0 formulas in $\mathcal{L} = \{\in\}$ alone, although it may be a bit tedious to write out the formulas. For example, the Δ_0 formula $\exists v \in z\, \exists w \in z\, [\mathrm{up}(v, w, z) \wedge \mathrm{up}(x, x, v) \wedge \mathrm{up}(x, y, w)]$ expresses the statement $\langle x, y\rangle = z$. For more details, see [57], which describes some tricks that make it somewhat easier to verify that a concept is Δ_0. Usually in set theory, Δ_0 is not an important object of study in its own right. The important concept is absoluteness, and Δ_0 is just one means to that end. Ordinal sum and product are absolute but not Δ_0:

Lemma II.4.5 *The functions $\alpha+\beta$ and $\alpha\cdot\beta$ are absolute for transitive models of ZF−P.*

Proof. Working in $ZF-P$, we already proved that ordinal sum and product are defined (Definition I.8.4); it was essential to use Replacement here to prove that every well-order is isomorphic to an ordinal. So, if M is a transitive model for $ZF-P$, and $\alpha, \beta \in M$, then $\alpha +^M \beta$ and $\alpha \cdot^M \beta$ are defined

For product, let $\gamma = \alpha \cdot^M \beta$. To prove absoluteness, we show that γ is indeed $\alpha \cdot \beta$. Applying our definition, fix $f \in M$ such that it is true in M that $f : \beta \times \alpha \xrightarrow[\mathrm{onto}]{1-1} \gamma$ and f is an isomorphism, where $\beta \times \alpha$ has the lexicographic order. Now "bijection" is absolute by Lemma II.4.2, and the same methods easily show that lexicographic order and "isomorphism" are absolute, so that f really *is* an isomorphism establishing that $\gamma = \mathrm{type}(\beta \times \alpha) = \alpha \cdot \beta$.

The argument for addition is similar. □

Neither $\alpha + \beta = \gamma$ nor $\alpha \cdot \beta = \gamma$ is provably equivalent in ZF to a Δ_0 formula $\varphi(\alpha, \beta, \gamma)$ of $\mathcal{L} = \{\in\}$. If φ were Δ_0, then it would relativize to ω; but on ω, neither sum nor product is first-order definable using just $<$ (i.e, $\in$). If either were so definable, then the set of even numbers would also be definable in $(\omega, <)$, and this is easily seen to be false by a model-theoretic argument (see [57] (Exercise II.16.8 and Lemma II.17.19)).

The models $R(\gamma)$ (for limit γ) show that one cannot replace the $ZF-P$ in Lemma II.4.5 by BST, or even the stronger Z. For example, $R(\omega+\omega) \models Z$ by Theorem II.2.2, but $\omega + \omega$ and $\omega \cdot \omega$ are not even defined in this model.

Exercise II.4.6 *Let γ be a limit ordinal such that $\forall \alpha < \gamma\, [\alpha^2 < \gamma]$. Show that ordinal sum and product are defined in $R(\gamma)$ and are absolute for $R(\gamma)$.*

Lemma II.4.7 *The notions "R well-orders A" and "R is well-founded on A" are absolute for transitive models of $ZF-P$.*

Proof. Since we already know that "total order" is absolute, it is sufficient to prove the absoluteness of "well-founded". So, fix $R, A \in M$. We prove that (R is well-founded on A) $\leftrightarrow$ (R is well-founded on A)^M:

For $\rightarrow$: Let $\psi(A, R, X)$ say that X is a non-empty subset of A with no R–minimal elements. If $\neg$(R is well-founded on A)^M, then there is an $X \in M$ such that $(\psi(A, R, X))^M$. But ψ is easily seen to be absolute, so X shows that R is really not well-founded on A.

For $\leftarrow$: The above argument does not reverse, since an X satisfying the property $\psi(A, R, X)$ need not be in M. Instead, if (R is well-founded on A)^M, apply, in M, the theorem of $ZF-P$ that there is a rank function (see Definition I.9.13). So, there is a $\Phi \in M$ such that M satisfies the statement, "Φ is a function mapping A into the ordinals and $xRy \rightarrow \Phi(x) < \Phi(y)$ for all $x, y \in A$". But, by what we have already proved, this quoted statement is absolute, and the existence of Φ implies that R is really well-founded by Lemma I.9.9. □

Exercise II.4.8 *The notions "R well-orders A" and "R is well-founded on A" are absolute for $R(\gamma)$ for any limit γ.*

Hint. Use the fact that $\mathcal{P}$ is absolute. □

Exercise II.4.9 *(ZFC) Prove that $R(\gamma) \models AC^+$ and $H(\kappa) \models AC^+$ for limit γ and regular κ.*

Hint. Applying Definition I.3.3, fix $A \in M$, where M is one of these models; we must find a $W \in M$ such that (W well-orders A)^M. By AC, fix any W such that W well-orders A. Then $W \in M$ (for *these* M). □

We next show that notions such as "finite" and "hereditarily finite" are absolute for transitive models of BST.

Lemma II.4.10 *Let M be a transitive model for BST. Then*

1. $[M]^{<\omega} \subseteq M$ *(see Definition I.13.18).*
2. $HF \subseteq M$.
3. ${}^{<\omega}M \subseteq M$.

Proof. For (1), prove that $[M]^n \subseteq M$ by induction on $n \in \omega$, using the absoluteness of the function $s, x \mapsto s \cup \{x\}$. Then, using (1), prove $R(n) \subseteq M$ by induction on $n \in \omega$; so (2) follows. For (3), if $f : n \to M$ then $f \subseteq n \times M$, so $f \subseteq M$ by absoluteness of the function $i, x \mapsto \langle i, x\rangle$; then $f \in M$ by (1). □

Lemma II.4.11 *The notions "x is finite" and "x is hereditarily finite" are absolute for transitive models of BST if they are defined by the following formulas:*

$$\mathrm{Fin}(x) \leftrightarrow \exists n, f\,[\mathrm{nat}(n) \wedge \mathrm{bij}(f, n, x)]$$
$$\mathrm{HrdFin}(x) \leftrightarrow \exists n, t, f\,[x \subseteq t \wedge \mathrm{trans}(t) \wedge \mathrm{nat}(n) \wedge \mathrm{bij}(f, n, t)]$$

Here $\mathrm{trans}(x)$ *says that x is transitive,* $\mathrm{nat}(x)$ *says that x is a natural number, and* $\mathrm{bij}(f, x, y)$ *says that f is a bijection from x onto y.*

Proof. Fix $x \in M$; we prove that $(\mathrm{Fin}(x)) \leftrightarrow (\mathrm{Fin}(x))^M$:

For $\leftarrow$: We have $n, f \in M$ such that $(\mathrm{nat}(n) \wedge \mathrm{bij}(f, n, x))^M$, and we already know that nat and bij are absolute.

For $\rightarrow$: Fix n, f such that $(\mathrm{nat}(n) \wedge \mathrm{bij}(f, n, x))$. Then $n, f \in M$ by Lemma II.4.10, so use absoluteness of nat and bij.

The argument for HrdFin is similar. □

Although our definitions of $\mathrm{Fin}(x)$ and $\mathrm{HrdFin}(x)$ are the "natural ones", we spelled them out because other plausible definition would not work here. Defining x to be hereditarily finite iff $x \in R(\omega)$ is fine in ZF, but it does not make sense in a model $M \models BST$ (such as $M = R(\omega)$) in which the object $R(\omega)$ does not exist. This issue was already discussed above after Lemma II.4.2.

Since $ZF-P$ does prove the existence of $HF = R(\omega)$ (Lemma I.10.16), we have the following, continuing in the spirit of Lemma II.4.3:

Lemma II.4.12 *The following are absolute for transitive models of $ZF-P$:*

1. *The 0-ary function HF.*
2. *The 0-ary function ω.*
3. *The 1-ary functions $[x]^{<\omega}$ and ${}^{<\omega}x$.*

We now have the absoluteness of all arithmetical relations.

Definition II.4.13 *An n-ary relation R is* arithmetical *iff $R(x_1, \ldots, x_n)$ is true only for $x_1, \ldots, x_n \in HF$ and all quantifiers in the definition of R are relativized to HF.*

Lemma II.4.14 *Every arithmetical relation is absolute for all transitive models of BST.*

Our Definition II.4.13 really just restated our earlier Definition I.17.1, now making it clear that the notion makes sense within BST. We do not need HF to be a set; we only require that all quantifiers in the definition of R to occur as $\forall y\,[\mathrm{HrdFin}(y) \to \cdots\cdots]$ and $\exists y\,[\mathrm{HrdFin}(y) \wedge \cdots\cdots]$.

As we pointed out in Section I.17, any decidable property of elements of HF is arithmetical, so it is absolute. Besides many examples from number theory and algebra (n is a prime, G is a finite simple group, etc.), we have all of logical syntax, including the notions of "formal proof" and "axiom of ZFC". Also, the set $\{\varphi : ZFC \vdash \varphi\}$ is absolute. This is not decidable, but it is clearly arithmetical (in fact Σ_1).

The 0-ary arithmetical relations are just the logical sentences φ that quantify only over HF. By Lemma II.4.14, such a φ is true in V iff φ is true in all transitive models of BST. Among such φ are many standard conjectures of number theory, such as the Twin Prime Conjecture, as well as the Riemann Hypothesis, which can be expressed as a statement about the distribution of primes. It is possible that the Twin Prime Conjecture is independent of ZFC, but if so, the independence cannot be established by the methods discussed in this book, which deal only with transitive models.

Using the absoluteness of logical syntax, we shall now show that the semantic notion $\mathfrak{A} \models \varphi$ is absolute along with related notions such as $\mathcal{D}^+(A)$ and $\mathcal{D}^-(A)$ (see Definition I.15.25). Now, $\models$ is defined by recursion (see Definition I.15.5), and the absoluteness will follow from a general lemma about the absoluteness of recursively defined functions. This lemma will also imply the absoluteness of ordinal exponentiation α^β (as in Table I.1 on page 40) and the function $\mathrm{rank}(x)$ (as in Definition I.9.20), both of which have recursive definitions. Recall our general Theorem I.9.11 justifying definitions by transfinite recursion on well-founded relation R on A. A could be a proper class, such as ON or V, in which case R must also be set-like on A.

Theorem II.4.15 *Assume that R is a defined 2-ary relation, G is a defined 2-ary function and A is a class (a defined 1-ary relation). Assume also that R is well-founded and set-like on A and let F be the defined 1-ary function, as in Theorem I.9.11, such that $\forall a \in A\,[F(a) = G(a, F\restriction(a\downarrow))]$; assume that $F(a) = \emptyset$ for $a \notin A$.*

Now, let M be a transitive model for $ZF-P$, and assume that R, A, G are all absolute for M, and that $(R$ is set-like on $A)^M$, and that $a\downarrow \subseteq M$ for all $a \in M$. Then $F^M(a)$ is defined for $a \in M$, and F is absolute for M.

Proof. Note that (R is well-founded on A)^M, since a non-empty $X \in M$ with $X \subseteq A$ and no R–minimal element would contradict well-foundedness of R in V. Also, the function $a \mapsto a{\downarrow}$ is absolute for M.

Now, Theorem I.9.11 was a theorem of $ZF-P$, so that F^M is defined, and note that by absoluteness of G, if $F{\restriction}(a{\downarrow}) = F^M{\restriction}(a{\downarrow})$ then $F(a) = F^M(a)$. If F were not absolute, then an R–minimal element of $\{a \in M : F(a) \neq F^M(a)\}$ would be contradictory. □

Since one may view ordinal sum and product as being defined recursively (see Table I.1), this theorem provides an alternate proof of Lemma II.4.5. As we remarked for that lemma, one cannot substitute the $ZF-P$ here with BST or even the stronger Z, and our proof of Theorem I.9.11 made essential use of the Replacement Axiom.

Theorem II.4.15 implies immediately:

Corollary II.4.16 *The functions α^β and* rank(x) *are absolute for transitive $M \models ZF-P$.*

Corollary II.4.17 *The notions "φ is a formula of $\mathcal{L}$" and "$\mathfrak{A} \models \varphi[\sigma]$" (see Definition I.15.5) are absolute for transitive models $M \models ZF-P$.*

Note that in model theory, $\mathcal{L}$ can have any cardinality, but any reasonable definition of logical syntax is defined by recursion using absolute notions.

Exercise II.4.18 *Show that Corollary II.4.17 holds even if we only have $M \models BST$.*

Hint. You can avoid using Replacement and Infinity in this case. □

In particular, these notions are absolute for $R(\gamma)$ for any limit γ.

Corollary II.4.19 *The functions $\mathcal{D}(A, P)$ and $\mathcal{D}^+(A)$ and $\mathcal{D}^-(A)$ from Definition I.15.25 are absolute for transitive models $M \models ZF-P$.*

Here, one cannot replace the $ZF-P$ by BST, since BST doesn't prove the existence of the sets $\mathcal{D}(A, P)$.

The models $H(\kappa)$ and $R(\gamma)$ are transitive and uncountable (for $\kappa, \gamma > \omega$). The next lemma shows that there are also countable transitive models for the same axioms, which will imply that "countable" is *not* absolute (although "finite" is absolute by Lemma II.4.11).

Lemma II.4.20 *Assume AC. Let B be any infinite set such that $(B, \in)$ satisfies the Axiom of Extensionality. Let κ be any infinite cardinal with $\kappa \leq |B|$. Fix $S \subseteq B$ such that S is transitive and $|S| \leq \kappa$. Then there is a transitive M such that $S \subseteq M$, $(M, \in) \equiv (B, \in)$, and $|M| = \kappa$. In particular, there is a countable transitive M such that $(M, \in) \equiv (B, \in)$.*

Proof. By the Downward Löwenheim–Skolem–Tarski Theorem (I.15.10), let $A \preccurlyeq B$ with $S \subseteq A$ and $|A| = \kappa$. Then A also satisfies the Axiom of Extensionality. Since $\in$ is well-founded on A (by the Foundation Axiom), the Mostowski function mos is defined on A (Definition I.9.31) and is an isomorphism from $(A, \in)$ onto $(M, \in)$ for some transitive set M (see Lemmas I.9.32 and I.9.35). Furthermore, since S is transitive, $\mathrm{mos}(y) = y$ for all $y \in S$ (see Lemma I.9.36), so $S \subseteq M$.

$|M| = \kappa$ because mos is a bijection and $|A| = \kappa$, and $(M, \in) \equiv (A, \in) \equiv (B, \in)$ because $M \cong A$ and $A \preccurlyeq B$ (see Definition I.15.19 and Lemma I.15.20). The "in particular" follows by letting $\kappa = \aleph_0$, and $S = \emptyset$. □

Now, working in *ZFC*, let B be either $R(\gamma)$ for some limit $\gamma > \omega$ or $H(\theta)$ for some regular $\theta \geq \aleph_2$. Then B satisfies some set theory (depending on γ, θ) plus the statement that there exists an uncountable set. Applying the lemma, we get a countable transitive $M \equiv B$. Then there is an $x \in M$ such that $(x \text{ is uncountable})^M$. But $x \in M$ implies that x is countable, so "countable" is not absolute for M; although the notions "ω" and "bijection" are absolute, an $f : \omega \xrightarrow[\text{onto}]{1\text{–}1} x$ need not lie in M. This non-absoluteness of "countable" leads to the Skolem "Paradox", as on page 98.

The next two exercises illustrate models for some failures of Replacement beyond what can be provided by the $R(\gamma)$. First, our system *ZC* is approximately the axioms of Zermelo [105] from 1908, but not exactly. He thought of $\mathbb{N}$ as the set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}$, rather than the von Neumann ordinal ω, and his Infinity Axiom was phrased accordingly.

Exercise II.4.21 *Let AI denote our Axiom of Infinity, and let AU denote the* Axiom des Unendlichen *of Zermelo:* $\exists x (\emptyset \in x \wedge \forall y \in x (\{y\} \in x))$. *Work in ZFC and produce transitive models for* $ZC + \neg AU$ *and for* $ZC - Inf + AU + \neg AI$.

Hint. Use the following general method. Fix a limit $\gamma > \omega$, and let $T \subseteq R(\gamma)$ be transitive. Now, let M be the set of all $x \in R(\gamma)$ such that $\mathrm{trcl}(x) \cap T$ is finite. Show that M satisfies all the *ZC* axioms, including AC^+, except possibly for Infinity. Now consider $T = \omega$ and $T = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}$. □

Actually, Zermelo independently discovered the von Neumann theory of ordinals and natural numbers somewhat after 1908 but before von Neumann, but this was not reflected in Zermelo's published work; see Ebbinghaus [26].

Note that $R(\gamma) \models AU$ for all limit $\gamma > \omega$. Also, using Replacement, AI and AU are equivalent (see [57], Exercise I.8.23). Some other pathological failures of Replacement require Power Set to fail also, so we need a slightly different method:

Exercise II.4.22 *Find a transitive* $M \models ZC - P$ *in which* $\omega \times \omega$ *and* $\omega^* := \{\{n\} : n \in \omega\}$ *don't exist.*

Hint. Recall from Sections I.5 and I.6 that these sets can be proved to exist using either the Power Set or the Replacement Axioms, but now we've deleted both axioms.

Let $M_0 = HF \cup \{\omega\}$, and let $M_{n+1} = M_n \cup [M_n]^{<\omega} \cup \{\bigcup x : x \in M_n\} \cup \{y : \exists x \in M_n\, [y \subseteq x]\}$. Let $M = \bigcup_n M_n$.

Note that M satisfies AC, not AC^+, since M does not contain a well-order of ω (or any other infinite subset of $\omega \times \omega$). □

Exercise II.4.23 *(ZFC) Show that $R(\omega+\omega)$ is a model for ZC plus "$R(\omega+\omega)$ doesn't exist".*

Hint. Obviously $R(\omega+\omega) \notin R(\omega+\omega)$, but you have to check that the function $\alpha \mapsto R(\alpha)$ is absolute where it is defined. This absoluteness is similar in spirit to Exercise II.3.1. □

Exercise II.4.24 *(ZFC) Assume that $\gamma > \omega$ and that the Replacement Axiom holds in $R(\gamma)$. Prove that $\gamma = \beth_\gamma$ and that $\{\delta < \gamma : \delta = \beth_\delta\}$ has size γ.*

Hint. Since $R(\gamma) \models ZFC$, you can use the *ZFC* theorem that $\{\delta : \delta = \beth_\delta\}$ is unbounded in ON, along with the absoluteness of $\delta \mapsto \beth_\delta$ (which you have to prove). □

This γ need not be strongly inaccessible. The first such γ has cofinality ω; see Exercise II.5.12.

Exercise II.4.25 *Work in the theory ZFC plus the assumption that there are at least 76 strongly inaccessible cardinals. Prove that there is a set model M for "ZFC plus there are exactly 75 strongly inaccessible cardinals".*

Hint. It's obvious what the model is, but you have to check a few absoluteness statements. □

Exercise II.4.26 *Let M be a transitive class, and assume that the Axioms of Extensionality, Comprehension, Pairing, Union, and Infinity hold in M. Prove that $\omega \in M$.*

Hint. Consider the proof of Lemma II.2.11. Fix $a \in M$ satisfying $\varphi(a)$. Then $\omega \subseteq a$, so use Comprehension in M plus absoluteness to prove that $\omega \in M$. □

The following lemma abstracts some of our arguments for *WF*, and will be reused later for L (Section II.6) and *HOD* (Section II.8).

Lemma II.4.27 *Suppose that M is a transitive class such that the Comprehension Axiom holds in M, and assume that for every subset $x \subseteq M$, there is a set $y \in M$ such that $x \subseteq y$. Then all the ZF axioms hold in M.*

Proof. Lemma II.2.4 applies to establish Axioms (1)(2)(4)(5)(6) in M. For example, for Axiom 5 (union), $\forall \mathcal{F} \in M\, [\bigcup \mathcal{F} \in M]$, because $\bigcup \mathcal{F} \subseteq y \in M$ for some $y \in M$, so $\bigcup \mathcal{F} \in M$ by Comprehension in M. Also, Lemma II.2.8 gives us Axiom 8. For Axiom 7 (Infinity), we have $\omega \subseteq M$ by Lemma II.4.10, so $\omega \subseteq y \in M$, so $\omega \in M$ using Comprehension in M. □

As we indicated briefly in Section I.17, the notions of Σ_1 and Π_1 (Definition I.17.3) are of great importance in recursion theory. They are also relevant to absoluteness questions in set theory. Briefly, we say that Σ_1 properties relativize up and Π_1 properties relativize down. To be more formal, note that Definition I.17.3 leave open the possibility that the formulas use non-logical symbols other than $\in$. Then, we have the following variant of the absoluteness of Δ_0 from Lemma I.16.2:

Lemma II.4.28 *Let M be a transitive model of BST. Let $\varphi_\Sigma(x_1, \ldots, x_n)$ and $\varphi_\Pi(x_1, \ldots, x_n)$ be two formulas that are, respectively, Σ_1 and Π_1 in $\in$ plus possibly other relations and functions known to absolute for M. Then for all $a_1, \ldots, a_n \in M$: $\varphi_\Sigma^M(\vec{a}) \to \varphi_\Sigma(\vec{a})$ and $\varphi_\Pi(\vec{a}) \to \varphi_\Pi^M(\vec{a})$.*

Proof. Let $\varphi_\Sigma(\vec{x})$ be $\exists y_1 \cdots \exists y_k\, \psi(\vec{x}, \vec{y})$, where ψ is Δ_0. If $\varphi_\Sigma^M(\vec{a})$ is true, then we can fix $b_1 \cdots b_k \in M$ such that $\psi^M(\vec{a}, \vec{b})$ is true. Then $\psi(\vec{a}, \vec{b})$ is true by the absoluteness of Δ_0, so $\varphi_\Sigma(\vec{a})$ is true. The result for φ_Π is similar. □

For example, let M be a transitive model of $ZF-P$, and let $\varphi_\Sigma(x)$ be $\exists f\, [f : x \xrightarrow{1-1} \omega]$, which says that x is countable. We just pointed out above that "countable" need not be absolute, but it does relativize up; that is if $(a \text{ is countable})^M$, then a really is countable.

As another example, we can give a briefer explanation of the proof that $(R \text{ is well-founded}) \leftrightarrow (R \text{ is well-founded})^M$ (i.e., well-foundedness is absolute; see Lemma II.4.7). For $\to$ we use the fact that "well-founded" is Π_1 in absolute concepts ($\forall X$: if X is non-empty then X has an R-minimal element). For $\leftarrow$ we use the fact that "well-founded" is Σ_1 in absolute concepts ($\exists \Phi$: Φ is a rank function).

Exercise II.4.29 *Let M be a transitive model for $ZF-P$. Let $\bigstar, \maltese \in M$ be two group operations on ω. Prove that the statement $(\omega, \bigstar) \cong (\omega, \maltese)$ is absolute for M.*

Hint. Let $\mathbb{P}$ be the set of "partial isomorphisms" p, where p is a finite bijection with $\mathrm{dom}(p), \mathrm{ran}(p) \subseteq \omega$ such that for some $n \in \omega$: $n \subseteq \mathrm{dom}(p)$, $n \subseteq \mathrm{ran}(p)$, $|p| \leq 2n$, and $\forall x, y, z \in \mathrm{dom}(p)\, [x \bigstar y = z \leftrightarrow p(x) \maltese p(y) = p(z)]$. Define $q < p$ iff $q \supsetneq p$. Then show that $\mathbb{P}$ is well-founded iff $(\omega, \bigstar) \ncong (\omega, \maltese)$. □

This exercise is a special case of the Suslin–Kleene analysis of Π_1^1 sets (see §7.10 of [89]), which proves that Π_1^1 and Σ_1^1 properties of countable objects are absolute for transitive models of $ZF-P$. A Σ_1^1 statement is one of the form

$\exists F \subseteq HF\, \varphi(F)$, where the quantifiers in φ all range over elements of HF. Note that the statement $(\omega, \bigstar) \cong (\omega, \maltese)$ is of this form.

Working in ZFC, the Completeness Theorem implies that $\mathrm{Con}(ZFC)$ holds iff there is a model for ZFC. But it does not give you a well-founded model:

Exercise II.4.30 *Let TM denote the statement that there is a transitive set M such that $M \models ZFC$. Show that $ZFC + \mathrm{Con}(ZFC) \nvdash TM$, assuming that $ZFC + \mathrm{Con}(ZFC)$ is consistent.*

Hint. Let $o(M)$ be the least ordinal not in M. Working in $ZFC + TM$, let M be a transitive model for ZFC with $o(M)$ minimal among all such M, and show that $M \models ZFC + \mathrm{Con}(ZFC) + \neg TM$. □

Note that any well-founded model will give you a transitive model by Lemma I.9.35. That lemma plus the Downward Löwenheim-Skolem-Tarski Theorem implies that this minimal $o(M)$ is countable. $ZFC + TM$ is "presumably" consistent, since $R(\kappa) \models ZFC$ whenever κ is strongly inaccessible; then $R(\kappa) \models ZFC + TM$, since $R(\kappa)$ contains all countable transitive models.

II.5 Reflection Theorems

Again, we work in ZF, flagging the use of AC when it is used in a lemma or theorem.

We have seen that working in ZFC, we can prove that $R(\gamma) \models ZC$ whenever $\gamma > \omega$ is a limit. We have also seen (Exercise II.4.24) that the Replacement Axiom fails for "most" $R(\gamma)$, although the specific instances of Replacement that fail depend on γ. We cannot prove in ZFC that there is a limit $\gamma > \omega$ such that $R(\gamma)$ satisfies all of Replacement; that is, $ZFC \nvdash \exists \gamma\, [R(\gamma) \models ZFC]$ (unless ZFC is inconsistent). However, the Reflection Theorem, proved below, shows that for any finite list Λ of instances of Replacement, $ZFC \vdash \exists \gamma\, [R(\gamma) \models ZC \cup \Lambda]$. Then, by Lemma II.4.20, we also obtain a countable transitive model $M \models ZC \cup \Lambda$.

Naively, working in ZFC, we could apply Lemma II.4.20 directly to V to get a countable $A \preccurlyeq V$ and then a countable transitive $M \cong A$, so that $M \models ZFC$. However, as we have already seen (e.g., in Section I.15), you can't apply model theory results directly to V, and you can't even *say* $A \preccurlyeq V$. However, we have also seen that many notions of model theory do make sense for proper classes if we restrict them to talk about only finitely many formulas at once. In particular, for a given list of formulas $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$, we can write the sentence $\exists A\, \left[\bigwedge_{i<n} (A \preccurlyeq_{\varphi_i} V)\right]$, and we shall see soon (Theorem II.5.3) that we can also prove this sentence. In fact, we can produce such an A of the form $R(\gamma)$ working in ZF, without any need of AC. Working in ZFC, we can then apply Lemma II.4.20 to the set $R(\gamma)$ to produce a countable A and a transitive M such that $M \cong A \preccurlyeq R(\gamma)$.

In order to prove a "class version" of Theorem I.15.10, we first need a "class version" of the Tarski–Vaught criterion (Lemma I.15.11) that only mentions finitely many formulas at a time. We shall state it only in the case $\mathcal{L} = \{\in\}$, although it is easily generalized to other $\mathcal{L}$.

Definition II.5.1 *A list of formulas $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ is* subformula–closed *iff every subformula of each φ_i is also on the list,* and *no formula on the list uses the universal quantifier $\forall$.*

Lemma II.5.2 *Let $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ be a subformula–closed list of formulas of $\mathcal{L} = \{\in\}$. Let A, B be classes with $\emptyset \neq A \subseteq B$. Then the following are equivalent:*

1. $\bigwedge_{i<n}(A \preccurlyeq_{\varphi_i} B)$.
2. *For all existential formulas $\varphi_i(x_1, \ldots, x_r)$, of the form $\exists y \varphi_j(\vec{x}, y)$, the following holds:*

$$\forall a_1, \ldots, a_r \in A \left[\varphi_i^B(\vec{a}) \to \exists b \in A\, \varphi_j^B(\vec{a}, b)\right] \quad .$$

Here, φ^B denotes relativization to the class B. Note that relativization to A is not mentioned. As with the original Lemma I.15.11, the importance of (2) is that it may be viewed as a closure property of A, which enables us, given a *class* B, to construct a *set* A for which (2) holds.

As usual with such statements about classes, this lemma is an abbreviation for an assertion in the metatheory — namely, given formulas $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ together with $\alpha(x), \beta(x)$ (defining classes A, B), one can prove from ZF that if $\emptyset \neq A \subseteq B$, then (1) $\leftrightarrow$ (2). Note that (2) is a finite conjunction of assertions, one for each existential φ_i. If there are no existential formulas on the list, then (2) is true vacuously and (1) is trivial.

Proof. (1) $\to$ (2): $\varphi_i^B(\vec{a}) \to \varphi_i^A(\vec{a}) \to \exists b \in A\, \varphi_j^A(\vec{a}, b) \to \exists b \in A\, \varphi_j^B(\vec{a}, b)$. Here, the middle $\to$ used the meaning of φ_i^A, and the other two used $A \preccurlyeq_{\varphi_i} B$ and $A \preccurlyeq_{\varphi_j} B$.

(2) $\to$ (1): Assume (2), and prove $A \preccurlyeq_{\varphi_i} B$ by induction on the length of φ_i, so assume that we have proved $A \preccurlyeq_{\varphi_j} B$ whenever φ_j is shorter than φ_i. The basis cases (φ_i is atomic) and the induction steps for propositional connectives are trivial, so now assume that $\varphi_i(\vec{x})$ is existential, of the form $\exists y \varphi_j(\vec{x}, y)$. Fix $a_1, \ldots, a_r \in A$. To show $\varphi_i^B(\vec{a}) \leftrightarrow \varphi_i^A(\vec{a})$, use

$$\varphi_i^B(\vec{a}) \leftrightarrow \exists b \in A\, \varphi_j^B(\vec{a}, b) \leftrightarrow \exists b \in A\, \varphi_j^A(\vec{a}, b) \leftrightarrow \varphi_i^A(\vec{a}) \quad .$$

Here, the middle $\leftrightarrow$ used $A \preccurlyeq_{\varphi_j} B$, the last $\leftrightarrow$ used the meaning of φ_i^A, and the first $\leftrightarrow$ used the meaning of φ_i^B for the $\leftarrow$ and (2) for the $\to$. $\square$

With $B = V$, the proof of the Reflection Theorem produces an $A = R(\eta)$ for which (1) holds, and hence (2) holds also. Actually, our argument uses very little about the $R(\eta)$; our statement of the theorem lists exactly the properties needed for the proof to work:

Theorem II.5.3 (Reflection Theorem) *Let $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ be any list of formulas of $\mathcal{L} = \{\in\}$. Assume that B is a non-empty class and $A(\xi)$ is a set for each $\xi \in ON$, and assume that:*

I. $\xi < \eta \to A(\xi) \subseteq A(\eta)$.
II. $A(\eta) = \bigcup_{\xi<\eta} A(\xi)$ *for limit* η.
III. $B = \bigcup_{\xi \in ON} A(\xi)$.

Then $\forall \xi \, \exists \eta > \xi \, \big[A(\eta) \neq \emptyset \wedge \bigwedge_{i<n} (A(\eta) \preccurlyeq_{\varphi_i} B) \wedge \eta \text{ is a limit ordinal}\big]$.

The most well-known application of this is where $B = V$ and $A(\xi) = R(\xi)$; then (I)(II)(III) are clear from Section I.9; (III) restates the Axiom of Foundation, which we are now always assuming.

We did not say that B is a *proper* class, but if B is a set then $A_\xi = B$ for ξ large enough, so the result is trivial. We also did not assume that all the A_ξ are non-empty (which is usually assumed when dealing with model-theoretic concepts), because A_ξ must be non-empty for ξ large enough.

As usual, this theorem is an abbreviation for an assertion in the metatheory — namely, given formulas $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ together with $\beta(x), \alpha(v, x)$ (defining B and the map $\xi \mapsto A(\xi)$), an appropriate sentence is a theorem of ZF.

Proof. We may assume that our list is subformula–closed; if not, replace each φ_i by a logically equivalent formula not using $\forall$ (replace the $\forall$ by $\neg\exists\neg$), and then add to the list all subformulas of formulas appearing on the list.

For each existential $\varphi_i(\vec{x})$ (of form $\exists y \varphi_j(\vec{x}, y)$, where $\vec{x}$ denotes an r–tuple; $r = r_i$), define $F_i : B^r \to ON$ as follows: If $\varphi_i^B(\vec{a})$, then $F_i(\vec{a})$ is the least ζ such that $\exists b \in A(\zeta) \, \varphi_j^B(\vec{a}, b)$. If $\neg\varphi_i^B(\vec{a})$, then $F_i(\vec{a}) = 0$. These F_i are analogous to the Skolem functions in the proof of Theorem I.15.10, but we are not assuming AC, so our functions don't *choose* some b; rather, we use the well-order on ON to *define* a ζ with $A(\zeta)$ containing at least one such b.

Next, define $G_i : ON \to ON$ by: $G_i(\xi) = \sup\{F_i(a_1, \ldots, a_r) : a_1, \ldots, a_r \in A(\xi)\}$ whenever φ_i is existential, with $r = r_i$. When φ_i is not existential, let $G_i(\xi) = 0$. Finally, let $K(\xi)$ be the larger of $\xi + 1$ and $\max\{G_i(\xi) : i < n\}$.

Now, fix ξ; it is sufficient to produce an $\eta > \xi$ such that $A(\eta) \neq \emptyset$ and (2) of Lemma II.5.2 holds for $A(\eta), B$. So, let ζ_0 be the least $\zeta > \xi$ such that $A(\zeta) \neq \emptyset$, and let $\zeta_{n+1} = K(\zeta_n)$. Then $\xi < \zeta_0 < \zeta_1 < \cdots$. Let $\eta = \sup\{\zeta_k : k \in \omega\}$. □

Corollary II.5.4 *Let Λ be a finite set of axioms of ZF. Then*

1. $ZF \vdash \exists \eta \, [R(\eta) \models Z \cup \Lambda]$.
2. $ZFC \vdash \exists \eta \, [R(\eta) \models ZC \cup \Lambda]$.
3. $ZFC \vdash \exists M \, [M \models ZC \cup \Lambda \wedge |M| = \aleph_0 \wedge M \text{ is transitive}]$.

Proof. For (1)(2), let $B = V$ and $A(\xi) = R(\xi)$. Let $\{\varphi_0, \ldots, \varphi_{n-1}\} = \Lambda$, and apply the theorem to get a limit $\eta > \omega$ such that $\bigwedge_{i<n} (R(\eta) \preccurlyeq_{\varphi_i} V)$. Since φ_i

is a sentence and an axiom of ZF, each $R(\eta) \prec_{\varphi_i} V$ yields $\varphi_i^{R(\eta)} \leftrightarrow \varphi_i$ and then $R(\eta) \models \varphi_i$. And, $ZF \vdash (R(\eta) \models Z)$ and $ZFC \vdash (R(\eta) \models ZC)$ by Theorem II.2.2. For (3), apply Lemma II.4.20 to $(R(\eta), \in)$. □

Note that since Z contains all of ZF except Replacement, it would be equivalent to state this Corollary with Λ a finite set of instances of Replacement.

Regarding (3): Although ZFC cannot produce a model for ZFC, it can produce a countable transitive model for enough of ZFC to prove all of known mathematics.

In the proof of Corollary II.5.4, the only place we needed the φ_i to be axioms of ZF is to conclude $R(\eta) \models \varphi_i$ from $\varphi_i^{R(\eta)} \leftrightarrow \varphi_i$. So, the same proof yields:

Corollary II.5.5 *Let Λ be a finite set of sentences of $\mathcal{L} = \{\in\}$. Then $ZFC \vdash \exists M \left[M \models ZC \wedge |M| = \aleph_0 \wedge M \text{ is transitive} \wedge \bigwedge_{i<n}(\varphi_i^M \leftrightarrow \varphi_i)\right]$.*

Of course, for sentences, $\varphi_i^M \leftrightarrow \varphi_i$ is the same as $M \prec_{\varphi_i} V$. We cannot in general get a small *transitive* M satisfying $M \prec_{\varphi_i} V$ for *formulas* φ_i:

Exercise II.5.6 *Assume AC. Find a formula φ such that every transitive M satisfying $M \prec_\varphi V$ is of the form $R(\gamma)$ for some $\gamma = \beth_\gamma$.*

Hint. φ can be $\varphi_0(x) \wedge \varphi_1$, where $\varphi_0(x)$ says that x is the set of the form $R(\alpha)$, and φ_1 is a sentence and a theorem of ZFC. □

On the other hand, there is no problem if we drop "transitive":

Corollary II.5.7 *Let Λ be a finite set of formulas of $\mathcal{L} = \{\in\}$. Then $ZFC \vdash \exists C \left[C \models ZC \wedge |C| = \aleph_0 \wedge \bigwedge_{i<n}(C \prec_{\varphi_i} V)\right]$.*

Proof. As in the proof of Corollary II.5.4, use the Reflection and Downward Löwenheim–Skolem–Tarski Theorems to get η with $\bigwedge_{i<n}(R(\eta) \prec_{\varphi_i} V)$, and then $C \prec R(\eta)$. □

If we apply the Mostowski isomorphism to C to get a transitive M, then C and M will satisfy the same *sentences*, so we are back to Corollary II.5.5, but we cannot claim that $M \prec_{\varphi_i} V$ for *formulas* φ_i by Exercise II.5.6.

If your goal is to work in ZFC and produce a countable model, as in Corollaries II.5.7 and II.5.5, then our proofs may seem a bit inelegant. We use one Skolem closure argument to get $\bigwedge_{i<n}(R(\eta) \prec_{\varphi_i} V)$, and a second Skolem closure argument to get $C \prec R(\eta)$. We said in our proof of Theorem II.5.3 that we were avoiding AC by not choosing Skolem functions, but the fact is that even in ZFC, one is not justified in using Skolem functions on V. Say $\varphi_0(x)$ is $\exists y(y \in x)$. Then our Skolem function $F : V \to V$ would be a choice function, as in Definition I.12.2; it would satisfy $F(x) \in x$ for all non-empty x. But the notion of a *function* on V only makes sense if it has been defined by a formula, so we would have produced a $\psi(x, y)$ for which we could prove

$\forall x\,[x \neq \emptyset \to \exists! y\,[\psi(x,y) \wedge y \in x]]$. But such a *global choice function* cannot be produced just in *ZFC* (see Exercise V.2.17). A global choice function, and a well-order of V, can be produced under Gödel's Axiom of Constructibility (see Theorem II.6.20).

Another view of this is that Corollary II.5.7 cannot be proved in ZFC^-. We really needed Foundation to assert that V is the increasing union of a well-ordered chain of *sets* (the $R(\xi)$):

Exercise II.5.8 *Work in ZFC^-, plus the assumption that $U := \{x : x = \{x\}\}$ is a proper class that has no infinite subsets. Write a finite list of formulas φ_i for $i < n$ and prove that there is no set C satisfying $\bigwedge_{i<n}(C \preccurlyeq_{\varphi_i} V)$.*

For the consistency of the hypotheses of this exercise, see Exercise II.9.11.

Theorem II.5.9 *If $\Gamma \supseteq ZF$ is consistent, then Γ is not finitely axiomatizable; that is, whenever $\Lambda \subset \Gamma$ is finite, there is a sentence ψ such that $\Gamma \vdash \psi$ but $\Lambda \nvdash \psi$.*

Proof. Let ψ say that $\exists \eta\,(R(\eta) \models \Lambda)$. We may assume, expanding Λ if necessary, that Λ proves basic facts about the $R(\eta)$ and $\models$. Then $\Gamma \vdash \psi$ by Reflection. But if $\Lambda \vdash \psi$ then Λ is inconsistent, since working in Λ, the least η such that $R(\eta) \models \Lambda$ would yield a contradiction (as in Exercise II.4.30). □

There is also a set version of the Reflection Theorem II.5.3, obtained by replacing *ON* by a regular uncountable cardinal. Since we are dealing with sets, we can use standard model-theoretic notions and deal with all formulas at once, and our theorem is really just (an abbreviation of) a single sentence in the language of set theory. For simplicity, we state this assuming *AC*, so that we can refer to standard cardinal arithmetic.

Theorem II.5.10 (*ZFC*) *Let $\kappa > \omega$ be a regular uncountable cardinal. Let $A(\xi)$ be sets for each $\xi \leq \kappa$, and assume that:*

I. $\xi < \eta \to A(\xi) \subseteq A(\eta)$.
II. $A(\eta) = \bigcup_{\xi<\eta} A(\xi)$ *for limit* $\eta \leq \kappa$.
III. $|A(\xi)| < \kappa$ *for all* $\xi < \kappa$, *and* $|A(\kappa)| = \kappa$

Then $\forall \xi < \kappa\, \exists \eta\, [\xi < \eta < \kappa \wedge A(\eta) \neq \emptyset \wedge A(\eta) \preccurlyeq A(\kappa) \wedge \eta$ is a limit ordinal].

Proof. Modify the proof of Theorem II.5.3, replacing *ON* by κ and B by $A(\kappa)$. Now, we let the φ_i (for $i < \omega$) list all formulas that do not use $\forall$. Each $F_i : (A(\kappa))^{r_i} \to \kappa$. Also, $G_i : \kappa \to \kappa$; to prove that $G_i(\xi) < \kappa$, use the regularity of κ and $|A(\xi)| < \kappa$. Finally, $K : \kappa \to \kappa$; $K(\xi) = \max(\xi+1, \sup\{G_i(\xi) : i < n\})$. To prove that $K(\xi) < \kappa$ and $\eta < \kappa$, use the regularity of κ and $\omega < \kappa$. □

Corollary II.5.11 (*ZFC*) *If κ is strongly inaccessible, then $\{\eta < \kappa : R(\eta) \preccurlyeq R(\kappa)\}$ is unbounded in κ.*

Note that $R(\eta) \preccurlyeq R(\kappa) \to R(\eta) \models ZFC$, but the $\leftarrow$ is false, since if γ is least such that $R(\gamma) \models ZFC$ then $R(\gamma) \not\preccurlyeq R(\kappa)$ because $R(\gamma) \models \neg\exists\eta\,[R(\eta) \models ZFC]$.

Exercise II.5.12 *Work in ZFC plus the assumption that $R(\gamma) \models ZFC$ for some γ. Prove that the first such γ has cofinality ω.*

Hint. Assume that $R(\gamma) \models ZFC$ and $\mathrm{cf}(\gamma) > \omega$. Prove that there is an $\eta < \gamma$ such that $R(\eta) \preccurlyeq R(\gamma)$, so that $R(\eta) \models ZFC$. To get η, use the proof of Theorem II.5.10, replacing κ by γ. Each $F_i : R(\gamma)^r \to \gamma$ can be defined within the model $R(\gamma)$, so its range is bounded on $R(\xi)$ when $\xi < \gamma$. □

The proof of Theorem II.5.9 shows that in axiomatizing ZF, one cannot replace the Replacement scheme by any finite set Λ of instances of Replacement, since $ZF \vdash \exists\eta\,[R(\eta) \models Z \cup \Lambda]$. This is not true for Comprehension:

Exercise II.5.13 *Show that there is a finite set Λ of instances of the Comprehension Axiom, such that Λ, together with the axioms of ZF other than Comprehension, proves all instances of Comprehension.*

Hint. Λ is large enough to prove basic facts of model theory and basic facts about the $R(\alpha)$, together with the statement that whenever $A \in R(\alpha)$, every subset of A in $\mathcal{D}^+(R(\alpha))$ exists. To prove Comprehension for arbitrary formulas, use the Replacement Axiom and a Reflection argument to reduce quantification over V to quantification over some $R(\alpha)$. □

II.6 The Constructible Sets

In this section, we shall work in ZF and define Gödel's class L of all *constructible* sets. We shall then show that L is a transitive (proper class) model for $ZFC + GCH$, obtaining Gödel's proof that $\mathrm{Con}(ZF) \to \mathrm{Con}(ZFC + GCH)$. We then shall have $ZF^- \sim ZF \sim ZFC^- \sim ZFC \sim ZFC + GCH$ in the terminology of Section II.1.

The results in this section all follow fairly easily from standard results about models of set theory and absoluteness. The reader should bear in mind that these "standard results", which we have already covered, were unknown in Gödel's day (the 1930s), and in many instances were developed only after the basic concepts appeared in Gödel's proof.

Around 1970, Jensen discovered that L satisfies a number of important combinatorial principles, such as his principle $\diamondsuit$, which do not follow from GCH alone. This will be discussed in Section III.7, where $\diamondsuit$ is discussed.

L is defined by replacing "$\mathcal{P}$" by "$\mathcal{D}^+$" (see Definition I.15.25) in the recursive "construction" of the $R(\alpha)$ and WF described in Lemma I.9.30:

Definition II.6.1 *Define $L(\delta)$ by recursion on δ by:*

1. $L(0) = \emptyset$.
2. $L(\beta + 1) = \mathcal{D}^+(L(\beta))$.
3. $L(\gamma) = \bigcup_{\alpha<\gamma} L(\alpha)$ *for limit ordinals* γ.

Then $L = \bigcup_{\alpha \in ON} L(\alpha)$.

The following basic properties are easy from the definition:

Lemma II.6.2 *For all ordinals* α, β*:*

1. $L(\alpha) \subseteq R(\alpha)$.
2. $L(\beta)$ *is transitive.*
3. $\alpha \leq \beta \to L(\alpha) \subseteq L(\beta)$.
4. $L(\beta) \cap ON = \beta$.

Proof. (1) is clear by induction, using $\mathcal{D}^+(A) \subseteq \mathcal{P}(A)$.

For (2), induct on β. The cases where $\beta = 0$ or β is a limit are obvious from the definitions, so assume that $L(\beta)$ is transitive, and we prove that $L(\beta + 1)$ is transitive. First note that $L(\beta) \subseteq L(\beta + 1)$, since if $b \in L(\beta)$, then $b = \{x \in L(\beta) : x \in b\}$ is a subset of $L(\beta)$ defined from the parameter $b \in L(\beta)$. Then, since $L(\beta + 1) \subseteq \mathcal{P}(L(\beta))$, $x \in L(\beta + 1) \to x \subseteq L(\beta) \to x \subseteq L(\beta + 1)$.

For (3), prove $\forall \beta \geq \alpha\, [L(\alpha) \subseteq L(\beta)]$ by induction on β. The cases where $\beta = \alpha$ or β is a limit are obvious, and the successor case follows from $L(\beta) \subseteq L(\beta + 1)$, which we just proved.

For (4), induct on β; again, only the successor case is non-trivial, so assume that $L(\beta) \cap ON = \beta$. Then $\beta \subseteq L(\beta + 1) \cap ON \subseteq R(\beta + 1) \cap ON = \beta + 1 = \beta \cup \{\beta\}$, so we need only prove that $\beta \in L(\beta + 1)$. But $\beta = ON \cap L(\beta) = \{a \in L(\beta) : L(\beta) \models \varphi[a]\} \in \mathcal{D}^-(L(\beta)) \subseteq L(\beta + 1)$, where $\varphi(x)$ is a Δ_0 formula which is equivalent to "x is an ordinal"; see Lemma II.4.2. □

Formally, the proof of (4) appeals to Lemma I.15.27 and the correspondence between the object φ in the metatheory and $\ulcorner \varphi \urcorner$ in the formal theory.

Recall that $R(\alpha) = \{x : \text{rank}(x) < \alpha\}$, so that for all x, $\text{rank}(x) = \alpha \leftrightarrow x \in R(\alpha + 1) \setminus R(\alpha)$. We can define an analogous L–rank, denoted by ρ. It is clear from Definition II.6.1 that for $x \in L$, the least δ such that $x \in L(\delta)$ is a successor ordinal, so

Definition II.6.3 *For* $x \in L$*, the* L–rank *of* x*,* $\rho(x)$*, is the least* α *such that* $x \in L(\alpha + 1)$.

Lemma II.6.4 *For all* α*:* $L(\alpha) = \{x \in L : \rho(x) < \alpha\}$ *and* $L(\alpha+1) \setminus L(\alpha) = \{x \in L : \rho(x) = \alpha\}$.

Lemma II.6.5 *Each* $L(\alpha) \in L$ *and* $\rho(L(\alpha)) = \rho(\alpha) = \alpha$.

Proof. $\alpha \in L(\alpha + 1) \setminus L(\alpha)$ follows from Lemma II.6.2, and $L(\alpha) \notin L(\alpha)$ is obvious. Also, $L(\alpha) \in L(\alpha + 1)$ is clear from $L(\alpha) \in \mathcal{D}^-(L(\alpha))$. □

To compare the $L(\alpha)$ and $R(\alpha)$, observe first that by Lemma I.15.26,

Lemma II.6.6 *Every finite subset of* $L(\alpha)$ *is in* $L(\alpha+1)$.

Then, we can see where the two sequences first diverge:

Lemma II.6.7 $L(\alpha) = R(\alpha)$ *for all* $\alpha \leq \omega$ *and* $L(\omega+1) \subsetneqq R(\omega+1)$.

Proof. $L(n) = R(n)$ holds by Lemma II.6.6 and induction on $n \in \omega$. Then $L(\omega) = R(\omega) = HF$ is clear by taking unions. $R(\omega+1) = \mathcal{P}(HF)$ is uncountable, while $L(\omega+1) = \mathcal{D}^+(HF)$ is countable because there are only countably many formulas and HF is countable. □

Actually, as we pointed out in Section I.15, $\mathcal{D}^+(HF) = \mathcal{D}^-(HF)$ (so we just call it $\mathcal{D}(HF)$) because every element of HF is definable in HF. Note that $L(\omega+1) = \mathcal{D}(HF)$ is the collection of *arithmetical* subsets of HF discussed in Section I.17; these include all the decidable sets, plus most other sets of finite objects that occur naturally in mathematics. So, although $R(\omega+1) \setminus L(\omega+1)$ is uncountable, it's hard to give a "simple" example of an element of this set. The best we can do is:

Exercise II.6.8 *Prove that* $\mathrm{Th}(HF) \in R(\omega+1) \setminus L(\omega+1)$, *and* $\rho(\mathrm{Th}(HF)) = \omega+1$, *while* $\mathrm{rank}(\mathrm{Th}(HF)) = \omega$.

Hint. Recall Definition I.15.15 of the theory of a model. $\mathrm{Th}(HF) \notin \mathcal{D}(HF)$ by Tarski's theorem on the undefinability of truth. To prove that $\mathrm{Th}(HF) \in \mathcal{D}(L(\omega+1))$, you use Tarski's definition of truth within the model $L(\omega+1)$, but it's not quite so simple. It's clear that $L(\omega+1)$ contains the set $HF = L(\omega)$, together with the set of all logical formulas, but $L(\omega+1)$ isn't a model of any reasonable set theory (e.g., $\omega \in L(\omega+1)$ but $\{\omega\} \notin L(\omega+1)$), so you can't just say "do Definition I.15.5 within the model". You need some tricky coding arguments to implement this definition inside of $L(\omega+1)$. □

The detailed study of the L–ranks of various subsets of HF is not relevant to our work in this book, but it is an interesting topic relating set theory to recursion theory. For sets whose L–ranks are small (isomorphic to decidable well-orders of ω), the L–ranks are related to the hyperarithmetic hierarchy; see Sacks [79] for more on this subject.

For our work in set theory, we shall only need to know the sizes of the $L(\alpha)$, given by Lemma II.6.10.

Lemma II.6.9 *Assume AC. Then* $|\mathcal{D}^+(A)| = |A|$ *for all infinite* A.

Proof. $|\mathcal{D}^+(A)| \geq |A|$ because all $\{a\} \in \mathcal{D}^+(A)$, and $|\mathcal{D}^+(A)| \leq |A|$ because there are only $\aleph_0$ formulas and only $|A|$ elements of $A^{<\omega}$ to be used as parameters in definitions. □

Lemma II.6.10 *Assume AC. Then* $|L(\alpha)| = |\alpha|$ *for all* $\alpha \geq \omega$.

Proof. $|L(\alpha)| \geq |\alpha|$ is obvious from $L(\alpha) \supseteq \alpha$. Now, prove $|L(\alpha)| = |\alpha|$ by induction on $\alpha \geq \omega$. For $\alpha = \omega$, it's clear from $L(\omega) = HF$.

For the successor step, if $|L(\alpha)| = |\alpha|$ then $|L(\alpha+1)| = |L(\alpha)| = |\alpha| = |\alpha+1|$ by Lemma II.6.9.

For a limit γ, if $|L(\alpha)| = |\alpha|$ for all infinite $\alpha < \gamma$, then $|L(\alpha)| \leq |\gamma|$ for all $\alpha < \gamma$, so that $L(\gamma)$ is the union of $|\gamma|$ sets, each of size $\leq |\gamma|$, so that $L(\gamma) \leq |\gamma|$ by AC. □

So, $|L(\omega_1)|$ is only $\aleph_1$, whereas $|R(\omega_1)| = \beth_{\omega_1}$, so it appears that the $R(\alpha)$ grow much faster than the $L(\alpha)$. In view of this, it may seem unlikely that L could be all of V, but in fact $V = L$ is consistent with ZF (as we shall see next). If $V = L$, then $AC + GCH$ holds and $L(\alpha) = R(\alpha)$ whenever $\alpha = \aleph_\alpha$ (see Exercise II.6.33).

To prove $\mathrm{Con}(ZF) \rightarrow \mathrm{Con}(ZF + V = L)$, we use the obvious model — namely, L. Of course, there's a few things to check, starting with:

Theorem II.6.11 *All axioms of ZF hold in L.*

Proof. We already know that L is transitive, so by Lemma II.4.27, it is enough to prove (1) that the Comprehension Axiom holds in L, and (2) that for every subset $x \subseteq L$, there is a $y \in L$ such that $x \subseteq y$.

For (2), let $\alpha = \sup\{\rho(z) + 1 : z \in x\}$, and let $y = L(\alpha)$.

For (1), we follow the discussion of Comprehension in Lemma II.2.4, although we cannot claim now that $\forall z \in L \forall y \subseteq z\, [y \in L]$,

Fix a formula φ without y free. φ may have x, z free, along with possibly other free variables $v_0, \ldots, v_{n-1}$, so write it as $\varphi(x, z, v_0, \ldots, v_{n-1})$. Then we must verify

$$\forall z, v_0, \ldots, v_{n-1} \in L\, \exists y \in L\, \forall x \in L\, [x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi^L(x, z, \vec{v})]$$

So, fix $z, v_0, \ldots, v_{n-1} \in L$. We can (and must) let $y = \{x \in z : \varphi^L(x, z, \vec{v})\}$, but we must verify that $y \in L$. First, fix α such that $z, v_0, \ldots, v_{n-1} \in L(\alpha)$. Then, by the Reflection Theorem II.5.3, fix $\beta > \alpha$ such that $L(\beta) \preccurlyeq_\varphi L$ (in this application, $A(\xi) = L(\xi)$ and $B = L$). Then $y = \{x \in L(\beta) : \psi^{L(\beta)}(x, z, \vec{v})\} \in \mathcal{D}^+(L(\beta)) = L(\beta+1) \subset L$, where $\psi(x, z, \vec{v})$ is the formula $\varphi(x, z, \vec{v}) \wedge x \in z$. □

It follows from this and absoluteness that $x, y \in L \rightarrow \{x, y\}, \langle x, y\rangle \in L$. It is occasionally useful to compute the L-ranks of such sets from $\rho(x), \rho(y)$, using the analog of Lemma I.9.26 for rank:

Lemma II.6.12 *Suppose that $x, y \in L$. Then:*

1. $\{x, y\} \in L$ *and* $\rho(\{x, y\}) = \max(\rho(x), \rho(y)) + 1$.
2. $\langle x, y\rangle \in L$ *and* $\rho(\langle x, y\rangle) = \max(\rho(x), \rho(y)) + 2$.
3. $\bigcup x \in L$ *and* $\rho(\bigcup x) \leq \rho(x)$.
4. $x \cup y \in L$ *and* $\rho(x \cup y) \leq \max(\rho(x), \rho(y))$.

Proof. (2) follows from (1) and we leave (4) as an exercise.

For (1), let $\alpha = \max(\rho(x), \rho(y))$; so $x, y \in L(\alpha + 1)$ and at least one of $x, y \notin L(\alpha)$. Then $\{x, y\} \not\subseteq L(\alpha)$, so $\{x, y\} \notin L(\alpha + 1)$. But $\{x, y\} \in L(\alpha + 2)$ because $\{x, y\} \in \mathcal{D}^+(L(\alpha + 1))$.

For (3): Let $\alpha = \rho(x)$, so $x \in L(\alpha + 1) = \mathcal{D}^+(L(\alpha))$. Say $x = \{a \in L(\alpha) : (\varphi(a, b_1, \ldots, b_n))^{L(\alpha)}\}$, where $b_1, \ldots, b_n \in L(\alpha)$. Then $\bigcup x \in \mathcal{D}^+(L(\alpha))$ because $\bigcup x = \{c \in L(\alpha) : (\exists v\, [c \in v \wedge \varphi(v, b_1, \ldots, b_n)])^{L(\alpha)}\}$. □

We next consider the assertion $V = L$, which is often referred to as the *Axiom of Constructibility.* Of course, $V = L$ is shorthand for $\forall x\, \exists \delta\, [x \in L(\delta)]$, so to see whether $V = L$ holds in a model M, we need to compute $(L(\delta))^M$. Fortunately, it is the same as $L(\delta)$ for the models we shall consider.

Lemma II.6.13 *The function $L(\delta)$ is absolute for transitive $M \models ZF-P$.*

Proof. We already know that the notion of "ordinal" is absolute for M, as is the function $\mathcal{D}^+(A)$ (see Corollary II.4.19). Next, observe that Definition II.6.1 of $L(\delta)$ requires only $ZF-P$, since $ZF-P$ justifies such recursions and $\mathcal{D}^+$ was defined in $ZF-P$. Thus, $(L(\delta))^M$ is defined for ordinals $\delta \in M$, and then $(L(\delta))^M = L(\delta)$ is easily proved by induction on δ. □

This last sentence is really just a special case of Theorem II.4.15 on the absoluteness of recursively defined functions.

Corollary II.6.14 $(V = L)^L$.

Proof. We must verify that $(\forall x\, \exists \delta\, [x \in L(\delta)])^L$. But since $ON \subset L$ and "ordinal" and $L(\delta)$ are absolute, this reduces to $\forall x \in L\, \exists \delta \in ON [x \in L(\delta)]$, which is obvious by the definition of L. □

We remark that although this corollary seems fairly easy, the absoluteness argument (Lemma II.6.13) is required and is not completely trivial.

The absoluteness of $L(\delta)$ lets us identify the *set* models of $V = L$.

Definition II.6.15 *For transitive sets M, $o(M) = M \cap ON$ = the set of ordinals in M = the first ordinal not in M.*

Note that $o(M)$ is a limit ordinal when $M \models ZF-P$ (or just Pairing, Union, and Comprehension) because $\alpha \in M \to (\alpha + 1) \in M$.

Lemma II.6.16 *Let M be a transitive set such that $M \models ZF-P$. Then $M \models V = L$ iff $M = L(o(M))$.*

Proof. Let $\gamma = o(M)$. By absoluteness, $L(\delta) \in M$ for all $\delta < \gamma$; then, since M is transitive and $L(\gamma) = \bigcup_{\delta < \gamma} L(\delta)$, we have $L(\gamma) \subseteq M$.

Now, as in the proof of Corollary II.6.14, the statement $(V = L)^M$ reduces to $\forall x \in M\, \exists \delta \in \gamma [x \in L(\delta)]$, which, since $L(\gamma) = \bigcup_{\delta < \gamma} L(\delta)$, is equivalent to $M \subseteq L(\gamma)$ and hence to $M = L(\gamma)$. □

We now turn to verifying $(AC + GCH)^L$. Since we have already verified $(ZF + V = L)^L$, it is sufficient to show that $ZF + V = L \vdash AC \wedge GCH$.

We begin with the proof of AC. Assuming $V = L$, it is sufficient to show that each $L(\alpha)$ can be well-ordered. We first explain how to well-order any $\mathcal{D}^+(A)$, given a well-order of A. Roughly, you can well-order the countable set of formulas and you can well-order $A^{<\omega}$, which is the set of all possible tuples of parameters occurring in Definition I.15.23 of "definable with parameters"; then, to compare $S_1 = \{a \in A : A \models \varphi_1[\vec{b}_1, a]\}$ with $S_2 = \{a \in A : A \models \varphi_2[\vec{b}_2, a]\}$, you first compare φ_1 and φ_2; if they are the same formula, then you compare $\vec{b}_1$ and $\vec{b}_2$. To spell this out in more detail, recall that in our definition of logical syntax, all symbols and formulas are in HF. Then,

Definition II.6.17 *Fix $\Gamma : HF \xrightarrow[\text{onto}]{1-1} \omega$, as in Lemma I.10.17. Working with $\mathcal{L} = \{\in\}$, list the set of all variables $\{v_r : r \in \omega\}$, where $r < s \to \Gamma(v_r) < \Gamma(v_s)$. Call a formula φ* good *iff for some $n \geq 0$, the free variables of φ are exactly $v_0, v_1, \ldots, v_n$. List the set of all good formulas as $\{\varphi_i : i \in \omega\}$ with $i < j \to \Gamma(\varphi_i) < \Gamma(\varphi_j)$; let φ_i have $n_i + 1$ free variables.*

Whenever $A \neq \emptyset$, $i \in \omega$ and $\vec{b} \in A^{n_i}$, let $D(A, i, \vec{b}) = \{a \in A : A \models \varphi[b_0, \ldots, b_{n_i - 1}, a]\}$. Then, for $S \in \mathcal{D}^+(A)$, let $i(S)$ be the least i such that $S = D(A, i, \vec{b})$ for some $\vec{b} \in A^{n_i}$. Write $n(S)$ for $n_{i(S)}$.

Of course, our definition of $i(S)$ uses the remark that $\mathcal{D}^+(A) = \{D(A, i, \vec{b}) : i \in \omega \wedge b \in A^{n_i}\}$.

Definition II.6.18 *Whenever $A \neq \emptyset$ and R is a well-order of A, let $R^{(n)}$ be the induced lexicographic order on A^n. Then, for $S \in \mathcal{D}^+(A)$, let $\vec{p}(S, R)$ be the $R^{(n(S))}$-first $\vec{b} \in A^{n(S)}$ such that $S = D(A, i(S), \vec{b})$. Then, let $W(A, R)$ be the well-order W of $\mathcal{D}^+(A)$ defined so that $S_1 \, W \, S_2$ iff either $i(S_1) < i(S_2)$ or $i(S_1) = i(S_2)$ and $\vec{p}(S_1, R) \, R^{(n(S_1))} \, \vec{p}(S_2, R)$.*

Let $W(\emptyset, \emptyset) = \emptyset$. Let $W(A, R) = \emptyset$ whenever R is not a well-order of A.

We leave it to the reader to check that $W(A, R)$ really is a well-order of $\mathcal{D}^+(A)$. Regarding $W(\emptyset, \emptyset)$, note that $\mathcal{D}^+(\emptyset) = \{\emptyset\}$, and the empty order is the only well-order of $\emptyset$ and of $\{\emptyset\}$.

Definition II.6.19 *By recursion on δ, define $\lhd_\delta \subseteq L(\delta) \times L(\delta)$ by: $x \lhd_\delta y$ iff $\rho(x) < \rho(y)$ or $\rho(x) = \rho(y)$ and $(x, y) \in W(L(\rho), \lhd_\rho)$, where $\rho = \rho(x) = \rho(y)$.*

Then define the relation $<_L$ on L by: $x <_L y$ iff $\rho(x) < \rho(y)$ or $\rho(x) = \rho(y)$ and $x \lhd_{\rho+1} y$, where $\rho = \rho(x) = \rho(y)$.

More informally, to compare $x, y \in L$: if $\rho(x) < \rho(y)$ then $x <_L y$ and if $\rho(y) < \rho(x)$ then $y <_L x$. If $\rho(x) = \rho(y) = \rho$, then $x, y \in L(\rho + 1) \backslash L(\rho)$, and we assume that the relation $<_L$ has already been computed on $L(\rho)$; then we compare x, y using $W\big(L(\rho), <_L \cap (L(\rho) \times L(\rho))\big)$. Our definitions yield:

Theorem II.6.20 *$<_L$ well-orders L, and under this well-order, each $L(\delta)$ is an initial segment, with $\lhd_\delta = <_L \cap (L(\delta) \times L(\delta))$. Assuming $V = L$: $<_L$ well-orders V and AC holds.*

This actually shows that $V = L$ implies *global choice*; that is, we have defined one relation that well-orders everything.

Our Definitions II.6.17 and II.6.18 were a bit wordy, and the precise details of $W(A, R)$ are not too important, but it is important that there is some *specific* definition here. We can't just say, "if A *can* be well-ordered then so can $\mathcal{D}(A)$". Then, if we try to prove by induction on α that $L(\alpha)$ can be well-ordered, what do we do at a limit ordinal? We actually have to *define*, for each α, a *specific* well-order of $L(\alpha)$, so that we can put the orderings together at limits. This is similar to the problem encountered in working Exercise I.12.17.

We remark that although the definition of $<_L$ is simple enough, it depends on specifying a precise definition of logical syntax, and there is no "one right way" of doing this. So, we know of no well-order of $L(\omega + 1)$ that is "natural", in the sense that Ackermann's well-order of $L(\omega) = HF$ from Lemma I.10.17 is natural. Does the set of primes come before or after the set of perfect numbers?

Exercise II.6.21 *Prove Lemma II.6.10 without using AC.*

Hint. Either re-do the proof, being a bit more careful, or use absoluteness plus the fact that the results are true in L. □

We now turn to *GCH*. First, we shall need:

Lemma II.6.22 *(AC) If κ is any regular uncountable cardinal, then $L(\kappa) \models ZF - P + V = L$.*

Proof. The proof of $L(\kappa) \models ZF - P$ is similar to the proof for $H(\kappa)$ (see Theorem II.2.1). For the Pairing and Union Axioms, use Lemma II.6.12 with Lemma II.2.4. We focus on Replacement and Comprehension, where the argument is somewhat different.

For Replacement, we modify the discussion in Lemma II.2.4. Let $M = L(\kappa)$. Assume that $A \in M$ and that $\forall x \in M\,[x \in A \to \exists! y \in M\, \varphi^M(x, y)]$. We need to produce a $B \in M$ such that $\big(\forall x \in A \exists y \in B\, \varphi(x, y)\big)^M$. Say $A \in L(\alpha)$, where $\alpha < \kappa$; and note that $|A| \leq |L(\alpha)| < \kappa$. Let f be the function with $\mathrm{dom}(f) = A$ such that $f(x)$ is the (unique) $y \in M$ such that $\varphi^M(x, y)$. Then each $\rho(f(x)) < \kappa$, so $\beta := \sup\{\rho(f(x)) + 1 : x \in A\} < \kappa$ because κ is regular and $|A| < \kappa$. Now, let $B = L(\beta)$; $B \in L(\kappa)$ because $\rho(B) = \beta + 1 < \kappa$.

We remark that also $f \in L(\kappa)$ and $\mathrm{ran}(f) \in L(\kappa)$ by Comprehension in $L(\kappa)$, which we shall prove next, but this is not necessary to verify Replacement.

For Comprehension, the proof is almost verbatim the same as the proof above for Theorem II.6.11 that Comprehension holds in L. Replace L by $L(\kappa)$, and α, β now range over κ, rather than ON. Also, instead of using

the Reflection Theorem II.5.3, use Theorem II.5.10, which is the version of Reflection obtained by replacing ON by a regular cardinal κ.

Finally, $L(\kappa) \models V = L$ by Lemma II.6.16. □

We said "(AC)" in stating this lemma to simplify the discussion of cardinalities, but in fact, as with Exercise II.6.21, one can avoid AC here, either by being a bit more careful or by absoluteness; that is, if κ is regular, then κ is regular in L, so one could apply this lemma in L, where AC holds.

The similarity of the proofs for $L(\kappa)$ and $H(\kappa)$ is not surprising, since $V = L$ implies that $L(\kappa)$ *is* $H(\kappa)$:

Theorem II.6.23 *$V = L$ implies that $L(\kappa) = H(\kappa)$ for all cardinals $\kappa \geq \omega$. Hence, $V = L$ implies GCH.*

Proof. Assume $V = L$ throughout. Since AC holds, we may use standard results on computing cardinalities. For the *"Hence"*, let λ be any infinite cardinal. Then $\mathcal{P}(\lambda) \subseteq H(\lambda^+) = L(\lambda^+)$, and $|L(\lambda^+)| = \lambda^+$ by Lemma II.6.10, so $2^\lambda \leq \lambda^+$, and hence $2^\lambda = \lambda^+$.

Next, two simple remarks, which do not require $V = L$. First, we already know that $L(\omega) = HF = H(\omega) = R(\omega)$, so we only need consider $\kappa > \omega$. Second, $L(\kappa) \subseteq H(\kappa)$ for all cardinals: if $x \in L(\kappa)$ with $\kappa > \omega$, fix α with $x \in L(\alpha)$ and $\omega \leq \alpha < \kappa$; then $\mathrm{trcl}(x) \subseteq L(\alpha)$, so that $|\mathrm{trcl}(x)| \leq |L(\alpha)| < \kappa$, so $x \in H(\kappa)$.

Now, to prove that $L(\kappa) = H(\kappa)$, note that it is sufficient to do this when κ is a successor cardinal, of form λ^+ for some λ. Then, when κ is an uncountable limit cardinal, we have $L(\kappa) = \bigcup_{\lambda<\kappa} L(\lambda^+) = \bigcup_{\lambda<\kappa} H(\lambda^+) = H(\kappa)$.

So, we are done if we can prove that $H(\lambda^+) \subseteq L(\lambda^+)$ whenever λ is an infinite cardinal; this is really the meat of the proof. Fix $b \in H(\lambda^+)$. Let $T = \mathrm{trcl}(\{b\})$. So $b \in T$ and $|T| \leq \lambda$. Applying $V = L$, fix any regular uncountable cardinal $\theta > \rho(T)$. Then $T \subseteq L(\theta)$ and $L(\theta) \models ZF\text{-}P + V = L$. By the Downward Löwenheim–Skolem–Tarski Theorem I.15.10, fix A such that $A \preccurlyeq L(\theta)$ and $T \subseteq A$ and $|A| \leq \lambda$. Then also $A \models ZF\text{-}P + V = L$.

Let mos be the Mostowski isomorphism defined on A, and let $B = \mathrm{ran}(\mathrm{mos})$. Then mos is an isomorphism from $(A, \in)$ onto $(B, \in)$ and B is transitive and $\mathrm{mos}(x) = x$ for all $x \in T$ (see Lemmas I.9.35 and I.9.36). In particular, $b = \mathrm{mos}(b) \in B$. Also, $(B, \in) \cong (A, \in)$ implies that $B \models ZF\text{-}P + V = L$. Since B is transitive, $B = L(\beta)$ where $\beta = o(B)$ by Lemma II.6.16. Then $|\beta| = |B| = |A| \leq \lambda$, so $\beta < \lambda^+$ and $b \in L(\lambda^+)$. □

To summarize, our results yield, in the finitistic metatheory,

Theorem II.6.24 $\mathrm{Con}(ZF) \to \mathrm{Con}(ZFC + V = L)$, *and hence* $\mathrm{Con}(ZF) \to \mathrm{Con}(ZFC + GCH)$. *Furthermore, if φ is any arithmetical sentence such that $ZFC + V = L \vdash \varphi$, then $ZF \vdash \varphi$.*

The "Furthermore" holds by absoluteness (Lemma II.4.14); that is, φ^L is equivalent to φ. So, if someone should prove the Twin Prime Conjecture using

$AC + GCH$, then one could convert it to a proof just in ZF by showing that the conjecture is true in L, and therefore true.

We next mention a few results on transitive models.

Lemma II.6.25 *(AC) If κ is any weakly inaccessible cardinal, then κ is a strongly inaccessible cardinal in L, and $L(\kappa) \models ZFC + V = L$.*

Proof. First note that being a cardinal and being a weakly inaccessible cardinal are Π_1 properties, so they relativize down (see Lemma II.4.28), so (κ is weakly inaccessible)^L. Then, use the fact that weak and strong inaccessibility are equivalent under GCH, which is true in L.

Working in L: κ is strongly inaccessible, so applying Theorem II.6.23 and Lemma II.6.22 and Corollary II.2.3: $L(\kappa) = H(\kappa) = R(\kappa) \models ZFC + V = L$.

Then $L(\kappa) \models ZFC + V = L$ is true in V by absoluteness of $\models$. □

We said "(AC)" in stating this lemma because our definition (I.13.30) of "inaccessible" used AC; but as in Exercise II.6.21, we could restate the lemma and then prove it without using AC.

Corollary II.6.26 *(AC) If there is a weakly inaccessible cardinal, then there is a countable transitive M such that $M \models ZFC + V = L$.*

Proof. Start with $L(\kappa)$, and apply Lemma II.4.20. □

A related fact:

Lemma II.6.27 *Suppose that M is a transitive model for ZF. Then $L(o(M))$ is a transitive model for $ZFC + V = L$.*

Proof. We have seen earlier in this section that working in ZF, we can prove that all axioms of $ZFC + V = L$ hold in L. Applied in the model M, this argument shows that all axioms of $ZFC + V = L$ hold in the model $L^M := \{x \in M : \exists \delta \in M(x \in (L(\delta))^M\}$. But $L^M = L(o(M))$ by absoluteness. □

Working just in ZFC, we cannot expect to produce a model for ZF, but we can use the reflection argument of Corollary II.5.4 to prove

Lemma II.6.28 *Let Λ be a finite set of axioms of ZFC. Then*

$$ZFC \vdash \exists M\,[M \models \Lambda + V = L \ \wedge\ |M| = \aleph_0 \ \wedge\ M \text{ is transitive}].$$

Proof. First apply the Reflection Theorem II.5.3 with L and the $L(\xi)$ to get some $L(\eta) \models \Lambda + V = L$, and then apply Lemma II.4.20. □

So, in ZFC we can produce a countable transitive model for enough of $ZFC + V = L$ to prove all of known mathematics, together with GCH.

There is a notion of relative constructibility, which is sometimes useful:

Definition II.6.29 *For any set A, define $L[A](\delta)$ by recursion on δ by:*

1. $L[A](0) = \{A\} \cup \mathrm{trcl}(A)$.
2. $L[A](\beta+1) = \mathcal{D}^+(L[A](\beta))$.
3. $L[A](\gamma) = \bigcup_{\alpha<\gamma} L[A](\alpha)$ *for limit ordinals* γ.

Then $L[A] = \bigcup_{\alpha\in ON} L[A](\alpha)$.

Exercise II.6.30 *For any A, $L[A]$ is a transitive model of ZF, and $A \in L[A]$. Also, $L[A] \models AC$ if A is a set of ordinals, and $L[A] \models GCH$ if $A \subseteq \omega$.*

Note that $L[\emptyset] = L$, although, since $L[\emptyset](0) = \{\emptyset\}$, we have $L[\emptyset](\alpha) = L(1+\alpha)$. We didn't say $L[A](0) = \mathrm{trcl}(A)$ because then, when A is any infinite subset of ω, we would have $L[A](0) = \omega$, so that $L[A]$ would be just L.

It is consistent to have $L[\mathcal{P}(\omega)] \models \neg AC$ (Exercise IV.8.17). It is also consistent to have an $A \subseteq \omega_1$ with $L[A] \models 2^{\aleph_0} = \aleph_2$ (Exercise IV.7.11), although that implies that $(\omega_1)^L < \omega_1$ by the following, due to Hajnal.

Exercise II.6.31 *Assume that $V = L[A]$, where $A \subseteq \omega_1$. Then GCH holds.*

Hint. For *CH*, show that $\mathcal{P}(\omega) \subseteq \bigcup\{L[A \cap \beta](\alpha) : \alpha, \beta < \omega_1\}$. □

The absoluteness of the function $L(\delta)$ puts some restrictions on how one might prove the consistency of $ZFC + \neg CH$; the simple "class model" method used with *WF* and L cannot work here:

Remark II.6.32 *Suppose that working in ZF (or even $ZFC + V = L$), one could define a transitive proper class M and prove that it is a model for $ZFC + \neg CH$ (or even just $ZF + V \neq L$). Then ZF is inconsistent.*

Proof. Working in $ZFC + V = L$: Since M is a proper class model, $ON \subseteq M$ by the absoluteness of rank (Corollary II.4.16). Then, $L \subseteq M$ by the absoluteness of $L(\delta)$. So, $M = L$ by $V = L$, so $M \models V = L$, which is a contradiction.

Then, the inconsistency of $ZFC + V = L$ implies the inconsistency of *ZF* (by using the model L). □

Section II.7 gives a *rough* idea of what does work.

Exercise II.6.33 *Assume $V = L$, and prove that $L(\alpha) = R(\alpha)$ iff $\alpha = \aleph_\alpha$ or $\alpha \leq \omega$.*

Exercise II.6.34 *Assume $V = L$, and let A be the set of definable (without parameters) elements of $L(\omega_1)$. Prove that $A \preccurlyeq L(\omega_1)$ and $A = L(\alpha)$ for some countable α.*

Hint. See Exercise I.15.28. A is transitive because if $a \in A$, then the $<_L$–first f such that $f : \omega \xrightarrow{\text{onto}} a$ is also in A, so that each $f(n) \in A$. □

Exercise II.6.35 *Assume that $V = L$. Let B be the set of all $\beta < \omega_1$ such that $L(\beta) \models ZF-P$ and every element of $L(\beta)$ is a definable (without parameters) element of $L(\beta)$. Prove that B is unbounded in ω_1.*

Hint. Let A, α be as obtained in Exercise II.6.34. Note that $\alpha \in B$. If B is bounded, let $\gamma = \sup\{\beta + 1 : \beta \in B\}$. Then $\gamma \in A$, so $\gamma < \alpha < \gamma$. □

Exercise II.6.36 *Assume $V = L$. Call a theory $\Gamma \supset ZF-P$* nice *iff Γ is complete and $\{\alpha < \omega_1 : L(\alpha) \models \Gamma\}$ is unbounded in ω_1. Prove that there are $\aleph_1$ nice theories.*

Hint. Let $\mathcal{N}$ be the set of nice theories. Observe that $\mathrm{Th}(L(\omega_1)) \in \mathcal{N}$ and that $\mathcal{N} \subseteq L(\omega_1)$ and $\mathcal{N} \in \mathcal{D}^-(L(\omega_1))$ (Definition I.15.25).

If $\mathcal{N}$ is countable, then each of its element is in $\mathcal{D}^-(L(\omega_1))$. For the element $\mathrm{Th}(L(\omega_1))$, this contradicts Tarski's theorem on undefinability of truth. □

II.7 The Forcing Idea

Say we wish to produce a model for $ZFC + \neg CH$. Using the "L" idea from Section II.6, we might work in ZF or ZFC and try to construct a transitive proper class model N that satisfies all the axioms of $ZFC + \neg CH$. Unfortunately, this cannot work; see Remark II.6.32.

Still, we are left with the following very naive and informal idea: In order to get a model for $\neg CH$, we need to extend our universe, getting $N \supseteq V$ rather than $N \subseteq V$. In our universe V, maybe CH holds, but perhaps in some ideal universe $N \supsetneq V$, CH fails. Although we mere humans only have $\aleph_1$ subsets of ω, the beings living in N have $\aleph_2$ of them, so these beings believe that $2^{\aleph_0} \geq \aleph_2$.

Formally, "$N \supsetneq V$" is nonsense. To make formal sense out of this naive idea, we deal with set models instead of proper class models.

Definition II.7.1 "ctm" *abbreviates "countable transitive model".*

We have already seen that there are ctms for various sub-theories of ZFC, since once we have produced a transitive model T, such as $H(\kappa)$ or $R(\lambda)$, we can apply the Downward Löwenheim–Skolem–Tarski Theorem and the Mostowski isomorphism to T to obtain a ctm M; see Lemma II.4.20. To get $M \models ZFC$, start with $T = H(\theta) = R(\theta)$, where θ is any strongly inaccessible cardinal, so that $T \models ZFC$ (Corollary II.2.3). To get a ctm for $ZFC + V = L$, and hence for GCH, start with $L(\theta)$ instead of $H(\theta)$ (see Corollary II.6.26); so, actually a weakly inaccessible cardinal is sufficient.

We shall show in Chapter IV that if M is a ctm for ZFC, then we can find a larger ctm $N \supseteq M$ that satisfies $ZFC + \neg CH$. Furthermore, if $M \models GCH$, then we can arrange for $2^{\aleph_0}$ in N to be any value not contradicting König's Theorem ($\mathrm{cf}(\mathfrak{c}) \geq \omega_1$), such as $\aleph_2$ or $\aleph_{17}$ or $\aleph_{\omega+1}$ or $\aleph_{\omega_1}$.

The idea for building N is: Since M is countable, $\gamma := o(M) = M \cap ON$ is a countable ordinal. We then have ordinals $\omega_1^M < \omega_2^M < \gamma$ that the people living in M think are the first and second uncountable cardinals, although we, in V, know that they are really countable ordinals. Now, the M–people may believe that CH is true, so they have listed their $(\mathcal{P}(\omega))^M = \mathcal{P}(\omega) \cap M$ in type ω_1^M, but since M is countable, there are plenty of subsets of ω outside of M; we can simply take ω_2^M of these and add these to M to form a larger ctm $N \supset M$, which will satisfy $ZFC + 2^{\aleph_0} = \aleph_2$.

There are obviously some technical details to be discussed here. We cannot just add a bunch of subsets of ω to M and hope to come up with a model of set theory, since along with these subsets we need to include all other sets built from them by set-theoretic operations, such as pairing, union, etc. We also need to make sure that $\omega_1^N = \omega_1^M$ and $\omega_2^N = \omega_2^M$ so that the N–people really believe that $2^{\aleph_0} = \aleph_2$. The method of *forcing*, covered in Chapter IV, handles all these details.

There is also a logical issue here, which is covered in Chapter IV. Working in ZFC, we can't really construct a ctm for ZFC (or prove that inaccessible cardinals exist), since this would contradict the Incompleteness Theorem. It turns out to be sufficient that we can produce a ctm for any desired finite subset of ZFC by using the Reflection Theorem (see Corollary II.5.4). Eventually, we shall indeed get finitistic proofs, in the metatheory, of statements such as $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + 2^{\aleph_0} = \aleph_2)$.

II.8 Ordinal Definable Sets

This notion yields a proof of $\mathrm{Con}(ZF) \to \mathrm{Con}(ZFC)$ that is shorter than the one via L. Unfortunately, the model we construct says nothing about CH or anything else of mathematical interest, so we shall only give the relevant definitions and leave most of the details as exercises. The notion was first suggested by Gödel in 1964, and was developed in detail by Myhill and Scott [72]. We work in ZF throughout.

Recall that by Definition I.15.23 the element $a \in A$ is *definable* (without parameters) in a model $\mathfrak{A}$ iff there is a formula $\varphi(x)$ of $\mathcal{L}$ that defines a in $\mathfrak{A}$. Assuming that $\mathcal{L}$ is countable, there are only countably many definable elements in A. This gives us the following well-known contradiction in ZF:

Paradox II.8.1 *Let $D \subset ON$ be the collection of all ordinals that are definable in $(V, \in)$. Since D is only a set (and countable), we can* define *α to be the least element in $ON \backslash D$. Then α is a non-definable ordinal, but I've just defined it, a contradiction.*

Of course, this exemplifies the usual error of treating the non-existent V as a model. In ZF, we could let α be the least ordinal not definable in $R(\omega_1)$.

This is no contradiction; we've just defined α, but we had to step outside of $R(\omega_1)$ to do so; α is not definable *in* (i.e. by quantifying over) $R(\omega_1)$.

Strangely, we can define, in *ZF*, a class called OD of *ordinal definable sets* – these are the sets definable *from* ordinals – so $ON \subseteq OD$ by definition. Of course, we're not justified in defining "$a \in OD$ iff a is definable in $(V, \in)$ from ordinals", but we can write down a more complicated definition and then prove that it "works".

Definition II.8.2 *For any transitive* **set** M, OD_M *is the set of all elements* $a \in M$ *that are definable in* $(M, \in)$ *with parameters in* $ON \cap M$.

So, $a \in OD_M$ iff there is a formula $\varphi(x_0, \ldots, x_{n-1}, y)$ and $\alpha_0, \ldots, \alpha_{n-1} \in ON \cap M$ such that $(\varphi(\vec{\alpha}, a))^M$ and $(\exists! y\, \varphi(\vec{\alpha}, y))^M$ both hold. Then, φ defines a in M *from* a finite sequence $\vec{\alpha}$ of ordinals.

Definition II.8.3 $OD = \bigcup\{OD_{R(\eta)} : \eta \in ON\}$

This definition of a class (i.e., a formula) OD makes sense in *ZF*, since we are applying model theory only to *set* models $R(\eta)$. The above paradox indicates that we should not be allowed to talk about OD_V, but in fact the next two exercises seem to say our definition "works", in the sense that our OD "really is the same as OD_V". The exercises are both really schemes in the metatheory. Formally, of course, we don't talk about OD_V, and we use Definition II.8.3 as our official definition of "ordinal definable".

Exercise II.8.4 *Show the following, which seems to say "*$OD_V \subseteq OD$*": For each formula* $\varphi(x_0, \ldots, x_{n-1}, y)$*, the sentence*

$$\forall \alpha_0, \ldots, \alpha_{n-1} \in ON\, \forall w\, [\varphi(\vec{\alpha}, w) \wedge \exists! y\, \varphi(\vec{\alpha}, y) \;\rightarrow\; w \in OD]$$

is provable in ZF.

Hint. Use Reflection to get an η such that $w \in OD_{R(\eta)}$. □

Exercise II.8.5 *Do the following, which seems to say "*$OD \subseteq OD_V$*": Write* one *formula* ψ *in two variables in* $\mathcal{L} = \{\in\}$ *such that the sentence*

$$\forall \alpha \exists! y \psi(\alpha, y) \;\wedge\; \forall w\, [w \in OD \leftrightarrow \exists \alpha \in ON\, \psi(\alpha, w)]$$

is provable in ZF.

Hint. First, modify the proof that $|\kappa \times \kappa| = \kappa$ (Theorem I.11.6) to define a function F from ON onto $ON^{<\omega}$; to do this, you first define a well-order of $ON^{<\omega}$ in "type" ON; for example, $s \lhd t$ iff $\sup(s) < \sup(t)$ or $\sup(s) = \sup(t) \wedge \mathrm{lh}(s) < \mathrm{lh}(t)$ or $\sup(s) = \sup(t) \wedge \mathrm{lh}(s) = \mathrm{lh}(t) \wedge s <_{\mathrm{lex}} t$. Also define a function g from ω onto HF (e.g., using Lemma I.10.17). Now, whenever $F(\alpha)$

is a sequence of the form $(k, \eta, \alpha_0, \ldots, \alpha_{n-1})$ where $k \in \omega$, all $\alpha_i < \eta$ and $g(k)$ is a formula φ in $n+1$ free variables such that $(\exists! y\, \varphi(\alpha_0, \ldots, \alpha_{n-1}, y))^{R(\eta)}$ holds, let $\psi(\alpha, a)$ say that a is the element of $R(\eta)$ such that $\varphi(\alpha_0, \ldots, \alpha_{n-1}, a))^{R(\eta)}$ holds. For other α, $\psi(\alpha, y) \leftrightarrow y = \emptyset$. □

So, ψ defines a map from ON onto OD. Using this *one* ψ, whenever $w \in OD$, there is an ordinal α such that ψ defines w from α in $(V, \in)$.

Since each $R(\alpha) \in OD$ (clear by Exercise II.8.4), OD is not transitive unless $OD = V$. To get a transitive model, use the class HOD of *hereditarily ordinal definable* sets:

Definition II.8.6 $HOD = \{x \in OD : \mathrm{trcl}(x) \subseteq OD\}$.

Exercise II.8.7 *$ON \subseteq HOD \subseteq OD$ and HOD is transitive.*

We shall see that HOD is always a transitive model of ZFC.

Exercise II.8.8 *For any set a: $a \in HOD$ iff $a \in OD$ and $a \subseteq HOD$.*

Exercise II.8.9 *Each $R(\alpha) \cap HOD \in HOD$.*

Hint. $R(\alpha) \cap HOD \in OD$ by Exercise II.8.4. □

Exercise II.8.10 *Show that all the ZFC axioms hold in HOD.*

Hint. For Comprehension, note that since HOD is definable, if A is definable from ordinals, then so is $\{x \in A : \varphi^{HOD}(x)\}$. For the rest of ZF, use Lemma II.4.27. For AC, note that using the ψ from Exercise II.8.5, if $A \in HOD$, then A has a well-order that is definable from ordinals. □

Not much more can be said about HOD, except for a bunch of counter-examples to any reasonable conjecture. CH may be true in HOD, since Gödel's Axiom of Constructibility implies both GCH and $HOD = V$:

Exercise II.8.11 *Show that $V = L$ implies $V = HOD$.*

Hint. Use $<_L$ to define a bijection from ON onto V. □

But also, CH may be false in HOD. The hypothesis of the next lemma is consistent by the method of Easton; see Exercise V.2.11.

Exercise II.8.12 *Working in ZFC, let $C_\alpha = \{n \in \omega : 2^{\aleph_{\alpha+n}} = \aleph_{\alpha+n+1}\}$. Assume that $|\{C_\alpha : \alpha \in ON\}| \geq \aleph_2$. Prove that CH is false in HOD.*

Note that we proved $(V = L)^L$ (Corollary II.6.14). We did *not* prove that $(V = HOD)^{HOD}$, since you *can't*; it's consistent with ZFC that $(HOD)^{HOD} \subsetneq HOD$ (see Exercise V.2.11). Both L and HOD were defined using the notion of definability, but the definition of L is more "constructive"; to build $L(\beta + 1)$, you use definability over $L(\beta)$ after you have constructed $L(\beta)$, allowing for an inductive proof of absoluteness; but for HOD, you use definability over V, and if $HOD \subsetneq V$, there is no reason to think that definability over HOD is related to definability over V. The axiom $V = HOD$ is consistent, since it holds in L, but it need not hold in HOD. Also, $V = HOD \not\to CH$ (see Exercise V.5.26).

II.9 The Independence of Foundation

Let AF denote the Axiom of Foundation. In this section, we prove $\mathrm{Con}(ZF) \to \mathrm{Con}(ZF^- + \neg AF)$ and $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC^- + \neg AF)$ by building a (proper class) model in which Foundation is false. So, combining this with the results of Sections II.2 and II.6, we shall have $ZF^- \sim ZF \sim ZF^- + \neg AF \sim ZFC^- \sim ZFC \sim ZFC^- + \neg AF$ in the terminology of Section II.1. We shall then, following the method of Fraenkel and Mostowski from the 1920s and 1930s, prove the consistency of various failure of AC with ZF^-.

As usual, we work in ZF, unless otherwise noted.

The main thrust of this book is the study of consistent extensions of ZFC. However, it is of some interest to point out that Foundation is not provable from the other axioms. Our models will show that some of the important applications of Foundation, such as the absoluteness of "ordinal", are not provable without Foundation.

However, as we have seen in Section II.3, failures of Foundation have nothing to do with mathematics, so we shall state all results as exercises, leaving the detailed proofs to the reader.

Note that our model cannot be an $\in$-model, $(M, \in)$, because working in ZF, all such models satisfy Foundation by Lemma II.2.4. In fact, our M will be V, but we shall interpret $\in$ as a binary relation E given by:

Definition II.9.1 *Suppose that $F : V \to V$ is a defined function. Define the relation $E = E_F$ by xEy iff $x \in F(y)$.*

As usual, F is really introduced by a formula $\psi(x, u)$ for which our axioms prove $\forall x \exists! u\, \psi(x, u)$, and then xEy abbreviates $\exists u\, [\psi(x, u) \wedge x \in u]$.

Exercise II.9.2 *Suppose that $F : V \to V$ is a permutation of V. Then $\varphi^{(V,E)}$ holds for each axiom φ of ZF^-.*

As usual, this is shorthand for saying that for each axiom φ of ZF^-, ZF^- proves that if F is a permutation then $\varphi^{(V,E)}$. Here, "permutation" is just shorthand for $\forall x \exists! u\, \psi(x, u) \wedge \forall u \exists! x\, \psi(x, u)$.

Since $M = V$ here, the quantifiers don't get bounded, but $\in$ is replaced by E. So, for example, the Pairing Axiom in (V, E) is $\forall x, y\, \exists z\, (xEz \wedge yEz)$, which is $\forall x, y\, \exists z\, (x \in F(z) \wedge y \in F(z))$, so we can let $z = F^{-1}(\{x, y\})$; i.e., $F(z) = \{x, y\}$.

Although we said that we are working in ZF, Exercise II.9.2 can be done in ZF^-. Assuming Foundation as part of our basic theory makes it a little easier to control some finer points about the model, as in Exercise II.9.3 below.

One can get interesting failures of Foundation by permutations that just move finitely many elements of V. Using standard group theory terminology, if F is $(0, 1)$, then $xE0$ iff $x \in 1$ iff $x = 0$, so the statement $\exists y (y = \{y\})$ is true in (V, E) (with $y = 0$). Since it is a theorem of set theory that an empty set exists, there must be an $\emptyset^{V,E}$, but that's $F^{-1}(\emptyset)$, which is 1 in this example.

If you want to get $(\{a\} = b \wedge \{b\} = c \wedge \{c\} = a)^{(V,E)}$, where a, b, c are all different, just let $F = (a, d)$, where $b = \{a\}$, $c = \{b\}$, and $d = \{c\}$. Then $F(b) = \{a\}$, $F(c) = \{b\}$, and $F(a) = \{c\}$.

For more spectacular failures of Foundation, F can move a proper class of elements. For example, we can get $\{y : y = \{y\}\}$ to be a proper class by making $\{y : F(y) = \{y\}\}$ a proper class (e.g., let F interchange α and $\{\alpha\}$ for all limit ordinals α). Even more spectacularly, we can get all possible failures of Foundation to happen at once (see Exercise II.9.4 below).

One may view our model as an extension of the universe V, obtained by adding some non-well-founded elements. Of course, formally this is nonsense, since there is nothing outside of V, but we can replace "extension" by "embedding", obtaining:

Exercise II.9.3 *Suppose that $F : V \to V$ is a permutation of V and $E = E_F$. Define $i : V \to V$ by recursion on $\in$ so that $i(x)$ is the y such that $F(y) = \{i(w) : w \in x\}$. Prove that i is 1-1, and is an isomorphism between $(V, \in)$ and $(WF^{(V,E)}, E)$.*

Also, $AC \leftrightarrow AC^{(V,E)}$.

Also, the model (V, E) satisfies the axiom that every relation R on a set A is isomorphic to a relation on a well-founded set. Don't *assume AC here.*

Hint. First show that the model satisfies the axiom that every A has a bijection onto a well-founded set. □

This last axiom is an obvious consequence of AC (biject A onto an ordinal), but is not provable without AC. For example (see Exercise II.9.10) it is consistent with ZF^- that AC is false in V but true in WF; then, if A cannot be well-ordered, there can be no bijection from A to a well-founded set.

Exercise II.9.4 *Prove that one can choose the permutation F such that (V, E) satisfies the following version of the Mostowski Lemma I.9.35: Whenever R is extensional on the set A, there is a transitive set T such that $(A, R) \cong (T, \in)$.*

Hint. If your were given a particular extensional relation R on a set A, you could arrange for E on A to be isomorphic to R by having $a \in F(b) \leftrightarrow aRb$; that is, $F(b) = \{a \in A : aRb\}$. This defines a 1-1 function on A (since R is extensional), and this function can then be extended to a permutation.

You might use Exercise II.9.3 to correspond relations (A, R) in V with relations in the model. □

Note that Lemma I.9.35 requires also that R is well-founded. This is necessary because in ZF, $\in$ is well-founded.

Exercise II.9.5 *Work in ZF^- plus the axiom proved consistent in Exercise II.9.4. Prove that*

1. *$\{y : y = \{y\}\}$ is a proper class.*

2. $\{(x,y) : y = \{x\} \wedge x = \{y\} \wedge y \neq x\}$ *is a proper class.*
3. *Whenever* R *totally orders a set* A*, there is a "fake ordinal"* t *that is a transitive set ordered by* $\in$ *isomorphically to* (A,R)*.*
4. *Assume* $\mathrm{Con}(\Gamma)$*, where* Γ *in* $\mathcal{L} = \{\in\}$ *contains the Axioms of Extensionality. Then there is a transitive model* $M \models \Gamma$ *such that* $\omega \notin M$*.*

Hint. For (1) apply the axiom with R the identity relation on A. So we get arbitrarily large transitive T such that $y = \{y\}$ for all $y \in T$.

For (4), first apply some model theory to get $(A,R) \models \Gamma$ in which $\{b \in A : bRa\}$ is either finite or uncountable for all $a \in A$. □

In (4), say $\Gamma = ZF-P$, which proves that "ω exists". Then the ω of M is a "fake ordinal"; M will also contain non-standard "fake natural numbers". This shows that the absoluteness results for "ordinal", "natural number", etc., proved in Section II.4 really needed Foundation.

By a completely different use of permutations, we can also get models in which the Axiom of Choice is false, although the failure will always be on non-well-founded sets. To do this, we work in the theory ZF^- plus an appropriate failure of Foundation (which we have just shown to be consistent), and define a sub-model of the universe in which some set cannot be well-ordered. First, we generalize the "construction" of the sets $R(\alpha)$ from Lemma I.9.30:

Definition II.9.6 (ZF^-) *For any set* T*, define*

1. $R(0,T) = T$.
2. $R(\alpha+1,T) = \mathcal{P}(R(\alpha,T))$.
3. $R(\gamma,T) = \bigcup_{\alpha<\gamma} R(\alpha,T)$ *for limit ordinals* γ.

Let $WF(T) = \bigcup_{\alpha \in ON} R(\alpha,T)$.

Exercise II.9.7 (ZF^-) *For any transitive set* T*,* $WF(T)$ *is a model of* ZF^- *and* $R(\alpha) \subseteq R(\alpha,T)$ *for each* α.

Definition II.9.8 *An* automorphism *of a set* T *is a permutation* π *of* T *such that* $\forall x,y \in T\,[x \in y \leftrightarrow \pi(x) \in \pi(y)]$.

Note that $\pi(x) = x$ for all $x \in T \cap WF$ (by induction on $\in$). On the other hand, if $x = \{x\}$ for all $x \in T$, then every permutation of T is an automorphism.

Exercise II.9.9 *For any transitive set* T*, every automorphism* π *of* T *extends uniquely to an automorphism* $\widehat{\pi}$ *of* $WF(T)$.

Exercise II.9.10 *Work in* ZFC^- *and assume that there is an infinite set* T *such that* $x = \{x\}$ *for all* $x \in T$*. Let* M *be the set of all* $y \in WF(T)$ *such that for some finite* $A \subseteq T$*,* $\widehat{\pi}(y) = y$ *for all permutations* π *of* T *such that* $\pi \restriction A = id_A$*. Prove that* $T \in M$*, and that* M *is a transitive model for* ZF^- *plus* $(AC)^{WF}$ *plus "*T *cannot be well-ordered".*

Exercise II.9.11 *Work in ZFC^-, plus the assumption that $U := \{x : x = \{x\}\}$ is a proper class. Form M from $WF(U)$ as in Exercise II.9.10. Now show that M is a transitive model for ZFC^- plus the statement that U is a proper class with no infinite subsets.*

Hint. Define $WF(U)$ to be the union of all $WF(T)$ such that T is a sub*set* of U. □

II.10 Set Theory with Classes

We have already seen that the notion of "class" is very useful, but also that it requires some ugly circumlocutions in the metatheory, since classes don't really exist in ZF set theory. For example, recursion on $\in$, which conceptually is the simple assertion

$$\forall G : V \to V \quad \exists F : V \to V \quad \forall x\,[F(x) = G(F\restriction x)] \quad , \qquad (\bigstar)$$

is really an ugly scheme in the metatheory, saying that for every formula defining a G, we can write another formula defining an F so that the formula asserting $\forall x\,[F(x) = G(F\restriction x)]$ is provable (see Section I.9). Why not make classes part of the formal theory, so that ($\bigstar$) is simply one sentence that we prove from the axioms?

There are in fact two well-known systems that do just this: the system *NGB* (von Neumann, Gödel, and Bernays; see Gödel [40]) and the system *MK* (Morse and Kelley; see Kelley [52]). These systems are very similar, so we shall discuss them together.

Informally, in ZF, all sets are classes, but sets x can be members of other classes (or even sets, such as $\{x\}$), whereas proper classes, such as V and ON, don't really exist and aren't members of anything.

To formalize classes in *NGB* or *MK*, work in predicate logic with $\mathcal{L} = \{\in\}$ as usual, but use capital letters $X, Y, Z, \ldots$ as variables; informally, they range over all classes. Now, *define* $\mathfrak{M}(X) \leftrightarrow \exists Y\, X \in Y$ (X is a set (Menge)). Then, let lower case letters "range" over sets; that is, $\forall x\, \varphi(x)$ and $\exists x\, \varphi(x)$ abbreviate, respectively, $\forall X\, [\mathfrak{M}(X) \to \varphi(X)]$ and $\exists X\, [\mathfrak{M}(X) \wedge \varphi(X)]$.

Axiom 1 (Extensionality) is $\forall z(z \in X \leftrightarrow z \in Y) \to X = Y$; two classes with the same members are equal. The z here could just as well have been a Z, since only sets can possibly be members of anything. Axioms (4)(5)(7)(8) are written in the obvious way with lower case letters; for example, Axiom 7 (Infinity) postulates the existence of an inductive *set*.

Axiom 3 (Comprehension) splits into two assertions. First, the intersection of a class with a set is a set: $\forall Z\, \forall w\, \exists y\, \forall x\, (x \in y \leftrightarrow x \in w \wedge x \in Z)$; this is a single sentence. Second, we have a scheme saying that every formula defines a class; that is, $\exists Z \forall x(x \in Z \leftrightarrow \varphi(x))$. Putting the two together, for any set w we can form the set $y = Z \cap w = \{x \in w : \varphi(x)\}$, so we have the ZF version

of Comprehension. As usual, φ may have other free variables (but not Z). Here (and only here), *MK* and *NGB* differ. In *NGB*, all *bound* variables of φ must be lower case (that is, range only over sets), whereas *MK* has no such restriction on φ.

In both theories, we can let $\varphi(x)$ be $x = x$ to define $V = \{x : x = x\}$, the class of all sets. Then, $\neg\mathfrak{M}(V)$ by Russell's Paradox.

Axiom 6 (Replacement) says $\forall F : V \to V \;\; \forall a\, \exists b \forall x \in a\, [F(x) \in b]$. The "$F : V \to V$" means that F is a class of ordered pairs with $\forall x \exists! y\, [(x, y) \in F]$. Replacement is a single sentence; we do not need a scheme for building F from a formula, since this is handled by Comprehension.

Axiom 2 (Foundation) says that every non-empty class has an $\in$–minimal element. It is actually enough to postulate this for sets, since, by the argument of Theorem I.9.3, one can then prove it for all classes.

ZF, *MK*, and *NGB* do not contain the Axiom of Choice, but of course Choice can be added. But there are two versions of Choice in both *MK* and *NGB*. *Local Choice* says that every set can be well-ordered. *Global Choice* says that every class can be well-ordered; equivalently, V can be well-ordered. In both *MK* and *NGB*, Local Choice does not imply Global Choice.

In the terminology of Section II.1, $ZF \leq NGB \leq MK$ is clear, but in fact $ZF \sim NGB \lhd MK$.

For $ZF \sim NGB$; that is, there is a finitistic proof of $\mathrm{Con}(ZF) \to \mathrm{Con}(NGB)$: In fact, *NGB* is a conservative extension of *ZF*, in the sense that every sentence φ with lower case (set) variables provable from *NGB* is provable from *ZF*; then $\mathrm{Con}(ZF) \to \mathrm{Con}(NGB)$ is the special case when φ is $0 \neq 0$. A finitistic proof of this is due to Shoenfield [88]. An earlier non-finitistic proof, due to Wang [103], is much easier to explain:

Exercise II.10.1 *Let $\mathfrak{A} = (A, E) \models ZF$. Show how to convert $\mathfrak{A}$ into a model of NGB by interpreting the classes as elements of $\mathcal{D}^+(\mathfrak{A})$ (see Definition I.15.25). Conclude from this and the Completeness Theorem that NGB is a conservative extension of ZF.*

For $ZF \lhd MK$:

Exercise II.10.2 *Work in MK and prove that $\forall \xi\, \exists \eta > \xi\, [R(\eta) \preccurlyeq V]$. Of course, first you have to define "$A \preccurlyeq V$". Conclude that $\exists \eta\, [R(\eta) \models ZF]$.*

Hint. Modify the proof of the Reflection Theorem II.5.3, now using all formulas at once. First, you have to verify that, unlike in *ZF*, you really can treat V as a model. To define "$V \models \varphi[\sigma]$", follow the discussion of Definition I.15.5. Let K be the class of all pairs (φ, σ) such that φ is a formula of $\mathcal{L} = \{\in\}$ and σ is an assignment for φ in V. Let $K_n = \{(\varphi, \sigma) \in K : \mathrm{lh}(\varphi) < n\}$. Prove by induction on n that there is a function $F : K_n \to \{0, 1\}$ that satisfies the recursive conditions that val_V should satisfy. Note that you can express these conditions on F in both *NGB* and *MK*, but to use induction, you have to form

$\{n \in \omega : \exists F \cdots\cdots\}$, which you can't do in *NGB* because the formula quantifies over classes.

Note that the recursion scheme (✯), which is provable in both *NGB* and *MK*, doesn't apply here. □

In both *NGB* and *MK*, the only infinite scheme is Comprehension. *NGB* is in fact finitely axiomatizable, since one can derive all instances of Comprehension from a finite subset thereof; see Chapter II of Gödel [40]. *MK* is not finitely axiomatizable (assuming it's consistent), since, as for *ZF*, one can use a reflection argument to show that $MK \vdash \mathrm{Con}(\Lambda)$ for all finite $\Lambda \subset MK$.

As a foundation for mathematics, there is no cogent philosophical argument to prefer any one of ZF, NGB, MK. Modern set theory prefers *ZF* (or *ZFC*) for the technical reason that it's somewhat easier to do the independence proofs for a theory that has only sets, because it is simpler to describe the models.

Also, any philosophical distinction among ZF, NGB, MK seems to be of little interest now, since it is completely overshadowed by questions about large cardinals:

Exercise II.10.3 *Work in the theory ZFC plus the assumption that there are at least* 76 *strongly inaccessible cardinals. Prove that there is a set model M for "MKC plus there are exactly* 75 *strongly inaccessible cardinals".*

Hint. Replace the $R(\kappa)$ by $R(\kappa + 1)$ in your solution to Exercise II.4.25. □

If you do a lot of work with category theory, then you might prefer to work in *NGB* or *MK*, since categories (of topological spaces, of groups, etc.) are often proper classes. Working in *NGB* or *MK*, general theorems about categories and functors can be expressed by single sentences, rather than by ugly schemes in the metatheory.

Chapter III

Infinitary Combinatorics

To *first approximation*, there are two aspects to set theory: *logical* and *mathematical*. The logical aspect involves axiomatics, models, and independence proofs. The mathematical aspect is called *infinitary combinatorics*. Here, we just work in *ZFC* and prove mathematical theorems, but we focus on results about infinite and uncountable sets, where one is likely to encounter problems that *could* be independent. You don't need any knowledge of logic or axioms to do this — you just prove theorems using "ordinary reasoning", in the spirit of Cantor. The early work in this area predates the independence results, but since the 1960s, people working on infinitary combinatorics had at least to be aware of the independence results, since they limit what you might hope to prove.

Infinitary combinatorics includes results of "pure set theory", such as cardinal arithmetic, but it also includes the more set-theoretical aspects of analysis, topology, and algebra. For example, you might ask set-theoretic questions about Lebesgue measure. You might consider the least cardinal κ such that there is a union of κ Lebesgue null sets that fails to be null. Infinitary combinatorics tells you that $\aleph_1 \leq \kappa \leq 2^{\aleph_0}$, and that κ must be regular (see Lemma III.1.7); but if you study this subject further, you should at least be aware that the independence proofs tell you not to waste your time trying to prove that $2^{\aleph_0} = \aleph_1$, or even that $\kappa = 2^{\aleph_0}$ or that $\kappa = \aleph_1$.

This *first approximation* has been seen, since the 1980s, to be not 100% correct, since methods from logic and model theory have become increasingly important in proving, in *ZFC*, purely mathematical theorems whose statement does not involve logic at all. In particular, the Downward Löwenheim–Skolem–Tarski Theorem, applied to models of the form $H(\theta)$, is frequently used to prove theorems in infinitary combinatorics; see Section III.8.

Most of the material in this chapter will be used somewhere in Chapters IV and V, which deal with the independence proofs and the method of forcing. However, the present chapter contains much more than what you need to un-

derstand the simpler applications of forcing, such as $\mathrm{Con}(ZFC + 2^{\aleph_0} = \aleph_2)$. To understand these, you can skip most of this chapter, reading only Section III.1 through Lemma III.1.24, Section III.2 through Corollary III.2.10, and Section III.3 through Lemma III.3.26.

III.1 Some Small Cardinals

This section will provide us with a collection of cardinals that *ZFC* proves lie somewhere between $\aleph_1$ and $2^{\aleph_0}$. Questions about the exact value of these cardinals provide many examples of statements independent of $ZFC + \neg CH$. For more on such cardinals, see Blass [11]. The cardinals we discuss in this section arise from considering various *ideals* or *filters*.

Definition III.1.1 *If A is any non-empty set, then an* ideal *on A is a subset $\mathcal{I} \subseteq \mathcal{P}(A)$ satisfying:*

1. $\emptyset \in \mathcal{I}$ *and* $A \notin \mathcal{I}$.
2. $\forall X, Y \in \mathcal{I}\,[X \cup Y \in \mathcal{I}]$
3. $\forall X, Y \subseteq A\,[X \subseteq Y \wedge Y \in \mathcal{I} \rightarrow X \in \mathcal{I}]$.

For example, $[A]^{<\kappa}$ (Definition I.13.18) is an ideal on A whenever κ is an infinite cardinal (so (2) holds) and $\kappa \leq |A|$ (so (1) holds).

Three specific ideals which will be used frequently:

Definition III.1.2

- ☞ $\mathsf{fin} = [\omega]^{<\omega}$, *the ideal of finite subsets of* ω.
- ☞ null *is the ideal of Lebesgue null subsets of* $\mathbb{R}$.
- ☞ meager *is the ideal of first category subsets of* $\mathbb{R}$.

Informally, we may think of the sets in the ideal as being "negligible" or "almost nothing". For example, in analysis, if we say that functions f, g on $\mathbb{R}$ are equal *almost everywhere*, we mean that $\{x : f(x) \neq g(x)\} \in \mathsf{null}$.

The dual notion to an ideal is a filter, which formalizes the notion of being "almost everything":

Definition III.1.3 *If A is any non-empty set, then a* filter *on A is a subset $\mathcal{F} \subseteq \mathcal{P}(A)$ satisfying:*

1. $A \in \mathcal{F}$ *and* $\emptyset \notin \mathcal{F}$.
2. $\forall X, Y \in \mathcal{F}\,[X \cap Y \in \mathcal{F}]$
3. $\forall X, Y \subseteq A\,[X \supseteq Y \wedge Y \in \mathcal{F} \rightarrow X \in \mathcal{F}]$.

The duality between the two notions is made explicit by:

Definition III.1.4 *If $\mathcal{E} \subseteq \mathcal{P}(A)$, then $\mathcal{E}^* = \{X \subseteq A : A \backslash X \in \mathcal{E}\}$. When $\mathcal{I} \subseteq \mathcal{P}(A)$ is an ideal, $\mathcal{I}^*$ is called the* dual filter. *When $\mathcal{F} \subseteq \mathcal{P}(A)$ is a filter, $\mathcal{F}^*$ is called the* dual ideal.

Lemma III.1.5 *If $\mathcal{E} \subseteq \mathcal{P}(A)$, then $\mathcal{E}^{**} = \mathcal{E}$. If $\mathcal{E}$ is an ideal then $\mathcal{E}^*$ is a filter. If $\mathcal{E}$ is a filter then $\mathcal{E}^*$ is an ideal.*

The study of ideals and filters leads to two different kinds of problems: The structure of the ideals or filters themselves, and the structure of various reduced powers (see Definition III.1.9). Regarding the structure of the ideals, consider the following three cardinals:

Definition III.1.6 *If $\mathcal{I}$ is an ideal on A with $[A]^{<\omega} \subseteq \mathcal{I}$ then:*

1. $\mathrm{add}(\mathcal{I})$ *is the least cardinal κ such that $\bigcup \mathcal{E} \notin \mathcal{I}$ for some $\mathcal{E} \in [\mathcal{I}]^\kappa$.*
2. $\mathrm{cov}(\mathcal{I})$ *is the least cardinal κ such that $A = \bigcup \mathcal{E}$ for some $\mathcal{E} \in [\mathcal{I}]^\kappa$.*
3. $\mathrm{non}(\mathcal{I})$ *is the least cardinal κ such that $X \notin \mathcal{I}$ for some $X \in [A]^\kappa$.*

The assumption that $[A]^{<\omega} \subseteq \mathcal{I}$ (all finite subsets of A are in the ideal) will be true of any ideal that we study, and implies that $\mathrm{add}(\mathcal{I})$ and $\mathrm{cov}(\mathcal{I})$ are well-defined. Note that these three cardinals are infinite; for example, $\mathrm{add}(\mathcal{I}) \geq \aleph_0$ because if κ is finite, then $\mathcal{E} \in [\mathcal{I}]^\kappa \to \bigcup \mathcal{E} \in \mathcal{I}$ by (2) of Definition III.1.1.

As an illustration and to explain the terminology, consider the ideal null. This ideal is countably ***add***itive; that is $\mathrm{add}(\mathsf{null}) \geq \aleph_1$, meaning that the union of $\aleph_0$ null sets is null. Also, $\mathrm{cov}(\mathsf{null}) \leq 2^{\aleph_0}$; that is, you can always ***cov***er $\mathbb{R}$ with $2^{\aleph_0}$ null sets — namely, singletons. It is consistent to have $\mathrm{add}(\mathsf{null}) = \aleph_1$ and $\mathrm{cov}(\mathsf{null}) = \aleph_2$; then there's a family of $\aleph_1$ null sets whose union isn't null, but it still takes $\aleph_2$ null sets to cover $\mathbb{R}$. Now, $\mathrm{non}(\mathsf{null})$ is the least size of a ***non***-null subset $X \subseteq \mathbb{R}$. Note that $\mathrm{non}(\mathsf{null}) \geq \mathrm{add}(\mathsf{null})$, since X is the union of $|X|$ singletons. This discussion clearly generalizes:

Lemma III.1.7 *If $\mathcal{I}$ is an ideal on A with $[A]^{<\omega} \subseteq \mathcal{I}$ then* $\mathrm{add}(\mathcal{I}) \leq \mathrm{non}(\mathcal{I}) \leq |A|$ *and* $\mathrm{add}(\mathcal{I}) \leq \mathrm{cov}(\mathcal{I}) \leq |A|$. *Also,* $\mathrm{add}(\mathcal{I})$ *is regular.*

Proof. Suppose that $\mathrm{add}(\mathcal{I}) = \lambda$, where λ is singular. Let $\theta = \mathrm{cf}(\lambda) < \lambda$, and fix $\kappa_\mu < \lambda$ for $\mu < \theta$ such that $\sup_\mu \kappa_\mu = \lambda$. Applying the definition of $\mathrm{add}(\mathcal{I})$, fix $X_\xi \in \mathcal{I}$ for $\xi < \lambda$ such that $\bigcup_{\xi<\lambda} X_\xi \notin \mathcal{I}$. But each $\kappa_\mu < \lambda$, so $Y_\mu := \bigcup_{\xi<\kappa_\mu} X_\xi \in \mathcal{I}$. Then $\theta < \lambda$ implies that $\bigcup_{\mu<\theta} Y_\mu \in \mathcal{I}$. But $\bigcup_{\mu<\theta} Y_\mu = \bigcup_{\xi<\lambda} X_\xi$, so we have a contradiction. □

It is easy to find counter-examples to possible "improvements" of this lemma. For example, if $\theta = \mathrm{cf}(\lambda) < \lambda$ and $\mathcal{I} = [\lambda]^{<\lambda}$, then $\mathrm{add}(\mathcal{I}) = \mathrm{cov}(\mathcal{I}) = \theta$, but $\mathrm{non}(\mathcal{I}) = \lambda$, which is singular. But if $\mathcal{I} = [\lambda]^{<\omega}$ then $\mathrm{add}(\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \aleph_0$, but $\mathrm{cov}(\mathcal{I}) = \lambda$, which is singular.

Considering these three cardinals applied to the ideals null and meager on $\mathbb{R}$, we already have six cardinals somewhere between $\aleph_1$ and $2^{\aleph_0}$. In Chapters

IV and V, we shall discuss some independence results regarding these six. Also, we shall, in Section III.3, discuss an axiom called Martin's Axiom (MA) which is consistent with $2^{\aleph_0}$ being large, and which implies that these six cardinals, plus quite a few more, equal $2^{\aleph_0}$.

The following well-known relation between the cardinals for null and meager was proved by Rothberger ([78], Lemma 1) in 1938. Of course, 1938 was long before these cardinals were given these fancy names, and his focus was really on $\aleph_1$, so he simply said, in plain German, that if there is a set of size $\aleph_1$ of positive outer measure, then $\mathbb{R}$ can be covered by $\aleph_1$ sets of first category.

Exercise III.1.8 $\mathrm{cov}(\mathsf{null}) \leq \mathrm{non}(\mathsf{meager})$ *and* $\mathrm{cov}(\mathsf{meager}) \leq \mathrm{non}(\mathsf{null})$.

Hint. Recall that you can find $A, B \subseteq \mathbb{R}$ so that $\mathbb{R} = A \cup B$ and $A \cap B = \emptyset$ and $A \in \mathsf{meager}$ and $B \in \mathsf{null}$; for example, let $A = \mathbb{Q} + K = \{q + x : q \in \mathbb{Q} \wedge x \in K\}$, where K is a Cantor set of positive measure. Now, if $X \notin \mathsf{null}$, then $X + A = \mathbb{R}$ (otherwise some translate of B would cover $-X$); but then $\mathbb{R}$ is a union of $|X|$ translates of A, and each translate is of first category. □

We turn now to the ideal fin on ω. The structure of fin is essentially trivial, and $\mathrm{add}(\mathsf{fin}) = \mathrm{non}(\mathsf{fin}) = \mathrm{cov}(\mathsf{fin}) = \aleph_0$. But, there are interesting questions involving reduced powers; that is; products modulo fin.

Definition III.1.9 *If R is a relation on B and $\mathcal{I}$ is an ideal with dual filter $\mathcal{F}$ on A and $f, g \in B^A$, then $f\ R_{\mathcal{I}}\ g$ iff $f\ R_{\mathcal{F}}\ g$ iff $\{a \in A : f(a)\ R\ f(b)\} \in \mathcal{F}$.*

This notion easily generalizes to n–ary relations, and it is used in model theory to define the concept of a reduced power of a structure, usually denoted by $\mathfrak{B}^A/\mathcal{I}$ or $\mathfrak{B}^A/\mathcal{F}$. For more on this, see [13].

In some applications, R is literally a subset of $B \times B$; in others, R is a defined property of pairs; for example, $f =_{\mathcal{I}} g$ iff $\{a \in A : f(a) = f(b)\} \in \mathcal{F}$. In the case of the ideal null on $\mathbb{R}$, with $f, g \in B^{\mathbb{R}}$, $f =_{\mathsf{null}} g$ iff $f(x) = g(x)$ *almost everywhere*; that is, except on a set of measure 0.

Here, we shall focus on the set-theoretic properties of $\omega^\omega/\mathsf{fin}$:

Definition III.1.10 *For $f, g \in \omega^\omega$: $f =^* g$ iff $f =_{\mathsf{fin}} g$ and $f \leq^* g$ iff $f \leq_{\mathsf{fin}} g$.*

Some easy facts: $=^*$ is an equivalence relation, and each equivalence class is countably infinite. $\leq^*$ is a pre-order (reflexive and transitive), and $f \leq^* g \leq^* f$ holds iff $f =^* g$. We shall avoid using the symbol $<^*$ here, since it could have two possible meanings: the order $<_{\mathsf{fin}}$ ($f <_{\mathsf{fin}} g$ iff $\{n : f(n) < g(n)\}$ is cofinite), or the strict partial order associated with the pre-order $\leq^*$ — i.e., $f \leq^* g \wedge \neg(g \leq^* f)$ — i.e., $\{n : f(n) \leq g(n)\}$ is cofinite and $\{n : f(n) < g(n)\}$ is infinite.

The study of $\omega^\omega/\mathsf{fin}$ suggests two new cardinals:

Definition III.1.11

- ☞ $\mathfrak{c} = 2^{\aleph_0}$, *the size of the* continuum.
- ☞ $\mathfrak{d}$ *is the least size of a* dominating family — *that is, a* $\mathcal{D} \subseteq \omega^\omega$ *such that* $\forall f \in \omega^\omega\ \exists g \in \mathcal{D}\ f \leq^* g$.
- ☞ $\mathfrak{b}$ *is the least size of an* unbounded family — *that is, a* $\mathcal{B} \subseteq \omega^\omega$ *such that* $\neg\exists g \in \omega^\omega\ \forall f \in \mathcal{B}\ f \leq^* g$.

For example, the set of all g such that all $g(n)$ are even is a dominating family, while the set of all f such that $f(2n) = 0$ for all n is unbounded but is not dominating. These two examples have size $\mathfrak{c}$.

Lemma III.1.12 $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

Proof. $\mathfrak{d} \leq \mathfrak{c}$ is obvious because $|\omega^\omega| = \mathfrak{c}$. $\mathfrak{b} \leq \mathfrak{d}$ follows from the fact that every dominating family $\mathcal{D}$ is also unbounded, since if $\forall f \in \mathcal{D}\ f \leq^* g$ then no function in $\mathcal{D}$ can dominate the function $n \mapsto g(n) + 1$.

To prove that $\mathfrak{b}$ is uncountable, a diagonal argument shows that a countable family $\mathcal{B} = \{f_n : n \in \omega\}$ cannot be unbounded: If $g(k) = \max\{f_n(k) : n \leq k\}$, then for each n, $f_n \leq^* g$ because $f_n(k) \leq g(k)$ for all $k \geq n$. □

The use of Fraktur letters for cardinals, and in particular $\mathfrak{c}$ for $2^{\aleph_0}$, goes back to antiquity; letters such as $\mathfrak{m}, \mathfrak{n}$, varying over "arbitrary" cardinals, were used where we now use κ, λ. The use of letters $\mathfrak{b}$, $\mathfrak{d}$, etc. to name specific cardinals $\leq \mathfrak{c}$ originates with van Douwen [22] in the 1980s, who also proved many of the inequalities between them that we shall discuss below.

Obviously, *CH* implies that $\mathfrak{b} = \mathfrak{d} = \aleph_1$. In models of $\neg CH$, essentially any values of $\mathfrak{b}$ and $\mathfrak{d}$ are possible, provided that they don't contradict Lemma III.1.12 and the next exercise:

Exercise III.1.13 *Prove that* $\mathfrak{b}$ *is regular and* $\mathrm{cf}(\mathfrak{d}) \geq \mathfrak{b}$.

The next lemma relates the cardinals for $\omega^\omega/\mathrm{fin}$ to the cardinals for the first category ideal:

Lemma III.1.14 $\mathfrak{b} \leq \mathrm{non}(\mathsf{meager})$.

Proof. We shall show that $\kappa < \mathfrak{b} \to \kappa < \mathrm{non}(\mathsf{meager})$, so fix $\kappa < \mathfrak{b}$ and fix $X \in [\mathbb{R}]^\kappa$; we prove that X is of first category — that is, X is contained in a countable union of closed nowhere dense sets. We shall actually prove the equivalent statement that X is disjoint from some countable intersection of dense open sets.

List $\mathbb{Q}$ without repetitions as $\{q_k : k \in \omega\}$. Whenever $f \in \omega^\omega$ and $n \in \omega$, let $U_f^n = \bigcup\left\{\left(q_k - 2^{-f(k)}, q_k + 2^{-f(k)}\right) : k \geq n\right\}$. Then U_f^n is open and dense, and $U_f^0 \supseteq U_f^1 \supseteq U_f^2 \supseteq \cdots$. Let $G_f = \bigcap_{n\in\omega} U_f^n$. We shall show that $X \cap G_g = \emptyset$ for a suitable g.

Observe that $f \leq^* g \to G_g \subseteq G_f$ because $U_g^n \subseteq U_f^n$ for all but finitely many n. For each $x \in X$, choose some $f_x \in \omega^\omega$ such that $x \notin G_{f_x}$; to do this, just make sure that $2^{-f_x(k)} < |x - q_k|$ for each k, unless $x = q_k$ (which happens for at most one k). Now, since $|X| < \mathfrak{b}$, fix g so that $f_x \leq^* g$ for all $x \in X$. Then $x \notin G_g$ for all $x \in X$ □

Many arguments in this subject have this flavor. $\mathfrak{b}$ and non(meager) were defined to be the least cardinality at which some bad thing happens (an unbounded family of functions and a non-meager set of real numbers). But in proving $\mathfrak{b} \leq$ non(meager), we accentuated the positive: if κ is good in the $\mathfrak{b}$ sense (all families of size κ can be bounded by some $g \in \omega^\omega$), then κ is also good in the non(meager) sense (all sets of size κ are meager).

We next consider $\leq^*$ on 2^ω. Here, it is more customary to deal with $\mathcal{P}(\omega)$ instead, identifying $Y \subseteq \omega$ with $\chi_Y : \omega \to \{0,1\}$.

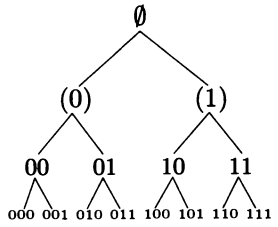
Definition III.1.15 *$X \subseteq^* Y$ iff $X \backslash Y$ is finite and $X \perp Y$ iff $X \cap Y$ is finite (X, Y are* almost disjoint*). $X =^* Y$ iff $X \subseteq^* Y \subseteq^* X$ iff their symmetric difference is finite. An* almost disjoint family *is a family of infinite sets $\mathcal{D}$ such that $X \perp Y$ whenever $X, Y \in \mathcal{D}$ and $X \neq Y$.*

These notions are most often used when X, Y are infinite subsets of ω or some other countable set. For $X, Y \subseteq \omega$, note that $X \subseteq^* Y$ iff $\chi_X \leq^* \chi_Y$. The obvious analogs of $\mathfrak{b}$ and $\mathfrak{d}$ are trivial here, since there is a largest function (namely, $n \mapsto 1$), but we shall see that there are other interesting cardinals.

Obviously, a family of disjoint subsets of ω must be countable, but

Lemma III.1.16 *There is an almost disjoint $\mathcal{D} \subseteq [\omega]^\omega$ such that $|\mathcal{D}| = \mathfrak{c}$.*

Proof. It is enough to get $\mathcal{D} \subseteq [T]^\omega$, where T is some other set of size $\aleph_0$; then, one can transfer $\mathcal{D}$ to ω using a bijection from T onto ω. In particular, let $T = 2^{<\omega}$, the set of all finite sequences of 0s and 1s. For $f \in 2^\omega$, let $X_f = \{f \restriction n : n \in \omega\}$, which is an infinite subset of T. Then $X_f \perp X_g$ whenever $f \neq g$, since if i is least such that $f(i) \neq g(i)$, then $X_f \cap X_g = \{f \restriction n : n \leq i\}$. So, let $\mathcal{D} = \{X_f : f \in 2^\omega\}$. You may picture the various X_f as paths through this tree, called the *Cantor tree*; each X_f contains one node at each level of the tree. For more on trees, see Section III.5. □



Definition III.1.17 *A* maximal almost disjoint family, *or* mad family, *is an almost disjoint family $\mathcal{D} \subseteq [\omega]^\omega$ that is maximal among all such families.*

By Zorn's Lemma or Tukey's Lemma, there is a mad family of size $\mathfrak{c}$, since one may take the $\mathcal{D}$ from Lemma III.1.16 and extend it to a maximal family. The family $\mathcal{D}$ constructed in proving Lemma III.1.16 is not maximal, since the

set $\{(1),(0,1),(0,0,1),(0,0,0,1),\ldots\}$ meets each element of $\mathcal{D}$ in a set of size 0 or 1. Less obviously, by a theorem of Mathias [66], one cannot produce any infinite mad family in ZF without using AC.

Definition III.1.18 $\mathfrak{a}$ *is the least size of an infinite* mad *family.*

Unfortunately, $\mathfrak{m}$ is used for the least cardinal at which Martin's Axiom fails; see Definition III.3.16. We have to say "infinite" because if $\mathcal{D}$ is a partition of ω into n disjoint infinite sets, then $\mathcal{D}$ is a mad family of size n.

Lemma III.1.19 $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$.

Proof. In view of Lemma III.1.12, we need only prove that $\mathfrak{b} \leq \mathfrak{a}$. As in the proof of Lemma III.1.14, we shall prove that $\kappa < \mathfrak{b} \to \kappa < \mathfrak{a}$, so fix an infinite $\kappa < \mathfrak{b}$ and fix an almost disjoint family $\mathcal{D} = \{X_\alpha : \alpha < \kappa\}$; we shall show that $\mathcal{D}$ is not maximal.

For each $n \in \omega$, let $\widehat{X}_n = X_n \setminus \bigcup_{i<n} X_i$. Observe that the $\widehat{X}_n$ are infinite and pairwise disjoint, and $\widehat{X}_n =^* X_n$. Whenever $g \in \omega^\omega$, let e^n_g be the least element of $\widehat{X}_n$ that is larger than $g(n)$, and let $E_g = \{e^n_g : n \in \omega\}$. Then E_g contains one element from each $\widehat{X}_n$, so that E_g is infinite and $E_g \perp X_n$ for each n (since $\widehat{X}_n =^* X_n$).

Now, when $\omega \leq \alpha < \kappa$: Each $X_\alpha \cap X_n$ is finite, so that we can fix an f_α so that all elements of $X_\alpha \cap X_n$ are less than $f_\alpha(n)$ for each n. Whenever $f_\alpha \leq^* g$, $E_g \cap X_\alpha$ is finite, since $f_\alpha(n) \leq g(n) \to e^n_g \notin X_\alpha$. Since $\kappa < \mathfrak{b}$, we can fix g so that $f_\alpha \leq^* g$ for all such α. Then $E_g \perp X_\alpha$ whenever $\omega \leq \alpha < \kappa$, and we have just noted that $E_g \perp X_n$ for all $n < \omega$, so that $\mathcal{D}$ is not maximal. □

It is consistent to have $\aleph_1 = \mathfrak{b} = \mathfrak{a} < \mathfrak{c}$; see Exercise IV.4.12.

We shall introduce one more cardinal, $\mathfrak{p}$, in this section. As with the other cardinals, $MA \to \mathfrak{p} = \mathfrak{c}$ (as we shall see in Section III.3). Informally, the axiom $\mathfrak{p} = \mathfrak{c}$ may be viewed as a weak form of MA, in that $\mathfrak{p} = \mathfrak{c}$ implies that most (but not all) of the cardinals already discussed are equal to $\mathfrak{c}$ also. This informal remark is made formal by Bell's Theorem III.3.61.

Definition III.1.20 *A family $\mathcal{E}$ of sets has the* FIP (Finite Intersection Property) *iff $\bigcap \mathcal{F} \neq \emptyset$ for all $\mathcal{F} \in [\mathcal{E}]^{<\omega}$. $\mathcal{E}$ has the* SFIP (Strong Finite Intersection Property) *iff $\bigcap \mathcal{F}$ is infinite for all $\mathcal{F} \in [\mathcal{E}]^{<\omega}$. A set K is a* pseudo-intersection *of $\mathcal{E}$ iff K is infinite and $K \subseteq^* Z$ for all $Z \in \mathcal{E}$.*

Note that unlike $\bigcap \mathcal{E}$, which is always uniquely defined, if there exists a pseudo-intersection of $\mathcal{E}$, then there are also many others. For example, on ω, let $Z_n = \{x \in \omega : n \mid x\}$ and let $\mathcal{E} = \{Z_n : 1 \leq n < \omega\}$. Then $\mathcal{E}$ has the SFIP. Also, $\bigcap \mathcal{E} = \emptyset$, but if $K = \{n! : n \in \omega\}$ then K is *one* pseudo-intersection of $\mathcal{E}$; others are $\{7n! : n \in \omega\}$ and $\{n!/7 : 7 \leq n < \omega\}$.

Definition III.1.21 $\mathfrak{p}$ *is the least size of a family* $\mathcal{E} \subseteq [\omega]^\omega$ *such that* $\mathcal{E}$ *has the SFIP and there does not exist any* $\mathfrak{p}$pseudo-intersection *of* $\mathcal{E}$.

It may not be immediately obvious that there is any such $\mathcal{E}$. For a simple example of one, let $\mathcal{D}$ be an infinite mad family, and let $\mathcal{E} = \{\omega \backslash x : x \in \mathcal{D}\}$. It is easy to see that $\mathcal{E}$ has the SFIP and has no pseudo-intersection. This example shows that $\mathfrak{p} \leq \mathfrak{a}$, which is also immediate from Lemma III.1.24 below, but the example used for that lemma is not so simple.

First,

Lemma III.1.22 $\aleph_1 \leq \mathfrak{p}$.

Proof. Let $\mathcal{E} \subseteq [\omega]^\omega$ have the SFIP, with $|\mathcal{E}| \leq \aleph_0$. List $\mathcal{E}$ as $\{Z_i : i \in \omega\}$. Recursively choose $k_n \in \omega$ so that $k_n \in \left(\bigcap_{i<n} Z_i \setminus \{k_i : i < n\}\right)$. Then $K := \{k_n : n \in \omega\}$ is infinite and $K \subseteq^* Z_i$ for each i. □

Lemma III.1.23 *Fix* $\mathcal{E} \subseteq [\omega]^\omega$ *with* $|\mathcal{E}| < \mathfrak{p}$. *Also, fix a non-empty* $\mathcal{H} \subseteq [\omega]^\omega$ *such that* $|\mathcal{H}| < \mathfrak{p}$ *and assume that for all* $H \in \mathcal{H}$, $\{Z \cap H : Z \in \mathcal{E}\}$ *has the SFIP. Then* $\mathcal{E}$ *has a pseudo-intersection* K *such that* $K \cap H$ *is infinite for all* $H \in \mathcal{H}$.

Proof. We remark that if $\mathcal{H} = \{H\}$, this is immediate from the definition of $\mathfrak{p}$, but the proof is a little tricky when $\mathcal{H}$ is infinite.

We shall apply the definition of $\mathfrak{p}$ to families of subsets of the countably infinite set $J := [\omega]^{<\omega} \backslash \{\emptyset\}$. For $Z \in \mathcal{E}$, let $\widehat{Z} = \{s \in J : s \subseteq Z\}$. For $H \in \mathcal{H}$, let $\widetilde{H} = \{s \in J : s \cap H \neq \emptyset\}$. For $\ell \in \omega$, let $T_\ell = \{s \in J : \min(s) > \ell\}$. Let

$$\mathcal{E}^* \;=\; \{\widehat{Z} : Z \in \mathcal{E}\} \cup \{\widetilde{H} : H \in \mathcal{H}\} \cup \{T_\ell : \ell \in \omega\} \;\subseteq\; \mathcal{P}(J) \quad .$$

Then $\mathcal{E}^*$ has the SFIP: that is, a finite intersection

$$I = \widehat{Z}_1 \cap \cdots \cap \widehat{Z}_m \cap \widetilde{H}_1 \cap \cdots \cap \widetilde{H}_m \cap T_\ell$$

is infinite because $Z := Z_1 \cap \cdots \cap Z_m$ meets each H_i in an infinite set, so there are $s \in J$ with arbitrarily large minimum so that $s \subseteq Z$ and s meets each H_i.

Since $|\mathcal{E}^*| < \mathfrak{p}$, fix an infinite $M \subseteq J$ such that M is a pseudo-intersection of $\mathcal{E}^*$. Let $K = \bigcup M \subseteq \omega$. Then K is infinite because M is infinite. For each $Z \in \mathcal{E}$: $M \subseteq^* \widehat{Z}$ implies that $K \subseteq^* Z$, so K is a pseudo-intersection of $\mathcal{E}$.

To prove that $K \cap H$ is infinite for $H \in \mathcal{H}$, fix any $\ell \in \omega$. $M \subseteq^* \widetilde{H} \cap T_\ell$, so fix an $s \in M$ with $s \in \widetilde{H} \cap T_\ell$. Then $s \cap H \neq \emptyset$ and $\min(s) > \ell$, so $s \cap H \subseteq \bigcup M = K$ implies that $K \cap H$ contains a number larger than ℓ. Since ℓ was arbitrary, $K \cap H$ is infinite. □

Lemma III.1.24 $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$ *and* $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

Proof. Given what we have already done, we need only prove that $\mathfrak{p} \leq \mathfrak{b}$. To do this, fix a cardinal $\kappa < \mathfrak{p}$; we shall prove that $\kappa < \mathfrak{b}$. So, we fix a $\mathcal{B} \subseteq \omega^\omega$ of size κ and show that $\mathcal{B}$ cannot be unbounded; that is, there is a g such that $f \leq^* g$ for all $f \in \mathcal{B}$. We shall apply Lemma III.1.23 now, but to families of subsets of the countably infinite set $\omega \times \omega$.

For each $f \in \mathcal{B}$, let $Z_f = \{(x,y) \in \omega \times \omega : y > f(x)\}$. Let $\mathcal{E} = \{Z_f : f \in \mathcal{B}\}$. Also, for $j \in \omega$, let $H_j = \{j\} \times \omega$, and let $\mathcal{H} = \{H_j : j \in \omega\}$. Note that for each j, $\{Z_f \cap H_j : Z_f \in \mathcal{E}\}$ has the SFIP because an intersection of the form $H_j \cap \bigcap_{i<m} Z_{f_i}$ contains all pairs (j,y) such that $y > \max_{i<m} f_i(j)$. Applying Lemma III.1.23, fix a $K \subseteq \omega \times \omega$ so that K is a pseudo-intersection of $\mathcal{E}$ and $K \cap H_j$ is infinite for each j. In particular, each $K \cap H_j \neq \emptyset$, so we may fix $g \subseteq K$ so that g is the graph of a function. Then, for each $f \in \mathcal{B}$, $g \subseteq^* Z_f$ implies that $f \leq^* g$. □

This lemma does not tell you how to order $\mathfrak{a}$ and $\mathfrak{d}$. In fact, $\mathfrak{d} < \mathfrak{a}$ is consistent by Shelah [85], but also $\mathfrak{a} < \mathfrak{d}$ is consistent (and holds in Cohen's original model of $\neg CH$; see Exercise V.2.23). By Lemma III.1.24, $\mathfrak{p} = \mathfrak{c}$ implies that $\mathfrak{d} = \mathfrak{a} = \mathfrak{b} = \mathfrak{c}$ also. By the next lemma, $\mathfrak{p} = \mathfrak{c}$ also implies that add(meager) = cov(meager) = non(meager) = $\mathfrak{c}$:

Lemma III.1.25 $\mathfrak{p} \leq$ add(meager).

Proof. Fix an infinite $\kappa < \mathfrak{p}$. We need to show that in $\mathbb{R}$ a union of κ first category sets is first category. By the definition of "first category", this is equivalent to showing that any union of κ closed nowhere dense sets is contained in a countable union of closed nowhere dense sets. Taking complements, this is equivalent to showing that any intersection of κ dense open sets is contained in a countable intersection of dense open sets.

So, let U_α be dense and open for $\alpha < \kappa$. We shall find dense open V_n for $n < \omega$ such that $\bigcap_n V_n \subseteq \bigcap_\alpha U_\alpha$. Let $\{B_i : i \in \omega\}$ be a countable base of non-empty open sets for the topology of $\mathbb{R}$; for example, the B_i can list all open intervals with rational endpoints.

Let $Z_\alpha = \{i \in \omega : B_i \subseteq U_\alpha\}$. Let $H_j = \{i \in \omega : B_i \subseteq B_j\}$. For each j, $\{Z_\alpha \cap H_j : \alpha < \kappa\}$ has the SFIP because given j and $\alpha_0, \ldots, \alpha_{n-1}$, $\bigcap_{\mu<n} U_{\alpha_\mu}$ is dense and open, so $B_j \cap \bigcap_{\mu<n} U_{\alpha_\mu}$ is non-empty and open, so there are infinitely many i such that $B_i \subseteq B_j \cap \bigcap_{\mu<n} U_{\alpha_\mu}$ and these i are all in $H_j \cap \bigcap_{\mu<n} Z_{\alpha_\mu}$.

Applying Lemma III.1.23, fix $K \subseteq \omega$ that is a pseudo-intersection of $\mathcal{E}$ and such that each $K \cap H_j$ is infinite. Let $V_n = \bigcup\{B_i : i \in K \wedge i > n\}$. Each V_n is dense because for each j, $K \cap H_j$ is infinite, so it contains some $i > n$, whence $B_i \subseteq V_n \cap B_j$; so V_n meets every basic open set B_j. Also, for each α, $K \subseteq^* Z_\alpha$ implies that $V_n \subseteq Z_\alpha$ for n large enough, so that $\bigcap_n V_n \subseteq U_\alpha$. □

Applying Rothberger's theorem (Exercise III.1.8), $\mathfrak{p} \leq$ non(null) also, so $\mathfrak{p} = \mathfrak{c}$ implies that all sets of reals of size less than $\mathfrak{c}$ are null sets. But this isn't true for add(null) or cov(null); it is consistent with $\mathfrak{p} = \mathfrak{c} \geq \aleph_2$ that $\mathbb{R}$ is a union of $\aleph_1$ null sets; see Corollary V.4.16.

The various cardinals we are discussing here in general do not affect the values of 2^κ for $\aleph_0 < \kappa < \mathfrak{c}$, but $\mathfrak{p}$ is an exception:

Lemma III.1.26 *If $\aleph_0 \leq \kappa < \mathfrak{p}$ then $2^\kappa = \mathfrak{c}$. In particular, if $\mathfrak{p} = \mathfrak{c}$ then $\mathfrak{c}$ is regular and $2^\kappa = \mathfrak{c}$ for all infinite $\kappa < \mathfrak{c}$.*

Proof. Fix an infinite $\kappa < \mathfrak{p}$. We prove that $2^\kappa \leq \mathfrak{c}$ (and hence $2^\kappa = \mathfrak{c}$) by producing a family of 2^κ almost disjoint infinite subsets of ω.

Let $T = {}^{\leqslant\kappa}2 = \bigcup\{{}^\xi 2 : \xi \leq \kappa\}$. We may view T as a tree (like the one pictured in the proof of Lemma III.1.16) whose elements are sequences of zeros and ones (see Definition I.7.24).

Using AC plus $\kappa < \mathfrak{p}$ plus recursion on $\mathrm{lh}(\tau)$, obtain $\Gamma : T \to [\omega]^\omega$ so that $\Gamma(\emptyset) = \omega$, $\Gamma(\tau^\frown 0)$ and $\Gamma(\tau^\frown 1)$ are disjoint infinite subsets of $\Gamma(\tau)$, and, when $\mathrm{lh}(\tau) \leq \kappa$ is a limit ordinal, $\Gamma(\tau)$ is some pseudointersection of $\{\Gamma(\tau{\restriction}\xi) : \xi < \mathrm{lh}(\tau)\}$. Now, by induction on $\xi \leq \kappa$, show that the sets $\Gamma(\tau)$ for $\tau \in {}^\xi 2$ are pairwise almost disjoint, so that $|{}^\xi 2| \leq \mathfrak{c}$. Setting $\xi = \kappa$ yields $2^\kappa \leq \mathfrak{c}$.

If $\mathfrak{p} = \mathfrak{c}$, then $\mathfrak{c}$ is regular by König's Theorem I.13.12 ($\mathrm{cf}(2^\kappa) > \kappa$). □

Actually, by a somewhat harder argument due to Szymański, $\mathfrak{p}$ must be regular, regardless of whether or not it equals $\mathfrak{c}$ (see Exercise III.1.38), whereas $\mathfrak{c}$ can be singular (see Chapter IV).

The cardinal $\mathfrak{p}$ also has applications to ultrafilters on ω.

Definition III.1.27 *If A is any non-empty set, then an* ultrafilter *$\mathcal{U}$ on A is a maximal filter on A; i.e., $\mathcal{U}$ is a filter and no proper superset of $\mathcal{U}$ is a filter.*

The dual to an ultrafilter is called a maximal ideal or *prime ideal*.

It is immediate from Zorn's Lemma that

Lemma III.1.28 *If $\mathcal{F}$ is a filter on A, then there is an ultrafilter $\mathcal{U}$ on A such that $\mathcal{F} \subseteq \mathcal{U}$.*

It is also easy to check that

Lemma III.1.29 *If $\mathcal{F}$ is a filter on A, then $\mathcal{F}$ is an ultrafilter iff for all $X, Y \subseteq A$: if $X \cup Y \in \mathcal{F}$ then either $X \in \mathcal{F}$ or $Y \in \mathcal{F}$.*

From this, it is easy to see that every ultrafilter on A is either principal or non-principal:

Definition III.1.30 *Let $\mathcal{U}$ be an ultrafilter on A. Then $\mathcal{U}$ is* principal *iff $\mathcal{U} = \{X \subseteq A : a \in X\}$ for some $a \in A$, and $\mathcal{U}$ is* non-principal *iff every set in $\mathcal{U}$ is infinite.*

If A is finite, then there are $|A|$ ultrafilters on A, all of which are principal. When A are infinite, there are still $|A|$ principal ultrafilters, and the non-principal ultrafilters are the ones that extend the filter of cofinite subsets of A.

These obviously exist by Lemma III.1.28, but in fact there are $2^{2^{|A|}}$ of them by a well-known theorem of Pospíšil [74]; see Exercise III.1.36. Note that ultrafilters are in $\mathcal{P}(\mathcal{P}(A))$, so there are clearly no more than $2^{2^{|A|}}$ of them.

Ultrapowers (or reduced powers using ultrafilters) are an important topic in model theory; see [13]. For uses of ultrafilters in topology, see [30]. Here, we focus on some set-theoretic properties of ultrafilters.

Definition III.1.31 *If $\mathcal{E} \subseteq \mathcal{P}(A)$ has the FIP, then the filter* generated by $\mathcal{E}$ *is the set of all $Y \subseteq A$ such that $Y \supseteq X_1 \cap \cdots \cap X_n$ for some $n \in \omega$ and some $X_1, \ldots, X_n \in \mathcal{E}$.*

Definition III.1.32 *If $\mathcal{U}$ is an ultrafilter on A, then the* character *of $\mathcal{U}$, or $\chi(\mathcal{U})$, is the least size of an $\mathcal{E} \subseteq \mathcal{U}$ that generates $\mathcal{U}$.*

The term "character" is borrowed from topology, where $\chi(x) = \chi(x, X)$ denotes the least size of a local base of the point x in the topological space X. Then, $\chi(\mathcal{U})$ is the topological character of $\mathcal{U}$ when viewed as a point in βA (the Čech compactification of A with the discrete topology, or the space of ultrafilters on A; see [30]).

If $\mathcal{U}$ is principal, then $\chi(\mathcal{U}) = 1$, since $\mathcal{U}$ is generated by some $\mathcal{E} = \{\{a\}\}$. If $\mathcal{U}$ is non-principal, then an easy diagonal argument shows that $\chi(\mathcal{U}) \geq \aleph_1$ Clearly, $\chi(\mathcal{U}) \leq 2^{|A|}$ (let $\mathcal{E} = \mathcal{U}$), and in fact, by Pospíšil [75] (or see Exercise III.1.36), when A is infinite there are $2^{2^{|A|}}$ ultrafilters on A of character $2^{|A|}$.

There are some interesting set-theoretic questions involving ultrafilters of character less than $2^{|A|}$. First, consider countable A:

Lemma III.1.33 *If $\mathcal{U}$ is a non-principal ultrafilter on ω, then $\mathfrak{p} \leq \chi(\mathcal{U}) \leq \mathfrak{c}$.*

Proof. Fix $\mathcal{E} \subseteq \mathcal{U}$ with $|\mathcal{E}| < \mathfrak{p}$. We show that $\mathcal{E}$ cannot generate $\mathcal{U}$.

Since $\mathcal{E}$ has the SFIP, let Z be a pseudo-intersection of $\mathcal{E}$. Then, fix $Y \subseteq Z$ with Y and $Z \backslash Y$ both infinite. Then one of $Y, \omega \backslash Y$ is in $\mathcal{U}$, but $Y \not\supseteq X_1 \cap \cdots \cap X_n$ and $\omega \backslash Y \not\supseteq X_1 \cap \cdots \cap X_n$ whenever $X_1, \ldots, X_n \in \mathcal{E}$. □

Also, $\chi(\mathcal{U}) \geq \max(\mathrm{cov}(\mathsf{meager}), \mathrm{cov}(\mathsf{null}))$ (Lemma V.4.26). It is consistent with $\mathfrak{c}$ being arbitrarily large to have a non-principal $\mathcal{U}$ on ω with $\chi(\mathcal{U}) = \aleph_1$ (see Lemma V.4.27). The corresponding question for ω_1 is still open:

Question III.1.34 *Is it consistent to have an ultrafilter $\mathcal{U}$ on ω_1 such that all sets in $\mathcal{U}$ are uncountable and $\chi(\mathcal{U}) < 2^{\aleph_1}$?*

Note that the question states that $\mathcal{U}$ is *uniform* (equivalently, $\mathcal{U}$ extends the filter of co-countable sets). If $\mathcal{U}$ contains a countable set S, then $\mathcal{U}$ is generated by a family of subsets of S, so that $\chi(\mathcal{U}) \leq \mathfrak{c}$, which is consistently $< 2^{\aleph_1}$.

The next two exercises let us construct ultrafilters of large character in *ZFC*. The first is a result of Hausdorff [45], the next of Pospíšil [75].

Exercise III.1.35 *For an infinite cardinal κ, $\mathcal{E} \subseteq \mathcal{P}(\kappa)$ is called an* independent family *iff for all $m, n \in \omega$ and* distinct *$X_1, \ldots, X_m, Y_1, \ldots, Y_n \in \mathcal{E}$:*

$$|X_1 \cap \cdots \cap X_m \cap (\kappa \backslash Y_1) \cap \ldots \cap (\kappa \backslash Y_n)| = \kappa \quad .$$

Prove that there is an independent $\mathcal{E} \subseteq \mathcal{P}(\kappa)$ of size 2^κ.

Hint. Let $I = \{(s, A) : s \in [\kappa]^{<\omega} \wedge A \subseteq \mathcal{P}(s)\}$. Then $|I| = \kappa$. For $X \subseteq \kappa$, let $E_X = \{(s, A) \in I : X \cap s \in A\}$, and show that $\{E_X : X \in \mathcal{P}(\kappa)\}$ is an independent family of subsets of I. □

Exercise III.1.36 *Let κ be an infinite cardinal. Prove that exactly 2^{2^κ} of the ultrafilters on κ have character 2^κ.*

Hint. For $X \subseteq \kappa$, let $X^0 = X$ and $X^1 = \kappa \backslash X$. Let $\mathcal{E} \subseteq \mathcal{P}(\kappa)$ be an independent family of size 2^κ. For each $f : \mathcal{E} \to 2$, let $\mathcal{U}_f$ be an ultrafilter such that $X^{f(X)} \in \mathcal{U}_f$ for each $X \in \mathcal{E}$ and $\kappa \setminus \bigcap_{n<\omega} X_n^{f(X_n)} \in \mathcal{U}_f$ whenever the X_n are distinct members of $\mathcal{E}$. Then show that $\chi(\mathcal{U}_f) = 2^\kappa$. □

This simple proof is due to Hajnal and Juhász. Pospíšil's proof was purely topological, viewing the ultrafilters as points in the Čech compactification of a discrete space of size κ. His papers [74, 75] (1937, 1939) do not mention independent sets or Hausdorff's [45] (1936).

Exercise III.1.37 *Let A be a non-empty set, and make $\mathcal{P}(A)$ into a ring in the standard way: Product is $\cap$ and sum is symmetric difference. $0 = \emptyset$ and $1 = A$. Prove that $\mathcal{I} \subseteq \mathcal{P}(A)$ is an ideal as per Definition III.1.1 iff $\mathcal{I}$ is an ideal in the sense of ring theory.*

Exercise III.1.38 (Szymański) *Prove that $\mathfrak{p}$ is regular.*

Hint. Say $\mathfrak{p} \geq \aleph_{\omega_1}$. To prove that $\mathfrak{p} > \aleph_{\omega_1}$, fix $\mathcal{E} \subseteq [\omega]^\omega$ of size $\aleph_{\omega_1}$ such that $\mathcal{E}$ has the SFIP, and produce a pseudo-intersection K of $\mathcal{E}$.

List $\mathcal{E}$ as $\{E_\alpha : \alpha < \omega_{\omega_1}\}$. For $\mu < \omega_1$, let A_μ be a pseudo-intersection of $\{E_\alpha : \alpha < \omega_\mu\}$. Note that $\{A_\mu : \mu < \omega_1\}$ need not have the SFIP. Now, using Lemma III.1.23 and recursion on $\mu < \omega_1$, choose B_μ so that $B_\mu \subseteq^* E_\alpha$ for all $\alpha < \omega_\mu$, and $B_\mu \subseteq^* B_\nu$ for all $\nu < \mu$, and $B_\mu \cap A_\pi$ is infinite for all $\pi \geq \mu$.

Then, let K be a pseudo-intersection of $\{B_\mu : \mu < \omega_1\}$. □

The proof in Lemma III.1.25 that $\mathfrak{p} \leq \mathrm{add}(\mathsf{meager})$ actually yields:

Exercise III.1.39 *Let X be any separable metric space, and fix $\kappa < \mathfrak{p}$. Prove that in X, a union of κ first category sets is of first category.*

Note that X need not be complete, so X might be first category in itself, but the result is still correct (although trivial in this case).

III.2 The Countable Chain Condition

This property of topological spaces was studied long before the independence results, but it has become a key concept for Martin's Axiom and forcing, so we discuss it in a separate section.

Definition III.2.1 *A topological space X has the* countable chain condition (ccc) *iff every family of disjoint non-empty open subsets of X is countable.*

The ccc is equivalent to a statement about *chains* of open sets (see Exercise III.2.20) but Definition III.2.1, in terms of disjoint families, turns out to be more useful in practice.

Recall that X is *separable* iff there is a countable $D \subseteq X$ that is dense in X; for example, $\mathbb{R}$ is separable, taking $D = \mathbb{Q}$. If D is dense and the U_α for $\alpha < \omega_1$ are non-empty and open and pairwise disjoint, then the $D \cap U_\alpha$ are non-empty and pairwise disjoint, which is impossible if D is countable, so

Lemma III.2.2 *Every separable space has the ccc.*

Exercise III.2.3 *Every ccc metric space is separable.*

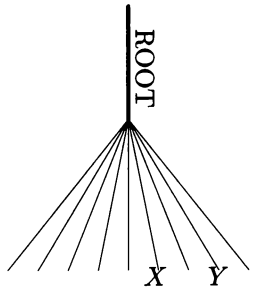
There are simple examples of ccc spaces that are not separable. In particular, consider the spaces 2^κ:

Definition III.2.4 *In topology, 2^κ denotes ${}^\kappa\{0,1\}$, where $2 = \{0,1\}$ is given the discrete topology and 2^κ has the Tychonov product topology.*

So, 2^κ is compact by the Tychonov Theorem. We shall see that 2^κ has the ccc for all κ but is separable iff $\kappa \leq \mathfrak{c}$. First, we prove the *Delta System Lemma* III.2.6, which appears frequently in proofs of ccc.

Definition III.2.5 *A family of sets $\mathcal{A}$ forms a* delta system *with* root *R iff $X \cap Y = R$ whenever $X, Y \in \mathcal{A}$ with $X \neq Y$.*

Note that R could be empty, in which case $\mathcal{A}$ is just a pairwise disjoint family.



Lemma III.2.6 (Delta System Lemma) *Let κ be an uncountable regular cardinal, and let $\mathcal{A}$ be a family of finite sets with $|\mathcal{A}| = \kappa$. Then there is a $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that $\mathcal{B}$ forms a delta system.*

Proof. Since $\mathrm{cf}(\kappa) = \kappa > \omega$ and there are only $\aleph_0$ possible $|X|$ for $X \in \mathcal{A}$, we may fix $n \in \omega$ and $\mathcal{D} \in [\mathcal{A}]^\kappa$ such that $|X| = n$ for all $X \in \mathcal{D}$. Now, we

induct on n. If $n = 1$, then $\mathcal{D}$ is already a delta system with empty root, so assume that $n > 1$. For each object p, let $\mathcal{D}_p = \{X \in \mathcal{D} : p \in X\}$. There are two cases:

Case I: $|\mathcal{D}_p| = \kappa$ for some p. Fix p, and let $\mathcal{E} = \{X \backslash \{p\} : X \in \mathcal{D}_p\}$, which is a family of κ sets of size $n-1$. Applying the lemma inductively, fix $\mathcal{C} \in [\mathcal{E}]^\kappa$ that forms a delta system with some root R. Then $\{Z \cup \{p\} : Z \in \mathcal{C}\} \in [\mathcal{D}]^\kappa$ forms a delta system with root $R \cup \{p\}$.

Case II: $|\mathcal{D}_p| < \kappa$ for all p. Then, for any set S with $|S| < \kappa$, $\{X \in \mathcal{D} : X \cap S \neq \emptyset\} = \bigcup_{p \in S} \mathcal{D}_p$ has size less than κ, since κ is regular (see Theorem I.13.11); so, there is an $X \in \mathcal{D}$ such that $X \cap S = \emptyset$. Then, by recursion on β, we may choose $X_\beta \in \mathcal{D}$ for $\beta < \kappa$ so that for each β, $X_\beta \cap \bigcup_{\alpha<\beta} X_\alpha = \emptyset$. But then $\{X_\beta : \beta < \kappa\}$ is a delta system with empty root. □

Different proofs of this lemma, and generalizations to families of countable sets, or sets of other sizes, will be discussed in Sections III.6 and III.8.

This lemma is false for $\kappa = \omega$, a simple counter-example being $\mathcal{A} = \omega$ (note that $n = \{0, \ldots, n-1\}$). However, *if* all elements of $\mathcal{A}$ have the same finite size, then the rest of the proof is correct, so we do get an infinite delta system. The situation is different for singular κ:

Exercise III.2.7 *Let κ be singular. Show that there is a family $\mathcal{A}$ of κ two-element subsets of κ, such that no $\mathcal{B} \in [\mathcal{A}]^\kappa$ forms a delta system.*

Hint. Let $\theta = \mathrm{cf}(\kappa) < \kappa$. Elements of $\mathcal{A}$ will contain one element below θ and one element above θ. □

One of the first questions to ask about any property of topological spaces is whether it is preserved under products. For example, the Tychonov Theorem says that every product of compact spaces is compact. Also, it is easily seen that countable products of metric spaces are metrizable, whereas this is false for uncountable products (2^{ω_1} being a simple counter-example). A product of $\mathfrak{c}$ or fewer separable spaces is separable (Exercise III.2.13), but this is false when there are more than $\mathfrak{c}$ spaces (Lemma III.2.11). Some of these results are non-trivial, but they are all done within *ZFC*.

For ccc spaces, the answer to this question is independent of *ZFC*. However, the answer for arbitrary products is settled once we know the answer for products of *two* spaces:

Theorem III.2.8 *Let X_i for $i \in I$ be any topological spaces. If the product $\prod_{i \in I} X_i$ is not ccc, then there is a finite $J \subseteq I$ such that $\prod_{i \in J} X_i$ is not ccc.*

Corollary III.2.9 *If every product of any two ccc spaces is ccc, than all products $\prod_{i \in I} X_i$ of ccc spaces are ccc.*

The theorem implies the corollary because we can prove by induction on $n \in \omega$ that the product of any n ccc spaces is ccc.

$MA + \neg CH$ implies that the ccc is preserved under products (see Theorem III.3.43), whereas CH implies that the ccc is not preserved under products (see Theorem III.8.19). The CH example is a bit complicated. A very simple example of a ccc X, called a Suslin line (Definition III.2.16 below), has the property that $X \times X$ is not ccc. The existence of a Suslin line is independent of $ZFC + GCH$, and also independent of $ZFC + \neg CH$.

Proof of Theorem III.2.8. Suppose that V^α, for $\alpha < \omega_1$, are pairwise disjoint non-empty subsets of $\prod_{i\in I} X_i$. Shrinking the V^α, we may assume that each one is a basic open set. Recall that in the standard (Tychonov) topology, basic open sets in the product are determined on a *finite* set of coordinates. So, $V^\alpha = \prod_{i\in I} U_i^\alpha$, where each U_i^α is open in X_i and, for each α, $U_i^\alpha = X_i$ for all but finitely many i. Let S^α be the finite set, $\{i \in I : U_i^\alpha \neq X_i\}$. By the Delta System Lemma, there is an uncountable $B \subseteq \omega_1$ and a finite root $J \subseteq I$ such that $S^\alpha \cap S^\beta = J$ whenever α, β are two different elements of B. Now, $\prod_{i\in I} U_i^\alpha \cap \prod_{i\in I} U_i^\beta = \emptyset$ (for $\alpha, \beta \in B$) implies that there is some $i \in I$ such that $U_i^\alpha \cap U_i^\beta = \emptyset$, and this i must be in J, since otherwise $U_i^\alpha = X_i$ or $U_i^\beta = X_i$. Thus, $\prod_{i\in J} X_i$ is not ccc, with disjoint open sets $\prod_{i\in J} U_i^\alpha$ for $\alpha \in B$. □

This proof illustrates a common change of perspective in delta system arguments. We are using Lemma III.2.6 with $\kappa = \aleph_1$. But Lemma III.2.6 talks about a *set* of $\aleph_1$ finite sets, and here we have an ω_1*-sequence* of finite sets, $\langle S^\alpha : \alpha < \omega_1 \rangle$. The *set* $\{S^\alpha : \alpha < \omega_1\}$ may be countable. The missing step in the argument is the observation that there are really two cases: Either there is an uncountable $A \subseteq \omega_1$ such that the S^α for $\alpha \in A$ are all different, in which case apply Lemma III.2.6 to the set $\{S^\alpha : \alpha \in A\}$ to get B; or there is an uncountable $A \subseteq \omega_1$ such that the S^α for $\alpha \in A$ are all the same set, J, in which case let $B = A$, since $S^\alpha \cap S^\beta = J \cap J = J$ whenever $\alpha, \beta \in A$.

Since a finite product of separable spaces is clearly separable and hence has the ccc, an immediate corollary of the theorem is

Corollary III.2.10 *Let X_i for $i \in I$ be separable topological spaces. Then the product $\prod_{i\in I} X_i$ is ccc.*

Actually, the product of $\mathfrak{c}$ separable spaces is separable (see Exercise III.2.13 below), but the product of more than $\mathfrak{c}$ is not:

Lemma III.2.11 *Let X_i for $i \in I$ be Hausdorff topological spaces, each with more than one point, and assume that $|I| > \mathfrak{c}$. Then the product $\prod_{i\in I} X_i$ is not separable.*

Proof. Choose any $x_n \in \prod_{i\in I} X_i$ for $n \in \omega$. We shall prove that $\{x_n : n \in \omega\}$ is not dense in the product.

For each X_i, choose disjoint non-empty subsets $U_i, V_i \subseteq X_i$. Each x_n is really a function with domain I, and each $x_n(i) \in X_i$. Define $g_n : I \to \{0, 1\}$

so that $g_n(i)$ is 0 if $x_n(i) \in U_i$ and 1 if $x_n(i) \in V_i$; if $x_n(i) \notin U_i \cup V_i$, let $g_n(i) = 0$, but this value won't matter.

For each $i \in I$, the ω–sequence $\langle g_n(i) : n \in \omega \rangle$ is in 2^ω, and there are only $\mathfrak{c}$ possibilities for what this ω–sequence can be, so, since $|I| > \mathfrak{c}$, we can fix $j \neq k$ in I that both have the same ω–sequence; that is, $g_n(j) = g_n(k)$ for each n. Now let $W = \{y \in \prod_{i \in I} X_i : y(j) \in U_j \ \wedge \ y(k) \in V_k\}$. Then W is open and non-empty, and contains no x_n, since $x_n \in W$ would imply that $g_n(j) = 0$ and $g_n(k) = 1$. So, $\{x_n : n \in \omega\}$ is not dense. □

To prove that the product of $\mathfrak{c}$ separable spaces is separable, first do:

Exercise III.2.12 *$\mathcal{F} \subseteq \omega^\omega$ is called an* independent family of functions *iff for all $m \in \omega$ and $j_1, \ldots, j_m \in \omega$ and* distinct *$f_1, \ldots, f_m \in \mathcal{F}$:*

$$|\{e \in \omega : f_1(e) = j_1 \wedge \ldots \wedge f_m(e) = j_m\}| = \aleph_0 \ .$$

Prove that there is an independent $\mathcal{F} \subseteq \omega^\omega$ of size $\mathfrak{c}$.

Hint. Let $E = \{(s,p) : s \in [\omega]^{<\omega} \ \& \ p : \mathcal{P}(s) \to \omega\}$. First get an independent $\mathcal{F} \subseteq \omega^E$. Let $\mathcal{F} = \{f_A : A \subseteq \omega\}$, where $f_A(s,p) = p(A \cap s)$. □

Exercise III.2.13 *Let X_i for $i \in I$ be separable topological spaces, and assume that $|I| \leq \mathfrak{c}$. Then the product $\prod_{i \in I} X_i$ is separable.*

Hint. For each $i \in I$, choose a countable $D_i \subseteq X_i$ that is dense in X_i, and let $D_i = \{d^i_j : j \in \omega\}$. Let $\{f_i : i \in I\} \subseteq \omega^\omega$ be an independent family of functions. Then $D = \{x_e : e \in \omega\}$ is dense in $\prod_{i \in I} X_i$, where $x_e(i) = d^i_{f_i(e)}$. □

Although the spaces 2^κ, for $\kappa > \mathfrak{c}$, provide *ZFC* examples of compact ccc non-separable spaces, the question of the existence of an *ordered* example is a well-known question asked by Suslin [95] in 1920, and the answer turns out to be independent of *ZFC*. His [95] consists of just the following sentence:

Question III.2.14 (Suslin) *Is it true that every dense total order without endpoints that is both ccc and connected in its order topology must be isomorphic to $(\mathbb{R}, <)$?*

We have translated [95] into modern terminology (and also into English), but he just asked the question, without any discussion. However, for some reason, *Suslin's Hypothesis*, or *SH*, has come to mean the assertion that the answer is "yes", while a *Suslin line* is a counter-example (which exists only if *SH* is false). We explain some of the terminology below, and point out that a Suslin line X is a ccc non-separable space, and also that $X \times X$ is not ccc.

Definition III.2.15 *If $(X, <)$ is a totally ordered set, then the* order topology *on X is the topology that has as a base the set of all intervals of the forms (x,y) and (x,∞) and $(-\infty,y)$ for $x,y \in X$. A* LOTS (Linearly Ordered Topological Space) *is a totally ordered set $(X, <)$ given its order topology.*

As usual, $\pm\infty$ are not members of X; $(-\infty, y)$ is a common shorthand for $\{z \in X : z < y\}$. Also, "linear order" is a common synonym for "total order". The order is called *dense* iff $|X| > 1$ and $(x, y) \neq \emptyset$ whenever $x < y$, and *without endpoints* iff $(x, \infty) \neq \emptyset$ and $(-\infty, y) \neq \emptyset$ for all $x, y \in X$.

Definition III.2.16 *A* Suslin line *is a LOTS that is ccc and not separable. SH* (Suslin's Hypothesis) *is the assertion that there are no Suslin lines.*

This definition focuses on the key set-theoretic properties (ccc plus non-separable), and omits some properties mentioned in Suslin's question (connected plus dense total order). This difference is explained by Exercises III.5.33 and III.2.17 below. Exercise III.5.33 says that if you have a Suslin line, then you can get another one that is connected and a dense total order. Exercise III.2.17, which was certainly known to Suslin, says that his question would have answer "yes" if you could prove that the order is separable. Thus, *SH* is equivalent to a "yes" answer to Suslin's original question.

Exercise III.2.17 *Let $(X, <)$ be a LOTS that is a dense total order without endpoints, and that is* separable *and connected in its order topology. Then $(X, <)$ is isomorphic to $(\mathbb{R}, <)$.*

Hint. Let $D \subseteq X$ be countable, and dense in the topological sense (i.e., $\overline{D} = X$). Then D as ordered by $<$ is dense (in the order sense) and without endpoints. By a well-known result of Cantor, $(D, <)$ is isomorphic to $(\mathbb{Q}, <)$ (see, e.g., [57], Exercise II.13.10). If $f : \mathbb{Q} \xrightarrow[\text{onto}]{1-1} D$ is an isomorphism, then show that f extends to an isomorphism from $\mathbb{R}$ onto X. □

Also, $\neg SH$ implies that products of ccc spaces are not always ccc:

Lemma III.2.18 *If $(X, <)$ is any Suslin line, then $X \times X$ is not ccc.*

Proof. We shall choose $a_\alpha, b_\alpha, c_\alpha \in X$ for $\alpha < \omega_1$ satisfying:

1. $a_\alpha < b_\alpha < c_\alpha$.
2. $(a_\alpha, b_\alpha) \neq \emptyset$ and $(b_\alpha, c_\alpha) \neq \emptyset$.
3. $\xi < \alpha \to b_\xi \notin (a_\alpha, c_\alpha)$.

Assuming that this can be done, the open sets $(a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha) \subseteq X \times X$ are non-empty by (2), and are pairwise disjoint, since whenever $\xi < \alpha$, (3) implies that either $b_\xi \leq a_\alpha$, so that $(a_\xi, b_\xi) \cap (a_\alpha, b_\alpha) = \emptyset$, or $b_\xi \geq c_\alpha$, so that $(b_\xi, c_\xi) \cap (b_\alpha, c_\alpha) = \emptyset$.

We choose the $a_\alpha, b_\alpha, c_\alpha$ by recursion on $\alpha < \omega_1$. But first, let I be the set of all isolated points; these are the $x \in X$ such that $\{x\}$ is open. Then I is countable by the ccc. Now, if we have chosen a_ξ, b_ξ, c_ξ for $\xi < \alpha$, then choose a_α, c_α such that $a_\alpha < c_\alpha$ and $(a_\alpha, c_\alpha) \neq \emptyset$ and $(a_\alpha, c_\alpha) \cap (I \cup \{b_\xi : \xi < \alpha\}) = \emptyset$; this is possible because the countable set $I \cup \{b_\xi : \xi < \alpha\}$ is not dense in

X. Then, (a_α, c_α) is non-empty and contains no isolated points, so it must be infinite, so choose $b_\alpha \in (a_\alpha, c_\alpha)$ so that $(a_\alpha, b_\alpha) \neq \emptyset$ and $(b_\alpha, c_\alpha) \neq \emptyset$. □

A LOTS $(X, <)$ is compact iff it is *Dedekind complete*; that is, every non-empty $S \subseteq X$ has both a greatest lower bound $\inf(S)$ and a least upper bound $\sup(S)$. In particular, X has a smallest element $\inf(X)$ and a largest element $\sup(X)$. To get a compact example of a ccc space whose square is not ccc (assuming $\neg SH$), use Lemma III.2.18 and

Exercise III.2.19 *If X is a Suslin line, then the Dedekind completion of X is a compact Suslin line.*

A *chain* of sets is a family of sets that is totally ordered by $\subseteq$. The countable chain condition is equivalent to a statement about well-ordered chains:

Exercise III.2.20 *Show that a space X has the ccc iff there is no family of open sets $\{U_\alpha : \alpha < \omega_1\}$ such that for all $\alpha < \beta$: $U_\alpha \subseteq U_\beta$ and $U_\beta \backslash \overline{U_\alpha} \neq \emptyset$ (U_α is not dense in U_β). Then show that although the space 2^{ω_1} is ccc, it does contain a family of open sets $\{U_\alpha : \alpha < \omega_1\}$ such that $U_\alpha \subsetneq U_\beta$ for all $\alpha < \beta$.*

Exercise III.2.21 *The space 2^ω is homeomorphic to the middle third Cantor set.*

Hint. Map $(1, 1, 0, 1, 0, 0, 1, 1, \ldots) \in 2^\omega$ to the number $0.22020022\cdots$ (base 3) in the Cantor set. □

Exercise III.2.22 *We say that κ is a* caliber *for a topological space X iff whenever $U_\alpha \subseteq X$ are open and non-empty for $\alpha < \kappa$, there is a $B \in [\kappa]^\kappa$ such that $\bigcap_{\alpha \in B} U_\alpha \neq \emptyset$. Now, let $X = \prod_{i \in I} X_i$, where each X_i is separable. Prove that $\aleph_1$ is a caliber for X.*

Hint. Modify the proof of Theorem III.2.8. □

The notion of caliber and the Delta System Lemma are due to Šanin (1946). One might ask whether $\aleph_1$ is a caliber for every compact ccc Hausdorff space. This is true under $MA + \neg CH$ and false under $\neg SH$; see Exercise III.3.45. If you delete "compact", there is an easy *ZFC* counter-example:

Exercise III.2.23 *Let X be the set of all $x \in 2^{\omega_1}$ such that $x(\alpha) = 0$ for all but finitely many α. Show that X is ccc, but that $\aleph_1$ is not a caliber for X.*

III.3 Martin's Axiom

In the 1960s, Martin, Solovay, Tennenbaum, and others greatly extended Cohen's method of forcing and proved many statements to be independent of $ZFC + \neg CH$. In particular, various values between $\aleph_1$ and $\mathfrak{c}$ for the cardinals

mentioned in Section III.1 are consistent. But then Martin and a few other people realized that one could encapsulate the results of some of these proofs into one axiom, now called Martin's Axiom or *MA*. Martin's Axiom is consistent with $\mathfrak{c}$ being arbitrarily large, and it implies that all the cardinals mentioned in Section III.1 are equal to $\mathfrak{c}$. Of course, *CH* implies that all these cardinals equal $\mathfrak{c}$, but also *CH* implies *MA*.

One does not need to know anything about forcing or logic to understand *MA*, and indeed *MA* would have been understandable in the 1920s, and the proof (Corollary III.3.44) that $MA + \neg CH \rightarrow SH$ would have been clear to Suslin, although without knowing forcing, it is a little hard to explain why $MA + \neg CH$ is a "natural" axiom. We shall define *MA* here and prove some basic results about *MA* and the cardinals mentioned in Section III.1. Some applications of *MA* to the study of trees will be discussed in Section III.5. The consistency of $ZFC + MA + \neg CH$ will be proved in Chapter V

Modern expositions of forcing usually cover *MA* first because *MA* introduces many of the combinatorial complexities found in forcing. Then, when discussing forcing, we can concentrate on the complexities involved in building models for set theory. We take this approach here. Actually, this section may tell you more about *MA* than you want to know; the material through Lemma III.3.26 is sufficient for understanding the basics of forcing; but also this section is just an introduction to the extensive literature on *MA*; see Fremlin's book [35] for more on the subject.

We shall explain *MA* as we would to Suslin, as a statement about ccc partial orders, but we shall use the modern terminology, which evolved from forcing:

Definition III.3.1 *A* forcing poset *is a triple,* $(\mathbb{P}, \leq, \mathbb{1})$ *such that* $\leq$ *is a pre-order on* $\mathbb{P}$ *and* $\mathbb{1} \in \mathbb{P}$ *is a largest element* $(\forall p \in \mathbb{P}\ p \leq \mathbb{1})$.

Notation III.3.2 *Elements of* $\mathbb{P}$ *are called* forcing conditions. $p \leq q$ *is read "*p extends q*".*

Recall (Definition I.6.2) that a *pre-order* is transitive and reflexive; it is a *partial order* iff it also satisfies $\neg\exists p, q \in \mathbb{P}\,[p \leq q \wedge q \leq p \wedge p \neq q]$. Most, but not all, of our examples will in fact be partial orders. The $\mathbb{1}$ is rarely used in *MA* applications, and in fact it could be omitted (Exercise III.3.75), but it will be useful in forcing, so we might as well get used to it. As usual, we shall abuse notation by referring to "the forcing poset $\mathbb{P}$" if $\leq$ and $\mathbb{1}$ are clear from context.

Definition III.3.3 *Let* $\mathbb{P}$ *be a forcing poset. Then* $p, q \in \mathbb{P}$ *are* compatible $(p \not\perp q)$ *iff they have a* common extension*; that is, there is an* $r \in \mathbb{P}$ *such that* $r \leq p$ *and* $r \leq q$*;* p, q *are* incompatible $(p \perp q)$ *iff they are not compatible. An* antichain *is a subset* $A \subseteq \mathbb{P}$ *whose elements are pairwise incompatible.* $\mathbb{P}$ *has the* countable chain condition (ccc) *iff, in* $\mathbb{P}$*, every antichain is countable.*

The following two examples illustrate the terminology in Definitions III.3.1 and III.3.3:

Definition III.3.4 *Let X be a non-empty topological space. Then $\mathbb{O}_X$ is the set of all non-empty open subsets of X; let $\leq$ be $\subseteq$, and let $\mathbb{1} = X$.*

Then $(\mathbb{O}_X, \leq, \mathbb{1})$ is a forcing poset and $\leq$ is a partial order. Since $p \perp q$ iff $p \cap q = \emptyset$, an antichain is a pairwise disjoint family, and $\mathbb{O}_X$ has the ccc iff the space X has the ccc as per Definition III.2.1

Example III.3.5 Let $\mathbb{P} = [\omega]^\omega$, and let $\leq$ be $\subseteq^*$. This is not a partial order, since $p \subseteq^* q \wedge q \subseteq^* p$ holds whenever $p =^* q$. The natural $\mathbb{1}$ is ω, but any cofinite subset of ω is also a largest element. Here, $p \perp q$ iff $p \cap q$ is finite; that is p, q are almost disjoint in the sense of Definition III.1.15, where we also used "$p \perp q$". $\mathbb{P}$ is not ccc, since it has an antichain of size $2^{\aleph_0}$ by Lemma III.1.16.

These two example are similar in that the elements of $\mathbb{P}$ are sets, and whenever $p \not\perp q$, they have a largest common extension, $p \cap q$ (that is $r \leq p \cap q$ whenever $r \leq p$ and $r \leq q$). This is not true in general. For a simple and somewhat artificial counter-example, let $\mathbb{P}$ be the set of all $p \subseteq 7 = \{0, 1, \ldots, 6\}$ such that $|p|$ is odd, and let $\leq$ be $\subseteq$, so $\mathbb{1} = 7$. Then $\{0, 1, 2, 3, 4\}$ and $\{0, 1, 2, 3, 5\}$ are compatible and they have eight common extensions, among which are $\{0, 1, 3\}$, $\{1, 2, 3\}$, and $\{2\}$, but there is no largest common extension.

The terminology in Notation III.3.2 arises from the fact that in may applications, we are trying to construct something, and the elements of $\mathbb{P}$ are approximations to what we are trying to construct. Consider

Definition III.3.6 *For any I, J:* $\mathrm{Fn}(I, J)$ *is the set of all* finite partial functions from I to J; *that is $p \in \mathrm{Fn}(I, J)$ iff $p \in [I \times J]^{<\omega}$ and p is the graph of a function. We make* $\mathrm{Fn}(I, J)$ *into a forcing poset by letting $\leq$ be $\supseteq$ and $\mathbb{1} = \emptyset$.*

Now, $p \leq q$ iff $p \supseteq q$ iff p *extends* q as a function. Also, p, q are compatible iff there is a function $r \in \mathrm{Fn}(I, J)$ such that $r \supseteq p$ and $r \supseteq q$; equivalently, if $p \cup q \in \mathrm{Fn}(I, J)$, in which case $p \cup q$ is a common extension of p and q. So, $p \not\perp q$ iff p, q are compatible as functions; that is, they agree on $\mathrm{dom}(p) \cap \mathrm{dom}(q)$.

We can think of $p \in \mathrm{Fn}(I, J)$ as a finite approximation to a function $f : I \to J$ that we are trying to construct. Informally, p expresses a *condition* that our f needs to satisfy, namely, p says "$f \supseteq p$". For a concrete example of this poset being used with *MA*, see Lemma III.3.13 below. Of course, first we need to define *MA*, which we shall do soon. But first, we point out that the ccc for $\mathrm{Fn}(I, J)$ is not completely trivial:

Lemma III.3.7 $\mathrm{Fn}(I, J)$ *has the ccc iff $I = \emptyset$ or J is countable.*

Proof. Note that if I or J is empty, then $\mathrm{Fn}(I, J) = \{\emptyset\}$, which is trivially ccc. From now on, assume that I and J are non-empty.

For $\rightarrow$: If J is uncountable, fix any $i \in I$. Then the singleton functions $\{(i,j)\}$ for $j \in J$ form an uncountable antichain.

For $\leftarrow$: Assume that J is countable. Fix any $p_\alpha \in \mathbb{P}$ for $\alpha < \omega_1$. We prove that they cannot be pairwise incompatible. Let $S_\alpha = \mathrm{dom}(p_\alpha) \in [I]^{<\omega}$. By the Delta System Lemma, there is an uncountable $B \subseteq \omega_1$ and a finite root $R \subseteq I$ such that $S_\alpha \cap S_\beta = R$ whenever α, β are two different elements of B. Now, J^R is countable, so we may fix $\alpha, \beta \in B$ with $\alpha \neq \beta$ and $p_\alpha \restriction R = p_\beta \restriction R$. Then the functions p_α, p_β agree on $R = \mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta)$, so $p_\alpha \not\perp p_\beta$. $\square$

This delta system argument is very similar to the proof of Theorem III.2.8. In fact, another proof of Lemma III.3.7 is: Give the countable set J the discrete topology. Then the space J^I is ccc by Theorem III.2.8 or its Corollary III.2.10. Each $p \in \mathrm{Fn}(I,J)$ defines the basic open set $N_p = \{f \in J^I : f \supseteq p\}$ and $p \perp q$ iff $N_p \cap N_q = \emptyset$, so the ccc for J^I implies the ccc for $\mathrm{Fn}(I,J)$.

To define MA, we first need to define "dense set" and "filter".

Definition III.3.8 *Let $\mathbb{P}$ be a forcing poset. Then $D \subseteq \mathbb{P}$ is* dense *in $\mathbb{P}$ iff* $\forall p \in \mathbb{P}\ \exists q \in D\ q \leq p$.

An example using $\mathbb{O}_X$ from Definition III.3.4: Let U be any open subset of X, and let $D = \{q \in \mathbb{O}_X : q \subseteq U\}$. Then D is dense in the sense of Definition III.3.8 iff U is dense in the topological sense.

Example III.3.9 *Assume that I is infinite and J is non-empty. Then sets of the form $\{q \in \mathrm{Fn}(I,J) : i \in \mathrm{dom}(q)\}$ and $\{q \in \mathrm{Fn}(I,J) : j \in \mathrm{ran}(q)\}$ are dense whenever $i \in I$ and $j \in J$.*

That is, any finite function p may be extended to a larger finite function q with j in its range (since $\mathrm{dom}(p) \subsetneq I$) or with i in its domain.

Definition III.3.10 *Let $\mathbb{P}$ be a forcing poset. Then $G \subseteq \mathbb{P}$ is a* filter *on $\mathbb{P}$ iff*

1. $\mathbb{1} \in G$.
2. $\forall p, q \in G\ \exists r \in G\ [r \leq p \wedge r \leq q]$.
3. $\forall p, q \in \mathbb{P}\ [p \geq q \wedge q \in G \rightarrow p \in G]$ (*G is* upward closed).

As a simple example, let A be any non-empty set, and let $\mathbb{P} = \mathcal{P}(A)\backslash\{\emptyset\}$, where $\leq$ is $\subseteq$ and $\mathbb{1} = A$. Then G is a filter in the sense of Definition III.3.10 iff G is a filter in the sense of Definition III.1.3.

Definition III.3.11 *$MA_{\mathbb{P}}(\kappa)$ is the statement that whenever $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, there exists a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.*

$MA(\kappa)$ is the statement that $MA_{\mathbb{P}}(\kappa)$ holds for all ccc posets $\mathbb{P}$.

MA is the statement $\forall \kappa < \mathfrak{c}\ MA(\kappa)$.

It is obvious from the "$|\mathcal{D}| \leq \kappa$" in this definition that

Lemma III.3.12 *When $\lambda \leq \kappa$: $MA_{\mathbb{P}}(\kappa) \to MA_{\mathbb{P}}(\lambda)$ and $MA(\kappa) \to MA(\lambda)$.*

The proof of the next lemma illustrates the use of filters and dense sets in $\mathrm{Fn}(I, J)$, along with MA, to construct a function $f : I \to J$ satisfying some desired properties. Since this proof is intended as an illustration, it is a little more verbose than it needs to be.

Lemma III.3.13 $MA(\kappa) \to \kappa < \mathfrak{c}$.

Proof. Let $\mathbb{P} = \mathrm{Fn}(I, J)$, where I is infinite and J is non-empty.

If G is any filter on $\mathbb{P}$, then in particular, any two functions $p, q \in G$ agree on $\mathrm{dom}(p) \cap \mathrm{dom}(q)$, so the union of all these functions, $f_G := \bigcup G$, is a function, with $\mathrm{dom}(f_G) \subseteq I$ and $\mathrm{ran}(f_G) \subseteq J$.

To force f_G to have some "desired" properties, we make G meet some dense sets. Following Example III.3.9, let $D_i = \{q \in \mathrm{Fn}(I, J) : i \in \mathrm{dom}(q)\}$ and $R_j = \{q \in \mathrm{Fn}(I, J) : j \in \mathrm{ran}(q)\}$, for $i \in I$ and $j \in J$. If $G \cap D_i \neq \emptyset$ for all $i \in I$ then $\mathrm{dom}(f_G) = I$, so $f_G : I \to J$. If in addition $G \cap R_j \neq \emptyset$ for all $j \in J$, then $\mathrm{ran}(f_G) = J$, so $f_G : I \xrightarrow{\text{onto}} J$. This would be a contradiction if $|I| = \aleph_0$ and $|J| = \aleph_1$, but we do not get this contradiction in MA applications because, by Lemma III.3.7, $\mathbb{P}$ is not ccc unless J is countable.

For a third kind of dense set, for each $h \in J^I$, let $E_h = \{q \in \mathbb{P} : q \not\subseteq h\}$. If G meets E_h, as well as all the D_i, then $f_G : I \to J$ and $f_G \neq h$, since at least one $q \in G$ differs from h somewhere. Also, E_h is dense whenever $|J| \geq 2$. To see this: given any $p \in \mathrm{Fn}(I, J)$, we can choose an $i \in I \backslash \mathrm{dom}(p)$ and then choose $j \in J \backslash \{h(i)\}$. Then $q := p \cup \{(i, j)\}$ is an extension of p and $q \not\subseteq h$.

Now assume that $MA(\kappa)$ holds, where $\kappa \geq \mathfrak{c}$. Let $|I| = \aleph_0$ and $2 \leq |J| \leq \aleph_0$. Then $\mathbb{P}$ is ccc by Lemma III.3.7 (or, trivially, since $\mathbb{P}$ is countable). Applying $MA(\kappa)$, let G meet all the $\aleph_0$ dense sets D_i for $i \in I$, plus all the $\mathfrak{c}$ dense sets E_h for $h \in J^I$. Then $f_G : I \to J$ and $f_G \neq h$ for all $h \in J^I$, a contradiction. □

But $MA(\aleph_0)$ is true by the next lemma, which actually says a bit more:

Lemma III.3.14 (Generic Filter Existence Lemma) *Let $\mathbb{P}$ be any forcing poset, let $\mathcal{D}$ be a countable family of dense subsets of $\mathbb{P}$, and fix any $p \in \mathbb{P}$. Then there exists a filter G on $\mathbb{P}$ such that $p \in G$ and $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.*

Proof. List $\mathcal{D}$ as $\{D_n : n \in \omega\}$. Using AC and recursion on ω, choose $r_n \in \mathbb{P}$ for $n \in \omega$ so that $r_0 = p$ and $r_{n+1} \leq r_n$ and $r_{n+1} \in D_n$; there is such an r_{n+1} because D_n is dense. Let $G = \{q \in \mathbb{P} : \exists n\, [r_n \leq q]\}$. Then G is a filter, $p \in G$, and each $r_{n+1} \in G$, so $G \cap D_n \neq \emptyset$. □

In particular, $CH \to MA$. The reason for the name of this lemma will be explained in Chapter IV.

The lemma gives us $MA_{\mathbb{P}}(\aleph_0)$ for *all* $\mathbb{P}$, but $MA(\aleph_1)$ would be become inconsistent if "ccc" were dropped:

Lemma III.3.15 *There is a non-ccc $\mathbb{P}$ such that $MA_{\mathbb{P}}(\aleph_1)$ is false.*

Proof. Follow the proof of Lemma III.3.13. Let $\mathbb{P} = \mathrm{Fn}(I, J)$, where $|I| = \aleph_0$ and $|J| = \aleph_1$, and let $\mathcal{D} = \{D_i : i \in I\} \cup \{R_j : j \in J\}$. As pointed out in that proof, if G meets all the sets in $\mathcal{D}$, then $\bigcup G : I \xrightarrow{\text{onto}} J$. □

Of course, this $\mathbb{P}$ is just one particular non-ccc poset. The Proper Forcing Axiom, *PFA*, is a consistent strengthening of $MA(\aleph_1)$ obtained by weakening the "ccc" in Definition III.3.11 to "proper". *PFA* settles some questions that are independent of $MA(\aleph_1)$. See Section V.7 for more on *PFA* and references to the literature.

Definition III.3.16 $\mathfrak{m}$ *is the least cardinal* κ *such that* $\neg MA(\kappa)$.

By what we have just done, it is clear that

Lemma III.3.17 $\aleph_1 \leq \mathfrak{m} \leq \mathfrak{c}$. *MA holds iff* $\mathfrak{m} = \mathfrak{c}$.

So, $\mathfrak{m}$ adds to the list of cardinals discussed in Section III.1 which are, in *ZFC*, somewhere between $\aleph_1$ and $2^{\aleph_0}$. We shall soon see that $\mathfrak{m} \leq$ all of those cardinals, so that *MA* implies that they all equal $\mathfrak{c}$. Then, in Chapter V, we shall see that *MA* is consistent with $\mathfrak{c}$ being arbitrarily large. But, *MA* does put some restrictions on what $\mathfrak{c}$ can be, since just from $\mathfrak{p} = \mathfrak{c}$, we get that $\mathfrak{c}$ is regular and $2^\kappa = \mathfrak{c}$ for all infinite $\kappa < \mathfrak{c}$ (see Lemma III.1.26), whereas it is consistent with *ZFC* to have $\mathfrak{c} = \aleph_{\omega_1}$ (which is singular), and $2^{\aleph_1}$ can be arbitrarily larger than $\mathfrak{c}$ (see Sections IV.3 and IV.7).

Next, a simple application of *MA* to compact spaces:

Lemma III.3.18 *Let X be any ccc compact Hausdorff space. Assume $MA(\kappa)$, and let $H_\alpha \subseteq X$, for $\alpha < \kappa$ be closed and nowhere dense. Then $\bigcup_{\alpha<\kappa} H_\alpha \neq X$.*

Proof. The poset $\mathbb{O}_X$ (see Definition III.3.4) is ccc because X is ccc. If G is any filter on $\mathbb{O}_X$, then $p, q \in G$ implies $p \cap q \in G$ because G contains an r such that $r \subseteq p \wedge r \subseteq q$ and then $r \subseteq p \cap q$ (see (2)(3) in Definition III.3.10). In particular G has the FIP, so by compactness, $F_G := \bigcap\{\overline{p} : p \in G\}$ is non-empty.

Let $D_\alpha = \{q \in \mathbb{O}_X : \overline{q} \cap H_\alpha = \emptyset\}$. Each D_α is dense because for every p, $p \backslash H_\alpha$ is open and non-empty, so, since compact Hausdorff spaces are regular, there is a non-empty open q such that $\overline{q} \subseteq p \backslash H_\alpha$. Then $q \in D_\alpha$ and $q \leq p$.

Note that $G \cap D_\alpha \neq \emptyset$ implies $F_G \cap H_\alpha = \emptyset$. So, apply $MA(\kappa)$ to choose G such that $G \cap D_\alpha \neq \emptyset$ for all α. □

This consequence of $MA(\kappa)$ is actually equivalent to $MA(\kappa)$ (Theorem III.4.7).

We may view our forcing conditions (elements of $\mathbb{P}$) as approximations to what we are trying to construct. Here, we are trying to construct a non-empty set F disjoint from all the H_α, and $p \in \mathbb{P}$ yields an outer approximation to F (p says "$F \subseteq \overline{p}$"). Also $q \leq p$ (q *extends* p, which here is $q \subseteq p$) means that q *says more*, or yields a better approximation.

For $\kappa = \aleph_0$: $MA_{\mathbb{P}}(\aleph_0)$ is true in *ZFC* for all $\mathbb{P}$ (see Lemma III.3.14), so Lemma III.3.18 holds for all compact Hausdorff spaces, not just ccc spaces. But this is just the standard Baire Category Theorem. In fact, the proof of Lemma III.3.14 is similar to the usual proof of the Baire Category Theorem, which involves choosing a decreasing ω–sequence of open sets.

Lemma III.3.18 is trivial when X contains an isolated point x; that is, X is not a union of κ nowhere dense sets because a nowhere dense set cannot contain x. This is echoed by something trivial about the poset $\mathbb{O}_X$ — namely, that it contains an atom. An atom is something that can't be split:

Definition III.3.19 *$r \in \mathbb{P}$ is an* atom *of the poset $\mathbb{P}$ iff there are no $p, q \leq r$ such that $p \perp q$. The poset $\mathbb{P}$ is* atomless *iff there are no atoms in $\mathbb{P}$.*

For example, if X is any Hausdorff space, then $r \in \mathbb{O}(X)$ is an atom iff $|r| = 1$; that is, $r = \{x\}$ for some isolated point x.

Lemma III.3.20 *If the poset $\mathbb{P}$ has an atom, then $MA_{\mathbb{P}}(\kappa)$ holds for all κ. If $\mathbb{P}$ is atomless, then $MA_{\mathbb{P}}(\kappa)$ is false for $\kappa = 2^{|\mathbb{P}|}$.*

Proof. If r is an atom and $G = \{q \in \mathbb{P} : q \not\perp r\}$, then G is a filter and G meets all dense sets. If $\mathbb{P}$ is atomless and G is a filter, then $D := \mathbb{P} \backslash G$ is dense, since whenever $r \in \mathbb{P}$ there are $p, q \leq r$ such that $p \perp q$, and at least one of these must be in D. Thus, G cannot meet all dense sets □

Actually, $MA_{\mathbb{P}}(\mathfrak{c})$ is false if $\mathbb{P}$ is both atomless and ccc; see Exercise III.3.90.

If $\mathbb{P}$ is atomless, then, by Definition III.3.19, below any $r \in \mathbb{P}$, there are two incompatible elements, but in fact there are $\aleph_0$:

Lemma III.3.21 *Assume that $\mathbb{P}$ is atomless. Then for each $r \in \mathbb{P}$, there is an infinite antichain $A \subseteq \{p : p \leq r\}$.*

Proof. Let $r_0 = r$. Given r_n: Since r_n is not an atom, choose $r_{n+1}, p_n \leq r_n$ such that $r_{n+1} \perp p_n$. Let $A = \{p_n : n \in \omega\}$. □

We mentioned above that $\mathfrak{m} \leq$ all the cardinals defined in Section III.1. In view of Lemmas III.1.24 and III.1.25, this follows from $\mathfrak{m} \leq \mathsf{add}(\mathsf{null})$ (Lemma III.3.28) and $\mathfrak{m} \leq \mathfrak{p}$:

Lemma III.3.22 $\mathfrak{m} \leq \mathfrak{p}$.

Proof. As usual, we prove $\kappa < \mathfrak{m} \to \kappa < \mathfrak{p}$. So, fix an infinite κ, assume that $MA(\kappa)$ holds, fix a family $\mathcal{E} \subseteq [\omega]^\omega$ such that $\mathcal{E}$ has the SFIP and $|\mathcal{E}| = \kappa$; then we must produce an infinite $K \subseteq \omega$ such that K is a pseudo-intersection of $\mathcal{E}$ — that is, $K \subseteq^* Z$ for all $Z \in \mathcal{E}$.

Define $\mathbb{P}$ to be the set of all pairs $p = (s_p, \mathcal{W}_p)$ such that $s_p \in [\omega]^{<\omega}$ and $\mathcal{W}_p \in [\mathcal{E}]^{<\omega}$. As usual, we view p as saying something about K. The s_p is a finite inner approximation; that is, p says "$s_p \subseteq K$". The $\mathcal{W}_p$ is a *promise* that for all $Z \in \mathcal{W}_p$, $K \backslash s_p \subseteq Z$, which will guarantee that $K \subseteq^* Z$. Now, $q \leq p$ should mean that q says more than p, so we define $q \leq p$ to hold iff

a. $s_q \supseteq s_p$ (q gives a better approximation to K).
b. $\mathcal{W}_q \supseteq \mathcal{W}_p$ (q promises more).
c. $\forall Z \in \mathcal{W}_p\, [(s_q \backslash s_p) \subseteq Z]$ (q does not break the promise made by p).

It is easy to verify directly from (a)(b)(c) that $\leq$ is a partial order on $\mathbb{P}$ with largest element $\mathbb{1} = (\emptyset, \emptyset)$. Informally, the promise "$K \backslash s_p \subseteq Z$" means that if we expand s_p to a larger s_q, then $s_q \backslash s_p$ should contain only elements of Z, since all elements of s_q will get put into K; this promise is made formal by (c).

Observe that (c) is vacuous if $s_q = s_p$, so that

$$s_p = s_r \quad \rightarrow \quad p \not\perp r \tag{$\star$}$$

because then $(s_p, \mathcal{W}_p \cup \mathcal{W}_r)$ extends both p and r. It follows that $\mathbb{P}$ is ccc, since if $A \subseteq \mathbb{P}$ is an antichain, then the s_p for $p \in A$ are all different, so A is countable because $[\omega]^{<\omega}$ is countable.

If G is a filter on $\mathbb{P}$, we put our finite inner approximations together to define $K_G = \bigcup\{s_p : p \in G\} \subseteq \omega$. K_G might be finite, but if G intersects all the dense sets $D_n = \{p \in \mathbb{P} : |s_p| \geq n\}$, for each $n \in \omega$, then K_G is infinite. To prove that D_n is really dense, fix any $p = (s_p, \mathcal{W}_p) \in \mathbb{P}$. Then $\bigcap \mathcal{W}_p$ is infinite by the SFIP, and if we fix $t \in \left[\bigcap \mathcal{W}_p\right]^n$, then $q := (s_p \cup t,\ \mathcal{W}_p) \leq p$ and $q \in D_n$.

For $Z \in \mathcal{E}$, let $D'_Z = \{p \in \mathbb{P} : Z \in \mathcal{W}_p\}$. Then D'_Z is dense because $(s_p, \mathcal{W}_p \cup \{Z\}) \leq p$. If G meets D'_Z, then $K_G \subseteq^* Z$. To prove this, fix $r \in G \cap D'_Z$. If $p \in G$ is arbitrary, there is some $q \in G$ such that $q \leq p$ and $q \leq r$ (applying (2) in Definition III.3.10). Then $s_q \supseteq s_p$ and $(s_q \backslash s_r) \subseteq Z$, so that $(s_p \backslash s_r) \subseteq Z$. Taking the union, we get $(K_G \backslash s_r) \subseteq Z$, so that $K_G \subseteq^* Z$.

So, if G meets all the κ dense sets D_n and D'_Z (for $n \in \omega$ and $Z \in \mathcal{E}$), then K_G is infinite and K_G is a pseudo-intersection of $\mathcal{E}$. □

The use of something like ($\star$) in a proof of ccc is fairly common, so we say a bit more about it:

Definition III.3.23 *A subset C of a poset $\mathbb{P}$ is* centered *iff for all $n \in \omega$ and all $p_1, \ldots, p_n \in C$, there is a $q \in \mathbb{P}$ (but* not *necessarily in C) such that $q \leq p_i$ for all i. $\mathbb{P}$ is σ–*centered *iff $\mathbb{P} = \bigcup_{\ell \in \omega} C_\ell$, where each C_ℓ is centered.*

Every antichain contains at most one element of each centered family, so

Lemma III.3.24 *Every σ–centered poset has the ccc.*

The converse is false; see Exercises III.3.27 and III.3.87.

The notion of "centered" is the poset analog of the FIP for families of sets. For example, if $C \subseteq \mathbb{O}_X$ (Definition III.3.4), then C is centered iff C has the FIP.

Another example:

Lemma III.3.25 *If G is a filter and $p_1, \ldots, p_n \in G$, then there is a $q \in G$ such that $q \leq p_i$ for all i. In particular, G is centered.*

Proof. By induction on n; for $n = 2$, use the definition of "filter". □

Another example, based on the verification of (☆) above:

Lemma III.3.26 *Let $\mathbb{P}$ be the poset used in the proof of Lemma III.3.22. Then $\mathbb{P}$ is a countable union of filters. In particular, $\mathbb{P}$ is σ–centered.*

Proof. For each $s \in [\omega]^{<\omega}$, let $C_s = \{p \in \mathbb{P} : s_p = s\}$. For each n,

$$\forall p_1, \ldots, p_n \in C_s \ \exists q \in C_s \ \forall i \ \ q \leq p_i \qquad (\bigstar)$$

because we can let q be $(s, \bigcup_i \mathcal{W}_{p_i})$. Each C_s is centered, but not a filter, but using (★), $G_s := \{r : \exists p \in C_s \ p \leq r\}$ is a filter. Then $\mathbb{P} = \bigcup_s C_s = \bigcup_s G_s$. □

So, Lemma III.3.22 proved $\kappa < \mathfrak{p}$ by using $MA_{\mathbb{P}}(\kappa)$ for a σ–centered poset $\mathbb{P}$. In fact, $\kappa < \mathfrak{p}$ *is equivalent* to the statement that $MA(\kappa)$ holds for all σ–centered posets; see Theorem III.3.61 below.

Note that the definition of "C_s is centered" did not require $q \in C_s$, which we got in (★), but the stronger (★) was needed to extend C_s to a filter. Most σ–centered posets occurring in practice are in fact countable unions of filters, but this is not true in general. For a counter-example, see [50].

Exercise III.3.27 *If X is a compact Hausdorff space, then X is separable iff $\mathbb{O}_X$ is σ–centered iff $\mathbb{O}_X$ is a countable union of filters. In particular, if $X = 2^\kappa$ for $\kappa > \mathfrak{c}$, then $\mathbb{O}_X$ is an example of a ccc poset that is not σ–centered.*

A more complicated example is the *amoeba order* used to prove:

Lemma III.3.28 $\mathfrak{m} \leq \mathrm{add}(\mathsf{null})$.

Proof. Fix an infinite $\kappa < \mathfrak{m}$. We shall prove that $\kappa < \mathrm{add}(\mathsf{null})$ by applying $MA_{\mathbb{P}}(\kappa)$ for a suitable ccc poset $\mathbb{P}$. We remark that you can't do this with a σ–centered $\mathbb{P}$ because $\mathfrak{p} > \mathrm{add}(\mathsf{null})$ is consistent (Corollary V.4.16) and $MA_{\mathbb{P}}(\kappa)$ holds for all $\kappa < \mathfrak{p}$ whenever $\mathbb{P}$ is σ–centered (Theorem III.3.61).

Let μ denote Lebesgue measure on $\mathbb{R}$, and fix μ–null sets N_α for $\alpha < \kappa$. We shall prove that $\bigcup_{\alpha<\kappa} N_\alpha$ is null. Recall that $N \subseteq \mathbb{R}$ is null iff for all $\varepsilon > 0$, there is an open $U \supseteq N$ such that $\mu(U) \leq \varepsilon$. So, fix $\varepsilon > 0$. We shall produce an open $U \supseteq \bigcup_{\alpha<\kappa} N_\alpha$ with $\mu(U) \leq \varepsilon$.

Let $\mathbb{P}$ be the set of all open $p \subseteq \mathbb{R}$ such that $\mu(p) < \varepsilon$. We view p as an inner approximation to the U we wish to construct. Then define $q \leq p$ (q extends p) iff $q \supseteq p$ (q is a better approximation); so $\mathbb{1} = \emptyset$. Observe that $p \not\perp q$ iff $\mu(p \cup q) < \varepsilon$, in which case $p \cup q \in \mathbb{P}$ and $p \cup q$ extends both p and q.

Next, we prove that $\mathbb{P}$ has the ccc; so assume that we have $p_\xi \in \mathbb{P}$ for $\xi < \omega_1$ so that $\xi \neq \eta \to p_\xi \perp p_\eta$, and we shall derive a contradiction. Then, the p_ξ are open sets such that each $\mu(p_\xi) < \varepsilon$ and $\xi \neq \eta \to \mu(p_\xi \cup p_\eta) \geq \varepsilon$. First, fix a $\delta > 0$ such that $E := \{\xi < \omega_1 : \mu(p_\xi) \leq \varepsilon - 3\delta\}$ is uncountable. Let $\mathcal{B}$ be the set of all finite unions of rational intervals. For each $\xi \in E$,

choose a $W_\xi \subseteq p_\xi$ such that $W_\xi \in \mathcal{B}$ and $\mu(p_\xi \backslash W_\xi) \leq \delta$. Then, for $\xi, \eta \in E$, $\xi \neq \eta \rightarrow \mu(W_\xi \cup W_\eta) \geq \varepsilon - 2\delta$, while each $\mu(W_\xi) \leq \mu(p_\xi) \leq \varepsilon - 3\delta$. But $\mathcal{B}$ is countable, so we may fix $\xi, \eta \in E$ with $\xi \neq \eta$ but $W_\xi = W_\eta$, which yields a contradiction.

For any filter G, let $U_G = \bigcup G$. Then U_G is clearly open, and we shall prove that $\mu(U_G) \leq \varepsilon$. First, since $U_G \subseteq \mathbb{R}$, it is also a second countable space, so it is Lindelöf; that is, there is a countable subset, $\{p_j : j \in \omega\} \subseteq G$, such that $U_G = \bigcup_{j\in\omega} p_j$. Since G is centered (Lemma III.3.25), for each n there is $q \in \mathbb{P}$ such that $q \leq p_j$ for all $j < n$. Since $\leq$ is $\supseteq$, this implies that each $\mu(\bigcup_{j<n} p_j) < \varepsilon$. Then $\mu(U_G) \leq \varepsilon$ follows from $U_G = \bigcup_{j\in\omega} p_j$ and the countable additivity of μ.

Let $D_\alpha = \{q \in \mathbb{P} : q \supseteq N_\alpha\}$. If $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$ then $U_G \supseteq \bigcup_{\alpha<\kappa} N_\alpha$, so we are done by applying $MA_{\mathbb{P}}(\kappa)$ if we can prove that each D_α is dense. So, fix α and fix $p \in \mathbb{P}$. Let V be open with $N_\alpha \subseteq V$ and $\mu(V) < \varepsilon - \mu(p)$, and let $q = p \cup V$. Then $\mu(q) < \varepsilon$, so $q \in \mathbb{P}$. Also, $q \leq p$ and $q \in D_\alpha$. □

This poset is called the *amoeba order* because in this last paragraph, the amoeba p reaches out a pseudopod V to engulf the small morsel of food N_α. The amoeba picture is more plausible in 2D or 3D, since the 1D amoebas $p \subset \mathbb{R}$ may fail to be connected:

Exercise III.3.29 *Let μ denote Lebesgue measure in $\mathbb{R}^n$, where $n \geq 2$. Let $\mathbb{P}$ be the set of all $p \subseteq \mathbb{R}^n$ such that $\mu(p) < \varepsilon$ and p is open* and *connected. Show that $\mathbb{P}$ is ccc and that $MA_{\mathbb{P}}(\kappa)$ implies that the union of κ μ-null sets is null.*

But these are still strange looking amoebas, since you can't assume that they are simply connected.

Exercise III.3.30 *Let μ be a σ-finite Borel measure on some separable metric space X. Show that $MA(\kappa)$ implies that the union of κ μ-null sets is null.*

Our main applications of *MA* so far (Lemmas III.3.18, III.3.22, and III.3.28) have been of the form $MA(\kappa) \rightarrow$ [something nice about κ], where the result for $\kappa = \aleph_0$ was already a classical *ZFC* fact. A different kind of result is $MA(\aleph_1) \rightarrow SH$ (Corollary III.3.44), where our proof yields nothing from $MA(\aleph_0)$. $MA(\aleph_1) \rightarrow SH$ will follow from a discussion of the influence of $MA(\aleph_1)$ on the pre-calibers of posets and on products of ccc spaces.

Definition III.3.31 *The infinite cardinal κ is a* pre-caliber *for the poset $\mathbb{P}$ iff whenever $p_\alpha \in \mathbb{P}$ for $\alpha < \kappa$, there is a $B \in [\kappa]^\kappa$ such that $\{p_\alpha : \alpha \in B\}$ is centered.*

This is related to Šanin's notion of *caliber* for a topological space X (see Exercise III.2.22). If $\mathbb{O}_X$ has κ as a pre-caliber, then we get $\{p_\alpha : \alpha \in B\}$ with the FIP, whereas the stronger notion of *caliber* would require $\bigcap_{\alpha\in B} p_\alpha \neq \emptyset$. Since an $\bigcap$ is meaningless in an arbitrary poset, it is natural to consider the notion of pre-caliber.

If $\mathbb{P}$ has $\aleph_1$ as a pre-caliber, then $\mathbb{P}$ must be ccc, since otherwise the p_α could be pairwise incompatible. Also, if $\mathbb{P}$ is σ–centered, then $\mathbb{P}$ must have $\aleph_1$ as a pre-caliber, since $\aleph_1$ of the p_α lie in the same centered family. But "σ–centered" is a strictly stronger property:

Exercise III.3.32 *Assume that* $2 \leq |J| \leq \aleph_0$. *Show, in ZFC, that* $\mathrm{Fn}(I, J)$ *has* $\aleph_1$ *as a pre-caliber for all* I, *and is* σ*–centered iff* $|I| \leq \mathfrak{c}$.

Hint. See the proofs of Lemmas III.3.7 and III.2.11 and Exercise III.2.13. □

Under *CH*, a ccc poset that does not have $\aleph_1$ as a pre-caliber is given by Exercise III.3.87. There is no such example under $MA(\aleph_1)$ by Lemma III.3.35.

Definition III.3.33 *Let* $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ *be a forcing poset with* $s \in \mathbb{P}$. *Then* $s{\downarrow}$ *denotes the forcing poset* $(\mathbb{S}, \leq_{\mathbb{S}}, \mathbb{1}_{\mathbb{S}})$, *where* $\mathbb{S} = \{p \in \mathbb{P} : p \leq s\}$, $\mathbb{1}_{\mathbb{S}} = s$, *and* $\leq_{\mathbb{S}}$ *is* $\leq_{\mathbb{P}}$ *restricted to* $\mathbb{S}$. *If* $D \subseteq \mathbb{P}$, *then* D *is* dense below s *iff* $D \cap s{\downarrow}$ *is dense in* $s{\downarrow}$; *equivalently, iff* $\forall p \leq s\ \exists q \in D\ q \leq p$.

Lemma III.3.34 *If* $\mathbb{P}$ *is ccc and* $s \in \mathbb{P}$ *then* $s{\downarrow}$ *is ccc. If* $G \subseteq s{\downarrow}$ *is a filter on the poset* $s{\downarrow}$ *and* $G^* = \{p \in \mathbb{P} : \exists q \in G\ q \leq p\}$, *then* G^* *is a filter on* $\mathbb{P}$ *and* $G^* \cap s{\downarrow} = G$.

This suggests that we might apply *MA* to some $s{\downarrow}$, rather than to $\mathbb{P}$ itself, as we do in the proof of:

Lemma III.3.35 $MA(\aleph_1)$ *implies that every ccc poset has* $\aleph_1$ *as a pre-caliber.*

Proof. Fix $p_\alpha \in \mathbb{P}$ for $\alpha < \omega_1$. We shall find a filter containing $\aleph_1$ of the p_α, which is sufficient, since filters are centered.

For each α, let $D_\alpha = \{q : \exists \beta \geq \alpha\, [q \leq p_\beta]\}$. Note that D_α gets smaller as α gets bigger.

Suppose that there is an s such that all D_α are dense below s. Then, applying Lemma III.3.34, there is a filter G^* on $\mathbb{P}$ that meets all the D_α. Since filters are upward closed, $G^* \cap D_\alpha \neq \emptyset$ implies that $\exists \beta \geq \alpha\, [p_\beta \in G^*]$. So, $\{\beta < \omega_1 : p_\beta \in G^*\}$ is unbounded in ω_1.

Now, suppose that there is no such s. Then, in particular, for each ξ there is an $\alpha_\xi > \xi$ such that D_{α_ξ} is not dense below p_ξ, so that there is an $r_\xi \leq p_\xi$ such that $\neg \exists q \leq r_\xi\, [q \in D_{\alpha_\xi}]$, which means $\neg \exists q \leq r_\xi\, \exists \beta \geq \alpha_\xi [q \leq p_\beta]$, so that $\forall \beta \geq \alpha_\xi [r_\xi \perp p_\beta]$, and hence $\forall \beta \geq \alpha_\xi [r_\xi \perp r_\beta]$, since $r_\beta \leq p_\beta$. But now we can, by recursion on $\nu < \omega_1$, choose $\xi_\nu < \omega_1$ so that $\mu < \nu \to \xi_\nu > \alpha_{\xi_\mu}$, so that $\mu < \nu \to r_{\xi_\mu} \perp r_{\xi_\nu}$, contradicting the ccc. □

The notions of *product* and *substructure* are important in model theory, as well as in algebra, where they are frequently applied to groups and rings. They are also important in forcing when applied to posets. We take up products first:

Definition III.3.36 *Let $(\mathbb{P}_i, \leq_i, \mathbb{1}_i)$, for $i \in I$, be forcing posets. Then their* product, $\prod_{i\in I}(\mathbb{P}_i, \leq_i, \mathbb{1}_i)$, *denotes the poset* $(\prod_{i\in I}\mathbb{P}_i, \leq, \mathbb{1})$ *where* $\vec{p} \leq \vec{q}$ *iff* $p_i \leq_i q_i$ *for all* i, *and* $\mathbb{1}$ *is the sequence* $\langle \mathbb{1}_i : i \in I\rangle$.

As usual, the *set* $\prod_{i\in I}\mathbb{P}_i$ denotes the set of functions f with $\mathrm{dom}(f) = I$ and each $f(i) \in \mathbb{P}_i$; and the sequence notation $\langle \mathbb{1}_i : i \in I\rangle$ is just a way of specifying one of these functions.

As usual, we often abuse notation by writing $\prod_{i\in I}\mathbb{P}_i$ for the product poset when the orders are clear from context. If there are only two posets, say $I = \{0,1\}$, we write the product as $\mathbb{P}_0 \times \mathbb{P}_1$. The question of whether the product of any two ccc posets is ccc is independent of *ZFC*, and the discussion parallels the discussion for ccc spaces in Section III.2. $MA(\aleph_1)$ implies that the product of any two ccc posets is ccc (see Theorem III.3.43), but this is false under *CH* (see Theorem III.8.19).

Spaces and posets are related by:

Lemma III.3.37 *If X, Y are any two topological spaces, then the space $X \times Y$ is ccc iff the poset $\mathbb{O}_X \times \mathbb{O}_Y$ is ccc.*

Proof. If $X \times Y$ is not ccc, then it has is a disjoint family of open sets $\{W_\alpha : \alpha < \omega_1\}$. Shrinking the W_α, we may assume that each W_α is a basic open set, $U_\alpha \times V_\alpha$; then $\{(U_\alpha, V_\alpha) : \alpha < \omega_1\}$ is an antichain in $\mathbb{O}_X \times \mathbb{O}_Y$. □

In particular, if X is a Suslin line, then $\mathbb{O}_X$ is ccc but $\mathbb{O}_X \times \mathbb{O}_X$ is not ccc; see Lemma III.2.18.

It is also true that if we have ccc posets $\mathbb{P}, \mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not ccc, then we can find ccc topological spaces X, Y such that $X \times Y$ is not ccc; see Exercise III.4.16.

Exercise III.3.38 *If there are ccc spaces X, Y such that $X \times Y$ is not ccc, then there is also one ccc space Z such that $Z \times Z$ is not ccc. If there are ccc posets $\mathbb{P}, \mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not ccc, then there is also one ccc poset $\mathbb{R}$ such that $\mathbb{R} \times \mathbb{R}$ is not ccc.*

Hint. Z is the disjoint sum of X, Y. □

Now consider infinite products. Theorem III.2.8 says that $\prod_{i\in I} X_i$ is ccc whenever all the finite sub-products are ccc. For posets this is false; infinite products are never ccc except in trivial cases:

Exercise III.3.39 *Let $\mathbb{P}_i$ for $i \in I$ be posets, and assume that we can find $p_i, q_i \in \mathbb{P}_i$ for each i with $p_i \perp q_i$. Then $\prod_{i\in I}\mathbb{P}_i$ has an antichain of size $2^{|I|}$.*

The difference is that the standard product topology does not use *all* products of open sets $\prod_{i\in I} U_i$ as a base, but only those such that $U_i = X_i$ for all but finitely many i. The corresponding notion for posets is:

Definition III.3.40 *Let $\mathbb{P}_i$, for $i \in I$, be forcing posets. Then their* finite support product, $\prod^{\mathsf{fin}}_{i\in I} \mathbb{P}_i$, *denotes* $\{\vec{p} \in \prod_{i\in I} \mathbb{P}_i : |\{i : p_i \neq \mathbb{1}_i\}| < \aleph_0\}$. *Both* $\prod^{\mathsf{fin}}_{i\in I} \mathbb{P}_i$ *and* $\prod_{i\in I} \mathbb{P}_i$ *have the same* $\leq$ *and* $\mathbb{1}$.

Of course, $\prod^{\mathsf{fin}}$ is the same as $\prod$ for finite products. With almost verbatim the same proof as for Theorem III.2.8, we get:

Theorem III.3.41 *Let $\mathbb{P}_i$ for $i \in I$ be any posets. If the product $\prod^{\mathsf{fin}}_{i\in I} \mathbb{P}_i$ is not ccc, then there is a finite $J \subseteq I$ such that $\prod_{i\in J} \mathbb{P}_i$ is not ccc.*

So, we now consider finite products. Observe first:

Lemma III.3.42 *Let $\mathbb{P}, \mathbb{Q}$ be posets, with $\mathbb{P}$ ccc and $\aleph_1$ a pre-caliber for $\mathbb{Q}$. Then $\mathbb{P} \times \mathbb{Q}$ is ccc.*

Proof. Suppose that $\{(p_\alpha, q_\alpha) : \alpha < \omega_1\}$ is an antichain in $\mathbb{P} \times \mathbb{Q}$. Then $\alpha \neq \beta$ implies $p_\alpha \perp p_\beta$ or $q_\alpha \perp q_\beta$, since otherwise we would have $r \in \mathbb{P}$ and $s \in \mathbb{Q}$ with $r \leq p_\alpha$, $r \leq p_\beta$, $s \leq q_\alpha$, and $s \leq q_\beta$, so that $(r, s) \leq (p_\alpha, q_\alpha)$ and $(r, s) \leq (p_\beta, q_\beta)$.

Now, fix an uncountable $B \subseteq \omega_1$ such that $\{q_\alpha : \alpha \in B\}$ is centered. Then for $\alpha, \beta \in B$, $\alpha \neq \beta$ implies $p_\alpha \perp p_\beta$, contradicting the ccc of $\mathbb{P}$. □

It follows by induction on n that every product of n posets, each with $\aleph_1$ as a pre-caliber, also is ccc (and, actually, has $\aleph_1$ a pre-caliber). So, applying Lemma III.3.35, we have:

Theorem III.3.43 *Assume $MA(\aleph_1)$. If $\mathbb{P}_i$, for $i \in I$ is a ccc poset, then $\prod^{\mathsf{fin}}_i \mathbb{P}_i$ is ccc. If X_i, for $i \in I$ is a ccc topological space, then $\prod_i X_i$ is ccc.*

Then, by Lemma III.2.18,

Corollary III.3.44 *$MA(\aleph_1)$ implies SH.*

Exercise III.3.45 *Prove that if X is a compact Hausdorff space X, then X has $\aleph_1$ as a caliber iff $\mathbb{O}_X$ has $\aleph_1$ as a pre-caliber. Conclude from this that the statement "Every ccc compact Hausdorff space X has $\aleph_1$ as a caliber" is true under $MA(\aleph_1)$ and false under $\neg SH$.*

Lemma III.3.46 *Every ccc poset of size less than $\mathfrak{m}$ is a countable union of filters, and hence is σ-centered.*

Proof. Let $\mathbb{P}$ be ccc, $\kappa = |\mathbb{P}|$, and assume $MA(\kappa)$. We shall prove that $\mathbb{P}$ is a countable union of filters. This is trivial if $\kappa \leq \aleph_0$, so assume that $\kappa \geq \aleph_1$. Let $\mathbb{Q} = \prod^{\mathsf{fin}}_{i\in\omega} \mathbb{P}_i$, where each $\mathbb{P}_i = \mathbb{P}$. Then $\mathbb{Q}$ is ccc by Theorem III.3.43.

Let $\pi_i : \mathbb{Q} \xrightarrow{\text{onto}} \mathbb{P}$ be the natural projection: $\pi_i(\vec{p}) = p_i$. Observe that if $G \subseteq \mathbb{Q}$ is any filter, then all the $\pi_i(G)$ are filters on $\mathbb{P}$.

Also, note that for each $r \in \mathbb{P}$, $D_r := \{\vec{p} \in \mathbb{Q} : \exists i\, [p_i = r]\}$ is dense in $\mathbb{Q}$. If G is a filter on $\mathbb{Q}$ that meets all the D_r, then $\bigcup_i \pi_i(G) = \mathbb{P}$. □

We next consider the notion of *substructure* as applied to posets:

Definition III.3.47 *A* subposet *of the forcing poset* $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ *is any poset* $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$ *such that* $\mathbb{1}_{\mathbb{Q}} = \mathbb{1}_{\mathbb{P}} \in \mathbb{Q} \subseteq \mathbb{P}$ *and* $\leq_{\mathbb{Q}} = \leq_{\mathbb{P}} \cap (\mathbb{Q} \times \mathbb{Q})$. *If* $q, q' \in \mathbb{Q}$ *then* $q \not\perp_{\mathbb{Q}} q'$ *means that* $\exists r \in \mathbb{Q}\,[r \leq q \;\&\; r \leq q']$ *and* $q \not\perp_{\mathbb{P}} q'$ *means that* $\exists r \in \mathbb{P}\,[r \leq q \;\&\; r \leq q']$.

As usual in algebra, we shall often just write "$\mathbb{Q} \subseteq \mathbb{P}$" when the orders and $\mathbb{1}$s are clear from context. Actually, we have just applied the model-theoretic notion of substructure from Definition I.15.7, letting $\mathcal{L}$ be $\{\leq, \mathbb{1}\}$. Note that $s{\downarrow}$ (Definition III.3.33) is not a subposet (unless $s = \mathbb{1}_{\mathbb{P}}$), since the $\mathbb{1}$s are different.

$q \perp_{\mathbb{Q}} q'$ need not imply $q \perp_{\mathbb{P}} q'$, leading to some pathological examples:

Exercise III.3.48 *Define a subposet of* $\mathrm{Fn}(\omega_1, 2)$ *that is not ccc.*

All the examples of interest to us will satisfy the following, which implies in particular that $q \perp_{\mathbb{P}} q' \leftrightarrow q \perp_{\mathbb{Q}} q'$:

Definition III.3.49 $\mathbb{Q} \subseteq_{\mathsf{ctr}} \mathbb{P}$ *iff* $\mathbb{Q} \subseteq \mathbb{P}$, *and for all* $n \in \omega$ *and all* $q_1, \ldots, q_n \in \mathbb{Q}$: *if* $\exists p \in \mathbb{P}\,[\forall i\, p \leq q_i]$ *then* $\exists q \in \mathbb{Q}\,[\forall i\, q \leq q_i]$.

For example, $\mathrm{Fn}(I, J) \subseteq_{\mathsf{ctr}} \mathrm{Fn}(L, J)$ whenever $I \subseteq L$.

Lemma III.3.50 *If* $\mathbb{Q} \subseteq_{\mathsf{ctr}} \mathbb{P}$ *then:*

a. If $\mathbb{P}$ *is ccc, then* $\mathbb{Q}$ *is ccc.*
b. If $\mathbb{P}$ *is* σ*-centered, then* $\mathbb{Q}$ *is* σ*-centered.*
c. If $\mathbb{P}$ *has* κ *as a pre-caliber, then* $\mathbb{Q}$ *has* κ *as a pre-caliber.*

In many *MA* applications, the natural poset to use has size $\mathfrak{c}$ or larger; see Lemma III.3.18, Lemma III.3.28, and Theorem III.3.43. Nevertheless, the following lemma shows that if $MA(\kappa)$ is false, then it fails for some poset of size $\leq \kappa$:

Lemma III.3.51 *Let* κ *be an infinite cardinal and let* $\mathbb{P}$ *be any poset for which* $MA_{\mathbb{P}}(\kappa)$ *is false. Then there is a* $\mathbb{Q} \subseteq_{\mathsf{ctr}} \mathbb{P}$ *for which* $MA_{\mathbb{Q}}(\kappa)$ *is false and* $|\mathbb{Q}| \leq \kappa$.

Proof. Fix dense sets $D_\alpha \subseteq \mathbb{P}$ for $\alpha < \kappa$ such that there is no a filter G on $\mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Consider $(\mathbb{P}, \leq, \mathbb{1}, D_\alpha)_{\alpha<\kappa}$ to be a structure for $\mathcal{L}$, where $\mathcal{L}$ has a symbol for $\leq$ and $\mathbb{1}$, along with κ unary relation symbols representing the D_α. By the Downward Löwenheim–Skolem–Tarski Theorem I.15.10, fix a $\mathbb{Q} \subseteq \mathbb{P}$ such that $|\mathbb{Q}| \leq \kappa$ and the substructure with universe $\mathbb{Q}$ is an elementary substructure of $\mathbb{P}$. In particular, each $D_\alpha \cap \mathbb{Q}$ is dense in $\mathbb{Q}$ because this is a first-order properties expressed in $\mathcal{L}$. Likewise, $\mathbb{Q} \subseteq_{\mathsf{ctr}} \mathbb{P}$ because $\exists x \bigwedge_i [x \leq q_i]$ is a first-order property of any given $q_1, \ldots, q_n \in \mathbb{Q}$.

If $MA_{\mathbb{Q}}(\kappa)$ holds, let $G \subseteq \mathbb{Q}$ be a filter such that $G \cap (D_\alpha \cap \mathbb{Q}) \neq \emptyset$ for all $\alpha < \kappa$. Let $G^+ = \{p \in \mathbb{P} : \exists q \in G\; q \leq p\}$. Then G^+ is a filter on $\mathbb{P}$ such that $G^+ \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$, contradicting our assumptions about the D_α. $\square$

Exercise III.3.52 *Prove Lemma III.3.51 without using any model theory.*

Hint. Replace the use of the Theorem I.15.10 by an explicit closure argument, using Lemma I.13.22. For example, $f_3(p_1, p_2, p_3)$ chooses a common extension of p_1, p_2, p_3 whenever there is one. □

The posets $s{\downarrow}$ (Definition III.3.33) suggest the following topology on $\mathbb{P}$:

Definition III.3.53 *If $\mathbb{P}$ is a forcing poset, the* poset topology *on the set $\mathbb{P}$ is defined by letting the family of open sets, $\mathcal{T}_{\mathbb{P}}$, be $\{U \subseteq \mathbb{P} : \forall s \in U\, [s{\downarrow} \subseteq U]\}$.*

Open is the same as downward closed, whereas filters are upward closed. One could also say that $\mathcal{T}_{\mathbb{P}}$ is the topology with $\{s{\downarrow} : s \in \mathbb{P}\}$ as a base. It is easy to see that $\mathcal{T}_{\mathbb{P}}$ is a topology — that is, it is closed under arbitrary unions and finite intersections — but it is actually closed under *arbitrary* intersections. It is also not Hausdorff unless $\mathbb{P} = \{\mathbb{1}\}$, since $\mathbb{P}$ is the only open set containing $\mathbb{1}$. Because of these strange features, it is not of much interest in general topology, but it arises naturally in discussions of *MA* and forcing. Note that $D \subseteq \mathbb{P}$ is dense in the topological sense ($\overline{D} = \mathbb{P}$) iff D is dense in the order sense (Definition III.3.8). Also, $\mathbb{P}$ is ccc in the topological sense (Definition III.2.1) iff $\mathbb{P}$ is ccc in the order sense (Definition III.3.3). A related definition:

Definition III.3.54 *For $S \subseteq \mathbb{P}$, define $S{\downarrow} = \bigcup\{p{\downarrow} : p \in S\}$; this is the* open hull *of S, or the smallest open superset of S.*

Of course, the existence of a smallest *open* superset of an arbitrary set is a bit strange in topology, but there is one here, since the family of open sets is closed under *arbitrary* intersections.

Since filters are upward closed,

Lemma III.3.55 *If G is a filter on $\mathbb{P}$ and $S \subseteq \mathbb{P}$, then $S \cap G = \emptyset$ iff $S{\downarrow} \cap G = \emptyset$.*

A finite intersection of dense open sets is dense open (in any topological space). Because of this, it is sometimes useful in verifying $MA_{\mathbb{P}}(\kappa)$ to note that it is sufficient to meet κ dense *open* sets (replacing each D by $D{\downarrow}$); see Theorem III.3.60, which lists a few equivalents of *MA* expressed in terms of dense open sets and/or linked families and/or maximal antichains. We first define the relevant notions.

Definition III.3.56 *$L \subseteq \mathbb{P}$ is* linked *iff $p_1 \not\perp p_2$ for all $p_1, p_2 \in L$.*

Obviously, every centered family is linked, but not conversely. There is now a natural weakening of "σ–centered" (called "σ–linked") and a weakening of pre-caliber $\aleph_1$ (called the Knaster property). For example, in the proof of Lemma III.3.42, the Knaster property for $\mathbb{Q}$ would have sufficed because we only needed $\{q_\alpha : \alpha \in B\}$ to be linked, not centered. For more on this topic, see Section V.4 and Fremlin's book [35].

In discussions of *MA* and forcing, it is sometimes convenient to replace the dense sets by maximal antichains.

Definition III.3.57 *A* maximal antichain *in a forcing poset $\mathbb{P}$ is an antichain $A \subseteq \mathbb{P}$ that is not a proper subset of any other antichain in $\mathbb{P}$.*

It is immediate from Zorn's Lemma or Tukey's Lemma that every antichain is a subset of a maximal antichain. Also, a mad family (Definition III.1.17) is just a maximal antichain in the poset $[\omega]^\omega$ (Example III.3.5).

Definition III.3.58 *For $E \subseteq \mathbb{P}$: $p \perp E$ means $\forall q \in E\, [p \perp q]$, and E is called* predense *iff $\neg\exists p\, [p \perp E]$.*

Lemma III.3.59 *E is predense iff $E{\downarrow}$ is dense iff $E{\downarrow}$ is dense open.*

Note that all dense sets and all maximal antichains are predense.

Theorem III.3.60 *For $\kappa \geq \aleph_0$ and forcing poset $\mathbb{P}$, consider the eight different statements we get by letting $\mathcal{X}$ be one of*

1. dense set 2. dense open set 3. maximal antichain 4. predense set

and letting $\mathcal{Y}$ be one of

a. filter b. linked family

and asserting:

Whenever $E_\alpha \subseteq \mathbb{P}$, for $\alpha < \kappa$, all have property $\mathcal{X}$, there exists an $F \subseteq \mathbb{P}$ with property $\mathcal{Y}$ such that $F \cap E_\alpha \neq \emptyset$ for all α.

Then these eight statements are all equivalent.

So, $(1a)$ is the usual statement of $MA_{\mathbb{P}}(\kappa)$. The others are sometimes useful in proving that $MA_{\mathbb{P}}(\kappa)$ follows from some other statement; see Theorem III.3.61 and Lemma III.3.68.

Proof. First observe the implications:

$$\begin{array}{ccccccc} (4a) & \longrightarrow & (3a) & \longrightarrow & (1a) & \longrightarrow & (2a) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (4b) & \longrightarrow & (3b) & \longrightarrow & (1b) & \longrightarrow & (2b) \end{array}$$

The vertical arrows follow from the fact that all filters are linked. For the horizontal arrows: $(4y) \to (3y)$ and $(1y) \to (2y)$ hold because property (3) implies property (4) and property (2) implies property (1). Also, $(3y) \to (1y)$ holds because every dense set contains a maximal antichain as a subset; to prove this: if E is dense and $A \subseteq E$ is maximal among all antichains that are subsets of E, then A is a maximal antichain.

We are now done if we prove that $(2a) \to (4a)$ and $(2b) \to (2a)$.

For $(2a) \to (4a)$: If E_α is predense, then $E_\alpha{\downarrow}$ is dense open, and if F is a filter then $F \cap E_\alpha \neq \emptyset$ iff $F \cap E_\alpha{\downarrow} \neq \emptyset$ (Lemma III.3.55).

For $(2b) \to (2a)$: Assume $(2b)$, and assume that all the D_α are dense open; we shall get a filter that meets all of them. Shrinking the D_α, we may assume that each $D_\alpha = A_\alpha{\downarrow}$, where A_α is a maximal antichain. Then, adding more

dense open sets, we may assume that $\forall \alpha, \beta\, \exists \gamma\, [A_\gamma \subseteq D_\gamma \subseteq A_\alpha{\downarrow} \cap A_\beta{\downarrow}]$. Now, let L be linked such that each $L \cap D_\alpha \neq \emptyset$. Choose $q_\alpha \in L \cap D_\alpha$, and then fix $p_\alpha \in A_\alpha$ such that $q_\alpha \leq p_\alpha$ (so, p_α is uniquely determined, even though q_α need not be). Note that $\{p_\alpha : \alpha < \kappa\}$ is linked.

For each α, β: There is a γ such that $A_\gamma \subseteq A_\alpha{\downarrow} \cap A_\beta{\downarrow}$. It follows that $p_\gamma \leq p_\alpha$ and $p_\gamma \leq p_\beta$, since if $p_\gamma \not\leq p_\alpha$, then (using $p_\gamma \in A_\alpha{\downarrow}$), there is a $\hat{p} \in A_\alpha$ such that $p_\gamma \leq \hat{p}$ and $\hat{p} \neq p_\alpha$, so $p_\gamma \perp p_\alpha$, contradicting $\{p_\alpha : \alpha < \kappa\}$ being linked.

Since $\forall \alpha, \beta\, \exists \gamma\, [p_\gamma \leq p_\alpha \wedge p_\gamma \leq p_\beta]$, $G := \{r : \exists \alpha\, p_\alpha \leq r\}$ is a filter, and G meets all the D_α. □

We now apply Theorem III.3.60 by proving the following theorem of Bell [8]. One application of Bell's theorem is that sometimes the easiest way to derive a result about cardinals $\kappa < \mathfrak{p}$ is to prove it first from $MA(\kappa)$, and then check that the poset used is σ–centered; see Exercises III.3.77 and III.3.79 and III.3.83. Our proof follows the exposition in Blass [11], which contains further results on the various cardinals below $\mathfrak{c}$.

Theorem III.3.61 *Define $\mathfrak{m}_\sigma$ to be the least cardinal κ such that $MA_{\mathbb{P}}(\kappa)$ is false for some σ–centered $\mathbb{P}$. Then $\mathfrak{m}_\sigma = \mathfrak{p}$.*

Proof. $\mathfrak{m}_\sigma \leq \mathfrak{p}$ is clear from Lemmas III.3.26 and III.3.22. To prove $\mathfrak{p} \leq \mathfrak{m}_\sigma$, fix an infinite $\kappa < \mathfrak{p}$ and fix a σ–centered poset $\mathbb{P}$. We must establish $MA_{\mathbb{P}}(\kappa)$. By Lemma III.3.51, we may assume that $|\mathbb{P}| \leq \kappa$. Fix dense open $D_\alpha \subseteq \mathbb{P}$ for $\alpha < \kappa$; by Theorem III.3.60, we are done if we can produce a linked family $L \subseteq \mathbb{P}$ such that $L \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.

Let $\mathbb{P} = \bigcup_{\ell \in \omega} C_\ell$, where each C_ℓ is centered. We may assume that $\mathbb{1} \in C_\ell$ for each ℓ, since $C_\ell \cup \{\mathbb{1}\}$ is centered if C_ℓ is. Also, assume that $\mathbb{P}$ is atomless, since otherwise $MA_\kappa(\mathbb{P})$ is trivial (Lemma III.3.20).

For $\alpha < \kappa$ and $p \in \mathbb{P}$, let $B_\alpha(p) = \{\ell \in \omega : D_\alpha \cap C_\ell \cap p{\downarrow} \neq \emptyset\}$.

Then for each $m \in \omega$, $\{B_\alpha(p) : p \in C_m \wedge \alpha < \kappa\}$ has the SFIP: To prove that $\bigcap_{i<n} B_{\alpha_i}(p_i)$ is infinite, let $E = \bigcap_{i<n} D_{\alpha_i}$, which is also dense open. Also, since C_m is centered, fix q such that $q \leq p_i$ for each i. Then $\bigcap_{i<n} B_{\alpha_i}(p_i) \supseteq I := \{\ell \in \omega : E \cap C_\ell \cap q{\downarrow} \neq \emptyset\}$. Since $\mathbb{P}$ is atomless, let $\{a_j : j \in \omega\} \subseteq q{\downarrow}$ be an infinite antichain (Lemma III.3.21), and then choose $b_j \leq a_j$ with $b_j \in E$. If $b_j \in C_{\ell_j}$, then the ℓ_j are all different, and all $\ell_j \in I$.

Since $\kappa < \mathfrak{p}$, choose $Z_m \in [\omega]^\omega$ such that $Z_m \subseteq^* B_\alpha(p)$ for all $p \in C_m$ and all $\alpha < \kappa$.

For $\tau \in \omega^{<\omega}$: If $\tau \neq \emptyset$, let $\Lambda(\tau) = \tau(\mathrm{lh}(\tau) - 1)$, the last element of the sequence τ. Let $\Lambda(\emptyset) = 0$.

Let $T = \left\{\tau \in \omega^{<\omega} : \forall n < \mathrm{lh}(\tau)\, [\tau(n) \in Z_{\Lambda(\tau \restriction n)}]\right\}$. We may picture T as a tree, and do recursion on T as in the proof of Lemma III.1.26. Note that, $\emptyset \in T$ (trivially), and $(3, 1, 4) \in T$ iff $3 \in Z_0 \wedge 1 \in Z_3 \wedge 4 \in Z_1$.

For $\alpha < \kappa$, fix $\Delta_\alpha : T \to \mathbb{P}$ so that $\Delta_\alpha(\emptyset) = \mathbb{1}$ and for each $\tau^\frown\ell \in T$:

1. $\ell \in B_\alpha(\Delta_\alpha(\tau)) \rightarrow \Delta_\alpha(\tau^\frown \ell) \in D_\alpha \cap C_\ell \cap \Delta_\alpha(\tau)\downarrow$.
2. $\ell \notin B_\alpha(\Delta_\alpha(\tau)) \rightarrow \Delta_\alpha(\tau^\frown \ell) = \mathbb{1}$.

Note that: $\Delta_\alpha(\tau) \in C_{\Lambda(\tau)}$ for all τ, since $\mathbb{1} \in C_\ell$.

For each α and each $\tau \in T$: If $\tau^\frown \ell \in T$ then $\ell \in Z_{\Lambda(\tau)} \subseteq^* B_\alpha(\Delta_\alpha(\tau))$ (since $\Delta_\alpha(\tau) \in C_{\Lambda(\tau)}$). So, there is $\Phi_\alpha(\tau) \in \omega$ such that $\ell \in B_\alpha(\Delta_\alpha(\tau))$ for all $\ell \geq \Phi_\alpha(\tau)$ such that $\tau^\frown \ell \in T$.

$\Phi_\alpha : T \rightarrow \omega$ for each $\alpha < \kappa$, and $\kappa < \mathfrak{p} \leq \mathfrak{b}$ (Lemma III.1.24), and $|T| = \aleph_0$, so fix $\Gamma : T \rightarrow \omega$ such that $\Phi_\alpha \leq^* \Gamma$ for all α. Then, by recursion on n, fix $g : \omega \rightarrow \omega$ so that all $g{\restriction}n \in T$ and $g(n)$ is the least $\ell \geq \Gamma(g{\restriction}n)$ such that $(g{\restriction}n)^\frown \ell \in T$.

For each α: $\Phi_\alpha \leq^* \Gamma$ implies that there is a $k_\alpha \in \omega$ such that $\Phi_\alpha(g{\restriction}n) \leq \Gamma(n)$ for all $n \geq k_\alpha$. When $n \geq k_\alpha$, we have $g(n) \in B_\alpha(\Delta_\alpha(g{\restriction}n)$ (set $\tau = g{\restriction}n$ and $\ell = g(n)$); so, by (1) above, $\Delta_\alpha(g{\restriction}(n+1)) \leq \Delta_\alpha(g{\restriction}n)$.

Let $p_\alpha = \Delta_\alpha(g{\restriction}(k_\alpha{+}1))$. Then $p_\alpha \in D_\alpha \cap C_{g(k_\alpha)} \cap \Delta_\alpha(g{\restriction}k_\alpha)\downarrow$. Let $L = \{p_\alpha : \alpha < \kappa\}$. Then clearly $L \cap D_\alpha \neq \emptyset$ for all α, but we must prove that L is linked. Fix α, β, and assume that $k_\alpha \leq k_\beta$. If $k_\alpha = k_\beta$, then p_α, p_β lie in $C_{g(k_\alpha)}$, which is centered, so $p_\alpha \not\perp p_\beta$. If $k_\alpha < k_\beta$, then applying $\Delta_\alpha(g{\restriction}(n+1)) \leq \Delta_\alpha(g{\restriction}n)$ for $n = k_\alpha + 1, k_\alpha + 2, \ldots$, we get $\Delta_\alpha(g{\restriction}(k_\beta + 1)) \leq \Delta_\alpha(g{\restriction}(k_\alpha + 1)) = p_\alpha$. Then $p_\alpha \not\perp p_\beta$ because $\Delta_\alpha(g{\restriction}(k_\beta + 1))$ and $\Delta_\beta(g{\restriction}(k_\beta + 1)) = p_\beta$ both lie in the centered family $C_{g(k_\beta)}$. □

Next, we consider the notion of subposet. Roughly, if $\mathbb{Q} \subseteq \mathbb{P}$, then we would expect $\mathbb{P}$ to "do more" than $\mathbb{Q}$, so that $MA_{\mathbb{P}}(\kappa)$ "should" imply $MA_{\mathbb{Q}}(\kappa)$. But this is false, even if we assume that $\mathbb{Q} \subseteq_{\mathsf{ctr}} \mathbb{P}$ (Definition III.3.49). We shall prove $MA_{\mathbb{P}}(\kappa) \rightarrow MA_{\mathbb{Q}}(\kappa)$ in some cases, but there are also easy counter-examples:

Example III.3.62 *Let* $\mathbb{Q} = \mathrm{Fn}(\omega, 2)$. *Let* $\mathbb{P}$ *be the set of* all $p \subseteq \omega \times \{0,1\}$ *such that* p *is the graph of a function; that is,* all *partial functions. Then* $\mathbb{Q} \subseteq_{\mathsf{ctr}} \mathbb{P}$, *and* $MA_{\mathbb{P}}(\kappa)$ *holds trivially for all* κ *because the total functions are atoms, whereas* $MA_{\mathbb{Q}}(\mathfrak{c})$ *is false (see Lemma III.3.20).*

But $MA_{\mathrm{Fn}(\omega_1,2)}(\kappa)$ does imply $MA_{\mathrm{Fn}(\omega,2)}(\kappa)$ (using Lemma III.3.68 below) because $\mathrm{Fn}(\omega, 2)$ is a *complete* subposet:

Definition III.3.63 $\mathbb{Q} \subseteq_{\mathsf{c}} \mathbb{P}$ ($\mathbb{Q}$ *is a* complete *subposet*) *iff* $\mathbb{Q} \subseteq_{\mathsf{ctr}} \mathbb{P}$ *and for all* $A \subseteq \mathbb{Q}$, *if* A *is a maximal antichain in* $\mathbb{Q}$ *then* A *is still maximal in* $\mathbb{P}$.

Observe that A is an antichain in $\mathbb{P}$ just by $\mathbb{Q} \subseteq_{\mathsf{ctr}} \mathbb{P}$, but it could fail to be maximal. In Example III.3.62, let $A = \{q_n : n \in \omega\}$, where $\mathrm{dom}(q_n) = n + 1$, $q_n(n) = 1$, and $q_n(i) = 0$ for all $i < n$. Then A is maximal in $\mathbb{Q}$ but not in $\mathbb{P}$ because if $p : \omega \rightarrow 2$ is the identically 0 function, then $q_n \perp p$ for all n.

Example III.3.64 *If* $I \subseteq L$ *then* $\mathrm{Fn}(I, J) \subseteq_{\mathsf{c}} \mathrm{Fn}(L, J)$.

Proof. Suppose that $A \subseteq \mathrm{Fn}(I,J)$ is a maximal antichain in $\mathrm{Fn}(I,J)$ and fails to remain maximal in $\mathrm{Fn}(L,J)$; so that there is some $p \in \mathrm{Fn}(L,J)$ such that $p \perp q$ for all $q \in A$. But then $p{\restriction}I \in \mathrm{Fn}(I,J)$ and $p{\restriction}I \perp q$ for all $q \in A$, contradicting maximality of A. □

It is sometimes useful to generalize the notion $\mathbb{Q} \subseteq_{\mathsf{c}} \mathbb{P}$ to the situation where we are mapping $\mathbb{Q}$ into $\mathbb{P}$:

Definition III.3.65 *Let* $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$ *and* $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ *be forcing posets and* $i : \mathbb{Q} \to \mathbb{P}$. *Then* i *is a* complete embedding *iff*

1. $i(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{\mathbb{P}}$.
2. $\forall q_1, q_2 \in \mathbb{Q}\,[q_1 \leq_{\mathbb{Q}} q_2 \to i(q_1) \leq_{\mathbb{P}} i(q_2)]$.
3. $\forall q_1, q_2 \in \mathbb{Q}\,[q_1 \perp_{\mathbb{Q}} q_2 \leftrightarrow i(q_1) \perp_{\mathbb{P}} i(q_2)]$.
4. *For all* $A \subseteq \mathbb{Q}$: *If* A *is a maximal antichain in* $\mathbb{Q}$ *then* $i(A)$ *is a maximal antichain in* $\mathbb{P}$.

Also, i *is a* dense embedding *iff* (1)(2)(3) *hold, along with*

5. $i(\mathbb{Q})$ *is a dense subset of* $\mathbb{P}$.

Lemma III.3.66 *Suppose that* $\mathbb{Q}$ *is a subposet of* $\mathbb{P}$ *and* $i : \mathbb{Q} \to \mathbb{P}$ *is just inclusion (the identity map). Then* i *is a complete embedding iff* $\mathbb{Q} \subseteq_{\mathsf{c}} \mathbb{P}$.

Proof. This is mostly trivial, but we need to verify that, assuming that i is a complete embedding, $\mathbb{Q} \subseteq_{\mathsf{ctr}} \mathbb{P}$; that is, if $q_1, \ldots, q_n$ have a common extension in $\mathbb{P}$, then they have a common extension in $\mathbb{Q}$. For $n = 2$, this is clear by (3), but for the general case: Applying Zorn's Lemma, A be a subset of $\mathbb{Q}$ satisfying

$$A \text{ is an antichain and } \forall a \in A\, \exists i\; a \perp q_i \qquad (*)$$

and A is maximal among subsets of $\mathbb{Q}$ satisfying $(*)$. As a subset of $\mathbb{P}$, A is not a maximal antichain, since if $p \leq q_i$ for all i, then $p \perp a$ for all $a \in A$. Applying (4), fix $r \in \mathbb{Q}$ such that $r \perp a$ for all $a \in A$. Then whenever $r' \in \mathbb{Q}$ and $r' \leq r$, $\forall i\; r' \not\perp q_i$, since otherwise $A \cup \{r'\}$ would contradict maximality of A. So, in $\mathbb{Q}$, we can get $r \geq r_1 \geq \cdots \geq r_n$, where each $r_i \leq q_i$. Then in $\mathbb{Q}$, r_n is a common extension of $q_1, \ldots, q_n$. □

Lemma III.3.67 *Every dense embedding is a complete embedding.*

Proof. To verify (4), fix an $A \subseteq \mathbb{Q}$ that is a maximal antichain in $\mathbb{Q}$. If $i(A)$ is not maximal in $\mathbb{P}$, fix $p_0 \in \mathbb{P}$ such that $p_0 \perp i(q)$ for all $q \in A$. Using (5), fix $q_0 \in \mathbb{Q}$ with $i(q_0) \leq p_0$. Then $i(q_0) \perp i(q)$, and hence by (3), $q_0 \perp q$, for all $q \in A$, contradicting maximality of A. □

Lemma III.3.68 *Let* $i : \mathbb{Q} \to \mathbb{P}$ *be a complete embedding. Then*

α. *If* $MA_{\mathbb{P}}(\kappa)$ *then* $MA_{\mathbb{Q}}(\kappa)$.
β. *If* $MA_{\mathbb{Q}}(\kappa)$ *and* i *is a dense embedding, then* $MA_{\mathbb{P}}(\kappa)$.

Proof. For (α): By Theorem III.3.60 it is sufficient to fix a family $\mathcal{A}$ of maximal antichains in $\mathbb{Q}$ with $|\mathcal{A}| \leq \kappa$, and produce a linked family $L \subseteq \mathbb{Q}$ such that $L \cap A \neq \emptyset$ for all $A \in \mathcal{A}$. Since each $i(A)$ is a maximal antichain in $\mathbb{P}$, we may fix a filter $G \subseteq \mathbb{P}$ such that $G \cap i(A) \neq \emptyset$ for all $A \in \mathcal{A}$. Then $i^{-1}(G) \subseteq \mathbb{Q}$ and $i^{-1}(G) \cap A \neq \emptyset$ for all $A \in \mathcal{A}$. This $i^{-1}(G)$ need not be a filter on $\mathbb{Q}$, but it is linked, using (3) in Definition III.3.65 of "complete embedding".

For (β): By Theorem III.3.60, it is sufficient to show that whenever $\mathcal{D}$ is a family of dense open subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, there is a linked $L \subseteq \mathbb{P}$ such that $L \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. Since i is a dense embedding, each $i^{-1}(D) \subseteq \mathbb{Q}$ is predense and open, and hence dense. So, fix a filter $H \subseteq \mathbb{Q}$ such that $H \cap i^{-1}(D) \neq \emptyset$, and hence $i(H) \cap D \neq \emptyset$, for all $D \in \mathcal{D}$. This $i(H)$ need not be a filter on $\mathbb{P}$, but it is linked by (3) in Definition III.3.65. □

Note that (β) can fail if i is only a complete embedding, since $\mathbb{Q}$ can be "much simpler" than $\mathbb{P}$. For trivial counter-examples, consider $\mathbb{Q} = \{\mathbb{1}\} \subseteq_c \mathbb{P}$.

A simple example of a dense embedding:

Example III.3.69 *Let* $\mathbb{P} = \mathrm{Fn}(\omega, \omega)$. *Let* $\mathbb{T} = \omega^{<\omega} = \{p \in \mathbb{P} : \mathrm{dom}(p) \in \omega\}$. *Then* $\mathbb{T}$ *is a dense subposet of* $\mathbb{P}$, *so that* $MA_{\mathbb{P}}(\kappa) \leftrightarrow MA_{\mathbb{T}}(\kappa)$ *for all* κ.

We use the letter $\mathbb{T}$ because we can think of $\mathbb{T}$ as a tree of finite sequences, with root node $\mathbb{1} = \emptyset = (\,)$, as in the proof of Theorem III.3.61. See Section III.5 for more on trees and using them as forcing posets.

Working a little harder, we can show that all countable atomless posets are "essentially alike" from the point of view of *MA*:

Exercise III.3.70 *Let* $\mathbb{P}$ *and* $\mathbb{Q}$ *be any countable atomless forcing posets. Then* $MA_{\mathbb{P}}(\kappa) \leftrightarrow MA_{\mathbb{Q}}(\kappa)$.

Hint. Show $MA_{\mathbb{T}}(\kappa) \leftrightarrow MA_{\mathbb{P}}(\kappa)$ by obtaining a dense embedding from $\mathbb{T}$ into $\mathbb{P}$. Let $i(\mathbb{1}) = \mathbb{1}$. Then, let $\{i(\,(m)\,) : m \in \omega\}$ enumerate some infinite maximal antichain in $\mathbb{P}$ (which exists by Lemma III.3.21). Then, for each $m \in \omega$, Let $\{i(\,(m,n)\,) : n \in \omega\}$ enumerate an infinite maximal antichain in $i(\,(m)\,)\!\downarrow$. Etc.

To make sure that $i(\mathbb{T})$ is dense in $\mathbb{P}$: List $\mathbb{P}$ as $\{p_k : k \in \omega\}$ and make sure that for $k = 0, 1, 2, \ldots$, you eventually put something below p_k into $i(\mathbb{T})$. □

The use of restriction to verify (4) in Example III.3.64 is actually completely general, as the next lemma shows. Besides providing a way of proving that a given i is a complete embedding, it gives a "first-order" equivalent to (4) — that is, one expressed by quantifying only over elements of $\mathbb{Q}$ and $\mathbb{P}$, whereas (4) talks about all subsets $A \subseteq \mathbb{Q}$. Conditions (1)(2)(3)(5) are already first-order. This will be relevant in Chapter IV when we do forcing, since it will imply that the notion of complete embedding is absolute for transitive models.

Definition III.3.71 *If $\mathbb{Q}, \mathbb{P}, i$ satisfy* (1)(2)(3) *of Definition III.3.65, and $p \in \mathbb{P}$, then $\boxed{p} \in \mathbb{Q}$ is a* reduction *of p to $\mathbb{Q}$ iff $\forall q \in \mathbb{Q}\,[i(q) \perp p \to q \perp \boxed{p}]$.*

This $\boxed{p}$ is not uniquely defined, but sometimes there is a natural choice. In Example III.3.64, where $I \subseteq L$ and i is the inclusion map from $\mathrm{Fn}(I, J)$ into $\mathrm{Fn}(L, J)$, if $p \in \mathrm{Fn}(L, J)$, then $\boxed{p} := p{\restriction}I$ is a reduction of p to $\mathrm{Fn}(I, J)$. In this case we also have $p \leq \boxed{p}$, but this is not required in general. Generalizing the argument from this example:

Lemma III.3.72 *If $\mathbb{Q}, \mathbb{P}, i$ satisfy* (1)(2)(3) *of Definition III.3.65, then i is a complete embedding iff for all $p \in \mathbb{P}$, there exists some reduction of p to $\mathbb{Q}$.*

Proof. For $\leftarrow$: We verify (4), so let $A \subseteq \mathbb{Q}$ be a maximal antichain in $\mathbb{Q}$. If $i(A)$ is not maximal in $\mathbb{P}$, fix $p \in \mathbb{P}$ such that $i(q) \perp p$ for all $q \in A$. If $\boxed{p}$ is a reduction of p to $\mathbb{Q}$, then $A \cup \{\boxed{p}\}$ contradicts maximality if A in $\mathbb{Q}$.

For $\rightarrow$: Fix $p \in \mathbb{P}$; we find a reduction of p to $\mathbb{Q}$. Call $A \subseteq \mathbb{Q}$ *nice* iff A is an antichain and $i(q) \perp p$ for all $q \in A$. Then $\emptyset$ is (trivially) nice. Fix a maximal nice $A \subseteq \mathbb{Q}$. Then $i(A)$ is an antichain in $\mathbb{P}$ by (3), but so is $i(A) \cup \{p\}$, so by (4), A is not a maximal antichain in $\mathbb{Q}$. So, fix $\boxed{p} \in \mathbb{Q}$ such that $q \perp \boxed{p}$ for all $q \in A$. Then $\boxed{p}$ is a reduction of p to $\mathbb{Q}$. To prove this, suppose we had some $q \in \mathbb{Q}$ with $i(q) \perp p$ but $q \not\perp \boxed{p}$. Then fix $q_1 \in \mathbb{Q}$ with $q_1 \leq q$ and $q_1 \leq \boxed{p}$. Then $q \perp q_1$ for all $q \in A$ and $i(q_1) \perp p$, so $A \cup \{q_1\}$ is nice, contradicting the fact that A is a maximal nice set. □

A product order gives a simple example of a complete embedding:

Lemma III.3.73 *Suppose that $\mathbb{P} = \mathbb{Q} \times \mathbb{S}$, and define $i : \mathbb{Q} \to \mathbb{P}$ by $i(q) = (q, \mathbb{1})$. Then i is a complete embedding.*

Proof. Let $\boxed{(q, s)} = q$. □

Note that Definition III.3.65 of "complete embedding" did not require i to be 1-1, although it has been 1-1 in our examples. We might have $i(q_1) = i(q_2)$ but $q_1 \neq q_2$ when the forcing conditions q_1, q_2 "say the same thing". For a trivial example, let $\mathbb{P} = \{\mathbb{1}\}$ and let $\mathbb{Q}$ be any poset in which any two forcing conditions are compatible (for example, a total order with largest element $\mathbb{1}$); then all forcing conditions in $\mathbb{Q}$ "say nothing"; let $i(q) = \mathbb{1}$. Then i is a complete embedding, and also a dense embedding. Section III.4 contains some less trivial examples.

We conclude this section with some assorted exercises.

Exercise III.3.74 *Let $\mathbb{Q} = \prod^{\mathrm{fin}}_{i \in I} \mathbb{P}_i$. Prove:*

a. If each $\mathbb{P}_i$ has $\aleph_1$ as a pre-caliber, then so does $\mathbb{Q}$.
b. If each $\mathbb{P}_i$ is σ-centered and $|I| \leq \mathfrak{c}$ then $\mathbb{Q}$ is σ-centered.
c. If $|I| > \mathfrak{c}$ and there are $p_i, q_i \in \mathbb{P}_i$ for each i with $p_i \perp q_i$, then $\mathbb{Q}$ is not σ-centered.

The $\mathbb{1}$ is rarely used in *MA* applications, and many texts (e.g., [49, 54]) omit it in the statement of *MA*. We included the $\mathbb{1}$ because it is used in forcing and in the definition of $\prod^{\mathsf{fin}}$, but omitting it yields an equivalent axiom:

Exercise III.3.75 *Let $MA^+(\kappa)$ be the "strengthening" of $MA(\kappa)$ obtained by allowing the $\mathbb{P}$ in Definition III.3.11 to be any ccc pre-order, which need not have a largest element. Note that the definitions of "dense" and "ccc" do not need a $\mathbb{1}$. Restate* (1) *in Definition III.3.10 of "filter" to say $G \neq \emptyset$. Then prove that $MA(\kappa)$ implies $MA^+(\kappa)$.*

Hint. Add a fake $\mathbb{1}$ on top of $\mathbb{P}$. □

Exercise III.3.76 (Fremlin and Miller) *Prove that* $\mathrm{cf}(\mathfrak{m}) > \omega$.

Hint. Use Lemma III.3.46 and Exercise III.1.38 and Theorem III.3.61. □

It is consistent to have $\mathrm{cf}(\mathfrak{m}) = \omega_1 < \mathfrak{m}$; see [56]. It is unknown whether $\mathrm{cf}(\mathfrak{m}) = \omega_2 < \mathfrak{m}$ is consistent.

Exercise III.3.77 *Prove that $MA(\kappa)$ implies that every total order of size κ can be embedded into $\mathcal{P}(\omega)/\mathsf{fin}$. That is, if $\lhd$ is a strict total order of X and $|X| = \kappa$, then there are $A_x \in [\omega]^\omega$ such that whenever $x \lhd y$: $A_x \backslash A_y$ is finite and $A_y \backslash A_x$ is infinite.*

Hint. Code the A_x by a function $F : X \times \omega \to 2$; so $A_x = \{n : F(x,n) = 1\}$. Then $p \in \mathbb{P}$ will be a finite approximation to F. So, let p be a triple, (S_p, k_p, σ_p), where $S_p \in [X]^{<\omega}$, $k_p \in \omega$, and $\sigma_p : S_p \times k_p \to 2$; then p determines F on the finite set $S_p \times k_p$. Define $q \leq p$ iff $S_q \supseteq S_p$, $k_q \geq k_p$, and $\sigma_q \supseteq \sigma_p$ (σ_q extends the function σ_p), *and* for all $x, y \in S_p$ with $x \lhd y$ and all n with $k_p \leq n < k_q$, $\sigma_q(x,n) \leq \sigma_q(y,n)$. This last requirement means that $p \in \mathbb{P}$ implicitly makes the *promise* that $A_x \backslash A_y \subseteq k_p$; this is analogous to the promises in the proof of Lemma III.3.22. Observe that the promise is irrelevant if $k_q = k_p$, so that

$$k_p = k_r \quad \to \quad (p \not\perp r \leftrightarrow \sigma_p \not\perp \sigma_r) \quad , \qquad (\star)$$

where the second "$\not\perp$" is as elements of $\mathrm{Fn}(X \times \omega, 2)$. Now, if there were an uncountable antichain A, you could assume that k_p is the same for all $p \in A$, and then apply Lemma III.3.7. □

As we said on page 180, in most theorems of the form $MA(\kappa) \to$ [something nice about κ], the result for $\kappa = \aleph_0$ is a classical fact, and the result for $\kappa \geq \aleph_1$ is not provable just from $ZFC + \mathfrak{c} > \kappa$. Here, the result for $\kappa = \aleph_1$ is also a classical fact, known from the early 1900s:

Exercise III.3.78 *Prove in ZFC that every total order of size $\leq \aleph_1$ can be embedded into $\mathcal{P}(\omega)/\mathsf{fin}$.*

Hint. WLOG, $\lhd$ is some total order on the set $X = \omega_1$. Then choose A_α by recursion on α. Given A_ξ for $\xi < \alpha$: Let $L_\alpha = \{\xi < \alpha : \xi \lhd \alpha\}$ and $R_\alpha = \{\xi < \alpha : \alpha \lhd \xi\}$. You need to choose A_α so that for all $\xi < \alpha$: $A_\alpha \neq^* A_\xi$, and $\xi \in L_\alpha \to A_\xi \subseteq^* A_\alpha$ and $\xi \in R_\alpha \to A_\alpha \subseteq^* A_\xi$.

There are several cases, depending on whether L_α has a largest element (under $\lhd$), and whether R_α has a smallest element, etc. In most cases, you use a diagonal argument similar to the proofs of $\mathfrak{b} > \aleph_0$ and $\mathfrak{p} > \aleph_0$ (Lemmas III.1.12 and III.1.22). This works because α is always countable. □

But the result for $\aleph_2$ is not provable from $ZFC + \mathfrak{c} > \aleph_2$. It is consistent with $\mathfrak{c}$ being arbitrarily large that the order type ω_2 is not embeddable in $\mathcal{P}(\omega)/\mathsf{fin}$; see Exercise IV.7.53.

Exercise III.3.79 *Prove that $\kappa < \mathfrak{p}$ implies that every total order of size κ can be embedded into $\mathcal{P}(\omega)/\mathsf{fin}$.*

Hint. Show that the poset used in Exercise III.3.77 is actually σ-centered. □

The diagonal argument from Exercise III.3.78 is easily modified to show:

Exercise III.3.80 *Assume that $A_n, B_n \in [\omega]^\omega$ for $n \in \omega$ and $A_m \perp B_n$ for each m, n. Prove that there is a $C \in [\omega]^\omega$ such that $A_n \subseteq^* C$ and $B_n \perp C$ for each n.*

So, C splits $\aleph_0$ sets from $\aleph_0$ almost disjoint sets. Since $MA(\kappa)$ implies that $\mathfrak{b} > \kappa$ and $\mathfrak{p} > \kappa$, one might hope that $MA(\kappa)$ implies that you can split $\aleph_1$ sets from $\aleph_1$ almost disjoint sets, but this is false by the following ZFC counterexample due to Luzin (1947). This shows that even if you assume $MA(\aleph_1)$, there are some non-trivial ways in which $\aleph_1$ does not behave like $\aleph_0$.

Exercise III.3.81 *Find almost disjoint $A_\alpha \in [\omega]^\omega$ for $\alpha < \omega_1$ such that whenever X, Y are two uncountable subsets of ω_1, there is* no *$C \in [\omega]^\omega$ such that $A_\alpha \subseteq^* C$ for all $\alpha \in X$ and $A_\beta \perp C$ for all $\beta \in Y$.*

Hint. Construct the A_α inductively. Make sure that for each α and each $n \in \omega$, $\{\xi < \alpha : |A_\xi \cap A_\alpha| \leq n\}$ is finite. □

$MA(\aleph_1)$ does imply that you can split $\aleph_0$ from $\aleph_1$; in fact, all you need is $\mathfrak{b} > \aleph_1$.

Exercise III.3.82 *Assume that $A_n, B_\alpha \in [\omega]^\omega$ for $n \in \omega$ and $\alpha < \kappa$, and $A_n \perp B_\alpha$ for each n, α. Assume that $\kappa < \mathfrak{b}$. Prove that there is a $C \in [\omega]^\omega$ such that $A_n \subseteq^* C$ and $B_\alpha \perp C$ for each n, α.*

Hint. This is a little like the proof that $\mathfrak{b} \leq \mathfrak{a}$ (Lemma III.1.19). First, reduce to the situation where $A_0 \subseteq A_1 \subseteq \cdots$ and each $A_{n+1} \setminus A_n$ is infinite. Now, for each α, let $f_\alpha(n)$ be the least element of $B_\alpha \cap (A_{n+1} \setminus A_n)$, and use the fact that $\{f_\alpha : \alpha < \kappa\}$ can be dominated. □

Exercise III.3.83 *Assume that $E \subseteq \mathbb{R} \times \mathbb{R}$ is the graph of a function and $|E| < \mathfrak{p}$. Prove that there are Cantor sets $K_n \subseteq \mathbb{R} \times \mathbb{R}$, for $n \in \omega$, such that each K_n is the graph of a function and $E \subseteq \bigcup_n K_n$.*

Hint. First derive this from $MA(|E|)$ and then note that the natural poset $\mathbb{Q}$ is σ–centered. $\mathbb{Q}$ can be the finite support product of ω copies of $\mathbb{P}$; the nth copy describes K_n. A Cantor set $K \subseteq \mathbb{R} \times \mathbb{R}$ can always be described by a tree of rational open boxes $I_s \times J_s$ for $s \in 2^{<\omega}$, where $\overline{I}_{s^\frown 0} \times \overline{J}_{s^\frown 0}$ and $\overline{I}_{s^\frown 1} \times \overline{J}_{s^\frown 1}$ are disjoint subsets of $I_s \times J_s$. Each $p \in \mathbb{P}$ can be a pair, (t^p, e^p), where t^p gives a finite amount of the tree and $e^p \in [E]^{<\omega}$ is a promise that the Cantor set will contain all the points of e. □

A related fact is easy to prove without using posets:

Exercise III.3.84 *Assume that $E \subseteq \mathbb{R} \times \mathbb{R}$ and $|E| <$ non(meager). Prove that there are Cantor sets $K_n \subseteq \mathbb{R} \times \mathbb{R}$, for $n \in \omega$, such that $E \subseteq \bigcup_n K_n$.*

Hint. Doing this in $\mathbb{R}$ instead of $\mathbb{R}^2$ is easy, since then every closed nowhere dense set is contained in a Cantor set. In $\mathbb{R}^2$, use this observation with E projected to the two coordinate axes. □

One cannot replace "$|E| < \mathfrak{p}$" by "$|E| <$ non(meager)" in Exercise III.3.83. For example, there is an $E \subseteq \mathbb{R}^2$ of positive Lebesgue outer measure such that E is the graph of a function and $|E| =$ non(null). Now, assume that non(null) $<$ non(meager); this is consistent by Exercise V.4.31. But a Cantor set that is the graph of a function must be a null set in the plane, so countably many such sets cannot cover E.

In Lemma III.3.18, you can't drop "Hausdorff": consider a countable T_1 space where the closed sets are X and the finite sets. More importantly, you can't drop the ccc when $\kappa > \aleph_0$. Here's two examples, one for set-theorists and one for analysts:

Exercise III.3.85 *Give the ordinal $\omega_1 + 1$ the order topology, which is compact. Let $X = (\omega_1 + 1)^\omega$. Let Y be the unit ball in a non-separable Hilbert space with the weak topology. Prove that both X and Y are unions of $\aleph_1$ closed nowhere dense sets.*

Hint. For X, consider the sets $((\alpha + 1) \cup \{\omega_1\})^\omega$. For Y, consider hyperplanes orthogonal to elements of an orthonormal base. □

Exercise III.3.85 and Lemma III.3.18 are related to the Baire Category Theorem for compact spaces. There is also a Baire Category Theorem for complete metric spaces. Exercise III.1.39 shows that one can replace the $\aleph_0$ in the Category Theorem by any cardinal less than $\mathfrak{p}$ *if* the metric space is separable, but this is not true for non-separable spaces. Here's two examples, one for set-theorists and one for analysts:

Exercise III.3.86 *Let $X = (\omega_1)^\omega$, where the set ω_1 has discrete topology. Let Y be any non-separable Hilbert space. Both X and Y are complete metric spaces. Prove that both X and Y are unions of $\aleph_1$ closed nowhere dense sets.*

Hint. For X, consider the closed subspaces α^ω. For Y, consider hyperplanes orthogonal to elements of an orthonormal base. □

We have seen (Lemma III.3.46) that MA implies every ccc poset of size less than $\mathfrak{c}$ is σ–centered. This is false for posets of size $\mathfrak{c}$:

Exercise III.3.87 *Let $\mathbb{P}$ be the set of all closed subsets of $[0,1]$ of positive Lebesgue measure; let $\leq$ be $\subseteq$ and $\mathbb{1} = [0,1]$. Show that $\mathbb{P}$ is ccc and not σ–centered. Furthermore, assuming CH, show that $\mathbb{P}$ does not have $\aleph_1$ as a pre-caliber.*

Hint. If $C_\ell \subseteq \mathbb{P}$ is centered for each $\ell < \omega$, let $F_\ell = \bigcap C_\ell$, and find a $p \in \mathbb{P}$ such that no F_ℓ is a subset of p. For the second part (due to Erdös), let $[0,1] = \{x_\xi : \xi < \omega_1\}$, and consider $p_\alpha \in \mathbb{P}$ such that $p_\alpha \cap \{x_\xi : \xi < \alpha\} = \emptyset$. □

This $\mathbb{P}$ does have the Knaster property described in the next exercise:

Exercise III.3.88 *Let μ be a probability measure defined on some σ–algebra $\mathcal{A}$ of subsets of X. Let $\mathbb{P} = \{p \in \mathcal{A} : \mu(p) > 0\}$; let $\leq$ be $\subseteq$ and $\mathbb{1} = X$. Let $E \in [\mathbb{P}]^{\aleph_1}$. Prove that there is a linked $L \in [E]^{\aleph_1}$.*

Hint. WLOG, $\mu(p) \geq 1/n$ for all $p \in E$. Now, induct on n; $n = 1$ is trivial. For $n > 1$, there are two cases. If $\exists p \in E \left[|\{q \in E : q \perp p\}| = \aleph_1\right]$, apply the result for $n-1$ to the subspace $X \backslash p$; note that $q \perp p$ implies that $\mu(q \backslash p)/\mu(X \backslash p) \geq 1/(n-1)$. If there is no such p, then every maximal linked sub-family of E is uncountable. □

Exercise III.3.89 *The amoeba order, used in the proof of Lemma III.3.28, is not σ–centered.*

Also, in the proof that $\mathfrak{m} \leq \mathrm{add}(\mathsf{null})$, one cannot expect to replace the amoeba order by some σ–centered poset, since $\mathfrak{m}_\sigma \leq \mathrm{add}(\mathsf{null})$ is not provable in *ZFC* (see Corollary V.4.16). Recall that $\mathfrak{m}_\sigma = \mathfrak{p}$ by Theorem III.3.61.

Lemma III.3.13 proved $\neg MA(\mathfrak{c})$ by using $\mathrm{Fn}(I,J)$, where $|I| = \aleph_0$ and $2 \leq |J| \leq \aleph_0$, because that was the simplest poset to describe, but actually we could have used any atomless ccc poset. Although $MA_{\mathbb{P}}(\mathfrak{c})$ holds trivially when $\mathbb{P}$ has an atom (Lemma III.3.20),

Exercise III.3.90 *$MA_{\mathbb{P}}(\mathfrak{c})$ is false whenever $\mathbb{P}$ is ccc and atomless.*

Hint. If $|\mathbb{P}| \leq \mathfrak{c}$, then there are at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ maximal antichains. If G meets all of them, then you get a contradiction from the observation that $\mathbb{P} \backslash G$ is dense. If $|\mathbb{P}| > \mathfrak{c}$, first get $\mathbb{Q} \subseteq \mathbb{P}$ somewhat like in the proof of Lemma III.3.51, so that $|\mathbb{Q}| \leq \mathfrak{c}$ and whenever $A \subseteq \mathbb{Q}$ is a (countable) antichain, if A is maximal in $\mathbb{Q}$ then A is maximal in $\mathbb{P}$ (see Lemma I.13.23). □

Definition III.3.91 *For a poset $\mathbb{P}$, the* Suslin number, *$S(\mathbb{P})$ is the least κ such that $|A| < \kappa$ for all antichains $A \subseteq \mathbb{P}$. For a topological space X, $S(X) = S(\mathbb{O}_X)$.*

Clearly, $S(\mathbb{P}) \leq |\mathbb{P}|^+$ for all $\mathbb{P}$, and $\mathbb{P}$ has the ccc iff $S(\mathbb{P}) \leq \aleph_1$.

Exercise III.3.92 *For any $\mathbb{P}$, call $p \in \mathbb{P}$ S–*minimal *iff $S(p\downarrow) = S(q\downarrow)$ for all $q \leq p$. Prove that the set of S–minimal elements is dense in $\mathbb{P}$.*

Hint. There is no decreasing ω–sequence of ordinals. □

It is trivial to construct $\mathbb{P}$ with $S(\mathbb{P})$ finite; and, $S(\mathbb{P}) \geq \aleph_1$ whenever $\mathbb{P}$ is atomless (see Lemma III.3.21). Also,

Exercise III.3.93 *There is no $\mathbb{P}$ such that $S(\mathbb{P}) = \aleph_0$.*

Hint. If all antichains are finite, consider a maximal antichain all of whose elements are S–minimal. □

A similar idea can be used to prove the following theorem of Erdös and Tarski (1943):

Exercise III.3.94 *If $S(\mathbb{P}) > \aleph_0$ then $S(\mathbb{P})$ is regular.*

For each regular $\kappa > \aleph_0$, there is a $\mathbb{P}$ with $S(\mathbb{P}) = \kappa$. If $\kappa = \lambda^+$, then use

Exercise III.3.95 *$S(\mathrm{Fn}(I, \lambda)) = \lambda^+$ for any infinite cardinal λ and any non-empty I.*

If κ is not a successor,

Exercise III.3.96 *Let κ be weakly inaccessible, and let $\mathbb{P} = \prod^{\mathrm{fin}}_{\alpha<\kappa} \mathrm{Fn}(\omega, \alpha)$. Show that $S(\mathbb{P}) = \kappa$.*

In some *MA* applications, there is a "natural" poset $\mathbb{P}$ to use that "obviously" works, except that the proof that $\mathbb{P}$ is ccc is non-trivial:

Exercise III.3.97 *Assume $MA(\kappa)$. Fix $\tau < \omega_1$, and fix $\mathcal{F} \subseteq [\omega_1]^{\aleph_0}$ such that $|\mathcal{F}| \leq \kappa$ and such that all elements of $\mathcal{F}$ have order type $< \tau$. Prove that there is an uncountable $J \subseteq \omega_1$ such that $J \cap x$ is finite for all $x \in \mathcal{F}$.*

Hint. This is like the proof that $\mathfrak{m} \leq \mathfrak{p}$, but the proof of ccc is harder. $p \in \mathbb{P}$ can be a pair $p = \langle A_p, B_p \rangle$, where $A_p \in [\omega_1]^{<\omega}$ and $B_p \in [\mathcal{F}]^{<\omega}$. A_p is an inner approximation to J. Informally, B_p "promises" that for all $x \in B_p$, all elements outside of A_p put into J will not lie in x (that is, $J \cap x \subseteq A_p$). Formally $q \leq p$ iff $A_q \supseteq A_p$ and $B_q \supseteq B_p$ and $(A_q \backslash A_p) \cap x = \emptyset$ for all $x \in B_p$. You can use the Delta System Lemma plus the bound on the order types to prove that $\mathbb{P}$ is ccc. □

Note that you need the bound on the order types here; otherwise, consider $\mathcal{F} = \omega_1 \backslash \omega$ (that is, $\mathcal{F}$ is the set of all infinite initial segments of ω_1).

III.4 Equivalents of *MA*

We discuss here some well-known equivalents of $MA(\kappa)$ in terms of topological spaces and boolean algebras. These results are important for general topology and boolean algebra, but most applications of $MA(\kappa)$ work directly from Definition III.3.11, as does the proof of consistency of $MA(\kappa)$. Because of that, this section is not needed for most of the results in this book, although it will occasionally be useful to express forcing results in terms of boolean algebras.

This section assumes that the reader is familiar with the classical facts relating boolean algebra and topology, such as regular open algebras and the Stone Representation Theorem; these are discussed in Halmos [44]. In particular, we assume that the reader knows the next four definitions, but we give them anyway to set the notation.

Definition III.4.1 *A* boolean algebra *is a model for the axioms for boolean algebras, written in* $\mathcal{L} = \{\leq, \wedge, \vee, \mathbb{0}, \mathbb{1}, '\}$.

As usual, we shall abuse notation by confusing a model $\mathfrak{B}$ with its universe $\mathbb{B}$. The axioms say that $\leq$ is a partial order, with least element $\mathbb{0}$ and greatest element $\mathbb{1}$, and that $\mathbb{B}$ is a complemented distributive lattice; so $x \vee x' = \mathbb{1}$ and $x \wedge x' = \mathbb{0}$. They also say that $\mathbb{0} \neq \mathbb{1}$, so that if $\mathbb{B}$ is a boolean algebra, then $\mathbb{B}\backslash\{\mathbb{0}\}$ is a forcing poset. Standard poset terminology, such as "ccc" or "$MA_{\mathbb{B}}(\kappa)$" or "σ–centered", when applied to $\mathbb{B}$, really refer to $\mathbb{B}\backslash\{\mathbb{0}\}$. For $p, q \in \mathbb{B}\backslash\{\mathbb{0}\}$, $p \perp q$ iff $p \wedge q = \mathbb{0}$. Note that Definition III.3.19 of "atom" and "atomless" in posets yield the standard boolean algebra notions when applied to boolean algebras.

Of course, $\mathbb{B}$ itself, with the $\mathbb{0}$, is also a forcing poset; but applications of *MA* and forcing are trivial when applied to posets that have an atom (such as $\mathbb{0}$); see Lemma III.3.20.

Standard examples of boolean algebras include $\mathcal{P}(A)$ for any non-empty set A, and various quotients $\mathcal{P}(A)/\mathcal{I}$ where $\mathcal{I}$ is an ideal on A. In particular, the cardinals $\mathfrak{p}$ and $\mathfrak{a}$ describe properties of $\mathcal{P}(\omega)/\mathrm{fin}$. Also, topological spaces yield boolean algebras:

Definition III.4.2 *Let X be a non-empty topological space. Then its* clopen algebra, $\mathrm{clop}(X)$, *is the set of all $U \subseteq X$ that are clopen (that is, both closed and open). The $\leq, \wedge, \vee, \mathbb{0}, \mathbb{1}$ are $\subseteq, \cap, \cup, \emptyset, X$, respectively, and $U' = X\backslash U$.*

Then X is connected iff $\mathrm{clop}(X)$ is the two element algebra, $\{\mathbb{0}, \mathbb{1}\} = \{\emptyset, X\}$. Most applications of $\mathrm{clop}(X)$ are for an X that is zero dimensional; that is, $\mathrm{clop}(X)$ forms a base for the topology.

$\mathrm{clop}(X)$ is a sub-algebra of $\mathcal{P}(X)$; that is, the boolean operations of $\mathrm{clop}(X)$ agree with the boolean operations on $\mathcal{P}(X)$. By the Stone Representation Theorem (below), every boolean algebra is isomorphic to the clopen algebra of its Stone space, and hence to a sub-algebra of some $\mathcal{P}(X)$. It follows that any

boolean equation valid in arbitrary $\mathcal{P}(X)$ (such as $(a \vee b)' = a' \wedge b'$) is valid in all boolean algebras.

Definition III.4.3 *If $\mathbb{B}$ is a boolean algebra, then* st($\mathbb{B}$) *denotes the* Stone space *of $\mathbb{B}$. The elements $x \in$ st($\mathbb{B}$) are the ultrafilters on $\mathbb{B}$, and the clopen sets of* st($\mathbb{B}$) *are of the form $N_b = \{x : b \in x\}$.*

We may regard st($\mathbb{B}$) as a closed subspace of $2^{\mathbb{B}}$ (identifying ultrafilters with their characteristic functions).

Theorem III.4.4 (Stone Representation Theorem) *For all boolean algebras $\mathbb{B}$:* st($\mathbb{B}$) *is a compact zero dimensional Hausdorff space and $\mathbb{B}$ is isomorphic to* clop(st($\mathbb{B}$)) *via the map $b \mapsto N_b$.*

Every non-empty compact zero dimensional Hausdorff space X is homeomorphic to st(clop(X)) *via the map $x \mapsto \{p \in \text{clop}(X) : x \in p\}$.*

Lemma III.4.5 *Let $\mathbb{B}$ be any boolean algebra and let $X =$ st($\mathbb{B}$). Then $MA_{\mathbb{B}}(\kappa)$ holds iff X is not the union of any $\leq\kappa$ closed nowhere dense sets.*

Proof. For $\rightarrow$: Fix closed nowhere dense $H_\alpha \subseteq X$ for $\alpha < \kappa$; we must show that $\bigcup_{\alpha<\kappa} H_\alpha \neq X$. This is similar to the proof of Lemma III.3.18, but we apply MA to $\mathbb{B}$, rather than to $\mathbb{O}_X$. By Theorem III.4.4, we identify $\mathbb{B}$ with clop(X). If G is any filter on $\mathbb{B}$, then G is a family of compact subsets of X with the FIP, so that $F_G := \bigcap\{p : p \in G\}$ is non-empty. Let $D_\alpha = \{q \in \text{clop}(X) : q \cap H_\alpha = \emptyset\}$. Then D_α is dense because X is zero dimensional. Note that $G \cap D_\alpha \neq \emptyset$ implies $F_G \cap H_\alpha = \emptyset$. So, apply $MA_{\mathbb{B}}(\kappa)$ to choose G such that $G \cap D_\alpha \neq \emptyset$ for all α.

For $\leftarrow$: To prove $MA_{\mathbb{B}}(\kappa)$, fix dense $D_\alpha \subseteq \mathbb{B}\backslash\{\mathbb{0}\}$ for $\alpha < \kappa$. We must find a filter x that meets all the D_α. In X, let $U_\alpha = \bigcup\{N_b : b \in D_\alpha\}$ and let $H_\alpha = X\backslash U_\alpha$. Each U_α is obviously open, and U_α is dense in the topological sense because D_α is dense in the order sense. So, H_α is closed and nowhere dense. Applying our assumption on X, fix $x \notin \bigcup_\alpha H_\alpha$, so that $x \in \bigcap_\alpha U_\alpha$. By definition of of st($\mathbb{B}$), x is an ultrafilter on $\mathbb{B}$. For each α, there is a $b \in D_\alpha$ such that $x \in N_b$, so that $b \in x$; so, $x \cap D_\alpha \neq \emptyset$. □

Definition III.4.6 *Let X be a non-empty topological space. Then its* regular open algebra, ro(X), *is the set of all $U \subseteq X$ that are both open and* regular *(that is, $U = \text{int}(\text{cl}(U))$). The $\leq, \wedge, \mathbb{0}, \mathbb{1}$ are $\subseteq, \cap, \emptyset, X$, respectively. $U \vee V = \text{int}(\text{cl}(U \cup V))$ and $U' = \text{int}(X\backslash U)$.*

Unlike clop(X), the algebra ro(X) is not in general a sub-algebra of $\mathcal{P}(X)$, since the boolean operations of ro(X) don't always agree with the boolean operations on $\mathcal{P}(X)$. For example, U' cannot be $X\backslash U$ unless U is clopen, since otherwise $X\backslash U$ will not be open. The interior of a *closed* set is always regular open, so in particular U', as defined above is regular open.

X is *extremally disconnected* iff every regular open set is clopen; equivalently, $\overline{U}$ is open whenever U is open. For these (and only these) X, $\mathrm{ro}(X) = \mathrm{clop}(X)$ and hence $\mathrm{ro}(X)$ is a sub-algebra of $\mathcal{P}(X)$.

$\mathrm{ro}(X)$ is always a complete boolean algebra; that is, if $\mathcal{E} \subseteq \mathrm{ro}(X)$, then $\bigvee \mathcal{E}$ exists (and equals $\mathrm{int}(\mathrm{cl}(\bigcup \mathcal{E}))$). Also, if $\mathbb{B}$ is a boolean algebra, then $\mathbb{B}$ is complete iff $\mathrm{clop}(\mathbb{B})$ is extremally disconnected.

Theorem III.4.7 *For any infinite cardinal κ, the following are equivalent:*

1. $MA(\kappa)$.
2. $MA_{\mathbb{B}}(\kappa)$ *holds for all ccc boolean algebras* $\mathbb{B}$.
3. $MA_{\mathbb{B}}(\kappa)$ *holds for all complete ccc boolean algebras* $\mathbb{B}$.
4. *No ccc compact Hausdorff space X is the union of $\leq \kappa$ closed nowhere dense sets.*
5. (4) *restricted to those X that are also zero dimensional.*
6. (4) *restricted to those X that are also extremally disconnected.*

We shall prove this below. Note that (4) $\rightarrow$ (5) $\rightarrow$ (6) and (1) $\rightarrow$ (2) $\rightarrow$ (3) are obvious, and (1) $\rightarrow$ (4) was Lemma III.3.18. Also, (6) $\rightarrow$ (3) is clear from Lemma III.4.5, since the compact Hausdorff extremally disconnected spaces are precisely the Stone spaces of complete boolean algebras. So, we shall be done if we establish (3) $\rightarrow$ (1); this will be proved by mapping an arbitrary poset into a complete boolean algebra:

Lemma III.4.8 *For every forcing poset $\mathbb{P}$, there is a complete boolean algebra $\mathbb{B}$ and a dense embedding $i : \mathbb{P} \rightarrow \mathbb{B}\backslash\{\mathbb{0}\}$.*

Proof. Give $\mathbb{P}$ the poset topology $\mathcal{T}_{\mathbb{P}}$ (Definition III.3.53). Let $\mathbb{B} = \mathrm{ro}(\mathbb{P}, \mathcal{T}_{\mathbb{P}})$, and let $i(p) = \mathrm{int}(\mathrm{cl}(p{\downarrow}))$. Now, we must verify (1)(2)(3)(5) of Definition III.3.65.

(1) is clear, since $\mathbb{1}_{\mathbb{B}} = \mathbb{P}$, and (2) is clear, since $\leq_{\mathbb{B}}$ is $\subseteq$. For (5), fix $W \in \mathbb{B}\backslash\{\mathbb{0}\}$; that is $W \subseteq \mathbb{P}$ is a non-empty regular open set. Fix $p \in W$; then $p{\downarrow} \subseteq W$ by definition of the topology, so that $i(p) = \mathrm{int}(\mathrm{cl}(p{\downarrow})) \subseteq \mathrm{int}(\mathrm{cl}(W)) = W$. For (3):

$$p \perp q \leftrightarrow p{\downarrow} \cap q{\downarrow} = \emptyset \leftrightarrow p{\downarrow} \cap \mathrm{int}(\mathrm{cl}(q{\downarrow})) = \emptyset \leftrightarrow$$
$$\mathrm{int}(\mathrm{cl}(p{\downarrow})) \cap \mathrm{int}(\mathrm{cl}(q{\downarrow})) = \emptyset \leftrightarrow i(p) \wedge i(q) = \mathbb{0}.$$

For the second and third $\leftrightarrow$s, use the fact that $p{\downarrow}$ and $\mathrm{int}(\mathrm{cl}(q{\downarrow}))$ are open. □

Proof of Theorem III.4.7. As remarked above, we need only prove (3) $\rightarrow$ (1), so assume (3), fix a ccc poset $\mathbb{P}$, and we must prove $MA_{\mathbb{P}}(\kappa)$.

Fix $i : \mathbb{P} \rightarrow \mathbb{B}\backslash\{\mathbb{0}\}$ as in Lemma III.4.8. Note that $\mathbb{B}$ has the ccc: if not, let $\{b_\alpha : \alpha < \omega_1\}$ be an antichain in $\mathbb{B}$. Since $i(\mathbb{P})$ is dense in $\mathbb{B}$, choose $p_\alpha \in \mathbb{P}$ with $i(p_\alpha) \leq b_\alpha$. Then $\{p_\alpha : \alpha < \omega_1\}$ contradicts the ccc for $\mathbb{P}$.

Then $MA_{\mathbb{P}}(\kappa)$ follows from $MA_{\mathbb{B}}(\kappa)$ by Lemma III.3.68. □

Exercise III.4.9 *Show that the* $\mathbb{B}$ *of Lemma III.4.8 is unique up to isomorphism. That is, if* $\mathbb{B}_1, i_1$ *and* $\mathbb{B}_2, i_2$ *both satisfy Lemma III.4.8, then there is an isomorphism* $h : \mathbb{B}_1 \to \mathbb{B}_2$ *such that* $i_2 = h \circ i_1$.

$\mathbb{B}$ is referred to as *the completion* of $\mathbb{P}$. In the case that $\mathbb{P}$ is also $\mathbb{A}\backslash\{\mathbb{0}\}$ for some boolean algebra, the completion $\mathbb{B}$ of $\mathbb{A}$ is discussed in texts [44] on boolean algebras; in that case, $\mathbb{B}$ is usually obtained as the regular open algebra of the Stone space of $\mathbb{A}$.

As pointed out on page 191, a complete (or even dense) embedding i is not required to be 1-1. Continuing the trivial example given there, say $\mathbb{P}$ is a total order with largest element $\mathbb{1}$. Then all p, q are compatible in $\mathbb{P}$, and $\mathbb{P}$ has a dense embedding into the poset $\{\mathbb{1}\}$. In our current context, note that in the poset topology for a total order, $\mathrm{ro}(\mathbb{P}) = \{\emptyset, \mathbb{P}\} = \{\mathbb{0}, \mathbb{1}\}$, so the $\mathbb{B}$ of Lemma III.4.8 is the two-element algebra and $\mathbb{B}\backslash\{\mathbb{0}\} = \{\mathbb{1}\}$.

In most uses in mathematics "embedding" does imply "1-1". In our current context, $i : \mathbb{P} \to \mathbb{B}$ may fail to be 1-1 when there are two different p, q that "*say the same thing*"; then $i(p) = i(q)$. This doesn't happen if $\mathbb{P}$ satisfies the quite reasonable condition of being *separative*:

Definition III.4.10 *The poset* $\mathbb{P}$ *is called* separative *iff* $\leq$ *is a partial order and, whenever* $p \not\leq q$, *there is an* $r \leq p$ *such that* $r \perp q$.

Exercise III.4.11 *Show that* $\mathbb{P}$ *is separative iff the* i *in Lemma III.4.8 is 1-1 and satisfies* $\forall p, q \in \mathbb{P}\,[p \leq q \leftrightarrow i(p) \leq i(q)]$. *Furthermore, for* any $\mathbb{P}$, *show that* $i(p) \leq i(q)$ *iff there is no* $r \leq p$ *such that* $r \perp q$.

A specific example: if $\mathbb{P}$ is the non-zero elements of a boolean algebra, then r can be $p \wedge q' \neq \mathbb{0}$ whenever $p \not\leq q$.

As mentioned above, a total order is a trivial example of a non-separative poset, but other examples occur naturally in practice, and we have already used some non-separative posets to prove non-trivial theorems. Consider $\mathbb{O}_X$:

Exercise III.4.12 *Let* x *be any non-isolated point in the Hausdorff space* X, *and fix* $p \in \mathbb{O}_X$ *with* $x \in p$. *Let* $q = p\backslash\{x\}$. *Show that* $i(p) = i(q)$.

When we used $\mathbb{O}_X$ to prove Lemma III.3.18, we used our filter G to form $F_G := \bigcap\{\overline{p} : p \in G\}$. When $\overline{p} = \overline{q}$, they both *say the same thing* about F_G.

Exercise III.4.13 *Show that with a suitable* $\mathcal{E}$, *the* $\mathbb{P}$ *used in the proof of Lemma III.3.22 is not separative. Display concrete* p, q *with* $q \leq p$ *and* $p \not\leq q$ *and* $i(p) = i(q)$.

Hint. Let $s_q = s_p$ and get $\mathcal{W}_q \supsetneq \mathcal{W}_p$ so that the promise made by the extra set(s) in $\mathcal{W}_q$ is redundant upon the promise already made by $\mathcal{W}_p$. □

Exercise III.4.14 *Show that* $\mathrm{Fn}(I,J)$ *is separative. Furthermore, the completion of* $\mathrm{Fn}(I,J)$ *can be identified with the regular open algebra of the space* J^I*, where* J *has the discrete topology; we identify* $i(p)$ *with* $\{x \in J^I : x \supseteq p\}$*, which is clopen, and hence regular open.*

Exercise III.4.15 *Show that* $\mathbb{P}$ *is separative iff* $\leq$ *is a partial order and each* $p{\downarrow}$ *is a regular open set in the poset topology. Then, when* $\mathbb{P}$ *is separative, show that the* $i(p)$ *in Lemma III.4.8 is just* $p{\downarrow}$*.*

Exercise III.4.16 *The following are equivalent:*

1. *There are ccc posets* $\mathbb{P},\mathbb{Q}$ *such that* $\mathbb{P}\times\mathbb{Q}$ *is not ccc.*
2. *There are ccc topological spaces* X,Y *such that* $X\times Y$ *is not ccc.*
3. *Same as (3), with* X,Y *compact Hausdorff.*

Hint. See Lemma III.3.37. □

Exercise III.4.17 *Let* $\mathbb{A},\mathbb{B}$ *be complete boolean algebras with* $\mathbb{A}$ *a sub-algebra of* $\mathbb{B}$*. Show that* $\mathbb{A}$ *is a complete sub-algebra of* $\mathbb{B}$ *in the forcing sense* ($\mathbb{A}\subseteq_{\mathsf{c}}\mathbb{B}$*; Definition III.3.63*) *iff* $\mathbb{A}$ *is a complete sub-algebra of* $\mathbb{B}$ *in the boolean algebra sense* (*all sups and infs of subsets of* $\mathbb{A}$ *have the same values in* $\mathbb{A}$ *as in* $\mathbb{B}$)*. Then, show that when* $\mathbb{A}\subseteq_{\mathsf{c}}\mathbb{B}$ *and* $b\in\mathbb{B}$*, one reduction of* b *to* $\mathbb{A}$ (*see Definition III.3.71 and Lemma III.3.72*) *is* $\bigwedge\{a\in\mathbb{A} : b\leq a\}$*.*

III.5 Trees

The informal notion of a *tree* occurs frequently in mathematics, as well as in computer science and biology. The following precise definition is most useful for applications in set theory:

Definition III.5.1 *A* tree *is a pair* $(T,\sqsubset)$ *such that* $\sqsubset$ *is a strict partial order of* T *and, for all* $y\in T$*,* $\{x\in T : x\sqsubset y\}$ *is well-ordered by* $\sqsubset$*. Then*

- ☞ $y{\downarrow}=\{x\in T : x\sqsubset y\}$ *and* $y{\uparrow}=\{x\in T : y\sqsubset x\}$.
- ☞ $\mathrm{height}(y)=\mathrm{height}(y,T)=\mathrm{type}(y{\downarrow})$.
- ☞ $\mathcal{L}_\alpha=\mathcal{L}_\alpha(T)=\{y\in T : \mathrm{height}(y)=\alpha\}=$ level α of T.
- ☞ $T_\alpha=\{y\in T : \mathrm{height}(y)<\alpha\}=\bigcup\{\mathcal{L}_\xi : \xi<\alpha\}$.
- ☞ $\mathrm{height}(T)$ *is the least* α *such that* $\mathcal{L}_\alpha(T)=\emptyset$.
- ☞ T *is* rooted *iff* $|\mathcal{L}_0(T)|=1$*, in which case the* root *of* T*, or* $\mathbb{1}_T$*, is the element of* $\mathcal{L}_0(T)$.
- ☞ $x,y\in T$ *are* compatible ($x\not\perp y$) *iff* $x\sqsubseteq y$ *or* $y\sqsubseteq x$*;* x,y *are* incompatible ($x\perp y$) *iff they are not compatible.*
- ☞ *An* antichain *is a subset* $A\subseteq T$ *whose elements are pairwise incompatible.*
- ☞ y *is an* immediate successor *of* x *iff* $x\sqsubset y$ *and* $\mathrm{height}(y)=\mathrm{height}(x)+1$.
- ☞ $x\in T$ *is a* leaf *iff* $x{\uparrow}=\emptyset$ *iff* x *has no immediate successors.*

As usual, we shall, as we just did, abuse notation by referring to the tree as "T" if $\sqsubset$ is clear from context. Our terminology $y{\downarrow}$ and $y{\uparrow}$ is consistent with our earlier use (see Definition I.9.2) of $y{\downarrow}$ for $\{x \in A : xRy\}$. Of course, the relation (here, $\sqsubset$) has to be clear from context when using $y{\downarrow}$ or $y{\uparrow}$.

A simple example of a tree is just a well-ordered set (a *stick*); this is a tree in which all pairs of elements are compatible. In particular, the ordinal β is a tree of height β, and $\mathcal{L}_\alpha(\beta) = \{\alpha\}$ when $\alpha < \beta$. If $\beta = 0$, then this is the empty tree, but if $\beta > 0$, then the tree β has a root $\mathbb{1} = 0$.

A tree that we have already mentioned (Definition I.13.15) without calling it a tree is a *sequence tree*; adding to that definition:

Definition III.5.2 $B^{<\alpha}$ *is the* complete B–ary tree *of height* α. *Here,* $\sqsubset$ *is* $\subsetneq$ *and* $\mathbb{1} = \emptyset$ *when* $\alpha > 0$.

It is easy to see that height$(B^{<\alpha})$ really is α (when $B \neq \emptyset$), and $\mathcal{L}_\xi(B^{<\alpha}) = B^\xi$ when $\xi < \alpha$. If α is a limit ordinal, then $B^{<\alpha}$ has no leaves, while if $\alpha = \xi + 1$, then the leaves are the elements of B^ξ. If $x \in B^{<\alpha}$ is not a leaf, then its immediate successors are of the form $x^\frown b$ (Definition I.7.24) for $b \in B$.

We have already used sequence trees a few times. The *Cantor tree* $2^{<\omega}$ was used to prove Lemma III.1.16 that there is an almost disjoint family in $[\omega]^\omega$ of size $\mathfrak{c}$. We used $2^{<(\kappa+1)}$ in the proof (Lemma III.1.26) that $2^\kappa = \mathfrak{c}$ whenever $\aleph_0 \leq \kappa < \mathfrak{p}$. The proof of Bell's Theorem III.3.61 used the tree $\omega^{<\omega}$.

Our trees grow upwards, with the root at the bottom, as they do in nature. In computer science applications (such as data structures and syntax trees), it is conventional to draw trees inverted, with the root at the top (as we did in the proof of Lemma III.1.16), although computer scientists still use the terms "root" and "leaf".

Our use of "compatible" and "$\mathbb{1}$" in Definition III.5.1 indicates that we shall sometimes use T as a forcing poset, but then the ordering is $x \leq y$ iff $y \sqsubseteq x$, so the root $\mathbb{1}$ is the largest element with respect to $\leq$; in that case it is natural to draw the tree inverted. Although the reverse of the tree order is useful for *MA* and forcing applications, the tree order itself is well-founded, and is useful when doing induction and recursion on T. Thinking of forcing, we should have defined $x \not\perp y$ to be $\exists z\,[x \sqsubseteq z \wedge y \sqsubseteq z]$. But that is equivalent to our definition, since $\{x \in T : x \sqsubseteq z\}$ is totally ordered by $\sqsubset$.

Viewed as a poset, every leaf of T is an atom (Definition III.3.19), but there may be others; for example, if T is a well-order, then all elements of T are compatible, so all elements are atoms. The tree $B^{<\beta}$ is atomless whenever β is a limit ordinal and $|B| \geq 2$.

A word of caution with the terminology $p{\downarrow}$ and $p{\uparrow}$: When discussing trees, this will always refer to the tree order $\sqsubset$ (*not* $\sqsubseteq$). When discussing *MA* and forcing, we also use the terminology $p{\downarrow}$, referring to the forcing order $\leq$ (see Definition III.3.33). We have an obvious interchange of "up" with "down". But also, in forcing it is more convenient to have p a member of $p{\downarrow} = \{q : q \leq p\}$,

so that p is the $\mathbb{1}$ of $p{\downarrow}$, whereas for trees it is more convenient to take $y{\downarrow}$ *not* to include y, so that the height of y is the type of $y{\downarrow}$.

Given a tree $(T, \sqsubset)$, every subset of T is a tree with the induced order, but the levels need not correspond. For example, each $\{x\} \cup x{\uparrow}$ is a tree with root x; that is, $\mathcal{L}_0(\{x\} \cup x{\uparrow}) = \{x\}$.

Definition III.5.3 *Given a tree $(T, \sqsubset)$, a* subtree *is any subset S such that $y{\downarrow} \subseteq S$ for all $y \in S$.*

For a subtree $S \subseteq T$, $\text{height}(y, S) = \text{height}(y, T) = \text{type}(y{\downarrow})$ whenever $y \in S$.

For example, the sequence tree $A^{<\alpha}$ is a subtree of $B^{<\alpha}$ whenever $A \subseteq B$. A trivial example of a subtree of T is any path through T:

Definition III.5.4 *A* path *through a tree T is a chain $C \subseteq T$ such that $C \cap \mathcal{L}_\alpha(T) \neq \emptyset$ for each $\alpha < \text{height}(T)$.*

For example, if $\text{height}(T) = \omega$, then the paths through T are countably infinite sets and are pairwise almost disjoint. In particular, as in the proof of Lemma III.1.16, the paths through the Cantor tree $T = 2^{<\omega}$ form a family of $2^{\aleph_0}$ almost disjoint subsets of T.

In the case of sequence trees $B^{<\alpha}$, one often identifies paths with functions from α into B. That is, if $f : \alpha \to B$ then $\{f|\xi : \xi < \alpha\}$ is a path through $B^{<\alpha}$, and if C is a path through $B^{<\alpha}$, then $\bigcup C = \bigcup_{x \in C} x$ is a function from α into B.

For general trees, the existence of paths is not always trivial. It is easy to see, by replacing C by $\bigcup\{x{\downarrow} : x \in C\}$, that

Lemma III.5.5 *If* $\text{height}(T)$ *is a cardinal κ, then there is a path through T iff there is a chain $C \subseteq T$ such that $|C| = \kappa$.*

A tree of height ω need not have any paths through it. For example, let T be the subtree of $\omega^{<\omega}$ consisting of all strictly decreasing sequences; so $(9, 5, 3, 1) \in T$ but $(9, 5, 1, 1) \notin T$. If C is a path through T then $\bigcup C : \omega \to \omega$ is strictly decreasing, which is impossible. Note that each $\mathcal{L}_n(T)$ is infinite (except for $n = 0$), as one might expect by the following lemma of König (the son of the König of Theorem I.13.12):

Lemma III.5.6 *If T is a tree such that* $\text{height}(T) = \omega$ *and each $\mathcal{L}_n(T)$ is finite, then there is a path through T.*

Proof. By recursion on n, choose $x_n \in \mathcal{L}_n(T)$ so that $x_n{\uparrow}$ is infinite and $x_{n+1} \in x_n{\uparrow}$. Then $\{x_n : n \in \omega\}$ is a path through T. □

Replacing ω in this lemma by a larger regular cardinal leads to some deep problems in set theory.

Definition III.5.7 *For any regular κ, a* κ–tree *is a tree T such that each level of T has size* less than κ *and* height$(T) = \kappa$.

Definition III.5.8 *For any regular κ, a* κ–Aronszajn tree *is a κ-tree in which all chains have size $< \kappa$. An* Aronszajn tree *is an ω_1-Aronszajn tree.*

By Lemma III.5.6, there are no ω–Aronszajn trees. The existence of ω_2–Aronszajn trees is independent of *ZFC*; see the remarks on page 204. But ω_1–Aronszajn trees exist, as Aronszajn showed in the 1930s. We give two different proofs of this. First:

Theorem III.5.9 *There is an Aronszajn tree.*

Proof. Let T be the subtree of $\omega^{<\omega_1}$ consisting of all 1-1 functions. So, $\mathcal{L}_\alpha(T)$, for a countable ordinal α, is the set of all $s : \alpha \xrightarrow{1-1} \omega$. If C is a path through T then $\bigcup C : \omega_1 \xrightarrow{1-1} \omega$, which is impossible. But, T is not Aronszajn because $|\mathcal{L}_\alpha(T)|$ is not less than ω_1 (i.e., countable), but rather $\mathfrak{c}$ for infinite α. But, we shall find a subtree $S \subseteq T$ all of whose levels are countable (and non-empty). First, by recursion on $\alpha < \omega_1$ choose $s_\alpha \in \mathcal{L}_\alpha(T)$ so that

$$\text{a. } \omega \backslash \text{ran}(s_\alpha) \text{ is infinite.} \qquad \text{b. } \xi < \alpha \to s_\alpha \restriction \xi =^* s_\xi.$$

Here, for $x, y \in \omega^\xi$, $x =^* y$ iff $\{\mu : x(\mu) \neq y(\mu)\}$ is finite. Assuming that this can be done, let $S = \bigcup_{\alpha<\omega_1} \{t \in \mathcal{L}_\alpha(T) : t =^* s_\alpha\}$. Then S is a subtree of T (using (b)), and all levels of S are countable, so S is an Aronszajn tree.

To choose the s_α: For $j \in \omega$, let s_j be the identity function. Given s_α, let $s_{\alpha+1} = s_\alpha \cup \{(\alpha, n)\}$ for some $n \in \omega \backslash \text{ran}(s_\alpha)$ (using (a)). Now, let $\gamma < \omega_1$ be a limit, and assume that we have s_α for $\alpha < \gamma$. Choose $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$ such that $\sup\{\alpha_\ell : \ell < \omega\} = \gamma$. Then, using (b), choose $t_\ell \in \mathcal{L}_{\alpha_\ell}(T)$ so that $t_0 = s_{\alpha_0}$ and $t_{\ell+1} =^* s_{\alpha_{\ell+1}}$ and $t_{\ell+1} \restriction \alpha_\ell = t_\ell$ (not $=^*$). Let $t = \bigcup_\ell t_\ell$. Then $t : \gamma \xrightarrow{1-1} \omega$, so $t \in \mathcal{L}_\gamma(T)$, and setting $s_\gamma = t$ would preserve (b) but might violate (a). So, we remove $\aleph_0$ elements from ran(t). Define $s_\gamma(\alpha_\ell) = t(\alpha_{2\ell})$ and $s_\gamma(\zeta) = t(\zeta)$ for $\zeta \in \gamma \backslash \{\alpha_\ell : \ell \in \omega\}$. Then $\omega \backslash \text{ran}(s_\gamma) \supseteq \{t(\alpha_{2\ell+1}) : \ell < \omega\}$, so that (a) holds for s_γ. □

A sequence of functions $\langle s_\alpha : \alpha < \omega_1 \rangle$ satisfying Condition (b) is sometimes called a *coherent sequence*; for more on these, see [99].

One may apply the proof of Theorem III.5.9 to obtain Aronszajn trees on larger cardinals, but there's a problem. For an ω_2–Aronszajn tree, replace ω_1 by ω_2 and ω by ω_1 One can construct a coherent sequence $\langle s_\alpha : \alpha < \omega_2 \rangle$; so now $s_\alpha : \alpha \xrightarrow{1-1} \omega_1$ and $x =^* y$ iff $\{\mu : x(\mu) \neq y(\mu)\}$ is countable. But then $\{t \in \mathcal{L}_\alpha(T) : t =^* s_\alpha\}$ has size $\mathfrak{c}$, not $\aleph_1$, so the construction works under *CH* (see Exercise III.5.44), but not in *ZFC*. In fact, Mitchell [68] showed that it is consistent with $\mathfrak{c} = \aleph_2$ that there are no ω_2–Aronszajn trees. Also, the Proper Forcing Axiom (*PFA*) implies that there are no ω_2–Aronszajn trees (see Exercise V.7.7). *PFA* also implies $MA + \mathfrak{c} = \aleph_2$. See Section V.7 for more on

PFA. It is consistent with $MA + \neg CH$ to have ω_2–Aronszajn trees; $\mathfrak{c}$ can be $\aleph_2$ or larger (Exercise V.4.21). There is always a κ–Aronszajn tree when κ is the least strongly inaccessible cardinal; see Exercise III.6.33.

The following construction of an Aronszajn tree is useful because it may be modified to construct such trees with additional properties, such as Suslin trees, under appropriate set-theoretic axioms (see Theorem III.7.3). Here, we decide ahead of time that the tree T as a set is just the ordinal ω_1, and we decide ahead of time what the levels are. Then, we construct the tree order $\sqsubset$. The proof of Theorem III.5.12 will become a model for similar constructions later (Theorems III.7.3 and III.8.16).

Definition III.5.10 *A map $\varphi : T \to \mathbb{R}$ from a tree to the real numbers is* order preserving *iff $x \sqsubset y \to \varphi(x) < \varphi(y)$ for all $x, y \in T$.*

Since there cannot be an increasing ω_1–sequence of real numbers,

Lemma III.5.11 *If T is an ω_1–tree and $\varphi : T \to \mathbb{R}$ is order preserving, then T is an Aronszajn tree.*

Theorem III.5.12 *There is an Aronszajn tree T that has an order preserving map φ into the rational numbers.*

Proof. As a set, T will be the ordinal ω_1, and the root will be 0. So, $\mathcal{L}_0 = \{0\}$, and $\mathcal{L}_\alpha$ will be countably infinite when $0 < \alpha < \omega_1$. Exactly which ordinals are placed in which $\mathcal{L}_\alpha$ is not important, but to be specific, let us say $\mathcal{L}_1 = \omega \backslash \{0\}$, $\mathcal{L}_{n+1} = \{\omega \cdot n + k : k \in \omega\}$ for $0 < n < \omega$, and $\mathcal{L}_\alpha = \{\omega \cdot \alpha + k : k \in \omega\}$ when $\omega \leq \alpha < \omega_1$. Then $T_\gamma = \bigcup_{\alpha<\gamma} \mathcal{L}_\alpha$ is always some ordinal. Specifically, $T_0 = \emptyset = 0$, $T_1 = \mathcal{L}_0 = 1$, $T_2 = \mathcal{L}_0 \cup \mathcal{L}_1 = \omega$, $T_3 = \omega + \omega = \omega \cdot 2$, $T_{n+1} = \omega \cdot n$ (for $n < \omega$), $T_\alpha = \omega \cdot \alpha$ (for $\omega \leq \alpha < \omega_1$), so that $T_\gamma = \gamma$ whenever $0 < \gamma < \omega_1$ and $\omega \cdot \gamma = \gamma$. We now need to define the tree order $\sqsubset$.

Each node (element of T) will have exactly $\aleph_0$ immediate successors. This is easily accomplished. The immediate successors of the root node 0 are the nodes in $\mathcal{L}_1$. Given $\mathcal{L}_\alpha$ where $0 < \alpha < \omega_1$, we split the set $\mathcal{L}_{\alpha+1}$ arbitrarily into $\aleph_0$ countably infinite subsets, $E^\xi_{\alpha+1}$ for $\xi \in \mathcal{L}_\alpha$, and put all the nodes in $E^\xi_{\alpha+1}$ directly above node ξ.

The meat of the construction is what happens at a limit γ. If we have already constructed $\sqsubset$ on T_γ, then we choose paths P_k through T_γ for $k \in \omega$, and then put the node $\omega \cdot \gamma + k$ directly above P_k. But, more needs to be said about the choice of P_k. In the construction at hand, we are also constructing $\varphi : T \to \mathbb{Q}$ by recursion, along with $\sqsubset$. Here, we must be a bit careful. If we had naively mapped all nodes in $\mathcal{L}_k$ to the rational number k, then there would be no way to define φ on $\mathcal{L}_\omega$ so that φ is order preserving.

To avoid such problems, we set $\varphi(0) = 0$ and make sure that φ satisfies:

$$\begin{aligned} &\forall x \in T\, \forall \alpha < \omega_1\, \forall q \in \mathbb{Q}\, [\alpha > \text{height}(x) \wedge q > \varphi(x) \to \\ &\quad \exists y \in \mathcal{L}_\alpha(T)\, [x \sqsubset y \wedge \varphi(y) = q]] \ . \end{aligned} \tag{$*$}$$

At successor levels, we easily preserve $(*)$ as follows. Given $\xi \in \mathcal{L}_\alpha$ and $E^\xi_{\alpha+1}$ as above, just make sure that $\varphi \restriction E^\xi_{\alpha+1}$ maps $E^\xi_{\alpha+1}$ *onto* $\mathbb{Q} \cap (\varphi(\xi), \infty)$.

At limit levels γ, given φ and $\sqsubset$ already defined on T_γ: Let $\{(x_k, q_k) : k \in \omega\}$ list all pairs (x, q) with $x \in T_\gamma$ and $q \in \mathbb{Q} \cap (\varphi(x), \infty)$. For each k, choose a path P_k through T_γ with $x_k \in P_k$ and $\sup\{\varphi(y) : y \in P_k\} = q_k$. Then, put the node $\omega \cdot \gamma + k$ directly above P_k, and let $\varphi(\omega \cdot \gamma + k) = q_k$.

Now, we are done if we can verify that such a P_k can indeed be chosen. So, fix k. Then, since γ is countable, choose ordinals α_n for $n \in \omega$ such that $\mathrm{height}(x_k) < \alpha_n \nearrow \gamma$; that is, $\mathrm{height}(x_k) < \alpha_0 < \alpha_1 < \cdots$ and $\sup_n \alpha_n = \gamma$. Also choose rationals r_n with $\varphi(x_k) < r_n \nearrow q_k$. Then, applying $(*)$ ω times, choose nodes z_n for $n \in \omega$ such that $x_k \sqsubset z_0 \sqsubset z_1 \sqsubset \cdots$, $\mathrm{height}(z_n) = \alpha_n$, and $\varphi(z_n) = r_n$. Finally, let P_k be the path $\bigcup_{n\in\omega}(z_n{\downarrow})$. □

We next note that the φ we constructed is *continuous* and that the tree we constructed is *special.*

Definition III.5.13 *If T is a tree and $\varphi : T \to \mathbb{R}$ is order preserving, then φ is* continuous *iff for each limit γ and each $t \in \mathcal{L}_\gamma(T)$, $\varphi(t) = \sup\{\varphi(x) : x \sqsubset t\}$.*

Lemma III.5.14 *The $\varphi : T \to \mathbb{Q}$ constructed in the proof of Theorem III.5.12 is continuous.*

For order preserving φ, Definition III.5.13 coincides with the usual topological notion of continuity if we define:

Definition III.5.15 *For any tree T, the* tree topology *is defined by: $U \subseteq T$ is open iff for all $t \in U$ with* height(t) *a limit ordinal, $\exists x \sqsubset t$ $[x{\uparrow} \cap t{\downarrow} \subseteq U]$.*

Note that all elements of T of non-limit height are isolated points. If T is just a well-ordered set, then the tree topology is the same as the order topology.

Definition III.5.16 *An Aronszajn tree T is* special *iff there are antichains $A_n \subseteq T$ for $n \in \omega$ such that $T = \bigcup_n A_n$.*

Lemma III.5.17 *Let T be any Aronszajn tree.*

1. *If there is an order preserving map $\varphi : T \to \mathbb{Q}$ then T is special.*
2. *If T is special, then there is an order preserving map $\varphi : T \to \mathbb{Q}$.*

Proof. For (1), note that each $\varphi^{-1}\{q\}$ is an antichain.

For (2), we may assume that the antichains A_n are disjoint, since we may replace A_n by $A_n \setminus \bigcup_{j<n} A_j$; let $a(x)$ be the n such that $x \in A_n$. Now define $\varphi : T \to \mathbb{Q}$ by defining $\varphi \restriction A_n$ by recursion on n. Let $\varphi(t) = 0$ for $t \in A_0$. In general, for $t \in A_n$, let $p_t = \max\big(\{-1\} \cup \{\varphi(x) : x \sqsubset t \wedge a(x) < n\}\big)$ and $q_t = \min\big(\{+1\} \cup \{\varphi(x) : t \sqsubset x \wedge a(x) < n\}\big)$, and let $\varphi(t) = (p_t + q_t)/2$. Observe that $p_t < q_t$; also, for $t \in A_n$, $\varphi(t)$ is of the form $k \cdot 2^{-n}$, where $k \in \mathbb{Z} \cap (-2^n, 2^n)$. □

In (2) (due to Kurepa, 1940); note that we may not get a *continuous* order preserving map into $\mathbb{Q}$. One cannot do this in *ZFC*; see the remarks after the proof of Theorem III.5.19, which shows that under $MA(\aleph_1)$, *every* Aronszajn tree has a continuous order preserving map into $\mathbb{Q}$. In particular, $MA(\aleph_1)$ implies that every Aronszajn tree is special, as first proved by Baumgartner in 1970. In the proof, the obvious poset (finite order preserving maps) works, but it is non-trivial to prove it has ccc, although that is easy from the following lemma:

Lemma III.5.18 *Let T be any tree with no uncountable chains. Fix an uncountable $\mathcal{E} \subseteq [T]^{<\omega}$ such that the sets in $\mathcal{E}$ are pairwise disjoint. Then there are $a, b \in \mathcal{E}$ such that $x \perp y$ (Definition III.5.1) for all $x \in a$ and $y \in b$.*

Proof. Since there are only $\aleph_0$ possibilities for $|a|$, we may assume that $|\mathcal{E}| = \aleph_1$ and, for some fixed $n \geq 1$: $|a| = n$ for all $a \in \mathcal{E}$. List $\mathcal{E}$ as $\{a^\alpha : \alpha < \omega_1\}$ and let $a^\alpha = \{x_i^\alpha : i < n\}$. We remark that the case $n = 1$ is trivial, since $\{x_0^\alpha : \alpha < \omega_1\}$ cannot form a chain.

Let $\mathcal{U}$ be any ultrafilter on ω_1 that extends the filter of co-countable sets, so that all sets in $\mathcal{U}$ are uncountable (see Definition III.1.27).

Assume that the lemma fails. Then, whenever $\alpha \neq \xi$, there are $i_{\alpha,\xi}, m_{\alpha,\xi} < n$ such that $x^\alpha_{i_{\alpha,\xi}} \not\perp x^\xi_{m_{\alpha,\xi}}$. There are only finitely many pairs (i, m), so for each α, there are $i_\alpha, m_\alpha < n$ such that $\{\xi \neq \alpha : (i_{\alpha,\xi}, m_{\alpha,\xi}) = (i_\alpha, m_\alpha)\} \in \mathcal{U}$. Then, we can fix $i, m < n$ such that $S := \{\alpha < \omega_1 : (i_\alpha, m_\alpha) = (i, m)\}$ is uncountable.

So, when $\alpha \in S$, $M_\alpha := \{\xi \neq \alpha : x_i^\alpha \not\perp x_m^\xi\} \in \mathcal{U}$. Now, consider any $\alpha, \beta \in S$. Then $M_\alpha \cap M_\beta$ is in $\mathcal{U}$ and hence uncountable. Since $x_i^\alpha{\downarrow} \cup x_i^\beta{\downarrow}$ is countable, we can choose $\xi \in M_\alpha \cap M_\beta$ so that $x_i^\alpha \sqsubset x_m^\xi$ and $x_i^\beta \sqsubset x_m^\xi$. But this implies that $x_i^\alpha \not\perp x_i^\beta$, so that $\{x_i^\alpha : \alpha \in S\}$ is an uncountable chain, which is a contradiction. □

Theorem III.5.19 *Let T be any tree with no uncountable chains and let $\kappa = |T|$. Then $MA(\kappa)$ implies that there is a continuous order preserving $\varphi : T \to \mathbb{Q}$. In particular, $MA(\aleph_1)$ implies that every Aronszajn tree is special.*

Proof. Let $\mathbb{P}$ be the set of all $p \in \mathrm{Fn}(T, \mathbb{Q})$ such that p is order preserving; that is $x \sqsubset y \to \varphi(x) < \varphi(y)$ for all $x, y \in \mathrm{dom}(p)$. As usual, $\leq$ is $\supseteq$ and $\mathbb{1} = \emptyset$.

To prove that $\mathbb{P}$ is ccc, we fix an uncountable $A \subseteq \mathbb{P}$, and find $p, q \in A$ such that $p \neq q$ and p, q are compatible in $\mathbb{P}$. Since $\{p \in \mathbb{P} : \mathrm{dom}(p) = a\}$ is countable for each $a \in [T]^{<\omega}$, there is an uncountable $A_1 \subseteq A$ such that the $\mathrm{dom}(p)$ for $p \in A_1$ are all different. Then, by the Delta System Lemma, there is an uncountable $A_2 \subseteq A_1$ such that $\{\mathrm{dom}(p) : p \in A_2\}$ form a delta system, with some root R. Then, there is an uncountable $A_3 \subseteq A_2$ such that the $p{\restriction}R$ are the same for all $p \in A_3$ Then, the $\mathrm{dom}(p)\backslash R$ for $p \in A_3$ are disjoint subsets of T, so by Lemma III.5.18, there are $p, q \in A_3$ such that $x \perp y$ for all $x \in \mathrm{dom}(p)\backslash R$ and $y \in \mathrm{dom}(q)\backslash R$. But then $p \cup q$ is an order preserving function, so $p \cup q \in \mathbb{P}$ and $p \cup q \leq p$ and $p \cup q \leq q$.

If G is a filter meeting the dense sets $\{p \in \mathbb{P} : x \in \mathrm{dom}(p)\}$ for each $x \in T$, and $\varphi = \bigcup G$, then $\varphi : T \to \mathbb{Q}$ is order preserving. To make φ continuous at each node $t \in T$ (when height(t) is a limit ordinal), also meet the dense sets $\{p \in \mathbb{P} : t \in \mathrm{dom}(p) \wedge \exists x \in \mathrm{dom}(p) \cap t{\downarrow}\, [p(x) \geq p(t) - 2^{-n}]\}$ for each $n \in \omega$. □

Some further remarks on mapping Aronszajn trees into $\mathbb{Q}$ or into $\mathbb{R}$: Under $MA(\aleph_1)$, Theorem III.5.19 shows that the best possible thing happens. However, $V = L$ yields the following three counter-examples to natural conjectures: First, a Suslin tree (see below), which exists by Theorem III.7.3, cannot have an order preserving map into $\mathbb{R}$ (see Lemma IV.6.5 and Exercise IV.6.7). Second, there is an Aronszajn tree T with an order preserving map into $\mathbb{R}$ but no such map into $\mathbb{Q}$ (that is, T is not special); this was first proved by Baumgartner; see Devlin [17] for a proof. Third, by Theorem III.8.16, there is a special Aronszajn tree with no *continuous* order preserving map into $\mathbb{Q}$. However, just in *ZFC*, an order preserving map into $\mathbb{R}$ automatically yields a continuous map:

Lemma III.5.20 *Let T be any tree with an order preserving $\varphi : T \to \mathbb{R}$. Then there is also a continuous order preserving $\psi : T \to \mathbb{R}$.*

Proof. Let $\psi(t)$ be $\varphi(t)$ unless height(t) is a limit ordinal, in which case $\psi(t) = \sup\{\varphi(x) : x \sqsubset t\}$. □

A Suslin tree is a kind of Aronszajn tree introduced by Kurepa, who showed that a Suslin tree exists iff a Suslin line (Definition III.2.16) exists.

Definition III.5.21 *For any regular κ, a* κ–Suslin tree *is a κ-tree in which all chains and all antichains have size $< \kappa$. A* Suslin tree *is an ω_1-Suslin tree.*

Lemma III.5.22 *Every κ-Suslin tree is a κ-Aronszajn tree.*

Before we build lines out of Suslin trees, we discuss three *niceness properties* that a tree might have. None of these properties is really vital to the basic theory, but they make the tree easier to visualize, and they are all easy to obtain in explicit constructions of trees:

Definition III.5.23 *For a tree T:*

- ☞ *T is* rooted *iff $|\mathcal{L}_0(T)| = 1$.*
- ☞ *T is* Hausdorff *iff for all limit γ and $x, y \in \mathcal{L}_\gamma(T)$, if $x{\downarrow} = y{\downarrow}$ then $x = y$.*
- ☞ *T is* well-pruned *iff T is a rooted κ-tree, where κ is regular, and $|x{\uparrow}| = \kappa$ for all $x \in T$.*

The notion of "rooted" just restates Definition III.5.1, and was discussed in the beginning of this section. All the non-empty trees constructed so far are rooted. For example the sequence tree $B^{<\alpha}$ has as root the empty sequence $\emptyset$.

Note that T is Hausdorff iff the tree topology (Definition III.5.15) is Hausdorff in the usual sense. Sequence trees are Hausdorff, as was the Aronszajn

tree from Theorem III.5.9, which was a subtree of a sequence tree. For a trivial example of a non-Hausdorff tree, let $T = \omega \cup \{\infty_1, \infty_2\}$, where ω is ordered in the usual way, with 0 as the root, and $\mathcal{L}_\omega = \{\infty_1, \infty_2\}$, so $\infty_1\downarrow = \infty_2\downarrow = \omega$. Less trivially, the Aronszajn tree from Theorem III.5.12 may or may not be Hausdorff, depending on how we choose the P_k. It is easy to make it non-Hausdorff, but also

Exercise III.5.24 *Show how to make the Aronszajn tree in Theorem III.5.12 Hausdorff.*

Hint. At limit levels γ, make sure that $k \neq \ell \to P_k \neq P_\ell$. □

The Aronszajn trees from Theorems III.5.9 and III.5.12 are both well-pruned; for the tree from Theorem III.5.12, this follows from condition (∗). One can modify them to make them not well-pruned by sprouting a leaf; that is, let $T' = T \cup \{v\}$, where v in T' is an immediate successor to some node in T; then $|v\uparrow| = 0$. This leaf will be removed by the following pruning process, which also removes all the deadwood and results in a well-pruned tree.

Definition III.5.25 *If T is a κ-tree, then $T^p = \{x \in T : |x\uparrow| = \kappa\}$.*

Lemma III.5.26 *For regular κ and κ-tree T:*

1. *T^p is a subtree of T.*
2. *T is well-pruned iff T is rooted and $T^p = T$.*
3. *T^p is a κ-tree.*
4. *$(T^p)^p = T^p$.*
5. *For each $y \in \mathcal{L}_0(T^p)$: $(\{y\} \cup y\uparrow) \cap T^p$ is a well-pruned κ-tree and is a subtree of T with root node y.*

Proof. (1) and (2) are obvious. (3) is clear if we prove that $T^p \cap \mathcal{L}_\alpha(T) \neq \emptyset$ for each $\alpha < \kappa$, and this holds because κ is regular and

$$T = \bigcup\{x\uparrow : x \in \mathcal{L}_\alpha(T)\} \cup \bigcup\{\mathcal{L}_\xi(T) : \xi \leq \alpha\} \quad ,$$

so one of these sets has size κ. By (3), $|T^p| = \kappa$. The same argument, applied to any $z\uparrow$ for $z \in T^p$ shows that $z\uparrow \cap T^p$ also has size κ, which implies (4). Then, (5) follows from (1 – 4). □

The definition of κ-Aronszajn and κ-Suslin trees does not require any of the niceness properties from Definition III.5.23, but it is easy to obtain them by modifying the tree. Note that the $(\{y\} \cup y\uparrow) \cap T^p$ in (5) is Aronszajn if T is Aronszajn and is Suslin if T is Suslin, so we have:

Lemma III.5.27 *For any regular κ: If there exists a κ-Aronszajn tree then there exists a well-pruned κ-Aronszajn tree. If there exists a κ-Suslin tree then there exists a well-pruned κ-Suslin tree.*

We shall not use Hausdorffness in this book, except in a few exercises, but

Exercise III.5.28 *Prove Lemma III.5.27 replacing "well-pruned" by "well-pruned and Hausdorff".*

Hint. A failure of Hausdorffness of T at a limit level γ can be cured by adding a new level γ. This forms a new tree T^* so that $\mathcal{L}_{\gamma+\alpha}(T) = \mathcal{L}_{\gamma+1+\alpha}(T^*)$. If $x, y \in \mathcal{L}_\omega(T)$ and $x{\downarrow} = y{\downarrow}$, then $x, y \in \mathcal{L}_{\omega+1}(T^*)$ and are immediate successors to some $z \in \mathcal{L}_\omega(T^*)$. □

Well-pruned-ness for a Suslin tree T gives an easy proof that T doesn't exist under $MA(\aleph_1)$. We may view T as a ccc forcing poset, and prove:

Lemma III.5.29 *If T is any well-pruned Suslin tree, then $MA_T(\aleph_1)$ is false. In particular, $MA(\aleph_1)$ implies that there are no Suslin trees.*

Proof. Recall that in the forcing order, $\mathbb{1}$ is the root and $x \leq y$ iff $y \sqsubseteq x$. For $\alpha < \omega_1$, let $D_\alpha = \{x \in T : \text{height}(x, T) > \alpha\}$. If T is well-pruned then each D_α is dense; that is $\forall p \in T\ \exists q \in D_\alpha\ p \sqsubseteq q$. Using $MA_T(\aleph_1)$, there is a filter G that meets each D_α. But then $|G| = \aleph_1$, and the elements of G are pairwise compatible, so that G is a chain in T, contradicting the definition of "Suslin".

The "In particular" follows by Lemma III.5.27. □

One cannot delete the "well-pruned" here. For example, if S is a Suslin tree, then we can form $T = S \cup \{v\}$ by sprouting a leaf, as described on page 209. Then T is also Suslin, but the leaf is an atom, so that $MA_T(\aleph_1)$ is true trivially (Lemma III.3.20).

Of course, we already know (Corollary III.3.44) that $MA(\aleph_1)$ implies that there are no Suslin lines, and it is true in *ZFC* that a Suslin line exists iff a Suslin tree exists. We shall split this "iff" into two separate theorems, since the methods of proof are completely unrelated.

First, regarding getting a tree from a line:

Definition III.5.30 *If $(X, <)$ is a totally ordered set, then $W \subseteq X$ is* convex *iff for all $x, z \in W$ and all $y \in X$, if $x < y < z$ then $y \in W$.*

Examples of convex sets are the various intervals, $(a, b]$, $[a, b]$, etc., but a convex set need not have a left or right endpoint. Also, X and $\emptyset$ and all singletons are convex. Note that if W is convex and W° is its interior in the order topology, then $|W \backslash W^\circ| \leq 2$; that is, the only possible elements of $W \backslash W^\circ$ are the smallest and largest elements of W (if they exist).

Our definition (III.2.16) of Suslin line (just ccc and not separable) allows for some very ugly lines; for example, X might have a two element convex set $\{x, y\}$ if x, y are neighboring points; (i.e., $x < y$ and $(x, y) = \emptyset$). It may also have a convex set that is isomorphic to $\mathbb{Z}$. The next lemma pretties the line up a bit; many authors (e.g, [49]) incorporate (1) of this lemma into the definition of "Suslin line".

Lemma III.5.31 *If there is a Suslin line X, then there is another Suslin line A such that:*

1. *A is a dense total order (i.e., $a < b \to (a,b) \neq \emptyset$).*
2. *In A, every non-empty interval is non-separable.*

Proof. In X, define $x \sim y$ iff $x = y$ or the interval $(\min(x,y), \max(x,y))$ is separable. Then $\sim$ is an equivalence relation and all the equivalence classes are convex. Let $A = X/\sim$ be the set of all equivalence classes. Give A the natural order: $a < b$ iff $a \neq b$ and $x < y$ for some (or any) $x \in a$ and $y \in b$. Note that if x, y are neighboring points in x, then they get identified to the same point in A.

Since the elements of A are pairwise disjoint, $a^\circ = \emptyset$ (equivalently, $|a| \leq 2$) holds for all but countably many elements of A, since X has the ccc. Depending on X, one or both of $\{a \in A : |a| = 1\}$ and $\{a \in A : |a| = 2\}$ may be uncountable.

Every $a \in A$ is a separable subset of X; the proof has $2 \cdot 2 = 4$ cases, depending on whether a has/hasn't a least/greatest element. Case I: $a = [x,y]$. By $x \sim y$, (x,y) is separable and hence so is $[x,y]$. Case II: $\min(a) = x$ and a has no largest element. By recursion, pick a sequence $\langle y_\xi : \xi < \gamma \rangle$ that is unbounded in a, where γ is a limit ordinal and $y_0 < y_1 < \cdots$. Then γ is countable because X has the ccc and the intervals $(y_0, y_2), (y_2, y_4), (y_4, y_6), \ldots$ are disjoint and non-empty. Each $[x, y_\xi]$ is separable because $x \sim y_\xi$, so $q = \bigcup_\xi [x, y_\xi]$ is separable. Cases III and IV are similar.

Also, each $a \in A$ is closed, but we don't need this fact in the proof.

To prove (1): Say $a < b$ and $(a,b) = \emptyset$. Fix $x \in a$ and $y \in b$. But then $(x,y) = ((x,y) \cap a) \cup ((x,y) \cap b)$, which is separable, contradicting $x \not\sim y$.

To prove that A has the ccc: If not, then (using (1)), there are $a_\alpha < b_\alpha$ for $\alpha < \omega_1$ such that the $[a_\alpha, b_\alpha]$ are pairwise disjoint in A. Choose $x_\alpha \in a_\alpha$ and $y_\alpha \in b_\alpha$. Then the $(x_\alpha, y_\alpha) \subset X$ contradict the ccc of X.

So, we are done if we prove (2), which implies that X is not separable.

To prove (2): Suppose that $a < b$ and $\{c_n : n \in \omega\} \subseteq [a,b]$ is dense in $[a,b]$ in the order topology of A. We may assume that the c_n include a and b and all the (countably many) $c \in (a,b)$ such that $|c| \geq 3$. For each n, choose a countable $D_n \subseteq c_n$ that is dense in c_n in the order topology of X; and let $D = \bigcup_n D_n$. Fix $x \in a$ and $y \in b$. If we show that $(x,y) \subseteq \overline{D}$ then (x,y) is separable, so we have a contradiction from $x \not\sim y$.

So, fix $t \in (x,y)$, and let c be its equivalence class. Then $t \in \overline{D}$ is clear from $D \supseteq D_n$ unless $c \neq a$ and $c \neq b$ and $|c| \leq 2$. Say $t \in c = \{u, v\}$ where $u \leq v$ (and possibly $u = v$). But then u is a limit from the left of points of D and v is a limit from the right of points of D, so $t \in \overline{D}$. □

Theorem III.5.32 *If there is a Suslin line, then there is a Suslin tree.*

Proof. Let $(X, <)$ be a Suslin line, which, by Lemma III.5.31, we may assume is a dense total order. We shall construct a Suslin tree. The construction is similar to the proof of Lemma III.2.18, which showed that $X \times X$ is not ccc. By recursion on $\alpha < \omega_1$, choose $a_\alpha, b_\alpha \in X$ for $\alpha < \omega_1$ satisfying:

1. $a_\alpha < b_\alpha$.
2. $\xi < \alpha \;\rightarrow\; a_\xi \notin [a_\alpha, b_\alpha] \;\wedge\; b_\xi \notin [a_\alpha, b_\alpha]$.

This is possible because the countable set $\{a_\xi : \xi < \alpha\} \cup \{b_\xi : \xi < \alpha\}$ is disjoint from some (c_α, d_α), and we can choose a_α, b_α satisfying $c_\alpha < a_\alpha < b_\alpha < d_\alpha$.

Note first that all the a_α, b_α are different points. Then note:

$$\xi < \alpha \;\rightarrow\; [a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi) \;\vee\; [a_\alpha, b_\alpha] \cap [a_\xi, b_\xi] = \emptyset \;, \qquad (*)$$

because either $a_\xi < a_\alpha < b_\alpha < b_\xi$ or $a_\alpha < b_\alpha < a_\xi < b_\xi$ or $a_\xi < b_\xi < a_\alpha < b_\alpha$; any other way of ordering these four points $a_\alpha, b_\alpha, a_\xi, b_\xi$ would contradict (1) or (2). Now, let the tree be the set ω_1, with the tree order $\sqsubset$ defined by:

$$\xi \sqsubset \alpha \;\leftrightarrow\; \xi < \alpha \;\wedge\; [a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi) \;.$$

Actually, by (1)(2), the "$\xi < \alpha$" is redundant, but it makes it clear that the relation $\sqsubset$ is well-founded. Also, $\sqsubset$ is transitive, so to prove that it really is a tree order, it is sufficient to fix $\xi, \eta \sqsubset \alpha$ with $\xi \neq \eta$ and prove that $\xi \sqsubset \eta$ or $\eta \sqsubset \xi$. WLOG, $\xi < \eta$. The intervals $[a_\eta, b_\eta]$ and $[a_\xi, b_\xi]$ cannot be disjoint, since they contain $[a_\alpha, b_\alpha]$, so $[a_\eta, b_\eta] \subseteq [a_\xi, b_\xi]$ by $(*)$, which implies that $\xi \sqsubset \eta$.

If $C \in [\omega_1]^{\aleph_1}$ were a chain in $\sqsubset$, let $C = \{\alpha_\mu : \mu < \omega_1\}$ (listed in increasing order); but then the sets $(a_{\alpha_\mu}, b_{\alpha_\mu}) \backslash [a_{\alpha_{\mu+1}}, b_{\alpha_{\mu+1}}]$ would contradict the ccc.

If $A \subseteq \omega_1$ is an antichain in $\sqsubset$, then by $(*)$, the intervals $[a_\alpha, b_\alpha]$ for $\alpha \in A$ are pairwise disjoint, so A is countable because X has the ccc. □

In this proof, there was a lot of flexibility in choosing the a_α, b_α, so the tree may or may not be well-pruned, or have leaves, etc. But, by Exercise III.5.34, we can give it the niceness properties from Definition III.5.23.

Exercise III.5.33 *Assume that there exists a Suslin line. Then there is a Suslin line $(X, <)$ that is a dense total order without endpoints and is connected in its order topology.*

Hint. Apply Lemma III.5.31 and take the Dedekind completion. □

Exercise III.5.34 *In the proof of Theorem III.5.32, assume that in the line X, all non-empty intervals in X are non-separable (this is possible by Lemma III.5.31). Show how to make the Suslin tree well-pruned and rooted. Furthermore, if X is connected, then we can make the tree Hausdorff.*

To construct a Suslin line from a Suslin tree, the following lemma will be useful:

Lemma III.5.35 *If κ is regular and T is a well-pruned κ-Aronszajn tree and $n < \omega$, then $\forall x \in T\, \exists \alpha > \text{height}(x)\, [|x{\uparrow} \cap \mathcal{L}_\alpha(T)| \geq n]$.*

Proof. Induct on n. The case $n = 2$ is clear from the fact that $x{\uparrow}$ has size κ and cannot form a chain. Assuming the lemma for n, first get $\beta > \text{height}(x)$ such that there are distinct $y_1, \ldots, y_n \in x{\uparrow} \cap \mathcal{L}_\beta(T)$. Then, using the $n = 2$ case, get $\alpha > \beta$ and distinct $z_0, z_1 \in y_1{\uparrow} \cap \mathcal{L}_\alpha(T)$. For $i = 2, \ldots, n$, choose $z_i \in y_i{\uparrow} \cap \mathcal{L}_\alpha(T)$. Then $x{\uparrow} \cap \mathcal{L}_\alpha(T)$ contains $z_0, z_1, \ldots, z_n$. □

In fact, this lemma holds if n is replaced by any regular cardinal less than κ; see Exercise III.6.32

Theorem III.5.36 *If there is a Suslin tree, then there is a Suslin line.*

Proof. Let T be a Suslin tree, which, by Lemma III.5.27, we may assume is well-pruned. We shall construct a Suslin line X.

Let X be the set of all $C \subseteq T$ such that C is a maximal chain. For $C \in X$, type(C) is some countable ordinal, since all chains in T are countable. If $x \in C$, then $C \cup x{\downarrow}$ is a chain, so $x{\downarrow} \subseteq C$ by maximality of C. It follows that for each $\alpha < \text{type}(C)$, $C \cap \mathcal{L}_\alpha(T)$ contains exactly one element, which we shall call $C(\alpha)$, while $C \cap \mathcal{L}_\alpha(T) = \emptyset$ when $\alpha \geq \text{type}(C)$. Also, type($C$) is a limit ordinal, since if $\text{type}(C) = \alpha + 1$, then $C(\alpha)$ is the largest element of C, which contradicts maximality of C (since T is well-pruned).

To order X, fix any strict total order $\lhd$ of the set T. If $C, D \in X$ and $C \neq D$, let $\delta(C, D)$ be the least α such that $C(\alpha) \neq D(\alpha)$; note that $\delta(C, D) < \min(\text{type}(C), \text{type}(D))$. Define $C < D$ iff $C \neq D$ and $C(\delta(C, D)) \lhd D(\delta(C, D))$. It is easy to check that $<$ really is a total order; the proof of transitivity $C < D < E \to C < E$ splits into cases: $\delta(C, D) = \delta(D, E)$ or $\delta(C, D) < \delta(D, E)$ or $\delta(C, D) > \delta(D, E)$.

To prove that $(X, <)$ is ccc, suppose that (C_ξ, E_ξ), for $\xi < \omega_1$, are disjoint non-empty intervals in X. Choose D_ξ with $C_\xi < D_\xi < E_\xi$, and then choose α_ξ with $\max(\delta(C_\xi, D_\xi), \delta(D_\xi, E_\xi)) < \alpha_\xi < \text{type}(D_\xi)$. Then $\{D_\xi(\alpha_\xi) : \xi < \omega_1\}$ is an antichain in T because $D_\xi(\alpha_\xi) \not\perp D_\eta(\alpha_\eta)$ would imply that $D_\xi, D_\eta \in (C_\xi, E_\xi) \cap (C_\xi, E_\xi)$.

To prove that $(X, <)$ is not separable, it is sufficient to fix $\alpha < \omega_1$ and prove that $\{H \in X : \text{type}(H) < \alpha\}$ is not dense in X. First, fix $s \in T$ with $\text{height}(s) > \alpha$. Then, applying Lemma III.5.35, fix distinct $c, d, e \in s{\uparrow}$ with $\text{height}(c) = \text{height}(d) = \text{height}(e) > \text{height}(s)$. Choose $C, D, E \in X$ with $c \in C$, $d \in D$, and $e \in E$, and assume they are ordered $C < D < E$. Then $(C, E) \neq \emptyset$, but $H \notin (C, E)$ whenever $\text{type}(H) < \alpha$. □

The following lemma will be useful later (Theorem III.7.3) when we construct a Suslin tree under $V = L$:

Lemma III.5.37 *For regular κ: Let T be a κ-tree in which all maximal antichains have size less then κ, and assume that $\forall x \in T\, \exists y, z \in x{\uparrow}\, [y \perp z]$. Then T is a κ-Suslin tree.*

Proof. Since every antichain is contained in a maximal one, *all* antichains have size less than κ, but we need to prove that there is no chain of size κ. If there were such a chain, then there would be a path P through T (Lemma III.5.5). Note that $\forall x \in T \exists y \in x\uparrow\ [y \notin P]$. Recursively choose x_α, y_α for $\alpha < \kappa$ such that $x_\alpha \in P$, $y_\alpha \in x_\alpha\uparrow\backslash P$, and $\alpha < \beta \to \text{height}(x_\alpha) < \text{height}(y_\alpha) < \text{height}(x_\beta)$. Then $\{y_\alpha : \alpha < \kappa\}$ would be an antichain of size κ. □

A κ–Aronszajn tree has no paths. It is easy to find a κ–tree with exactly one path (such as the ordinal κ), or with exactly κ paths (such as $\{s \in 2^{<\kappa} : |\{\alpha \in \text{dom}(s) : s(\alpha) \neq 0| < \aleph_0\}$). But κ–trees with more than κ paths, first studied by Kurepa, are non-trivial.

Definition III.5.38 *For any regular κ, a* κ–Kurepa tree *is a κ–tree T with at least κ^+ paths through T. κ–KH is the statement that there exists a κ–Kurepa tree. A* Kurepa tree *is an ω_1–Kurepa tree, and KH, or* (Kurepa's Hypothesis), *is ω_1–KH.*

For some unknown historical reason, *KH* says that a Kurepa tree exists, while *SH* says that a Suslin tree doesn't exist. For $\kappa = \omega$ or κ strongly inaccessible, κ–*KH* is trivially true, since the tree $2^{<\kappa}$ is a κ–Kurepa tree.

For $\kappa = \omega_1$: *KH* follows from $V = L$ (see Theorem III.7.6). But, *KH* does not follow from *GCH* alone (see Theorem V.2.25 and Exercise V.2.27).

The existence of a Kurepa tree is equivalent to the existence of a *Kurepa family* of sets:

Definition III.5.39 *For $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ and $A \subseteq \kappa$, let $\mathcal{F}\restriction A = \{X \cap A : X \in \mathcal{F}\}$.*

For any regular κ, a κ–Kurepa family *is an $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{F}| \geq \kappa^+$ and $|\mathcal{F}\restriction\alpha| < \kappa$ for all $\alpha < \kappa$.*

Lemma III.5.40 *For any regular κ, there exists a κ–Kurepa tree iff there exists a κ–Kurepa family.*

Proof. If $\mathcal{F}$ is a Kurepa family, define a subtree T of $2^{<\kappa}$ so that each $\mathcal{L}_\alpha(T) = \{\chi_{X\cap\alpha} : X \in \mathcal{F}\}$; here, $\chi_{X\cap\alpha} \in 2^\alpha \subset 2^{<\kappa}$ denotes the characteristic function. Then T is a Kurepa tree: T is a κ–tree because each $|\mathcal{F}\restriction\alpha| < \kappa$, and T is Kurepa because it has at least $|\mathcal{F}|$ paths.

Before proving the converse, observe that for $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ and $A \subseteq B \subseteq \kappa$, $|\mathcal{F}\restriction A| \leq |\mathcal{F}\restriction B|$ because the function $Z \mapsto Z \cap A$ maps $\mathcal{F}\restriction B$ onto $\mathcal{F}\restriction A$.

Now, assume that T is a Kurepa tree. Since $|T| = \kappa$, we may assume WLOG that the set T is the ordinal κ. Then, let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ be the set of all paths through T, so $|\mathcal{F}| \geq \kappa^+$. For each $\alpha < \kappa$, fix $\delta < \kappa$ so that $\alpha \subseteq T_\delta$. Then $|\mathcal{F}\restriction T_\delta| < \kappa$ since $|\mathcal{L}_\delta(T)| < \kappa$, and $|\mathcal{F}\restriction\alpha| \leq |\mathcal{F}\restriction T_\delta|$, so $|\mathcal{F}\restriction\alpha| < \kappa$. □

Exercise III.5.41 *For any regular κ: If there exists a κ–Kurepa tree then there exists one that is Hausdorff and well-pruned.*

Exercise III.5.42 *A tree T is Hausdorff and rooted iff T is isomorphic to a subtree of some sequence tree $B^{<\alpha}$.*

In the literature, the term "normal tree" is often used to describe a κ–tree that satisfies some version of the niceness properties in Definition III.5.23. This usage is unfortunately a bit confusing, since it is unrelated to the normality of the tree as a topological space, which is also discussed in the literature. Every tree is T_1 in the tree topology, and it is easy to see that every Hausdorff (T_2) tree is also T_3 and zero dimensional (that is, has a base of clopen sets), but the normality property (T_4) is non-trivial. Consider the case for ω_1–Aronszajn trees: Under $MA(\aleph_1)$, every T_2 Aronszajn tree is T_4 (Fleissner, 1975) (and also special by Theorem III.5.19); whereas under CH, every special Aronszajn tree is not T_4 (Devlin and Shelah, 1979). Every T_2 Suslin tree is T_4 (see Exercise IV.6.12). For further details, see Todorčević [97].

There are two reasons for discussing trees in this book. First, there are set-theoretic independence proofs regarding specific kinds of trees, such as Suslin trees, Aronszajn trees, and Kurepa trees. Second, trees are a useful tool for proving theorems, as we have seen in Lemmas III.1.16 and III.1.26, and Theorem III.3.61. Another such theorem, due to essentially Hausdorff in 1914, generalizes the Cantor tree and the ordering on the rational numbers:

Exercise III.5.43 *Let κ be an uncountable regular cardinal. Let T be the tree $\{-1,+1\}^{<\kappa}$. If $x \in T$, extend x to $\tilde{x} \in \{-1,0,+1\}^{<\kappa}$ so that $\tilde{x}_\xi = 0$ for $\xi \in \kappa \backslash \mathrm{dom}(x)$. Then, order T by saying that $x < y$ iff $\tilde{x}$ precedes $\tilde{y}$ lexicographically. Prove that $<$ is a dense total order without endpoints and that it has the following $<\kappa$ separation property: Whenever $A, B \in [T]^{<\kappa}$ with $\forall x \in A\ \forall z \in B\ x < z$, there is a $y \in T$ such that $\forall x \in A\ \forall z \in B\ x < y < z$.*

Hausdorff called this total order "an η_ξ set" (where $\kappa = \aleph_\xi$). When $\kappa = \aleph_0$, the ordering may be obtained by associating the nodes in the tree with the dyadic rationals in $(0,1)$ via $x \mapsto 1/2 + \sum\{2^{-\ell-2} x_\ell : \ell < \mathrm{lh}(x)\}$. The letter η is often used for the "order type" of the rationals.

Exercise III.5.44 *Show that CH implies that there is an ω_2–Aronszajn tree.*

Hint 1. In the proof of Theorem III.5.12, replace ω_1 by ω_2 and replace $\mathbb{Q}$ by Hausdorff's η_1 set. □

Hint 2. Let J be a set of size $\aleph_1$. Modify the proof of Theorem III.5.9, using also the remarks following the proof. It is sufficient to find $s_\alpha : \alpha \xrightarrow{1-1} J$ for $\alpha < \omega_2$ such that $\xi < \alpha \to s_\alpha | \xi =^* s_\xi$. But if we simply assume inductively that $J \backslash \mathrm{ran}(s_\alpha)$ is uncountable, there may be a problem at stage γ where $\gamma < \omega_2$ is a limit of countable cofinality. If $\alpha_\ell \nearrow \gamma$ as before, we are stuck if $J \backslash \bigcup_\ell (\mathrm{ran}(s_\alpha))$ is countable. Instead, let $J = \omega_1 \times \omega_1$, and maintain the stronger inductive assumption that $\mathrm{ran}(s_\alpha) \subseteq \{(\pi, \rho) : \rho < f_\alpha(\pi)\}$ for some $f_\alpha : \omega_1 \to \omega_1$. □

Exercise III.5.45 *Let T be a κ-Suslin tree. For $x \in T$ and $D \subseteq T$, call D* dense above x *iff $\forall y \sqsupseteq x \exists z \sqsupseteq y\, [z \in D]$. Then prove that given any $D \subseteq T$, there is an $\alpha < \kappa$ such that for all $x \in \mathcal{L}_\alpha$: either (1) D is dense above x or (2) $D \cap (\{x\} \cup x\uparrow) = \emptyset$.*

Hint. Find a maximal antichain A such that (1) or (2) holds for all $x \in A$. □

Exercise III.5.46 *Referring to Definition III.5.25, if T is a κ-Suslin tree then $|T \backslash T^p| < \kappa$.*

Hint. Let $D = T^p$ in Exercise III.5.45. □

Exercise III.5.47 *If S, T are κ-trees, define their* tree product $S \circledast T$ *to be $\{s \times t \in S \times T : \text{height}(s) = \text{height}(t)\}$; define $(s_0, t_0) \sqsubset (s_1, t_1)$ iff $s_0 \sqsubset s_1$ and $t_0 \sqsubset t_1$. Prove that if S, T are κ-Aronszajn trees then $S \circledast T$ is a κ-Aronszajn tree, and if S, T are κ-Kurepa trees then $S \circledast T$ is a κ-Kurepa tree.*

Note that if S, T are well-pruned and we view them as forcing posets, then $S \circledast T$ is dense in the product poset $S \times T$. Then, the following exercise provides another proof (similar to Corollary III.3.44) that $MA(\aleph_1) \to SH$:

Exercise III.5.48 *$T \circledast T$ can never be a κ-Suslin tree.*

Hint. To produce an antichain of size κ, note that if $\text{height}(x_0) < \text{height}(y_0) = \text{height}(z_0) < \text{height}(x_1) < \text{height}(y_1) = \text{height}(z_1)$ and $x_i \sqsubset y_i$ and $x_i \sqsubset z_i$ and $y_i \neq z_i$ (for $i = 0, 1$), then $(y_0, z_0) \perp (y_1, z_1)$. □

If $V = L$, then there are two *different* ω_1-Suslin trees S, T such that $S \circledast T$ is Suslin (Exercise III.7.10).

Exercise III.5.49 *Let $(X, <)$ be any uncountable ccc LOTS. Prove that X has a dense subset of size $\aleph_1$.*

Hint. Modify the Suslin tree construction (Theorem III.5.32). □

Exercise III.5.50 *Show (in ZFC) that there exists a densely ordered LOTS $(X, <)$ such that every separable subspace is nowhere dense, and such that there are no increasing or decreasing ω_1-sequences in X.*

Hint. Use an Aronszajn tree to build an Aronszajn line. □

Note that under $\neg SH$, there is such an X that is a Suslin line; see Lemma III.5.31. The proof explains why a ccc LOTS cannot have increasing or decreasing ω_1-sequences (see "Case II"), but we can't demand that the X in Exercise III.5.50 be ccc, since then it would be a Suslin line.

Two independence results mentioned above lead to set theories that are strictly stronger proof-theoretically than *ZFC* (see Definition II.1.1). Let φ be the statement that there are no ω_1–Kurepa trees (i.e., $\neg KH$), and let ψ be the statement that there are no ω_2–Aronszajn trees. Both φ and ψ imply that the cardinal ω_2 is an inaccessible cardinal in Gödel's model L (for φ, see Exercise III.7.22; for ψ, this is a result of Silver; see [68], Theorem 5.9). It follows that $ZFC + \varphi \vdash \mathrm{Con}(ZFC)$ and $ZFC + \psi \vdash \mathrm{Con}(ZFC)$.

Under *GCH*, Exercise III.5.44 is easily generalized to prove that there is a κ–Aronszajn tree for every cardinal κ that is the successor of a regular cardinal. This leaves the inaccessibles and the successors of singular cardinals. For more on this, see Exercise III.6.33 and the remarks at the end of Section III.6; roughly there must be a κ–Aronszajn for most, but not all, strongly inaccessible cardinals.

III.6 Club Filters

Here, we discuss a family of filters and ideals (Definitions III.1.1 and III.1.3) that has great importance in set theory.

Definition III.6.1 *Let γ be any limit ordinal and fix $A \subseteq \gamma$. A is* unbounded *in γ iff $\sup(A) = \gamma$. A is* closed *in γ iff for all limit $\nu < \gamma$: if $\sup(A \cap \nu) = \nu$ then $\nu \in A$. A is* club *in γ iff A is both* ***cl****osed and* ***unb****ounded.*

Note that $A \subseteq \gamma$ is closed iff A is closed in the order topology on γ (Definition III.2.15). If $\gamma = \omega$, then the order topology is discrete, so that every subset is closed, and, for $A \subseteq \omega$: A is unbounded iff A is club iff A is infinite.

If γ is any limit ordinal and C is the set of limit ordinals less than γ, then C is closed in γ; C is club iff γ is a limit of limits (such as ω^2 or ω^ω or ω_1).

On ω_1, the intersection of any two clubs is a club. More generally:

Lemma III.6.2 *Let γ be any limit ordinal, and assume that $\kappa := \mathrm{cf}(\gamma) > \omega$. Then the intersection of any family of fewer than κ club subsets of γ is club.*

Proof. Fix clubs $C_\xi \subseteq \gamma$ for $\xi < \lambda$, and let $D = \bigcap_\xi C_\xi$. Every intersection of closed sets is closed, but we must show that D is unbounded, assuming that $\lambda < \kappa$. So, fix $\delta_0 < \gamma$; we shall prove that $\sup(D) > \delta_0$.

First, for $\xi < \lambda$, define $f_\xi : \gamma \to \gamma$ so that $f_\xi(\alpha)$ is the least ordinal in C_ξ that is greater than α. Then, by recursion on n, define $\delta_n < \gamma$ so that $\delta_{n+1} = \sup\{f_\xi(\delta_n) : \xi < \lambda\}$; note that $\delta_{n+1} < \gamma$ because $\mathrm{cf}(\gamma) > \lambda$. Also, $\delta_0 < \delta_1 < \delta_2 \cdots$. Define $\delta_\omega = \sup_n \delta_n$; note that $\delta_\omega < \gamma$ because $\mathrm{cf}(\gamma) > \omega$.

For each $\xi < \gamma$, $\delta_\omega \in C_\xi$ because $\delta_\omega = \sup\{f_\xi(\delta_n) : n < \omega\}$ and each $f_\xi(\delta_n) \in C_\xi$ and C_ξ is closed. Thus, $\delta_\omega \in D$, so $\sup(D) \geq \delta_\omega > \delta_0$. □

Two obvious "improvements" of this lemma are easily seen to be false:

Exercise III.6.3 *Let γ be any limit ordinal, with $\kappa = \mathrm{cf}(\gamma)$. Then there is a family of κ club subsets of γ whose intersection is empty. Furthermore, if $\kappa = \omega$, then there are two disjoint club subsets of γ.*

When $\mathrm{cf}(\gamma) > \omega$, the fact that the intersection of any two clubs is a club implies that the clubs generate a filter:

Definition III.6.4 *Fix a limit γ with $\mathrm{cf}(\gamma) > \omega$. Then,*

$$\begin{aligned} \mathsf{club}_\gamma &= \{X \subseteq \gamma : \exists C \subseteq \gamma\, [C \text{ is club} \wedge C \subseteq X]\} \\ \mathsf{nonstat}_\gamma &= \{X \subseteq \gamma : \exists C \subseteq \gamma\, [C \text{ is club} \wedge C \cap X = \emptyset]\} \quad . \end{aligned}$$

club_γ *is called the* club filter *and its dual ideal,* $\mathsf{nonstat}_\gamma$*, is called the* nonstationary ideal. *$X \subseteq \gamma$ is called* stationary *iff $X \notin \mathsf{nonstat}_\gamma$; equivalently, iff X meets every club subset of γ.*

It is easily seen from Definitions III.6.4, III.1.1, III.1.3, III.1.4, III.1.6, Lemma III.6.2, and Exercise III.6.3 that:

Lemma III.6.5 *For a limit γ with $\kappa = \mathrm{cf}(\gamma) > \omega$: club_γ is a filter on γ and $\mathsf{nonstat}_\gamma$ is its dual ideal. Furthermore,* $\mathrm{add}(\mathsf{nonstat}_\gamma) = \mathrm{cov}(\mathsf{nonstat}_\gamma) = \mathrm{non}(\mathsf{nonstat}_\gamma) = \kappa$.

In practice, $\mathrm{cov}(\mathsf{nonstat}_\gamma)$ and $\mathrm{non}(\mathsf{nonstat}_\gamma)$ are rarely mentioned. Also, the "$\mathrm{add}(\mathsf{nonstat}_\gamma) = \kappa$" is usually applied to the dual filter: the intersection of fewer than κ elements of club_γ lies in club_γ. Then, we say that club_γ is *κ–complete* but not *κ^+–complete*, where

Definition III.6.6 *A filter $\mathcal{F}$ on a set A is* κ–complete *iff the intersection of* fewer *than κ elements of $\mathcal{F}$ always lies in $\mathcal{F}$.*

We remark that Lemma III.6.5 requires the Axiom of Choice: to prove that $\bigcap_{\xi<\lambda} X_\xi \in \mathsf{club}_\gamma$ (assuming that each $X_\xi \in \mathsf{club}_\gamma$ and $\lambda < \kappa$), you *choose* clubs $C_\xi \subseteq X_\xi$ and then apply Lemma III.6.2. Lemma III.6.2 did not use AC, although without AC, one cannot prove that there are any γ with $\mathrm{cf}(\gamma) > \omega$ (see Gitik [39]).

From now on, we shall focus on the situation where $\kappa = \gamma$; that is, we study the club filter and nonstationary ideal on uncountable regular cardinals κ. This is the situation most often encountered in practice. Also, Exercise III.6.31 reduces the study of club_γ to the study of $\mathsf{club}_{\mathrm{cf}(\gamma)}$ when $\mathrm{cf}(\gamma) < \gamma$.

The statement $\mathrm{add}(\mathsf{nonstat}_\kappa) = \kappa$ is often applied via the following:

Lemma III.6.7 *Assume that $\kappa > \omega$ is regular, $T \subseteq \kappa$ is stationary, and $f : T \to E$, where $|E| < \kappa$. Then some $f^{-1}\{e\}$ is stationary.*

Proof. If all the $f^{-1}\{e\}$ were non-stationary, then we would contradict Lemma III.6.5 ($\mathrm{add}(\mathsf{nonstat}_\kappa) = \kappa$). □

It is not immediately obvious that club_κ isn't an ultrafilter (Definition III.1.27); equivalently, that there are two disjoint stationary subsets of κ. When $\kappa \geq \omega_2$, the following definition and lemma provide a concrete example:

Definition III.6.8 *Whenever θ is a regular cardinal, S^κ_θ is the set of all limit ordinals $\gamma < \kappa$ such that $\mathrm{cf}(\gamma) = \theta$.*

Lemma III.6.9 *S^κ_θ is a stationary subset of κ whenever $\omega \leq \theta < \kappa$ and θ and κ are both regular.*

That is, every club $C \subseteq \kappa$ contains limit ordinals of cofinality θ (for example, the θth element of C). Setting $\theta = \omega$ and $\theta = \omega_1$ produces two disjoint stationary subsets of κ whenever $\kappa \geq \omega_2$. If κ is a regular limit cardinal (i.e., weakly inaccessible), then the various S^κ_θ provide κ disjoint stationary subsets. But in fact, there are also κ disjoint stationary subsets of any successor cardinal by the following argument due to Ulam (1930):

Theorem III.6.10 *Let κ be a successor cardinal and $\mathcal{I}$ any ideal on κ such that $\mathrm{add}(\mathcal{I}) = \kappa$ and $\mathcal{I}$ contains all singleton sets. Then there are disjoint $X_\delta \subseteq \kappa$ for $\delta < \kappa$ such that each $X_\delta \notin \mathcal{I}$.*

Proof. Note that $[\kappa]^{<\kappa} \subseteq \mathcal{I}$. Let $\mathcal{F}$ be the dual filter, which is κ–complete.

Let $\kappa = \lambda^+$. For $\rho < \kappa$, choose a function $f_\rho : \rho \xrightarrow{1\text{–}1} \lambda$. Then define, for $\xi < \lambda$ and $\alpha < \kappa$, $X^\xi_\alpha = \{\rho : \alpha < \rho < \kappa \wedge f_\rho(\alpha) = \xi\}$. Observe that:

1. For each $\xi < \lambda$: $\alpha \neq \beta \to X^\xi_\alpha \cap X^\xi_\beta = \emptyset$.
2. For each $\alpha < \kappa$: $|\kappa \setminus \bigcup\{X^\xi_\alpha : \xi < \lambda\}| < \kappa$.

For (1), use the fact that each f_ρ is 1-1. For (2), note that the union is $\kappa\backslash(\alpha+1)$.

By (2) and $\mathrm{add}(\mathcal{I}) = \kappa$, for each $\alpha < \kappa$ we can choose $\xi_\alpha < \lambda$ such that $X^{\xi_\alpha}_\alpha \notin \mathcal{I}$. Then, since κ is regular, we can fix a $\xi < \lambda$ such that $A := \{\alpha < \kappa : \xi_\alpha = \xi\}$ has size κ. Then, by (1), $\{X^\xi_\alpha : \alpha \in A\}$ is a disjoint family of κ subsets of κ, none of which is in $\mathcal{I}$. □

A $\kappa \times \lambda$ matrix of sets X^ξ_α satisfying (1)(2) is called an *Ulam matrix.*

This theorem applied to $\mathsf{nonstat}_\kappa$ shows that there are κ disjoint stationary subsets of any successor cardinal κ. In addition, every stationary $S \subseteq \kappa$ contains κ disjoint stationary subsets, since we may apply the theorem to the ideal $\{X \subseteq \kappa : X \cap S \in \mathsf{nonstat}_\kappa\}$. This latter fact is true also when κ is weakly inaccessible, as was shown by Solovay [94] in the 1960s.

As a result about arbitrarily ideals, Theorem III.6.10 can fail for weakly inaccessible κ; a counter-example is known as a *κ–saturated* ideal. This leads to the study of large cardinals, since such a κ must also be the κth weak inaccessible, and also weakly Mahlo (defined in Exercise III.6.30), and much more; see [94] and the texts of Jech [49] and Kanamori [51].

A special feature of the club filter is that it is closed under diagonal intersections:

Definition III.6.11 *Given a κ-sequence of subsets of κ, $\langle A_\alpha : \alpha < \kappa \rangle$, their* diagonal intersection *is:*

$$\triangle_{\alpha<\kappa} A_\alpha = \{\beta < \kappa : \beta \in \bigcap_{\alpha<\beta} A_\alpha\} \ .$$

Lemma III.6.12 *Let $\kappa > \omega$ be a regular uncountable cardinal.*

1. *If $C_\alpha \subseteq \kappa$ is club for each α, then $\triangle_{\alpha<\kappa} C_\alpha$ is club.*
2. *If $A_\alpha \subseteq \kappa \in \mathsf{club}_\kappa$ for each α, then $\triangle_{\alpha<\kappa} A_\alpha \in \mathsf{club}_\kappa$.*

Proof. (2) follows easily from (1). For (1), $D := \triangle_{\alpha<\kappa} C_\alpha$ is easily seen to be closed. To prove that D is unbounded, fix any $\delta_0 < \kappa$. Recursively choose $\delta_{n+1} > \delta_n$ so that $\delta_{n+1} \in \bigcap\{C_\alpha : \alpha < \delta_n\}$ (which is club by Lemma III.6.2). Let $\delta_\omega = \sup_n \delta_n$; then $\delta_\omega \in \bigcap\{C_\alpha : \alpha < \delta_\omega\}$, so $\delta_\omega \in D$. □

Using diagonal intersections, we see that the set of "almost fixed-points" of an $f : \kappa \to \kappa$ form a club:

Lemma III.6.13 *Assume that $\kappa > \omega$ is a regular uncountable cardinal and $f : \kappa \to \kappa$. Then $D := \{\beta < \kappa : \forall \alpha < \beta\, [f(\alpha) < \beta]\}$ is a club.*

Proof. $D = \triangle_{\alpha<\kappa} C_\alpha$, where $C_\alpha = \{\beta : f(\alpha) < \beta < \kappa\}$. □

Trivially, $0 \in D$, but usually one is interested in the non-zero elements. Some examples: If $f(\alpha) = \alpha + 1$, then $D \backslash \{0\}$ is the set of limit ordinals less than κ. If $f(\alpha) = \alpha + \alpha + 1$, then $D \backslash \{0\}$ is the set of indecomposable ordinals (see Exercise I.9.53) less than κ. If $f(\alpha) = \beth_\alpha$ and κ is strongly inaccessible, then $D \backslash \{0\} = \{\alpha < \kappa : \alpha = \beth_\alpha\}$.

Diagonal intersections also yield the *Pressing Down Lemma* (Fodor, 1956):

Lemma III.6.14 *Assume that $\kappa > \omega$ is a regular uncountable cardinal, $S \subseteq \kappa$ is stationary, and $f : S \to \kappa$ is such that $f(\xi) < \xi$ for all $\xi \in S$. Then there is some $\alpha < \kappa$ such that $f^{-1}\{\alpha\}$ is stationary.*

Proof. If not, then each $A_\alpha := \kappa \backslash f^{-1}\{\alpha\} \in \mathsf{club}_\kappa$, so $\triangle_{\alpha<\kappa} A_\alpha \in \mathsf{club}_\kappa$, so fix $\beta \in S \cap \triangle_{\alpha<\kappa} A_\alpha$. Then $f(\beta) \neq \alpha$ for all $\alpha < \kappa$, contradicting $f(\beta) < \alpha$. □

The function f is called a *pressing down function* or a *regressive function*. The Pressing Down Lemma is frequently used in topology; for examples, see Exercises III.6.28 and III.6.29. It is also used in proving a generalization of the Delta System Lemma III.2.6, which was the special case of the following when $\lambda = \aleph_0$; note that the proof is shorter than our original proof even for this case.

Lemma III.6.15 (Delta System Lemma) *Let λ and κ be regular cardinals such that $\omega \leq \lambda < \kappa$. Assume that $\forall \theta < \kappa\, [\theta^{<\lambda} < \kappa]$. Let $\mathcal{A}$ be a family of sets with $|\mathcal{A}| = \kappa$. such that $|A| < \lambda$ for all $A \in \mathcal{A}$. Then there is a $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that $\mathcal{B}$ forms a delta system.*

Proof. List $\mathcal{A}$ as $\{A_\alpha : \alpha < \kappa\}$. Since $\bigcup_\alpha A_\alpha$ is some set of size $\leq \kappa$, we may assume WLOG that each $A_\alpha \subseteq \kappa$.

Let $S = S^\kappa_\lambda$, which is a stationary subset of κ (Lemma III.6.9). For $\alpha \in S$, $\sup(\alpha \cap A_\alpha) < \alpha$ (since $|A_\alpha| < \mathrm{cf}(\alpha)$), so by the Pressing Down Lemma, fix a stationary $T \subseteq S$ and a $\nu < \kappa$ such that $\sup(\alpha \cap A_\alpha) < \nu$ for all $\alpha \in T$. Also, let $D = \{\beta < \kappa : \forall \alpha < \beta\, [\sup(A_\alpha) < \beta]\}$; this is a club by Lemma III.6.13.

Now, if $\alpha, \beta \in T \cap D$ with $\alpha < \beta$, then $\sup(\beta \cap A_\beta) < \nu$ plus $\sup(A_\alpha) < \beta$ implies that $A_\alpha \cap A_\beta \subseteq \nu$. Since T is stationary, and hence of size κ, while $\left|[\nu]^{<\lambda}\right| < \kappa$, there is, by Lemma III.6.7, a stationary $W \subseteq T$ such that the $A_\alpha \cap \nu$, for $\alpha \in W$, are all the same set R. Then $\{A_\alpha : \alpha \in W\}$ forms a delta system with root R. □

Stationary sets are used in the proof that *GCH* cannot first fail at a singular cardinal of uncountable cofinality; see Exercise III.6.19. This theorem was first proved by Silver in 1974. We outline here a simpler proof found by Baumgartner, Jensen, and Prikry, and published in the American Mathematical Monthly in 1977; see [7]. Further information on powers of singular cardinals was given by Galvin and Hajnal [37], and later by the pcf theory of Shelah [83].

We begin with a modification of the notion of almost disjoint sets:

Exercise III.6.16 *Let $\lambda = \omega_{\omega_1}$. If $\mathcal{F} \subseteq {}^{\omega_1}\lambda$, call $\mathcal{F}$ an* eventually different family (edf) *iff $\{\alpha : f(\alpha) = g(\alpha)\}$ is countable whenever $f, g \in \mathcal{F}$ and $f \neq g$. Assuming that $\forall \alpha < \omega_1\, [2^{\aleph_\alpha} = \aleph_{\alpha+1}]$, prove that there is an edf of size 2^λ such that $f(\alpha) < \omega_{\alpha+1}$ for all $f \in \mathcal{F}$ and all $\alpha < \omega_1$.*

Hint. For $X \subseteq \lambda$, define $f_X(\alpha) = X \cap \omega_\alpha \in \mathcal{P}(\omega_\alpha)$. Then use $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ to associate $f_X(\alpha)$ with an ordinal less than $\omega_{\alpha+1}$. □

Exercise III.6.17 *Assume that $\lambda = \omega_{\omega_1}$, $\forall \alpha < \omega_1\, [2^{\aleph_\alpha} < \lambda]$, $\mathcal{F} \subseteq {}^{\omega_1}\lambda$ is an edf, and $\{\alpha < \omega_1 : f(\alpha) < g(\alpha)\}$ is stationary for all $f \in \mathcal{F}$, where $g : \omega_1 \to \lambda$ and $g(\alpha) < \omega_{\alpha+1}$ for all α. Prove that $|\mathcal{F}| \leq \lambda$.*

Hint. First consider the simple case when $g(\alpha) = \omega_\alpha$. Let $|f(\alpha)| = \omega_{f'(\alpha)}$. Use the pressing down lemma applied to f', plus the fact that there's fewer than λ stationary sets. For the general g, first inject each $g(\alpha)$ into ω_α. □

Exercise III.6.18 *Let $\lambda = \omega_{\omega_1}$. Assume that $\forall \alpha < \omega_1\, [2^{\aleph_\alpha} = \aleph_{\alpha+1}]$. Prove that $2^\lambda = \lambda^+$.*

Hint. Let $\mathcal{F}$ be as in Exercise III.6.16. Partially order $\mathcal{F}$ by: $f <^* g$ iff $\{\alpha : f(\alpha) < g(\alpha)\} \in \mathsf{club}_{\omega_1}$. Let $\lhd$ be any total order of $\mathcal{F}$ extending $<^*$. Then show that each $g \in \mathcal{F}$ has no more than λ predecessors under $\lhd$. □

Now, one can replace ω_{ω_1} by another singular cardinal λ and replace the ω_α by some other cofinal sequence in λ to get:

Exercise III.6.19 *Assume that λ is a singular cardinal with $\mathrm{cf}(\lambda) > \omega$, and assume that $\{\theta < \lambda : 2^\theta = \theta^+\}$ is stationary in λ. Prove that $2^\lambda = \lambda^+$.*

The club filter is related to the Downward Löwenheim–Skolem–Tarski Theorem I.15.10. First, a variant of Lemma I.13.22:

Exercise III.6.20 *Let $\mathcal{F}$ be a countable family of finitary functions on ω_1, and let C be the set of all $\alpha < \omega_1$ such that α is closed under $\mathcal{F}$ (Definition I.13.20). Then C is a club subset of ω_1.*

Then, the Löwenheim–Skolem–Tarski variant:

Exercise III.6.21 *Let $\mathfrak{B}$ be a structure for a countable $\mathcal{L}$ with universe B, where $B = \omega_1$. Then $C := \{\alpha < \omega_1 : \alpha \preccurlyeq \mathfrak{B}\}$ is a club subset of ω_1.*

Hint. Of course, "$\alpha \preccurlyeq \mathfrak{B}$" includes the assumption that the set α is closed under all the functions of $\mathcal{L}$, so that we may regard α as a sub-structure of $\mathfrak{B}$.

Adding Skolem functions and applying Exercise III.6.20, it is easy to see that C contains a club, but you need to prove that C *is* a club. To prove that C is closed, use the Tarski–Vaught criterion (Lemma I.15.11). □

If you view a filter (club_{ω_1}) as being a notion of "almost everything", then you can paraphrase this exercise by saying that almost every α is an elementary submodel. Now, say in this exercise we drop the "$B = \omega_1$", so $\mathfrak{B}$ is an arbitrary structure for $\mathcal{L}$. There are still countable $\mathfrak{A} \preccurlyeq \mathfrak{B}$ by the Downward Löwenheim–Skolem–Tarski Theorem, but to say that "$A \preccurlyeq \mathfrak{B}$ for almost every A", we need the following filter on $[B]^{\leq \aleph_0}$, introduced by Jech and Kueker in the 1970s:

Definition III.6.22 *Let B be any set. $[B]^{\leq \aleph_0}$ denotes the family of countable subsets of B. Fix $\mathcal{C} \subseteq [B]^{\leq \aleph_0}$.*

$\mathcal{C}$ is unbounded *in $[B]^{\leq \aleph_0}$ iff $\forall A \in [B]^{\leq \aleph_0} \exists M \supseteq A\, [M \in \mathcal{C}]$.*

$\mathcal{C}$ is closed *in $[B]^{\leq \aleph_0}$ iff whenever $A_n \in \mathcal{C}$ for $n \in \omega$ and $A_0 \subseteq A_1 \subseteq \cdots$, then $\bigcup_n A_n \in \mathcal{C}$.*

$\mathcal{C}$ is club *in $[B]^{\leq \aleph_0}$ iff $\mathcal{C}$ is both closed and unbounded.*

One can also generalize this by considering $[B]^{<\kappa}$ instead of $[B]^{\leq \aleph_0}$, but we shall not do this here.

Exercise III.6.23 *A countable intersection of club subsets of B is a club.*

Because of this, the clubs generate a filter. Analogously to Definition III.6.4:

Definition III.6.24 *Fix any set B. Then,*

$$\begin{aligned} \mathsf{club}^B &= \{\mathcal{X} \subseteq [B]^{\leq \aleph_0} : \exists \mathcal{C} \subseteq [B]^{\leq \aleph_0}\, [\mathcal{C} \text{ is club} \wedge \mathcal{C} \subseteq \mathcal{X}]\} \\ \mathsf{nonstat}^B &= \{\mathcal{X} \subseteq [B]^{\leq \aleph_0} : \exists \mathcal{C} \subseteq [B]^{\leq \aleph_0}\, [\mathcal{C} \text{ is club} \wedge \mathcal{C} \cap \mathcal{X} = \emptyset]\} \ . \end{aligned}$$

club^B *is called the* club filter *and its dual ideal,* $\mathsf{nonstat}^B$*, is called the* nonstationary ideal. *$\mathcal{X} \subseteq [B]^{\leq \aleph_0}$ is called* stationary *iff $\mathcal{X} \notin \mathsf{nonstat}^B$; equivalently, iff $\mathcal{X}$ meets every club subset of $[B]^{\leq \aleph_0}$.*

If B is countable, then $\mathcal{X} \in \mathsf{club}^B$ iff $\mathcal{X}$ is stationary iff $B \in \mathcal{X}$, so these notions are only interesting when B is uncountable.

Consider the special case $B = \omega_1$, where we have the filters club^{ω_1} defined here and club_{ω_1} from Definition III.6.4. These are "essentially the same". Let $\mathcal{D}$ be the set of all $A \in [\omega_1]^{\leq \aleph_0}$ such that A is an initial segment of ω_1. Note that $\mathcal{D}$ is a club, and then note that $\mathcal{D}$ really equals the set ω_1. Then note that for any $\mathcal{C} \subseteq [\omega_1]^{\leq \aleph_0}$, $\mathcal{C} \in \mathsf{club}^{\omega_1}$ iff $\mathcal{C} \cap \omega_1 \in \mathsf{club}_{\omega_1}$.

When $|B| > \aleph_1$ we really do have a new notion here; for example, club^{ω_2} is a filter on the countable subsets of ω_2, whereas club_{ω_2} is a filter on the elements of ω_2, most of which are uncountable.

Now we can say that "almost every $A \in [B]^{\leq \aleph_0}$ is an elementary submodel":

Exercise III.6.25 *Let $\mathcal{F}$ be a countable family of finitary functions on B, where B is any set, and let $\mathcal{C}$ be the set of all $A \in [B]^{\leq \aleph_0}$ such that A is closed under $\mathcal{F}$ (Definition I.13.20). Then $\mathcal{C}$ is a club subset of $[B]^{\leq \aleph_0}$.*

Exercise III.6.26 *Let $\mathfrak{B}$ be a structure for a countable $\mathcal{L}$ with universe B. Then $\mathcal{C} := \{A \in [B]^{\leq \aleph_0} : A \preccurlyeq \mathfrak{B}\}$ is a club subset of $B^{\leq \aleph_0}$.*

Hint. Of course, "$A \preccurlyeq \mathfrak{B}$" includes the assumption that the set A is closed under all the functions of $\mathcal{L}$. □

One can also prove a Pressing Down Lemma like Lemma III.6.14:

Exercise III.6.27 *Assume that $\mathcal{S} \subseteq [B]^{\leq \aleph_0}$ is stationary, and $f : \mathcal{S} \to B$ is such that $f(A) \in A$ for all $A \in \mathcal{S}$. Then there is some $b \in B$ such that $f^{-1}\{b\}$ is stationary.*

Returning now to the "standard" Pressing Down Lemma III.6.14:

Exercise III.6.28 *Let κ be a regular uncountable cardinal, X a metric space, and $h : \kappa \to X$ a continuous function (giving κ its order topology). Prove that there is a $\xi < \kappa$ such that $h(\alpha) = h(\xi)$ for all $\alpha \geq \xi$.*

Hint. Let S be the set of limit ordinals less than κ. For each $n \in \omega$, get a regressive function f on S so that for all $\xi \in S$, the diameter of $\{h(\mu) : f(\xi) \leq \mu \leq \xi\}$ is less than $1/n$, and apply the Pressing Down Lemma. □

In particular, the order topology on κ itself is not metrizable. For $\kappa > \omega_1$, this is obvious, since κ is not even first countable. For $\kappa = \omega_1$, this may be slightly surprising, since the topological space ω_1 can be embedded in the *long line*, which resembles the real numbers. Informally, we get the long line by connecting adjacent ordinals in ω_1 by line segments. More formally:

Exercise III.6.29 *Let $L = \omega_1 \times [0, 1)$. Order L lexicographically, and give L the order topology. L is called the* long line. *Prove that:*

1. *In L, $\omega_1 \times \{0\}$ is homeomorphic to ω_1.*
2. *Whenever X is a metric space and $h : L \to X$ is continuous, there is an $x \in L$ such that $h(y) = h(x)$ for all $y \geq x$.*
3. *$L\backslash\{(0,0)\}$ is connected and is a 1-manifold — that is, every point has a neighborhood homeomorphic to $\mathbb{R}$.*
4. *With a suitable atlas, $L\backslash\{(0,0)\}$ is a C^∞ manifold.*
5. *Up to homeomorphism, the only connected 1-manifolds are the circle, $\mathbb{R}$, $L\backslash\{(0,0)\}$, and the doubly long line, obtained by gluing two copies of L together at the endpoint $(0,0)$.*

Exercise III.6.30 *A cardinal κ is called* weakly Mahlo *iff κ is weakly inaccessible and the set of weakly inaccessible cardinals less than κ is a stationary subset of κ. Then κ is called* strongly Mahlo *iff κ is weakly Mahlo and strongly inaccessible.*

1. *Prove that a strongly inaccessible κ is strongly Mahlo iff the set of strongly inaccessible cardinals less than κ is a stationary subset of κ.*
2. *Let E_0 be the class of weakly inaccessible cardinals. Then, let $E_{\alpha+1} = \{\kappa \in E_\alpha : |\kappa \cap E_\alpha| = \kappa\}$, and, for limit γ, let $E_\gamma = \bigcap\{E_\alpha : \alpha < \gamma\}$. Prove that if κ is weakly Mahlo, then $\kappa \in E_\kappa$ and is the κth element of $\{\alpha : \alpha \in E_\alpha\}$.*
3. *Same as (2), replacing "weakly" by "strongly".*
4. *If κ is weakly Mahlo, then κ is weakly (and hence strongly) Mahlo in L.*

By (2)(3), the Mahlo cardinals transcend what you can get by counting inaccessibles. It is consistent that $\mathfrak{c}$ is weakly Mahlo and that *MA* holds; see Exercise V.4.23.

The following exercise reduces the study of club_γ to the case where γ is a regular cardinal.

Exercise III.6.31 *Assume that γ is a limit ordinal with $\omega < \kappa = \mathrm{cf}(\gamma) < \gamma$. Then there is club $C \subset \gamma$ such that $\mathrm{type}(C) = \kappa$. Furthermore, if $f : C \xrightarrow[\text{onto}]{1-1} \kappa$ is the isomorphism from C onto κ, then $Y \in \mathsf{club}_\kappa$ iff $f^{-1}(Y) \in \mathsf{club}_\gamma$ whenever $Y \subseteq \kappa$; and, $X \in \mathsf{club}_\gamma$ iff $f(X \cap C) \in \mathsf{club}_\kappa$ whenever $X \subseteq \gamma$.*

For finite θ, the following exercise was Lemma III.5.35.

Exercise III.6.32 *Let κ be regular with T a well-pruned κ-Aronszajn tree. Fix a cardinal $\theta < \kappa$. Then $\forall x \in T\, \exists \alpha > \mathrm{height}(x)\, [|x{\uparrow} \cap \mathcal{L}_\alpha(T)| \geq \theta]$.*

Hint. Passing to a subtree, it is enough to derive a contradiction from the assumption that $|\mathcal{L}_\alpha| < \theta$ for all α. Note that the map $\alpha \mapsto |\mathcal{L}_\alpha|$ is nondecreasing, so WLOG θ is regular. Let $S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \theta\}$. For $\alpha \in S$, let $f(\alpha) < \alpha$ be such that for all $x, y \in \mathcal{L}_\alpha$, either $x{\downarrow} = y{\downarrow}$ or $x{\downarrow} \cap \mathcal{L}_{f(\alpha)} \neq y{\downarrow} \cap \mathcal{L}_{f(\alpha)}$. Now, apply the Pressing Down Lemma. □

Exercise III.6.33 *Let κ be strongly inaccessible cardinal, and assume that there are no κ–Aronszajn trees. Prove that whenever $S \subseteq \kappa$ is stationary, there is an uncountable regular $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ. In particular, κ is strongly Mahlo, and not the first strongly Mahlo cardinal.*

Hint. Intersecting S with a club, WLOG all elements of S are uncountable cardinals. Assuming the result is false, we define a κ–Aronszajn tree T as follows: T is a subtree of $\kappa^{<\kappa}$. For $t \in \kappa^\alpha$, $t \in T$ iff $t(\xi) < \xi$ for all $\xi \in \alpha \cap S$, and t is 1-1 on $\alpha \cap S$, and $t(\xi) = 0$ for all $\xi \in \alpha \backslash S$. A path through T would yield a 1-1 regressive function on S, contradicting the Pressing Down Lemma.

To show that height(T) is really κ, so that T is Aronszajn, you use the assumption that $S \cap \lambda$ is non-stationary in λ for all uncountable regular $\lambda < \kappa$. By induction on ordinals $\alpha < \kappa$, show that $\mathcal{L}_\alpha(T) \neq \emptyset$. This is slightly tricky when α is a limit cardinal. There are two cases, both using a club of infinite cardinals $C \subset \alpha$. If α is singular, let type$(C) < \alpha$. If α is regular, let $C \cap S = \emptyset$.

To prove that κ is Mahlo, just consider the case where S is a club. □

Strongly inaccessible cardinals κ for which there are no κ–Aronszajn trees are called *weakly compact.* See Kanamori [51] for more on such large cardinals.

As pointed out at the end of Section III.5, *GCH* implies that there are κ–Aronszajn trees whenever κ is a successor of a regular cardinal. Successors of singular cardinals present some difficult set-theoretic questions. Jensen showed that all such κ have κ–Aronszajn trees (and even κ–Suslin trees) if $V = L$; see [20]. On the other hand, Magidor and Shelah [63] show that there are no λ^+–Aronszajn trees if λ is singular and the strongly compact cardinals are unbounded below λ. Strongly compact cardinals are much larger than weakly compact cardinals, and their existence is (relatively) consistent with GCH.

III.7 $\diamondsuit$ and $\diamondsuit^+$

Around 1970, Jensen discovered that Gödels's Axiom of Constructibility $V = L$ (see Section II.6) implies $\neg SH$ (there exists a Suslin tree). He then defined a simple set-theoretic principle, $\diamondsuit$, and proved that $V = L \rightarrow \diamondsuit$ and $\diamondsuit \rightarrow \neg SH$. This $\diamondsuit$ now has many other applications besides the Suslin problem. Also, Solovay showed that $V = L \rightarrow KH$ (there exists a Kurepa tree). This proof led to the stronger principle $\diamondsuit^+$ and proofs that $V = L \rightarrow \diamondsuit^+$ and $\diamondsuit^+ \rightarrow KH$. Using forcing, one can show that $GCH + \diamondsuit$ does not imply KH (see Theorem V.2.25 and Exercise V.2.27). Also, *GCH* does not imply $\neg SH$; this was first proved by Jensen (see [21]), and later, using a different method, by Shelah [81].

Definition III.7.1 $\diamondsuit$ *is the statement: There exists sets $A_\alpha \subseteq \alpha$ for $\alpha < \omega_1$ such for every $A \subseteq \omega_1$, the set $\{\alpha : A \cap \alpha = A_\alpha\}$ is stationary. The sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ is called a* $\diamondsuit$ sequence.

Lemma III.7.2 $\diamondsuit$ *implies CH.*

Proof. If $\langle A_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit$ sequence, then $\mathcal{P}(\omega) \subseteq \{A_\alpha : \alpha < \omega_1\}$; to see this, fix $A \subseteq \omega$, fix $\alpha > \omega$ with $A \cap \alpha = A_\alpha$, and note that $A \cap \alpha = A$. □

We may view $\diamondsuit$ as a strengthening of *CH*. A $\diamondsuit$ sequence easily gave us a a listing of $\mathcal{P}(\omega)$ in type ω_1, but the sequence also "*captures*" all subsets of ω_1 as well. Of course, you can't list $\mathcal{P}(\omega_1)$ in type ω_1, but the $\diamondsuit$ sequence lets you construct an object in ω_1 steps, building in some desired property with respect to every subset of ω_1. We shall see this in the construction of a Suslin tree T. As in the proof of Theorem III.5.12, T will be the ordinal ω_1 as a set, so we need to construct the tree order $\sqsubset$. We construct $\sqsubset$ in ω_1 steps. Roughly, to make sure that every antichain A is countable, we "capture" A at stage α in the construction when $A \cap \alpha = A_\alpha$. Note that by Lemma III.5.37, we don't have to worry about chains; it is sufficient to guarantee that all maximal antichains are countable.

Theorem III.7.3 $\diamondsuit$ *implies that there exists a Suslin tree.*

Proof. We begin verbatim as in the first two paragraphs of the proof of Theorem III.5.12, which produced an Aronszajn tree in *ZFC*. In particular, T is ω_1 as a set, and $T_\gamma = \bigcup_{\alpha<\gamma} \mathcal{L}_\alpha$ is always some ordinal; $T_\gamma = \omega \cdot \gamma$ when $\gamma \geq \omega$. We are constructing $\sqsubset$ by recursion. As before, each node will have exactly $\aleph_0$ immediate successors.

As before, the meat of the construction is what happens at a limit γ. If we have already constructed $\sqsubset$ on T_γ, then we choose paths P_k through T_γ for $k \in \omega$, and then put the node $\omega \cdot \gamma + k$ directly above P_k. But, more needs to be said about the choice of P_k. To guarantee that there even exist such paths, we make sure inductively that $\sqsubset$ satisfies:

$$\forall x \in T \, \forall \alpha < \omega_1 \, [\alpha > \text{height}(x) \rightarrow \exists y \in \mathcal{L}_\alpha(T) \, [x \sqsubset y]] \ . \qquad (*)$$

Note that $(*)$ is automatically preserved by our construction at successor levels.

Let $\{x_k : k \in \omega\}$ list the elements of T_γ (i.e., the ordinal $\omega \cdot \gamma$). For each k, choose a path P_k through T_γ with $x_k \in P_k$. Then, put the node $\omega \cdot \gamma + k$ directly above P_k. This is possible exactly as in the proof of Theorem III.5.12, and guarantees that $(*)$ is preserved, so that we shall construct an ω_1–tree.

To make the tree Suslin, let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$ sequence, with each $A_\alpha \subseteq \alpha$. Then, in addition to the above, make sure that whenever $\omega \cdot \gamma = \gamma > 0$ (so $T_\gamma = \gamma$ as a set): *If* A_γ is a maximal antichain in T_γ, then $P_k \cap A_\gamma \neq \emptyset$ for each k. To see that this is possible: First, since A_γ is maximal, choose $w_k \in A_\gamma$ such that $x_k \not\perp w_k$; so $w_k \sqsubseteq x_k$ or $x_k \sqsubseteq w_k$ (or both, if $w_k = x_k$). Then, choose $\alpha_n = \alpha_{k,n}$ for $n \in \omega$ such that $\max(\text{height}(x_k), \text{height}(w_k)) < \alpha_n \nearrow \gamma$. Then, using $(*)$ ω times, choose nodes z_n for $n \in \omega$ such that $x_k, w_k \sqsubset z_0 \sqsubset z_1 \sqsubset \cdots$ and $\text{height}(z_n) = \alpha_n$; and let P_k be the path $\bigcup_n (z_n{\downarrow})$.

Observe that every node at level γ is above some node in A_γ, so that A_γ remains a maximal antichain in T.

To conclude the proof, we fix a maximal antichain $A \subseteq T$ and prove that A is countable (and, equal to some A_γ). First, note that there is a club $C \subseteq \omega_1$ such that for all $\gamma \in C$: $\omega \cdot \gamma = \gamma > 0$ and $A \cap \gamma$ is a maximal antichain in T_γ. To see this, define $f(\alpha)$ to be the least ordinal $\delta > \omega \cdot \alpha$ such that $\forall x \in T_\alpha \, \exists w \in A \cap T_\delta \, [x \not\perp w]$. Then, by Lemma III.6.13, let C be a club such that $\forall \alpha < \gamma \, [f(\alpha) < \gamma]$ whenever $\gamma \in C$.

Now, fix $\gamma \in C$ such that $A \cap \gamma = A_\gamma$. Our construction already guaranteed that A_γ is maximal, so that $A = A_\gamma$, which is countable. □

Exercise III.7.4 *Show how to make the tree in Theorem III.7.3 Hausdorff; see Definition III.5.23.*

We next define a stronger principle, $\diamondsuit^+$, and show that $\diamondsuit^+ \to KH$. $\diamondsuit$ does not imply KH; see Exercise V.2.27.

To first approximation, we obtain $\diamondsuit^+$ by strengthening "is stationary" to "contains a club" in Definition III.7.1. But this would be inconsistent, since if A, B are two different subsets of ω_1, then the two sets $\{\alpha : A \cap \alpha = A_\alpha\}$ and $\{\alpha : B \cap \alpha = A_\alpha\}$ cannot both contain clubs, since their intersection is countable. So, we also weaken the principle by replacing A_α by a countable set $\mathcal{A}_\alpha$ of possible candidates for capturing $A \cap \alpha$:

Definition III.7.5 *Let $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ be a sequence of sets, where each $\mathcal{A}_\alpha$ is a countable subset of $\mathcal{P}(\alpha)$. This sequence is a* $\diamondsuit^*$ sequence *iff*

$$\forall A \subseteq \omega_1 \, \exists C \subseteq \omega_1 \, [C \text{ is club } \wedge \forall \alpha \in C \, [A \cap \alpha \in \mathcal{A}_\alpha]] \ .$$

This sequence is a $\diamondsuit^+$ sequence *iff*

$$\forall A \subseteq \omega_1 \, \exists C \subseteq \omega_1 \, [C \text{ is club } \wedge \forall \alpha \in C \, [A \cap \alpha \in \mathcal{A}_\alpha \wedge C \cap \alpha \in \mathcal{A}_\alpha]] \ .$$

Then, $\diamondsuit^*$ is the statement that there exists a $\diamondsuit^*$ sequence, and $\diamondsuit^+$ is the statement that there exists a $\diamondsuit^+$ sequence.

Clearly, $\diamondsuit^+ \to \diamondsuit^*$; the $\diamondsuit^*$ sequence captures every $A \subseteq \omega_1$ on a club, and the $\diamondsuit^+$ sequence has the stronger property of also capturing the club on itself. Also, $\diamondsuit^* \to \diamondsuit$, as we shall show below (Theorem III.7.8), after proving that $\diamondsuit^+ \to KH$. $\diamondsuit^* \not\to \diamondsuit^+$, since there are models in which $\diamondsuit^* \wedge \neg KH$ holds (see [18]). Also, $\diamondsuit \not\to \diamondsuit^*$ (see Exercise V.2.24).

In verifying that $\diamondsuit^+ \to KH$, it is somewhat simpler to produce a Kurepa family rather than a Kurepa tree; these are equivalent by Lemma III.5.40.

Theorem III.7.6 $\diamondsuit^+$ *implies that there is an ω_1-Kurepa family $\mathcal{F} \subseteq \mathcal{P}(\omega_1)$ such that*

$$\forall A \in [\omega_1]^{\aleph_1} \, \exists X \in [A]^{\aleph_1} \, [X \in \mathcal{F}] \ . \qquad (*)$$

Proof. By Definition III.5.39, our $\mathcal{F}$ needs to satisfy two things: $|\mathcal{F}| \geq \aleph_2$, and $|\mathcal{F}\restriction\alpha| \leq \aleph_0$ for all $\alpha < \omega_1$.

First note that $|\mathcal{F}| \geq \aleph_2$ follows from $(*)$: If $|\mathcal{F}| \leq \aleph_1$, then we could list $\mathcal{F} \cap [\omega_1]^{\aleph_1}$ as $\{B_\xi : \xi < \omega_1\}$, and then let $A = \{\alpha_\xi : \xi < \omega_1\} \subseteq \omega_1$, where the α_ξ are chosen recursively so that $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$ and, for each ξ, $\exists \beta \in B_\xi\, [\alpha_\xi < \beta < \alpha_{\xi+1}]$. Then A is uncountable and contains no uncountable subset from $\mathcal{F}$. Actually, $|\mathcal{F}| = 2^{\aleph_1}$ (see Exercise III.7.11).

Next, we shall build countable sets $\mathcal{F}_\beta \subseteq \mathcal{P}(\beta)$ and *define* $\mathcal{F} = \{X \subseteq \omega_1 : \forall \beta < \omega_1\, [X \cap \beta \in \mathcal{F}_\beta]\}$. Then all $|\mathcal{F}\restriction\beta| \leq \aleph_0$, so we shall be done if we can obtain $(*)$. Let $\langle \mathcal{A}_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit^+$ sequence. We can almost use $\mathcal{A}_\alpha$ for our $\mathcal{F}_\alpha$, but we need to modify it. If A is *any* subset of ω_1, then $\{\alpha : A \cap \alpha \in \mathcal{A}_\alpha\}$ contains a club, but we require $A \cap \beta \in \mathcal{F}_\beta$ for *all* β. To handle the ordinals between the ordinals of the club, we thin A down a bit.

For ordinals $\xi < \eta$, let $(\xi, \eta) = \{\mu : \xi < \mu < \eta\}$ and $[\xi, \eta) = \{\mu : \xi \leq \mu < \eta\}$. For $C \subseteq \omega_1$, call ξ, η *adjacent* in C iff $\xi < \eta$, $\xi, \eta \in C$, and $C \cap (\xi, \eta) = \emptyset$. Then, if $C, A \subseteq \omega_1$, let $\mathrm{thin}(A, C)$ be the set of all ordinals $\mathrm{min}(A \cap [\xi, \eta))$ such that ξ, η are adjacent elements of C and $A \cap [\xi, \eta) \neq \emptyset$. Note that $\mathrm{thin}(A, C) \subseteq A$, and that if C is club and $|A| = \aleph_1$, then $|\mathrm{thin}(A, C)| = \aleph_1$.

Now, for $\beta < \omega_1$, let

$$\mathcal{F}_\beta = \left\{ a \cup \mathrm{thin}(A, C) : \ a \in [\beta]^{<\omega} \ \wedge \ A, C \in \left(\{\emptyset\} \cup \bigcup \{\mathcal{A}_\alpha : \alpha \leq \beta\}\right) \right\} .$$

To verify $(*)$, fix an uncountable $A \subseteq \omega_1$, then fix a club C such that $A \cap \alpha \in \mathcal{A}_\alpha$ and $C \cap \alpha \in \mathcal{A}_\alpha$ for all $\alpha \in C$, and let $X = \mathrm{thin}(A, C)$. Now, fix $\beta < \omega_1$, and we show that $X \cap \beta \in \mathcal{F}_\beta$. If $C \cap \beta$ is finite, then $X \cap \beta \in [\beta]^{<\omega}$, so $X \cap \beta \in \mathcal{F}_\beta$ because $\mathrm{thin}(\emptyset, \emptyset) = \emptyset$. If $C \cap \beta$ is infinite, let α be the largest ordinal $\leq \beta$ that is a limit point of C. Then $X \cap \beta = a \cup \mathrm{thin}(A \cap \alpha, C \cap \alpha)$ for some finite a, and $A \cap \alpha, C \cap \alpha \in \mathcal{A}_\alpha$, so $X \cap \beta \in \mathcal{F}_\beta$. □

Next, to prove that $\diamondsuit^* \to \diamondsuit$, we interpolate another principle that obviously follows from $\diamondsuit^*$:

Definition III.7.7 *Let $\langle \mathcal{A}_\alpha : \alpha < \omega_1\rangle$ be a sequence of sets, where each $\mathcal{A}_\alpha$ is a countable subset of $\mathcal{P}(\alpha)$. This sequence is a* $\diamondsuit^-$ sequence *iff $\{\alpha < \omega_1 : A \cap \alpha \in \mathcal{A}_\alpha\}$ is stationary for all $A \subseteq \omega_1$. Then, $\diamondsuit^-$ is the statement that there exists a $\diamondsuit^-$ sequence.*

Theorem III.7.8 $\diamondsuit^+ \to \diamondsuit^* \to \diamondsuit^- \leftrightarrow \diamondsuit$.

Proof. All the implications are clear except for $\diamondsuit^- \to \diamondsuit$. To prove this, we fix a $\diamondsuit^-$ sequence, $\langle \mathcal{A}_\alpha : \alpha < \omega_1\rangle$, and produce a $\diamondsuit$ sequence.

When A is any set of ordinals and $n \in \omega$, let $\psi_n(A) = \{\xi : \omega \cdot \xi + n \in A\}$. We think of A as coding an ω–sequence of sets, $\langle \psi_n(A) : n \in \omega\rangle$. Note that $A \subseteq \alpha \to \psi_n(A) \subseteq \alpha$.

We may assume that all $\mathcal{A}_\alpha \neq \emptyset$, and then let $\mathcal{A}_\alpha = \{A^n_\alpha : n \in \omega\}$. Let $B^n_\alpha = \psi_n(A^n_\alpha)$; then $B^n_\alpha \subseteq \alpha$. We shall prove that there is some n such that $\langle B^n_\alpha : \alpha < \omega_1\rangle$ is a $\diamondsuit$ sequence.

If this fails, then for each n, fix $B^n \subseteq \omega_1$ such that $\{\alpha : B^n \cap \alpha = B^n_\alpha\}$ is non-stationary. Then, let $C \subseteq \omega_1$ be club such that $B^n \cap \alpha \neq B^n_\alpha$ for all $\alpha \in C$ and all $n \in \omega$. Shrinking C, we may assume that $\omega \cdot \alpha = \alpha > 0$ for all $\alpha \in C$ (since $\{\alpha < \omega_1 : \omega \cdot \alpha = \alpha > 0\}$ is club). Then, for $\alpha \in C$, $\xi < \alpha \to \omega \cdot \xi + n < \alpha$.

Define $B = \{\omega \cdot \xi + n : n \in \omega \wedge \xi \in B^n\}$. Then $B \subseteq \omega_1$ and $\psi_n(B) = B^n$ and $\psi_n(B \cap \alpha) = B^n \cap \alpha$, for $n \in \omega$ and $\alpha \in C$. Then, for these n, α, $\psi_n(B \cap \alpha) \neq B^n_\alpha = \psi_n(A^n_\alpha)$, so $B \cap \alpha \neq A^n_\alpha$. But then $B \cap \alpha \notin \mathcal{A}_\alpha$ for all $\alpha \in C$, contradicting the definition of $\diamondsuit^-$ sequence. □

The coding arguments used in this proof are quite common. They can be modified to allow $\diamondsuit$ sequences or $\diamondsuit^+$ sequences to capture other objects of size $\aleph_1$, such as subsets of $\omega_1 \times \omega_1$ or functions from ω_1 into ω_1. For example,

Exercise III.7.9 *Assume $\diamondsuit$, and prove the following:*

1. *There exists sets $A_\alpha \subseteq \alpha \times \alpha$ for $\alpha < \omega_1$ such for every $A \subseteq \omega_1 \times \omega_1$, the set $\{\alpha : A \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary.*
2. *There exists functions $g_\alpha : \alpha \to \alpha$ for $\alpha < \omega_1$ such for every $g : \omega_1 \to \omega_1$, the set $\{\alpha : g{\restriction}\alpha = g_\alpha\}$ is stationary.*

Hint. For (1), fix any bijection $f : \omega_1 \xrightarrow[\text{onto}]{1-1} \omega_1 \times \omega_1$, and note that $\{\alpha < \omega_1 : f(\alpha) = f(\alpha \times \alpha)\}$ is a club.

For (2), use that fact that any $g : \omega_1 \to \omega_1$ is really a subset of $\omega_1 \times \omega_1$, and $\{\alpha : g{\restriction}\alpha = g \cap (\alpha \times \alpha)\}$ is a club. □

Exercise III.7.10 *Assume $\diamondsuit$ and prove that there are two Suslin trees S, T such that $S \circledast T$ (as in Exercise III.5.47) is Suslin.*

Hint. Use $\diamondsuit$ to capture maximal antichains in $S \circledast T$. Note that $S \neq T$ by Exercise III.5.48. □

Exercise III.7.11 *Modifying Definition III.1.15: For $A, B \in [\omega_1]^{\aleph_1}$, $A \perp B$ iff $|A \cap B| \leq \aleph_0$ (A, B are* almost disjoint*). An* almost disjoint family *is a family $\mathcal{D} \subset [\omega_1]^{\aleph_1}$ such that $A \perp B$ whenever $A, B \in \mathcal{D}$ and $A \neq B$. Prove that there is an almost disjoint family of size $\aleph_2$, and that CH implies that there is one of size $2^{\aleph_1}$. Conclude from this and $\diamondsuit^+ \to CH$ that the family $\mathcal{F}$ satisfying $(*)$ in Theorem III.7.6 has size $2^{\aleph_1}$.*

Hint. Use a diagonal argument to show that $|\mathcal{D}| \geq \aleph_2$ if $\mathcal{D}$ is an uncountable maximal almost disjoint family. To get $|\mathcal{D}| = 2^{\aleph_1}$ under CH, replace $2^{<\omega}$ by the tree $T = 2^{<\omega_1}$ in the proof of Lemma III.1.16. □

This exercise is best possible, in the sense that it is consistent with *ZFC* to have $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$ and no almost disjoint family of size $\aleph_3$; see Exercise IV.7.51.

We now turn to the proof that $\diamondsuit^+$ holds in Gödel's class L of constructible sets. This is a modification of the proof of *GCH* in L (Theorem II.6.23), again using the Downward Löwenheim–Skolem–Tarski Theorem (I.15.10). First, a simple lemma about countable elementary submodels, which we shall use here, and later in Section III.8.

Lemma III.7.12 *Let θ be an uncountable cardinal, and let M be countable with $M \prec H(\theta)$. Then*

1. *If $a \in M$ and a is countable, then $a \subseteq M$.*
2. *$M \cap \omega_1$ is a countable limit ordinal.*
3. *If $\theta = \omega_1$, then M is transitive.*
4. *If $\theta > \omega_1$: $\omega_1 \in M$ and $\omega_1 \not\subseteq M$; if $\beta = M \cap \omega_1$ and* mos *is the Mostowski isomorphism from M onto a transitive T, then $\mathrm{mos}(\omega_1) = \beta$ and $\mathrm{mos}(\xi) = \xi$ for all $\xi < \beta$. Also, $T \models ZFC-P$ (if θ is regular), and $\beta = (\omega_1)^T$.*

Proof. For (1), note that ω and all smaller ordinals are definable in $H(\theta)$, so $\omega \in M$ and $\omega \subseteq M$. Assume that $a \neq \emptyset$ (otherwise the result is trivial); so there is a function $f : \omega \xrightarrow{\text{onto}} a$. *All* such f are in $H(\theta)$, so $M \prec H(\theta)$ implies that *some* such f is in M. Now every element of a is of the form $f(n)$ for some n, and $f, n \in M \to f(n) \in M$ (by $M \prec H(\theta)$), so $a \subseteq M$.

(3) is immediate from (1). For (2), note that (1) implies that $\beta := M \cap \omega_1$ is a countable initial segment of ω_1, and hence is an ordinal (applying Lemma I.7.13); then β is a limit ordinal because $\xi \in \beta \to \xi + 1 \in \beta$ by $M \prec H(\theta)$.

For (4): $\omega_1 \not\subseteq M$ is clear, and $\omega_1 \in M$ because ω_1 is definable in $H(\theta)$ (and in V) as the first uncountable ordinal. Using $M \cap (\omega_1 \cup \{\omega_1\}) = \beta \cup \{\omega_1\}$, we see that $\mathrm{mos}(\xi) = \xi$ for all $\xi < \beta$ and $\mathrm{mos}(\omega_1) = \beta$. The rest follows from the fact that $T \cong M \prec H(\theta) \models ZFC-P$. □

We now prove that $V = L \to \diamondsuit$. Formally, this proof is redundant because we shall then prove that $V = L \to \diamondsuit^+$, which is stronger. We give the $\diamondsuit$ proof because it introduces some of the ideas of the $\diamondsuit^+$ proof in a much simpler setting, and because it easily generalizes to establish $\diamondsuit$ at all regular uncountable cardinals, whereas $\diamondsuit^+$ can fail at some of these cardinals (see Exercises III.7.16, III.7.18, and III.7.19).

Theorem III.7.13 *Assume $V = L$. Then $\diamondsuit$ is true.*

Proof. Recall the relation $<_L$ (Definition II.6.19). This well-orders L (Theorem II.6.20), and is easily seen, by the methods of Lemma II.6.13, to be absolute for all transitive $T \models ZF-P + V = L$. Also recall (Lemma II.6.16) that any

such T is of the form $L(\delta)$. Also, $H(\theta) = L(\theta)$ for all cardinals $\theta \geq \omega$ (Theorem II.6.23).

We define our $\diamondsuit$ sequence $\vec{\mathcal{A}} = \langle A_\alpha : \alpha < \omega_1 \rangle$, together with sets C_α for $\alpha < \omega_1$, recursively as follows: For limit ordinals α, define

$$P(\alpha, A, C) \iff A, C \subseteq \alpha \wedge C \text{ is club in } \alpha \wedge \neg\exists \xi \in C\,[A \cap \xi = A_\xi] \ .$$

Then let (A_α, C_α) be the $<_L$ first pair such that $P(\alpha, A, C)$. If there is no such pair, or if α is not a limit, let $A_\alpha = C_\alpha = \emptyset$. Observe that A_α, C_α are really defined for all ordinals α, and the definition is absolute for transitive models of $ZF-P+V=L$, although $\vec{\mathcal{A}}$ uses only the first ω_1 of the A_α.

To prove that $\vec{\mathcal{A}}$ really is a $\diamondsuit$ sequence, assume that it is not. Applying Definition III.6.4 (of "stationary"), $\exists A, C\,[P(\omega_1, A, C)]$, so that $P(\omega_1, A_{\omega_1}, C_{\omega_1})$.

Fix a countable $M \prec H(\omega_2) = L(\omega_2)$. Let $\beta = M \cap \omega_1$, which is some countable limit ordinal. Let mos be the Mostowski isomorphism from M onto some transitive model T. Then $\mathrm{mos}(\omega_1) = \beta$, and $\mathrm{mos}(\xi) = \xi$, for all $\xi < \beta$, and $\beta = (\omega_1)^T$. Also, T is a countable transitive model for $ZF-P+V=L$, so $T = L(\gamma)$ for some countable γ.

Applying $M \prec H(\omega_2)$ and absoluteness: For each ordinal $\alpha \in M$, $A_\alpha, C_\alpha \in M$ and $\mathrm{mos}(A_\alpha) = A_{\mathrm{mos}(\alpha)}$ and $\mathrm{mos}(C_\alpha) = C_{\mathrm{mos}(\alpha)}$. In particular, $A_{\omega_1}, C_{\omega_1} \in M$ and $\mathrm{mos}(A_{\omega_1}) = A_\beta$ and $\mathrm{mos}(C_{\omega_1}) = C_\beta$. But also, applying Definition I.9.31 (of mos), $\mathrm{mos}(A_{\omega_1}) = \{\mathrm{mos}(\xi) : \xi \in A_{\omega_1} \cap M\} = \{\xi : \xi \in A_{\omega_1} \cap \beta\} = A_{\omega_1} \cap \beta$. So, $A_{\omega_1} \cap \beta = A_\beta$. Likewise, $C_{\omega_1} \cap \beta = C_\beta = \mathrm{mos}(C_{\omega_1})$.

Similarly, C_{ω_1} is club in ω_1 (by $P(\omega_1, A_{\omega_1}, C_{\omega_1})$), so $\mathrm{mos}(C_{\omega_1})$ is club in $\mathrm{mos}(\omega_1)$, so $C_{\omega_1} \cap \beta = C_\beta$ is club in β. But then $\beta \in C_{\omega_1}$ because C_{ω_1} is closed. Then $\beta \in C_{\omega_1} \wedge A_{\omega_1} \cap \beta = A_\beta$ contradicts $P(\omega_1, A_{\omega_1}, C_{\omega_1})$. □

Theorem III.7.14 *Assume $V = L$. Then $\diamondsuit^+$, and hence also $\diamondsuit$, is true.*

Proof. We first define the $\diamondsuit^+$ sequence, $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$: For each $\alpha < \omega_1$, let α^* be the least ordinal in $\{\delta : \alpha < \delta < \omega_1 \wedge L(\delta) \prec L(\omega_1)\}$. To see that there is some such δ, let $M \prec L(\omega_1) = H(\omega_1)$ be countable, with $\alpha \cup \{\alpha\} \subseteq M$. Then $M \models ZF-P+V=L$, and M is transitive by Lemma III.7.12, so $M = L(\delta)$ for some δ with $\alpha < \delta < \omega_1$.

Define $\mathcal{A}_\alpha = L(\alpha^*) \cap \mathcal{P}(\alpha)$. Clearly, $\mathcal{A}_\alpha$ is a countable subset of $\mathcal{P}(\alpha)$. To prove that we have a $\diamondsuit^+$ sequence, fix $A \subseteq \omega_1$.

$A \in L(\omega_2)$ by $V = L$. For $\sigma < \omega_1$, let $M_{A,\sigma} = \mathfrak{H}(L(\omega_2), \{A\} \cup \sigma)$. As in Exercise I.15.28, $M_{A,\sigma}$ is the set of elements of $L(\omega_2)$ that are definable in $(L(\omega_2), \in)$ using parameters in $\{A\} \cup \sigma$. Then $M_{A,\sigma} \prec L(\omega_2)$ (since $L(\omega_2)$ has a definable well-order), and $M_{A,\sigma}$ is countable and $A \in M_{A,\sigma}$ and $\sigma \subseteq M_{A,\sigma}$. Also, by Lemma III.7.12, $M_{A,\sigma} \cap \omega_1$ is a countable limit ordinal. Clearly, $M_{A,\sigma} \cap \omega_1 \geq \sigma$. Let $C = C_A = \{\sigma < \omega_1 : M_{A,\sigma} \cap \omega_1 = \sigma\}$. We leave it as an exercise to check that C is club in ω_1.

Now, fix $\alpha \in C$. We need to show that $A \cap \alpha \in \mathcal{A}_\alpha$ and $C \cap \alpha \in \mathcal{A}_\alpha$. Let mos be the Mostowski isomorphism from $M_{A,\alpha}$ onto some transitive $T = L(\gamma)$.

Then $\mathrm{mos}(\omega_1) = \alpha$, and $\mathrm{mos}(\xi) = \xi$ for all $\xi < \alpha$, and $\alpha = (\omega_1)^T$. Note that $\alpha < \gamma < \alpha^*$, since $L(\alpha^*)$ contains some $f : \omega \xrightarrow{\text{onto}} \alpha$ (using $L(\alpha^*) \not\preccurlyeq H(\omega_1)$), while $L(\gamma)$ contains no such function (using $\alpha = (\omega_1)^T$). It follows that $A \cap \alpha = \mathrm{mos}(A) \in L(\gamma) \subset L(\alpha^*)$, so $A \cap \alpha \in \mathcal{A}_\alpha$.

Finally, we show that $C \cap \alpha \in L(\alpha^*)$ and hence $C \cap \alpha \in \mathcal{A}_\alpha$. Working in the model $L(\alpha^*) \models ZF-P$, we can do standard model theory, all of which is absolute. In particular, we have the sets $L(\gamma), \alpha, A \cap \alpha$, so we can define $\widehat{C} = \{\sigma < \alpha : \mathfrak{H}(L(\gamma), \{A \cap \alpha\} \cup \sigma) \cap \alpha = \sigma\}$. Then $\widehat{C} \in L(\alpha^*)$, so $\widehat{C} \in \mathcal{A}_\alpha$, so we need to show that $\widehat{C} = C \cap \alpha$. So, fix $\sigma < \alpha$. We need to show that $\mathfrak{H}(L(\gamma), \{A \cap \alpha\} \cup \sigma) \cap \alpha = \sigma$ iff $\mathfrak{H}(L(\omega_2), \{A\} \cup \sigma) \cap \omega_1 = \sigma$.

We prove the $\rightarrow$ direction; the $\leftarrow$ direction is similar. Suppose that $\zeta < \omega_1$ and $\zeta \in \mathfrak{H}(L(\omega_2), \{A\} \cup \sigma)$ and $\zeta \geq \sigma$. Then ζ is definable in the model $L(\omega_2)$ using parameters in $\{A\} \cup \sigma$. Since $\sigma < \alpha \in C$, we know that $\zeta < \alpha$. Replacing $L(\omega_2)$ by its elementary submodel $\mathfrak{H}(L(\omega_2), \{A\} \cup \alpha) = M_{A,\alpha}$, this ζ is definable in $M_{A,\alpha}$ using parameters in $\{A\} \cup \sigma$. Since mos fixes ζ and all the parameters in σ, and $\mathrm{mos}(A) = A \cap \alpha$, this ζ is also definable in the model $L(\gamma)$ using parameters in $\{A \cap \alpha\} \cup \sigma$, so $\mathfrak{H}(L(\gamma), \{A \cap \alpha\} \cup \sigma) \cap \alpha \neq \sigma$. □

$\diamondsuit$ and related notions can be generalized as follows:

Definition III.7.15 *Let κ be any uncountable regular cardinal and let S be a stationary subset of κ.*

A $\diamondsuit_S(\kappa)$ sequence *is a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that each $A_\alpha \subseteq \alpha$ and for every $A \subseteq \kappa$, the set $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary.*

Let $\langle \mathcal{A}_\alpha : \alpha \in S\rangle$ be a sequence of sets, where each $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{A}_\alpha| \leq |\alpha|$. This sequence is, respectively, a $\diamondsuit^*_S(\kappa)$ sequence, *a $\diamondsuit^+_S(\kappa)$* sequence, *a $\diamondsuit^-_S(\kappa)$* sequence, *iff, respectively:*

$$\forall A \subseteq \kappa\, \exists C \subseteq \kappa\, [C \text{ is club } \wedge\, \forall \alpha \in C \cap S\, [A \cap \alpha \in \mathcal{A}_\alpha]]\ ,$$
$$\forall A \subseteq \kappa\, \exists C \subseteq \kappa\, [C \text{ is club } \wedge\, \forall \alpha \in C \cap S\, [A \cap \alpha \in \mathcal{A}_\alpha \wedge C \cap \alpha \in \mathcal{A}_\alpha]]\ ,$$
$$\forall A \subseteq \kappa\, [\{\alpha \in S : A \cap \alpha \in \mathcal{A}_\alpha\} \text{ is stationary}]\ .$$

Then $\diamondsuit_S(\kappa)$, $\diamondsuit^*_S(\kappa)$, $\diamondsuit^+_S(\kappa)$, and $\diamondsuit^-_S(\kappa)$ are, respectively, the statements there exists a $\diamondsuit_S(\kappa)$ sequence, there exists a $\diamondsuit^*_S(\kappa)$ sequence, there exists a $\diamondsuit^+_S(\kappa)$ sequence, and there exists a $\diamondsuit^-_S(\kappa)$ sequence. The principles $\diamondsuit(\kappa)$, $\diamondsuit^*(\kappa)$, $\diamondsuit^+(\kappa)$, $\diamondsuit^-(\kappa)$ are, respectively, the principles $\diamondsuit_\kappa(\kappa)$, $\diamondsuit^*_\kappa(\kappa)$, $\diamondsuit^+_\kappa(\kappa)$, $\diamondsuit^-_\kappa(\kappa)$.

Observe that $\diamondsuit_S(\kappa) \rightarrow \diamondsuit(\kappa)$ and $\diamondsuit^-_S(\kappa) \rightarrow \diamondsuit^-(\kappa)$, while $\diamondsuit^*(\kappa) \rightarrow \diamondsuit^*_S(\kappa)$ and $\diamondsuit^+(\kappa) \rightarrow \diamondsuit^+_S(\kappa)$. If $\kappa = \lambda^+$, we would get equivalent notions for $\diamondsuit^*_S(\kappa)$, $\diamondsuit^+_S(\kappa)$, and $\diamondsuit^-_S(\kappa)$ if we said "$|\mathcal{A}_\alpha| \leq \lambda$" instead of "$|\mathcal{A}_\alpha| \leq |\alpha|$" but the "$|\mathcal{A}_\alpha| \leq |\alpha|$"is needed when κ is weakly inaccessible.

Exercise III.7.16 *Prove that $\diamondsuit^+(\kappa) \rightarrow \diamondsuit^*(\kappa) \rightarrow \diamondsuit^-(\kappa) \leftrightarrow \diamondsuit(\kappa)$, and that $\diamondsuit(\kappa) \rightarrow 2^{<\kappa} = \kappa$.*

Exercise III.7.17 *Assume that $\kappa = \lambda^+$, where λ is regular and $2^{<\lambda} = \lambda$. Let S be the set of limit ordinals below κ of cofinality λ, and assume that $\diamondsuit_S(\kappa)$ holds. Prove that there is a κ-Suslin tree*

Hint. Make T $<\lambda$-closed; that is, when $\gamma < \kappa$ is a limit and $\mathrm{cf}(\gamma) < \lambda$, put *all* paths through T_γ into $\mathcal{L}_\gamma(T)$. □

Exercise III.7.18 *Assume $V = L$, and let κ be an uncountable regular cardinal. Prove that $\diamondsuit_S(\kappa)$ holds for all stationary $S \subseteq \kappa$ and that $\diamondsuit^+(\kappa)$ holds if κ is a successor cardinal.*

The situation for inaccessible cardinals is trickier. For large enough κ, $\diamondsuit(\kappa)$ and $\neg\diamondsuit^*(\kappa)$ are both provable in *ZFC*:

Exercise III.7.19 *A cardinal κ is called* ineffable *iff κ is regular and uncountable, and for all sequences $\langle A_\alpha : \alpha < \kappa\rangle$ with each $A_\alpha \subseteq \alpha$, there is a stationary $S \subseteq \kappa$ such that $A_\alpha = A_\beta \cap \alpha$ for all $\alpha < \beta$ with $\alpha, \beta \in S$. Prove in ZFC that if κ is ineffable then κ is strongly inaccessible and $\diamondsuit(\kappa)$ and $\neg\diamondsuit^*(\kappa)$.*

Exercise III.7.20 *Let κ be ineffable. Prove that κ is weakly compact; that is, κ is strongly inaccessible and there are no κ-Aronszajn trees; see also Exercise III.6.33. Also, prove that κ is ineffable in L.*

For more on ineffable cardinals, see Devlin [20].

Returning to $\diamondsuit^+$ on ω_1, the following extends Exercise II.6.31.

Exercise III.7.21 *If $V = L[X]$ for some $X \subseteq \omega_1$, then $\diamondsuit^+$ holds.*

As an application of this:

Exercise III.7.22 *If there are no ω_1-Kurepa trees, then the cardinal ω_2 is an inaccessible cardinal in L.*

Hint. If ω_2 is not inaccessible in L, find an $X \subseteq \omega_1$ such that $(\omega_1)^{L[X]} = \omega_1$ and $(\omega_2)^{L[X]} = \omega_2$, and then apply Exercise III.7.21. □

III.8 Elementary Submodels

Since the definition of L is model-theoretic (iterating $\mathcal{D}^+$), it is not surprising that model theory is used to prove consequences of $V = L$, such as *GCH* (Theorem II.6.23) and $\diamondsuit$ and $\diamondsuit^+$ (Theorem III.7.13 and III.7.14). The primary tool for these proofs was the Downward Löwenheim–Skolem–Tarski Theorem I.15.10 ($\searrow$LST), which was applied to produce small elementary submodels of various $H(\theta)$.

But also, there are many $\searrow$LST proofs of purely set-theoretic results just from *ZFC*. One example was Lemma III.3.51, using $\searrow$LST to produce a "small" subposet $\mathbb{Q}$ of a forcing poset $\mathbb{P}$. These results can always be proved without using any model theory (see Exercise III.3.52) by replacing the $\searrow$LST by an explicit closure argument that lists the relevant Skolem functions. But as the results get more complicated, it gets harder to list these Skolem functions, and the method of elementary submodels saves you from having to think about this.

In the examples of this section, we do not have a "nice traditional" first-order structure, such as $(\mathbb{P}, \leq)$, to which to apply $\searrow$LST. Rather, we stick everything of interest into some $H(\theta)$, and apply $\searrow$LST directly to $(H(\theta), \in)$ to get a "small" $M \preccurlyeq H(\theta)$. In these arguments, unlike arguments from $V = L$, we deal directly with M, not with the transitive isomorph of M. Although M is usually not transitive, it is worth noting:

Lemma III.8.1 *If $M \preccurlyeq H(\theta)$ with $\theta \geq \aleph_1$, then $R \in M \to R \subseteq M$ for countable R, and $R \subseteq M \to R \in M$ for finite R.*

Proof. The first statement is from Lemma III.7.12, and the second statement holds because R is definable from a finite set of elements of M (namely, the elements of R). □

To illustrate the method, we reprove the Delta System Lemma III.2.6. To focus on the model theory, we just prove the simplest non-trivial version:

Lemma III.8.2 (Delta System Lemma) *Let $\mathcal{A}$ be an uncountable family of finite sets. Then there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ that forms a delta system.*

Proof. For any finite set R, and any $\mathcal{D} \subseteq \mathcal{A}$, let $\Delta(R, \mathcal{D})$ abbreviate the statement "$R \subseteq X$ for all $X \in \mathcal{D}$, and $X \cap Y = R$ for all $X, Y \in \mathcal{D}$ with $X \neq Y$". Then $\Delta(R, \emptyset)$ is true trivially. Let $\max\Delta(R, \mathcal{B})$ say that $\Delta(R, \mathcal{B})$ and $\mathcal{B}$ is a maximal member of $\{\mathcal{D} \subseteq \mathcal{A} : \Delta(R, \mathcal{D})\}$; such maximal sets exist by Zorn's Lemma.

Now, let θ be an uncountable regular cardinal such that $\mathcal{A} \in H(\theta)$, and fix a countable $M \preccurlyeq H(\theta)$ with $\mathcal{A} \in M$. Since $[M]^{<\omega}$ is countable, we may fix $X_0 \in \mathcal{A}$ with $X_0 \notin M$. Let $R = X_0 \cap M$; then $R \in M$ (Lemma III.8.1). If every $\mathcal{B}$ satisfying $\max\Delta(R, \mathcal{B})$ is uncountable, we are done, so assume that $\exists \mathcal{B}\, [\max\Delta(R, \mathcal{B}) \wedge |\mathcal{B}| \leq \aleph_0]$. Since this statement is expressible in $H(\theta)$ and $M \preccurlyeq H(\theta)$, we can fix a $\mathcal{B} \in M$ such that $\mathcal{B}$ is countable and $\max\Delta(R, \mathcal{B})$. Then, $\mathcal{B} \subseteq M$, and then $\bigcup_{Y \in \mathcal{B}} Y \subseteq M$ (Lemma III.7.12). Since $X_0 \cap M = R$ and $X_0 \notin M$, $\Delta(R, \mathcal{B} \cup \{X_0\})$ holds and contradicts maximality of $\mathcal{B}$. □

This proof actually produced an R such that *every* $\mathcal{B}$ satisfying $\max\Delta(R, \mathcal{B})$ is uncountable, a fact that is not apparent from our earlier proofs in Lemmas III.2.6 and III.6.15.

As is typical in such elementary submodel proofs, we never said which cardinal θ to use; it just has to be large enough to contain everything relevant

to the proof. In Lemma III.8.2, the elements of $\mathcal{A}$ can be arbitrary finite sets, of arbitrarily large rank, so θ might have to be very large. We could have said, WLOG $|\mathcal{A}| = \aleph_1$, and then WLOG $\mathcal{A} \subseteq [\omega_1]^{<\omega}$, in which case θ could be $\aleph_2$, but a precise value for θ is never important in these arguments. These proofs implicitly use the fact that $H(\theta) \models ZFC-P$ (by Theorem II.2.1, since θ is regular and uncountable), so that standard mathematical arguments can be done within $H(\theta)$. The failure of the Power Set Axiom in $H(\theta)$ is never a problem, since we can choose θ to be large enough to contain whatever power sets we're interested in.

To prove a delta system lemma for collections of countable or larger sets (as in Lemma III.6.15), one needs M with some additional closure properties. For example,

Definition III.8.3 *M is* countably closed *iff* $[M]^\omega \subseteq M$.

If $[M]^\omega \subseteq M$, then M satisfies other types of countable closure as well. For example, $M^\omega \subseteq M$, since if $f : \omega \to M$, then f is a countable subset of $\omega \times M \subseteq M$, so $f \in [M]^\omega \subseteq M$. Clearly, such an M has size at least $\mathfrak{c} = 2^{\aleph_0}$, and we obtain M of size exactly $\mathfrak{c}$ by an easy modification of the $\searrow$LST proof:

Lemma III.8.4 *Fix an uncountable cardinal θ and fix $S \subseteq H(\theta)$ with $|S| \leq \mathfrak{c}$. Then there is a countably closed $M \preccurlyeq H(\theta)$ such that $|M| = \mathfrak{c}$ and $S \subseteq M$.*

Proof. Modify the proof of Theorem I.15.10. Again, we have a countable family of Skolem functions $\mathcal{F}$; each function is n–ary for some $n < \omega$. We form M by closing S under $\mathcal{F} \cup \{g\}$, where g is the ω–ary function given by $\langle y_n : n \in \omega \rangle \mapsto \{y_n : n \in \omega\}$. Then $M \preccurlyeq H(\theta)$, and $[M]^\omega \subseteq M$, and $|M| \leq \mathfrak{c}$ by Lemma I.13.23; then, as noted above $|M| = \mathfrak{c}$. □

For delta systems, this gives us:

Exercise III.8.5 *Modify the proof of Lemma III.8.2 above, now using a countably closed elementary submodel, to prove: If $\mathcal{A}$ is a family of $\mathfrak{c}^+$ countable sets, then there is a $\mathcal{B} \subseteq \mathcal{A}$ of size $\mathfrak{c}^+$ that forms a delta system. Also, produce a countable R such that every $\mathcal{B}$ satisfying* $\max\Delta(R, \mathcal{B})$ *has size $\mathfrak{c}^+$.*

As another example of this method:

Theorem III.8.6 *Let X be a first countable compact Hausdorff space. Then $|X| \leq \mathfrak{c}$.*

Proof. Let $\mathcal{T}$ be the topology on X (i.e., the family of all open subsets). Let θ be a suitably large regular cardinal with $X, \mathcal{T} \in H(\theta)$, and fix a countably closed M with $X, \mathcal{T} \in M \preccurlyeq H(\theta)$ and $|M| = \mathfrak{c}$. It is now sufficient to prove that $X \subseteq M$.

$X \cap M$ is closed in X. To see this, let $a \in X$ be a limit point of $X \cap M$; then, since X is first countable, fix $f : \omega \to (X \cap M)$ so that f (viewed as an ω–sequence) converges to a. So, $f \in M$ because M is countably closed. Then "f has a limit point" is a statement about $f, X, \mathcal{T}$ that is true in $H(\theta)$, so it is true in M, so the (unique) limit point of f, which is a, must be in M.

For $y \in X \cap M$, let $\mathcal{U}_y$ be a countable local base at y. Choose this $\mathcal{U}_y \in M$; this is possible because $y \in M \prec H(\theta)$. Then $\mathcal{U}_y \subseteq M$ by Lemma III.7.12.

If $X \not\subseteq M$, fix $b \in X \backslash M$. For each $y \in X \cap M$, choose $V_y \in \mathcal{U}_y$ such that $b \notin V_y$. By compactness, fix $n \in \omega$ and $y_i \in X \cap M$ for $i \leq n$ such that $X \cap M \subseteq \bigcup_{i \leq n} V_{y_i}$. Then $\exists x \in X\, [x \notin V_{y_0} \wedge \cdots \wedge x \notin V_{y_n}]$ is a statement about elements of M and is true in $H(\theta)$ but false in M, contradicting $M \prec H(\theta)$. □

This result is the famous theorem of Arhangel'skiĭ, proved in 1969; it settled a question of Aleksandrov that had been open since 1922. Some specific examples, such as compact metric spaces and compact Suslin lines (Exercise III.2.19), were already well-known to have size no more than $\mathfrak{c}$.

The original proof was rather complicated, but a much simpler proof, using a closure argument, was found by Pol in 1974. Pol's proof resembles the above proof, but without the use of elementary submodels. The advantage of using elementary submodels is that a person familiar with this approach would write down the first paragraph of the proof of Theorem III.8.6 without even thinking. It is then not hard to come up with the details to complete the proof.

For more examples of the use of elementary submodels to prove theorems in general topology, see Dow [23].

Given a proof with elementary submodels, one can construct another proof that avoids the $\searrow$LST. So, if you discover a new theorem using this method, you have two choices: You can publish a short proof using elementary submodels, or you can publish a longer proof that does not use any model theory. Which choice you take depends on the intended audience. In fact, with the exception of Theorem III.8.16, all the results mentioned in this section are fairly old, and the original proof did not use elementary submodels. But the recent literature on set theory and general topology (such as [59], proving Theorem III.8.16) normally assumes that the reader is familiar with elementary submodels, and presents the shorter proof.

Exercise III.8.7 *Prove Theorem III.8.6 without using any model theory.*

Hint. This is essentially Pol's proof. Get a $Y \subseteq X$ such that $|Y| \leq \mathfrak{c}$ and Y is closed under sequential limits and ....., and then prove that Y is closed in X and then prove that $Y = X$. □

Exercise III.8.8 *Prove Lemma III.8.2, including the fact that there is an R such that* every $\mathcal{B}$ *satisfying* $\max\Delta(R, \mathcal{B})$ *is uncountable, without using any model theory.*

Hint. If there is no such R, construct countable $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ such that for all $R \in [E_n]^{<\omega}$, there is a countable $\mathcal{B} \subseteq \mathcal{A}$ such that $\max\Delta(R, \mathcal{B})$ and $\bigcup \mathcal{B} \subseteq E_{n+1}$. Use $\bigcup_n E_n$ instead of the M in the proof of Lemma III.8.2. □

We next use elementary submodels to prove the Erdös–Rado Theorem. First, some notation, due to Erdös:

Definition III.8.9 *For cardinals $\kappa, \lambda, \sigma, n$, with $n \in \omega$: $\kappa \to (\lambda)^n_\sigma$ abbreviates the assertion that for all $f : [\kappa]^n \to \sigma$, there is a $Z \in [\kappa]^\lambda$ such that f is constant on $[Z]^n$. We say that Z is* homogeneous *for the* partition *f.*

Observe that if $\kappa \to (\lambda)^n_\sigma$ and $\kappa' \geq \kappa$ and $\lambda' \leq \lambda$ then $\kappa' \to (\lambda')^n_\sigma$. We usually assume that $n \geq 2$, since the case $n = 0$ is trivial, and the case $n = 1$ reduces to questions about cofinalities; for example, $\kappa \to (\kappa)^1_\omega$ holds iff either $\mathrm{cf}(\kappa) \geq \omega_1$ or $\kappa \in \{0, 1\}$. The classical Ramsey Theorem says that $\omega \to (\omega)^n_k$ for all $n, k < \omega$.



In the following, we focus on the case when κ, λ are infinite, although the subject for finite κ, λ is a non-trivial area of finite combinatorics. It is an easy exercise to prove that $6 \to (3)^2_2$, whereas the pentagram shows that $5 \not\to (3)^2_2$.

Since $\omega \to (\omega)^2_2$ by Ramsey's Theorem, it is natural to ask: for which κ does $\kappa \to (\omega_1)^2_2$? By an old example of Sierpiński:.

Lemma III.8.10 $\mathfrak{c} \not\to (\omega_1)^2_2$.

Proof. We replace $\mathfrak{c}$ by $\mathbb{R}$, which has size $\mathfrak{c}$. Let $<$ be the usual order on $\mathbb{R}$, and let $\lhd$ well-order $\mathbb{R}$. Define $f(\{x, y\})$ to be 0 if $<$ and $\lhd$ agree on $\{x, y\}$ and 1 if they disagree. If $Z \subseteq \mathbb{R}$ is homogeneous for f, then Z is well-ordered either by $<$ or by $>$, so Z is countable. □

But $\kappa \geq \mathfrak{c}^+$ works:

Lemma III.8.11 $\mathfrak{c}^+ \to (\omega_1)^2_\omega$ *(and hence* $\mathfrak{c}^+ \to (\omega_1)^2_2$*).*

Proof. Fix $f : [I]^2 \to \omega$, where $|I| \geq \mathfrak{c}^+$. We shall produce a $Z \subseteq I$ such that $|Z| = \aleph_1$ and f is constant on $[Z]^2$.

Fix a regular θ with $I, f \in H(\theta)$. Then, fix a countably closed M with $I, f \in M \prec H(\theta)$ and $|M| = \mathfrak{c}$. Since $|I| > \mathfrak{c}$, fix $j \in I \backslash M$.

Then, recursively choose $i_\xi \in I \cap M$ for $\xi < \omega_1$ so that for all $\eta < \xi$: $i_\xi \neq i_\eta$ and $f(\{i_\eta, i_\xi\}) = f(\{i_\eta, j\})$. To prove that there is such an i_ξ: Let $D = \{i_\eta : \eta < \xi\}$ and define $g : D \to \omega$ so that $g(i) = f(\{i, j\})$. Then $g, D \in M$ (since M is countably closed), and the statement $\exists y\,[y \notin D \wedge \forall i \in D\,[f(\{i, y\}) = g(i)]]$ is a statement about the objects $f, g, D \in M$ that is true in $H(\theta)$ (setting $y = j$), so it is true in M, so use such a y for i_ξ.

Now, fix $k \in \omega$ such that $Z := \{\eta < \omega_1 : f(\{i_\eta, j\}) = k\}$ is uncountable. Then each $p \in [Z]^2$ is of the form $\{i_\eta, i_\xi\}$ for some $\eta < \xi < \omega_1$, and $f(p) = f(\{i_\eta, j\} = k)$. □

More generally,

Exercise III.8.12 *Prove that* $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$.

Hint. Now, $[M]^\kappa \subseteq M$ and $|M| = 2^\kappa$. □

Exercise III.8.13 *Let* $\exp_0(\kappa) = \kappa$ *and* $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$. *Prove the general Erdös–Rado Theorem,* $(\exp_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa$ *for any infinite* κ *and any* $n \in \omega$.

Hint. Induct on n. The $n = 1$ case is Exercise III.8.12. The $n = 0$ case, $\kappa^+ \to (\kappa^+)^1_\kappa$ is essentially the pigeonhole principle (which is true because κ^+ is regular), and this was used implicitly in the proof of the $n = 1$ case (choosing k, Z in the proof of Lemma III.8.11). The general idea in the induction is to choose a "large" set $\{i_\xi : \xi < \cdots\cdots\}$ on which f applied to an $n+1$–tuple only depends on the first n elements of the tuple. □

The original proof of Erdös and Rado in 1956 was purely combinatorial. A simpler model-theoretic argument was given by S. Simpson in 1970. Further information on related partition theorems is contained in [31].

We next prove two results using a "club of elementary submodels". The precise meaning of this term varies in the literature (a different meaning is suggested by Exercise III.6.26); here we mean:

Definition III.8.14 *Let* θ *be a regular uncountable cardinal. The sequence* $\langle M_\xi : \xi < \omega_1 \rangle$ *is a* nice chain *of elementary submodels of* $H(\theta)$ *iff:*

1. $M_0 = \emptyset$.
2. $M_\xi \preccurlyeq H(\theta)$ *when* $0 < \xi < \omega_1$.
3. *All* M_ξ *are countable.*
4. *When* $\xi < \eta$, $M_\xi \in M_\eta$ *and* $M_\xi \subseteq M_\eta$.
5. $M_\eta = \bigcup_{\xi<\eta} M_\xi$ *for limit* η

For $x \in \bigcup_\xi M_\xi$, *let* $\mathrm{ht}(x)$, *the* height *of* x, *be the* ξ *such that* $x \in M_{\xi+1} \backslash M_\xi$. *The* associated club *is* $C = \{M_\xi \cap \omega_1 : 0 < \xi < \omega_1\}$.

Lemma III.8.15 *Such a nice chain always exists. Furthermore,* $C \subseteq \omega_1$ *is a club, and* $\xi < \eta$ *implies* $\xi \leq M_\xi \cap \omega_1 < M_\eta \cap \omega_1$.

Proof. Build the M_ξ by recursion. Let $M_{\xi+1}$ be any countable elementary submodel such that $M_\xi \cup \{M_\xi\} \subseteq M_{\xi+1}$. For limit η, take (5) as the definition of M_η. Then $M_\eta \preccurlyeq H(\theta)$ follows by the Tarski–Vaught criterion.

Each $M_\xi \cap \omega_1 \in C$ is a countable ordinal by Lemma III.7.12, and $\xi < \eta$ implies $M_\xi \cap \omega_1 < M_\eta \cap \omega_1$ by (4). Then, it is easy to see that C is a club. □

By letting M_0 be $\emptyset$ rather than an elementary submodel, we ensure that $\mathrm{ht}(x)$ is really defined for all $x \in \bigcup_\xi M_\xi$; otherwise the $x \in M_0$ would have to be handled as a special case.

Our first result using these chains involves the construction of an Aronszajn tree, and uses C for an application of $\diamondsuit$. This is a little like the Suslin tree construction in Theorem III.7.3, but there the club was easy to describe directly without using elementary submodels.

We have seen (Lemma III.5.17) that every special Aronszajn tree has an order preserving map into $\mathbb{Q}$. Under $MA(\aleph_1)$, there is a continuous map (Theorem III.5.19), but this is not a theorem of *ZFC*:

Theorem III.8.16 *Assuming $\diamondsuit$, there is a special Aronszajn tree T that has no continuous order preserving map into $\mathbb{Q}$.*

Proof. We begin verbatim as in the first four paragraphs of the proof of Theorem III.5.12, which produced an Aronszajn tree in *ZFC* together with an order preserving $\varphi : T \to \mathbb{Q}$. We shall use $\diamondsuit$ to defeat all continuous order preserving $\psi : T \to \mathbb{Q}$.

In particular, T is ω_1 as a set, and $T_\gamma = \bigcup_{\alpha<\gamma} \mathcal{L}_\alpha$ is always some ordinal; $T_\gamma = \omega \cdot \gamma$ and $\mathcal{L}_\gamma(T) = \{\omega \cdot \gamma + k : k \in \omega\}$ when $\gamma \geq \omega$. We construct $\sqsubset$ by recursion, making sure that

$$\begin{aligned} &\forall x \in T\, \forall \alpha < \omega_1\, \forall q \in \mathbb{Q}\, [\alpha > \text{height}(x) \wedge q > \varphi(x) \to \\ &\quad \exists y \in \mathcal{L}_\alpha(T)\, [x \sqsubset y \wedge \varphi(y) = q]] \end{aligned} \tag{$*$}$$

holds. As before, each node will have exactly $\aleph_0$ immediate successors.

Let $\langle \psi_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$ sequence, where each $\psi_\alpha : \alpha \to \mathbb{Q}$, and for every $\psi : \omega_1 \to \omega_1$, the set $\{\alpha : \psi{\restriction}\alpha = \psi_\alpha\}$ is stationary. To prove that such a sequence exists under $\diamondsuit$, use the method of Exercise III.7.9.

In the recursive construction, build each $\mathcal{L}_\gamma(T)$ to preserve $(*)$, as in the proof of Theorem III.5.12. But in addition, whenever $\omega \cdot \gamma = \gamma > 0$ (so $T_\gamma = \gamma$ as a set and $\psi_\gamma : T_\gamma \to \mathbb{Q}$): *if* ψ_γ is order preserving, then *if* it is possible, extend $\sqsubset$ such that the node $\gamma \in \mathcal{L}_\gamma(T)$ satisfies:

$$\sup\{\psi_\gamma(x) : x \sqsubset \gamma\} \notin \mathbb{Q} \quad \wedge \quad \sup\{\varphi(x) : x \sqsubset \gamma\} \leq 1 \ . \tag{$\dagger$}$$

Note that both these sups lie in $\mathbb{R} \cup \{\infty\}$; but they might be ∞, or they might be finite and rational. Use the nodes $\gamma + 1, \gamma + 2, \ldots$ to preserve $(*)$, putting node $\gamma + k$ above path P_k. If $(\dagger)$ is possible (by an appropriate choice of P_0), we may let $\varphi(\gamma) = 1$ (or any other value in $\mathbb{Q} \cap [1, \infty)$). If $(\dagger)$ is impossible, then ignore it and just preserve $(*)$.

Lemma III.8.17 (Main Lemma) *Suppose $\psi : T \to \mathbb{Q}$ is order preserving. Then there is a club $C \subseteq \omega_1$ so that for all limit points γ of C: $\omega \cdot \gamma = \gamma$, and,* if $\psi_\gamma = \psi{\restriction}\gamma$ then $(\dagger)$ *is possible at level γ.*

The theorem follows immediately, since choosing such a γ for which $\psi_\gamma = \psi{\restriction}\gamma$, we see that ψ cannot be continuous at node $\gamma \in \mathcal{L}_\gamma(T)$. So, we proceed to prove the Main Lemma. Let $\langle M_\xi : \xi < \omega_1 \rangle$ be nice chain of elementary submodels of $H(\aleph_2)$ with $\varphi, \psi, \sqsubset \in M_1$. Let C be the associated club.

In the following, let $S \subseteq_s T$ mean that S is a subtree of T (Definition III.5.3). Given $S \subseteq_s T$, let $S^p = \{x \in S : |x{\uparrow} \cap S| = \aleph_1\}$; that is, we prune S as in Lemma III.5.26. Note that $S^p \subseteq_s S$ and, if $|S| = \aleph_1$ then $|S^p| = \aleph_1$ and $(S^p)^p = S^p$. Also, by $(*)$, $T^p = T$.

When $S \subseteq_s T$, $\Gamma : S \to \mathbb{Q}$ is order preserving, $q \in \mathbb{Q}$, and $s \in S$, define:

$$W(S, \Gamma, x, q) = \{x\} \cup x{\downarrow} \cup \{y \in x{\uparrow} : \Gamma(y) < q\} \quad .$$

Then $W(S, \Gamma, x, q) \subseteq_s S$. If $\Gamma(x) \geq q$, then $W(S, \Gamma, x, q) = \{x\} \cup x{\downarrow}$. If $\Gamma(x) < q$, then $W(S, \Gamma, x, q)$ could still be countable, or even just $\{x\} \cup x{\downarrow}$; this will depend on Γ. But, by $(*)$, if $\varphi(x) < q$ then $W(T, \varphi, x, q) = (W(T, \varphi, x, q))^p$ is uncountable.

Now, fix a limit point γ of C, with $\psi_\gamma = \psi{\restriction}\gamma$. Let $\alpha_n \nearrow \gamma$, with $\alpha_0 = 0$ and $\alpha_n \in C$ for all $n > 0$. List $\mathbb{Q}$ as $\{q_n : n \in \omega\}$. We shall construct nodes $0 = x_0 \sqsubset x_1 \sqsubset x_2 \cdots$ with $x_n \in \mathcal{L}_{\alpha_n}(T)$. We shall also get subtrees of T: $T \supseteq_s T_0 \supseteq_s T_1 \supseteq_s T_2 \cdots$; with each $x_n \in T_n$. Node γ will be above the path $P_0 = \bigcup_{n \in \omega}(x_n{\downarrow})$. We shall also have $|T_n| = \aleph_1$ and $(T_n)^p = T_n$ for all n.

Let $T_0 = W(T, \varphi, 0, 1)$. Then $\sup\{\varphi(x) : x \sqsubset \gamma\} \leq 1$ will be immediate. The other part of $(\dagger)$ will follow by obtaining, for each n, *either*

A. $\psi(x_{n+1}) > q_n$ *or*
B. For some $\varepsilon_n > 0$: $\forall y \in T_{n+1}\, [\psi(y) < q_n - \varepsilon_n]$.

Then (B) must happen for some n because $\psi(\gamma)$ is finite. Note that having (A) or (B) for each n guarantees that $\sup\{\psi_\gamma(x) : x \sqsubset \gamma\}$ is irrational, so that ψ cannot be continuous.

In the construction, assume that we have x_n, T_n, and we obtain x_{n+1}, T_{n+1} as follows, where we let ξ_n be such that $\alpha_n = M_{\xi_n} \cap \omega_1$.

If $\exists y \in T_n\, [x_n \sqsubset y \wedge \psi(y) \geq q_n]$: Choose such a y with height$(y) < \alpha_{n+1}$ (using $M_{\xi_{n+1}} \preccurlyeq H(\aleph_2)$). Then choose $x_{n+1} \in T_n$ with $y \sqsubset x_{n+1}$ and height$(x_{n+1}) = \alpha_{n+1}$; such an x_{n+1} exists by $(T_n)^p = T_n$. Then $\psi(x_{n+1}) > q_n$. Let $T_{n+1} = T_n$.

If $\forall y \in T_n\, [x_n \sqsubset y \to \psi(y) < q_n]$: Since $|x_n{\uparrow} \cap T_n| = \aleph_1$, fix $\varepsilon_n > 0$ so that $|x_n{\uparrow} \cap \{y \in T_n : y < q_n - \varepsilon_n\}| = \aleph_1$. Let $T_{n+1} = (W(T_n, \psi, x_n, q_n - \varepsilon_n))^p$, which is uncountable. Then, choose $x_{n+1} \in \mathcal{L}_{\alpha_{n+1}}(T_{n+1})$. □

Exercise III.8.18 *Show how to make the tree in Theorem III.8.16 Hausdorff.*

Hint. See Exercise III.5.24. □

Our second result using clubs of elementary submodels illustrates their use, under *CH*, as a technique for defining ccc posets. Further examples are in Section V.6. Here, we shall prove:

Theorem III.8.19 *CH implies that there are ccc posets $\mathbb{P}, \mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not ccc.*

It follows then by Exercise III.4.16 that there are ccc compact Hausdorff spaces X, Y such that $X \times Y$ is not ccc.

Recall that $MA(\aleph_1)$ implies that ccc *is* productive (Theorem III.3.43). On the other hand, $\neg SH$ (i.e., a Suslin line X exists) also implies the existence of such $\mathbb{P}, \mathbb{Q}$ ($\mathbb{P} = \mathbb{Q} = \mathbb{O}_X$; see Lemmas III.2.18 and III.3.37); also, a Suslin tree itself provides an example (see Exercise III.5.48).

It takes a completely different argument to produce such $\mathbb{P}, \mathbb{Q}$ from *CH* (which is consistent with *SH*; see [21, 81]). This was first done by Laver in the 1970s (see Galvin [36]). There are now a number of such examples, mainly variations on the same idea, expressed by the following definition and lemma, which grew out of a consideration of partition theorems such as Lemma III.8.11.

Definition III.8.20 *Suppose that* $f : [I]^2 \to 2$. *Define forcing posets* $\mathbb{P}_\mu = \mathbb{P}^f_\mu$ *for* $\mu = 0, 1$ *as follows:* $\mathbb{P}_\mu = \{p \in [I]^{<\omega} : \forall i, j \in p\, [i \neq j \to f(\{i,j\}) = \mu]\}$. *Define* $p \leq q$ *iff* $p \supseteq q$, *so that* $\mathbb{1} = \emptyset$.

Note that $\mathbb{P}_\mu$ trivially contains $\emptyset$ together with all singleton $\{i\} \subseteq I$. A doubleton $\{i,j\}$ is in $\mathbb{P}_\mu$ iff $f(\{i,j\}) = \mu$).

One might try to use these $\mathbb{P}_\mu$ under *MA* to produce a large homogeneous $Z \subseteq I$, except that there is no reason to think that $\mathbb{P}_\mu$ is ccc when I is uncountable. For example, if f has constant value 0 on some $[Z]^2$, then $\{\{i\} : i \in Z\}$ is an antichain of size $|Z|$ in $\mathbb{P}_1$. Somewhat less trivially,

Exercise III.8.21 *Fix* $I \in [\mathbb{R}]^{\aleph_1}$, *and let* f *be the Sierpiński partition from Lemma III.8.10 restricted to* $[I]^2$. *Prove that neither* $\mathbb{P}_0$ *nor* $\mathbb{P}_1$ *has the ccc, but neither has an uncountable antichain of the form* $\{\{i\} : i \in Z\}$.

Hint. Get antichains of the form $\{\{i_\alpha, j_\alpha\} : \alpha < \omega_1\}$. □

In any case

Lemma III.8.22 *If* $f : [I]^2 \to 2$ *and* I *is uncountable then* $\mathbb{P}_0 \times \mathbb{P}_1$ *is not ccc.*

Proof. $\{(\{i\}, \{i\}) : i \in I\}$ is an antichain of size $|I|$ in $\mathbb{P}_0 \times \mathbb{P}_1$, since $f(\{i,j\}) = \mu$ implies that $\{i\} \perp \{j\}$ in $\mathbb{P}_{1-\mu}$. □

So, we just need to find an f such that each $\mathbb{P}_\mu$ is ccc. We already have a non-trivial example under $\neg SH$, by applying Lemmas III.5.18 and III.8.25:

Exercise III.8.23 *Let* I *be a Suslin tree, and let* $f(\{i,j\}) = 0$ *iff* $i \perp j$. *Show that* $\mathbb{P}_0$ *and* $\mathbb{P}_1$ *are both ccc.*

Under *CH*, there are various examples. The one we describe here is roughly from [36]. Another is in Section V.6 (see page 374). As a preliminary lemma:

Lemma III.8.24 *Assume CH, and let* $\langle M_\xi : \xi < \omega_1 \rangle$ *be a nice chain of elementary submodels of* $H(\theta)$, *where* $\theta > \aleph_1$ *is regular. Then* $H(\aleph_1) \subseteq \bigcup_\xi M_\xi$.

Proof. $|H(\aleph_1)| = \mathfrak{c} = \aleph_1$ (Lemma I.13.28), so $H(\aleph_1) \in H(\theta)$ and $H(\theta)$ contains all maps from ω_1 onto $H(\aleph_1)$. By $M_1 \prec H(\theta)$, some such map, Φ, is a member of M_1. Then $\Phi(\zeta) \in M_\xi$ whenever $\zeta \in M_\xi \cap \omega_1$, and this is true whenever $\zeta < \xi$ because $\xi \leq M_\xi \cap \omega_1$ by Lemma III.8.15. □

As an aid to verifying the ccc:

Lemma III.8.25 *With the notation of Definition III.8.20, if $\mathbb{P}_\mu$ is not ccc, then it has an uncountable antichain whose members are pairwise disjoint.*

Proof. Let A be an uncountable antichain. By the Delta System Lemma, there is an uncountable $A' \subseteq A$ such that A' forms a delta system with some root r. Now, if $p, q \in A'$ with $p \neq q$, then $p \perp q$ implies that $p \cup q \notin \mathbb{P}_\mu$, so that $f(\{i,j\}) \neq \mu$ for some $i, j \in p \cup q$ with $i \neq j$. But then $i, j \notin r$ (since $p, q \in \mathbb{P}_\mu$), so that $(p \backslash r) \perp (q \backslash r)$. Then $\{p \backslash r : p \in A'\}$ is also an antichain. □

Proof of Theorem III.8.19. Let $I = \omega_1$. It is sufficient to build a function $f : [I]^2 \to 2$ so that $\mathbb{P}_\mu = \mathbb{P}_\mu^f$ is ccc for both $\mu = 0$ and $\mu = 1$.

By *CH*, there are only $\aleph_1$ countably infinite disjoint sub-families of $[I]^{<\omega}$, so we can list them all in an ω_1–sequence and make sure, one-by-one, that each such family can't be contained in an uncountable antichain. Exactly how we make sure of this is a little complicated, and the elementary submodels simplify the combinatorics somewhat.

Let θ be a suitably large regular cardinal, and let $\langle M_\xi : \xi < \omega_1 \rangle$ be a nice chain of elementary submodels of $H(\theta)$. For $\zeta < \xi < \omega_1$, let $f(\{\zeta, \xi\}) = g_\xi(\zeta)$, where each g_ξ satisfies:

a. Whenever $E \in M_\xi$ and E is an infinite disjoint family of finite subsets of ξ: For each $\mu \in \{0,1\}$, there are infinitely many $q \in E$ such that $\forall \zeta \in q\, [g_\xi(\zeta) = \mu]$.
b. $g_\xi \in M_{\xi+1}$.

Observe that (a) is vacuous when ξ is finite, and then g_ξ can be chosen arbitrarily. To see that (a) is possible for infinite ξ: Since M_ξ is countable, list all the relevant E as $\{E_m : m < \omega\}$; make sure that each $\{m : E_m = E\}$ contains infinitely many even numbers and infinitely many odd numbers. Then, recursively choose $q_m \in E_m$ such that $q_m \cap q_\ell = \emptyset$ whenever $\ell < m$. Then, for $\zeta \in q_m$, let $g_\xi(\zeta)$ be 0 if m is even and 1 if m is odd. For $\zeta \in \xi \setminus \bigcup_m q_m$, $g_\xi(\zeta)$ is arbitrary. To obtain (b), use $M_{\xi+1} \prec H(\theta)$ plus the fact, by Definition III.8.14 and Lemma III.8.15, that $M_{\xi+1}$ contains ξ and M_ξ.

Now, to prove that $\mathbb{P}_\mu$ is ccc, we apply Lemma III.8.25. Assume that A is an uncountable antichain whose members are pairwise disjoint, and derive a contradiction. Let E be any countably infinite subset of A. Applying Lemma III.8.24, fix $\beta < \omega_1$ such that $E \in M_\beta$. Note that then $\sup \bigcup E < M_\beta \cap \omega_1$. Fix $p \in A$ such that $\min(p) > \beta$. List p in increasing order as $\{\xi_0, \ldots, \xi_{n-1}\}$. Let $E_0 = E$, and for $\ell < n$, let $E_{\ell+1} = \{q \in E_\ell : \forall \zeta \in q\, [f(\{\zeta, \xi_\ell\}) = \mu]\}$.

We check, inductively, that each E_ℓ is infinite (using (a)), and that $E_\ell \in M_{\xi_\ell}$ (which implies, by (b), that $E_{\ell+1} \in M_{\xi_\ell+1}$). In particular, $E_n \neq \emptyset$. If $q \in E_n$, then $f(\{\zeta, \xi_\ell\}) = \mu$ for all $\ell < n$ and all $\zeta \in q$, so that $q \not\perp p$, contradicting the assumption that A is an antichain. □

These partition examples are often discussed in terms of *graphs*. The set $[I]^2$ is the complete undirected graph whose set of nodes is I; each $i, j \in I$ with $i \neq j$ is connected by an edge, which we have labeled $\{i, j\}$. Then $f : [I]^2 \to 2$ colors all the edges in two colors, red (0) and blue (1) (or solid line and dashed line in the pentagram on page 237). Then $\mathbb{P}_0^f$ can be used to build a subset Z of I all of whose edges are colored red. This may be viewed in terms of forcing (that is, Z exists in an extension of the universe; see Sections II.7 and IV.1), or as an *MA* application:

Exercise III.8.26 *Assume $MA(\aleph_1)$, and assume that $|I| = \aleph_1$. Fix $f : [I]^2 \to 2$, fix $\mu \in \{0, 1\}$, and assume that $\mathbb{P}_\mu^f$ is ccc. Prove that there is an uncountable $Z \subseteq I$ such that $f(\{i, j\}) = \mu$ for all $\{i, j\} \in [Z]^2$.*

Hint. Apply the fact that $\mathbb{P}_\mu^f$ has $\aleph_1$ as a pre-caliber to $\{\{i\} : i \in I\}$. □

A different hint is given by:

Exercise III.8.27 *Assume $MA(\aleph_1)$, and assume that $|I| = \aleph_1$. Fix $f : [I]^2 \to 2$, fix $\mu \in \{0, 1\}$, and assume that $\mathbb{P}_\mu^f$ is ccc. Prove that there are subsets $Z_n \subseteq I$ for $n \in \omega$ such that $I = \bigcup_n Z_n$ and, for each n, $f(\{i, j\}) = \mu$ for all $\{i, j\} \in [Z_n]^2$. In particular, some Z_n has size $\aleph_1$.*

Hint. Let $\mathbb{Q}_\mu \subseteq (\mathbb{P}_\mu)^\omega$ be the finite support product of $\aleph_0$ copies of $\mathbb{P}$ (Definition III.3.40). If $G \subseteq \mathbb{Q}_\mu$ is a filter meeting appropriate dense sets, then Z_n can be $\{i \in I : \exists \vec{p} \in G\, [p_n = \{i\}]\}$. □

Note that to apply *MA* to $\mathbb{Q}_\mu$, we need first to use *MA* to conclude that the product of ccc posets is ccc (Theorem III.3.43). As we have just seen, there are counter-examples to this under *CH*, but

Exercise III.8.28 *Assume CH and define $\mathbb{Q}_\mu$ as above, where $I = \omega_1$ and f is as in the proof of Theorem III.8.19. Prove that $\mathbb{Q}_0$ and $\mathbb{Q}_1$ are both ccc.*

Hint. Use the following property of that f: Whenever $E \in [[\omega_1]^{<\omega}]^{\aleph_0}$ is a disjoint family, there is a β with $\sup(E) < \beta < \omega_1$ such that for $\mu = 0, 1$, $\forall p \in [\omega_1 \backslash \beta]^{<\omega}\ \exists q \in E\ \forall \zeta \in q\ \forall \xi \in p\, [f(\{\zeta, \xi\}) = \mu]$. □

Applying $MA_{\mathbb{Q}_0}(\aleph_1)$ and $MA_{\mathbb{Q}_1}(\aleph_1)$, we would get $I = \bigcup_m Z_m = \bigcup_n Y_n$, where all edges between nodes in each Z_m are colored red and all edges between nodes in each Y_n are colored blue. Then each $|Z_m \cap Y_n| \leq 1$, which is contradictory because I is uncountable. Of course, we have *CH*, not $MA(\aleph_1)$, but this "contradiction" is of some interest in forcing; see Proposition V.2.29.

Chapter IV

Forcing

This chapter describes the basic logic behind forcing and provides relative consistency proofs of statements such as $2^{\aleph_0} = \aleph_7$ and $2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_9$. The method of forcing was introduced by Cohen [14, 15], although the treatment here in terms of posets and filters is due to Solovay. Modern expositions of forcing owe a lot to the exposition in Shoenfield's [90].

IV.1 The Forcing Idea

This section continues Section II.7. Say we wish to produce a model N for $ZFC + \neg CH$. *Very naively*, we step outside of our universe and create some ideal universe $N \supsetneq V$ that has sufficiently many subsets of ω so that CH fails in N. As explained in Section II.7, we make formal sense out of this naive idea by using countable set models. We start with a ctm M for ZFC and prove that there is a larger ctm $N \supseteq M$ that satisfies $ZFC + \neg CH$.

To build N, fix some forcing poset $\mathbb{P}$ that happens to be in M. Any such $\mathbb{P}$ will give us a ctm $N \supseteq M$ that satisfies ZFC. Exactly what N will satisfy beyond ZFC will depend on which $\mathbb{P}$ we choose.

A filter $G \subseteq \mathbb{P}$ is called $\mathbb{P}$*–generic over* M iff G meets all dense $D \subseteq \mathbb{P}$ with $D \in M$. Such filters exist by the Generic Filter Existence Lemma III.3.14 because M is countable. If $\mathbb{P}$ is atomless, then G will not be in M. Our model N will be called $M[G]$; this notation indicates that N contains G plus whatever other sets are required to get $N \models ZFC$ and $N \supseteq M$. N and M will have the same ordinals; that is $o(N) = o(M)$.

For a simple example of this, follow the notation in the proof of Lemma III.3.13, and let $\mathbb{P} = \mathrm{Fn}(I, J)$, where $I, J \in M$. Then $\mathbb{P} \in M$ by an easy absoluteness argument. Also, by absoluteness, the dense sets $D_i = \{q \in \mathrm{Fn}(I, J) : i \in \mathrm{dom}(q)\}$ will all be in M, so that G will meet these, which implies that the set $f_G = \bigcup G \in M[G]$ will be a function from I to J.

Now, say $I = \kappa \times \omega$ and $J = 2 = \{0,1\}$, where $\kappa < o(M)$, so κ is some ordinal in M. Then $f_G : \kappa \times \omega \to 2$. But we can think of f_G as coding κ functions, $h_\alpha \in 2^\omega$, for $\alpha < \kappa$, where $h_\alpha(n) = f_G(\alpha, n)$. The dense sets

$$E_{\alpha,\beta} = \{p \in \mathbb{P} : \exists n\,[(\alpha,n),(\beta,n) \in \mathrm{dom}(p) \wedge p(\alpha,n) \neq p(\beta,n)]\} \ ,$$

for $\alpha < \beta < \kappa$, are also in M, so G meets these too, which implies that the h_α are all different functions.

Next, in the spirit of Section II.7, we have ordinals $\omega_1^M < \omega_2^M < o(M)$ that the people living in M think are the first and second uncountable cardinals, although we, living in V, know that these are really countable ordinals. Say $\kappa = \omega_2^M$. Then people living in $N = M[G]$ have this ω_2-sequence $\langle h_\alpha : \alpha < \omega_2 \rangle$ of distinct elements of 2^ω, so they will believe that $2^{\aleph_0} \geq \aleph_2$; so $M[G] \models ZFC + \neg CH$. If also $M \models GCH$, then $M[G]$ will satisfy $2^{\aleph_0} = \aleph_2$.

By varying κ and refining this argument, we can arrange for $2^{\aleph_0}$ in N to be any value not contradicting König's Theorem. Of course, one of the many technical details that need to be verified in the next two sections is that, using this particular poset, $\omega_1^N = \omega_1^M$ and $\omega_2^N = \omega_2^M$, so that our κ-sequence really looks like an ω_2-sequence to the people living in N.

Details for the construction of $N \supseteq M$ will be given in Sections IV.2 and IV.3. This is all completely rigorous, but it does not quite give us a finitistic relative consistency proof of, say, $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \neg CH)$. As we remarked in Section I.1, the Incompleteness Theorem implies that working in ZFC alone, we cannot actually produce any set models M for ZFC (unless ZFC is inconsistent), so how do we even get started? As we noted in Section II.7, it is easy to get a ctm M satisfying $ZFC + V = L$, and hence GCH, if there exists an inaccessible cardinal, but ZFC does not prove that such cardinals exist.

Sections IV.2 and IV.3 will show that $\mathrm{Con}(ZFC + \neg CH)$ is a theorem of ZFC plus the existence of an inaccessible cardinal. This may be sufficient for some platonists, but it does not establish a finitistic relative consistency proof. Two ways of getting such proofs are discussed in Section IV.5. One such way, briefly, is the following. By the Reflection Theorem (Corollary II.5.4), whenever Λ is a finite set of axioms for ZFC, one can, in ZFC, produce a ctm M for Λ. Then, the arguments in Sections IV.2 and IV.3 will show that given any finite $\Omega \subset ZFC$, there is a larger finite $\Lambda \subset ZFC$ such that every ctm for Λ may be extended to a ctm for $\Omega + \neg CH$. So, for each finite $\Omega \subset ZFC$, $ZFC \vdash \exists N\,[N \models \Omega + \neg CH]$. So, to prove $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \neg CH)$: If we can derive an inconsistency from $ZFC + \neg CH$, then it must arise from some finite $\Omega + \neg CH$, where $\Omega \subset ZFC$. Then, in ZFC, we can produce a model N for the inconsistent theory $\Omega + \neg CH$, so ZFC is inconsistent.

We shall return to these metamathematical considerations in Section IV.5, and for now we concentrate on the mathematical complexities of extending models of set theory. In the meantime, the reader may, if desired, take a statement such as "let M be a ctm for ZFC" to be an abbreviation for "let M be a ctm for enough of ZFC to carry out the argument at hand".

IV.2 Generic Extensions

Let M be a ctm. As indicated in Section IV.1, we shall extend M to a larger ctm $N = M[G]$ by adding to M a filter G on a forcing poset $\mathbb{P} \in M$ that meets appropriately many dense sets. For the beginning of this section, it will be enough that $M \models ZF-P$, so that we can use the basic absoluteness results of Section II.4. Eventually (e.g., when we prove that $N \models ZFC$), it will become important that $M \models ZFC$.

Notation IV.2.1 *Let $(\mathbb{P}, \leq, \mathbb{1})$ be a forcing poset. Then "$\mathbb{P} \in M$" abbreviates "$(\mathbb{P}, \leq, \mathbb{1}) \in M$".*

This continues our standard abuse of notation by referring to "$\mathbb{P}$" when we really mean "$(\mathbb{P}, \leq, \mathbb{1})$". Of course, $\mathbb{P} \in M \rightarrow \mathbb{1} \in M$ because M is transitive, but it is always important to assume that M contains the order relation $\leq$, not just the set $\mathbb{P}$. Note that by basic absoluteness results, the notion of $(\mathbb{P}, \leq, \mathbb{1})$ *being* a forcing poset (Definition III.3.1) is absolute for transitive models of $ZF-P$. In many (but not all!) cases, the order $\leq$ will be in M just by absoluteness; for example, if $I, J \in M$, then M contains the set $\mathrm{Fn}(I, J)$ as well as its order relation $\{(p, q) \in \mathrm{Fn}(I, J) \times \mathrm{Fn}(I, J) : p \supseteq q\}$.

Definition IV.2.2 *For a forcing poset $\mathbb{P}$, G is* $\mathbb{P}$–generic *over M iff G is a filter on $\mathbb{P}$ and $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ such that $D \in M$.*

These generic filters exist when M is countable and hence contains only countably many dense sets. Lemma III.3.14 implies immediately:

Lemma IV.2.3 (Generic Filter Existence Lemma) *Let M be a ctm for $ZF-P$ and let $\mathbb{P} \in M$ be a forcing poset. Then for every $p \in \mathbb{P}$, there exists a filter G on $\mathbb{P}$ such that $p \in G$ and G is is $\mathbb{P}$–generic over M.*

Note that the notion of being dense is absolute, but the enumeration of the dense sets in type ω (needed for the proof of Lemma III.3.14) must take place outside of M, so G is usually not in M:

Lemma IV.2.4 *If $\mathbb{P} \in M$ is atomless and the filter G is $\mathbb{P}$–generic over M, then $G \notin M$.*

Proof. As in the proof of Lemma III.3.20, let $D = \mathbb{P} \backslash G$. Then D is dense. If $G \in M$, then $D \in M$, contradicting $G \cap D \neq \emptyset$. □

If $\mathbb{P}$ has an atom r, then one can find generic filters in M, such as the filter $\{q \in \mathbb{P} : q \not\perp r\}$ (see Lemma III.3.20), but posets with atoms are of little interest in forcing or in MA applications.

We begin by defining $M[G]$. Roughly, this will be the set of all sets that can be built from G and elements of M using "simple set-theoretic processes". Each element of $M[G]$ will have a *name* in M that describes how it is built:

Definition IV.2.5 *τ is a $\mathbb{P}$–*name *iff τ is a relation and*

$$\forall \langle \sigma, p \rangle \in \tau \, [\sigma \text{ is a } \mathbb{P}\text{–name} \wedge p \in \mathbb{P}] \ .$$

Then, $V^{\mathbb{P}}$ is the class of all $\mathbb{P}$–names.

This is a definition by recursion. In the terminology of our basic Theorem I.9.11, we are defining the characteristic function $F : V \to \{0,1\}$ of the class of $\mathbb{P}$ names. The recursion specifies that: $F(\tau) = 1$ iff τ is a relation and $\forall \langle \sigma, p \rangle \in \tau \, [F(\sigma) = 1 \wedge p \in \mathbb{P}]$; $F(\tau) = 0$ otherwise. The recursion is on the well-founded set-like relation R, where xRy iff $x \in \mathrm{trcl}(y)$.

Definition IV.2.5 does not mention the order on $\mathbb{P}$. It also does not mention models, but we plan to relativize it to our ctm M. Since R and the concepts used in the recursion are absolute for transitive models of $ZF-P$, F and the notion of $\mathbb{P}$–name are also absolute (see Theorem II.4.15).

Definition IV.2.6 *If M is a transitive model of $ZF-P$ and $\mathbb{P} \in M$, then $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M = \{\tau \in M : (\tau \text{ is a } \mathbb{P}\text{–name})^M\}$.*

So, people living in M can decide which sets are $\mathbb{P}$–names and discuss some of their properties, but they do not have access to the generic filter G, which is needed to compute what the names name and to define $M[G]$:

Definition IV.2.7 *If τ is a $\mathbb{P}$–name and $G \subseteq \mathbb{P}$, then by recursion define:*

$$\mathrm{val}(\tau, G) = \tau_G = \{\mathrm{val}(\sigma, G) : \exists p \in G \, [\langle \sigma, p \rangle \in \tau]\} \ .$$

Then $M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$ whenever M is a transitive model of $ZF-P$ and $\mathbb{P} \in M$.

This definition of name and value is related to the concept that by the Foundation Axiom, we can obtain all sets from nothing by iterating *collection*; that is, taking a bunch of sets and putting them between brackets, $\{\cdots\cdots\}$ (see the discussion of Lemma I.9.30). Some examples: $\emptyset$ is a $\mathbb{P}$–name (since Definition IV.2.5 is vacuously true of $\emptyset$), and $\emptyset_G = \{\} = \emptyset$. If we have three names $\sigma^1, \sigma^2, \sigma^2$ and we want to collect the things they name, then we let $\tau = \{\langle \sigma^1, \mathbb{1} \rangle, \langle \sigma^2, \mathbb{1} \rangle, \langle \sigma^3, \mathbb{1} \rangle\}$. When G is any filter, $\mathbb{1} \in G$, so $\tau_G = \{\sigma^1_G, \sigma^2_G, \sigma^3_G\}$. But if $p^1, p^2, p^3 \in \mathbb{P}$ and $\pi = \{\langle \sigma^1, p^1 \rangle, \langle \sigma^2, p^2 \rangle, \langle \sigma^3, p^3 \rangle\}$, then we may view this as collection "*conditional* on G"; that is, π_G is some subset of $\{\sigma^1_G, \sigma^2_G, \sigma^3_G\}$, and we can't tell which subset this is without knowing which of p^1, p^2, p^3 are in G.

Exercise IV.2.8 *Let $\tau = \{\langle \emptyset, p \rangle, \ \langle \{\langle \emptyset, q \rangle\}, r \rangle\}$. Compute τ_G for each of the 8 possibilities for p, q, r being $\in$ or $\notin G$.*

Just using "*unconditional*" collection, employing only the forcing condition $\mathbb{1}$, we see that $M[G]$ contains all elements of M:

Definition IV.2.9 *For a forcing poset* $(\mathbb{P}, \leq, \mathbb{1})$ *and any set* x, *define* $\check{x} = \{\langle \check{y}, \mathbb{1}\rangle : y \in x\}$. *"*$\check{x}$*" is read in English "*$x$ *check".*

This is another recursive definition, and it is easily seen by induction that $\check{x}$ is always a $\mathbb{P}$–name. Of course, the definition of $\check{x}$ depends on the object $\mathbb{1}$, but the poset $(\mathbb{P}, \leq, \mathbb{1})$ will always be clear from context. As an example, $\check{2} = \{\langle \check{1}, \mathbb{1}\rangle, \langle \check{0}, \mathbb{1}\rangle\} = \{\langle\{\langle\emptyset, \mathbb{1}\rangle\}, \mathbb{1}\rangle, \langle\emptyset, \mathbb{1}\rangle\}$.

Lemma IV.2.10 *Assume that* M *is a transitive model of* $ZF-P$ *with* $\mathbb{P} \in M$ *and* G *a filter on* $\mathbb{P}$. *Then:*

1. $\forall x \in M\, [\check{x} \in M^{\mathbb{P}} \wedge \mathrm{val}(\check{x}, G) = x]$.
2. $M[G] \supseteq M$.

Proof. (2) is immediate from (1). For (1): $\check{x} \in M$ by absoluteness, and $\mathrm{val}(\check{x}, G) = x$ is proved by induction, using $\mathrm{val}(\check{x}, G) = \{\mathrm{val}(\check{y}, G) : y \in x\}$. □

Now, usually $G \notin M$, but people living in M can cook up a name, Γ, which they know must name G:

Definition IV.2.11 *Given a forcing poset* $\mathbb{P}$, *define* $\Gamma = \{\langle \check{p}, p\rangle : p \in \mathbb{P}\}$.

Again, Γ depends on $\mathbb{P}$, which will always be clear from context. Since $\Gamma_G = \{\check{p}_G : p \in G\} = \{p : p \in G\} = G$, we have

Lemma IV.2.12 *Under the hypotheses of Lemma IV.2.10,* Γ *is a* $\mathbb{P}$*–name and* $\Gamma_G = G$. *Hence,* $G \in M[G]$.

The names Γ and the $\check{x}$ have the same flavor, in that they have been cooked up to name a desired object, but there's an important difference. If we think of the function $G \mapsto \tau_G$, then $\check{x}$ yields the *constant* function $G \mapsto x$, and Γ yields the *identity* function $G \mapsto G$. We can also cook up names for unordered and ordered pairs:

Definition IV.2.13 $\mathrm{up}(\sigma, \tau) = \{\langle \sigma, \mathbb{1}\rangle, \langle \tau, \mathbb{1}\rangle\}$ *and* $\mathrm{op}(\sigma, \tau) = \mathrm{up}(\mathrm{up}(\sigma, \sigma), \mathrm{up}(\sigma, \tau))$.

Lemma IV.2.14 *Under the hypotheses of Lemma IV.2.10, if* $\sigma, \tau \in M^{\mathbb{P}}$, *then* $\mathrm{up}(\sigma, \tau), \mathrm{op}(\sigma, \tau) \in M^{\mathbb{P}}$, $\mathrm{val}(\mathrm{up}(\sigma, \tau), G) = \{\sigma_G, \tau_G\}$, *and* $\mathrm{val}(\mathrm{op}(\sigma, \tau), G) = \langle \sigma_G, \tau_G\rangle$.

Lemma IV.2.15 *Under the hypotheses of Lemma IV.2.10,* $M[G]$ *is transitive and is a model for the Axioms of Extensionality, Foundation, Pairing, and Union.*

Proof. $M[G]$ is transitive because every possible element of τ_G (where $\tau \in M^{\mathbb{P}}$) is of the form σ_G (where $\sigma \in M^{\mathbb{P}}$). Then $M[G] \models$ Extensionality because $M[G]$ is transitive, and Foundation is true in every class. Lemma IV.2.14 implies that $M[G] \models$ Pairing (since $a, b \in M[G] \to \{a, b\} \in M[G]$). To prove that $M[G] \models$ Union, note that using our statement of the axiom in Section I.3, it is sufficient to prove that whenever $a \in M[G]$, there is a $b \in M[G]$ such that $\bigcup a \subseteq b$. Fix $\tau \in M^{\mathbb{P}}$ such that $a = \tau_G$, let $\pi = \bigcup \mathrm{dom}(\tau)$, and let $b = \pi_G$.

Note that τ is a set of pairs of the form $\langle \sigma, p \rangle$, and $\mathrm{dom}(\tau)$ is just the set of all these σ's. Then, our Definition IV.2.5 (of "name") implies that the union of any family of names is also a name, so π is a $\mathbb{P}$–name, and $\pi \in M$ by absoluteness of $\bigcup$, so $b \in M[G]$.

If $c \in a = \tau_G$, then $c = \sigma_G$ for some $\sigma \in \mathrm{dom}(\tau)$. Then $\sigma \subseteq \pi$, so, applying Definition IV.2.7, $c = \sigma_G \subseteq \pi_G = b$. Thus, $\bigcup a \subseteq b$. □

Note that we did not prove that $a \in M[G] \to \bigcup a \in M[G]$. This will follow once we show that $M[G] \models$ Comprehension, which is *much* harder and requires G to be generic, whereas so far, we have only needed G to be a filter. As an exercise in constructing names, one can actually prove directly that $\bigcup a \in M[G]$ whenever G is a filter:

Exercise IV.2.16 *Using the notation of Lemma IV.2.15, replace the definition of* π *by:* $\pi = \{\langle \vartheta, p \rangle : \exists \langle \sigma, q \rangle \in \tau \, \exists r \, [\langle \vartheta, r \rangle \in \sigma \wedge p \leq r \wedge p \leq q]\}$. *Let* $b = \pi_G$ *and prove that* $\bigcup a = b$.

For another example, say $\mathbb{P} = \mathrm{Fn}(I, J)$, where $I, J \in M$. Then $\bigcup G$ is a function, and in this case there is a simpler and more natural name for it, namely $\mathring{f} = \{\langle \mathrm{op}(\check{\imath}, \check{\jmath}), p \rangle : p \in \mathbb{P} \wedge \langle i, j \rangle \in p\}$. Since $\mathrm{val}(\mathrm{op}(\check{\imath}, \check{\jmath}), G) = \langle i, j \rangle$, we have $\mathrm{val}(\mathring{f}, G) = \{\langle i, j \rangle : \exists p \in G \, \langle i, j \rangle \in p\} = \bigcup G$.

Our use of $\mathring{f}$ as a name for something that we know is a function, rather than using a Greek letter like π, is fairly common notation:

Notation IV.2.17 *We use Greek letters such as* τ, σ, π *for names in abstract discussions. In concrete discussions, we often use Latin letters with a circle over them. In particular,* $\mathring{G}$ *is often used for* Γ.

The following simple example illustrates where it becomes important that G be generic, and not merely a filter. As always, we assume that $\mathbb{P} \in M$. Then $\mathbb{P} \in M[G]$ (named by $\check{\mathbb{P}}$) and $G \in M[G]$ (named by $\mathring{G}$). Let C be the complement, $\mathbb{P} \backslash G$. If $M[G]$ satisfies any non-trivial set theory at all, then C must be in $M[G]$, but what is its name? A natural choice would be $\mathring{C} = \{\langle \check{p}, q \rangle : p, q \in \mathbb{P} \wedge p \perp q\}$. Then $\mathring{C}_G = \{p \in \mathbb{P} : \exists q \in G \; p \perp q\}$. Since G is a filter, its elements are pairwise compatible, so $G \cap \mathring{C}_G = \emptyset$. But why is $G \cup \mathring{C}_G = \mathbb{P}$? For each $p \in \mathbb{P}$, let $D_p = \{q \in \mathbb{P} : p \perp q \vee q \leq p\}$. It is easy to check that D_p is dense, and $D_p \in M$ by absoluteness. If G is *generic*, then fix $q \in G \cap D_p$. If $p \perp q$, then $p \in \mathring{C}_G$; if $q \leq p$ then $p \in G$ because filters are upward closed.

So, $G \cup \mathring{C}_G = \mathbb{P}$. If we merely assumed that G is a filter, then this $\mathring{C}$ need not work; for example, if $G = \{\mathbb{1}\}$ then $\mathring{C}_G = \emptyset$. In fact, for mere filters G, the complement $\mathbb{P} \backslash G$ need not be in $M[G]$ at all; see Exercise IV.2.47

Although $M[G]$ is bigger than M, it is not too much bigger:

Lemma IV.2.18 *Under the hypotheses of Lemma IV.2.10:*

1. $\text{rank}(\tau_G) \leq \text{rank}(\tau)$ *for all* $\tau \in M^{\mathbb{P}}$.
2. $o(M[G]) = o(M)$.
3. $|M[G]| = |M|$.

Proof. (1) is proved by induction on τ, using

$$\begin{aligned} \text{rank}(\tau_G) = \sup\{\text{rank}(x) + 1 : x \in \tau_G\} \leq \\ \sup\{\text{rank}(\sigma_G) + 1 : \sigma \in \text{dom}(\tau)\} \leq \text{rank}(\tau) \quad . \end{aligned}$$

The "=" used Lemma I.9.24. The second "$\leq$" used $\sigma \in \text{dom}(\tau) \to \text{rank}(\sigma) < \text{rank}(\tau)$, plus $\text{rank}(\sigma_G) \leq \text{rank}(\sigma)$, which we are assuming inductively.

For (2), if $\alpha \in M[G] \cap ON$, then $\alpha = \tau_G$ for some $\tau \in M^{\mathbb{P}}$, and then $\alpha = \text{rank}(\alpha) \leq \text{rank}(\tau) \in M \cap ON$ by absoluteness of rank. Then $M[G] \cap ON = M \cap ON$ follows from $M[G] \supseteq M$.

For (3), $M^{\mathbb{P}} \subseteq M \subseteq M[G]$ implies $|M[G]| \leq |M^{\mathbb{P}}| \leq |M| \leq |M[G]|$; the first "$\leq$" applies Definition IV.2.7 of $M[G]$. □

We plan to have $|M| = \aleph_0$ (it's a ctm), although we don't actually use that until we need the fact that generic filters exist.

Before proving that $M[G]$ is a model for *ZFC* (assuming that M is), we note that it is the least such model:

Lemma IV.2.19 *Under the hypotheses of Lemma IV.2.10, let N be another transitive model of ZF−P with $M \subseteq N$ and $G \in N$. Then $M[G] \subseteq N$.*

Proof. The recursive Definition IV.2.7 of $\text{val}(\tau, G)$ shows that val is absolute, so $\tau, G \in N$ implies $\tau_G \in N$. □

Now, in verifying that $M[G]$ is a model for set theory, the hardest axiom to verify is Comprehension. Once that is done, it will be fairly easy to show that $M[G] \models ZFC$ (assuming that $M \models ZFC$).

To illustrate the problem with Comprehension, say $\varphi(x, y)$ is a formula and $\sigma \in M^{\mathbb{P}}$. Why should the set $S = \{n \in \omega : (\varphi(n, \sigma_G))^{M[G]}\}$ be in $M[G]$?

This is not immediately obvious even in very simple cases, such as when φ is $x \in y$ (so $S = \omega \cap \sigma_G$), or when φ is $x \notin y$ (so $S = \omega \backslash \sigma_G$). Recall that our original justification for set intersection and difference (see Definition I.4.5) was by such a use of Comprehension.

Of course, we must cook up a name $\tau \in M^{\mathbb{P}}$ and prove that $S = \tau_G$. We have already cooked up names for some special purposes, such as $\text{op}(\sigma, \pi)$

(which names $\langle \sigma_G, \pi_G \rangle$) and $\check{3}$ (which names 3). This cooking illustrates our intuitive idea that people living in M can talk about the names and what they name, even though they do not have access to the filter G. But now we must push our intuition further and deal with an arbitrary φ. Our τ will in fact be:

$$\{\langle \check{n}, p \rangle : n \in \omega \wedge p \in \mathbb{P} \wedge p \Vdash \varphi(\check{n}, \sigma)\} \quad . \tag{❀}$$

The symbol "$\Vdash$" is read "forces"; roughly, it says that $\varphi(\check{n}, \sigma)$ is true in $M[G]$ whenever G is generic and $p \in G$. More precisely, we have the next three definitions:

Definition IV.2.20 *For a poset $\mathbb{P}$, the* $\mathbb{P}$ forcing language *$\mathcal{FL}_{\mathbb{P}}$ is the class of logical formulas formed using the binary relation $\in$ and all the names in $V^{\mathbb{P}}$ as constant symbols.*

Since $V^{\mathbb{P}}$ is a proper class, so is $\mathcal{FL}_{\mathbb{P}}$. In discussing $M[G]$, we use only $\mathcal{FL}_{\mathbb{P}} \cap M$, which, by absoluteness of logical syntax (Corollary II.4.17) is the set of logical formulas formed using the binary relation $\in$ and all the names in $M^{\mathbb{P}}$ as constant symbols. For example, the $\varphi(\check{n}, \sigma)$ in (❀) is a sentence of $\mathcal{FL}_{\mathbb{P}} \cap M$ if $\sigma \in M^{\mathbb{P}}$. Then the notion "$M[G] \models \varphi(\check{n}, \sigma)$" has its natural meaning:

Definition IV.2.21 *Let ψ be a sentence of $\mathcal{FL}_{\mathbb{P}} \cap M$. Then $M[G] \models \psi$ has its usual model-theoretic meaning from Definition I.15.5, interpreting $\in$ as membership, and interpreting each name τ as τ_G.*

Definition IV.2.22 *Assume that M is a ctm for $ZF-P$, $\mathbb{P} \in M$ is a forcing poset, and ψ is a* sentence *of $\mathcal{FL}_{\mathbb{P}} \cap M$. Then $p \Vdash_{\mathbb{P},M} \psi$ holds iff: $M[G] \models \psi$ for all filters G on $\mathbb{P}$ such that $p \in G$ and G is $\mathbb{P}$-generic over M. We omit the subscripts $\mathbb{P}, M$ on the $\Vdash$ when they are clear from context. "$p \Vdash \psi$" is read "p* forces *ψ".*

Here, we are assuming that M is countable, so that we know that generic filters always exists. Note that we only define $p \Vdash \psi$ for sentences ψ, not arbitrarily formulas. Two simple examples, using, as above, names $\dot{G}$ and $\dot{C}$ for G and its complement: If $p \leq q$ then $p \Vdash \check{q} \in \dot{G}$, since for all filters G, if $p \in G$ then $q \in G$. If $E \subseteq \mathbb{P}$ and E is not centered (Definition III.3.23), then $\mathbb{1} \Vdash \exists x\, [x \in \check{C} \wedge x \in \check{E}]$, since whenever G is generic, $\dot{C}_G$ is the complement of G, and this complement must meet E (i.e., $E \not\subseteq G$), or we would contradict Lemma III.3.25. Note that $\mathbb{1} \Vdash \psi$ iff $M[G] \models \psi$ for *all* filters G that are $\mathbb{P}$-generic over M. For example, if ψ is the Union Axiom, then $\mathbb{1} \Vdash \psi$ by Lemma IV.2.15. Also, $q \in G \wedge q \leq p \to p \in G$ when G is a filter, so Definition IV.2.22 yields the following simple fact about forcing:

Lemma IV.2.23 *If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$.*

Now (❀) defines τ, but two major problems remain. First, why is $\tau \in M^{\mathbb{P}}$? To define $\Vdash$, we needed to step outside of M and consider not only one generic filter, but *all* of them. Second, why does $\tau_G = S$? The first problem is addressed by the *Definability Lemma*, which says roughly that forcing is actually definable in M; that is, people living in M can reason about possible generic G and decide whether $p \Vdash \psi$ without ever seeing a generic G, as we just did in the above two examples. The second problem is addressed by the *Truth Lemma*, which says that anything true in $M[G]$ is actually forced by something. We state the Truth Lemma first, since it is simpler:

Lemma IV.2.24 (The Truth Lemma) *Let M be a ctm for $ZF-P$, let $\mathbb{P} \in M$ be a forcing poset, let ψ be a* sentence *of $\mathcal{FL}_{\mathbb{P}} \cap M$, and let G be $\mathbb{P}$-generic over M. Then $M[G] \models \psi$ iff there is a $p \in G$ such that $p \Vdash \psi$.*

Observe that the $\leftarrow$ of this "iff" is obvious from the definition of "$p \Vdash \psi$". But, the $\rightarrow$ direction is non-trivial. A simple example: Say $r_1, r_2 \in G$, and let ψ be $\check{r}_1 \in \mathring{G} \wedge \check{r}_2 \in \mathring{G}$. Then $M[G] \models \psi$. Now choose $p \in G$ with $p \leq r_1$ and $p \leq r_2$. Then $p \Vdash \psi$ because if H is *any* generic filter and $p \in H$, then $r_1, r_2 \in H$. Note that r_1 and r_2 need not force ψ because there may be generic H containing just one of r_1, r_2.

Now, the naive statement of the *Definability Lemma* would be something like $\{(p, \psi) : p \Vdash \psi\} \in \mathcal{D}^+(M)$ (see Definition I.15.25). We are saying $\mathcal{D}^+$ not $\mathcal{D}^-$ because $\mathbb{P}$ might be used as a parameter in the definition, but even with the $\mathcal{D}^+$, this is false. Consider the trivial $\mathbb{P} = \{\mathbb{1}\}$, so every name is of the form $\check{a}$, and $\mathbb{1} \Vdash \psi$ iff ψ is true in M (interpreting each $\check{a}$ as a). If this notion were in $\mathcal{D}^+(M)$, then we would contradict Tarski's theorem on undefinability of truth (see [57], Theorem IV.5.29). A correct statement is:

Lemma IV.2.25 (The Definability Lemma) *Let M be a ctm for $ZF-P$. Let $\varphi(x_1, \ldots, x_n)$ be a formula in $\mathcal{L} = \{\in\}$, with all free variables shown. Then*

$$\big\{(p, \mathbb{P}, \leq, \mathbb{1}, \vartheta_1, \ldots, \vartheta_n) : (\mathbb{P}, \leq, \mathbb{1}) \text{ is a forcing poset } \wedge\ p \in \mathbb{P} \wedge (\mathbb{P}, \leq, \mathbb{1}) \in M \wedge \vartheta_1, \ldots, \vartheta_n \in M^{\mathbb{P}} \wedge p \Vdash_{\mathbb{P}, M} \varphi(\vartheta_1, \ldots, \vartheta_n)\big\}$$

is in $\mathcal{D}^-(M)$.

Using this lemma, we can use the definability of forcing $p \Vdash \varphi(\vartheta_1, \ldots, \vartheta_n)$ with a *fixed* φ, plus the Comprehension Axiom in M, to cook up names that must lie in M. For example, the Definability Lemma implies that the name τ defined above by (❀) is in M. We now check that it works; that is, τ_G really is $S := \{n \in \omega : (\varphi(n, \sigma_G))^{M[G]}\}$. The definition of τ_G implies that $\tau_G = \{n \in \omega : \exists p \in G\ p \Vdash \varphi(\check{n}, \sigma)\}$, and then the definition of $\Vdash$ implies that $\tau_G \subseteq S$. To prove that $S \subseteq \tau_G$, fix $n \in S$. Then $M[G] \models \varphi(\check{n}, \sigma)$, so by the Truth Lemma applied to the sentence $\varphi(\check{n}, \sigma)$ of $\mathcal{FL}_{\mathbb{P}} \cap M$, we can fix $p \in G$ such that $p \Vdash \varphi(\check{n}, \sigma)$. Then $\langle \check{n}, p \rangle \in \tau$ so $n \in \tau_G$.

We have two tasks remaining for this section. First, prove the Truth and Definability Lemmas, and second, use them to prove that $M[G] \models ZFC$, assuming that $M \models ZFC$. These may be done in either order. We shall take up the second task first because it is somewhat simpler, and it illustrates how these two lemmas are used in practice.

Lemma IV.2.26 *Let M be a ctm for $ZF-P$, let $\mathbb{P} \in M$ be a forcing poset, and let G be $\mathbb{P}$-generic over M. Then $M[G]$ is a model for all axioms of $ZF-P$ except possibly Replacement.*

Proof. We begin with the Comprehension Axiom, which generalizes our discussion above of (❀). We start as we did in the proof of Comprehension in L (Theorem II.6.11), although the argument is different. Let $N = M[G]$.

Fix a formula φ of $\mathcal{L} = \{\in\}$ without y free. φ may have x, z free, along with possibly other free variables $v^0, \ldots, v^{n-1}$, so write it as $\varphi(x, z, v^0, \ldots, v^{n-1})$. Then we must verify

$$\forall z, v^0, \ldots, v^{n-1} \in N \, \exists y \in N \, \forall x \in N \, [x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi^N(x, z, \vec{v})]$$

Since $N = M[G]$, fix $\pi_G, \sigma_G^0, \ldots, \sigma_G^{n-1} \in N$ (corresponding to $z, v^0, \ldots, v^{n-1}$), where $\pi, \sigma^0, \ldots, \sigma^{n-1} \in M^{\mathbb{P}}$. Let $S = \{x \in \pi_G : \varphi^N(x, \pi_G, \sigma_G^0, \ldots, \sigma_G^{n-1})\}$. As with the L proof, it is sufficient (and necessary) to prove that $S \in N$, so we must cook up a $\tau \in M^{\mathbb{P}}$ such that $\tau_G = S$. To shorten our notation, let $\tilde{\varphi}(x)$ denote the *formula* $\varphi(x, \pi, \sigma^0, \ldots, \sigma^{n-1})$ of $\mathcal{FL}_{\mathbb{P}} \cap M$. Now, π is a set of pairs of the form $\langle \vartheta, r \rangle$ and every element of π_G is among these ϑ_G, so $S = \{\vartheta_G : \vartheta \in \mathrm{dom}(\pi) \wedge M[G] \models (\vartheta \in \pi \wedge \tilde{\varphi}(\vartheta))\}$. Define

$$\tau = \{\langle \vartheta, p \rangle : \ \vartheta \in \mathrm{dom}(\pi) \ \wedge \ p \in \mathbb{P} \ \wedge \ p \Vdash (\vartheta \in \pi \wedge \tilde{\varphi}(\vartheta))\} \ .$$

Then $\tau_G = \{\vartheta_G : \vartheta \in \mathrm{dom}(\pi) \wedge \exists p \in G \; p \Vdash (\vartheta \in \pi \wedge \tilde{\varphi}(\vartheta))\}$, and $\tau \in M^{\mathbb{P}}$ by the Definability Lemma. The definition of $\Vdash$ implies that $\tau_G \subseteq S$. To prove that $S \subseteq \tau_G$, fix $\vartheta_G \in S$, where $\vartheta \in \mathrm{dom}(\tau)$. Then $M[G] \models (\vartheta \in \pi \wedge \tilde{\varphi}(\vartheta))$, so by the Truth Lemma, fix $p \in G$ such that $p \Vdash (\vartheta \in \pi \wedge \tilde{\varphi}(\vartheta))$. Then $\langle \vartheta, p \rangle \in \tau$, so $\vartheta_G \in \tau_G$.

In view of Lemma IV.2.15, the only remaining Axiom is Infinity, and that holds because $\omega \in N$ (see Lemma II.2.11). □

Theorem IV.2.27 *Let M be a ctm for ZF, let $\mathbb{P} \in M$ be a forcing poset, and let G be $\mathbb{P}$-generic over M. Then $M[G] \models ZF$. Furthermore, $M[G] \models ZFC$ if $M \models ZFC$.*

Proof. The remaining axioms are Power Set, Replacement, and Choice.

For Power Set (similarly to Union above), it is sufficient to prove that whenever $a \in M[G]$, there is a $b \in M[G]$ such that $\mathcal{P}(a) \cap M[G] \subseteq b$. Fix $\tau \in M^{\mathbb{P}}$ such that $\tau_G = a$. Let $Q = (\mathcal{P}(\mathrm{dom}(\tau) \times \mathbb{P}))^M$. This is the set of all

names $\vartheta \in M^{\mathcal{P}}$ such that $\mathrm{dom}(\vartheta) \subseteq \mathrm{dom}(\tau)$. Let $\pi = Q \times \{\mathbb{1}\}$ and let $b = \pi_G = \{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle \sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle \sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

For Replacement, we assume that $a \in M[G]$ and $(\forall x \in a\, \exists y\, \tilde{\varphi}(x,y))^{M[G]}$, and produce a $b \in M[G]$ such that $(\forall x \in a\, \exists y \in b\, \tilde{\varphi}(x,y))^{M[G]}$. Here, we are shortening our notation, as in the proof of Comprehension above; so $\tilde{\varphi}(x,y)$ is a formula in $\mathcal{FL}_{\mathbb{P}} \cap M$, which may mention some names as fixed parameters. Say $a = \tau_G \in M[G]$. So, $M[G] \models \forall x \in \tau\, \exists y\, \tilde{\varphi}(x,y)$. Working in M (using the Definability Lemma), let Q be a set of names such that for all $p \in \mathbb{P}$ and all $\sigma \in \mathrm{dom}(\tau)$, if there is some name ϑ such that $p \Vdash \varphi(\sigma, \vartheta)$, then at least one such ϑ is in Q. We obtain such a Q using the Reflection Theorem in M; so Q can be $M^{\mathbb{P}} \cap (R(\alpha))^M$ for a suitably large $\alpha < o(M)$. Let $\pi = Q \times \{\mathbb{1}\}$ and let $b = \pi_G = \{\vartheta_G : \vartheta \in Q\}$.

To prove $(\forall x \in a\, \exists y \in b\, \tilde{\varphi}(x,y))^{M[G]}$, fix $x \in a$. So, $x = \sigma_G$ for some $\sigma \in \mathrm{dom}(\tau)$. Then $M[G] \models \exists y\, \tilde{\varphi}(\sigma, y)$, so there is a $\vartheta \in M^{\mathbb{P}}$ such that $M[G] \models \tilde{\varphi}(\sigma, \vartheta)$, and then, by the Truth Lemma, there is a $p \in G$ such that $p \Vdash \varphi(\sigma, \vartheta)$. Then there is a $\vartheta \in Q$ such that $p \Vdash \varphi(\sigma, \vartheta)$. If $y = \vartheta_G$, then $y \in b$ and $(\tilde{\varphi}(x,y))^{M[G]}$.

To prove $M[G] \models AC$, fix $a = \tau_G \in M[G]$; we prove that a has a well-order in $M[G]$. Working in M, use AC and list $\mathrm{dom}(\tau)$ as $\{\sigma^\xi : \xi < \alpha\}$, and let $\mathring{f}$ be the name $\{\langle \mathrm{op}(\check{\xi}, \sigma^\xi), \mathbb{1}\rangle : \xi < \alpha\}$. In $M[G]$, we have $f := \mathring{f}_G$. By Lemma IV.2.14, $f = \{\langle \xi, \sigma^\xi_G\rangle : \xi < \alpha\}$, so f is a function with $\mathrm{dom}(f) = \alpha$ and $a \subseteq \mathrm{ran}(f)$. Then, working in $M[G]$, we can well-order a by saying $x \lhd y$ iff $\min\{\xi < \alpha : f(\xi) = x\} < \min\{\xi < \alpha : f(\xi) = y\}$. □

Some remarks on this proof. For Power Set, the key to the proof is that there is a *set* $Q \in M$ that contains enough names to represent all possible subsets of τ_G in $M[G]$. We shall find a smaller set Q in Section IV.3, which will be useful for computing cardinal exponentiation in $M[G]$. Note that there need not be a set $Q \in M$ that contains *all possible* names for subsets of τ_G; for example if $p \in G$ and $q \perp p$ then $\varkappa_G = \emptyset$ whenever $\varkappa = \{\langle \check{x}, q\rangle\}$ (for $x \in M$), and the collection of all these $\varkappa$ is a proper class of M.

For Replacement, it looks like we have proved a stronger form of the axiom that weakens the usual hypothesis $\forall x \in a\, \exists! y \cdots\cdots$ in Section I.3 to just $\forall x \in a\, \exists y \cdots\cdots$. But in fact, the stronger form is provable in ZF by the Reflection Theorem; see also [57], Exercise I.14.17.

We now list a few additional forcing facts that will be useful in later arguments. First, a remark:

Definition IV.2.21 of "$M[G] \models \psi$" is standard in the literature, but it is actually a bit misleading. Note that this is really is a *ternary* relation between M, G, ψ, not a binary relation between the set $M[G]$ and ψ:

Exercise IV.2.28 *Let M be a ctm for ZFC. Find a poset $\mathbb{P}$ and a sentence $\psi \in \mathcal{FL}_{\mathbb{P}} \cap M$ and two different generic filters G, H with $M[G] = M[H]$ and $M[G] \models \psi$ and $M[H] \models \neg\psi$ because some τ_G differs from τ_H.*

Hint. $\mathbb{P}$ can be finite, so $M[G] = M[H] = M$, and τ can be $\mathring{G}$. □

Lemma IV.2.29 *Assume that the filter G is $\mathbb{P}$-generic over M. Assume that $D \subseteq \mathbb{P}$, $D \in M$, and D is dense below $p \in \mathbb{P}$. If $p \in G$ then $G \cap D \neq \emptyset$.*

Proof. Note that the case $p = \mathbb{1}$ just restates the definition of "generic". For the general case, define $D^+ = D \cup \{q \in \mathbb{P} : q \perp p\}$. It is easy to see that D^+ is dense and $D^+ \in M$, so that $G \cap D^+ \neq \emptyset$. But then $G \cap D \neq \emptyset$ because $p \in G$ and G is a filter. □

Lemmas IV.2.30, IV.2.32, and IV.2.33 show that one can always analyze the forcing of a compound sentence in terms of the parts of that sentence.

Lemma IV.2.30 *For any forcing poset $\mathbb{P} \in M$ and sentences $\varphi, \psi \in \mathcal{FL}_{\mathbb{P}} \cap M$:*

1. *No p can force both φ and $\neg\varphi$.*
2. *If φ, ψ are logically equivalent, then $p \Vdash \varphi$ iff $p \Vdash \psi$.*
3. *If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$.*
4. *$p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$.*
5. *$p \Vdash \neg\varphi$ iff $\neg\exists q \leq p\, [q \Vdash \varphi]$ and $p \Vdash \varphi$ iff $\neg\exists q \leq p\, [q \Vdash \neg\varphi]$.*
6. *$p \Vdash \varphi \to \psi$ iff $\neg\exists q \leq p\, [q \Vdash \varphi \wedge q \Vdash \neg\psi]$.*
7. *$p \Vdash \varphi \vee \psi$ iff $\{q \leq p : [q \Vdash \varphi] \vee [q \Vdash \psi]\}$ is dense below p.*
8. *$p \Vdash \varphi \leftrightarrow \psi$ iff $\neg\exists q \leq p\, [q \Vdash \varphi \wedge q \Vdash \neg\psi]$ and $\neg\exists q \leq p\, [q \Vdash \psi \wedge q \Vdash \neg\varphi]$.*

Proof. (1–4) are clear from the definition (IV.2.22) of forcing; (1) also uses the fact that there exists a generic G with $p \in G$. (3) restates Lemma IV.2.23.

For the first $\leftrightarrow$ of (5): $\to$ follows from (3) and (1). For $\leftarrow$, assume that $p \not\Vdash \neg\varphi$, and fix a generic G with $p \in G$ and $M[G] \models \varphi$. Then, by the Truth Lemma, fix an $r \in G$ such that $r \Vdash \varphi$. Since G is a filter, fix $q \in G$ such that $q \leq r$ and $q \leq p$. Then $q \leq p$ and, by (3), $q \Vdash \varphi$. For the second $\leftrightarrow$, apply the first to $\neg\neg\varphi$ and use (2).

Now, (6)(7)(8) follow from (4)(5) by using (2) and expressing $\varphi \to \psi$ as $\neg(\varphi \wedge \neg\psi)$, $\varphi \vee \psi$ as $\neg\varphi \to \psi$, and $\varphi \leftrightarrow \psi$ as $(\varphi \to \psi) \wedge (\psi \to \varphi)$. For (7): $p \not\Vdash \neg\varphi \to \psi \Leftrightarrow \exists r \leq p\, [[r \Vdash \neg\varphi] \wedge [r \Vdash \neg\psi]] \Leftrightarrow \exists r \leq p \forall q \leq r\, [[q \not\Vdash \varphi] \wedge [q \not\Vdash \psi]]$, so $p \Vdash \varphi \vee \psi$ iff $\forall r \leq p\, \exists q \leq r\, [q \Vdash \varphi \vee q \Vdash \psi]$. □

In the proof of (5), this use of the Truth Lemma to obtain $q \leq p$ with $q \Vdash \varphi$ is so common that it is worth singling out. The same argument yields:

Lemma IV.2.31 *If G is $\mathbb{P}$-generic over M and $M[G] \models \varphi$ and $p \in G$, then there is a $q \in G$ such that $q \Vdash \varphi$ and $q \leq p$.*

Lemma IV.2.30 is an indication (*not a proof*) of why the Definability Lemma is true. Working in M, we might view (4–8) as the propositional steps in a recursive definition of $\Vdash$. Of course, we need a similar lemma for quantification:

Lemma IV.2.32 *For any forcing poset $\mathbb{P} \in M$ and formula $\varphi(x) \in \mathcal{FL}_{\mathbb{P}} \cap M$, with no variable other than x free:*

1. *$p \Vdash \forall x\, \varphi(x)$ iff $p \Vdash \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$.*
2. *$p \Vdash \exists x\, \varphi(x)$ iff $\{q \leq p : \exists \tau \in M^{\mathbb{P}}\, [q \Vdash \varphi(\tau)]\}$ is dense below p.*

Proof. For (1), $p \Vdash \forall x\, \varphi(x)$ iff for all generic $G \ni p$, $M[G] \models \forall x\, \varphi(x)$; and $M[G] \models \forall x\, \varphi(x)$ iff $M[G] \models \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$. So, $p \Vdash \forall x\, \varphi(x)$ holds iff for all $\tau \in M^{\mathbb{P}}$ and all generic $G \ni p$, $M[G] \models \varphi(\tau)$, which is equivalent to $\forall \tau \in M^{\mathbb{P}}\, [p \Vdash \varphi(\tau)]$.

We can derive (2) from (1), using Lemma IV.2.30, since $\exists x\, \varphi(x)$ is equivalent to $\neg \forall x\, \neg \varphi(x)$. By IV.2.30(5), $p \Vdash \neg \forall x\, \neg \varphi(x)$ iff for all $r \leq p$, $r \not\Vdash \forall x\, \neg \varphi(x)$. But $r \not\Vdash \forall x\, \neg \varphi(x)$ holds iff $r \not\Vdash \neg \varphi(\tau)$ for some $\tau \in M^{\mathbb{P}}$, so applying IV.2.30(5) again, $p \Vdash \neg \forall x\, \neg \varphi(x)$ iff $\forall r \leq p\, \exists \tau \in M^{\mathbb{P}}\, \exists q \leq r\, q \Vdash \varphi(\tau)$. □

Note that (1) and (2) of Lemma IV.2.32 are the natural analogs of (4) and (7) of Lemma IV.2.30, if we view $\forall x\, \varphi(x)$ and $\exists x\, \varphi(x)$ as asserting $\bigwedge_{\tau \in M^{\mathbb{P}}} \varphi(\tau)$ and $\bigvee_{\tau \in M^{\mathbb{P}}} \varphi(\tau)$.

There is also a similar lemma for atomic sentences:

Lemma IV.2.33 *For any forcing poset $\mathbb{P} \in M$ and names $\tau, \vartheta, \pi \in M^{\mathbb{P}}$:*

1. *$p \Vdash \tau = \vartheta$ iff $\forall \sigma \in \operatorname{dom}(\tau) \cup \operatorname{dom}(\vartheta)\, \forall q \leq p\, [q \Vdash \sigma \in \tau \leftrightarrow q \Vdash \sigma \in \vartheta]$.*
2. *$p \Vdash \pi \in \tau$ iff $\{q \leq p : \exists \langle \sigma, r \rangle \in \tau\, [q \leq r \wedge q \Vdash \pi = \sigma]\}$ is dense below p.*

We leave a proof of this directly from the Truth and Definability Lemmas as an exercise for the reader; we shall give another proof later. As an example, to prove (2) $\leftarrow$: Assume that $D := \{q \leq p : \exists \langle \sigma, r \rangle \in \tau\, [q \leq r \wedge q \Vdash \pi = \sigma]\}$ is dense below p. Fix any generic $G \ni p$; we must show that $\pi_G \in \tau_G$. By the Definability Lemma, $D \in M$, so by Lemma IV.2.29, fix $q \in D \cap G$. Then fix $\langle \sigma, r \rangle \in \tau$ such that $q \leq r$ and $q \Vdash \pi = \sigma$. Then $\pi_G = \sigma_G$ (because $q \in G$) and $r \in G$ (because G is a filter), so $\sigma_G \in \tau_G$ (by definition of τ_G), so $\pi_G \in \tau_G$.

Note that the proofs of Lemmas IV.2.30, IV.2.31, IV.2.32, and IV.2.33 used the Truth and Definability Lemmas, whereas the proof of Lemma IV.2.29 did not. We now proceed to prove the Truth and Definability Lemmas. Obviously, we cannot quote Lemmas IV.2.30, IV.2.32, IV.2.31, IV.2.33 in our proof, but they do provide some useful motivation:

For the Definability Lemma: The people living in M might temporarily ignore the Truth and Definability Lemmas, as well as anything outside their

universe M, and simply take the various clauses of Lemmas IV.2.30, IV.2.32, and IV.2.33 as a recursive definition of a relation that we shall call $\Vdash^*$. That is, working from the axioms of set theory, and without any discussion of models, we define a relation $p \Vdash^* \varphi$. Then, if we have a model M, we shall prove by induction that $p \Vdash \varphi$ iff $(p \Vdash^* \varphi)^M$, which will establish the Definability Lemma. Actually, to make the induction work, we shall need to prove the Truth Lemma at the same time, so we shall have proved both of them.

We shall start with the atomic sentences, which actually constitute the hardest part of the argument.

Definition IV.2.34 *$\mathcal{AL}_{\mathbb{P}}$ is the class of atomic sentences of $\mathcal{FL}_{\mathbb{P}}$; these are of the form $\tau = \vartheta$ and $\pi \in \tau$.*

Lemma IV.2.33 suggests that we define:

Definition IV.2.35 *For any forcing poset $\mathbb{P}$ and names $\tau, \vartheta, \pi \in V^{\mathbb{P}}$:*

1. *$p \Vdash^* \tau = \vartheta$ iff $\forall \sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\vartheta)\, \forall q \leq p\, [q \Vdash^* \sigma \in \tau \leftrightarrow q \Vdash^* \sigma \in \vartheta]$.*
2. *$p \Vdash^* \pi \in \tau$ iff $\{q \leq p : \exists \langle \sigma, r\rangle \in \tau\, [q \leq r \wedge q \Vdash^* \pi = \sigma]\}$ is dense below p.*

This recursion requires some justification, but first two simple examples:

Example IV.2.36

a. $p \Vdash^* \tau = \tau$ because the "$\leftrightarrow$" in (1) is tautological.
b. If $\langle \sigma, r\rangle \in \tau$ and $p \leq r$ then $p \Vdash^* \sigma \in \tau$ because $q \Vdash^* \sigma = \sigma$ for all q.

This recursion is justified in $ZF-P$ using Theorem I.9.11. As usual with the recursive definition of a predicate, we are really defining the characteristic function of that predicate, $F : \mathbb{P} \times \mathcal{AL}_{\mathbb{P}} \to \{0, 1\}$. We view $\mathbb{P}$ and its order $\leq$ as "fixed parameters" in the recursion (see the remarks on Theorem I.9.11). But, we need to show that this definition of F is recursion *on* some well-founded relation R. Roughly, we use the observation that, as with our discussion of Definition IV.2.5 (of "τ is a $\mathbb{P}$-name"), the relation $\in$ is well-founded and $\langle \sigma, r\rangle \in \tau \to \sigma \in \mathrm{trcl}(\tau)$. But, it's a little more complicated here. F is defined on pairs (p, φ), where $\varphi \in \mathcal{AL}_{\mathbb{P}}$. To define R on these pairs, fix $\sigma_1, \tau_1, \sigma_2, \tau_2 \in V^{\mathbb{P}}$, and fix $p_1, p_2 \in \mathbb{P}$. Let $x \lhd y$ abbreviate $x \in \mathrm{trcl}(y)$. Then:

1. $(p_1,\ \sigma_1 \in \tau_1)\ R\ (p_2,\ \sigma_2 = \tau_2)$ iff
   $(\sigma_1 \lhd \sigma_2$ or $\sigma_1 \lhd \tau_2)$ and $(\tau_1 = \sigma_2$ or $\tau_1 = \tau_2)$.
2. $(p_1,\ \sigma_1 = \tau_1)\ R\ (p_2,\ \sigma_2 \in \tau_2)$ iff
   $\sigma_1 = \sigma_2$ and $\tau_1 \lhd \tau_2$.
3. $(p_1,\ \sigma_1 \in \tau_1)\ \not R\ (p_2,\ \sigma_2 \in \tau_2)$ and $(p_1,\ \sigma_1 = \tau_1)\ \not R\ (p_2,\ \sigma_2 = \tau_2)$.

Then R is set-like because $\mathbb{P}$ is a set. R is not a transitive relation, but that is not required in Theorem I.9.11. It is easy to see that the recursion is *on* R in the sense that the definition of $p \Vdash^* \varphi$ depends on knowing whether

$q \Vdash^* \psi$ for various $(q,\psi)\ R\ (p,\varphi)$. To prove that R is well-founded, first define $\rho(p,\ \sigma=\tau) = \rho(p,\ \sigma\in\tau) = \max(\mathrm{rank}(\sigma),\mathrm{rank}(\tau)) \in ON$. Note that $(p_1,\varphi_1)\ R\ (p_2,\varphi_2) \to \rho(p_1,\varphi_1) \le \rho(p_2,\varphi_2)$. Also, define $\mathrm{type}(p,\ \sigma=\tau)=1$, and let $\mathrm{type}(p,\ \sigma\in\tau)$ be 0 if $\mathrm{rank}(\sigma)<\mathrm{rank}(\tau)$ and 2 if $\mathrm{rank}(\sigma)\ge\mathrm{rank}(\tau)$. Define $\Gamma : \mathbb{P}\times\mathcal{AL}_{\mathbb{P}} \in ON$ by $\Gamma(p,\varphi) = 3\cdot\rho(p,\varphi)+\mathrm{type}(p,\varphi)$. Observe that $(p_1,\varphi_1)\ R\ (p_2,\varphi_2) \to \Gamma(p_1,\varphi_1) < \Gamma(p_2,\varphi_2)$, so R is well-founded by Lemma I.9.9.

Before we try to relate $\Vdash^*$ to $\Vdash$, we note the following facts about $\Vdash^*$:

Lemma IV.2.37 *For $\varphi\in\mathcal{AL}_{\mathbb{P}}$:*

a. If $p\Vdash^*\varphi$ and $p_1\le p$ then $p_1\Vdash^*\varphi$.
b. $p\Vdash^*\varphi$ iff $\{p_1\le p : p_1\Vdash^*\varphi\}$ is dense below p.

Proof. Part (a) is clear from the definition of $\Vdash^*$. For (b), the $\to$ implication is clear from (a). For the $\leftarrow$ implication: When φ is $\pi\in\tau$, use the definition of $\Vdash^*$ and the fact that if $A\subseteq\mathbb{P}$ and $\{q : A \text{ is dense below } q\}$ is dense below p then A is dense below p. When φ is $\tau=\vartheta$, use (b) for $\sigma\in\tau$ and $\sigma\in\vartheta$. □

Definition IV.2.38 *For $\varphi\in\mathcal{AL}_{\mathbb{P}}$ and $p\in\mathbb{P}$: $p\Vdash^*\neg\varphi$ iff $\neg\exists q\le p\,[q\Vdash^*\varphi]$.*

This will later (Definition IV.2.42) become part of our recursive definition of $\Vdash^*$ for general φ. It is easy to see from Lemma IV.2.37 that

Lemma IV.2.39 *For $\varphi\in\mathcal{AL}_{\mathbb{P}}$ and $p\in\mathbb{P}$: $p\Vdash^*\varphi$ iff $\neg\exists q\le p\,[q\Vdash^*\neg\varphi]$.*

The following lemma relates $\Vdash^*$ to models and generic filters. It does not mention $\Vdash$, but the Truth and Definability Lemmas will follow easily from it.

Lemma IV.2.40 *Let the set M be a transitive model of $ZF-P$, with $\mathbb{P}\in M$. Fix $\varphi\in\mathcal{AL}_{\mathbb{P}}\cap M$. Let G be $\mathbb{P}$-generic over M. Then*

a. If $p\in G$ and $(p\Vdash^*\varphi)^M$ then $M[G]\models\varphi$.
b. If $M[G]\models\varphi$ then there is some $p\in G$ such that $(p\Vdash^*\varphi)^M$.

Note that M need not be countable, although if M is uncountable, there need not be any generic G. Also, Definition IV.2.35 of $\Vdash^*$ for *atomic* φ is easily seen to be absolute, so that we could just as well have written $p\Vdash^*\varphi$ instead of the relativized $(p\Vdash^*\varphi)^M$. We wrote the relativized form because later (Lemma IV.2.44), we shall have verbatim the same statement for arbitrary sentences of $\mathcal{FL}_{\mathbb{P}}\cap M$, where the definition is not absolute. But, we shall delete the relativization in the proof, which is specific to atomic φ:

Proof. We induct on the relation R used to justify Definition IV.2.35.

For (a): Fix $p\in G$.

If $p\Vdash^*\pi\in\tau$ then $D := \{q\le p : \exists\langle\sigma,r\rangle\in\tau\,[q\le r\wedge q\Vdash^*\pi=\sigma]\}$ is dense below p, and $D\in M$, so by Lemma IV.2.29, fix $q\in G\cap D$, and

then fix $\langle \sigma, r\rangle \in \tau$ with $q \leq r$ and $q \Vdash^* \pi = \sigma$. Applying (a) and induction, $M[G] \models \pi = \sigma$, so $\pi_G = \sigma_G$. Also, $r \in G$ because G is a filter, so $\sigma_G \in \tau_G$ because $\langle \sigma, r\rangle \in \tau$. Thus, $M[G] \models \pi \in \tau$.

If $p \Vdash^* \tau = \vartheta$ then we shall prove that $\tau_G \subseteq \vartheta_G$; the proof that $\vartheta_G \subseteq \tau_G$ is essentially the same. So, fix an element of τ_G, which must be of the form σ_G, where $\langle \sigma, r\rangle \in \tau$ and $r \in G$; we shall show that $\sigma_G \in \vartheta_G$. Fix $q \in G$ such that $q \leq p$ and $q \leq r$. Then $q \Vdash^* \sigma \in \tau$ (see Example IV.2.36) so $q \Vdash^* \sigma \in \vartheta$ (by definition of $\Vdash^*$). Applying (a) and induction, $M[G] \models \sigma \in \vartheta$, so $\sigma_G \in \vartheta_G$.

Now, for (b):

If $M[G] \models \pi \in \tau$, we must find a $p \in G$ such that $D := \{q \leq p : \exists \langle \sigma, r\rangle \in \tau\, [q \leq r \wedge q \Vdash^* \pi = \sigma]\}$ is dense below p. By definition of τ_G, fix $\langle \sigma, r\rangle \in \tau$ with $r \in G$ such that $\pi_G = \sigma_G$. Applying (b) and induction, fix $p \in G$ such that $p \Vdash^* \pi = \sigma$. Since G is a filter, Lemma IV.2.37 implies that we may assume that $p \leq r$; then Lemma IV.2.37 again implies that $D \supseteq p{\downarrow}$.

When φ is $\tau = \vartheta$, let D be the set of all $p \in G$ such that at least one of the following three is true:

1. $p \Vdash^* \tau = \vartheta$
2. For some $\sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\vartheta)$: $p \Vdash^* \sigma \in \tau$ and $p \Vdash^* \sigma \notin \vartheta$.
3. For some $\sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\vartheta)$: $p \Vdash^* \sigma \notin \tau$ and $p \Vdash^* \sigma \in \vartheta$.

Then $D \in M$, and D is dense by the definition of "$p \Vdash^* \tau = \vartheta$" and Definition IV.2.38 and Lemma IV.2.39. If $M[G] \models \tau = \vartheta$, fix $p \in G \cap D$. If (1) holds, we are done. If (2) holds, then $p \Vdash^* \sigma \in \tau$ implies that $M[G] \models \sigma \in \tau$ (applying (a)), so $\sigma_G \in \tau_G = \vartheta_G$, so applying (b) and induction, fix $q \in G$ such that $q \Vdash^* \sigma \in \vartheta$. Then fix $r \in G$ such that $r \leq p$ and $r \leq q$. Then $r \Vdash^* \sigma \in \vartheta$ (by Lemma IV.2.37), and then $r \leq p$ contradicts $p \Vdash^* \sigma \notin \vartheta$ (by Definition IV.2.38). A similar contradiction arises from (3). □

Lemma IV.2.41 *Let M be a ctm of ZF−P, with $\mathbb{P} \in M$. For $p \in \mathbb{P}$ and $\varphi \in \mathcal{AL}_{\mathbb{P}} \cap M$, $p \Vdash \varphi$ iff $(p \Vdash^* \varphi)^M$.*

Proof. Again, when discussing atomic sentences, we can drop the relativizations to M. For $\leftarrow$: If $p \Vdash^* \varphi$ then Lemma IV.2.40 (a) plus the definition of $\Vdash$ implies that $p \Vdash \varphi$.

For $\rightarrow$: Assume that $p \Vdash \varphi$ and $(p \not\Vdash^* \varphi)^M$. By Lemma IV.2.39, fix $q \leq p$ such that $q \Vdash^* \neg\varphi$, then, by Definition IV.2.38, $\neg\exists r \leq q\, [r \Vdash^* \varphi]$. Now, let G be generic with $q \in G$. Then $p \in G$, so $M[G] \models \varphi$ by the definition of $\Vdash$. By Lemma IV.2.40, fix $s \in G$ such that $s \Vdash^* \varphi$. Since $p, s \in G$, fix r such that $r \leq p$ and $r \leq s$. Then $r \Vdash^* \varphi$ by Lemma IV.2.37, which is a contradiction. □

Note that our Lemmas IV.2.40 and IV.2.41 have so far been proved just for atomic sentences, although it was convenient to use $\Vdash^*$ for negated atomic sentences, together with Lemma IV.2.39, to aid the proof.

Applying Lemma IV.2.41, we now have, for *atomic* sentences, the Definability Lemma (since the notion $(p \Vdash^* \varphi)^M$ is obviously definable over M) and

the Truth Lemma (replacing $\Vdash^*$ by $\Vdash$ in Lemma IV.2.40). To do this for *all* sentences, we first need to define "$p \Vdash^* \varphi$" for arbitrary sentences $\varphi \in \mathcal{FL}_{\mathbb{P}}$. Informally, the definition is "obvious" by using the appropriate clauses of Lemmas IV.2.30 and IV.2.32. For example, we should say $p \Vdash^* \forall x\, \varphi(x)$ iff $\forall \tau \in V^{\mathbb{P}}\, [p \Vdash^* \varphi(\tau)]$. Here, we do not expect our $\Vdash^*$ to be absolute. As before, the definition of $\Vdash^*$ does not mention models, but when we relativize to a model M, the $\forall \tau \in V^{\mathbb{P}}$ becomes $\forall \tau \in M^{\mathbb{P}}$, as we would expect from Lemma IV.2.32.

However, there is one further technicality, related to Tarski's theorem on the undefinability of truth, as we pointed out before stating the Definability Lemma IV.2.25. Even with the trivial $\mathbb{P} = \{\mathbb{1}\}$, where every name is of the form $\check{a}$, an attempt to define the notion $\mathbb{1} \Vdash^* \varphi$ for all sentences $\varphi \in \mathcal{FL}_{\mathbb{P}}$ would fail for the same reason that we cannot define the notion $(V, \in) \models \varphi$; namely, the natural R to recurse on is not set-like; see the discussion of Definition I.15.5. Specifically, when we define $p \Vdash^* \forall x\, \varphi(x)$ iff $\forall \tau \in V^{\mathbb{P}}\, [p \Vdash^* \varphi(\tau)]$, we are evaluating $\Vdash^*$ for $\forall x\, \varphi(x)$ in terms of a *proper class* of sentences $\varphi(\tau)$. So, formally, the recursion takes place in the metatheory. By recursion on $\mathrm{lh}(\varphi)$, we are defining, in the metatheory, the notion $p \Vdash^*_{\mathbb{P}} \varphi(\varkappa_1, \ldots, \varkappa_n)$ for formulas $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L} = \{\in\}$, and names $\varkappa_1, \ldots, \varkappa_n \in V^{\mathbb{P}}$. Then, the Definability Lemma will follow by proving, for a ctm M, that $p \Vdash_{\mathbb{P},M} \varphi(\varkappa_1, \ldots, \varkappa_n)$ is equivalent to $(p \Vdash^*_{\mathbb{P}} \varphi(\varkappa_1, \ldots, \varkappa_n))^M$.

Still, for readability, we suppress mention of these logical niceties in our official definition:

Definition IV.2.42 *For any forcing poset $\mathbb{P}$, we define the notion $p \Vdash^*_{\mathbb{P}} \varphi$ for sentences φ of $\mathcal{FL}_{\mathbb{P}}$ by the following clauses:*

1. *For atomic φ, as given by Definition IV.2.35.*
2. *$p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.*
3. *$p \Vdash^* \neg\varphi$ iff $\neg\exists q \leq p\, [q \Vdash^* \varphi]$.*
4. *$p \Vdash^* \varphi \to \psi$ iff $\neg\exists q \leq p\, [q \Vdash^* \varphi \wedge q \Vdash^* \neg\psi]$.*
5. *$p \Vdash^* \varphi \vee \psi$ iff $\{q : [q \Vdash^* \varphi] \vee [q \Vdash^* \psi]\}$ is dense below p*
6. *$p \Vdash^* \varphi \leftrightarrow \psi$ iff $\neg\exists q \leq p\, [q \Vdash^* \varphi \wedge q \Vdash^* \neg\psi]$ and $\neg\exists q \leq p\, [q \Vdash^* \psi \wedge q \Vdash^* \neg\varphi]$.*
7. *$p \Vdash^* \forall x\, \varphi(x)$ iff $p \Vdash^* \varphi(\tau)$ for all $\tau \in V^{\mathbb{P}}$.*
8. *$p \Vdash^* \exists x\, \varphi(x)$ iff $\{q : \exists \tau \in V^{\mathbb{P}}\, [q \Vdash^* \varphi(\tau)]\}$ is dense below p.*

Some remarks: In the case of $\neg$, our definition is consistent with Definition IV.2.38 above. One might make our definitions and proofs slightly shorter by using only $\wedge$ and $\neg$ as propositional connectives, rather than the $\wedge, \vee, \neg, \to, \leftrightarrow$ from Section I.2. But this makes little difference, since the propositional steps are easy anyway; the main difficulties are in forcing for the atomic sentences, and in correctly understanding the logic of this recursive definition. As noted above, in the metatheory, for each formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L} = \{\in\}$, with n free variables, we are associating another formula $Forces_{\varphi}(\mathbb{P}, \leq, \mathbb{1}, p, \varkappa_1, \ldots, \varkappa_n)$ with $n + 4$ free variables asserting that $(\mathbb{P}, \leq, \mathbb{1})$ is a forcing poset, $p \in \mathbb{P}$,

$\varkappa_1, \ldots, \varkappa_n \in V^{\mathbb{P}}$, and $p \Vdash^*_{\mathbb{P}} \varphi(\varkappa_1, \ldots, \varkappa_n)$. This is a simple recursion on $\mathrm{lh}(\varphi)$, and the inductions, also on $\mathrm{lh}(\varphi)$, take place in the metatheory. We now retrace what we did for atomic sentence, again suppressing these logical niceties in the statements of our lemmas.

First, we extend Lemmas IV.2.37 and IV.2.39 from atomic φ to all φ:

Lemma IV.2.43 *For $\varphi \in \mathcal{FL}_{\mathbb{P}}$:*

a. If $p \Vdash^* \varphi$ and $p_1 \leq p$ then $p_1 \Vdash^* \varphi$.
b. $p \Vdash^* \varphi$ iff $\{p_1 \leq p : p_1 \Vdash^* \varphi\}$ is dense below p.
c. $p \Vdash^* \varphi$ iff $\neg \exists q \leq p\, [q \Vdash^* \neg\varphi]$.

Proof. (a) is an easy induction on φ, and (c) is immediate from (b).

For (b), the $\rightarrow$ direction is clear from (a), and the $\leftarrow$ direction is proved by induction on φ. All seven cases are straightforward. We give two as examples.

If φ is $\forall x\, \psi(x)$ and $\{p_1 \leq p : \forall \tau \in V^{\mathbb{P}}\, [p_1 \Vdash^* \psi(\tau)]\}$ is dense below p, then for each τ, $\{p_1 \leq p : p_1 \Vdash^* \psi(\tau)]\}$ is dense below p, so $p \Vdash^* \psi(\tau)$ by (b) for $\psi(\tau)$. Hence, $p \Vdash^* \forall x\, \psi(x)$.

If φ is $\chi \vee \psi$, let $A = \{q : [q \Vdash^* \chi] \vee [q \Vdash^* \psi]\}$. If $\{p_1 : A \text{ is dense below } p_1\}$ is dense below p, then A is dense below p, so $p \Vdash^* \chi \vee \psi$. In this case, as in a number of others, we do not use the inductive hypotheses. □

Again, this lemma was an abbreviation of a scheme in the metatheory. For example, (a) asserts that for each formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L} = \{\in\}$, one can prove that for all $\varkappa_1, \ldots, \varkappa_n \in V^{\mathbb{P}}$, if $p \Vdash^* \varphi(\vec{\varkappa})$ and $p_1 \leq p$ then $p_1 \Vdash^* \varphi(\vec{\varkappa})$.

The next lemma extends Lemma IV.2.40 from atomic φ to all φ.

Lemma IV.2.44 *Let the set M be a transitive model of $ZF-P$, with $\mathbb{P} \in M$. Fix $\varphi \in \mathcal{FL}_{\mathbb{P}} \cap M$. Let G be $\mathbb{P}$-generic over M. Then*

a. If $p \in G$ and $(p \Vdash^* \varphi)^M$ then $M[G] \models \varphi$.
b. If $M[G] \models \varphi$ then there is some $p \in G$ such that $(p \Vdash^* \varphi)^M$.

Proof. We prove (a)(b) together by induction on φ. Let $L(\varphi)$ abbreviate the assertion that the lemma holds for φ. There are seven cases in the induction, corresponding to clauses (2–8) in Definition IV.2.42, but they are all fairly easy, so we shall only do two to illustrate the method and some points of logic.

$L(\varphi) \rightarrow L(\neg\varphi)$: For (a), if $p \in G$ and $(p \Vdash^* \neg\varphi)^M$ and $M[G] \models \varphi$, then by $L(\varphi)$, fix $r \in G$ such that $(r \Vdash^* \varphi)^M$, and then fix $q \in G$ such that $q \leq p$ and $q \leq r$. Then $(r \Vdash^* \varphi)^M$ by Lemma IV.2.43 (applied in M) and $q \leq p$, contradicting the definition of "$p \Vdash^* \neg\varphi$" (applied in M). Here, we are using the fact that $M \models ZF-P$, which is enough set theory to justify Definition IV.2.42 and to carry out proofs of lemmas such as Lemma IV.2.43. For (b), note that $D := \{p \in \mathbb{P} : (p \Vdash^* \varphi)^M \vee (p \Vdash^* \neg\varphi)^M\} \in M$, and D is dense by the definition of "$p \Vdash^* \neg\varphi$". Assume that $M \models \neg\varphi$,and fix $p \in G \cap D$. If $(p \Vdash^* \neg\varphi)^M$ we are done, and if $p \Vdash^* \varphi)^M$, we contradict (a) of $L(\varphi)$.

$[\forall \tau \in M^{\mathbb{P}}\, L(\varphi(\tau))] \to L(\exists x\, \varphi(x))$: For (a): If $p \in G$ and $(p \Vdash^* \exists x\, \varphi(x))^M$, then $D := \{q : \exists \tau \in M^{\mathbb{P}}[q \Vdash^* \varphi(\tau)]^M\}$ is dense below p (by Definition IV.2.42(8) relativized to M). Since $p \in G$ and G is generic, fix $q \leq p$ and $\tau \in M^{\mathbb{P}}$ such that $q \in G$ and $(q \Vdash^* \varphi(\tau))^M$. Then $M[G] \models \varphi(\tau)$ by (a) for $\varphi(\tau)$, so $M \models \exists x\, \varphi(x)$. For (b), if $M[G] \models \exists x\, \varphi(x)$ then we can fix $\tau \in M^{\mathbb{P}}$ such that $M[G] \models \varphi(\tau)$. Applying (b) for $\varphi(\tau)$, fix $p \in G$ such that $(p \Vdash^* \varphi(\tau))^M$. Then $(p \Vdash^* \exists x\, \varphi(x))^M$ is clear from Definition IV.2.42(8). □

Actually, one can prove that if $M \models ZFC$ and $p \Vdash \exists x\, \varphi(x)$, then there is a $\tau \in M^{\mathbb{P}}$ such that $p \Vdash \varphi(\tau)$. This fact is known as the *Maximal Principle* (Theorem IV.7.1). Using this, one might simplify Definition IV.2.42(8) to $p \Vdash^* \exists x\, \varphi(x)$ iff $p \Vdash^* \varphi(\tau)$ for some $\tau \in V^{\mathbb{P}}$. However, the proof of the Maximal Principle relies on the material covered in this section, so appealing to it here would make our arguments circular. Also, the Maximal Principle assumes that $M \models AC$, whereas the basic forcing results here apply to all ctms for ZF.

Now, in analogy with Lemma IV.2.41, we prove the following lemma. The proof is verbatim the same, replacing the use of Lemmas IV.2.37, IV.2.39, and IV.2.40, by Lemmas IV.2.43 and IV.2.44.

Lemma IV.2.45 *Let M be a ctm of $ZF-P$, with $\mathbb{P} \in M$. For $p \in \mathbb{P}$ and $\varphi \in \mathcal{FL}_{\mathbb{P}} \cap M$, $p \Vdash \varphi$ iff $(p \Vdash^* \varphi)^M$.*

Proof of Lemmas IV.2.24 and IV.2.25 (Truth and Definability). For the Truth Lemma, replace $(p \Vdash^* \varphi)^M$ by $p \Vdash \varphi$ in Lemma IV.2.44(b). For the Definability Lemma, the notion $(p \Vdash^* \varphi(\varkappa_1, \ldots, \varkappa_n))^M$ is clearly definable over M (see Definition IV.2.42 and the discussion thereof). □

Proof of Lemma IV.2.33. This is obvious if one replaces $\Vdash$ by $\Vdash^*$. □

We conclude with two pathologies involving *non*-generic filters. Exercise IV.2.46 indicates a fundamental problem. Exercise IV.2.47 is both more difficult and less fundamental, since it is specific to our definition of "$\mathbb{P}$–name"; it could be "cured" by revising the definition of "$\mathbb{P}$–name" to guarantee that every $M[G]$ is closed under relative complements.

Exercise IV.2.46 *Assume that M is a ctm for ZFC, and let $\mathbb{P} = Fn(\omega, 2)$. Then there is a filter G on $\mathbb{P}$ such that there is* no *transitive $N \supseteq M$ such that $G \in N$ and $N \models ZF-P$ and $o(N) = o(M)$.*

Hint. G can code a well-order of ω of type larger than $o(M)$. □

Exercise IV.2.47 *Assume that M is a ctm for ZFC. Give an example of a $\mathbb{P} \in M$ and a (non-generic) filter G on $\mathbb{P}$ for which $(\mathbb{P} \backslash G) \notin M[G]$.*

Hint. One could let $\mathbb{P}$ be HF, and let $\leq$ be an appropriate forcing order such that $p \leq q \leq p$ whenever $\mathrm{rank}(p) = \mathrm{rank}(q)$. Then G can be built by a diagonal argument. Observe that for filters, $G \subseteq H \to \mathrm{rank}(\tau_G) \leq \mathrm{rank}(\tau_H)$. □

IV.3 Computing Cardinal Exponentiation

Throughout this section, M denotes a ctm for *ZFC*. We shall prove a few general results about computing with cardinal arithmetic in $M[G]$. In particular, we shall use posets of the form $\mathrm{Fn}(I, J)$ to get models $M[G]$ in which $2^{\aleph_0}$ is any value not contradicted by König's theorem (that $\mathrm{cf}(2^{\aleph_0}) > \omega$).

Before discussing cardinal arithmetic in $M[G]$, we must first decide what the cardinals are. This is sometimes non-trivial. Note that although M and $M[G]$ have the same ordinals ($o(M[G]) = o(M)$; see Lemma IV.2.18), they need not have the same cardinals.

Example IV.3.1 *Let $\mathbb{P} = \mathrm{Fn}(I, J)$, where $I, J \in M$ and I, J are infinite. Let G be $\mathbb{P}$-generic over M, and let $f = \bigcup G$. Then $f \in M[G]$ and f is a function from I onto J.*

Proof. See the proof of Lemma III.3.13, which proved that $f : I \xrightarrow{\text{onto}} J$ assuming that G met certain dense sets. Since $I, J \in M$, all these dense sets are in M by an absoluteness argument, so G meets them all. □

For example, let $I = \omega$ and $J = \omega_5^M$. Now, $\omega^M = \omega$ by absoluteness, but ω_5^M is some countable ordinal, although people living in M think that it is the fifth uncountable cardinal. But $M[G]$ contains a map from ω onto ω_5^M, so people living in $M[G]$ know that ω_5^M is countable. Since $M[G] \models ZFC$, it has its own uncountable cardinals $\omega_1^{M[G]}, \omega_2^{M[G]}, \ldots$, but these are clearly all above ω_5^M. We shall see later (Exercise IV.7.10) that these are in fact $\omega_6^M, \omega_7^M, \ldots$. For now, we concentrate on the simpler situations where M and $M[G]$ have the same cardinals.

Definition IV.3.2 *For a forcing poset $\mathbb{P} \in M$:*

1. $\mathbb{P}$ preserves cardinals *if and only if for all generic G:* $(\beta \text{ is a cardinal})^M$ *iff* $(\beta \text{ is a cardinal})^{M[G]}$ *for all $\beta < o(M)$.*
2. $\mathbb{P}$ preserves cofinalities *if and only if for all generic G:* $\mathrm{cf}^M(\gamma) = \mathrm{cf}^{M[G]}(\gamma)$ *for all limit $\gamma < o(M)$.*

Of course, this notion depends on M as well as on $\mathbb{P}$.

Regarding (1): Note that the $\leftarrow$ direction of the "iff" always holds because Π_1 properties relativize down (see Lemma II.4.28) and "β is a cardinal" is Π_1 in absolute notions ($\forall \alpha < \beta \forall f \neg[f : \alpha \xrightarrow{\text{onto}} \beta]$). The $\rightarrow$ direction must hold for $\beta \leq \omega$ (by absoluteness of ω), but could fail for $\beta \geq \omega_1^M$, as we just saw.

Regarding (2): Note that $\mathrm{cf}^M(\gamma) \geq \mathrm{cf}^{M[G]}(\gamma)$ follows by Definition I.13.9 (of $\mathrm{cf}(\gamma)$) and the absoluteness of type(X). If $\gamma = \omega_1^M$ and γ becomes countable in $M[G]$, then $\gamma = \mathrm{cf}^M(\gamma) > \omega = \mathrm{cf}^{M[G]}(\gamma)$.

Lemma IV.3.3 *For a forcing poset $\mathbb{P} \in M$:*

1. $\mathbb{P}$ *preserves cofinalities if and only if for all generic* G:
   $(*)$: *for all limit* β *with* $\omega < \beta < o(M)$,
   $(\beta$ *is regular*$)^M \to (\beta$ *is regular*$)^{M[G]}$.
2. *If* $\mathbb{P}$ *preserves cofinalities then* $\mathbb{P}$ *preserves cardinals.*

Proof. For the non-trivial part of (1), we assume $(*)$, fix a limit $\gamma < o(M)$, let $\beta = \mathrm{cf}^M(\gamma)$, and prove that $\mathrm{cf}^{M[G]}(\gamma) = \beta$. Fix $X \in \mathcal{P}(\gamma) \cap M$ with $\mathrm{type}(X) = \beta$ and $\sup(X) = \gamma$. Applying Lemma I.13.10 in M and in $M[G]$: $(\beta$ is regular$)^M$, so $(\beta$ is regular$)^{M[G]}$; then, $\mathrm{cf}^{M[G]}(\gamma) = \mathrm{cf}^{M[G]}(\beta) = \beta$. The "so" used $(*)$ when $\beta > \omega$ and the absoluteness of ω when $\beta = \omega$.

For (2): By (1), M and $M[G]$ have the same regular cardinals. Now, use the fact that in *ZFC*, every cardinal is either regular or $\leq \omega$ or a supremum of regular (successor) cardinals. □

The converse of (2) is true if $M \models V = L$, but is known to be false for some other M. For example, a measurable cardinal is a kind of (very large) inaccessible cardinal. If $(\kappa$ is a measurable cardinal$)^M$, then Prikry forcing preserves cardinals, but $\mathrm{cf}^{M[G]}(\kappa) = \omega$. For more on these issues, see Jech [49].

In this book, all posets that we discuss that preserve cardinals will also preserve cofinalities. In this section, these will be just the ccc posets. It is common to say that "ccc posets preserve cofinalities and cardinals". Of course, since M is countable, all $\mathbb{P} \in M$ are countable, and hence trivially ccc, but they need not preserve cardinals. The more formally correct assertion is:

Theorem IV.3.4 *If* $\mathbb{P} \in M$ *and* $(\mathbb{P}$ *is ccc*$)^M$, *then* $\mathbb{P}$ *preserves cofinalities and hence* $\mathbb{P}$ *preserves cardinals.*

We shall see many other theorems of this nature, relating combinatorial properties of $\mathbb{P}$ in M (*not* in V) to properties of the $\mathbb{P}$-generic extensions.

Theorem IV.3.4 is proved using the following "approximation lemma" for ccc forcing. If $f : A \to B$ in $M[G]$, then f need not be in M, but we can, within M, pick out a countable set of possible values for each $f(a)$:

Lemma IV.3.5 *Assume that* $\mathbb{P} \in M$, $(\mathbb{P}$ *is ccc*$)^M$, *and* $A, B \in M$. *Let* G *be* $\mathbb{P}$*-generic over* M. *Fix* $f \in M[G]$ *with* $f : A \to B$. *Then, there is an* $F : A \to \mathcal{P}(B)$ *with* $F \in M$ *such that for all* $a \in A$, $f(a) \in F(a)$ *and* $(|F(a)| \leq \aleph_0)^M$.

Proof. Fix a name $\mathring{f} \in M^{\mathbb{P}}$ such that $\mathring{f}_G = f$. Then "$\mathring{f} : \check{A} \to \check{B}$" is a sentence of $\mathcal{FL}_{\mathbb{P}} \cap M$ about the three names $\mathring{f}, \check{A}, \check{B}$, and this sentence is true in $M[G]$, so by the Truth Lemma, fix $p \in G$ such that $p \Vdash \mathring{f} : \check{A} \to \check{B}$.

Now define $F : A \to \mathcal{P}(B)$ by $F(a) = \{b \in B : \exists q \leq p\, [q \Vdash \mathring{f}(\check{a}) = \check{b}\}$. Then $F \in M$ by the Definability Lemma.

To prove that each $f(a) \in F(a)$: Let $b = f(a)$. Then $M[G] \models \mathring{f}(\check{a}) = \check{b}$, so by Lemma IV.2.31, there is a $q \leq p$ in G such that $q \Vdash \mathring{f}(\check{a}) = \check{b}$. Then $b \in F(a)$.

To prove that each $(|F(a)| \leq \aleph_0)^M$: For each $b \in F(a)$, choose $q_b \leq p$ such that $q \Vdash \mathring{f}(\check{a}) = \check{b}$. Then the q_b are pairwise incompatible, since if $b \neq c$ and $r \leq q_b$ and $r \leq q_c$, then if H is a generic filter containing r (and hence also p, q_b, q_c), $M[H] \models \mathring{f} : \check{A} \to \check{B} \wedge \mathring{f}(\check{a}) = \check{b} \wedge \mathring{f}(\check{a}) = \check{c} \wedge \check{b} \neq \check{c}$, which is impossible. Furthermore, the function $b \mapsto q_b$ may be assumed to lie in M, using the Definability Lemma plus AC^M. Now, working completely in M, we conclude that $|F(a)| \leq \aleph_0$ by using $b \neq c \to q_b \perp q_c$ plus the ccc. □

A remark on this proof: Previously, whenever we have used the terminology $p \Vdash \varphi$, the φ denoted either a general sentence (as in the proof of the Comprehension Axiom in $M[G]$) or a short explicit sentence (such as $\tau \in \sigma$). But here, when we said "$p \Vdash \mathring{f} : \check{A} \to \check{B}$", we are talking about forcing one specific sentence, but we never said explicitly what that sentence is. We have in mind that one writes a formula $\psi(x, y, z)$ in $\mathcal{L} = \{\in\}$ saying that x is a function from y to z, and we intend to assert that "$p \Vdash \psi(\mathring{f}, \check{A}, \check{B})$". But what exactly is ψ? You could write down an explicit ψ by tracing through Definition I.6.8 and earlier definitions, but no one ever does, and if they did, probably no two people would come up with exactly the same ψ. It is important for the theory that there is such a ψ, so that we can talk about forcing it. The fact that any "reasonable" ψ is okay here is made formal by:

Lemma IV.3.6 *Suppose that $\psi(x_1, \ldots, x_n)$ and $\varphi(x_1, \ldots, x_n)$ are two formulas of $\mathcal{L} = \{\in\}$. Fix $\mathbb{P} \in M$ and $p \in \mathbb{P}$ and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. Then*

If $ZFC \vdash \forall \vec{x}\,[\psi(\vec{x}) \to \varphi(\vec{x})]$ *then* $(p \Vdash \psi(\vec{\tau})) \to (p \Vdash \varphi(\vec{\tau}))$.

If $ZFC \vdash \forall \vec{x}\,[\psi(\vec{x}) \leftrightarrow \varphi(\vec{x})]$ *then* $(p \Vdash \psi(\vec{\tau})) \leftrightarrow (p \Vdash \varphi(\vec{\tau}))$.

Proof. For the first statement, use the definition of $\Vdash$ and the fact that $M[G] \models ZFC$, so that $(M[G] \models \psi(\vec{\tau})) \to M[G](\models \varphi(\vec{\tau}))$. The second statement is similar. □

Proof of Theorem IV.3.4. By Lemma IV.3.3, it is sufficient to fix a limit ordinal β with $\omega < \beta < o(M)$, assume that $(\beta \text{ is regular})^M$, and prove that $(\beta \text{ is regular})^{M[G]}$. If this is false, then fix $X \subseteq \beta$ with $X \in M[G]$ and $\sup(X) = \beta$ and $\alpha := \text{type}(X) < \beta$. Let $f : \alpha \xrightarrow[\text{onto}]{1\text{-}1} X$ be the unique order preserving bijection. Then $f : \alpha \to \beta$, so by Lemma IV.3.5, fix $F : \alpha \to \mathcal{P}(\beta)$ with $F \in M$ such that for all $\xi < \alpha$, $f(\xi) \in F(\xi)$ and $(|f(\xi)| \leq \aleph_0)^M$. Let $Y = \bigcup_{\xi<\alpha} F(\xi)$. Then $Y \subseteq \beta$ and $\sup(Y) = \beta$.

Working completely in M: We have an uncountable regular cardinal β and $|Y| < \beta$ (since Y is a union of fewer than β countable sets), so $\sup(Y) = \beta$, contradicting the definition of "regular". □

We have now completed all the details for producing models of $ZFC + \neg CH$:

Lemma IV.3.7 *Fix $\alpha < o(M)$ and let $\kappa = (\aleph_\alpha)^M$. Then let $\mathbb{P} = \text{Fn}(\kappa \times \omega, 2)$, and let G be $\mathbb{P}$-generic over M. Then $(2^{\aleph_0} \geq \aleph_\alpha)^{M[G]}$*

Proof. Since $(\mathbb{P} \text{ is ccc})^M$ by Lemma III.3.7 applied in M, we have also $\kappa = (\aleph_\alpha)^{M[G]}$ by Theorem IV.3.4. Furthermore, as explained in Section IV.1, $M[G]$ contains a κ–sequence $\langle h_\alpha : \alpha < \kappa \rangle$ of distinct elements of 2^ω, so that the statement $2^{\aleph_0} \geq \kappa = \aleph_\alpha$ is true in $M[G]$. □

But now, we wish to prove that $2^{\aleph_0}$ can be *exactly* $\aleph_\alpha$ in $M[G]$ whenever this does not contradict König's theorem. To do this, we need an *upper* bound to $(2^{\aleph_0})^{M[G]}$ to go along with the *lower* bound given by Lemma IV.3.7. This upper bound is produced by re-doing the proof of the Power Set Axiom in $M[G]$ (Theorem IV.2.27), replacing the Q there by a smaller set of "*nice*" names.

Definition IV.3.8 *For $\tau \in V^{\mathbb{P}}$, a* nice name for a subset of τ *is a name of the form $\bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \mathrm{dom}(\tau)\}$, where each A_σ is an antichain in $\mathbb{P}$.*

The names in Q from the proof of Theorem IV.2.27 had a similar form, but did not require A_σ to be an antichain.

Lemma IV.3.9 *Fix $\tau \in V^{\mathbb{P}}$, let $\kappa = |\mathbb{P}|$ and $\lambda = |\mathrm{dom}(\tau)|$, and assume that $\mathbb{P}$ is ccc and that κ and λ are infinite. Then there are no more than κ^λ nice names for subsets of τ.*

Proof. By the ccc, there are no more than $\kappa^{\aleph_0}$ antichains, and hence no more than $(\kappa^{\aleph_0})^\lambda = \kappa^{\aleph_0 \cdot \lambda} = \kappa^\lambda$ nice names. □

Definition IV.3.8 and Lemma IV.3.9 did not mention forcing or models, but now we plan to relativize them to M to bound cardinal exponentiation in $M[G]$. First, we show that every subset if τ is named by a nice name.

Lemma IV.3.10 *If $\mathbb{P} \in M$ and $\tau, \mu \in M^{\mathbb{P}}$, then there is a nice name $\vartheta \in M^{\mathbb{P}}$ for a subset of τ such that $\mathbb{1} \Vdash (\mu \subseteq \tau \to \mu = \vartheta)$.*

Proof. Working entirely in M, let $\vartheta = \bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \mathrm{dom}(\tau)\}$, where each A_σ is chosen so that

1. A_σ is an antichain in $\mathbb{P}$.
2. For all $p \in A_\sigma$, $p \Vdash \sigma \in \mu$.
3. A_σ is maximal with respect to (1)(2).

This can be done in M by using AC in M and the Definability Lemma.

Now, let G be $\mathbb{P}$–generic over M. We must prove that $M[G] \models (\mu \subseteq \tau \to \mu = \vartheta)$, so assume that $\mu_G \subseteq \tau_G$ and we prove that $\mu_G = \vartheta_G$.

For $\vartheta_G \subseteq \mu_G$: Fix $a \in \vartheta_G$. Then $a = \sigma_G$, where $\langle \sigma, p \rangle \in \vartheta$ for some $p \in G$. By definition of ϑ, $p \Vdash \sigma \in \mu$, so $a \in \mu_G$.

For $\mu_G \subseteq \vartheta_G$: Suppose that $a \in \mu_G \backslash \vartheta_G$. Then $a \in \mu_G \subseteq \tau_G$, so $a = \sigma_G$ for some $\sigma \in \mathrm{dom}(\tau)$. $M[G] \models (\sigma \in \mu \wedge \sigma \notin \vartheta)$, so, by the Truth Lemma, fix $q \in G$ such that $q \Vdash (\sigma \in \mu \wedge \sigma \notin \vartheta)$. Then $q \Vdash \sigma \in \mu$ and $q \Vdash \sigma \notin \vartheta$, so $q \perp p$ for all $p \in A_\sigma$ (see Lemma IV.2.30(1) and note that $\langle \sigma, p \rangle \in \vartheta \to p \Vdash \sigma \in \vartheta$). But then q contradicts maximality of A_σ. □

We now have an upper bound:

Lemma IV.3.11 *Fix $\mathbb{P} \in M$. Assume that in M, $\mathbb{P}$ is ccc, κ, λ, δ are infinite cardinals, $\kappa = |\mathbb{P}|$, and $\delta = \kappa^\lambda$ (i.e., the preceding statements all hold relativized to M). Let G be $\mathbb{P}$-generic over M. Then $(2^\lambda \leq \delta)^{M[G]}$.*

Proof. Working in M: The name $\check{\lambda} = \{\langle \check{\xi}, \mathbb{1}\rangle : \xi < \lambda\}$ has size λ, so applying Lemma IV.3.9, list all the nice names for subsets of $\check{\lambda}$ as $\{\vartheta_\zeta : \zeta < \delta\}$. Then, let $\mathring{f}$ be the name $\{\langle \mathrm{op}(\check{\zeta}, \theta_\zeta), \mathbb{1}\rangle : \zeta < \delta\}$.

Working in $M[G]$: $\mathring{f}_G$ is a function with domain δ and each $\mathring{f}_G(\zeta) = \mathrm{val}(\theta_\zeta, G)$. Furthermore, if s is any subset of λ in $M[G]$, then $s = \mu_G$ for some μ, and then, by Lemma IV.3.10, there is some ζ such that $\mathbb{1} \Vdash (\mu \subseteq \check{\lambda} \to \mu = \vartheta_\zeta)$, so that $\mathring{f}_G(\zeta) = s$. Thus, $\mathrm{ran}(\mathring{f}_G) \supseteq \mathcal{P}(\lambda)$ so that $2^\lambda \leq \delta$ (in $M[G]$). Note that (δ is a cardinal $)^{M[G]}$ by Theorem IV.3.4. □

Exercise IV.3.12 *Prove that actually $\mathrm{ran}(\mathring{f}_G) = \mathcal{P}(\lambda)$ in the proof of Lemma IV.3.11. But also, give a simple example to show that when ϑ is a nice name for a subset of τ, it need not always be true that $\vartheta_G \subseteq \tau_G$*

Hint. Your τ must contain some pair $\langle \sigma, p\rangle$ with $p \neq \mathbb{1}$. □

Combining the upper and lower bounds for the cardinal $\mathfrak{c} = 2^{\aleph_0}$:

Theorem IV.3.13 *Assume that $(\kappa = \aleph_\alpha \wedge \kappa^{\aleph_0} = \kappa)^M$. Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$, and let G be $\mathbb{P}$-generic over M. Then $(2^{\aleph_0} = \aleph_\alpha = \kappa)^{M[G]}$.*

Proof. Lemma IV.3.7 implies that $(2^{\aleph_0} \geq \aleph_\alpha = \kappa)^{M[G]}$, and Lemma IV.3.11 (with $\lambda = \aleph_0$ and $\delta = \kappa$) implies that $(2^{\aleph_0} \leq \kappa)^{M[G]}$. □

Informally, it follows that $\mathfrak{c}$ is consistently any cardinal of uncountable cofinality. To make this into a formal statement, we replace "any cardinal" by a specific cardinal. At the same time, we can get cardinal exponentiation to be minimal given the value of $\mathfrak{c}$:

Corollary IV.3.14 *Assume that there is a ctm for ZFC. Then one can find ctms for ZFC satisfying $\forall \lambda \geq \mathfrak{c}\,[2^\lambda = \lambda^+]$ plus each of the following statements:*

1. $\mathfrak{c} = \aleph_2$.
2. $\mathfrak{c} = \aleph_7$.
3. $\mathfrak{c} = \aleph_{\omega+1}$.
4. $\mathfrak{c} = \aleph_{\omega_1}$.
5. $\mathfrak{c} = \aleph_{\omega_1+\omega^3+\omega+1}$.
6. $\mathfrak{c} = \aleph_{\mathfrak{c}}$.

Proof. Let N be a ctm for *ZFC*, and let $M = L(o(M))$, which is a ctm for $ZFC + V = L$ (see Lemma II.6.27). Since $V = L \to GCH$ and *GCH* implies that $\kappa^{\aleph_0} = \kappa$ whenever $\mathrm{cf}(\kappa > \omega)$, we have $(\kappa^{\aleph_0} = \kappa)^M$ whenever $(\kappa = \aleph_\alpha)^M$ and α is any one of 2, 7, $\omega+1$, $(\omega_1)^M$, $(\omega_1)^M+\omega^3+\omega+1$. This, plus Theorem IV.3.13, will give us (1–5). For (6), work in M and choose any α such that $\alpha = \aleph_\alpha = \beth_\alpha \wedge \mathrm{cf}(\alpha) > \omega$ holds in M (see Exercise I.11.8).

For all six, to verify that $(2^\lambda = \lambda^+)^{M[G]}$ when $\lambda = (\aleph_\beta)^M$ and $\beta \geq \alpha$: $(2^\lambda \geq \lambda^+)^{M[G]}$ is clear, and $(2^\lambda \leq \lambda^+)^{M[G]}$ follows from Lemma IV.3.11 applied with $(\delta = \kappa^\lambda = \lambda^+)^M$ (using GCH^M). □

Exercise IV.3.15 *Show that the model constructed in the proof of Corollary IV.3.14 also satisfies the statement that for all cardinals λ with $\aleph_0 < \lambda < \mathfrak{c}$, 2^λ is $\mathfrak{c}$ if $\mathrm{cf}(\mathfrak{c}) > \lambda$ and $\mathfrak{c}^+$ if $\mathrm{cf}(\mathfrak{c}) \geq \lambda$.*

To restate these and similar facts about ctms as relative consistency results (e.g., $\mathrm{Con}(ZFC) \rightarrow \mathrm{Con}(ZFC + \mathfrak{c} = \aleph_7)$), see Section IV.5. For now, we state a few more results about ctms. First, *GCH* does not imply $V = L$:

Lemma IV.3.16 *If there is a ctm for ZFC, then there is a ctm for ZFC + GCH + $V \neq L$.*

Proof. As in the proof of Corollary IV.3.14, let M be any ctm for $ZFC + GCH$. Fix $\mathbb{P} \in M$ such that $\mathbb{P}$ is atomless and $(\mathbb{P} \text{ is ccc})^M$ and $(|\mathbb{P}| \leq \aleph_1)^M$. For example, $\mathbb{P}$ can be $\mathrm{Fn}(\omega, 2)$ or $\mathrm{Fn}((\omega_1)^M, 2)$. Let G be $\mathbb{P}$–generic over M.

Since $\mathbb{P}$ is atomless, $G \notin M$ (Lemma IV.2.4), so $M \subsetneq M[G]$. But $o(M[G]) = o(M)$, so $M \models V \neq L$ by Lemma II.6.16.

To prove $(GCH)^{M[G]}$, use Lemma IV.3.11 with $(\kappa = |\mathbb{P}| \in \{\aleph_0, \aleph_1\})^M$ and λ any infinite cardinal of M and $(\delta = \kappa^\lambda = \lambda^+)^M$ to get $(2^\lambda \leq \lambda^+)^{M[G]}$. □

We can also get models in which $\mathfrak{c}$ is weakly inaccessible:

Lemma IV.3.17 *The following are equivalent:*

1. *There is a ctm $M \models ZFC + GCH + \exists \kappa\, [\kappa$ is strongly inaccessible$]$.*
2. *There is a ctm $M \models ZFC + \exists \kappa\, [\kappa$ is weakly inaccessible$]$.*
3. *There is a ctm $M \models ZFC + \mathfrak{c}$ is weakly inaccessible.*
4. *There is a ctm $M \models ZFC + \exists \kappa < \mathfrak{c}\, [\kappa$ is weakly inaccessible$]$.*

Proof. (1) $\rightarrow$ (2) is trivial. To prove (1) $\rightarrow$ (3) and (1) $\rightarrow$ (4), start with $(\kappa \text{ strongly inaccessible})^M$; *GCH* is irrelevant here. Note that as long as $(\mathbb{P} \text{ is ccc})^M$, we shall have $(\kappa \text{ is weakly inaccessible})^{M[G]}$ because cofinalities and cardinals are preserved. To get (3), let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$, and to get (4), let $\mathbb{P} = \mathrm{Fn}((\kappa^+)^M \times \omega, 2)$; and proceed as in the proof of Corollary IV.3.14.

(3) $\rightarrow$ (2) and (4) $\rightarrow$ (2) are trivial. To get (2) $\rightarrow$ (1), note that if $(\kappa \text{ weakly inaccessible})^M$, then $(\kappa \text{ strongly inaccessible})^{L(o(M))}$; see Lemmas II.6.25 and II.6.27. □

These results on ctms lead to relative consistency proofs; here we have that the following five theories are proof-theoretically equivalent (Definition II.1.2):

1. $ZFC + GCH + \exists \kappa\, [\kappa$ is strongly inaccessible$]$.
2. $ZFC + \exists \kappa\, [\kappa$ is weakly inaccessible$]$.
3. $ZFC + \mathfrak{c}$ is weakly inaccessible.
4. $ZFC + \exists \kappa < \mathfrak{c}\, [\kappa$ is weakly inaccessible$]$.
5. $ZFC +$ there are no Kurepa trees .

that $\mathrm{Con}(1) \leftrightarrow \mathrm{Con}(2) \leftrightarrow \mathrm{Con}(3) \leftrightarrow \mathrm{Con}(4) \leftrightarrow \mathrm{Con}(5)$ is provable in the finitistic metatheory. For the first four, this follows from Lemma IV.3.17, as will be explained in Section IV.5. For $\mathrm{Con}(5) \rightarrow \mathrm{Con}(1)$, use the fact that $\neg KH$ implies

that there is an inaccessible cardinal in the model L (Exercise III.7.22). For Con(1) $\rightarrow$ Con(5), use the forcing result to be proved later (Theorem V.2.25) that from a ctm with a strong inaccessible we can build another ctm satisfying $\neg KH$. Each of these five theories is then strictly stronger proof-theoretically (Definition II.1.1) than *ZFC* alone, since from a strong inaccessible κ one can produce the model $R(\kappa) \models ZFC$.

Exercise IV.3.18 *Assume that $\mathbb{P}, J \in M$ and, in M, $\mathbb{P}$ is countable and J is a set of size $\aleph_1$. Let G be $\mathbb{P}$-generic over M. In $M[G]$, let E be an uncountable subset of J. Prove that there is an $E' \in M$ such that $E' \subseteq E$ and E' is uncountable in M. Also, give a counter-example to the existence of such an E' when $\mathbb{P} = \mathrm{Fn}(J, 2)$.*

Hint. $E' = \{j \in J : p \Vdash j \in \mathring{E}\}$ for some $p \in \mathbb{P}$, where $\mathring{E}$ is a name for E. □

IV.4 Embeddings of Posets

Let $\mathbb{P}_\kappa = \mathrm{Fn}(\kappa \times \omega, 2)$. Informally, as κ gets bigger, the $\mathbb{P}_\kappa$-generic extension gets bigger also; for example, as we just saw in Section IV.3, the size of $\mathfrak{c}$ in $M[G]$ increases. We can make this informal remark formal by observing that $\kappa \leq \kappa' \rightarrow \mathbb{P}_\kappa \subseteq_{\mathsf{c}} \mathbb{P}_{\kappa'}$ (see Example III.3.64), and then proving a general result (Lemma IV.4.4 below) that $\mathbb{Q} \subseteq_{\mathsf{c}} \mathbb{P}$ implies that the $\mathbb{Q}$ extension is contained in the $\mathbb{P}$ extension. This Lemma IV.4.4 is the "generic extension" version of the fact that $MA_{\mathbb{P}}(\lambda) \rightarrow MA_{\mathbb{Q}}(\lambda)$ (Lemma III.3.68). The proof of that lemma used Theorem III.3.60, which gave a number of equivalences of $MA_{\mathbb{P}}(\lambda)$ in terms of maximal antichains and predense sets. So, before proving Lemma IV.4.4, we shall prove the "generic extension" version of Theorem III.3.60. Besides switching from *MA* to models, Theorem IV.4.1 is a result about properties of a specific G, while Theorem III.3.60 was a result about the existence of a G.

Theorem IV.4.1 *Let M be a transitive model of ZFC, let $\mathbb{P} \in M$ be a forcing poset, and fix $G \subseteq \mathbb{P}$. Now, consider the eight different statements about G that we get by letting $\mathcal{X}$ be one of*

1. dense set 2. dense open set 3. maximal antichain 4. predense set

and letting $\mathcal{Y}$ be one of

a. filter b. upward closed linked family

and asserting:

G has property $\mathcal{Y}$, and $G \cap E \neq \emptyset$ for all $E \subseteq \mathbb{P}$ such that $E \in M$ and E has property $\mathcal{X}$.

Then these eight statements are all equivalent.

So, (1*a*) is the usual definition of "$\mathbb{P}$-generic over M". The other statements are sometimes useful in proving general statements about generic extensions, such as Lemma IV.4.2 below.

Proof. Note that each of the four possibilities for $\mathcal{X}$ is absolute for M, as are related notions, such as $E{\downarrow}$. We now almost copy the proof of Theorem III.3.60. We first verify the vertical and horizontal arrows in the diagram there. In proving $(3y) \to (1y)$, use $M \models AC$ to conclude that whenever $E \in M$, there is an $A \in M$ such that $A \subseteq E$ and A is is maximal among all antichains that are subsets of E; then, when E is dense, this A is a maximal antichain.

We finish by proving $(2a) \to (4a)$ and $(2b) \to (2a)$. The proof of $(2a) \to (4a)$ is again the same, but the proof of $(2b) \to (2a)$ is much simpler:

Assume that G is linked and upward closed, and assume that G meets all dense open D such that $D \in M$. We need to show that G is a filter, so fix $p, q \in G$ and we need to produce an $r \in G$ such that $r \leq p \wedge r \leq q$. Let $D = \{r \in \mathbb{P} : (r \perp p) \vee (r \perp q) \vee (r \leq p \wedge r \leq q)\}$ and note that D is dense open and $D \in M$, so fix $r \in G \cap D$. Then each of $r \perp p$ and $r \perp q$ contradicts G being linked, so $r \leq p \wedge r \leq q$. □

Next, we verify that whenever $\mathbb{Q} \subseteq_c \mathbb{P}$ and G is $\mathbb{P}$–generic, $G \cap \mathbb{Q}$ is $\mathbb{Q}$–generic. But as with Lemma III.3.68, this generalizes to the situation where $i : \mathbb{Q} \to \mathbb{P}$ is a complete embedding (Definition III.3.65). Then $\mathbb{Q} \subseteq_c \mathbb{P}$ is the special case where the inclusion (identity) map is a complete embedding (see Lemma III.3.66).

Lemma IV.4.2 *Let M be a transitive model of ZFC, with forcing posets $\mathbb{Q}, \mathbb{P} \in M$, and assume that $i : \mathbb{Q} \to \mathbb{P}$ is a complete embedding, with $i \in M$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$–generic over M. Then $i^{-1}(G)$ is $\mathbb{Q}$–generic over M.*

Proof. First note that the notion of i *being* a complete embedding is absolute for M, since it can be expressed by quantifying only over elements of $\mathbb{Q}$ and $\mathbb{P}$; see Lemma III.3.72. Now, the fact that $i^{-1}(G)$ is linked and upward closed is immediate from the definition of "complete embedding", so by Theorem IV.4.1, it is sufficient to show that $i^{-1}(G) \cap A \neq \emptyset$ whenever $A \subseteq \mathbb{Q}$ and $A \in M$ and A is a maximal antichain. But then, $i(A)$ is a maximal antichain in $\mathbb{P}$, and $i(A) \in M$ because $i \in M$, so $G \cap i(A) \neq \emptyset$, and hence $i^{-1}(G) \cap A \neq \emptyset$. □

If $H = i^{-1}(G)$, then $M[H] \subseteq M[G]$. This is clear from Lemma IV.2.19 applied to $M[H]$ with $N = M[G]$, since $M \subseteq N$ and $H \in N$ (since $i, G \in N$). However, it is often useful to know that this inclusion of $M[H]$ into $M[G]$ is induced by the natural transcription of names:

Definition IV.4.3 *Let $\mathbb{Q}, \mathbb{P}$ be forcing posets, with $i : \mathbb{Q} \to \mathbb{P}$. Define $i_* : V^{\mathbb{Q}} \to V^{\mathbb{P}}$ by: $i_*(\tau) = \{\langle i_*(\sigma), i(q)\rangle : \langle \sigma, q\rangle \in \tau\}$.*

This is defined in V, using all names, but we plan to relativize it to models of set theory. The definition was made for an arbitrary $i : \mathbb{Q} \to \mathbb{P}$; clearly, it is a recursion, and it is easy to prove by induction that $i_*(\tau)$ really is a $\mathbb{P}$ name; but all non-trivial results will require i to be a complete embedding. Note that in the case that $\mathbb{Q} \subseteq \mathbb{P}$ and i is inclusion, $V^{\mathbb{Q}} \subseteq V^{\mathbb{P}}$ and each $i_*(\tau) = \tau$. Also note that i_* is absolute for transitive models.

Lemma IV.4.4 *Let M be a transitive model of ZFC, with forcing posets $\mathbb{Q}, \mathbb{P} \in M$, and assume that $i : \mathbb{Q} \to \mathbb{P}$ is a complete embedding, with $i \in M$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over M, and let $H = i^{-1}(G)$. Then*

1. *For each $\tau \in M^{\mathbb{Q}}$, $i_*(\tau) \in M^{\mathbb{P}}$ and $\mathrm{val}(i_*(\tau), G) = \mathrm{val}(\tau, H)$.*
2. *$M[H] \subseteq M[G]$.*

Proof. (1) is an easy induction, and does not actually require i to be a complete embedding, although that was needed to prove that H is generic. (2) is immediate from (1). □

In the special case of a dense embedding $i : \mathbb{Q} \to \mathbb{P}$, we shall see (Lemma IV.4.7) that $M[H] = M[G]$; furthermore, every $\mathbb{Q}$-generic H determines a $\mathbb{P}$-generic $G = \tilde{i}(H)$ such that $H = i^{-1}(G)$. So in this case, $\mathbb{Q}$ and $\mathbb{P}$ are equivalent for forcing; this is the "generic extension" analog of the fact that $MA_{\mathbb{P}}(\lambda) \leftrightarrow MA_{\mathbb{Q}}(\lambda)$ (Lemma III.3.68). See the end of this section for a more formal discussion of what "equivalent for forcing" means.

Definition IV.4.5 *If $i : \mathbb{Q} \to \mathbb{P}$ and $H \subseteq \mathbb{Q}$, let $\tilde{i}(H) = \{p \in \mathbb{P} : \exists q \in H \; i(q) \leq p\}$.*

Lemma IV.4.6 *If G_1, G_2 are both $\mathbb{P}$-generic over M and $G_1 \subseteq G_2$ then $G_1 = G_2$.*

Proof. Fix $p \in G_2$, and let $D = \{r \in \mathbb{P} : r \leq p \vee r \perp p\}$. D is dense and in M, so fix $r \in G_1 \cap D$. Then $r \perp p$ would contradict G_2 being a filter, so $r \leq p$, which implies that $p \in G_1$. □

Lemma IV.4.7 *Let M be a transitive model of ZFC, with $\mathbb{Q}, \mathbb{P}, i \in M$, and assume that $i : \mathbb{Q} \to \mathbb{P}$ is a dense embedding. Then*

1. *If $H \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over M and $G = \tilde{i}(H)$ then G is $\mathbb{P}$-generic over M and $H = i^{-1}(G)$.*
2. *If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over M and $H = i^{-1}(G)$, then H is $\mathbb{Q}$-generic over M and $G = \tilde{i}(H)$.*
3. *In (1) and in (2), $M[H] = M[G]$.*
4. *$q \Vdash_{\mathbb{Q}} \varphi(\tau_1, \ldots, \tau_n)$ iff $i(q) \Vdash_{\mathbb{P}} \varphi(i_*(\tau_1), \ldots, i_*(\tau_n))$, where $\varphi(x_1, \ldots, x_n)$ is a formula of $\mathcal{L} = \{\in\}$, $q \in \mathbb{Q}$, and $\tau_1, \ldots, \tau_n \in M^{\mathbb{Q}}$.*

Proof. For (1): It is clear from the definition that G is a filter. To prove that G is generic, it is enough, by Theorem IV.4.1, to show that $G \cap D \neq \emptyset$ whenever $D \in M$ is dense and open in $\mathbb{P}$. Since i is a dense embedding, $i^{-1}(D)$ is predense and open, and hence dense in $\mathbb{Q}$, and if $q \in H \cap i^{-1}(D)$ then $i(q) \in G \cap D$. Now, $H \subseteq i^{-1}(G)$ is clear from the definitions, and both of them are generic by Lemma IV.4.2, so $H = i^{-1}(G)$ follows from Lemma IV.4.6.

For (2): H is generic by Lemma IV.4.2, so $\tilde{i}(H)$ is generic by (1), and $G \subseteq \tilde{i}(H)$ is clear from the definitions, so $G = \tilde{i}(H)$ by Lemma IV.4.6.

For (3): $M[H] \subseteq M[G]$ by Lemma IV.4.4. Also, $G \in M[H]$ by because $H, i \in M[H]$, so $M[G] \subseteq M[H]$ by Lemma IV.2.19 applied with $N = M[H]$.

For (4): First note that for all generic G, H as in (1)(2), $q \in H \leftrightarrow i(q) \in G$; the $\leftarrow$ holds because $H = i^{-1}(G)$ and the $\rightarrow$ holds because $G = \tilde{i}(H)$. Also, $M[H] \models \varphi[(\tau_1)_H, \ldots, (\tau_n)_H]$ iff $M[G] \models \varphi[(i_*(\tau_1))_G, \ldots, (i_*(\tau_n))_G]$ because $M[H] = M[G]$ and each $(\tau_i)_H = (i_*(\tau_i))_G$. The result now follows by the definition of $\Vdash$. □

A special case of a dense embedding is an isomorphism $i : \mathbb{Q} \xrightarrow[\text{onto}]{1\text{–}1} \mathbb{P}$; of course, this means an isomorphism of the *structures* $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}}), (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ (see Definition I.15.18). In that case, $\tilde{i}(H)$ is just $i``H$ when H is a filter. So, not surprisingly, isomorphic posets are equivalent for forcing, *provided* that the isomorphism is in M. If the isomorphism is not in M, the isomorphism yields nothing; for example, if M is a ctm and $\omega \leq \kappa, \lambda < o(M)$ then $\mathrm{Fn}(\kappa \times \omega, 2) \cong \mathrm{Fn}(\lambda \times \omega, 2)$, since in V, $|\kappa| = |\lambda| = \omega$, but we have just seen that the two posets need not be equivalent for forcing, since the generic extensions may have different values for $\mathfrak{c}$ when κ, λ are different cardinals of M.

Now consider $\mathrm{Fn}(\kappa \times \omega, 2) \cong \mathrm{Fn}(\kappa, 2)$. Here, there *is* an isomorphism in M, since $(|\kappa \times \omega| = |\kappa|)^M$, so these do yield equivalent extensions. We used $\mathrm{Fn}(\kappa \times \omega, 2)$ to make it slightly simpler to describe κ different elements $h_\alpha \in {}^{\omega}2$ (see Section IV.1); although with $\mathrm{Fn}(\kappa, 2)$, we could have said $h_\alpha(n) = f_G(\omega \cdot \alpha + n)$. In the literature, both $\mathrm{Fn}(\kappa \times \omega, 2)$ and $\mathrm{Fn}(\kappa, 2)$ are referred to as *Cohen forcing*, since in his original papers [14, 15] on $\neg CH$, he used essentially $\mathrm{Fn}(\kappa \times \omega, 2)$ (although he did not present his results in terms of posets).

There are often dense embeddings between non-isomorphic posets. Example III.3.69 points out that the poset $\mathbb{T} = \omega^{<\omega}$ is dense in $\mathrm{Fn}(\omega, \omega)$. So, these two are equivalent for forcing, although they are clearly not isomorphic. Since $\mathbb{T}$ densely embeds into any countable atomless poset (see Exercise III.3.70), it follows that any two atomless posets that are countable in M are equivalent for forcing over M.

Now consider $\mathrm{Fn}(\kappa \times \omega, \omega) \cong \mathrm{Fn}(\kappa, \omega)$. These are also referred to as Cohen forcing; although they are not isomorphic to $\mathrm{Fn}(\kappa, 2)$, they are equivalent for forcing by the next exercise applied within M:

Exercise IV.4.8 *For any infinite I: $\mathrm{Fn}(I, \omega) \not\cong \mathrm{Fn}(I, 2)$, but the finite support product (Definition III.3.40) of $|I|$ copies of $\omega^{<\omega}$ embeds densely into both of them.*

We sometimes use $\mathrm{Fn}(\kappa \times \omega, \omega)$ rather than $\mathrm{Fn}(\kappa \times \omega, 2)$, since that makes it easier to describe a "generic" sequence of functions from ω to ω; namely, $h_\alpha(n) = f(\alpha, n)$. See Exercise IV.4.13 for an example.

The values of all the cardinals discussed in Section III.1 are known in the Cohen forcing extensions of models of GCH; they are all either $\mathfrak{c}$ or $\aleph_1$. For more on this, see Section V.2. The next two lemmas show that $\mathfrak{b} = \aleph_1$.

Lemma IV.4.9 *Let M be a ctm of ZFC and fix $\mathbb{Q} \in M$ such that $(|\mathbb{Q}| \leq \aleph_0)^M$. Let G be $\mathbb{Q}$-generic over M. Then there is no $h \in \omega^\omega \cap M[G]$ such that $f \leq^* h$ for all f in $\omega^\omega \cap M$.*

Proof. Suppose that there were such an h. Fix $\mathring{h} \in M^{\mathbb{Q}}$ such that $\mathring{h}_G = h$. Let $W = \omega^\omega \cap M = (\omega^\omega)^M$. Then $M[G] \models \forall x \in \check{W}[x \leq^* \mathring{h}]$, so fix $p \in G$ such that $p \Vdash \mathring{h} : \omega \to \omega \wedge \forall x \in \check{W}[x \leq^* \mathring{h}]$.

Now, working in M, list $p{\downarrow}$ as $\{r_j : j \in \omega\}$. For each n, let $E_n = \{\ell \in \omega : \exists j < n\, [r_j \Vdash \check{\ell} \leq \mathring{h}(n)]\}$. Note that E_n is finite, and define $f(n) = \max(E_n)+1$; so $f \in \omega^\omega$. This is all done in M using the Definability Lemma, so $f \in W$.

Now, $p \Vdash \forall x \in \check{W}[x \leq^* \mathring{h}]$ and $\mathbb{1} \Vdash \check{f} \in \check{W}$, so, by Lemma IV.2.32 and the definition of $\leq^*$, $p \Vdash \exists m \forall n \geq m\, [\check{f}(n) \leq^* \mathring{h}(n)]$. Then, by Lemma IV.2.32 again, we can fix $m \in \omega$ and a $q \leq p$ such that $q \Vdash \forall n > \check{m}\, [\check{f}(n) \leq^* \mathring{h}(n)]$. Say $q = r_j$. Fix n with $n > r_j$ and $n > m$, and let $\ell = f(n)$. Then $r_j \Vdash \check{\ell} \leq \mathring{h}(n)$, so $f(n) > \ell$ by the definition of f, which is a contradiction. □

Lemma IV.4.10 *Let M be a ctm of ZFC and let $\mathbb{P} = \mathrm{Fn}(I, J)$, where $I, J \in M$ and $(|J| \leq \aleph_0)^M$. Let G be $\mathbb{P}$-generic over M. Then there is no $h \in \omega^\omega \cap M[G]$ such that $f \leq^* h$ for all $f \in \omega^\omega \cap M$. In particular, if $M \models CH$ then $M[G] \models \mathfrak{b} = \aleph_1$.*

Proof. Suppose that there were such an h. Fix a nice $\mathbb{P}$-name (Definition IV.3.8) $\mathring{h}$ for a subset of $(\omega \times \omega)^\check{}$ such that $\mathring{h}_G = h$.

Working in M: the antichains in the name $\mathring{h}$ give us a countable set $S \subseteq \mathbb{P}$. The forcing conditions here are finite functions, so $K := \bigcup\{\mathrm{dom}(q) : q \in S\}$ is a countable subset of I, so if $\mathbb{Q} = \mathrm{Fn}(K, J)$, then $\mathring{h}$ is also a $\mathbb{Q}$-name.

Since $\mathbb{Q} \subseteq_c \mathbb{P}$ (Example III.3.64), we can apply Lemma IV.4.4 to the inclusion of $\mathbb{Q}$ into $\mathbb{P}$. So, if $H = G \cap \mathbb{Q}$, then H is $\mathbb{Q}$-generic over M, and we have $M \subseteq M[H] \subseteq M[G]$ with $h = \mathring{h}_G = \mathring{h}_H \in M[H]$ and $f \leq^* h$ for all f in $\omega^\omega \cap M$. But this contradicts Lemma IV.4.9.

The "in particular" follows because if $M \models CH$, then in $M[G]$, we have $W = \omega^\omega \cap M$, which is an unbounded family of size $\aleph_1$. □

Note the "check" in the expression $(\omega \times \omega)^\check{}$. For typographical reasons, we write, e.g., $\check{7}$ but $65537^\check{}$. In the current literature, the "checks" are usually omitted anyway, leaving it to the reader to fill them in; see Section IV.6.

By a somewhat harder argument, we can get $\mathfrak{a} = \aleph_1$ in the above $M[G]$; see Exercise IV.4.12. This implies $\mathfrak{b} = \aleph_1$, since always $\mathfrak{b} \leq \mathfrak{a}$ (Lemma III.1.19).

Now, suppose that we want a model where $\mathfrak{b} > \aleph_1$. As a first step, we can describe a ccc forcing such that, unlike in Lemma IV.4.10, there really is an $h \in \omega^\omega \cap M[G]$ such that $f \leq^* h$ for all $f \in \omega^\omega \cap M$. A well-known way of doing this is to use the *dominating functions order* (*dfo*) (Exercise IV.4.11). This is only a first step because we want a model in which *every* set of size $\aleph_1$ is dominated, not just the set $\omega^\omega \cap M[G]$. We can get such a model by iterating

forcing ω_2 times with the dfo, using the methods of Chapter V, producing an ascending ω_2–chain of models.

The dfo can also be used to prove that $\mathfrak{m} \leq \mathfrak{b}$. We did not do that in this book because we instead proved $\mathfrak{m} \leq \mathfrak{p}$ (Lemma III.3.22) by using a somewhat different poset, and we proved $\mathfrak{p} \leq \mathfrak{b}$ (Lemma III.1.24) without using posets at all. In Chapter V, we shall use iterated forcing to produce models of $MA+\neg CH$; so they will satisfy $\mathfrak{c} = \mathfrak{b} = \mathfrak{p} = \mathfrak{m} > \aleph_1$. Again, we have no need for the dfo. However, the dfo is still of some interest in producing models of $\neg CH$ where full MA does not hold. For example, one can iterate the dfo to produce models where $\mathfrak{c} = \mathfrak{b} = \mathfrak{d} > \aleph_1$ but $\mathfrak{p} = \mathfrak{m} = \aleph_1$ (Baumgartner and Dordal [6]).

Exercise IV.4.11 *Fix an infinite $\mathcal{E} \subseteq \omega^\omega$. We define the* dominating functions order $\mathbb{P}$ *obtained from $\mathcal{E}$. The intent is to "force" an h such that $f \leq^* h$ for all $h \in \mathcal{E}$. The elements of $\mathbb{P}$ are pairs $p = (s_p, \mathcal{Y}_p)$ such that $s_p \in \mathrm{Fn}(\omega, \omega)$ and $\mathcal{Y}_p \in [\mathcal{E}]^{<\omega}$. Informally, s_p is a finite approximation to h and $\mathcal{Y}_p$ is a* promise *that for all $f \in \mathcal{Y}_p$, $h(n) > f(n)$ for all $n \notin \mathrm{dom}(s_p)$. Formally, in analogy with Lemma III.3.22, define $q \leq p$ iff $s_q \supseteq s_p$, $\mathcal{Y}_q \supseteq \mathcal{Y}_p$, and, for all $f \in \mathcal{Y}_p$, $s_q(n) > f(n)$ for all $n \in \mathrm{dom}(s_q) \backslash \mathrm{dom}(s_p)$.*

Then prove that $\mathbb{P}$ is σ–centered, and describe a family of $|\mathcal{E}|$ dense sets such that whenever G is a filter meeting all of them and $h = \bigcup\{s_p : p \in G\}$, then $h \in \omega^\omega$ and $f \leq^* h$ for all $h \in \mathcal{E}$.

We could apply this to a ctm M by letting $\mathcal{E} = \omega^\omega \cap M$, so that $\mathbb{P} \in M$. Then, if G is generic, $M[G]$ contains an h dominating $\omega^\omega \cap M$.

Exercise IV.4.12 *Let M be a ctm of ZFC + CH. Construct an $\mathcal{A} \in M$ such that $\mathcal{A} \subseteq [\omega]^\omega$ and $(\mathcal{A}$ is a mad family$)^M$ and $(|\mathcal{A}| = \aleph_1)^M$ and $\mathcal{A}$ remains mad in all Cohen forcing extensions, so that for all such extensions, $M[G] \models \mathfrak{a} = \aleph_1$. Here, "Cohen forcing" means $\mathrm{Fn}(I, J)$, where $I, J \in M$ and $(2 \leq |J| \leq \aleph_0)^M$.*

Hint. Applying the method used in proving Lemma IV.4.10, it is enough to do this when $(|I| \leq \aleph_0)^M$. Then, by the method used for Exercise III.3.70, it is enough to do this when G is $\mathbb{T}$–generic over M, where $\mathbb{T} = \omega^{<\omega}$.

Working completely in M: $\mathcal{A} = \{A_\xi : \xi < \omega_1\} \subset [\omega]^\omega$. The A_n, for $n < \omega$ are arbitrary disjoint infinite sets. Let $\{(\tau_\xi, p_\xi) : \omega \leq \xi < \omega_1\}$ list all pairs (τ, p) such that τ is a nice name for a subset of $\check{\omega}$ and $p \in \mathbb{T}$ and $p \Vdash |\tau_\xi| = \aleph_0$. For $\omega \leq \xi < \omega_1$, recursively choose A_ξ so that $\forall \eta < \xi\, [A_\xi \perp A_\eta]$ *and* if $\forall \eta < \xi\, [p_\xi \Vdash \tau_\xi \perp \check{A}_\eta]$ then

$$\forall m \in \omega\, \forall q \leq p_\xi\, \exists r \leq q\, \exists n > m\, [n \in A_\xi \wedge r \Vdash \check{n} \in \tau_\xi] \ .$$

This is possible because $\mathbb{T}$ and ξ are both countable. □

Lemma IV.4.10 and Exercise IV.4.12 introduce one half of iterated forcing, which, in the special case of Cohen extensions, shows that if $K \subseteq I$, then $M[H] \subseteq M[G]$, where G is $\mathrm{Fn}(I, J)$–generic over M and $H = G \cap \mathrm{Fn}(K, J)$.

The second half, discussed in Chapter V, shows that $M[G] = (M[H])[H']$, where H' is $\mathrm{Fn}(I\backslash K, J)$–generic over $M[H]$. The next exercise is a bit tricky now, but will become very easy using the second half; see Lemma V.2.19.

Exercise IV.4.13 *Let M, I, J be as in Exercise IV.4.12, and assume also that $(|I|^{\aleph_0} = |I|)^M$. Prove that $M[G] \models \mathfrak{d} = \mathfrak{c}$.*

Hint. Say $(|I| = \kappa)^M$. Use $\mathrm{Fn}(\kappa \times \omega, \omega)$, so in $M[G]$ we have $f = \bigcup G$ and then $h_\alpha \in \omega^\omega$ for $\alpha < \kappa$, where $h_\alpha(n) = f(\alpha, n)$. Prove that in $M[G]$, no subfamily of ω^ω of size less than κ dominates all the h_α. □

We next pursue the following observation. In our specific forcing results so far, when we showed that $M[G]$ satisfied some set-theoretic statement φ, our G was an *arbitrary* generic filter; that is, $M[G] \models \varphi$ for *all* generic G, so $\mathbb{1} \Vdash \varphi$. For example, we stated the conclusion of Lemma IV.3.16 as "there is a ctm for $ZFC + GCH + V \neq L$", but the proof actually showed, for appropriate $M, \mathbb{P}$, that $ZFC + GCH + V \neq L$ holds in $M[G]$ for all generic G, so that $\mathbb{1} \Vdash GCH$ and $\mathbb{1} \Vdash V \neq L$. This phenomenon illustrates a general *zero-one law*, which says that for some (not all!) posets $\mathbb{P}$, either $\mathbb{1} \Vdash \varphi$ or $\mathbb{1} \Vdash \neg\varphi$ whenever φ is a sentence in $\mathcal{L} = \{\in\}$ (that is, φ doesn't mention any names).

Definition IV.4.14 *For a poset $\mathbb{P}$, let $\mathrm{Aut}(\mathbb{P})$ be the set of isomorphisms from $\mathbb{P}$ onto $\mathbb{P}$. Call $\mathbb{P}$* weakly homogeneous *iff for all $p \in \mathbb{P}$, $\{i(p) : i \in \mathrm{Aut}(\mathbb{P})\}$ is predense in $\mathbb{P}$ (Definition III.3.58).*

For example, $\mathrm{Fn}(I, J)$ is weakly homogeneous, since automorphisms can permute I and also permute J, so given p, there cannot be a q such that $q \perp \{i(p) : i \in \mathrm{Aut}(\mathbb{P})\}$. It follows that the zero-one law applies to all the forcings we have discussed using various $\mathrm{Fn}(I, J)$.

Theorem IV.4.15 (zero-one law) *Let $\mathbb{P}$ be weakly homogeneous and φ a sentence in $\mathcal{L} = \{\in\}$. Then $\mathbb{1} \Vdash \varphi$ or $\mathbb{1} \Vdash \neg\varphi$.*

Proof. If $\mathbb{1} \not\Vdash \varphi$ and $\mathbb{1} \not\Vdash \neg\varphi$, then by Lemma IV.2.30(5), there are p, q such that $p \Vdash \neg\varphi$ and $q \Vdash \varphi$. Fix $i \in \mathrm{Aut}(\mathbb{P})$ such that $q \not\perp i(p)$. But $i(p) \Vdash \neg\varphi$ by Lemma IV.4.7(4), so $q \perp i(p)$, which is a contradiction. □

Exercise IV.4.16 *Give an explicit example showing that Theorem IV.4.15 can fail if we delete the assumption that $\mathbb{P}$ is weakly homogeneous.*

Hint. Form $\mathbb{P}$ be putting two posets $\mathbb{Q}, \mathbb{R}$ together, side by side, and adding a new $\mathbb{1}$ on top. □

Exercise IV.4.17 *Fix $\mathcal{E} \subseteq \omega^\omega$ such that $f \in \mathcal{E} \leftrightarrow f' \in \mathcal{E}$ whenever $f' =^* f$. Prove that the dominating functions order defined from $\mathcal{E}$ as in Exercise IV.4.11 is weakly homogeneous.*

We conclude this section with some remarks on forcing with complete boolean algebras. That is, ($\mathbb{B}$ is a complete boolean algebra)^M and $\mathbb{P} = \mathbb{B}\backslash\{\mathbb{0}\}$. The discussion parallels the discussion in Section III.4 of *MA* and complete boolean algebras, and is similarly not central to the theme of this book. On the one hand, posets of this form $\mathbb{B}\backslash\{\mathbb{0}\}$ are rather unusual and rarely occur naturally in *MA* and forcing applications. On the other hand, the posets $\mathbb{B}\backslash\{\mathbb{0}\}$ are completely general, since for every poset $\mathbb{P}$, there is a complete boolean algebra $\mathbb{B}$ (called the *completion* of $\mathbb{P}$) together with a dense embedding $i : \mathbb{P} \to \mathbb{B}\backslash\{\mathbb{0}\}$ (Lemma III.4.8); then, by Lemma IV.4.7, $\mathbb{P}$ and $\mathbb{B}\backslash\{\mathbb{0}\}$ are equivalent for forcing.

We shall see now that the abstract theory of forcing is simpler, and is easier to motivate, when restricted to complete boolean algebras. This suggests an alternate approach to the exposition of forcing. One can develop the general theory of $M[G]$ *only* in the case of forcing with complete boolean algebras of M. Then, when we need to force with a specific poset $\mathbb{P}$, we may, in M, form the completion $\mathbb{B}$ of $\mathbb{P}$ and use $\mathbb{B}$ to form the generic extension. This is the approach taken in Jech [49].

To simplify the notation: In the following, M always denotes a ctm for *ZFC*, and, if we don't say otherwise, $\mathbb{B} \in M$ and ($\mathbb{B}$ is a complete boolean algebra)^M. Then $\mathbb{B}\backslash\{\mathbb{0}\}$ is the actual poset we use, but we shall refer to this as "forcing with $\mathbb{B}$". Note that filters G on $\mathbb{B}\backslash\{\mathbb{0}\}$ are also filters on $\mathbb{B}$ in the usual boolean algebra sense, and filters on a boolean algebra never contain $\mathbb{0}$. We shall say that $\mathbb{0} \Vdash \varphi$ is always true, since Definition IV.2.22, which would require "for all filters G on $\mathbb{B}$ such that $\mathbb{0} \in G$ and $\cdots\cdots$", is true vacuously. Note also that $p \perp q$ iff $p \wedge q = \mathbb{0}$

Forcing with complete boolean algebras can be viewed as constructing a *boolean valued model*; that is, a model of set theory where the truth values of sentences lie in $\mathbb{B}$, rather than in the two-element algebra $\{\mathbb{0}, \mathbb{1}\} = \{F, T\}$. First, we define the truth value:

Definition IV.4.18 *If φ is any sentence of $\mathcal{FL}_{\mathbb{B}} \cap M$, then the* truth value *of φ is* $[\![\varphi]\!] = \bigvee\{p \in \mathbb{B} : p \Vdash \varphi\}$.

This $[\![\varphi]\!]$ is well-defined because ($\mathbb{B}$ is complete)^M and the set whose supremum we are taking lies in M by the Definability Lemma. By the next lemma, $[\![\varphi]\!]$ is the largest forcing condition that forces φ:

Lemma IV.4.19 *$p \Vdash \varphi$ iff $p \leq [\![\varphi]\!]$.*

Proof. The $\to$ direction is clear from the definition, and for the $\leftarrow$, it is sufficient to prove that $[\![\varphi]\!] \Vdash \varphi$ (see Lemma IV.2.30(3)). But if $[\![\varphi]\!] \not\Vdash \varphi$, then fix a non-$\mathbb{0}$ $q \leq [\![\varphi]\!]$ such that $q \Vdash \neg\varphi$ (Lemma IV.2.30(5)). Then $q \perp p$ for all p such that $p \Vdash \varphi$; then, taking the $\bigvee$, $q \perp [\![\varphi]\!]$, contradicting $q \leq [\![\varphi]\!]$. □

We remark that when $\mathbb{B}\backslash\{\mathbb{0}\}$ is weakly homogeneous, the zero-one law, Theorem IV.4.15, implies that $[\![\varphi]\!] \in \{\mathbb{0}, \mathbb{1}\}$ whenever φ a sentence in $\mathcal{L} = \{\in\}$.

Exercise IV.4.20 *For sentences $\varphi, \psi \in \mathcal{FL}_{\mathbb{B}} \cap M$:*

4. $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$
5. $[\![\neg\varphi]\!] = [\![\varphi]\!]'$
6. $[\![\varphi \to \psi]\!] = [\![\varphi]\!]' \vee [\![\psi]\!]$
7. $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$
8. $[\![\varphi \leftrightarrow \psi]\!] = ([\![\varphi]\!]' \vee [\![\psi]\!]) \wedge ([\![\varphi]\!] \vee [\![\psi]\!]')$

This exercise is the boolean analog of (4–8) of Lemma IV.2.30, which described forcing for boolean combinations of sentences. It can be derived from that lemma, or proved directly by the method of proof of that lemma. Note that the boolean (4–8) are not only much simpler than the corresponding (4–8) of Lemma IV.2.30, but they also have a clearer motivation.

There is also the analog of Lemma IV.2.32, (for quantification). Again, this are much simpler in terms of complete boolean algebras:

Exercise IV.4.21 *For $\varphi(x) \in \mathcal{FL}_{\mathbb{B}} \cap M$, with no variable other than x free:*

1. $[\![\forall x\, \varphi(x)]\!] = \bigwedge\{[\![\varphi(\tau)]\!] : \tau \in M^{\mathbb{B}}\}$.
2. $[\![\exists x\, \varphi(x)]\!] = \bigvee\{[\![\varphi(\tau)]\!] : \tau \in M^{\mathbb{B}}\}$.

One can also simplify the definition of $\mathbb{B}$–name by making them functions, rather than relations; so we can replace Definition IV.2.5 by

Definition IV.4.22 *τ is a* boolean $\mathbb{B}$–name *iff τ is a function and all elements of* $\mathrm{dom}(\tau)$ *are boolean $\mathbb{B}$–names.*

Note that in forcing with arbitrary posets, each $\check{x}$ is already a function (and $\mathrm{ran}(\check{x}) = \{\mathbb{1}\}$) and our standard name $\mathring{G}$ for the generic filter is also a function: $\mathring{G} = \{\langle \check{p}, p\rangle : p \in \mathbb{P}\}$. But the natural name for the complement $\mathbb{P}\backslash G$ is $\{\langle \check{p}, q\rangle : q \perp p\}$, and there is no way in general to represent $\mathbb{P}\backslash G$ by a function. But, for boolean algebras, this can be replaced by the function $\{\langle \check{p}, p'\rangle : p \in \mathbb{P}\}$. We might say that for $p \in \mathbb{P}$, the truth value of "$p \in \mathring{G}$" is p and the truth value of "$p \notin \mathring{G}$" is p'.

Exercise IV.4.23 *If $\tau \in M^{\mathbb{P}}$ is a $\mathbb{B}$–name in the original sense of Definition IV.2.5, then there is a boolean $\mathbb{B}$–name τ' such that $\mathbb{1} \Vdash \tau = \tau'$.*

Hint. You have to construct τ' by recursion. Roughly, τ is a relation, and there may be some $\sigma \in \mathrm{dom}(\tau)$ such that $\langle \sigma, p_i\rangle \in \tau$ for all $i \in I$. But just replace these pairs by the one pair $\langle \sigma, \bigvee_i p_i\rangle$. □

Using these boolean names, the analog of Lemma IV.2.33 (forcing for atomic sentences) becomes:

Exercise IV.4.24 *For boolean names $\tau, \vartheta, \pi \in M^{\mathbb{B}}$:*

1. $[\![\tau = \vartheta]\!] = \bigwedge\{[\![\sigma \in \tau]\!] \leftrightarrow [\![\sigma \in \vartheta]\!] : \sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\vartheta)\}$.
2. $[\![\pi \in \tau]\!] = \bigvee\{[\![\pi = \sigma]\!] \wedge \tau(\sigma) : \sigma \in \mathrm{dom}(\tau)\}$.

In (1), $p \leftrightarrow q$ denotes $(p' \vee q) \wedge (p \vee q')$ in a boolean algebra.

For example, if $\mathrm{dom}(\tau) = \{\check{n} : n \in \omega\}$, then we may think of τ as a "boolean valued subset of ω". Here, it is easy to see that $[\![\check{n} \in \tau]\!] = \tau(\check{n})$; this is the boolean truth value that n is in τ.

In this book, it is intended that the above exercises be derived from what has already been proved about forcing with arbitrary posets. But, in expositions based on boolean valued models, the generic extension is defined only for complete boolean algebras. Then, instead of defining a $\Vdash^*$, the clauses in these exercises are used, in V, to define a $[\![\varphi]\!]^*$ for sentence φ of $\mathcal{FL}_{\mathbb{B}}$; for example, $[\![\varphi \vee \psi]\!]^* := [\![\varphi]\!]^* \vee [\![\psi]\!]^*$. This is again much simpler and clearer than the corresponding clauses in Definition IV.2.42, which greatly simplifies the proof of the corresponding Truth and Definability Lemmas, as well as the proof that $([\![\varphi]\!]^*)^M = [\![\varphi]\!]$ (corresponding to Lemma IV.2.45). Of course, you pay for this simplicity by having to embed your posets $\mathbb{P}$ into complete boolean algebras before doing any forcing.

Finally, we can ask what it means for two posets, $\mathbb{P}, \mathbb{Q}$ to be "forcing equivalent". This has more than one possible answer, but the following notion, suggested by boolean valued models, is sufficient for anything we do here:

Definition IV.4.25 *Let $\mathbb{P} \hookrightarrow_d \mathbb{R}$ abbreviate the statement that there exists a dense embedding from $\mathbb{P}$ to $\mathbb{R}$. Posets $\mathbb{P}, \mathbb{Q}$ are* strongly forcing equivalent *($\mathbb{P} \cong \mathbb{Q}$) iff there is a third poset $\mathbb{R}$ such that $\mathbb{P} \hookrightarrow_d \mathbb{R}$ and $\mathbb{Q} \hookrightarrow_d \mathbb{R}$.*

The following lemma is easy from the definition of boolean completion (obtained as in Lemma III.4.8); see also Exercise III.4.9:

Lemma IV.4.26 *For posets $\mathbb{P}, \mathbb{Q}$, (1) $\leftrightarrow$ (2) and (3) $\rightarrow$ (1)(2).*

1. *$\mathbb{P} \cong \mathbb{Q}$.*
2. *$\mathbb{P}, \mathbb{Q}$ have isomorphic completions.*
3. *There is a third poset $\mathbb{S}$ such that $\mathbb{S} \hookrightarrow_d \mathbb{P}$ and $\mathbb{S} \hookrightarrow_d \mathbb{Q}$.*

It follows from (2) that relation $\cong$ is an equivalence relation.

We have already used this notion of equivalence without officially defining it. It is clear from Lemma III.3.68 that $\mathbb{P} \cong \mathbb{Q}$ implies $MA_{\mathbb{P}}(\kappa) \leftrightarrow MA_{\mathbb{Q}}(\kappa)$ for all κ. Also, we saw that the various flavors of Cohen forcing are forcing equivalent; for example, $\mathrm{Fn}(I, \omega) \cong \mathrm{Fn}(I, 2)$ by Exercise IV.4.8, which verified condition (3). Also, Lemma IV.4.6 shows that if $(\mathbb{P} \cong \mathbb{Q})^M$, then we may view every $\mathbb{P}$–generic extension as a $\mathbb{Q}$–generic extension and vice versa.

Although condition (3) is a very common way of verifying $\mathbb{P} \cong \mathbb{Q}$, it does not follow from $\mathbb{P} \cong \mathbb{Q}$:

Exercise IV.4.27 *Assuming $\diamondsuit$, construct a well-pruned Suslin tree $\mathbb{T}$, together with $\mathbb{P}, \mathbb{Q} \subseteq \mathbb{T}$, such that $\mathbb{P}, \mathbb{Q}$ are dense in $\mathbb{T}$ (viewing $\mathbb{T}$ as a forcing poset) but (3) is false for $\mathbb{P}, \mathbb{Q}$. Note that $\mathbb{P} \cong \mathbb{Q}$ is obvious. One way to do this is to make sure that $p{\uparrow} \circledast q{\uparrow}$ (in the tree sense) is Suslin whenever p, q are incompatible elements of $\mathbb{T}$; see also Exercise III.7.10.*

There are also notions weaker than $\mathbb{P} \cong \mathbb{Q}$ that one might regard as forcing equivalence. Consider the following simple example: If $\mathbb{P}_\xi$ for $\xi < \gamma$ are forcing posets, then we define their sum $\mathbb{P} = \bigoplus_{\xi<\gamma} \mathbb{P}_\xi$ as follows: Assuming that the $\mathbb{P}_\xi$ are disjoint as sets, $\mathbb{P} = \{\mathbb{1}\} \cup \bigcup_\xi \mathbb{P}_\xi$. Order $\mathbb{P}$ by putting $\mathbb{1}$ at the top; directly below $\mathbb{1}$ are the $\mathbb{1}_\xi = \mathbb{1}_{\mathbb{P}_\xi}$, with $\{\mathbb{1}_\xi : \xi < \gamma\}$ forming a maximal antichain. Below $\mathbb{1}_\xi$, the elements of $\mathbb{P}_\xi$ are ordered in $\mathbb{P}$ as they were in $\mathbb{P}_\xi$.

If this $\mathbb{P} = \bigoplus_{\xi<\gamma} \mathbb{P}_\xi$ occurs within a ctm M and G is $\mathbb{P}$–generic over M, then G contains exactly one of the $\mathbb{1}_\xi$, and $G = \{\mathbb{1}\} \cup (G \cap \mathbb{P}_\xi)$, so $M[G]$ is really a $\mathbb{P}_\xi$–generic extension.

As a special case, if all the $\mathbb{P}_\xi$ are isomorphic, then $\mathbb{P} = \bigoplus_{\xi<\gamma} \mathbb{P}_\xi$ is the same for forcing as is $\mathbb{P}_0$; it is also easy to see that $MA_{\mathbb{P}}(\kappa) \leftrightarrow MA_{\mathbb{P}_0}(\kappa)$. But we cannot claim that $\mathbb{P} \cong \mathbb{P}_0$, since, for example, $\mathbb{P}_0$ may be ccc, while $\mathbb{P}$ will never be ccc if γ is uncountable.

IV.5 The Metamathematics of Forcing

If one views forcing purely as the study of ctms for *ZFC*, then the results of the previous sections explain how to use one ctm for *ZFC* to get another ctm satisfying *ZFC* plus some additional mathematical statements, such as $\mathfrak{c} = \aleph_7$ (see Corollary IV.3.14). In this section, we explain why these manipulations produce relative consistency results, such as $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \mathfrak{c} = \aleph_7)$, that are provable in the finitistic metatheory. To follow this explanation, one needs to pay close attention to the formal logic behind the set theory.

More generally than relative consistency results, we wish to assert, in the metatheory, analogously to remarks on pages 8 and 141, theorems such as:

Theorem IV.5.1 *If* $ZFC + \mathfrak{c} = \aleph_7 \vdash \Xi$, *where* Ξ *is any arithmetical statement, then* $ZFC \vdash \Xi$.

For example, Ξ could be $0 = 1$, which will then yield $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \mathfrak{c} = \aleph_7)$. But it could also be the Twin Prime Conjecture; so, if someone should establish this conjecture with the aid of the axiom $\mathfrak{c} = \aleph_7$, then one could also establish this conjecture in *ZFC*. Of course, the "$\mathfrak{c} = \aleph_7$" can be replaced by any one of the other statements for which we have provided models using forcing and/or constructibility. If you replace "$\mathfrak{c} = \aleph_7$" by "*GCH*", then this was discussed on page 141, but there the metamathematics was more transparent, since we have this very explicit model L, and if $GCH \to \Xi$, then Ξ is true in L and hence true by absoluteness. But using "$\mathfrak{c} = \aleph_7$", we don't have an explicit model; rather, we have some general methods for manipulating ctms for *ZFC*, and we know that working just in *ZFC*, we can't actually produce *any* such ctms.

The following two subsections discuss two different approaches to handling the formal logic.

IV.5.1 Countable Transitive Models

This, as outlined in Section IV.1, is our "official approach". Working in the metatheory we say: Suppose that we have a formal proof establishing that $ZFC + \mathfrak{c} = \aleph_7 \vdash \Xi$. This proof mentions only a finite set Ω of axioms for ZFC. Then, examine the proofs of Corollary IV.3.14 and the results leading up to it to see that for each finite $\Omega \subset ZFC$, one can find a larger finite $\Lambda \subset ZFC$ such that ZFC proves the implication, "if there is a ctm for Λ, then there is a ctm for $\Omega + \mathfrak{c} = \aleph_7$". The results to be examined include not only the theorems on forcing, but also the theorems on constructibilty, so we can first get our model of GCH. The results also include the absoluteness facts we used; that is, each time we needed a specific property to be absolute for transitive models of ZFC, we note that actually a finite subset of ZFC suffices. Once this is all done, we obtain a contradiction in ZFC by first producing a ctm M for Λ using the Reflection Theorem (see Corollary II.5.4), and then producing a ctm N for $\Omega + \mathfrak{c} = \aleph_7$, so that we have $N \models \Xi$, and hence Ξ is true by absoluteness.

This idea was described in Cohen's original work [15] on the independence of CH. A variant on this approach, followed by Shoenfield [89], seems a bit more elegant, since it avoids talking about unspecified subsets $\Omega, \Lambda \subset ZFC$. Let $\mathcal{L} = \{\in, C, F\}$, where C and F are constant symbols. Let Σ in $\mathcal{L}$ be the axioms of ZFC (written only in $\{\in\}$), plus the axiom that C is a transitive set and $F \subset \omega \times C$ is a bijection from ω onto C, plus the relativization φ^C for each ZFC axiom φ.

Then Σ is a conservative extension of ZFC; that is, if $\Sigma \vdash \psi$, where ψ is a sentence of $\{\in\}$, then $ZFC \vdash \psi$. To prove this (in the metatheory), first note that if $\Sigma \vdash \psi$, then $ZFC + \gamma(C, F) \vdash \psi$, where $\gamma(x, y)$ is some formula in $\mathcal{L}$ asserting that "x is a transitive set and y is a bijection from ω onto x and $\bigwedge_{i=1}^{n}((\alpha_i)^x)$ holds", where the α_i are all ZFC axiom. Then, an elementary proof theory argument yields $ZFC \vdash \exists x, y\, \gamma(x, y) \to \psi$, and the Reflection Theorem yields $ZFC \vdash \exists x, y\, \gamma(x, y)$, so $ZFC \vdash \psi$.

To prove Theorem IV.5.1:

First, working in Σ: One can explicitly define $M = L(o(C))$ and $\mathbb{P} = \mathrm{Fn}((\omega_7)^M \times \omega,\ 2)$, and then, using F, explicitly define a G such that G is $\mathbb{P}$–generic over M. Then, one can prove $\varphi^{M[G]}$ for each φ of $ZFC + \mathfrak{c} = \aleph_7$. We then have $\Xi^{M[G]}$, and hence Ξ by absoluteness. So, $\Sigma \vdash \Xi$

Hence, $ZFC \vdash \Xi$ because Σ is a conservative extension of ZFC.

Note that the set $\{\varphi^C : \varphi \in ZFC\}$ is an infinite set of axioms, described in the metatheory. Working in ZFC, we can define the notion "axiom of set theory" and then write the single sentence $\forall \varphi \in ZFC\, [C \models \varphi]$ (i.e., $C \models ZFC$), but Σ does not prove this sentence, since then $\Sigma \vdash \mathrm{Con}(ZFC)$, which is impossible (assuming that ZFC is consistent), since Σ is a conservative extension of ZFC. Then, of course, $\Sigma \nvdash (C \models ZF{-}P)$, because $\Sigma \vdash ((C \models ZF{-}P) \leftrightarrow (C \models ZFC))$, since Σ proves that C satisfies the two single axioms of Power Set and Choice.

So, when using our absoluteness results with C, we do need to note that each such result used only finitely many axioms at a time. For example, Lemma II.4.5 says that $\alpha+\beta$ is absolute for transitive models for $ZF-P$. This applies to C because it only requires that C satisfies enough of $ZF-P$ to justify the proof of that lemma, not the model-theoretic statement $C \models ZF-P$. Likewise, we have already applied our absoluteness results to the proper class L, where the analogous statement $L \models ZF-P$ cannot even be written in set theory.

IV.5.2 Forcing over the Universe

Here, we avoid set models completely.

In Section IV.2, we used the notion "$p \Vdash^*_{\mathbb{P}} \varphi$" for sentences φ (Definition IV.2.42). In this definition, there was no mention of models. φ was an arbitrary sentence of $\mathcal{FL}_{\mathbb{P}}$; that is, it used names from $V^{\mathbb{P}}$, not some $M^{\mathbb{P}}$. We then relativized $\Vdash^*$ to a model M and proved that $p \Vdash \varphi$ iff $(p \Vdash^* \varphi)^M$ (Lemma IV.2.45). So $\Vdash$ was our primary notion, and $\Vdash^*$ was just a trick for proving the Definability Lemma.

But now, we have no M, and we never define $\Vdash$, so $\Vdash^*$ is our primary notion. So we, living in V, are in the position of the M–people described in Section IV.2. As in Section IV.1, we can dream about some ideal universe $V[G] \supset V$ and use these dreams to motivate our discussion of $\Vdash^*$, but these dreams cannot be mentioned in rigorous proofs of theorems about $\Vdash^*$. This motivation suggests that we verify the following, for general posets $\mathbb{P}$:

a. If $\varphi_1, \dots, \varphi_n \vdash \psi$ and $p \Vdash^* \varphi_i$ for each i, then $p \Vdash^* \psi$.
b. $p \Vdash^* \varphi$ for each axiom φ of ZFC.
c. If $\varphi(x_1, \dots, x_n)$ is a formula known to be absolute for transitive models by the methods of Section II.4, then for every p and all sets $a_1, \dots, a_n$: $p \Vdash^* \varphi(\check{a}_1, \dots, \check{a}_n)$ iff $\mathbb{1} \Vdash^* \varphi(\check{a}_1, \dots, \check{a}_n)$ iff $\varphi(a_1, \dots, a_n)$ is true in V.

Of course, this is a scheme in the metatheory about what can be proved in ZFC. The statement "$\varphi(\vec{a})$ is true in V" is really just the same as "$\varphi(\vec{a})$".

Now, to prove Theorem IV.5.1: First note that it is enough to prove $ZFC + GCH \vdash \Xi$, since then $ZFC \vdash \Xi$ follows by Theorem II.6.24. Now, working in $ZFC + GCH$, one can define $\mathbb{P} = \mathrm{Fn}(\omega_7 \times \omega, 2)$, and then prove that $\mathbb{1} \Vdash^* \mathfrak{c} = \aleph_7$. Applying this and (a)(b), we get $\mathbb{1} \Vdash^* \Xi$, and then (c) implies that Ξ is true. So, $ZFC + GCH \vdash \Xi$.

Note that (a) and (c) require a bit of work to prove. The analogous results for $\Vdash$ are essentially trivial. Forcing over models, (a) (replacing $\Vdash^*$ with $\Vdash$) is clear from the fact that if $M[G] \models \varphi_i$ for each i, then $M[G] \models \psi$ by the Soundness Theorem; see also Lemma IV.3.6. Also, the $\Vdash$ analog of (c) is clear from the fact that we really have transitive models, so for $a_1, \dots, a_n \in M$, $\varphi(a_1, \dots, a_n)^M$ iff $\varphi(a_1, \dots, a_n)$ iff $\varphi(a_1, \dots, a_n)^{M[G]}$. Statement (b) for $\Vdash^*$ also requires some work, as was true for $\Vdash$ as well.

One can verify statement (a) using (in the metatheory) an induction over formal proofs. With a Hilbert-style system, it is similar to the proof of the Soundness Theorem I.15.12 outlined in Section I.15. You need to show that $\mathbb{1} \Vdash^* \varphi$ whenever φ is a logical axiom, and, for Modus Ponens, that if $p \Vdash^* \varphi \to \psi$ and $p \Vdash^* \varphi$, then $p \Vdash^* \psi$.

For statement (c), you have to go through all the absoluteness proofs and see that they work here. Informally, we are establishing results of the form $V \preccurlyeq_\varphi V[G]$ for various φ, such as Δ_0 formulas, as discussed in Section I.16, where we discussed how to interpret absoluteness statements for proper class models (e.g., $L \preccurlyeq_\varphi V$). But "$V \preccurlyeq_\varphi V[G]$" is a statement about a generic set G which doesn't really even exist, so it is interpreted formally by statement (c).

The exposition of this approach is much simpler, and has a clearer motivation, if it restricted to the case of forcing with a complete boolean algebra $\mathbb{B}$, as indicated in Section IV.4. Then we can use the clauses of Exercises IV.4.20, IV.4.21, and IV.4.24 as the basis of a recursive definition of $[\![\varphi]\!]^*$ for sentences φ in the forcing language. Now, we are getting closer to standard model theory, except that we are building a *boolean valued model*. We have a proper class model whose universe is the set of names $V^{\mathbb{B}}$, but the truth values lie in $\mathbb{B}$, rather than in $2 = \{\mathbb{0}, \mathbb{1}\} = \{F, T\}$. This model "extends" V in the sense that the sub-class $\{\check{x} : x \in V\}$ forms an isomorphic copy of V.

In expositions such as Rosser [77], where these boolean-valued extensions of V are used as the basic approach to the independence proofs, one always writes $[\![\varphi]\!]$, not $[\![\varphi]\!]^*$, since there are no set models being discussed at all, so we do not need to distinguish $[\![\varphi]\!]$ from $[\![\varphi]\!]^*$, as we did at the end of Section IV.4.

These boolean extensions of V are a little like the results on the independence of Foundation in Section II.9; there we worked in ZF and constructed an "extension" of the universe in which Foundation failed. These were ordinary (2–valued) models, but "extension" was again formalized by defining an isomorphic embedding of V into the model (Exercise II.9.3).

IV.6 Current Forcing Notation

Journal articles on forcing assume that the reader has seen some presentation of the method, such as the one given here. As a result, the terminology is a bit more informal and blurs over some distinctions that are important to understand when first learning the subject. These articles do not specify a precise formulation of basic forcing, and readers of the papers presumably know how to decode the notation into one of the presentations discussed in Section IV.5. Different readers may have different presentations in mind when reading the article. In this book, the ctm approach is our "official" presentation, but we also comment on translations into the "forcing over V" approach.

In the ctm approach, we usually refer to M as the *ground model*, and $M[G]$ as the *generic extension*. Articles frequently call the ground model V and refer

to $V[G]$ as the extension, in line with the informal motivation described in Sections II.7 and IV.1. Also, the checks are usually omitted when it is clear from context how to fill them in.

Thus, in discussing Theorem IV.3.13, we might say: In the ground model V, assume that $\kappa^{\aleph_0} = \kappa$, and let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$. Then $2^{\aleph_0} = \kappa$ must hold in the generic extension $V[G]$. Thus, $\mathbb{1} \Vdash 2^{\aleph_0} = \kappa$. In particular, if $V \models GCH$, then κ could be $\aleph_7$, in which case $\mathbb{1} \Vdash 2^{\aleph_0} = \aleph_7$.

Readers following the ctm approach would realize that really the ground model is a ctm M and we should really say $(2^{\aleph_0} = \kappa)^{M[G]}$; of course, in V, κ is really some countable ordinal. Also, one should distinguish between the ordinal κ and its name $\check{\kappa}$ in the forcing language, and say $\mathbb{1} \Vdash 2^{\aleph_0} = \check{\kappa}$.

Readers following the "forcing over V" approach don't use any set models, and would realize that there is no real meaning to $V[G]$. But, they do have a forcing language, which, *informally*, talks about $V[G]$, and they can define forcing. So, they should write $\mathbb{1} \Vdash^* 2^{\aleph_0} = \check{\kappa}$.

Here, as before, we are using $\Vdash$ for the forcing notion defined in terms of models and generic extensions (Definition IV.2.22) and $\Vdash^*$ for the forcing notion defined by recursion, without regard to models (Definition IV.2.42). But in the literature, the term "$\Vdash^*$" is never used; both forcing notions are denoted by "$\Vdash$", and the reader presumably knows Lemma IV.2.45 relating the two.

Regarding the checks: Even our terminology used before this section was not 100% formal. In the statement $\mathbb{1} \Vdash 2^{\aleph_0} = \aleph_7$, which of the numbers $0, 2, 7$ should actually be $\check{0}, \check{2}, \check{7}$? Actually, you could make a case for all $2^3 = 8$ answers. If you say "none", you are thinking $\mathbb{1} \Vdash \varphi$, where φ is a sentence in $\mathcal{L} = \{\in\}$ asserting that the size of the power set of the natural numbers is the seventh uncountable cardinal. If you say "all", you are thinking $\mathbb{1} \Vdash \varphi(\check{0}, \check{2}, \check{7})$, where $\varphi(x, y, z)$ is a formula in $\mathcal{L} = \{\in\}$ asserting that y is a cardinal, x, z are ordinals, and $y^{\aleph_x} = \aleph_z$.

On a less pedantic note, sometimes checks really are necessary to remove ambiguities. Consider the assertion: if $E \subseteq \mathbb{R}$ and $\mathbb{P} = \mathrm{Fn}(\omega, E)$, then E must become countable in $V[G]$, so that $\mathbb{1} \Vdash (|E| \leq \aleph_0)$. Anyone reading this in a set theory paper will realize that this is true (see Example IV.3.1), and that to be pedantic one should really write $\mathbb{1} \Vdash (|\check{E}| \leq \aleph_0)$. Now, this is correct even if E is the set of all real numbers, but you *can't* write $\mathbb{1} \Vdash (|\mathbb{R}| \leq \aleph_0)$, since that asserts that the generic extension satisfies "the set of all real numbers is countable", which is impossible. You have to say $\mathbb{1} \Vdash (|\check{\mathbb{R}}| \leq \aleph_0)$; here, the $\check{\mathbb{R}}$ is the standard name for the set of real numbers of the ground model V, and this set becomes countable in $V[G]$. In the ctm approach, you might want to be a bit more pedantic and say $\mathbb{1} \Vdash (|\check{E}| \leq \aleph_0)$, where $E = \mathbb{R}^M$; since M is countable, the true $\mathbb{R}$ isn't even in M.

In some cases, you want to leave the check *off* the $\mathbb{R}$. For example, if $\mathbb{1} \Vdash 2^{\aleph_0} = \aleph_7$, then also $\mathbb{1} \Vdash |\mathbb{R}| = \aleph_7$, since $ZFC \vdash |\mathbb{R}| = 2^{\aleph_0}$. You do *not* want a check on the $\mathbb{R}$, since you are really talking about the set of all real numbers in $V[G]$. Writing $\mathbb{1} \Vdash |\check{\mathbb{R}}| = \aleph_7$ would be wrong, since that asserts

that the set of real numbers of the ground model has size $\aleph_7$ in the generic extension.

The next lemma is sometimes useful, and also illustrates our notation.

Lemma IV.6.1 *If $r \Vdash \exists x \in \pi\ \varphi(x)$, then there is a $q \leq r$ and a $\sigma \in \mathrm{dom}(\pi)$ such that $q \Vdash [\sigma \in \pi \wedge \varphi(\sigma)]$.*

Readers following the "forcing over V" approach will realize that the "$\Vdash$" here really is the recursively defined "$\Vdash^*$", and they would then have to prove it using properties of $\Vdash^*$, which takes a bit of work; note that the definition (IV.2.42) of $\Vdash^*$ for existential statements gives you a σ, but not one in $\mathrm{dom}(\pi)$. Readers following the official ctm approach of this book will see that the proof is fairly simple, and is similar to other proofs we have given. But they will have to realize that for this lemma to make sense, we must have $\mathbb{P} \in M$ and $\varphi \in \mathcal{FL}_{\mathbb{P}} \cap M$ and $\pi \in M^{\mathbb{P}}$.

Proof. Let G be generic with $r \in G$. Then $M[G] \models \exists x \in \pi\ \varphi(x)$. Using the definition of π_G, fix $\sigma \in \mathrm{dom}(\pi)$ such that $M[G] \models [\sigma \in \pi \wedge \varphi(\sigma)]$. Then, by Lemma IV.2.31, there is a $q \in G$ such that $q \leq r$ and $q \Vdash [\sigma \in \pi \wedge \varphi(\sigma)]$. □

Another example of common forcing terminology:

Definition IV.6.2 *Let $Card(\alpha)$ abbreviate the statement that α is a cardinal. If $\mathbb{P}$ is a forcing poset and κ is a cardinal then* $\mathbb{P}$ preserves κ *iff* $\mathbb{1} \Vdash_{\mathbb{P}} Card(\kappa)$ *and* $\mathbb{P}$ collapses κ *iff* $\mathbb{1} \Vdash_{\mathbb{P}} \neg Card(\kappa)$.

Lemma IV.6.3 *If $\mathbb{P}$ is ccc, then $\mathbb{P}$ preserves all cardinals. If $\kappa > \omega$ is a cardinal and $\mathbb{P} = \mathrm{Fn}(\omega, \kappa)$, then $\mathbb{P}$ collapses κ.*

This lemma is clear from what we have already done (Theorem IV.3.4 and Example IV.3.1) -- but our focus here is on the notation. As before, readers following the "forcing over V approach" would realize that the $\Vdash$ is really $\Vdash^*$. Readers using the ctm approach have to realize in Definition IV.6.2 that $\mathbb{P}$ is in M and $(\kappa \text{ is a cardinal})^M$, and we are talking about what happens in $M[G]$. In either case, the κ in the forcing statements "$\mathbb{1} \Vdash \cdots \kappa \cdots$" is really a $\check{\kappa}$.

Note that because of the "$\mathbb{1} \Vdash$" in Definition IV.6.2, it can happen that $\mathbb{P}$ neither preserves nor collapses a particular κ; for a counter-example, use the method of Exercise IV.4.16.

So far, the notational conventions discussed in this section are not very deep; one just has to be aware of where the checks should go, and one has to know that sometimes a "V" in a paper becomes an "M" in the ctm approach.

But, now we consider some cases where decoding our notation into meaningful statements requires a bit of thought. Consider the following two lemmas about Suslin trees. The first is easily decoded:

Lemma IV.6.4 *In V, let T be a well-pruned Suslin tree and view T as a forcing poset. Then T is ccc, and, in $V[G]$, G is a path through T, so that T is not even Aronszajn.*

Proof. This is like the proof of Lemma III.5.29. Recall from page 202 that the forcing order is $x \leq y$ iff $y \sqsubseteq x$, and $\mathbb{1}$ is the root. T is ccc because all antichains are countable by Definition III.5.21 (of Suslin). For each $\alpha < \omega_1$, the set $D_\alpha := \{x \in T : \text{height}(x, T) > \alpha\}$ is dense because T is well-pruned. Then G is a chain that meets unboundedly many levels, so G meets all levels (because G is upward closed). □

Readers following the ctm approach will replace "V" by "M" and relativize the relevant hypotheses to M: $(T \text{ is Suslin, etc.})^M$. Readers following the "forcing over V" approach will, as before, realize that G doesn't really exists, but they have a name $\mathring{G}$ and can write $\mathbb{1} \Vdash^* (\mathring{G}$ is a path through $\check{T})$.

Lemma IV.6.5 *If T is a Suslin tree, then there is no order preserving map (Definition III.5.10) $\varphi : T \to \mathbb{R}$.*

Proof. Assume that φ is such a map. We may also assume that T is well-pruned; if not, replace it by a well-pruned subtree using Lemma III.5.26. Now, work in $V[G]$ and apply Lemma IV.6.4. Note that ω_1 is the same in V and $V[G]$, since the forcing is ccc. So, in $V[G]$, we have an order preserving map from a set G of order type ω_1 into $\mathbb{R}$, which is clearly impossible. □

Note that the *statement* of the lemma does not mention forcing or models, and is easily understood once Suslin trees have been defined (by Kurepa, in the 1930s). One can also prove the lemma by purely combinatorial means (Exercise IV.6.7), but the proof is longer and requires some thought, whereas people familiar with forcing would discover the above proof immediately.

Readers who have gone through the trouble of learning the "forcing over V" approach will easily see that the above is indeed a rigorous proof, although it requires (a)(b)(c) from Subsection IV.5.2. Readers who only know the ctm approach will see that the above proof shows that Lemma IV.6.5 is true in every ctm M for *ZFC*. They then need a reflection argument to see that Lemma IV.6.5 is a consequence of *ZFC*; that is, working in *ZFC*, if ¬(Lemma IV.6.5), then we could get a ctm for ¬(Lemma IV.6.5) plus a large enough finite subset of *ZFC* to get a contradiction via the given forcing proof.

To get a purely combinatorial proof, use the following purely combinatorial fact about the poset topology (Definition III.3.53):

Lemma IV.6.6 *Let T be a well-pruned Suslin tree and view T as a forcing poset. Then in the poset topology, every countable intersection of dense open sets is dense and open.*

Proof. Note that this is *not* the tree topology used in Definition III.5.13. In the poset topology, $U \subseteq T$ is open iff $\forall s \in U\, [s{\uparrow} \subseteq U]$. If each U_n is dense open, let $V = \bigcap_{n\in\omega} U_n$. Since every intersection of open sets is open in this topology, we need only show that V is dense. Since co-countable sets are dense (since T is well-pruned), it is sufficient to show that each U_n is co-countable.

So, fix n, and then let A be a maximal antichain in T such that $A \subseteq U_n$ (see the proof of Theorem III.3.60). By maximality, $T = A \cup \bigcup_{s \in A}(s\uparrow \cup s\downarrow)$. U_n is open, so $T \backslash U_n \subseteq \bigcup_{s \in A} s\downarrow$, which is countable because A is countable. □

Exercise IV.6.7 *Prove Lemma IV.6.5 by a purely combinatorial argument.*

Hint. For each rational q, let $U_q = \{x \in T : \varphi(x) > q \vee \neg\exists y\,[x \sqsubset y \wedge \varphi(y) > q]$. WLOG, T is well-pruned. Then U_q is dense and open in the poset topology. □

The property described in Lemma IV.6.6 occurs so frequently that it is given a name:

Definition IV.6.8 *A topological space X is* Baire *iff in X, every countable intersection of dense open sets is dense.*

So, all complete metric spaces and all compact Hausdorff spaces are Baire spaces by the Baire Category Theorem. Note that we didn't say that the intersection had to be open; this happens automatically in poset topologies, but is rather unusual in "reasonable" topological spaces.

By now, the reader is probably wondering why the above definition, exercise, and lemma were put in this section, rather than in Section III.5 on trees. The reason is that the Baire property occurs very frequently in forcing because Baire forcing posets don't add ω–sequences; more precisely:

Lemma IV.6.9 *For a forcing poset $\mathbb{P}$, (1) $\rightarrow$ (2) and (2) $\leftrightarrow$ (3):*

1. *$\mathbb{P}$ is Baire in the poset topology.*
2. *For each set $E \in V$ and each generic G, V and $V[G]$ have the same elements of E^ω.*
3. *For each set $E \in V$, name τ, and $p \in \mathbb{P}$: If $p \Vdash \tau : \omega \rightarrow E$, then there is a function $h : \omega \rightarrow E$ and some $q \leq p$ such that $q \Vdash \tau = \check{h}$.*

Note that in (3), we might have left the check off the h, but we put it there to emphasize that h comes from the ground model. We leave the proof under the "forcing over V" approach as an exercise, but we explain how to translate the lemma into our ctm approach, and then we prove it:

Proof. For a ctm M, (2)(3) are naturally expressed by writing "M" for "V". Of course, $\tau \in M^{\mathbb{P}}$ and h should come from M. For (2) $\rightarrow$ (3), let G be generic with $p \in G$. Then $h := \tau_G \in E^\omega$, so by (2), $h \in M$. Then, by Lemma IV.2.31, there is a $q \leq p$ with $q \in G$ such that $q \Vdash \tau = \check{h}$. For (3) $\rightarrow$ (2), fix an element $\tau_G \in E^\omega \cap M[G]$, and then fix $r \in G$ such that $r \Vdash \tau : \check{\omega} \rightarrow \check{E}$. Working in M, (3) implies that $D := \{q : q \perp r \vee (q \leq r \wedge \exists h \in E^\omega\,[q \Vdash \tau = \check{h}])\}$ is dense. Then, in $M[G]$, fix $q \in G \cap D$. Then $q \leq r$ and $q \Vdash \tau = \check{h}$, so $\tau_G = h \in M$.

Statement (1) should read, $(\mathbb{P}$ is Baire$)^M$. To prove (1) $\rightarrow$ (3), we work in M: Assume that $p \Vdash \tau : \check{\omega} \rightarrow \check{E}$. Let $U_n = \{q : q \perp p \vee \exists e \in E\,[q \Vdash \tau(\check{n}) = \check{e}]\}$.

Then U_n is dense and open in $\mathbb{P}$. To see that U_n is dense, observe that if $r \Vdash \tau : \omega \to E$, then there is a $q \leq r$ and an $e \in E$ such that $r \Vdash \tau(\check{n}) = \check{e}$ (use Lemma IV.6.1, where $\varphi(x)$ says $\langle \check{n}, x\rangle \in \tau$).

Applying (1), fix $q \in \bigcap_n U_n$ such that $q \leq p$. Define $h : \omega \to E$ so that $h(n)$ is the $e \in E$ such that $q \Vdash \tau(\check{n}) = \check{e}$. Then $q \Vdash \tau = \check{h}$.

The assertion $q \Vdash \tau = \check{h}$ is most easily verified by stepping out of M and noting that $\tau_G = h$ whenever G is generic and $q \in G$. □

The implication (3) $\to$ (1) can fail: Let $\mathbb{P} = \omega$, which we view as a tree, with root 0. So, in the forcing order, $\mathbb{1} = 0 > 1 > 2 > 3 > \cdots$. If $U_n = \{n, n+1, n+2, \ldots\}$, then each U_n is dense open, and $\bigcap_n U_n = \emptyset$. But (2)(3) hold, since $M[G] = M$ for all filters G. But, for less pathological posets:

Exercise IV.6.10 *In Lemma IV.6.9, assume that $\mathbb{P}$ is separative (see Definition III.4.10). Then* (1) $\leftrightarrow$ (2) $\leftrightarrow$ (3).

Hint. To prove (3) $\to$ (1): Let U_n, for $n \in \omega$, be dense open sets. Choose $A_n \subseteq U_n$ such that A_n is a maximal antichain. Let $E = \mathbb{P}$, and define τ so that $\mathbb{1} \Vdash \tau : \omega \to E$, and so that in every $M[G]$, $G \cap A_n = \{\tau_G(n)\}$.

To prove that $\bigcap_n U_n$ is dense, fix p and get $q \leq p$ such that $q \Vdash \tau = \check{h}$ for some $h \in M$ (applying (3)), and prove that $q \in \bigcap_n U_n$. To prove that $q \in U_n$, note that $h(n) \in U_n$ and $\neg\exists r \leq q\, [r \perp h(n)]$, so $q \leq h(n)$. □

Exercise IV.6.11 *There exists an atomless ccc Baire poset iff there exists a Suslin tree.*

Hint. For the $\to$ direction, get $T \subseteq \mathbb{P}$, where the tree order is the reverse of the poset order. Each level of T is a maximal antichain in $\mathbb{P}$. The Baire property ensures that the construction doesn't die at limit ordinals. □

Non ccc Baire posets occur frequently when we are trying to get the generic extension to satisfy $CH + \neg GCH$. We start with GCH in the ground model and devise $\mathbb{P}$ to make, e.g., $2^{\aleph_1} > \aleph_2$. If $\mathbb{P}$ is Baire, then V and $V[G]$ will have the same ${}^{\omega}2$, so CH will continue to hold in $V[G]$. For specific examples of this, along with some other results relating combinatorial and forcing properties of posets, see Section IV.7.

Exercise IV.6.12 *Let T be a Hausdorff Suslin tree. Let $H, K \subseteq T$ be disjoint and closed in the tree topology. Prove that there is an $\alpha < \omega_1$ such that for all $x \in \mathcal{L}_\alpha$: either $x{\uparrow} \cap H = \emptyset$ or $x{\uparrow} \cap K = \emptyset$. Conclude from this that T is a normal topological space.*

Hint. If this is false, then applying Exercises III.5.45 and III.5.46, reduce to the situation where T is well-pruned and both H and K are forcing-dense above $\mathbb{1}$. Then, as in Lemma IV.6.5, in $V[G]$ both $G \cap H$ and $G \cap K$ are uncountable, which is impossible; in the space ω_1, there cannot be two disjoint clubs. □

Exercise IV.6.13 *If $\mathbb{P}$ is ccc and $\mathbb{P}$ doesn't add new real numbers, then $\mathbb{P}$ doesn't add ω-sequences.*

Hint. Use Lemma IV.3.5. As usual, you may interpret this exercise in the ctm approach. Also, see Notation IV.7.31. The "natural" proof views adding a real number as $\omega^\omega \cap M \subsetneq \omega^\omega \cap M[G]$, but that's equivalent to $2^\omega \cap M \subsetneq 2^\omega \cap M[G]$ and to $\mathbb{R} \cap M \subsetneq \mathbb{R} \cap M[G]$. □

There are examples of (non ccc) posets that don't add reals but do add an ω-sequence. Most well-known are Prikry forcing (which requires a measurable cardinal) and Namba forcing; see Jech [49] (Theorems 21.10 and 28.10).

IV.7 Further Results and Posets

We collect here some further results that can be obtained by non-iterated forcing. As always, M denotes a ctm for *ZFC*, and G denotes a filter that is generic over M. Following Section IV.6, we shall occasionally state results using an informal "forcing over V" terminology, but rigorous proofs will be given using M. The material past Exercise IV.7.20 is a bit technical; although much of this will be used eventually, it is not needed for understanding basic iterated forcing in Chapter V

We begin with the Maximal Principle, which we shall state officially using our ctm M. It is a strengthening and simplification of Lemma IV.2.32(2).

Theorem IV.7.1 (Maximal Principle) *For any forcing poset $\mathbb{P} \in M$ and formula $\varphi(x) \in \mathcal{FL}_\mathbb{P} \cap M$, with no variable other than x free: $p \Vdash \exists x\, \varphi(x)$ iff $p \Vdash \varphi(\tau)$ for some $\tau \in M^\mathbb{P}$.*

The $\leftarrow$ direction is clear from the definition of $\Vdash$. But the $\rightarrow$ direction is a significant strengthening of the $\rightarrow$ of Lemma IV.2.32(2), which made it seem that we might need multiple τ's, depending on the various $q \leq p$. Using this theorem, we can simplify Lemma IV.2.32(2) to: $p \Vdash^* \exists x\, \varphi(x)$ iff $p \Vdash^* \varphi(\tau)$ for some $\tau \in V^\mathbb{P}$.

First,

Lemma IV.7.2 *Working in M: Let $A \subseteq \mathbb{P}$ be an antichain, and, for each $q \in A$, let σ_q be a $\mathbb{P}$-name. Then there is a $\mathbb{P}$-name τ such that $q \Vdash \tau = \sigma_q$ for each $q \in A$.*

Proof. Working in M, let

$$\tau = \bigcup_{q \in A} \left\{ \langle \pi, r \rangle \in \mathrm{dom}(\sigma_q) \times q{\downarrow} : r \Vdash \pi \in \sigma_q \right\} .$$

Now, fix $q \in A$. To prove that $q \Vdash \tau = \sigma_q$, we fix a generic G with $q \in G$ and prove that $\tau_G = (\sigma_q)_G$. Since A is an antichain,

$$\tau_G = \{ \pi_G : \pi \in \mathrm{dom}(\sigma_q) \wedge \exists r \in G \cap q{\downarrow} [r \Vdash \pi \in \sigma_q] \} \subseteq (\sigma_q)_G$$

(since $r \in G \ \wedge \ r \Vdash \pi \in \sigma_q \ \rightarrow \ \pi_G \in (\sigma_q)_G$). To prove $(\sigma_q)_G \subseteq \tau_G$, fix $\pi_G \in (\sigma_q)_G$ where $\pi \in \mathrm{dom}(\sigma_q)$. Then, fix $r \in G$ such that $r \Vdash \pi \in \sigma_q$ and $r \leq q$. Then $\langle \pi, r \rangle \in \tau$, so $\pi_G \in \tau_G$. □

Proof of Theorem IV.7.1. Assume that $p \Vdash \exists x\, \varphi(x)$. Working in M: Let A be such that

1. A is an antichain and $A \subseteq p{\downarrow}$.
2. $\forall q \in A\, \exists \sigma \in M^{\mathbb{P}}\, [q \Vdash \varphi(\sigma)]$.
3. A is maximal with respect to (1)(2).

For $q \in A$, choose σ_q such that $q \Vdash \varphi(\sigma_q)$. Then, by Lemma IV.7.2, fix a $\mathbb{P}$–name τ such that $q \Vdash \tau = \sigma_q$ for each $q \in A$. Since also $q \Vdash \varphi(\sigma_q)$, we have $q \Vdash \varphi(\tau)$ for each $q \in A$. We now prove that $p \Vdash \varphi(\tau)$. If not, fix $r \leq p$ such that $r \Vdash \neg\varphi(\tau)$. Since $r \Vdash \exists x\, \varphi(x)$, apply Lemma IV.2.32(2) and fix $s \leq r$ and $\sigma \in M^{\mathbb{P}}$ such that $s \Vdash \varphi(\sigma)$. Now $s \leq r$ implies that $s \Vdash \neg\varphi(\tau)$, so $s \perp q$ for all $q \in A$. But then, $A \cup \{s\}$ contradicts maximality of A. □

We next consider a generalization of the Cohen forcing $\mathrm{Fn}(I, 2)$ that lets us construct models where *GCH* first fails at a cardinal greater than $\aleph_0$.

Definition IV.7.3 *For any sets I, J and cardinal λ: $\mathrm{Fn}_\lambda(I, J)$ is the set of all $p \in [I \times J]^{<\lambda}$ such that p is the graph of a function. We make $\mathrm{Fn}_\lambda(I, J)$ into a forcing poset by letting $\leq$ be $\supseteq$ and $\mathbb{1} = \emptyset$.*

So, $\mathrm{Fn}(I, J) = \mathrm{Fn}_\omega(I, J)$; these are finite partial functions; $\mathrm{Fn}_{\aleph_1}(I, J)$ is the poset of *countable* partial functions from I to J.

When $\lambda > \omega$, the notion $\mathrm{Fn}_\lambda(I, J)$ is *not* absolute. In forcing, we take $I, J \in M$ and force with $(\mathrm{Fn}_\lambda(I, J))^M$. The true $\mathrm{Fn}_\lambda(I, J)$ will be uncountable, and hence not in M, if $\lambda > \omega$, I is infinite, and $J \neq \emptyset$.

But, in discussing combinatorial properties, we work in V and prove theorems about $\mathrm{Fn}_\lambda(I, J)$; then, we shall relativize the theorems to M and see how they affect the forcing.

First, consider the ccc. This is false when $\lambda \geq \aleph_1$, I is infinite, and $|J| \geq 2$, since it is easy to find an antichain of size $2^{\aleph_0}$. But, the next lemma provides a weakened chain condition; applied with $\lambda = \aleph_1$, it shows that $\mathrm{Fn}_{\aleph_1}(I, J)$ has no antichains of size larger than $2^{\aleph_0}$ as long as $|J| \leq 2^{\aleph_0}$; we say the poset has the $(2^{\aleph_0})^+$-cc:

Definition IV.7.4 *For a cardinal θ, the poset $\mathbb{P}$ has the θ* chain condition *(θ-cc) iff, in $\mathbb{P}$, every antichain has size less than θ.*

So, the ccc is the $\aleph_1$-cc. Also, $\mathbb{P}$ has the θ-cc iff $S(\mathbb{P}) \leq \theta$, where $S(\mathbb{P})$ is the Suslin number (Definition III.3.91).

Lemma IV.7.5 *For any infinite λ, $\mathrm{Fn}_\lambda(I, J)$ has the $(|J|^{<\lambda})^+$-cc. In particular, $\mathrm{Fn}_\lambda(I, J)$ has the $(2^{<\lambda})^+$-cc whenever $|J| \leq 2^{<\lambda}$.*

Proof. Let $\kappa = (|J|^{<\lambda})^+$. We must prove that there is no antichain of size κ. This is trivial if $|J|$ is 0 or 1, so assume that $|J| \geq 2$, so that κ is regular and $\kappa > \lambda$. Now, suppose that $W \subseteq \mathrm{Fn}_\lambda(I, J)$ is an antichain of size κ.

We may assume that λ is regular. If not, then since κ is regular and each $|p| < \lambda$, there is a $\sigma < \lambda$ such that $W' = \{p \in W : |p| \leq \sigma\}$ has size κ, so we may replace W by W' and replace λ by the regular σ^+.

Now, list W as $\{p_\alpha : \alpha < \kappa\}$, and let $S_\alpha = \mathrm{dom}(p_\alpha)$. Then, by the Delta System Lemma III.6.15, there is a $B \in [\kappa]^\kappa$ and a root R such that $S_\alpha \cap S_\beta = R$ whenever α, β are two different elements of B. Since $|J^R| < \kappa$, fix $\alpha, \beta \in B$ with $\alpha \neq \beta$ and $p_\alpha \restriction R = p_\beta \restriction R$. Then p_α, p_β agree on $R = \mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta)$, so $p_\alpha \not\perp p_\beta$, which is a contradiction. □

For regular λ, we shall use the posets $\mathrm{Fn}_\lambda(I, J)$ to make 2^λ big without changing the powers of smaller cardinals. In particular, if we assume *GCH* in M and force with $(\mathrm{Fn}_{\aleph_1}(\aleph_8, 2))^M$, then $M[G]$ will satisfy $CH + 2^{\aleph_1} = \aleph_8$.

To study $\mathrm{Fn}_\lambda(I, J)$ further, we retrace the discussion in Section IV.3, which dealt with $\mathrm{Fn}(I, J)$. We first consider preservation of cardinals. By Theorem IV.7.9 below, θ-cc posets must preserve cardinals $\geq \theta$. For example, using Lemma IV.7.5 with $\lambda = \aleph_1$, we see that under *CH*, $\mathrm{Fn}_{\aleph_1}(I, J)$ has the $\aleph_2$-cc whenever $|J| \leq \aleph_1$. So, forcing over $M \models GCH$ with $(\mathrm{Fn}_{\aleph_1}(\aleph_8, 2))^M$, we know that cardinals of M that are $\geq \aleph_2^M$ remain cardinals in $M[G]$. To prove that $\aleph_1^M$ remains a cardinal in $M[G]$, we shall introduce a completely different argument after proving Theorem IV.7.9.

Definition IV.7.6 *When $\mathbb{P} \in M$ is a forcing poset and $(\theta$ is a cardinal$)^M$:*

1. *$\mathbb{P}$ preserves cardinals $\geq \theta$ if and only if:*
   $(\beta$ is a cardinal $)^M$ iff $(\beta$ is a cardinal $)^{M[G]}$ whenever $\theta \leq \beta < o(M)$.
2. *$\mathbb{P}$ preserves cofinalities $\geq \theta$ if and only if:*
   $\mathrm{cf}^M(\gamma) = \mathrm{cf}^{M[G]}(\gamma)$ for all limit $\gamma < o(M)$ such that $\mathrm{cf}^M(\gamma) \geq \theta$.

This generalizes Definition IV.3.2, which was the case where $\theta = (\omega_1)^M$. As we pointed out there, in (1), the $\leftarrow$ direction of the "iff" always holds, and in (2), $\mathrm{cf}^M(\gamma) \geq \mathrm{cf}^{M[G]}(\gamma)$ always holds, regardless of $\mathbb{P}$. Note that for $\beta, \gamma < (\omega_1)^M$, statements (1)(2) hold automatically by absoluteness, so the definition is only of interest when $\theta \geq (\omega_1)^M$.

Lemma IV.7.7 *If $\mathbb{P} \in M$ is a forcing poset and $(\theta$ is a regular cardinal$)^M$:*

1. *$\mathbb{P}$ preserves cofinalities $\geq \theta$ if and only if:*
   $(*)$: for all limit β with $\theta \leq \beta < o(M)$,
   $(\beta$ is regular$)^M \rightarrow (\beta$ is regular$)^{M[G]}$.
2. *If $\mathbb{P}$ preserves cofinalities $\geq \theta$ then $\mathbb{P}$ preserves cardinals $\geq \theta$.*

The proof is the same as for Lemma IV.3.3, which was the case where $\theta = (\omega_1)^M$. The regularity of θ is needed in deriving (2) from (1). We need

to assert (in M) that every singular cardinal $\beta \geq \theta$ is the supremum of a set of regular cardinals $\geq \theta$. This is trivial for $\beta > \theta$, but is false for $\beta = \theta$ if θ is singular.

We now generalize the approximation Lemma IV.3.5:

Lemma IV.7.8 *Assume that $\mathbb{P} \in M$, $(\theta$ is an uncountable cardinal$)^M$, and $(\mathbb{P}$ is θ-cc$)^M$. Fix $A, B \in M$. Let G be $\mathbb{P}$-generic over M. Fix $f \in M[G]$ with $f : A \to B$. Then, there is an $F : A \to \mathcal{P}(B)$ with $F \in M$ such that for all $a \in A$, $f(a) \in F(a)$ and $(|F(a)| < \theta)^M$.*

The proof is the same as for Lemma IV.3.5, and the proof of the following theorem is the same as for Theorem IV.3.4.

Theorem IV.7.9 *If $\mathbb{P} \in M$ and $(\theta$ is a regular cardinal$)^M$ and $(\mathbb{P}$ is θ-cc$)^M$, then $\mathbb{P}$ preserves cofinalities $\geq \theta$ and hence $\mathbb{P}$ preserves cardinals $\geq \theta$.*

Of course, the "and hence" quotes Lemma IV.7.7. This Lemma IV.7.7 could fail for singular θ. For example, say $\theta = (\aleph_\omega)^M$ and $\mathbb{P} = \mathrm{Fn}(\omega, \theta)$. Then, in M, $\mathbb{P}$ is θ^+-cc (trivially, since $|\mathbb{P}| = \theta$), which implies that $\mathbb{P}$ preserves cardinals and cofinalities $\geq \theta^+$ (of course, this means θ^+ computed in M). But θ becomes countable in $M[G]$, so $\mathbb{P}$ does not preserve cardinals $\geq \theta$. However, $\mathbb{P}$ preserves cofinalities $\geq \theta$; this holds because $(\theta$ is singular$)^M$, so, referring to Definition IV.7.6(2), if $\mathrm{cf}^M(\gamma) \geq \theta$ then $\mathrm{cf}^M(\gamma) > \theta$.

But, Theorem IV.7.9 is true for singular θ because if $\mathbb{P}$ is θ-cc for a singular θ, then $\mathbb{P}$ actually is θ'-cc for some regular $\theta' < \theta$; see Exercise III.3.94.

Theorem IV.7.9 implies that the posets $\mathrm{Fn}(I, J)$, where I is infinite and J is uncountable, don't collapse any more cardinals than they "obviously do" by Example IV.3.1. In discussing this, we can replace $\mathrm{Fn}(I, J)$ by the isomorphic poset $\mathrm{Fn}(\kappa, \lambda)$, where $\kappa = |I|$ and $\lambda = |J|$; see Lemma IV.4.7 and the following discussion. Then,

Exercise IV.7.10 *In M, let $\mathbb{P} = \mathrm{Fn}(\kappa, \lambda)$, where $\aleph_0 \leq \kappa < \lambda$. Then λ is countable in $M[G]$ and all cardinals of M above λ remain cardinals in $M[G]$. Also, prove that $M[G] \models GCH$, assuming that $M \models GCH$.*

Hint. If $f = \bigcup G$, then $f \restriction \omega$ maps ω onto λ. □

So, for example, say $M \models V = L$ and $\mathbb{P} = \mathrm{Fn}(\omega, \omega_5^M)$. Then $M[G]$ is a model for *GCH* plus $\aleph_1 = \aleph_6^L$, $\aleph_2 = \aleph_7^L$, ... ; in general, $\aleph_\alpha = \aleph_{5+\alpha}^L$ for $\alpha \geq 1$.

Exercise IV.7.11 *Obtain a ctm for ZFC plus $\exists A \subseteq \omega_1 \, [L[A] \models \neg CH]$.*

Hint. Start with $M \models V = L$ and add $\aleph_2$ Cohen reals; so in $M[G]$ there's an $A \subseteq \omega_2$ such that $L[A] \models 2^{\aleph_0} = \aleph_2$. Then force over $M[G]$ with $\mathrm{Fn}(\omega, \omega_1)$ to change the ω_2 to ω_1.

Note that you have to collapse cardinals here; ω_1 cannot be $(\omega_1)^L$ since by Exercise II.6.31, $V = L[A] \wedge A \subseteq \omega_1 \to GCH$. □

If $\mathbb{P}$ is θ-cc for some $\theta > \aleph_1$, then although Theorem IV.7.9 implies that cardinals $\geq \theta$ are preserved, we cannot claim that cardinals $\kappa < \theta$ are preserved without assuming something more about $\mathbb{P}$. In some cases, this "something more" is very simple:

Lemma IV.7.12 *Assume that $\omega < \lambda < o(M)$ and that for all ordinals $\delta < \lambda$, ${}^{\delta}\lambda \cap M = {}^{\delta}\lambda \cap M[G]$. Then $\mathrm{cf}^M(\gamma) = \mathrm{cf}^{M[G]}(\gamma)$ for all limit $\gamma \leq \lambda$, and (β is a cardinal)^M iff (β is a cardinal)$^{M[G]}$ for all $\beta \leq \lambda$.*

This is easily proved from the definitions of "cardinal" and "cofinality" and has nothing to do with forcing. Just observe that β fails to be a cardinal iff $\exists \delta < \beta \exists f [f : \delta \xrightarrow[\text{onto}]{} \beta]$ and that $\mathrm{cf}(\gamma) \leq \delta$ iff $\sup(\mathrm{ran}(f)) = \gamma$ for some $f : \delta \to \gamma$. Of course, we now need some posets $\mathbb{P}$ for which the hypotheses of the lemma hold. The simplest examples are given by:

Definition IV.7.13 *$\mathbb{P}$ is λ-closed iff whenever $\delta < \lambda$ and $\langle p_\xi : \xi < \delta \rangle$ is a decreasing sequence of elements of $\mathbb{P}$ (i.e., $\xi < \eta \to p_\eta \leq p_\xi$), then there is a $q \in \mathbb{P}$ such that $q \leq p_\xi$ for all $\xi < \delta$. "countably closed" = "ω_1-closed".*

Lemma IV.7.14 *If λ is regular, then $Fn_\lambda(I, J)$ is λ-closed.*

Proof. The q in Definition IV.7.13 is $\bigcup_{\xi<\delta} p_\xi$. □

Lemma IV.7.15 *Assume that ($\mathbb{P}$ is λ-closed)^M and assume that A, E in M, with $(|A| < \lambda)^M$. Fix $f : A \to E$ with $f \in M[G]$. Then $f \in M$.*

Proof. Let $\delta = |A|^M$. First, assume that $A = \delta$. As with (3) $\to$ (2) of Lemma IV.6.9, it is sufficient to prove that whenever $p \Vdash \mathring{f} : \delta \to E$, then there is a function $h : \delta \to E$ and some $q \leq p$ such that $q \Vdash \mathring{f} = \check{h}$.

Working in M, recursively choose a decreasing sequence of elements of $\mathbb{P}$, $\langle p_\xi : \xi \leq \delta \rangle$. Let $p_0 = p$. If $\eta \leq \delta$ is a limit, just choose p_η such that $p_\eta \leq p_\xi$ for all $\xi < \eta$; this is possible since $\mathbb{P}$ is λ-closed. Given p_ξ where $\xi < \delta$, choose $p_{\xi+1} \leq p_\xi$ and an $e_\xi \in E$ so that $p_{\xi+1} \Vdash \mathring{f}(\xi) = e_\xi$ (the reader may install the missing checks here). The $p_{\xi+1}$ and e_ξ exist by applying Lemma IV.6.1, where $\varphi(x)$ says $\langle \check{\xi}, x \rangle \in \mathring{f}$. Now, let $h(\xi) = e_\xi$, and let $q = p_\delta$. Then $q \Vdash \mathring{f}(\xi) = h(\xi)$ for each ξ so $q \Vdash \mathring{f} = h$.

Now, if $A \neq \delta$, fix a bijection $k : \delta \xrightarrow[\text{onto}]{1-1} A$ with $k \in M$. Then $f \circ k : \delta \to E$, so $f \circ k \in M$, so $f \in M$. □

We can now prove that $\mathrm{Fn}_\lambda(I, J)$ preserves all cardinals and cofinalities, with appropriate assumptions on λ, I, J:

Theorem IV.7.16 *In M, assume that $\mathbb{P} = \mathrm{Fn}_\lambda(I, J)$, where $\lambda \geq \aleph_0$ is regular, $2^{<\lambda} = \lambda$, and $|J| \leq \lambda$. Then $\mathbb{P}$ preserves cofinalities and (hence) cardinals.*

Proof. By Lemma IV.3.3, we just need to show that $(\beta \text{ is regular})^M \rightarrow (\beta \text{ is regular})^{M[G]}$ for all limit β with $\omega < \beta < o(M)$.

If $\beta \leq \lambda$, then the implication holds by Lemmas IV.7.15 and IV.7.12 because $\mathbb{P}$ is λ-closed in M. If $\beta > \lambda$, then the implication holds because $\mathbb{P}$ has the λ^+-cc in M by Lemma IV.7.5, and hence preserves all cardinals and cofinalities $\geq \lambda^+$ by Theorem IV.7.9. □

The special case where $\lambda = \aleph_0 \geq |J|$ was already covered in Section IV.3. We can now generalize Theorem IV.3.13, which used $\mathrm{Fn}(\kappa \times \omega, 2)$ to make $2^{\aleph_0} = \kappa$:

Theorem IV.7.17 *In M, assume that $\mathbb{P} = \mathrm{Fn}_\lambda(\kappa \times \lambda, 2)$, where κ, λ are cardinals such that $\kappa > \lambda \geq \aleph_0$, λ is regular, $\kappa^\lambda = \kappa$, and $2^{<\lambda} = \lambda$. Then $\mathbb{P}$ preserves cofinalities and cardinals, and $2^\lambda = \kappa$ in $M[G]$.*

Proof. Preservation of cofinalities and cardinals holds by Theorem IV.7.16.

$(2^\lambda \geq \kappa)^{M[G]}$ is proved as was Lemma IV.3.7, which handled the case $\lambda = \omega$. In $M[G]$, we have $\bigcup G : \kappa \times \lambda \rightarrow 2$, and this encodes a κ–sequence $\langle h_\alpha : \alpha < \kappa \rangle$ of elements of 2^λ, where $h_\alpha(\xi) = (\bigcup G)(\alpha, \xi)$. Furthermore, when $\alpha \neq \beta$, the set $E_{\alpha,\beta} = \{p \in \mathbb{P} : \exists \xi\, [(\alpha,\xi),(\beta,\xi) \in \mathrm{dom}(p) \wedge p(\alpha,\xi) \neq p(\beta,\xi)]\}$ is dense and in M, which implies that these h_α are all different. So, $(2^\lambda \geq \kappa)$.

$(2^\lambda \leq \kappa)^{M[G]}$ is proved by counting nice names. In M, all antichains in $\mathbb{P}$ have size no more than λ, and $|\mathbb{P}| = \kappa^{<\lambda} = \kappa$, so there are no more than $\kappa^\lambda = \kappa$ antichains, and hence no more than $\kappa^\lambda = \kappa$ nice names for subsets of λ. Then $(2^\lambda \leq \kappa)^{M[G]}$ follows as in the proof of Lemma IV.3.11. □

The cardinal arithmetic is somewhat simpler if $M \models GCH$, where the $\kappa^\lambda = \kappa$ can be replaced by $\mathrm{cf}(\kappa) > \lambda$ and the $2^{<\lambda} = \lambda$ follows from λ being regular.

Corollary IV.7.18 *Assume that there is a ctm for ZFC. Then one can find ctms for ZFC satisfying each of the following statements:*

1. $CH \wedge 2^{\aleph_1} = \aleph_8 \wedge \forall \theta \geq \aleph_1\, [2^\theta = \max(\theta^+, \aleph_8)]$.
2. $CH \wedge 2^{\aleph_1} = \aleph_5 \wedge 2^{\aleph_2} = \aleph_{\omega+1} \wedge \forall \theta \geq \aleph_2\, [2^\theta = \max(\theta^+, \aleph_{\omega+1})]$.
3. $2^{\aleph_0} = \aleph_3 \wedge 2^{\aleph_1} = \aleph_5 \wedge 2^{\aleph_2} = \aleph_7$.

Proof. As in the proof of Corollary IV.3.14, we may start with $M \models GCH$.

For (1), let $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\omega_8 \times \omega_1, 2)$ in M, and apply Theorem IV.7.17 to get $2^{\aleph_1} = \aleph_8$ in $M[G]$. Also, 2^θ in $M[G]$ for $\theta > \aleph_1$ is easily computed by counting nice names in M. $M[G] \models CH$ because $M \models CH$ and $(\mathbb{P} \text{ is } \aleph_1\text{-closed})^M$, so that $({}^\omega 2)^{M[G]} = ({}^\omega 2)^M$ by Lemma IV.7.15.

For (2), we force twice. Let $\mathbb{P} = (\mathrm{Fn}_{\omega_2}(\omega_{\omega+1} \times \omega_2, 2))^M$, and $N = M[G]$. Then $N \models CH \wedge 2^{\aleph_1} = \aleph_2 \wedge 2^{\aleph_2} = \aleph_{\omega+1} \wedge \forall \theta \geq \aleph_2\, [2^\theta = \max(\theta^+, \aleph_{\omega+1})]$; this is similar to (1).

Now, N is another ctm for *ZFC*, so we can view N as the ground model and let $\mathbb{Q} = (\mathrm{Fn}_{\omega_1}(\omega_5 \times \omega_1, 2))^N$, and apply Theorem IV.7.17 with $\lambda = \aleph_1^N = \aleph_1^M$

and $\kappa = \aleph_5^N = \aleph_5^M$. We have $\kappa^\lambda = \kappa$ and $2^{<\lambda} = \lambda$ in N because these statements were true in M by the *GCH* and M, N have the same ${}^\lambda\kappa$ since $(\mathbb{P}$ is $\aleph_2$-closed$)^M$. So, $\mathbb{Q}$ preserves cofinalities and cardinals, and, if H is $\mathbb{Q}$–generic over N, then $(2^{\aleph_1} = \aleph_5)^{N[H]}$. *CH* holds in $N[H]$ because $N \models CH$ and $({}^\omega 2)^{N[H]} = ({}^\omega 2)^N$. $N[H] \models 2^{\aleph_2} \geq \aleph_{\omega+1}$ because $N \models 2^{\aleph_2} = \aleph_{\omega+1}$ and $N[H] \supseteq N$ and $N, N[H]$ have the same cardinals. Finally, we prove that $\forall \theta \geq \aleph_2\, [2^\theta = \max(\theta^+, \aleph_{\omega+1})]$ holds in $N[H]$ by using cardinal arithmetic in N to count nice names in N.

For (3), we force three times, obtaining models $M = M_0 \subset M_1 \subset M_2 \subset M_3$:

$$\begin{aligned} M_1 &\models 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2 \wedge 2^{\aleph_2} = \aleph_7 \\ M_2 &\models 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_5 \wedge 2^{\aleph_2} = \aleph_7 \\ M_3 &\models 2^{\aleph_0} = \aleph_3 \wedge 2^{\aleph_1} = \aleph_5 \wedge 2^{\aleph_2} = \aleph_7 \ , \end{aligned}$$

using $(\mathrm{Fn}_{\omega_2}(\omega_7 \times \omega_2, 2))^{M_0}$, $(\mathrm{Fn}_{\omega_1}(\omega_5 \times \omega_1, 2))^{M_1}$, and $(\mathrm{Fn}_{\omega_0}(\omega_3 \times \omega_0, 2))^{M_2}$, to build M_1, M_2, M_3. Of course $\mathrm{Fn}_{\omega_0}(\omega_3 \times \omega_0, 2) = \mathrm{Fn}(\omega_3 \times \omega, 2)$. □

In proving (2) and (3), it is important that we proceed *top-down*, dealing with the largest cardinals first. This ensures that when we get around to forcing with $\mathrm{Fn}_\lambda(I, J)$, we have not yet added any λ–sequences, so that we have not messed up the hypothesis of Theorems IV.7.16 and IV.7.17 that $2^{<\lambda} = \lambda$. If this hypothesis fails, then cardinals won't be preserved. For example, consider $\lambda = \aleph_1$, so $2^{<\lambda} = \lambda \leftrightarrow CH$. Say, in proving (3), we first form M_1 using $(\mathrm{Fn}(\omega_3 \times \omega, 2))^M$. Then $M_1 \models 2^{\aleph_0} = \aleph_3$. If then we force over M_1 using $(\mathrm{Fn}_{\omega_1}(\omega_5 \times \omega_1, 2))^{M_1}$, we will collapse cardinals $\aleph_2$ and $\aleph_3$, making *CH* true in the generic extension. More generally,

Exercise IV.7.19 *Assume that $M \models \neg CH$, and let $\mathbb{P} = (\mathrm{Fn}_{\aleph_1}(I, 2))^M$, where $(|I| > \aleph_0)^M$. Then $M[G] \models CH$ and all cardinals κ of M with $(\aleph_1 < \kappa \leq \mathfrak{c})^M$ cease to be cardinals in $M[G]$.*

Hint. Since $\mathbb{P}$ is countably closed in M, the models M and $M[G]$ have the same ${}^\omega 2$ and the same ω_1. It is thus sufficient to see that $M[G]$ contains a map from $(\omega_1)^M$ onto $\mathfrak{c}^M$. Now, working in M, there is a complete embedding from $\mathrm{Fn}_{\aleph_1}(\omega_1, 2)$ into $\mathbb{P}$, and $\mathrm{Fn}_{\aleph_1}(\omega_1, 2) \approxeq \mathrm{Fn}_{\aleph_1}(\omega_1, \mathfrak{c})$ (Definition IV.4.25). But $\mathrm{Fn}_{\aleph_1}(\omega_1, \mathfrak{c})$ is the "standard way" of forcing a generic map from ω_1 onto $\mathfrak{c}$, similarly to Example IV.3.1. □

We are left with two natural questions:

a. How do we adjust the powers of infinitely many regular cardinals?

b. How do we adjust the powers of singular cardinals?

Regarding (a), say $M \models GCH$ and we wish to improve (3) of Corollary IV.7.18 to get $M[G] \models \forall n \in \omega\ [2^{\aleph_n} = \aleph_{3+2n}]$. Naively, we iterate forcing ω times rather than three times. This presents two sub-problems. First, as we just pointed out, to preserve cardinals we must iterate *top-down*, dealing with the largest cardinals first, which is nonsense if there are ω cardinals. Second,

even if we could somehow figure out how to get an ω–chain of models, what do we do at the limit? Readers who have studied algebra have probably already noted an analogy with field extensions, such as $\mathbb{Q} \subset \mathbb{Q}[\sqrt{2}] \subset \mathbb{Q}[\sqrt{2}][\sqrt[5]{3}] \subset \cdots$. Continuing through the transfinite (say, in constructing the algebraic closure), you just take unions at limit ordinals, since the union of a chain of fields is a field. This doesn't work with models of set theory, since the union of a chain of models of *ZFC* almost never satisfies *ZFC*:

Exercise IV.7.20 *Suppose that M_n is a ctm for ZFC for each $n \in \omega$ and $M = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$. Let $M_\omega = \bigcup_n M_n$. Further, suppose that there is an $A \in M$ such that each $\mathcal{P}(A) \cap M_n \subsetneq \mathcal{P}(A) \cap M_{n+1}$. Then the Power Set Axiom is false in M_ω.*

If your goal is $M_\omega \models \forall n \in \omega\, [2^{\aleph_n} = \aleph_{3+2n}]$, then the A in this exercise will be $(\omega_\omega)^{M_0}$. The correct way to iterate forcing to achieve this goal is discussed in Section V.2. For now, we remark that one can achieve this particular goal by forcing *once* over $M \models GCH$, using the poset $\prod_{n\in\omega} \mathrm{Fn}_{\omega_n}(\omega_{3+2n} \times \omega_n, 2)$ (evaluated in M, using Definition III.3.36), but the methods of iterated forcing will help us prove that this poset really works.

Regarding (b), say $M \models GCH$ and λ is an infinite cardinal of M and we wish to get $M[G] \models 2^\lambda = \lambda^{++} \wedge \forall \theta < \lambda\, [2^\theta = \theta^+]$. If λ is regular in M, then we just force with $(\mathrm{Fn}_\lambda(\lambda^{++} \times \lambda, 2))^M$ and apply Theorem IV.7.17. In $M[G]$, *GCH* remains true below λ because $\mathbb{P}$ is λ-closed in M. When λ is singular, we cannot expect to succeed in general because *GCH* cannot first fail at a singular cardinal of uncountable cofinality (see Exercise III.6.19). Also, when λ is singular, posets of the form $\mathrm{Fn}_\lambda(I, J)$ won't work because λ is not even a cardinal in the generic extension:

Exercise IV.7.21 *Assume in M that λ is a singular cardinal, $|I| \geq \lambda$, $|J| \geq 2$, and $\mathbb{P} = \mathrm{Fn}_\lambda(I, J)$. Then λ is not a cardinal in $M[G]$.*

Hint. As an example, suppose that $\lambda = (\omega_\omega)^M$, $I = \lambda$, and $J = 2$. In $M[G]$, we have $f = \bigcup G : \lambda \to 2$, and f codes a map $h : \omega \xrightarrow{\text{onto}} \lambda$ as follows:

For each n: Let $\alpha_n < \lambda$ be the least α such that $\forall \xi < \omega_n^M\, [f(\alpha + \xi) = 0]$. Let $g(n)$ be the least ξ such that $f(\alpha_n + \xi) = 1$; then $\omega_n^M \leq \xi < \lambda$. Define $h(n)$ by: $g(n) = \omega_n^M + h(n)$. Informally, the first occurrence of ω_n^M zeros in a row is followed by $h(n)$ more zeros. □

There is actually a fundamental problem here; we cannot simply modify $\mathbb{P}$ and expect it to work. Let Γ be *ZFC* plus $\forall n < \omega\, [2^{\aleph_n} = \aleph_{n+1}]$ plus $2^{\aleph_\omega} \geq \aleph_{\omega+2}$. Continuing our remarks on page 109, Γ is strictly stronger proof-theoretically than *ZFC*, so that we cannot expect an arbitrary ctm for *ZFC* to have any forcing extension satisfying Γ, and in fact if $M \models V = L$, then no forcing extension of M satisfies Γ. For more on this, see Chapter V of [20].

We leave these $\mathrm{Fn}_\lambda(I,J)$ for now and consider some other issues. First, we point out that one can modify our forcing language so that it can talk about the ground model, using the symbol $\check{V}$ to denote the class of all ground model sets. For what we do in this book, this use of $\check{V}$ may be considered to be an informal abbreviation, so we just give two illustrations and leave a formal treatment to the exercises.

Example IV.7.22 *If $\mathbb{P}$ is λ-closed and $\delta < \lambda$ then $\mathbb{1} \Vdash \forall f \in {}^{\delta}\check{V}\, [f \in \check{V}]$.*

This is easily derived from Lemma IV.7.15, once we decode what it *says*. It is an example of the informal notation described in Section IV.6, but we are now using $\check{V}$ as a name for the ground model.

In the ctm approach, we are assuming that $(\mathbb{P}$ is λ-closed$)^M$, and $f \in M[G]$ is a function with $\mathrm{dom}(f) = \delta$ and $\mathrm{ran}(f) \subseteq M$, and we wish to prove that $f \in M$. To use Lemma IV.7.15 here, we need to know that $f : \delta \to E$, where E is some element of M. But that follows by:

Lemma IV.7.23 *Suppose that $f \in M[G]$ and f is a function. Then* $\mathrm{ran}(f) \subseteq M$ *iff there is an $E \in M$ such that* $\mathrm{ran}(f) \subseteq E$.

Proof. The $\leftarrow$ direction follows from the fact that M is transitive. For the $\rightarrow$ direction, let $\alpha = \mathrm{rank}(f)$, so that $\mathrm{ran}(f) \subseteq R(\alpha)$. Applying absoluteness of rank, $\alpha < o(M[G]) = o(M)$, so let $E = R(\alpha) \cap M = (R(\alpha))^M \in M$. □

This issue didn't arise earlier because in our applications, we already had an explicit E anyway. But, one often uses "$\mathbb{P}$ doesn't add δ–sequences" as shorthand for $\mathbb{1} \Vdash \forall f\, [(f : \delta \to \check{V}) \to f \in \check{V}]$, which, in the "forcing over V" approach, we might view as an abbreviation for the more verbose statement that for each set E, if we set $W = {}^{\delta}E$, then $\mathbb{1} \Vdash^* \forall f\, [(f : \delta \to \check{E}) \to f \in \check{W}]$.

Exercise IV.7.24 *Let $\mathbb{P}$ be an atomless poset. Then*
$\mathbb{1} \Vdash\ \forall x\, \exists y \supseteq x\, [|y \cap \check{V}| = |y \backslash \check{V}|]$.

This is also easily proved, once it is properly decoded. In the ctm approach, since $\Vdash$ refers to truth in $M[G]$ and $\check{V}$ refers to M, we are saying that whenever G is generic, $\forall x \in M[G]\ \exists y \in M[G]\ \left[y \supseteq x \wedge \left(|y \cap M|^{M[G]} = |y \backslash M|^{M[G]}\right)\right]$; that is, we are computing the cardinals in $M[G]$ — not in M (which would be meaningless if $y \cap M \notin M$), and not in V (where everything in $M[G]$ is countable). To do the exercise, if $|x \cap M| < |x \backslash M|$, then $y = x \cup (\{G\} \times \alpha)$ for a suitable α. But this all doesn't make any sense unless we know that $x \cap M \in M[G]$. We need:

Lemma IV.7.25 $\mathbb{1} \Vdash\ \forall x [x \cap \check{V}$ *exists*$]$*; i.e., if $x \in M[G]$ then $x \cap M \in M[G]$.*

Proof. Let $\alpha = \mathrm{rank}(x)$, so that $x \subseteq R(\alpha)$. Then $\alpha < o(M[G]) = o(M)$, so let $E = R(\alpha) \cap M = (R(\alpha))^M \in M$. Then $x \cap M = x \cap E$ and $x, E \in M[G]$, so $x \cap E \in M[G]$. □

This was just like Lemma IV.7.23. In specific applications, one can always replace the ground model V by a specific $R(\alpha)$ and then avoid talking about V. Still, it is of some interest to know that one *can* actually talk about V. In the ctm approach, this may be stated formally by:

Exercise IV.7.26 *Let $\mathcal{L} = \{\in, P\}$, where P is a 1–ary predicate symbol, informally denoting a class. Let ZFC_P be the axioms of ZFC, except that in the Comprehension and Replacement schemes, the φ can use P as well as $\in$. Now, regard $M[G]$ as a structure for $\mathcal{L}$ by interpreting P as M, and prove that $M[G] \models ZFC_P$. Also, prove the Truth and Definability Lemmas for sentences using P. You can view "$\pi \in \check{V}$" as an abbreviation for "$P(\pi)$". Analogously to Definition IV.2.35, $p \Vdash^* \pi \in \check{V}$ iff $\{q \leq p : \exists x\, [q \Vdash^* \pi = \check{x}]\}$ is dense below p.*

In the boolean valued models approach $[\![\pi \in \check{V}]\!] = \bigvee\{[\![\pi = \check{x}]\!] : x \in V\}$.

Exercise IV.7.27 *Let (in the ground model) $\theta = |\delta|$. Then prove that $\mathbb{1} \Vdash \forall f \in {}^{\delta}\check{V}\, [f \in \check{V}]$ iff $\mathbb{1} \Vdash \forall f \in {}^{\theta}\check{V}\, [f \in \check{V}]$, after interpreting this statement correctly in the ctm approach.*

Informally, the property "$\mathbb{P}$ doesn't add δ–sequences" only depends on $|\delta|$.

The notion that $\mathbb{P}$ doesn't add θ–sequences can be expressed topologically. Generalizing Lemma IV.6.9 and Exercise IV.6.10,

Exercise IV.7.28 *For a forcing poset $\mathbb{P}$ and cardinal θ, prove that* (1) $\rightarrow$ (2) *and that* (2) $\rightarrow$ (1) *if $\mathbb{P}$ is separative:*

1. *In $\mathbb{P}$, every intersection of θ dense open sets is dense ($\mathbb{P}$ is θ^+-*Baire *in the poset topology).*
2. *$\mathbb{P}$ doesn't add θ–sequences.*

Of course, first interpret these statements correctly in the ctm approach.

As we pointed out in discussing Lemma IV.6.9, the implication (2) $\rightarrow$ (1) can fail, and the counter-example was even a tree order. Trees will be separative if they branch at every node.

Exercise IV.7.29 *Assume that κ is regular and T is a well-pruned κ–Suslin tree. Then in the poset topology, every intersection of* fewer *than κ dense open sets is dense. Conclude from this that forcing with $\mathbb{P}$ preserves cardinals and cofinalities. Also show that in $V[G]$, G is a path through T.*

This exercise, which extends Lemma III.5.29, shows that it is easy to kill *one* κ–Suslin tree in a generic extension. But, that doesn't give us a model of κ–*SH* (the assertion that there are *no* κ–Suslin trees). For $\kappa = \omega_1$, we shall obtain a model of *SH* (i.e., ω_1–*SH*) in Chapter V by iterated forcing, killing *all* Suslin trees; this forcing in fact produces a model of $MA + \neg CH$, which implies *SH* (Corollary III.3.44). It is much harder to obtain a model of

$SH + CH$. This was done first by Jensen (see [21]), and later, using a different argument, by Shelah (see [81, 84]). It is unknown whether ω_2–$SH + GCH$ is consistent. Mitchell [68] constructed a model in which there are not even any ω_2–Aronszajn trees, but in his model, $\mathfrak{c} = \aleph_2$. Under CH, there is an ω_2–Aronszajn tree (Exercise III.5.44). Laver and Shelah [62] constructed a model for ω_2–$SH + CH$, but $2^{\aleph_1} = \aleph_3$ in their model. As an aside, we remark that in general, for getting a model for the κ–$SH + GCH$, it is enough to get GCH below κ; e.g., if you have a model for $SH + CH$, then you easily get one for $SH + GCH$; see Exercise IV.7.54.

One can also *create* Suslin trees in generic extensions. For ω_1–Suslin trees, this was done by Jech and by Tennenbaum in the 1960s, using two different methods. Tennenbaum forced with finite approximations to the tree order, so his poset was ccc. Jech's forcing, which we describe below (Exercise IV.7.41), used countable approximations to the tree, and is ω_1-closed. But it turns out (Exercise IV.7.43) that his forcing also makes $\diamondsuit$ true, and we already know that $\diamondsuit \to \neg SH$ (Theorem III.7.3). Knowing all this, it is simpler just to force $\diamondsuit$, using $\mathrm{Fn}_{\aleph_1}(I, 2)$. The next lemma overlaps somewhat Exercise IV.7.19, which shows that $\mathrm{Fn}_{\aleph_1}(I, 2)$ makes CH true:

Lemma IV.7.30 *If* $\mathbb{P} = (\mathrm{Fn}_{\aleph_1}(I, 2))^M$, *where* $(|I| > \aleph_0)^M$, *then* $M[G] \models \diamondsuit$.

Proof. Replacing $\mathbb{P}$ by an isomorphic poset, we may assume that $I = J \cup K$, where $J = \{(\xi, \alpha) : \xi < \alpha < (\omega_1)^M\}$ and K is an arbitrary set. In $M[G]$, we have $\bigcup G : I \to 2$, and $\bigcup G \restriction J$ codes the $\diamondsuit$ sequence (Definition III.7.1). Define $f_\alpha : \alpha \to 2$ by $f_\alpha(\xi) = \bigcup G(\xi, \alpha)$; then f_α is the characteristic function of $A_\alpha := f_\alpha^{-1}\{1\}$, and we shall prove that $\vec{A} := \langle A_\alpha : \alpha < \omega_1^M \rangle$ is a $\diamondsuit$ sequence in $M[G]$. Note that $(\omega_1)^M = (\omega_1)^{M[G]}$ because $\mathbb{P}$ is countably closed in M.

Working in $M[G]$: If $\vec{A}$ is not a $\diamondsuit$ sequence, then fix $h : \omega_1 \to 2$ such that $\{\alpha < \omega_1 : h\restriction\alpha = f_\alpha\}$ is non-stationary, so then fix a club $C \subseteq \alpha$ such that $h\restriction\alpha \neq f_\alpha$ for all $\alpha \in C$. Now, fix names $\mathring{h}, \mathring{C} \in M^{\mathbb{P}}$ such that $h = \mathring{h}_G$ and $C = \mathring{C}_G$. Then, fix $p \in G$ that forces all the bad things about $\mathring{h}, \mathring{C}$:

$$p \Vdash \left[\mathring{C} \subseteq \check{\omega}_1 \text{ is club } \wedge\, \mathring{h} : \check{\omega}_1 \to 2 \wedge \forall \alpha \in \mathring{C}\, \exists \xi < \alpha \left[\bigcup \dot{G}(\xi, \alpha) \neq \mathring{h}(\xi)\right]\right] \ . \quad (*)$$

The statement being forced is a sentence in $\mathcal{FL}_{\mathbb{P}}$ using the names $\check{\omega}_1, \mathring{C}, \mathring{h}$, as well as $\dot{G}$, the canonical name for the generic filter.

We now forget about G and work in M to derive a contradiction from $(*)$. For $p \in \mathbb{P}$, let $\mathrm{supt}(p)$ be the least $\beta < \omega_1$ such that $\mathrm{dom}(p) \cap J \subseteq \{(\xi, \alpha) : \xi < \alpha < \beta\}$. Then, recursively choose $p_n, \beta_n, \delta_n, k_n$ for $n < \omega$ such that:

1. Each $p_n \in \mathbb{P}$ and $p = p_0 \geq p_1 \geq p_2 \geq \cdots$.
2. $\beta_n = \mathrm{supt}(p_n)$ and $\beta_n < \delta_n < \beta_{n+1}$.
3. $k_n \in {}^{\beta_n}2$ and $p_{n+1} \Vdash (\check{\delta}_n \in \mathring{C}) \wedge \mathring{h}\restriction\check{\beta}_n = \check{k}_n$.

To prove that we can indeed choose such $p_n, \beta_n, \delta_n, k_n$: p_0 is given by (1), and β_n is determined from p_n by (2). Now, fix n, assume that we have p_n, β_n, and we

shall obtain $\delta_n, k_n, p_{n+1}, \beta_{n+1}$. First, $p_n \leq p$, which forces $\mathring{C}$ to be unbounded, so $p_n \Vdash \exists \xi \in \check{\omega}_1 [\xi \in \mathring{C} \wedge \check{\beta}_n < \xi]$. Applying Lemma IV.6.1, we can choose $q \leq p_n$ and a $\delta_n \in \omega_1$ such that $q \Vdash [\check{\delta}_n \in \mathring{C} \wedge \check{\beta}_n < \check{\delta}_n]$. Then $\beta_n < \delta_n$ is true (since it is true in some generic extension) and $q \Vdash \check{\delta}_n \in \mathring{C}$. Now, let $W = {}^{\beta_n}2$. Since $\aleph_1$-closed forcings don't add β_n-sequences, $q \Vdash \exists x \in \check{W} [\mathring{h}{\restriction}\check{\beta}_n = x]$. Applying Lemma IV.6.1 again, we can choose $k_n \in W$ and $p_{n+1} \leq q$ such that $p_{n+1} \Vdash \mathring{h}{\restriction}\check{\beta}_n = \check{k}_n$. Extending p_{n+1} further if necessary, we may assume that $\beta_{n+1} := \mathrm{supt}(p_{n+1}) > \delta_n$.

Still within M, let $r = \bigcup_n p_n$, and let $\gamma = \sup_n \beta_n = \sup_n \delta_n$. Then $r \Vdash (\check{\delta}_n \in \mathring{C})$ for each n and $r \Vdash \mathring{C}$ is club , so $r \Vdash \check{\gamma} \in \mathring{C}$ (consider any generic extension by a filter containing r). Similarly, $r \Vdash \mathring{h}{\restriction}\check{\beta}_n = \check{k}_n$ for each n, so if $k = \bigcup_n k_n$, then $r \Vdash \mathring{h}{\restriction}\check{\gamma} = \check{k}$ (so, in particular, k is a function from γ to 2). Now, $\mathrm{supt}(r) = \gamma$, so we can define $s \leq r$ so that $s(\gamma, \xi) = k(\xi)$ for all $\xi < \gamma$. But then $s \Vdash \check{\gamma} \in \mathring{C}$ and $s \Vdash \forall \xi < \check{\gamma} [\bigcup \mathring{G}(\xi, \check{\gamma}) = \mathring{h}(\xi)]$ which, since $s \leq p$, contradicts $(*)$ (consider any generic extension by a filter containing s). □

Note that in the last two paragraphs, although we said "work in M", three of our parenthetical remarks said to step outside of M temporarily (and look at some generic extension) to justify some statement about forcing. This is typical of forcing arguments, although eventually such parenthetical remarks are left to the reader.

We next give some results about random real forcing.

Notation IV.7.31 *A* real *is a real number (element of* $\mathbb{R}$*), but "real" also often indicates an element of* 2^ω *or* ω^ω *or* $\mathcal{P}(\omega)$.

For example, the poset $\mathrm{Fn}(I, 2)$ is often called "Cohen real forcing" because $\mathrm{Fn}(\kappa \times \omega, 2)$ adds a set $\{h_\alpha : \alpha < \kappa\} \subset 2^\omega$ (see the proof of Lemma IV.3.7). We often view h_α as the binary expansion of a real number $r_\alpha = \sum_{n \in \omega} h_\alpha(n) 2^{-n-1}$. Then $\{r_\alpha : \alpha < \kappa\} \subset \mathbb{R}$ has some interesting properties as a set or real numbers. For example, it has size κ and is a Luzin set (see Exercise V.2.23).

Random real forcing also adds κ reals in a generic extension. This provides a model of $\mathfrak{c} \geq \kappa$, but the reals have different properties from Cohen reals (see Exercises V.4.30 and V.4.31). As with Cohen reals, we think of our reals at first as being in 2^ω; we can obtain elements of $\mathbb{R}$ later by using binary expansions.

Let κ be any uncountable cardinal. *Informally*, perform the following experiment: Toss a fair coin κ times, and identify heads with 0 and tails with 1, so our experiment determines a κ-sequence F of 0s and 1s; so $F \in 2^\kappa$. For $\alpha < \kappa$, we now define a real, $h_\alpha \in 2^\omega$, by: $h_\alpha(n) = F(\omega \cdot \alpha + n)$. Furthermore, the probability $\Pr(h_\alpha = h_\beta) = 0$ when $\alpha \neq \beta$, since for each n, $\Pr(h_\alpha{\restriction}n = h_\beta{\restriction}n) = \Pr(\forall m < n [F(\omega \cdot \alpha + m) = F(\omega \cdot \beta + m)]) = 2^{-n}$. So, we are 100% sure that these reals are all different, so that $2^{\aleph_0} \geq \kappa$. This may be taken as a platonistic argument that *CH* is false; but also, since κ is arbitrarily large, it may be an argument that the Power Set Axiom is false.

We can make formal sense out of this naive idea by "doing" the experiment in some forcing extension $V[G] \supseteq V$, where G is $\mathbb{P}$–generic over V. The poset $\mathbb{P}$ is motivated by classical probability theory, where you don't actually *do* any experiments — you just define the sample space X, which may be viewed informally as the set of all possible outcomes of the experiment. Here, $X = 2^\kappa$. You also need a probability measure μ on X. It is natural to let μ be the standard product measure (see Halmos [43]) of κ copies of the counting measure μ_0 on 2 (since the coin is fair, $\mu_0(\{0\}) = \mu_0(\{1\}) = 1/2$).

There are now several possible precise definitions for $\mathbb{P}$, all of which lead to equivalent forcings. We shall choose one, and leave the others to Exercise IV.7.38. Given a probability measure, a complete boolean algebra appears naturally, and we shall use it for our forcing, using the terminology in Section IV.4. Measure algebras are defined in [43]; we shall use the following notation:

Definition IV.7.32 *If μ is a probability measure on X, then $\mathbb{MB} = \mathbb{MB}(X, \mu)$ denotes the measure algebra. Elements of $\mathbb{MB}$ are equivalence classes $[S]$ of measurable sets modulo null sets. $[S] \leq [T]$ iff $\mu(S \backslash T) = 0$. $\mathbb{1} = [X]$ and $\mathbb{0} = [\emptyset]$. $\mu([S])$ denotes $\mu(S)$.*

As in Section IV.4, we shall say that we are forcing with $\mathbb{MB}$, but the forcing poset is really $\mathbb{MB} \backslash \{\mathbb{0}\}$.

Definition IV.7.33 *Let κ be any infinite cardinal. The poset for* adding κ random reals *is $\mathbb{MB}(2^{\kappa \times \omega}, \mu)$, where μ is the standard product measure.*

Note that we have replaced the exponent κ by $\kappa \times \omega$. Now our notation will correspond to the $\mathrm{Fn}(\kappa \times \omega, 2)$ which we used to force κ Cohen reals. Some basic facts about random real forcing hold for every $\mathbb{MB}(X, \mu)$:

Lemma IV.7.34 *Let $\mathbb{MB} = \mathbb{MB}(X, \mu)$, where μ is a probability measure on X. Then $\mathbb{MB}$ is ccc. Also, $\mathbb{1} \Vdash \forall f \in \omega^\omega \ \exists h \in \omega^\omega \cap \check{V} \ \forall n \in \omega \ f(n) \leq h(n)$.*

Proof. For the ccc: if $A \subseteq \mathbb{MB}$ is an antichain, then $|\{p \in A : \mu(p) \geq 1/n\}| \leq n$ for each $n \in \omega$. The second statement says that every function in ω^ω in the generic extension is dominated by a function in the ground model. We have expressed it using our "$\check{V}$" terminology, (see page 296), but we can decode it into our official ctm approach:

If M is our ctm for *ZFC*, then working in M, we have a probability measure μ on X and we have defined $\mathbb{MB} = \mathbb{MB}(X, \mu)$. Let $W = (\omega^\omega)^M$. We wish to prove that for each generic G, if $f \in \omega^\omega \cap M[G]$, then there is an $h \in \omega^\omega \cap M$ such that $\forall n \in \omega \ f(n) \leq h(n)$. Note that "measure algebra" is not absolute, so our forcing poset is really $(\mathbb{MB}(X, \mu))^M$, and any measure theory that we do with X is really done within the model M.

It is sufficient to prove within M that whenever $p \Vdash \mathring{f} : \omega \to \omega$, there is an $h : \omega \to \omega$ and some $q \leq p$ such that $q \Vdash \forall n \in \omega \ \mathring{f}(n) \leq \check{h}(n)$ (see the proof of (3) $\to$ (2) in Lemma IV.6.9). To obtain h, we make use of the boolean

truth value $[\![\varphi]\!]$, which is the largest forcing condition that forces φ (see Lemma IV.4.19). For each n: $p \leq [\![\mathring{f}(n) \in \omega]\!]$, so if we define $r_{n,m} = p \wedge [\![f(n) = m]\!]$, then the $r_{n,m}$ are pairwise incompatible and $\bigvee_m r_{n,m} = p$. Thus, we can choose $h(n) \in \omega$ large enough such that

$$\mu\big(\bigvee\{r_{n,m} : m \leq h(n)\}\big) \;\geq\; \mu(p) \cdot (1 - 2^{-n-2}) \quad .$$

Let $q = \bigwedge_{n\in\omega} \bigvee\{r_{n,m} : m \leq h(n)\}$. Then $q \leq p$ and $q \neq \mathbb{0}$ because

$$\mu(q) \geq \mu(p) \cdot (1 - \textstyle\sum_n 2^{-n-2}) = \mu(p)/2 > 0 \quad .$$

Also, for each $n \in \omega$, $q \Vdash \exists m \leq h(n)\ \mathring{f}(n) = m$, so $q \Vdash \mathring{f}(n) \leq \check{h}(n)$; so $q \Vdash \forall n \in \omega\ \mathring{f}(n) \leq \check{h}(n)$. □

This "approximation lemma" for random real forcing is analogous to Lemma IV.3.5 for general ccc posets; in M, h picks out a *finite* set of possible values for each $f(n)$. However, Lemma IV.7.34 is false for "most" ccc posets. For example, by the following lemma, it is false for Cohen forcing. We remark that Lemma IV.4.9 showed that no one function in $\omega^\omega \cap M[G]$ dominates $\omega^\omega \cap M$, while this lemma shows that $\omega^\omega \cap M$ does not dominate $\omega^\omega \cap M[G]$.

Lemma IV.7.35 *Let $\mathbb{P} = \mathrm{Fn}(\omega, \omega)$ and let G be $\mathbb{P}$-generic over M. In $M[G]$, let $f = \bigcup G : \omega \to \omega$. Then there is no $h \in \omega^\omega \cap M$ such that $f \leq^* h$.*

Proof. For each $m \in \omega$, $\{p \in \mathbb{P} : \exists n \in \mathrm{dom}(p)\ [n > m \wedge p(n) > h(n)]\}$ is dense, so $\forall m\, \exists n > n\, [f(n) > h(n)]$ whenever $h \in M$. □

Lemma IV.7.34 lets us get models of $\mathfrak{b} = \mathfrak{d} = \aleph_1 < \mathfrak{c}$, whereas the Cohen real models satisfy $\aleph_1 = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$ (see Lemmas IV.4.10 and Lemma V.2.19).

Lemma IV.7.36 *Suppose M is a ctm for ZFC+CH, and let G be $\mathbb{MB}$-generic over M, where $\mathbb{MB}$ is a measure algebra in M. Then $M[G] \models \mathfrak{b} = \mathfrak{d} = \aleph_1$.*

Proof. $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d}$ is always true (Lemma III.1.12), and $\mathfrak{d} \leq \aleph_1$ holds because $\omega^\omega \cap M$ is a dominating family of size $\aleph_1$. □

Of course, we now need to get $M[G] \models \aleph_1 < \mathfrak{c}$ for a suitable $\mathbb{MB}$, and it is here that we use $\mathbb{MB} = \mathbb{MB}(2^{\kappa\times\omega}, \mu)$. This random real forcing produces the same cardinal arithmetic as does the Cohen forcing $\mathrm{Fn}(\kappa \times \omega, 2)$. Thus, the two forcings are "equally good" for producing models of $\neg CH$. We presented Cohen real forcing first because it can be understood without any knowledge of measure theory, and because it has a natural generalization to $\mathrm{Fn}_\lambda(I, J)$, which is lacking for random real forcing. The Cohen real analog for the following lemma is easy from the results of Section IV.3. For simplicity, we just state what happens when $M \models GCH$:

Lemma IV.7.37 *Let M be a ctm for GCH. Working within M, let κ be an infinite cardinal and let $\mathbb{MB} = \mathbb{MB}(2^{\kappa\times\omega}, \mu)$. Let G be $\mathbb{MB}$-generic over M. Then M and $M[G]$ have the same cardinals and cofinalities. In $M[G]$, $2^{\aleph_0}$ is κ if $\mathrm{cf}(\kappa) > \omega$ and κ^+ if $\mathrm{cf}(\kappa) = \omega$; also $2^\theta = \theta^+$ for all $\theta \geq \kappa$.*

Proof. Cofinalities and cardinalities are preserved by the ccc.

To prove that $M[G] \models 2^{\aleph_0} \geq \kappa$, we produce "reals" $h_\alpha \in 2^\omega$ for $\alpha < \kappa$. Informally, as in our discussion above, we toss a coin $\kappa \times \omega$ times to obtain $F : \kappa \times \omega \to 2$; then $h_\alpha(n) = F(\alpha, n)$. More formally, we get a name $\mathring{F}$ such that $[\![\mathring{F} : \kappa \times \omega \to 2]\!] = \mathbb{1}$ and each $[\![\mathring{F}(\alpha, n) = i]\!] = p_i^{\alpha,n}$, where $p_i^{\alpha,n} = [\{x \in 2^{\kappa\times\omega} : x(\alpha, n) = i\}]$. So, in $M[G]$, G is a filter on $\mathbb{MB}$ and we compute $F(\alpha, n)$ to be the $i \in \{0, 1\}$ such that $p_i^{\alpha,n} \in G$. One of the two must be in G because $\{p_0^{\alpha,n}, p_1^{\alpha,n}\}$ is a maximal antichain and in M (more generally, the generic filter is always an ultrafilter). The h_α are all distinct because if $\alpha \neq \beta$,

$$[\![\forall n\big(F(\alpha, n) = F(\beta, n)\big)]\!] = \big[\{x \in 2^{\kappa\times\omega} : \forall n \left(x(\alpha, n) = x(\beta, n)\right)\}\big] = \mathbb{0} \quad .$$

Then, $2^{\aleph_0} \geq \kappa^+$ in the case that $\mathrm{cf}(\kappa) = \omega$.

To get the claimed upper bounds on $2^{\aleph_0}$ and on 2^θ for all $\theta \geq \kappa$, we count nice names in M. Working in M, note that $|\mathbb{MB}| = \kappa^{\aleph_0}$ because there are κ clopen sets in the space $2^{\kappa\times\omega}$, and the boolean σ–algebra $\mathcal{B}$ generated by the clopen sets has size $\kappa^{\aleph_0}$, and every measurable set is equal to an element of $\mathcal{B}$ modulo a null set. The rest of the computation just uses the size of the poset $\mathbb{MB}$, and follows from Lemma IV.3.11 and the *GCH*. □

When $\kappa = \aleph_0$, our $\mathbb{MB}$ is called either the poset for adding ω random reals, or the poset for adding one random real, since $\mathbb{MB}(2^{\omega\times\omega}, \mu) \cong \mathbb{MB}(2^\omega, \mu)$. Now, $M[G] \models GCH + V \neq L$.

Although we used the measure algebra to define random real forcing, the next exercise describes a number of equivalent posets that yield equivalent forcings.

Exercise IV.7.38 *Let I be any non-empty set and let μ be the natural probability measure on 2^I. Let $\mathcal{BA} \subseteq \mathcal{P}(2^I)$ be the boolean σ–algebra $\mathcal{BA}$ generated by the clopen sets (these are the Baire sets). Let $\mathcal{BO} \subseteq \mathcal{P}(2^I)$ be the boolean σ–algebra $\mathcal{BA}$ generated by the open sets (these are the Borel sets). Let $\mathcal{M}$ be the set of all measurable sets. Let $\mathcal{F}$ be the set of all closed G_δ sets. So, $\mathcal{F} \subseteq \mathcal{BA} \subseteq \mathcal{BO} \subseteq \mathcal{M}$. Define $\mathbb{P}_\mathcal{F} = \{p \in \mathcal{F} : \mu(p) > 0\}$. Likewise define $P_{\mathcal{BA}}, P_{\mathcal{BO}}, P_\mathcal{M}$. Order all four posets by $\subseteq$. For all four of these, prove that the natural embedding into $\mathbb{MB}(2^I, \mu)$ is a dense embedding, so that they all yield equivalent forcings, in the sense of Definition IV.4.25.*

Note that the boolean algebra $\mathbb{MB}(2^I, \mu)$ is isomorphic to the algebra of Borel sets modulo null sets. Then, if you replace "null" by "first category", you get Cohen forcing:

Exercise IV.7.39 *Let $\mathbb{B}$ be the boolean algebra of Borel subsets of 2^I modulo the first category sets. Prove that $\mathbb{B} \cong \mathrm{ro}(2^I)$ (Definition III.4.6). Define $i : \mathrm{Fn}(I,2) \to \mathbb{B}$ so that $i(p)$ is the equivalence class of the clopen set $\{x \in 2^I : x \supseteq p\}$. Prove that i is a dense embedding.*

Hint. First prove that every Borel set U has the *property of Baire*; that is, it is equal to a regular open set R modulo a first category set. For open U, $R = \mathrm{int}(\mathrm{cl}(U))$; and now proceed by induction. □

The property of Baire is the category analog of "measurable", and should not be confused with the property of being a Baire space (Definition IV.6.8). For more on the analogy between Cohen and random real forcing, see [55].

The following exercise generalizes Lemma IV.7.30, which showed that the poset $\mathrm{Fn}_{\aleph_1}(I,2)$ makes $\diamondsuit$ true in the generic extension.

Exercise IV.7.40 *Assume that in M, κ is an uncountable regular cardinal, $|I| \geq \kappa$, and $\mathbb{P} = \mathrm{Fn}_\kappa(I,2)$. Then $M[G] \models \diamondsuit(\kappa)$ (Definition III.7.15).*

The next four exercises give some properties of Jech's poset for adding a κ–Suslin tree in a generic extension. For $\kappa = \omega_1$, it turns out to be equivalent to the poset $\mathrm{Fn}_{\aleph_1}(\omega_1, 2)$, which makes $\diamondsuit$ true (Lemma IV.7.30). But it is still of interest for $\kappa > \omega_1$, where it is not equivalent to any $\mathrm{Fn}_\kappa(I,2)$, and in any case $\diamondsuit(\kappa)$ alone is not sufficient for producing a κ–Suslin tree.

Exercise IV.7.41 *Let $\kappa > \omega$ be regular. Define Jech's κ–Suslin tree poset $\mathbb{P}$ as follows. Elements of $\mathbb{P}$ are either the empty tree $\mathbb{1}$ or a subtree p of ${}^{<\kappa}2$ such that $|p| < \kappa$ and $\mathrm{height}(p) = \alpha + 1$ for some limit $\alpha < \kappa$, and for each $s \in p$ with $\mathrm{height}(s) < \alpha$: $s^\frown 0 \in p$ and $s^\frown 1 \in p$ and $\exists t \in \mathcal{L}_\alpha(p)\,[s \sqsubset t]$.*

Define $q \leq p$ iff q is an end extension of p; that is, if $\mathrm{height}(p) = \alpha + 1$ and $\mathrm{height}(q) = \beta + 1$, then $\alpha \leq \beta$ and $p = \{s \in q : \mathrm{height}(s) \leq \alpha\}$.

Prove that $\mathbb{P}$ is ω_1-closed but not ω_2-closed.

Hint. There are $p_\xi \in \mathbb{P}$ for $\xi < \omega_1$ such that each $|p_\xi| = \aleph_0$ and $p_0 > p_1 > \cdots$ and $\bigcup_\xi p_\xi$ is an ω_1–Aronszajn tree. Then, there is no q extending all the p_ξ. □

κ remains regular in the generic extension. To prove this when $\kappa > \omega_1$, use the following, due to Prikry and Silver:

Exercise IV.7.42 *Let $\kappa, \mathbb{P}$ be as in Exercise IV.7.41. Then, in $\mathbb{P}$, every intersection of fewer than κ dense open sets dense ($\mathbb{P}$ is κ-Baire in the poset topology). Hence, by Exercise IV.7.28, $\mathbb{P}$ doesn't add $< \kappa$-sequences.*

Hint. Say D_ξ is dense open for $\xi < \omega_1$. Starting with an arbitrary p_0, we can recursively get $p_\xi \in \mathbb{P}$ for $\xi < \omega_1$ such that $p_0 > p_1 > \cdots$ and each $p_{\xi+1} \in D_\xi$. But, as Exercise IV.7.41 shows, we must do something more to ensure that there is a q extending all the p_ξ.

Say height$(p_\xi) = \alpha_\xi + 1$. Along with the p_ξ, construct a function f so that whenever $s \in p_\xi$, $f(s,\xi) \in L_{\alpha_\xi}(p_\xi)$ and $s \sqsubseteq f(s,\xi)$. Make sure that $\xi < \eta \to f(s,\xi) \sqsubset f(s,\eta)$. Then, when you get to stage ω_1, you can let $\tilde{q} = \bigcup_\xi p_\xi$. This $\tilde{q}$ has height $\alpha_{\omega_1} = \sup_\xi \alpha_\xi$, so $\tilde{q} \notin \mathbb{P}$, but you can extend $\tilde{q}$ to a $q \in \mathbb{P}$ because for each $s \in \tilde{q}$, the function f gives you a path through $\tilde{q}$ going through s. □

Thus, $\mathbb{P}$ preserves cofinalities at κ and below. Also, if $2^{<\kappa} = \kappa$, then $|\mathbb{P}| = \kappa$, so it is clearly κ^+-cc and thus preserves cofinalities above κ as well.

Exercise IV.7.43 *Let $\mathbb{P}$ be as in Exercise IV.7.41, with $\kappa = \omega_1$. Prove that $\mathbb{P} \cong \mathrm{Fn}_{\aleph_1}(\omega_1, 2)$ (Definition IV.4.25), so $\diamondsuit$ holds in the generic extension.*

Hint. Let $\mathbb{Q}$ be the set of all $p \in \mathrm{Fn}_{\aleph_1}(\omega_1, \mathfrak{c})$ such that dom(p) is a successor ordinal or 0. Prove that $\mathbb{Q}$ has a dense embedding into both of these posets. □

Finally, we must show that our $\mathbb{P}$ really does add a κ-Suslin tree even when $\kappa > \aleph_1$:

Exercise IV.7.44 *Let M be a ctm for ZFC. Fix $\kappa \in M$ such that $\kappa > \omega$ and (κ is regular)^M, and within M, define $\mathbb{P}$ as described in Exercise IV.7.41.*

In $M[G]$, define $T = \bigcup G$; so $T \subseteq {}^{<\kappa}2$. Prove that T is a κ-Suslin tree in $M[G]$.

Hint. To prove that T is a κ-tree (Definition III.5.7), show that for each $\delta < \kappa$, $\{p \in \mathbb{P} : \mathrm{height}(p) > \delta\}$ is dense.

The proof that T is Suslin is a little like the proof that $\diamondsuit$ implies that there is a Suslin tree (Theorem III.7.3). Say $p \Vdash (\mathring{A}$ is a maximal antichain in $\mathring{T})$. Then, we need to produce a $q \leq p$ such that $q \Vdash |\mathring{A}| < \kappa$.

First, get p_n for $n \in \omega$ with $p = p_0 > p_1 > \cdots$ such that for each $s \in p_n$, either $p_{n+1} \Vdash \check{s} \in \mathring{A}$ or $p_{n+1} \Vdash \check{s} \notin \mathring{A}$. Let $\tilde{q} = \bigcup_n p_n$; then height$(\tilde{q})$ is some limit ordinal γ. Let $\tilde{A} = \{s \in q : \exists m\, [p_m \Vdash \check{s} \in \mathring{A}]\}$. Make sure that $\tilde{A}$ winds up being a maximal antichain in $\tilde{q}$. Now, extend $\tilde{q}$ to a forcing condition q with height$(q) = \gamma + 1$ so that every node in level γ of q is above some element of $\tilde{A}$. Then $q \Vdash (\mathring{A} = \tilde{A})$. □

The next exercise is too easy, but it suggests two harder ones that follow it.

Exercise IV.7.45 *Let κ be regular and let T be a well-pruned κ-Aronszajn tree. Then, as forcing poset, T is not κ-closed.*

Now, T kills itself in the T-forcing extension, and if T is Suslin, this extension preserves cardinals and cofinalities (Exercise IV.7.29). By an argument due to Silver, this killing cannot be accomplished by any κ-closed poset.

Exercise IV.7.46 *Assume in M that κ is regular and uncountable and $\mathbb{P}$ is some κ-closed poset. Then every κ-Suslin tree in M remains κ-Suslin in $M[G]$,*

every κ-Aronszajn tree in M remains κ-Aronszajn in $M[G]$, every $\diamondsuit(\kappa)$ sequence in M remains a $\diamondsuit(\kappa)$ sequence in $M[G]$, and every stationary subset of κ remains stationary in $M[G]$.

Hint. Suppose that T is a κ–Aronszajn tree and p forces that $\mathring{C}$ is a path through T. Working in M, get $p = p_0 \geq p_1 \geq \cdots \geq p_\xi \geq \cdots$ for $\xi < \kappa$, along with nodes $x_\xi \in T$, such that $p_\xi \Vdash x_\xi \in \mathring{C}$ and the x_ξ are all different. Now $\mathbb{P}$ need not be κ^+-closed, so there need not be any forcing condition extending all the p_ξ, but, in M, $\{x_\xi : \xi < \kappa\}$ is a chain in T of size κ. □

Despite Exercise IV.7.45, a κ–Suslin tree can have some non-trivial closure property weaker than κ-closure. For example:

Exercise IV.7.47 *Prove that $V = L$ implies that there is a well-pruned ω_2–Suslin tree that is countably closed as a forcing poset. Also, prove that if $M \models CH$, then such a tree can be added in a forcing extension of M that preserves cofinalities and cardinals.*

Hint. For the forcing, modify the Jech order so that the forcing conditions are countably closed. Then, you have to modify the proof that the generic tree is Suslin (Exercise IV.7.44), using $\langle p_\xi : \xi < \omega_1 \rangle$ instead of $\langle p_n : n < \omega \rangle$.

For the L proof, see Exercise III.7.17. □

We next make some remarks on *cardinal collapsing*; that is, situations where cardinals of M cease to be cardinals in $M[G]$. As explained after Exercise IV.7.10, we can collapse cardinals to produce models for statements such as $\aleph_1 = \aleph_6^L$, $\aleph_2 = \aleph_7^L$, ... One can also get a model in which the true $\aleph_1$ is an inaccessible cardinal of L by the following argument, due to A. Lévy:

Exercise IV.7.48 *Assume that $(\kappa$ is weakly inaccessible$)^M$; so, $\kappa = \aleph_\kappa^M$. Let $\mathbb{P}$ be the poset defined in Exercise III.3.96. Then prove that $\aleph_1^{M[G]} = \kappa$, $\aleph_2^{M[G]} = (\kappa^+)^M$, and in general, $\aleph_{1+\alpha}^{M[G]} = (\aleph_{\kappa+\alpha})^M$.*

Converting this to a relative consistency proof, as in Section IV.5, we would get $\mathrm{Con}(ZFC+\varphi) \rightarrow \mathrm{Con}(ZFC+\psi)$, where φ is the statement that inaccessible cardinals exist and ψ is the statement that the true $\aleph_1$ is an inaccessible cardinal of L. Note that weak and strong inaccessibles are equiconsistent, in the sense of Lemma IV.3.17. Also, $ZFC + \psi \vdash \mathrm{Con}(ZFC)$, so we cannot expect to get a finitistic proof of $\mathrm{Con}(ZFC) \rightarrow \mathrm{Con}(ZFC + \psi)$ because that would yield $ZFC + \psi \vdash \mathrm{Con}(ZFC + \psi)$.

The following is clear from what we have already done (see Definition IV.6.2 and Lemmas IV.6.3, IV.7.15, and IV.7.12):

Remark IV.7.49 *If $\mathbb{Q}$ is either ccc or doesn't add ω-sequences, then $\mathbb{Q}$ does not collapse ω_1.*

As we noted after Lemma IV.6.3, this remark can be understood formally either in the "forcing over V" approach or the ctm approach. But the ccc is a straightforward combinatorial property, and can be understood without any knowledge of forcing at all. The property "doesn't add ω–sequences" is a little more complicated, but fairly straightforward. If $\mathbb{Q}$ is separative, then it is equivalent to $\mathbb{Q}$ being Baire (Lemma IV.6.9 and Exercise IV.6.10). A non-separative $\mathbb{Q}$ can be densely mapped into a separative $\mathbb{P}$ (such as its boolean completion; see Lemma III.4.8); then $\mathbb{Q}$ doesn't add ω–sequences iff $\mathbb{P}$ is Baire.

Now, the property of collapsing ω_1 also has a combinatorial equivalent, whereupon Remark IV.7.49 could be stated and proved without reference to forcing. But, the combinatorial equivalent is a bit ugly, so detailed results on this property usually presume some knowledge of forcing over V, or else presume that everything is relativized to some ctm. The equivalent is slightly less ugly in the case of complete boolean algebras:

Exercise IV.7.50 *Let $\mathbb{B}$ be a complete boolean algebra. Prove that $\mathbb{B}$ collapses ω_1 iff there are $b^\alpha_n \in \mathbb{B}$ for $n < \omega$ and $\alpha < \omega_1$ such that each $\{b^\alpha_n : \alpha < \omega_1\}$ is an antichain and each $\bigvee_{n\in\omega} b^\alpha_n = \mathbb{1}$.*

Hint. In the ctm approach, we have $\mathbb{B} \in M$ and "ω_1" really means $(\omega_1)^M$. We may view b^α_n as $[\![\mathring{f}(\check{n}) = \check{\alpha}]\!]$, where $\mathring{f} \in M^{\mathbb{B}}$ is forced by $\mathbb{1}$ to be a function with domain $= \omega$ and range $\supseteq (\omega_1)^M$. Note that by the Maximal Principle, if $\mathbb{1}$ forces that $(\omega_1)^M$ becomes countable, then there is a name $\mathring{f}$ forced by $\mathbb{1}$ to be a map from ω onto $(\omega_1)^M$. □

The next exercise involves almost disjoint families $\mathcal{D} \subset [\omega_1]^{\aleph_1}$. By Exercise III.7.11, there is always such a family of size $\aleph_2$, and *CH* implies that there is one of size $2^{\aleph_1}$. However, by the following argument of Baumgartner, one can have $2^{\aleph_0} = 2^{\aleph_1}$ arbitrarily large and no almost disjoint family of size $\aleph_3$:

Exercise IV.7.51 *Assume that in M, GCH holds and $\mathbb{P}$ is ccc. Then in $M[G]$, $|\mathcal{D}| \leq \aleph_2$ for all almost disjoint families $\mathcal{D} \subset [\omega_1]^{\aleph_1}$.*

Hint. Say p forces that $\mathring{E}$ is a subset of $\omega_3 \times \omega_1$ such that for $\alpha < \omega_3$, the sets $\mathring{E}_\alpha := \{\xi < \omega_1 : (\alpha, \xi) \in \mathring{E}\}$ are uncountable and pairwise almost disjoint. Then, there is (in M) a function $f : [\omega_3]^2 \to \omega_1$ such that whenever $\alpha < \beta < \omega_3$, $p \Vdash \mathring{E}_\alpha \cap \mathring{E}_\beta \subseteq f(\alpha, \beta)$; this is where the ccc is used. Since $\aleph_3 > 2^{\aleph_1}$ (in M), you can now use the Erdös–Rado Theorem (in M). □

Exercise IV.7.52 *Show that the existence of an almost disjoint $\mathcal{D} \subseteq [\omega_1]^{\aleph_1}$ of size $\aleph_3$ is neither provable nor refutable from ZFC plus $2^{\aleph_0} = 2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$.*

Hint. Start with $M \models GCH$. To get no such family in $M[G]$, use Exercise IV.7.51. To get a model where there is such a family, first extend M to get *CH* plus $2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$, and then add $\aleph_3$ Cohen reals. □

Exercise IV.7.53 *In Exercise IV.7.51, assume also that* $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$. *Show that in* $M[G]$, *the order type* ω_2 *is not embeddable in* $\mathcal{P}(\omega)/\mathrm{fin}$.

Hint. Say $\mathring{E}_\alpha$ is a nice name for a subset of ω for $\alpha < \omega_2$ and, for $\alpha < \beta < \omega_2$, $\mathbb{1}$ forces that $\mathring{E}_\alpha \subseteq^* \mathring{E}_\beta$ and $\neg(\mathring{E}_\alpha =^* \mathring{E}_\beta)$. Fix $J_\alpha \in [\kappa]^{\aleph_0}$ such that $\mathring{E}_\alpha$ is really an $\mathrm{Fn}(J_\alpha, 2)$–name. Working in M and applying *GCH* and the Delta System Lemma III.6.15, WLOG the J_α form a delta system. You can now derive a contradiction from the fact that for a countable J, there are only $2^{\aleph_0} = \aleph_1$ different nice $\mathrm{Fn}(J, 2)$–names for subsets of ω. □

Remarks. See also Exercise III.3.78 and the remark following it. Unlike in Exercise IV.7.51, $\mathbb{P}$ cannot be an arbitrary ccc poset because then (by Theorem V.4.1) $M[G]$ might satisfy $MA(\aleph_2)$; see Exercise III.3.77.

Exercise IV.7.54 *Work in ZFC and assume that* $\kappa > \omega$ *is regular,* κ*–SH holds, and* $2^\theta = \theta^+$ *for all infinite* $\theta < \kappa$. *Produce a transitive submodel* $N \subseteq V$ *with* $ON \subseteq N$ *and* $N \models \kappa$*–SH +* GCH *and* $\mathcal{P}(\alpha) \subseteq N$ *for all* $\alpha < \kappa$.

Hint. Modify the L construction. N can contain all bounded subsets of κ, which will guarantee *GCH* below κ. Also, make sure that whenever $T \in N$ is a κ–tree, N contains either a chain or an antichain in T of size κ. □

Exercise IV.7.55 *Assume that in* M, κ *is strongly inaccessible and* $|\mathbb{P}| < \kappa$. *Then* κ *is strongly inaccessible in* $M[G]$.

Note that weak inaccessibility is preserved by all ccc forcing; this is immediate from the fact that cofinalities are preserved. So is the weak Mahlo property (defined in Exercise III.6.30). This follows from the next exercise, which also shows that "stationary" is preserved:

Exercise IV.7.56 *Let* G *be* $\mathbb{P}$*–generic over* M, *where* $\mathbb{P}$ *is ccc in* M. *Let* $\kappa > \omega$ *be regular in* M. *If* $C \in \mathcal{P}(\kappa) \cap M[G]$ *is club in* κ, *then there is a* $C' \subseteq C$ *such that* C' *is club in* κ *and* $C' \in M$. *Hence, if* $S \in \mathcal{P}(\kappa) \cap M$ *and* $(S \text{ is stationary})^M$ *then* $(S \text{ is stationary})^{M[G]}$. *Also, if* κ *is weakly Mahlo in* M, *then* κ *remains weakly Mahlo in* $M[G]$.

Hint. If p forces $\mathring{C}$ to be club, let $C' = \{\alpha < \kappa : p \Vdash \alpha \in \mathring{C}\}$. □

Exercise IV.7.57 *Assume that* $\mathbb{P}$ *is ccc in* M *and* $M[G] \models \diamondsuit$. *Then* $M \models \diamondsuit$.

Hint. Use a $\mathbb{P}$–name for a $\diamondsuit$ sequence to construct a $\diamondsuit^-$ sequence in M. □

So, you can't make $\diamondsuit$ true by ccc forcing (but you can by a non-ccc forcing; see Lemma IV.7.30): You can make $\diamondsuit$ false by ccc forcing. This is trivial, since you can make *CH* false. But getting a model of $CH + \neg\diamondsuit$ is non-trivial. One might start with *CH* in the ground model and force with a ccc poset of size $\aleph_1$. This will guarantee that $M[G] \models CH$, but it doesn't kill $\diamondsuit$:

Exercise IV.7.58 *Assume that in M: $\mathbb{P}$ is ccc in M and $\diamondsuit$ holds and $|\mathbb{P}| \leq \aleph_1$. Then $M[G] \models \diamondsuit$.*

Hint. Use $\diamondsuit$ in M to capture $\mathbb{P}$–names for subsets of ω_1. Note that stationary sets remain stationary by Exercise IV.7.56. □

It is consistent with *GCH* to have $\neg\diamondsuit$ and even *SH*; see Devlin and Johnsbråten [21]. This is done with a ccc poset of size $\aleph_2$.

The notion of "stationary" in the Jech–Kueker sense (Definition III.6.24) is preserved both by ccc forcing and by countably closed forcing, although for different reasons. Of course, we already know that both ccc and countably closed forcing preserve ω_1, again for different reasons.

For ccc forcing, use this analog of Exercise IV.7.56:

Exercise IV.7.59 *In the ground model, assume that B is any set and $\mathbb{P}$ is a ccc poset. Also assume that $p \Vdash$ "$\mathring{C} \subseteq [B]^{\leq\aleph_0} \wedge \mathring{C}$ is club". Then, in the ground model, define $\mathcal{C}' = \{A \in [B]^{\leq\aleph_0} : p \Vdash B \in \mathring{C}\}$. Then $\mathcal{C}'$ is club and $p \Vdash \mathcal{C}' \subseteq \mathring{C}$.*

Conclude from this that if $\mathcal{S} \subseteq [B]^{\leq\aleph_0}$ is stationary in the ground model then $\mathbb{1} \Vdash$ "$\mathcal{S}$ is stationary".

Note that unlike the case for club subsets of κ, we cannot claim that $\mathcal{C}'$ remains club in the extension, since $\mathbb{P}$ may add new increasing ω–sequences from $\mathcal{C}'$ whose union is not even in the ground model.

For countably closed forcing, the following exercise refutes the existence of such a $\mathcal{C}'$; it may be applied both to club^{ω_1} and to club_{ω_1}.

Exercise IV.7.60 *In the ground model M, let $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\omega_1, 2)$. In $M[G]$, we have $\bigcup G : \omega_1 \to 2$ and $X := (\bigcup G)^{-1}\{1\}$, which is an unbounded subset of ω_1. Let C be the closure of X, which is club in ω_1.*

Now, suppose that $E \subseteq C$ and $E \in M$. Prove that E is countable.

Nevertheless, stationary sets remain stationary by a different argument, vaguely akin to Exercise IV.7.46.

Exercise IV.7.61 *In the ground model, assume that B is any set and $\mathbb{P}$ is a countably closed poset. Also assume that $\mathcal{S} \subseteq [B]^{\leq\aleph_0}$ is stationary. Prove that $\mathbb{1} \Vdash$ "$\mathcal{S}$ is stationary".*

Hint. If not, then fix $p \in \mathbb{P}$ and a name $\mathring{C}$ such that $p \Vdash$ "$\mathring{C} \cap \mathcal{S} = \emptyset \ \wedge\ \mathring{C}$ is club". Then, in the ground model, obtain a club $\mathcal{C}' \subseteq [B]^{\leq\aleph_0}$ such that for all $A \in \mathcal{C}'$, there exist $p_n \in \mathbb{P}$ and $A_n \subseteq A$ for $n \in \omega$ such that $p \geq p_0 \geq p_1 \geq \cdots$ and $A_0 \subseteq A_1 \subseteq \cdots$ and $\bigcup_n A_n = A$ and all $p_n \Vdash A_n \in \mathring{C}$. Then, since $\mathbb{P}$ is countably closed, $\mathcal{C}' \cap \mathcal{S} = \emptyset$. □

A forcing poset is called *proper* iff for all B, it preserves stationary subsets of $[B]^{\leq\aleph_0}$. The *Proper Forcing Axiom* (*PFA*) is the assertion that $MA_{\mathbb{P}}(\aleph_1)$ holds for all proper $\mathbb{P}$. See Section V.7 for more on *PFA*.

IV.8 Independence of the Axiom of Choice

As Cohen showed in his original work [14, 15], the method of forcing can also be used to produce models of $ZF + \neg AC$. We describe some such models in this section. Jech's book [48] has much more on this topic.

Most of mathematics works in ZFC, but models where AC fails are still of some interest, since models of $ZF + \neg AC$ are often paired with models of ZFC in which some sorts of sets are not definable. For example, we shall produce a model of ZF in which $\mathcal{P}(\omega)$ cannot be well-ordered. But the same method produces a model of $ZFC + GCH$ in which there is no definable well-order of $\mathcal{P}(\omega)$. This differs from the situation under $V = L$, where there is a definable well-order of $\mathcal{P}(\omega)$, since $<_L$ (Definition II.6.19) well-orders the universe, and in particular $\mathcal{P}(\omega)$.

Actually, our models will refute something weaker than a well-order of $\mathcal{P}(\omega)$. Just for this section, we define:

Definition IV.8.1 *A* nice choice set *is an* $\mathcal{E} \subseteq \mathcal{P}(\omega)$ *such that*

$$\forall x \in \mathcal{P}(\omega)\,[x \in \mathcal{E} \leftrightarrow (\omega \backslash x) \notin \mathcal{E}] \ \wedge \ \forall x, y \in \mathcal{P}(\omega)\,[x =^* y \rightarrow [x \in \mathcal{E} \leftrightarrow y \in \mathcal{E}]] \ .$$

That is, for each $x \in \mathcal{P}(\omega)$, $\mathcal{E}$ chooses one of x and $\omega \backslash x$, and that choice is the same for all $y =^* x$. It is easy to define a nice choice set from a well-order of ω, so if there are no definable nice choice sets, then there are also no definable well-orders of ω. There are also no definable non-principal ultrafilters on ω (Definition III.1.30), since each such ultrafilter is a nice choice set.

Before describing the model, some remarks on "definable". If we claim that there *is* a definable nice choice set, then we prove our claim by writing down a definition of such an $\mathcal{E}$. Under $V = L$, our definition will use $<_L$; for example, we could say $x \in \mathcal{E}$ iff $x =^* z$, where z is $<_L$-least in $\{y \in \mathcal{P}(\omega) : y =^* x \vee y =^* (\omega \backslash x)\}$. If we claim that there *are no* definable nice choice sets, then there is no way to assert this claim by one sentence in the language of set theory. We could assert our claim as an infinite set of axioms: namely for each formula $\varphi(x)$, we could assert that $\{x \in \mathcal{P}(\omega) : \varphi(x)\}$ is not a nice choice set. But in fact, we shall prove a somewhat stronger claim, where φ is allowed to mention, as fixed parameters, a finite number of ordinals and a finite number of subsets of ω. Now, the fact that we are allowing ordinal parameters means that we can indeed assert our claim by *one* sentence. Recall, from Section II.8, that one can define in ZF the class OD of ordinal definable sets — these are the sets definable from ordinals. One can likewise define the larger class, called $OD^{\mathbb{R}}$, of sets definable from ordinals and "reals" (i.e., subsets of ω; see Notation IV.7.31). Then, one can write as one sentence the statement that no nice choice set is a member of $OD^{\mathbb{R}}$.

We now have two tasks: to explain an appropriate forcing that avoids definable nice choice sets, and to explain $OD^{\mathbb{R}}$. We do the forcing first. The poset is simply $\mathrm{Fn}(\omega_1, 2)$, and the main lemma is:

Lemma IV.8.2 *Let M be a ctm for ZFC and let $\mathbb{P} = \mathrm{Fn}((\omega_1)^M, 2)$. Let $\varphi(x)$ be a formula with one free variable in $\mathcal{FL}_{\mathbb{P}} \cap M$ such that all names mentioned are either nice names for subsets of ω, or names of the form $\check{b}$, where $b \in M$. Let G be $\mathbb{P}$-generic over M. Then $M[G]$ satisfies the sentence asserting that $\{x \in \mathcal{P}(\omega) : \varphi(x)\}$ is* not *a nice choice set.*

Proof. The proof uses automorphisms of $\mathbb{P}$ and some results from Section IV.4. If $S \subseteq (\omega_1)^M$, let i^S be the automorphism of $\mathbb{P}$ that swaps 0/1 outside of S; that is, $\mathrm{dom}(i^S(r)) = \mathrm{dom}(r)$ and $(i^S(r))(\xi) = r(\xi)$ when $\xi \in S$ and $(i^S(r))(\xi) = 1 - r(\xi)$ when $\xi \notin S$. Then $i^S \in M$ whenever $S \in M$. We shall apply i^S to the names $\vartheta_\alpha := \{\langle \check{n}, \{(\alpha + n, 1)\}\rangle : n < \omega\}$, where $\alpha < (\omega_1)^M$. Note that $\{(\alpha + n, 1)\}$ is in $\mathbb{P}$, as is its complementary condition $\{(\alpha + n, 0)\}$. So, $\mathbb{1} \Vdash \vartheta_\alpha \subseteq \omega$, and for each n:

$$\{(\alpha + n, 1)\} \Vdash \check{n} \in \vartheta_\alpha \quad ; \quad \{(\alpha + n, 0)\} \Vdash \check{n} \notin \vartheta_\alpha \quad .$$

Consider the name $i^S_*(\vartheta_\alpha)$ (Definition IV.4.3), and note that $\mathbb{1} \Vdash i^S_*(\vartheta_\alpha) \subseteq \omega$. Also, $\mathbb{1} \Vdash i^S_*(\vartheta_\alpha) =^* (\omega \backslash \vartheta_\alpha)$ whenever $S \cap \{\alpha + n : n < \omega\}$ is finite.

Roughly, we want to say that since lots of automorphisms exchange ϑ_α with its complement, there is no definable way to choose between the two of them.

More formally, assume that $\{x \in \mathcal{P}(\omega) : \varphi(x)\}$ is a nice choice set in some $M[G]$. Then there is a $p \in \mathbb{P}$ that forces the sentence:

$$\forall x \in \mathcal{P}(\omega)\, [\varphi(x) \leftrightarrow \neg\varphi(\omega \backslash x)] \ \wedge \ \forall x, y \in \mathcal{P}(\omega)\, [x =^* y \rightarrow [\varphi(x) \leftrightarrow \varphi(y)]] \quad .$$

Now, working in M, write out φ as $\psi(x, \tau_1, \ldots, \tau_m, \check{b}_1, \ldots, \check{b}_n)$, where the τ_ℓ are nice names for subsets of ω and the b_ℓ are arbitrary sets in M. By Lemma IV.4.7, for any $q \in \mathbb{P}$ and name σ: $q \Vdash \psi(\sigma, \tau_1, \ldots, \tau_m, \check{b}_1, \ldots, \check{b}_n)$ iff $i(q) \Vdash \psi(i_*(\sigma), i_*(\tau_1), \ldots, i_*(\tau_m), i_*(\check{b}_1), \ldots, i_*(\check{b}_n))$, whenever $i \in M$ is any automorphism. But $i_*(\check{b}) = \check{b}$. Also, a nice name for a subset of ω only mentions countably many forcing conditions, and we may fix $\alpha < \omega_1$ such that all these conditions really come from $\mathrm{Fn}(\alpha, 2)$. If $i(r) = r$ for all $r \in \mathrm{Fn}(\alpha, 2)$, then each $i_*(\tau_\ell) = \tau_\ell$, so that $q \Vdash \varphi(\sigma)$ iff $i(q) \Vdash \varphi(i_*(\sigma))$.

Next, fix $\sigma = \vartheta_\alpha$. Then, we may fix $q \leq p$ such that either $q \Vdash \varphi(\sigma)$ or $q \Vdash \neg\varphi(\sigma)$. Then, let $S = \alpha \cup \mathrm{dom}(q)$, and fix $i = i^S$. Then $i(q) = q$, so that $q \Vdash \varphi(\sigma)$ iff $q \Vdash \varphi(i_*(\sigma))$, so either $q \Vdash \varphi(\sigma) \wedge \varphi(i_*(\sigma))$, or $q \Vdash \neg\varphi(\sigma) \wedge \neg\varphi(i_*(\sigma))$. By the above, $\mathbb{1} \Vdash i_*(\sigma) =^* (\omega \backslash \sigma)$. Also, $q \leq p$, so q forces whatever p forces, which implies that $q \Vdash \varphi(\sigma) \leftrightarrow \neg\varphi(i_*(\sigma))$, which contradicts both $q \Vdash \varphi(\sigma) \wedge \varphi(i_*(\sigma))$, and $q \Vdash \neg\varphi(\sigma) \wedge \neg\varphi(i_*(\sigma))$. □

This proof is similar in spirit to the Fraenkel–Mostowski proof of consistency of $\neg AC$ with ZF^- (see Exercise II.9.10). There, in models where the Axiom of Foundation is false, one can use automorphisms of the $\in$ relation (Definition II.9.8) to construct models of $\neg AC$. Under Foundation, there are no non-trivial automorphisms of $\in$, but in the forcing, we use automorphisms of $\mathbb{P}$ instead. See Jech [48] for more on the relationship between Fraenkel–Mostowski models and forcing models of $\neg AC$.

Of course, we haven't yet produced a model for $ZF + \neg AC$. Our $M[G]$, is, as always, a model for ZFC, but it also satisfies the statement that no nice choice set is definable using elements of M (which include all the ordinals of $M[G]$) and from reals. It will follow that $HOD^{\mathbb{R}}$ (as computed within $M[G]$) is a model for ZF plus the statement that no nice choice set exists at all. To verify this, one must check some some properties of $OD^{\mathbb{R}}$, which we do next, following the rough outline of Section II.8.

Definition IV.8.3 *For any transitive set N, $OD_N^{\mathbb{R}}$ is the set of all $a \in N$ such that for some $n \geq 0$, there is a formula $\varphi(x_0, \ldots, x_{n-1}, y)$ in $n+1$ free variables and $b_0, \ldots, b_{n-1} \in N \cap (\mathcal{P}(\omega) \cup ON)$ such that $(\varphi(\vec{b}, a))^N$ and $(\exists! y\, \varphi(\vec{b}, y))^N$ both hold.*

So, each $a \in OD_N^{\mathbb{R}}$ is definable in N using ordinals and "reals" from N as parameters. As with Definition II.8.2, N needs to be a set for this definition to be expressible in set theory. But, we can "cheat" and write the following definition, which captures the "essence" of $OD_V^{\mathbb{R}}$, and then use Reflection to prove that it "works":

Definition IV.8.4 $OD^{\mathbb{R}} = \bigcup \{ OD_{R(\eta)}^{\mathbb{R}} : \eta \in ON \}$.

We are following Notation IV.7.31 in calling elements of $\mathcal{P}(\omega)$ "reals". We don't want to use the real $\mathbb{R}$ here, since the set theoretic definition of $\mathbb{R}$ is different in different texts. Of course, under any reasonable definition of $\mathbb{R}$ (such as Definition I.14.2), each element of $\mathbb{R}$ will be explicitly definable from some subset of ω so that $\mathbb{R} \subseteq OD^{\mathbb{R}}$.

Exercise IV.8.5 *Show the following, which seems to say "$OD_V^{\mathbb{R}} \subseteq OD^{\mathbb{R}}$": For each formula $\varphi(x_0, \ldots, x_{n-1}, y)$, the sentence*

$$\forall b_0, \ldots, b_{n-1} \in (\mathcal{P}(\omega) \cup ON)\ \forall w\ \Big[\varphi(\vec{b}, w) \wedge \exists! y\, \varphi(\vec{b}, y)\ \rightarrow\ w \in OD^{\mathbb{R}} \Big]$$

is provable in ZF.

Exercise IV.8.6 *Do the following, which seems to say "$OD^{\mathbb{R}} \subseteq OD_V^{\mathbb{R}}$": Write* one *formula ψ in three variables in $\mathcal{L} = \{\in\}$ such that the sentence*

$$\begin{aligned} &\forall \alpha \in ON\ \forall z \in \mathcal{P}(\omega)\ \exists! y\ \psi(\alpha, z, y) \quad \wedge \\ &\quad \forall w \Big[w \in OD^{\mathbb{R}}\ \leftrightarrow\ \exists \alpha \in ON\ \exists z \in \mathcal{P}(\omega)\ \psi(\alpha, z, w) \Big] \end{aligned}$$

is provable in ZF.

So, ψ defines a map from $ON \times \mathcal{P}(\omega)$ onto $OD^{\mathbb{R}}$. Using this *one* ψ, whenever $w \in OD$, there is an ordinal α and a real z such that ψ defines w from α, z in $(V, \in)$. To do this exercise, you need to know that you can define a map from ON onto $ON^{<\omega}$ (as in the hint to Exercise II.8.5), and that you can define a map from $\mathcal{P}(\omega)$ onto $\mathcal{P}(\omega)^{<\omega}$.

Definition IV.8.7 $HOD^{\mathbb{R}} = \{x \in OD^{\mathbb{R}} : \mathrm{trcl}(x) \subseteq OD^{\mathbb{R}}\}$.

Exercise IV.8.8 *Show that*

1. $ON \subseteq HOD^{\mathbb{R}} \subseteq OD^{\mathbb{R}}$ *and* $HOD^{\mathbb{R}}$ *is transitive.*
2. *For any set* a: $a \in HOD^{\mathbb{R}}$ *iff* $a \in OD^{\mathbb{R}}$ *and* $a \subseteq HOD^{\mathbb{R}}$.
3. *Each* $R(\alpha) \cap HOD^{\mathbb{R}} \in HOD^{\mathbb{R}}$.
4. *All the ZF axioms hold in* $HOD^{\mathbb{R}}$.

Note that HOD satisfies ZFC but need not contain all reals, whereas $HOD^{\mathbb{R}}$ satisfies ZF and obviously contains all reals, but need not satisfy AC.

Exercise IV.8.9 $HOD^{\mathbb{R}}$ *satisfies* AC *iff there is a well-order of* $\mathbb{R}$ *that is in* $HOD^{\mathbb{R}}$.

Theorem IV.8.10 *If there is a ctm for ZFC, then there are ctms* N_1, N_2 *such that* N_1 *satisfies* $ZFC + GCH$ *plus the statement that there are no nice choice sets in* $HOD^{\mathbb{R}}$, *and* N_2 *satisfies ZF plus the statement that there are no nice choice sets at all.*

Proof. As in the proofs of Corollary IV.3.14, we may start with a ctm $M \models ZFC + GCH$. Then, let $\mathbb{P} = \mathrm{Fn}((\omega_1)^M, 2)$. As in the proof of Lemma IV.3.16, $M[G] \models ZFC + GCH$. Then $N_1 = M[G]$ and $N_2 = (HOD^{\mathbb{R}})^{N_1}$. □

As usual, this theorem can be converted to a finitistic relative consistency proof by the methods of Section IV.5. We conclude this section with a few more remarks on our model.

Working in ZF, it is easy to define a *total* order on $\mathcal{P}(\omega)$, but

Exercise IV.8.11 *If there is no nice choice set then there is no total order of* $\mathcal{P}(\mathcal{P}(\omega))$.

Exercise IV.8.12 *Working in ZFC, prove that the* countable AC *always holds in* $HOD^{\mathbb{R}}$. *This is the statement that whenever* $E_n \neq \emptyset$ *for all* $n \in \omega$, *the product* $\prod_n E_n \neq \emptyset$.

In particular, our model N_2 satisfies the countable AC, which is sufficient to develop elementary real analysis. This analysis includes basic facts about measure theory, such as the equivalence of the various definitions of Lebesgue measure μ, and the fact that $\mu(X) = 0$ for countable $X \subset \mathbb{R}$, and that μ is countably additive. This development cannot be done in ZF alone, since it is consistent with ZF that $\mathbb{R}$ is a countable union of countable sets (see, e.g., [48]), which would imply that $\mu(\mathbb{R}) = 0$ if μ had its usual properties.

The countable AC does not imply that there is a Lebesgue non-measurable set of reals. Unfortunately, one cannot use our model N_2 to prove this (see Exercise IV.8.16). Solovay [93] did, in 1970, construct a model of ZF plus the statement that all sets of reals are measurable. His model is like our N_2, but

he used a different forcing poset. In M, we assume that κ is an inaccessible cardinal. Then, $\mathbb{P}$ is the Lévy collapse (see Exercise IV.7.48), so κ is the ω_1 of $M[G]$. It is fairly easy to see that this poset will also produce the results in Theorem IV.8.10, but the rest of the argument is much more sophisticated.

Our N_2 does show that the standard Vitali set example of a non-measurable set need not work:

Exercise IV.8.13 *Show that the model N_2 from Theorem IV.8.10 satisfies the statement that there is no set $X \subseteq \mathbb{R}$ such that X contains precisely one element from each coset of $\mathbb{Q}$.*

Hint. Inject $\mathcal{P}(\omega)$ into $\mathbb{R}$ by $x \mapsto \sum\{3^{-j} : j \in x\}$, and now use a permutation argument. When proving forcing results about actual real numbers (not elements of $\mathcal{P}(\omega)$), one presumes some treatment of $\mathbb{R}$ like the one in Definition I.14.2 that makes basic properties of the real numbers absolute. □

A different way of producing a non-measurable set is given by:

Exercise IV.8.14 *Work in ZF plus the countable AC. Assume that for some ordinal γ, there is a sequence of Lebesgue null sets $\{Z_\xi : \xi < \gamma\}$ such that $\bigcup_\xi Z_\xi = \mathbb{R}$. Show that there is a Lebesgue non-measurable set of reals.*

Hint. WLOG, the Z_ξ are disjoint, so we have $f : \mathbb{R} \to \gamma$ such that each $\mu(f^{-1}\{\xi\}) = 0$. If every subset of $\mathbb{R}$ is measurable then every subset of $\mathbb{R}^2$ is measurable. Now, contradict Fubini's theorem using $\{(x, y) : f(x) < f(y)\}$. □

Working in *ZFC*, the Z_ξ could be singletons, with $|\gamma| = |\mathbb{R}|$; that is, every well-order of $\mathbb{R}$ is a non-measurable subset of the plane. The Vitali set example is preferred in elementary analysis texts because it does not require a knowledge of well-orders. Returning to Cohen forcing and our model N_2, the Z_ξ will come from:

Exercise IV.8.15 *Define $W_f = \bigcap_n \bigcup_{m>n}(f(m) - 2^{-m}, f(m) + 2^{-m})$, where $f : \omega \to \mathbb{Q}$. Then W_f is a null G_δ set in $\mathbb{R}$. Now, let G be $\mathbb{P}$-generic over M, where $\mathbb{P} = \mathrm{Fn}(I, 2)$. Prove that every real number in $M[G]$ is in W_f for some $f \in \mathbb{Q}^\omega \cap M$.*

Hint. This is a little like Lemmas IV.4.9 and IV.4.10. □

Exercise IV.8.16 *In the proof of Theorem IV.8.10, suppose that we started with $M \models ZFC + V = L$. Show that the model N_2 for $ZF + \neg AC$ that we constructed also satisfies the statement that there is a set of reals that is not Lebesgue measurable.*

Hint. By applying Exercise IV.8.15, we see that working in N_2, we have the hypotheses of Exercise IV.8.14 with $\gamma = \omega_1$ and $Z_\xi = W_{f_\xi}$, where the f_ξ list the constructible functions from ω into $\mathbb{Q}$. □

Under any reasonable definition of $\mathbb{R}$ (such as Definition I.14.2), $L[\mathbb{R}] = L[\mathcal{P}(\omega)]$ (see Definition II.6.29). It is easy to see that $L[\mathbb{R}] \subseteq HOD^{\mathbb{R}}$. Also,

Exercise IV.8.17 *Assume that $V = L$ holds in the ground model M, and let G be $\mathbb{P}$-generic over M, where $\mathbb{P} = \mathrm{Fn}(I, 2)$. Then the statement $L[\mathbb{R}] = HOD^{\mathbb{R}}$ holds in $M[G]$. Hence, it is consistent with ZFC that AC is false in $L[\mathbb{R}]$.*

Chapter V

Iterated Forcing

Here, we start with a ctm M for ZFC and iterate the forcing procedure α times, producing a chain of models,

$$M = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\xi \subseteq \cdots \subseteq M_\alpha \ .$$

For each $\xi < \alpha$, we have $\mathbb{P}_\xi \in M_\xi$ and $M_{\xi+1} = M_\xi[G_\xi]$, where G_ξ is $\mathbb{P}_\xi$–generic over M_ξ. For finite α, we have already done this in constructing models with various failures of GCH. For example, we used $\alpha = 3$ in proving Corollary IV.7.18; we forced three times to get $M_3 \models 2^{\aleph_0} = \aleph_3 \wedge 2^{\aleph_1} = \aleph_5 \wedge 2^{\aleph_2} = \aleph_7$. We also pointed out that there is a problem when $\alpha \geq \omega$, since it is not clear what M_ω should be. If we naively set $M_\omega = \bigcup_n M_n$, then M_ω will usually not satisfy ZFC (Exercise IV.7.20).

To solve our problem, we shall study the $\alpha = 2$ case more carefully. Here, $M_1 = M[G]$ and $M_2 = M[G][H]$, where $\mathbb{P} \in M$ and $\mathbb{Q} \in M[G]$ and G is $\mathbb{P}$–generic over M and H is $\mathbb{Q}$–generic over $M[G]$. In Section V.3, we show how to do this two step iteration in one step; this is done by obtaining another poset $\mathbb{R} \in M$ and a filter K that is $\mathbb{R}$–generic over M such that $M[G][H] = M[K]$. Furthermore, $\mathbb{P} \subseteq_c \mathbb{R}$ (Definition III.3.63). So, we have $\mathbb{P} = \mathbb{P}_0 \subseteq_c \mathbb{P}_1 = \mathbb{R}$.

Now, when we do an α step iteration, we shall actually work in M and construct a chain of posets:

$$\mathbb{P}_0 \subseteq_c \mathbb{P}_1 \subseteq_c \cdots \subseteq_c \mathbb{P}_\xi \subseteq_c \cdots \subseteq_c \mathbb{P}_\alpha \ .$$

There is still a question of what to do at stage ω and other limit ordinals. The naive answer, setting $\mathbb{P}_\omega = \bigcup_n \mathbb{P}_n$, is now reasonable, since the union of a chain of posets is always a poset, and this answer has many applications, discussed in Section V.4; among these applications is the consistency of $MA + \neg CH$. There are many other options for what to do at limit ordinals; see Section V.5 for some examples.

Section V.1 discusses a special case of the $\alpha = 2$ case, where the posets $\mathbb{P}, \mathbb{Q}$ are both in M. We still have $M \subseteq M[G] \subseteq M[G][H]$, where G is $\mathbb{P}$–generic

over M and H is $\mathbb{Q}$-generic over $M[H]$. Now, the poset $\mathbb{P} \times \mathbb{Q}$ is also in M, and $M[G][H]$ can be obtained in one step by forcing with the product poset $\mathbb{P} \times \mathbb{Q}$. This special case has many applications, discussed in Section V.2. One of these applications is general Easton forcing, producing models of statements such as $\forall n \in \omega\ [2^{\aleph_n} = \aleph_{3+2n}]$. The examples in Section V.2 are not needed to understand the proof of $\mathrm{Con}(MA + \neg CH)$.

V.1 Products

We first observe that forcing with $\mathbb{P} \times \mathbb{Q}$ adds both a $\mathbb{P}$-generic filter and a $\mathbb{Q}$-generic filter:

Lemma V.1.1 *In the ground model M, let $\mathbb{P}$ and $\mathbb{Q}$ be posets, with $i : \mathbb{P} \to \mathbb{P} \times \mathbb{Q}$ and $j : \mathbb{Q} \to \mathbb{P} \times \mathbb{Q}$ defined by $i(p) = (p, \mathbb{1})$ and $i(q) = (\mathbb{1}, q)$. Let K be $\mathbb{P} \times \mathbb{Q}$-generic over M. In $M[K]$, let $G = i^{-1}(K)$ and $H = j^{-1}(K)$. Then G is $\mathbb{P}$-generic over M and H is $\mathbb{Q}$-generic over M and $K = G \times H$.*

Proof. The maps i and j are complete embeddings by Lemma III.3.73, and then the genericity of G, H follows from Lemma IV.4.2. To prove that $K \subseteq G \times H$, fix $(p, q) \in K$. Then $(p, \mathbb{1}), (\mathbb{1}, q) \in K$, so $p \in G$ and $q \in H$. To prove that $K \supseteq G \times H$, fix $p \in G$ and $q \in H$, so that $(p, \mathbb{1}), (\mathbb{1}, q) \in K$. Since K is a filter, choose $(p', q') \in K$ such that $(p', q') \leq (p, \mathbb{1})$ and $(p', q') \leq (\mathbb{1}, q)$. Then $(p', q') \leq (p, q)$, so $(p, q) \in K$. □

Of course, in the above lemma and proof, the symbols $\leq$ and $\mathbb{1}$ refer to the $\leq$ and $\mathbb{1}$ in the appropriate poset, which is either $\mathbb{P}$ or $\mathbb{Q}$ or $\mathbb{P} \times \mathbb{Q}$.

We need to say somewhat more to identify $M[K]$ with the iterated extension $M[G][H]$; in particular, we need to show that H is $\mathbb{Q}$-generic over $M[G]$, which is a stronger assertion, since H must meet all dense sets in $\mathbb{Q}$ from $M[G]$, not just those from M.

Theorem V.1.2 *In M, fix $\mathbb{P}, \mathbb{Q}, i, j$ as in Lemma V.1.1. Outside of M, choose any $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$. Then the following are equivalent:*

1. *$G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over M.*
2. *G is $\mathbb{P}$-generic over M and H is $\mathbb{Q}$-generic over $M[G]$.*
3. *H is $\mathbb{Q}$-generic over M and G is $\mathbb{P}$-generic over $M[H]$.*

Furthermore, if (1 – 3) hold, then $M[G \times H] = M[G][H] = M[H][G]$.

Proof. For (1) → (2): We already know from Lemma V.1.1 that G is $\mathbb{P}$-generic over M and H is $\mathbb{Q}$-generic over M. To prove that H is $\mathbb{Q}$-generic over $M[G]$, fix a dense $D \subseteq \mathbb{Q}$ such that $D \in M[G]$. Then fix a name $\mathring{D} \in M^{\mathbb{P}}$ such that $D = \mathring{D}_G$, and then fix $p_0 \in G$ such that $p_0 \Vdash_{\mathbb{P}} (\mathring{D}$ is dense in $\mathbb{Q})$. Now let

$$D' = \{(p_1, q_1) \in \mathbb{P} \times \mathbb{Q} : p_1 \leq p_0 \ \wedge\ p_1 \Vdash (q_1 \in \mathring{D})\}\ .$$

As usual, we are leaving it to the reader to fill in a few checks here (see page 283). Note that $D' \in M$.

We first verify that D' is dense below $(p_0, \mathbb{1})$: Fix $(p_2, q_2) \leq (p_0, \mathbb{1})$. Then $p_2 \leq p_0$, so p_2 also forces $\mathring{D}$ to be dense, so p_2 forces $\exists y \in \mathbb{Q}\,[y \in \mathring{D} \wedge y \leq q_2]$, so we can fix $p_3 \leq p_2$ and $q_3 \in \mathbb{Q}$ such that $p_3 \Vdash [q_3 \in \mathring{D} \wedge q_3 \leq q_2]$ (see Lemma IV.6.1). Then $(p_3, q_3) \in D'$ and $q_3 \leq q_2$, so $(p_3, q_3) \leq (p_2, q_2)$.

Now, since $G \times H$ is generic and $(p_0, \mathbb{1}) \in G \times H$, we may fix $(p_4, q_4) \in (G \times H) \cap D'$ (see Lemma IV.2.29). Then $p_4 \in G$ and $q_4 \in H$ and $p_4 \Vdash q_4 \in \mathring{D}$, so $q_4 \in \mathring{D}_G = D$. Thus, $H \cap D \neq \emptyset$.

For (2) $\rightarrow$ (1): $G \times H$ is a filter because it is the product of two filters. To prove that it is generic, fix a dense $D \subseteq \mathbb{P} \times \mathbb{Q}$ such that $D \in M$. Now let

$$D^* = \{q \in \mathbb{Q} : \exists p \in G\,[(p, q) \in D]\} \ .$$

Then $D^* \in M[G]$. Also, $H \cap D^* \neq \emptyset \rightarrow (G \times H) \cap D \neq \emptyset$, so we shall be done if we verify that D^* is dense in $\mathbb{Q}$. To do this, fix $q_0 \in \mathbb{Q}$. Then $\tilde{D} := \{p_1 : \exists q_1 \leq q_0\,[(p_1, q_1) \in D]\}$ is dense and in M, so fix $p_1 \in G \cap \tilde{D}$ and then fix $q_1 \leq q_0$ such that $(p_1, q_1) \in D$.

(1) $\leftrightarrow$ (3) is proved similarly. Now, the "furthermore" is easy using the minimality of the generic extension (Lemma IV.2.19): For $M[G \times H] \subseteq M[G][H]$, use the facts that $G \times H \in M[G][H]$ and $M \subseteq M[G][H]$.

For $M[G][H] \subseteq M[G \times H]$, use $H \in M[G \times H]$ and $M[G] \subseteq M[G \times H]$. Here, $M[G] \subseteq M[G \times H]$ is verified either by quoting Lemma IV.4.4, or by using minimality a third time: since $G \in M[G \times H]$ and $M \subseteq M[G \times H]$. $\square$

V.2 Applications of Products

For our first application, we discuss the method of Easton [25] (1970) for building models in which the map $\kappa \mapsto 2^\kappa$ for regular κ is anything "not obviously contradictory". As pointed out earlier (see page 109 and Exercises III.6.18 and III.6.19), the situation for exponents of singular cardinals is much more complicated, and is mostly beyond the scope of this book.

We begin by continuing the discussion in Section IV.7. Say *GCH* holds in the ground model M, and we wish the generic extension to satisfy $2^{\aleph_1} = \aleph_5$ as well as $2^{\aleph_2} = \aleph_7$. The natural posets that accomplish these two tasks separately would be $\mathbb{P} = (\mathrm{Fn}_{\omega_1}(\omega_5 \times \omega_1, 2))^M$ and $\mathbb{Q} = (\mathrm{Fn}_{\omega_2}(\omega_7 \times \omega_2, 2))^M$. We can now force with $\mathbb{P} \times \mathbb{Q}$, which accomplishes both tasks with one extension. Note that Theorem V.1.2 seems to present a symmetry between $\mathbb{P}$ and $\mathbb{Q}$; the $\mathbb{P} \times \mathbb{Q}$–generic filter is of the form $G \times H$, and $M[G \times H] = M[G][H] = M[H][G]$. However, on page 294, we said that it is important to proceed *top-down*, forcing with $\mathbb{Q}$ and then with $\mathbb{P}$. This seeming contradiction is resolved by noting that the notion $\mathrm{Fn}_\lambda(I, J)$ is not absolute, except in the case $\lambda = \omega$. Both $\mathbb{P}$ and $\mathbb{Q}$ were defined in M. If we view $M[G \times H]$ as $M[G][H]$, then H is $\mathbb{Q}$–generic over $M[G]$, but $\mathbb{Q}$ is *not* the same as $(\mathrm{Fn}_{\omega_2}(\omega_7 \times \omega_2, 2))^{M[G]}$, so we are not justified

in using any of our results about $\mathrm{Fn}_\lambda(I,J)$ preserving cardinals when passing from $M[G]$ to $M[G][H]$. On the other hand, if we view $M[G\times H]$ as $M[H][G]$, then G is $\mathbb{P}$–generic over $M[H]$, and now $\mathbb{P}$ *is* the same as $(\mathrm{Fn}_{\omega_1}(\omega_5\times\omega_1,2))^{M[H]}$ because $\mathbb{Q}$ is ω_2-closed in M, so that $\mathbb{Q}$ does not add ω_1–sequences, so we are justified in using theorems about $\mathrm{Fn}_\lambda(I,J)$ in passing from $M[H]$ to $M[H][G]$. So, if you want to explain the extension without discussing product posets, it is better to view the extension as $M[H][G]$

If you just want to violate *GCH* at finitely many cardinals, it is simpler to ignore the theory of products and just extend finitely many times, as we did in proving Corollary IV.7.18. But, as we indicated in the discussion following this corollary, there are problems in violating *GCH* at infinitely many cardinals. For example, if we want the generic extension to satisfy $\forall n\in\omega\ [2^{\aleph_n}=\aleph_{3+2n}]$, it is not clear how to do this by an infinite iteration. As we said then, this can be accomplished in one step using the poset $\mathbb{P}=\prod_{n\in\omega}\mathrm{Fn}_{\omega_n}(\omega_{3+2n}\times\omega_n,2)$. We shall now prove that this $\mathbb{P}$ really works. For the proof, we do not need to develop a theory of infinite products; we just use the fact that for each $j\in\omega$, $\mathbb{P}\cong\mathbb{P}^-\times\mathbb{P}^+$, where $\mathbb{P}^-=\prod_{n\le j}\mathrm{Fn}_{\omega_n}(\omega_{3+2n},2)$ and $\mathbb{P}^+=\prod_{n>j}\mathrm{Fn}_{\omega_n}(\omega_{3+2n},2)$, and we shall view the $\mathbb{P}$–extension as first forcing with $\mathbb{P}^+$ and then forcing with $\mathbb{P}^-$. Note that we are simplifying the notation here slightly by replacing each $\mathrm{Fn}_{\omega_n}(\omega_{3+2n}\times\omega_n,2)$ by the isomorphic $\mathrm{Fn}_{\omega_n}(\omega_{3+2n},2)$. Before presenting our proof that $\mathbb{P}$ works, we shall state the general framework:

Definition V.2.1 *An* index function *is a function E such that* $\mathrm{dom}(E)$ *is a set of regular cardinals. An* Easton index function *is an index function E such that each $E(\kappa)$ for $\kappa\in\mathrm{dom}(E)$ is an infinite cardinal with* $\mathrm{cf}(E(\kappa))>\kappa$, *and such that E satisfies* $\forall\kappa,\kappa'\in\mathrm{dom}(E)\,[\kappa<\kappa'\to E(\kappa)\le E(\kappa')]$.

An Easton index function is a possibility for what $\kappa\mapsto 2^\kappa$ "might be"; it incorporates the restrictions imposed by König's theorem ($\mathrm{cf}(2^\kappa)>\kappa$) and monotonicity ($\kappa<\kappa'\to 2^\kappa\le 2^{\kappa'}$). To make 2^κ actually equal to $E(\kappa)$ in some generic extension, we shall assume *GCH* in the ground model and force with the poset $\mathbb{P}(E)$:

Definition V.2.2 *If E is an index function with $I=\mathrm{dom}(E)$, and $\mathbb{R}$ is the product poset $\prod_{\kappa\in I}\mathrm{Fn}_\kappa(E(\kappa),2)$ (as in Definition III.3.36), then the* Easton poset $\mathbb{P}(E)$ *is the subposet of $\mathbb{R}$ defined by: $p\in\mathbb{P}(E)$ iff $p\in\mathbb{R}$ and*

$$\text{for all regular cardinals } \lambda:\ \ |\{\kappa\in\lambda\cap I : p(\kappa)\neq\mathbb{1}\}|<\lambda\quad. \tag{$*$}$$

Note that $(*)$ is trivial unless $|\lambda\cap I|=\lambda$; so, $\mathbb{P}(E)=\prod_{\kappa\in I}\mathrm{Fn}_\kappa(E(\kappa),2)$ unless there is a weakly inaccessible cardinal λ such that $|\lambda\cap\mathrm{dom}(E)|=\lambda$. But then $(*)$ becomes important for the proof that $\mathbb{P}(E)$ preserves cardinals. Specifically, it is used in the following lemma, which gives us a chain condition:

Lemma V.2.3 *Suppose that E is an index function with* $\mathrm{dom}(E)\subseteq\lambda^+$, *where λ is a regular cardinal such that $2^{<\lambda}=\lambda$. Then $\mathbb{P}(E)$ has the λ^+-cc.*

Proof. This is a little like Lemma IV.7.5. Fix $W = \{p_\alpha : \alpha < \lambda^+\} \subseteq \mathbb{P}(E)$; we must prove that W is not an antichain. Each p_α is really a function, with $\mathrm{dom}(p_\alpha) = I := \mathrm{dom}(E)$ and each $p_\alpha(\kappa) \in \mathrm{Fn}_\kappa(E(\kappa), 2)$. Let $D_\alpha = \bigcup\{\{\kappa\} \times \mathrm{dom}(p_\alpha(\kappa)) : \kappa \in I\}$. Applying $(*)$, each $|D_\alpha| < \lambda$ (observe that $p_\alpha(\kappa) = \mathbb{1} \to \mathrm{dom}(p_\alpha(\kappa)) = \emptyset$). Then, applying $2^{<\lambda} = \lambda$ and the Delta System Lemma III.6.15, there is a $B \in [\lambda^+]^{\lambda^+}$ and a root R such that $D_\alpha \cap D_\beta = R$ whenever α, β are two different elements of B. Since $2^{|R|} \leq 2^{<\lambda} = \lambda$, there are $\alpha, \beta \in B$ such that $\alpha \neq \beta$ and $(p_\alpha(\kappa))(s) = (p_\beta(\kappa))(s)$ for all $(\kappa, s) \in R$. But then $p_\alpha \not\perp p_\beta$. □

Even when $\mathrm{dom}(E) \not\subseteq \lambda^+$, we can apply this lemma by viewing $\mathbb{P}(E)$ as a product as follows:

Definition V.2.4 *If E is an index function and λ is any ordinal, then define $E^+_\lambda = E\restriction\{\kappa : \kappa > \lambda\}$ and $E^-_\lambda = E\restriction\{\kappa : \kappa \leq \lambda\}$.*

Lemma V.2.5 $\mathbb{P}(E) \cong \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$.

Lemma V.2.6 *Assuming GCH, if E is any index function then $\mathbb{P}(E)$ preserves cofinalities and cardinals.*

Proof. As usual, this statement means that if M is a ctm for *ZFC + GCH* and (E is an index function)^M and $(\mathbb{P} = \mathbb{P}(E))^M$, then $\mathbb{P}$ preserves cofinalities (and hence, cardinals).

By Lemma IV.3.3, we just need to show that every uncountable regular cardinal θ of M remains regular in $M[K]$ whenever K is $\mathbb{P}$–generic over M. Assume that this is false. Let $\lambda = (\mathrm{cf}(\theta))^{M[K]} < \theta$; so, λ is regular in $M[K]$ and hence in M. Fix $f \in M[K]$ such that $f : \lambda \to \theta$ and $\sup(\mathrm{ran}(f)) = \theta$.

Applying Lemma V.2.5 (and the fact that the isomorphism is in M), Theorem V.1.2 implies that $M[K] = M[H][G]$, where H is $(\mathbb{P}(E^+_\lambda))^M$–generic over M and G is $(\mathbb{P}(E^-_\lambda))^M$–generic over $M[H]$.

Now, $(\mathbb{P}(E^+_\lambda))^M$ is λ^+-closed in M, so it doesn't add λ–sequences (Lemma IV.7.15). Thus, $2^{<\lambda} = \lambda$ remains true in $M[H]$ and $(\mathbb{P}(E^-_\lambda))^{M[H]} = (\mathbb{P}(E^-_\lambda))^M$. It follows by Lemma V.2.3 applied in $M[H]$ that $(\mathbb{P}(E^-_\lambda))^M$ is λ^+-cc in $M[H]$. Thus, by Lemma IV.7.8, there is an $F : \lambda \to \mathcal{P}(\theta)$ with $F \in M[H]$ such that for all $\xi < \lambda$, $f(\xi) \in F(\xi)$ and $(|F(\xi)| \leq \lambda)^{M[H]}$.

Again, since $(\mathbb{P}(E^+_\lambda))^M$ is λ^+-closed in M, $F \in M$ and each $(|F(\xi)| \leq \lambda)^M$. But then, working in M, we have the set $\bigcup_{\xi<\lambda} F(\xi)$ of size $\leq \lambda$ cofinal in θ, contradicting the regularity of θ. □

Note that so far, our results hold for any index function. The *Easton* index function is used to control the map $\kappa \mapsto 2^\kappa$:

Theorem V.2.7 *Let M be a ctm for ZFC + GCH. In M, let E be any Easton index function and let $\mathbb{P} = \mathbb{P}(E)$. Let K be $\mathbb{P}$–generic over M. Then $\mathbb{P}$ preserves cofinalities and cardinals, and $M[K] \models \forall \kappa \in \mathrm{dom}(E)\,[2^\kappa = E(\kappa)]$.*

Proof. Preservation of cofinalities and cardinals was Lemma V.2.6.

Now, fix $\kappa \in \mathrm{dom}(E)$. Then $(2^\kappa \geq E(\kappa))^{M[K]}$ is proved in the standard way by using the factor $\mathrm{Fn}_\kappa(E(\kappa), 2)$. This factor lets us define a function $F_\kappa : E(\kappa) \to \{0,1\}$ by saying that $F_\kappa(\delta) = i$ iff there is a $p \in K$ such that $\delta \in \mathrm{dom}(p(\kappa))$ and $(p(\kappa))(\delta) = i$. Then, from F_κ we read off $E(\kappa)$ elements $h_\alpha \in {}^\kappa 2$ by defining $h_\alpha(\xi) = F_\kappa(\kappa \cdot \alpha + \xi)$. It is easily seen that the h_α, for $\alpha < E(\kappa)$, are all different, proving that $2^\kappa \geq E(\kappa)$.

We prove that $(2^\kappa \leq E(\kappa))^{M[K]}$ by counting nice names. As in the proof of Lemma V.2.6, we have $\mathbb{P}(E) \cong \mathbb{P}(E^-_\kappa) \times \mathbb{P}(E^+_\kappa)$, and hence $M[K] = M[H][G]$, where H is $(\mathbb{P}(E^+_\kappa))^M$-generic over M and G is $(\mathbb{P}(E^-_\kappa))^M$-generic over $M[H]$.

Working in M and using GCH, $|\mathbb{P}(E^-_\kappa)| = E(\kappa)$ and $\mathbb{P}(E^-_\kappa)$ has the κ^+-cc, so that there are only $E(\kappa)$ nice $\mathbb{P}(E^-_\kappa)$-names for subsets of $\check{\kappa}$. Then forcing with $\mathbb{P}(E^+_\kappa)$ doesn't add κ-sequences, so that it is also true in $M[H]$ that there are only $E(\kappa)$ nice $\mathbb{P}(E^-_\kappa)$-names for subsets of $\check{\kappa}$. But then, viewing $M[H]$ as the ground model, $(2^\kappa \leq E(\kappa))^{M[H][G]}$ follows as in the proof of Lemma IV.3.11. □

Corollary V.2.8 *If GCH holds in the ground model, then* $\forall n \in \omega\ [2^{\aleph_n} = \aleph_{3+2n}]$ *holds in the generic extension by* $\prod_{n\in\omega} \mathrm{Fn}_{\omega_n}(\omega_{3+2n}, 2)$.

Two natural extensions of this are left to the exercises. First, one can show (Exercises V.2.9 and V.2.10) that in Theorem V.2.7, the size of each 2^κ for $\kappa \notin \mathrm{dom}(E)$ is minimal consistent with $\forall \kappa \in \mathrm{dom}(E)\,[2^\kappa = E(\kappa)]$. Second (Exercise V.2.15), one can also control the map $\kappa \mapsto 2^\kappa$ on the *class* of *all* regular cardinals, so that one can produce a ctm for the statement that $2^\kappa = \kappa^{+++}$ for all regular κ.

Exercise V.2.9 *Working in ZFC, let E be an Easton index function and assume that* $\forall \kappa \in \mathrm{dom}(E)\,[2^\kappa = E(\kappa)]$. *For any infinite cardinal θ, let* $E'(\theta) = \max(\theta^+, \sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dom}(E)\})$. *Then let $E^*(\theta)$ be $E'(\theta)$ when* $\mathrm{cf}(E'(\theta)) > \theta$ *and $E'(\theta)^+$ when* $\mathrm{cf}(E'(\theta)) \leq \theta$. *Prove that* $2^\theta \geq E^*(\theta)$.

Exercise V.2.10 *Show that in the model of Theorem V.2.7,* $2^\theta = E^*(\theta)$ *for all infinite θ.*

As McAloon [67] showed in the 1970s, the Easton extensions can be used to verify the statements made at the end of Section II.8 about the consistency of various properties of *HOD*. For example,

Exercise V.2.11 *Starting with $V = L$ in the ground model, follow the suggestion of Exercise II.8.12 and obtain a generic extension satisfying the statements* $(2^{\aleph_0} = \aleph_2)^{HOD}$ *and* $L = (HOD)^{HOD} \subsetneqq HOD \subsetneqq V$.

Hint. You can use an automorphism argument, as in Section IV.8, to prove that a set is not in *HOD*. Also, the "zero-one" law (Theorem IV.4.15) shows

that if you start with $V = L$ and force with $\mathrm{Fn}(\omega_2, 2)$, the generic extension will satisfy $2^{\aleph_0} = \aleph_2$ and $HOD = L$. □

Easton [25] also showed how to violate *GCH* on a proper class of regular cardinals. For example, letting *Reg* denote the class of all regular cardinals, we can prove the consistency of $ZFC + \forall \kappa \in Reg\, [2^\kappa = \kappa^{+++}]$. To do this, we need to allow our index function E in Definition V.2.1 to be a proper class of ordered pairs, so that $I := \mathrm{dom}(E)$ is also a proper class; for example, we could say $\mathrm{dom}(E) = Reg$ with $E(\kappa) = \kappa^{+++}$ for all $\kappa \in Reg$. This much makes sense, using the standard conventions (as in Section I.4) for dealing with proper classes. But we can't copy Definition V.2.2 and define the Easton poset to be a subposet of $\mathbb{R} := \prod_{\kappa \in I} \mathrm{Fn}_\kappa(E(\kappa), 2)$, since then the elements of $\mathbb{R}$ would be proper classes (functions with domain I), which seems to be nonsense. However, we might regard (informally) *ON* as a strongly inaccessible cardinal, and condition $(*)$ applied with $\lambda = ON$ says that for $p \in \mathbb{P}(E)$, $\{\kappa \in I : p(\kappa) \neq \mathbb{1}\}$ is a set. Then we may replace p by a function whose domain is a subset of I, with the understanding that $p(\kappa) = \mathbb{1}$ when $\kappa \notin \mathrm{dom}(p)$. So,

Definition V.2.12 *If E is an index function with $I = \mathrm{dom}(E)$ a proper class, then define $\mathbb{P}(E)$ as in Definition V.2.2, except that $\mathbb{R}$ is replaced by the class of all p such that $\mathrm{dom}(p)$ is a subset of I and each $p(\kappa) \in \mathrm{Fn}_\kappa(E(\kappa), 2) \setminus \{\mathbb{1}\}$, and order $\mathbb{R}$ by making believe that $p(\kappa) = \mathbb{1}$ when $\kappa \notin \mathrm{dom}(p)$.*

Exercise V.2.13 *Verify that some* (but not all!) *of the basic results of forcing in Section IV.2 hold when $\mathbb{P}$ is a proper class of M.*

In particular, one can define $M[G]$, prove that it is transitive, etc. You have to make some modifications in the presentation. Names are still sets, not classes, so you don't have a name for G. The proof of the Power Set Axiom in $M[G]$ in Theorem IV.2.27 fails, as it should:

Exercise V.2.14 *Let $\mathbb{P} = \mathrm{Fn}(\omega, o(M))$. Prove that the Power Set Axiom is false in $M[G]$; specifically, $\mathcal{P}(\omega)$ doesn't exist.*

In the case of $\mathbb{P}(E)$ where E is an index function, one can avoid this problem using the splitting $\mathbb{P}(E) \cong \mathbb{P}(E^-_\kappa) \times \mathbb{P}(E^+_\kappa)$. For example, all subsets of ω in $M[G]$ really occur in the extension by $\mathbb{P}(E^-_\omega)$.

Exercise V.2.15 *In M, let $\mathbb{P} = \mathbb{P}(E)$ where E is some proper class index function. Show that $M[G] \models ZFC$. Then, if $M \models GCH$ and E is an Easton index function, show that $M[K] \models \forall \kappa \in \mathrm{dom}(E)\, [2^\kappa = E(\kappa)]$.*

Exercise V.2.16 *Prove* $\mathrm{Con}(ZFC) \rightarrow \mathrm{Con}(ZFC + GCH)$ *by using forcing, rather than L.*

Hint. Recall (Exercise IV.7.19) that $\mathrm{Fn}_{\aleph_1}(\omega_1, 2)$ makes *CH* true. □

Exercise V.2.17 *Assuming* Con(*ZFC*), *prove the consistency of the failure of* global choice. *This failure is axiomatized by the statements "$\varphi(x,y)$ does not define a well-order of V" for each formula $\varphi(x,y)$ with x,y and possibly other variables free.*

Hint. Start with $M \models V = L$, and do Exercise V.2.15 where $E(\kappa) = \kappa^+$ for all regular κ. Then $M[G] \models ZFC$. Then, use the method of Section IV.8 to prove the failure of global choice. Since $\varphi(x,y)$ may contain a tuple $\vec{t}$ of free variables (viewed as "fixed" parameters in a purported definition of a well-order of V), it is important that G be a proper class. If G were a set, then in the model $M[G]$, there would be a well-order of the universe definable from the set G. □

In most applications of products, we start with a given poset (such as $\mathbb{P}(E)$ in Easton forcing) and then utilize the fact that it may be viewed in many different ways as a product of two other posets (such as $\mathbb{P}(E_\lambda^-) \times \mathbb{P}(E_\lambda^+)$)

Another example of this is Cohen forcing, $\mathrm{Fn}(I,J)$. We have already used the observation that $\mathrm{Fn}(W,J) \subseteq_c \mathrm{Fn}(I,J)$ whenever $W \subseteq I$. For example, consider Lemma IV.4.10, that if $M \models CH$ and K is $\mathrm{Fn}(I,J)$–generic over M, then $\mathfrak{b} = \aleph_1$ holds in $M[K]$ (when J is countable in M). The proof used the fact than every $h \in \omega^\omega \cap M[K]$ lies in an extension of M by some $\mathrm{Fn}(W,J)$, where $W \subseteq I$ and W is countable in M, and we then used the countability of $\mathrm{Fn}(W,J)$ to prove that h cannot dominate $\omega^\omega \cap M$.

Further properties of the Cohen extension use the further observation that $\mathrm{Fn}(I,J) \cong \mathrm{Fn}(W,J) \times \mathrm{Fn}(I\backslash W,J)$, so that we may view the extension by $\mathrm{Fn}(I,J)$ as a two step iteration, first forcing with $\mathrm{Fn}(W,J)$ and then with $\mathrm{Fn}(I\backslash W,J)$. The next lemma expresses some of Section V.1 in this framework:

Lemma V.2.18 *In M, assume that $W \subseteq I$ and $\mathbb{P} = \mathrm{Fn}(I,J)$. Let K be $\mathbb{P}$–generic over M. Let $G = K \cap \mathrm{Fn}(W,J)$ and let $H = K \cap \mathrm{Fn}(I\backslash W,J)$. Then G is $\mathrm{Fn}(W,J)$–generic over M and H is $\mathrm{Fn}(I\backslash W,J)$–generic over $M[G]$ and $M[K] = M[G][H]$.*

Proof. In M, define $\zeta : \mathrm{Fn}(W,J) \times \mathrm{Fn}(I\backslash W,J) \xrightarrow[\text{onto}]{1\text{–}1} \mathrm{Fn}(I,J)$ by $\zeta(p,q) = p \cup q$, and observe that ζ is an isomorphism, with inverse $r \mapsto (r{\restriction}W, r{\restriction}(I\backslash W))$. Let $\widetilde{K} = \zeta^{-1}(K)$. By Lemma IV.4.7, $\widetilde{K}$ is $\mathrm{Fn}(W,J) \times \mathrm{Fn}(I\backslash W,J)$–generic over M and $M[K] = M[\widetilde{K}]$. By Lemma V.1.1 and Theorem V.1.2, $\widetilde{K} = G \times H$, where G is $\mathrm{Fn}(W,J)$–generic over M and H is $\mathrm{Fn}(I\backslash W,J)$–generic over $M[G]$ and $M[\widetilde{K}] = M[G][H]$. Then $G = \{p \in \mathrm{Fn}(W,J) : (p,\mathbb{1}) \in \widetilde{K}\} = \{p \in \mathrm{Fn}(W,J) : p \cup \emptyset \in K\} = K \cap \mathrm{Fn}(W,J)$, and likewise $H = K \cap \mathrm{Fn}(I\backslash W,J)$. □

Note that here, unlike in the Easton extension, there is no problem with absoluteness, since $\mathrm{Fn}(I,J)$ is always absolute, so there is a complete symmetry between $\mathrm{Fn}(W,J)$ and $\mathrm{Fn}(I\backslash W,J)$, but we shall phrase our arguments so that $\mathrm{Fn}(W,J)$ is what we plan to extend by first. A typical application is:

Lemma V.2.19 *In M, let $\mathbb{P} = \mathrm{Fn}(\kappa,\omega)$, where κ is any cardinal. Let K be $\mathbb{P}$–generic over M. Then $M[K] \models \mathfrak{d} \geq \kappa$.*

Proof. If the result is false, then, working in $M[K]$, there would be a cardinal $\theta < \kappa$ and a function $\vec{h} : \theta \times \omega \to \omega$ such that $\{h_\alpha : \alpha < \theta\}$ is a dominating family, where $h_\alpha(n) = \vec{h}(\alpha, n)$. Of course, $\theta \geq \omega_1^M = \omega_1^{M[K]}$. Let $\tau \in M^{\mathbb{P}}$ be a nice name for a subset of $(\theta \times \omega) \times \omega$ such that $\tau_K = \vec{h}$.

Working in M, fix $W_0 \subset \kappa$ such that $|W_0| \leq \theta$ and all forcing conditions mentioned in τ come from $\mathrm{Fn}(W_0, \omega)$; then choose W with $W_0 \subset W \subset \kappa$ and $|\kappa \backslash W| = \aleph_0$. Note that τ is a $\mathrm{Fn}(W, \omega)$–name and $\mathrm{Fn}(\kappa \backslash W, \omega) \cong \mathrm{Fn}(\omega, \omega)$.

By Lemma V.2.18, $M[K] = M[G][H]$, where $\vec{h} \in M[G]$. By Lemma IV.7.35 (replacing M by $M[G]$ and $\mathrm{Fn}(\omega, \omega)$ by $\mathrm{Fn}(\kappa \backslash W, \omega)$), there is an $f \in M[G][H]$ such that $\neg(f \leq^* h_\alpha)$ for all α, which is a contradiction. □

Exercise IV.4.13 said essentially that one may do this, using a more complicated argument, but without using products or Lemma V.2.18. Instead, work only with $\mathbb{P} = \mathrm{Fn}(\kappa, \omega)$, and say, WLOG, $\kappa \backslash W = \omega$, and let $\mathring{h}$ be the $\mathbb{P}$–name for $\bigcup K \restriction \omega$, and show that no p can force $\mathring{h} \leq^* \mathring{f}_\alpha$.

Similarly to Lemma V.2.19, Lemma V.2.22 will prove that $\mathrm{cov}(\mathsf{meager}) \geq \kappa$ in the extension by $\mathrm{Fn}(\kappa, \omega)$. Here, there is an additional problem: $\mathfrak{d}$ talks about elements of ω^ω, which are countable objects, while $\mathrm{cov}(\mathsf{meager})$ talks about subsets of $\mathbb{R}$, which have size $\mathfrak{c}$. But, we can code an open set of reals by a countable object that tells how that set is built:

Definition V.2.20 *For $c \subseteq \mathbb{Q} \times \mathbb{Q}$, define $U_c = \bigcup\{(x, y) : \langle x, y\rangle \in c\} \subseteq \mathbb{R}$. Here, (x, y) denotes an open interval in $\mathbb{R}$, with $(x, y) = \emptyset$ when $x \geq y$.*

Note that every open subset of $\mathbb{R}$ is U_c for some (and, infinitely many) c. Also, note that $\mathbb{Q}$ is an absolute concept if it is defined in any reasonable way (see Definition I.14.1), as are the statements "$c \subseteq \mathbb{Q} \times \mathbb{Q}$" and "$U_c$ is dense in $\mathbb{R}$" and "$z \in U_c$". Now, analogously to Lemmas IV.7.35 and V.2.19, we prove:

Lemma V.2.21 *Let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$ and let H be $\mathbb{P}$–generic over M. In $M[H]$, let $h = \bigcup H : \omega \to 2$, and let $r = \sum_j h(j) \cdot 2^{-j} \in [0, 2] \subset \mathbb{R}$. Then $r \in U_c$ for all $c \in \mathcal{P}(\mathbb{Q} \times \mathbb{Q}) \cap M$ such that U_c is dense in $\mathbb{R}$.*

Proof. Let D_c be the set of all $p \in \mathbb{P}$ such that for some $n \in \omega$ and $\langle x, y\rangle \in c$, $\mathrm{dom}(p) = n$ and $x < \sum_{j<n} p(j) \cdot 2^{-j} < y - 2^{1-n}$. If U_c is dense in $\mathbb{R}$, then D_c is dense in $\mathbb{P}$, and $G \cap D_c \neq \emptyset$ implies that $r \in U_c$. □

Lemma V.2.22 *In M, let $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$, where κ is any cardinal. Let K be $\mathbb{P}$–generic over M. Then $M[K] \models \mathrm{cov}(\mathsf{meager}) \geq \kappa$.*

Proof. If the result is false, then, working in $M[K]$, there would be a cardinal $\theta < \kappa$ and a subset $\vec{c} \subseteq \theta \times (\mathbb{Q} \times \mathbb{Q})$ such that $\bigcap_{\alpha<\theta} U_{c_\alpha} = \emptyset$, where $c_\alpha = \{\langle x, y\rangle : \langle \alpha, \langle x, y\rangle\rangle \in c\}$ and each U_{c_α} is dense in $\mathbb{R}$. Now, proceed as in the proof of Lemma V.2.19. □

Comparing Lemmas V.2.19 and V.2.22, note that by Exercise IV.4.8, we can use interchangeably $\mathrm{Fn}(\kappa, 2)$, $\mathrm{Fn}(\kappa, \omega)$, or $\mathrm{Fn}(\kappa \times \omega, 2)$ — whichever seems more convenient.

For many results in the literature, one needs to code arbitrary Borel sets, not just open sets. It is easy to check that the basic cardinal functions, such as cov(meager), remain the same if we replace the space $\mathbb{R}$ by either the Cantor set 2^ω or the space ω^ω (which is homeomorphic to $\mathbb{R}\backslash\mathbb{Q}$). Once this is done the coding becomes a bit simpler. For example, open subsets of 2^ω can be coded by subsets of $\mathrm{Fn}(\omega, 2)$. See, e.g., [55].

We can now, in the Cohen extension, compute the values of all the cardinals discussed in Section III.1. We confine ourselves to the situation where the cardinal arithmetic is simple:

Exercise V.2.23 *Assume that in M, CH holds and $\kappa > \aleph_0$ and $\kappa^{\aleph_0} = \kappa$. Let K be $\mathrm{Fn}(\kappa, 2)$-generic over M. Then in $M[K]$:*

1. $\mathfrak{p} = \mathfrak{b} = \mathfrak{a} = \aleph_1$ *and* $\mathfrak{d} = \mathfrak{c} = \kappa$.
2. $\mathrm{cov}(\mathsf{null}) = \mathrm{add}(\mathsf{null}) = \mathrm{non}(\mathsf{meager}) = \mathrm{add}(\mathsf{meager}) = \aleph_1$.
3. $\mathrm{cov}(\mathsf{meager}) = \mathrm{non}(\mathsf{null}) = \mathfrak{c}$.
4. *There is a set $L \in [\mathbb{R}]^{\mathfrak{c}}$ that is a Luzin set; that is, every $X \in [L]^{\aleph_1}$ is not of first category.*

Hint. For (1), use Theorem IV.3.13, Lemma III.1.24, Exercise IV.4.12, and Lemma V.2.19. By Rothberger's Theorem (Exercise III.1.8), $\mathrm{cov}(\mathsf{meager}) \leq \mathrm{non}(\mathsf{null})$, so (3) follows from $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c}$.

By Lemma III.1.7, (2) will follow from $\mathrm{cov}(\mathsf{null}) = \mathrm{non}(\mathsf{meager}) = \aleph_1$, and $\mathrm{cov}(\mathsf{null}) = \aleph_1$ holds by Exercise IV.8.15, so we need only show that $\mathrm{non}(\mathsf{meager}) = \aleph_1$, which follows from (4).

For (4), view the poset as $\mathrm{Fn}(\kappa \times \omega, 2)$, so in the generic extension we get $\{h_\alpha : \alpha < \kappa\} \subseteq 2^\omega\}$ in the standard way. Let $r_\alpha = \sum_j h_\alpha(j) \cdot 2^{-j}$ and let $L = \{r_\alpha : \alpha < \kappa\}$. The proof that L is Luzin uses the fact that every code c for a dense open set only depends on $\mathrm{Fn}(W \times \omega, 2)$ for some countable $W \subseteq \kappa$. □

Similar product arguments using $\mathrm{Fn}_{\omega_1}(\kappa, 2)$ yield:

Exercise V.2.24 *In M, assume that GCH holds and let $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\omega_2, 2)$. Let G be $\mathbb{P}$-generic over M. Then $M[G] \models GCH \wedge \diamondsuit \wedge \neg\diamondsuit^*$.*

Hint. For $\diamondsuit$, see Lemma IV.7.30. For $\neg\diamondsuit^*$, show that any potential $\diamondsuit^*$ sequence lives in the extension by some $\mathrm{Fn}_{\omega_1}(W, 2)$, where $W \in [\omega_2]^{\aleph_1}$, and this sequence is destroyed by the poset $\mathrm{Fn}_{\omega_1}(\omega_2\backslash W, 2)$. □

We next produce a model where Kurepa's Hypothesis (KH) is false. We already know that KH follows from $\diamondsuit^+$ (Theorem III.7.6) and hence is true in L. The next theorem establishes the equiconsistency of $\neg KH$ with the existence of an inaccessible cardinal (see the remarks on page 268):

Theorem V.2.25 *If* $(\kappa$ *is strongly inaccessible*$)^M$*, where* M *is a ctm for ZFC, then for some generic extension* $M[G]$*:* $\kappa = (\omega_2)^{M[G]}$ *and* $M[G] \models \neg KH$.

In the proof (due to Silver, 1971), the forcing will be countably closed, so that $(\omega_1)^{M[G]} = \omega_1^M$. Thus, if T is any ω_1-tree in M, it remains an ω_1-tree in $M[G]$. We first explain how to make sure that the trees in M are not Kurepa in $M[G]$; then we iterate that forcing to kill all Kurepa trees.

Lemma V.2.26 *In* M*, let* $(T, \sqsubset)$ *be an* ω_1*-tree and let* $\mathbb{P}$ *be any countably closed poset. If* $C \in M[G]$ *is a path through* T*, then* $C \in M$*. In particular, if* $\theta \geq 2^{\aleph_1}$ *and* $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\omega_1, \theta)$ *hold in* M*, then* T *is not a Kurepa tree in* $M[G]$.

Proof. For the "in particular": Note that $\bigcup G$ is a map from $(\omega_1)^{M[G]} = \omega_1^M$ onto θ. In M, let $\mathcal{C}$ be the set of all paths through T, so $(|\mathcal{C}| \leq \theta)^M$. Once we have proved that T has no paths besides these in $M[G]$, we shall know that in $M[G]$: $|\mathcal{C}| \leq |\theta| = \aleph_1$, so that T has $\leq \aleph_1$ paths, so that T is not Kurepa.

To prove that $M[G]$ adds no new paths to T, assume that this is false. Then there is a name $\mathring{C}$ and a $p \in \mathbb{P}$ such that $p \Vdash [\mathring{C} \notin \check{\mathcal{C}} \wedge \mathring{C}$ is a path through $T]$.

Now, working in M, we derive a contradiction by showing that some level of T is uncountable. First, we get a tree of extensions of p indexed by the Cantor tree $2^{<\omega}$. We shall get p_s, x_s for $s \in 2^{<\omega}$ and α_n for $n \in \omega$ satisfying:

1. $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \omega_1$.
2. $p_s \in \mathbb{P}$ and $p_s \leq p$.
3. When $\mathrm{lh}(s) = n$: $x_s \in \mathcal{L}_{\alpha_n}(T)$.
4. $p_s \Vdash x_s \in \mathring{C}$.
5. $p_{s^\frown 0} \leq p_s$ and $p_{s^\frown 1} \leq p_s$.
6. $x_{s^\frown 0} \neq x_{s^\frown 1}$.

Assuming that this can all be done, let $\gamma = \sup\{\alpha_n : n < \omega\} < \omega_1$. For $f \in 2^\omega$. use (5) and the fact that $\mathbb{P}$ is countably closed to choose p_f such that $p_f \leq q_{f\restriction n}$ for all n. Since $p_f \Vdash [\mathring{C}$ is a path through $T]$ (applying (2)), $p_f \Vdash \exists y \in \mathcal{L}_\gamma [y \in \mathring{C}]$, so (by Lemma IV.6.1), there is a $q_f \leq p_f$ and an $x_f \in \mathcal{L}_\gamma$ such that $q_f \Vdash x_f \in \mathring{C}$. For each n, $q_f \Vdash x_{f\restriction n} \in \mathring{C}$ (by (4)), and $\mathrm{height}(x_{f\restriction n}) = \alpha_n < \gamma$ (by (3)), so $x_{f\restriction n} \sqsubset x_f$. But then, applying (6), the x_f for $f \in 2^\omega$ are all different, so that $|\mathcal{L}_\gamma| \geq 2^{\aleph_0}$, which is a contradiction.

To construct the p_s, x_s and α_n: Fix $p_\emptyset \leq p$ and $x_0 \in \mathcal{L}_0$ such that such that $p_\emptyset \Vdash x \in \mathring{C}$ (if T isn't rooted, there may be more than one choice for x_0).

Now, fix n and assume that we have constructed p_s, x_s for all s with $\mathrm{lh}(s) = n$. For each such s: Let $E_s = \{y \in T : \exists q \leq p_s [q \Vdash y \in \mathring{C}]\}$. Observe that E_s is a subtree of T and E_s meets every level of T and $E_s \cap T_{\alpha_n+1} = \{x_s\} \cup x_s{\downarrow}$. E_s is also well-pruned (even if T isn't). To see this: If $y \in E_s$ and $q \Vdash y \in \mathring{C}$ and $q \leq p_s$ and $\mathrm{height}(y) < \beta < \omega_1$, then, since q forces $\mathring{C}$ to be a path, there is an $r \leq q$ and a $z \in \mathcal{L}_\beta \cap y{\uparrow}$ such that $r \Vdash z \in \mathring{C}$.

But E_s cannot be a path. To see this: If it were a path, then $E_s \in \mathcal{C}$, whereas $p_s \Vdash [\mathring{C} \notin \mathcal{C}]$, so that $p_s \Vdash \mathring{C} \neq E_s$, so $p_s \Vdash \mathring{C} \not\subseteq E_s$ (since they are both paths), so there would be a $y \in T \backslash E_s$ and a $q \leq p_s$ such that $q \Vdash y \in \mathring{C}$, contradicting the definition of E_s.

Since the E_s are well-pruned and not paths, we can fix $\alpha_{n+1} > \alpha_n$ so that for each s of length n, there are $x_{s^\frown 0}, x_{s^\frown 1} \in E_s \cap \mathcal{L}_{\alpha_{n+1}}$ with $x_{s^\frown 0} \neq x_{s^\frown 1}$. There are then $p_{s^\frown i}$ for $i = 0, 1$ such that $p_{s^\frown i} \leq p_s$ and $p_{s^\frown i} \Vdash x_{s^\frown i} \in \mathring{C}$. $\square$

This $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\omega_1, \theta)$ is similar to the cardinal collapsing posets of the form $\mathrm{Fn}(\omega, \theta)$ described in Example IV.3.1 and Exercise IV.7.10, except that now, forcing with countable conditions rather than finite ones, θ becomes of size $\aleph_1$, not $\aleph_0$ in the generic extension.

$\mathbb{P}$ kills all the Kurepa trees in M, but we cannot claim that $M[G] \models \neg KH$ because $M[G]$ will have trees that do not lie in M. To get a model for $\neg KH$, we have to iterate this forcing, collapsing larger and larger cardinals, obtaining the analog of the Lévy collapse (see Exercises III.3.96 and IV.7.48), but here we use the *countable Lévy collapse*, using countable forcing conditions and a countable support product, so that the inaccessible κ becomes $\aleph_2$, not $\aleph_1$.

Proof of Theorem V.2.25. Working in M: For $S \subseteq \kappa$, let:

$$\mathbb{P}_S = \left\{\vec{p} \in \textstyle\prod_{\alpha \in S} \mathrm{Fn}_{\omega_1}(\omega_1, \alpha) : |\{\alpha : p_\alpha \neq \mathbb{1}\}| \leq \aleph_0\right\} \ .$$

We shall force with $\mathbb{P} := \mathbb{P}_\kappa$, and we shall make use of the observation that $\mathbb{P} \cong \mathbb{P}_S \times \mathbb{P}_{\kappa \setminus S}$ for all $S \subseteq \kappa$.

Since κ is strongly inaccessible, a delta system argument (similarly to Lemma V.2.3) shows that $\mathbb{P}$ has the κ-cc. So, $\mathbb{P}$ preserves cofinalities and cardinals $\geq \kappa$ (Theorem IV.7.9). Also, $\mathbb{P}$ is countably closed, so it does not add ω–sequences

So, in $M[G]$, κ remains a cardinal and $(\omega_1)^{M[G]} = (\omega_1)^M$. Also, for each $\alpha < \kappa$, there is a complete embedding defined in M from $(\mathrm{Fn}_{\omega_1}(\omega_1, \alpha))^M$ into $\mathbb{P}$, so that in $M[G]$, there is an $(\mathrm{Fn}_{\omega_1}(\omega_1, \alpha))^M$–generic filter, whose union is a map from ω_1 onto α, so that $(\omega_2)^{M[G]} = \kappa$.

Now, suppose that $M[G] \models KH$. Then, in $M[G]$, there is a Kurepa tree, which we may assume is the set ω_1, with some tree order $\sqsubset$; so $\sqsubset \subseteq \omega_1 \times \omega_1$. Then $\sqsubset = \mathring{\sqsubset}_G$, where $\mathring{\sqsubset}$ is a nice name for a subset of $\omega_1 \times \omega_1$.

Working in M: For $\vec{p} \in \mathbb{P}$, let $\mathrm{supt}(\vec{p}) = \{\alpha : p_\alpha \neq \mathbb{1}\}$, which is always countable. Let B be the set of all forcing conditions mentioned in $\mathring{\sqsubset}$. Then $|B| < \kappa$ because all antichains have size less than κ, which is regular and $> \aleph_1$. Let $S = \bigcup\{\mathrm{supt}(p) : p \in B\}$; so $|S| < \kappa$. Let $\mathbb{P}^- = \mathbb{P}_S$ and $\mathbb{P}^+ = \mathbb{P}_{\kappa \setminus S}$, so that $\mathbb{P} \cong \mathbb{P}^- \times \mathbb{P}^+$.

Using this isomorphism and applying Theorem V.1.2, we see that $M[G] = M[G^-][G^+]$, where G^- is $\mathbb{P}^-$–generic over M and G^+ is $\mathbb{P}^+$–generic over $M[G^-]$ and $\sqsubset \in M[G^-]$. Furthermore, since $\mathbb{P}^-$ is countably closed, it doesn't add ω–sequences, so that ($\mathbb{P}^+$ is countably closed)$^{M[G^-]}$, and hence Lemma V.2.26 applies in passing from $M[G^-]$ to $M[G^-][G^+] = M[G]$, so that every path through the tree $(\omega_1, \sqsubset)$ that lies in $M[G]$ already lies in $M[G^-]$. Let $\mathcal{C}$ be the set of all these paths.

Using the inaccessibility of κ, $(|\mathbb{P}^-| < \kappa)^M$, which implies that κ remains strongly inaccessible in $M[G^-]$ (see Exercise IV.7.55). Working in $M[G^-]$, fix

$\alpha < \kappa$ such that $\alpha \notin S$ and $\alpha \geq |\mathcal{C}|$. Then, working in $M[G]$, $|\mathcal{C}| \leq |\alpha| \leq \omega_1$, so that the tree $(\omega_1, \sqsubset)$ is not Kurepa. □

Exercise V.2.27 *In the model used for the proof of Theorem V.2.25, show that* $M[G] \models \diamondsuit \wedge \neg\diamondsuit^*$. *Furthermore, if* $M \models GCH$ *then* $M[G] \models GCH$.

Hint. See Exercise V.2.24. □

So, $\diamondsuit \not\to KH$. By a harder argument [18], $\diamondsuit^* \not\to KH$.

We now describe two kinds of examples of posets $\mathbb{P}, \mathbb{Q}$ such that both $\mathbb{P}$ and $\mathbb{Q}$ preserve ω_1 but $\mathbb{P} \times \mathbb{Q}$ collapses ω_1 (see Definition IV.6.2). It is easy to see that if $\mathbb{P}, \mathbb{Q}$ are both countably closed, then so is $\mathbb{P} \times \mathbb{Q}$, so $\mathbb{P} \times \mathbb{Q}$ preserves ω_1, but in Jensen's example, $\mathbb{P}, \mathbb{Q}$ have the weaker property of not adding ω–sequences:

Exercise V.2.28 *Let* $S \subseteq \omega_1$ *be stationary, and let* $\mathbb{P}_S$ *be Jensen's poset for shooting a club through* S. $\mathbb{P}_S$ *is the set of all countable* $p \subseteq S$ *such that* p *is closed in* ω_1. *So, either* $p = \emptyset = \mathbb{1}$ *or* $\sup(p) = \max(p) \in p$. *Define* $q \leq p$ *iff* $q \supseteq p$ *and* q *is an end extension — that is, either* $p = \emptyset$ *or* $q \cap (\max(p)+1) = p$. *Show that* $\mathbb{P}_S$ *is Baire (so, it preserves* ω_1*) and, in the generic extension,* $\bigcup G$ *is a club and is a subset of* S. *So, if* S, T *are two disjoint stationary sets, then* $\mathbb{P}_S \times \mathbb{P}_T$ *collapses* ω_1.

Hint. The hard part, proving that $\mathbb{P}_S$ is Baire, does not require any knowledge of forcing. If D_n is dense open for $n \in \omega$, and p is given, then first produce a club C such that all $\gamma \in C$ are limit ordinals with the property that there exists an ω–chain $p = p_0 > p_1 > \cdots$ with all $p_n \in D_n$ and $\sup(\bigcup_n p_n) = \gamma$. Then, when $\gamma \in C \cap S$, $q := \bigcup_n p_n \cup \{\gamma\} \in \mathbb{P}_S$ and $q \in \bigcap_n D_n$. You can get C using a club of elementary submodels (as in Section III.8) or by a direct combinatorial argument.

The easy part: If G is $(\mathbb{P}_S \times \mathbb{P}_T)$–generic over M, then in $M[G]$ there are two disjoint club subsets of $(\omega_1)^M$. But it is a theorem of *ZFC* that the intersection of two club subsets of ω_1 is a club (Lemma III.6.2). So $(\omega_1)^M \neq (\omega_1)^{M[G]}$. □

Note that the logic is trivial in the hard part but non-trivial in the easy part. Our discussion above presumes the ctm approach, but one can also do it using the "forcing over V" approach; see the discussion of Definition IV.6.2.

The second kind of example is where $\mathbb{P}$ and $\mathbb{Q}$ are both ccc. This can happen in several different ways. We have already covered most of the next proposition.

Proposition V.2.29 *Let* H *denote the assertion that there are ccc posets* $\mathbb{P}$, $\mathbb{Q}$ *such that* $\mathbb{P} \times \mathbb{Q}$ *is not ccc. Let* H^+ *denote the assertion that there are ccc posets* $\mathbb{P}$, $\mathbb{Q}$ *such that* $\mathbb{P} \times \mathbb{Q}$ *collapses* ω_1. *Then*

$$H^+ \to H \ ; \ MA(\aleph_1) \to \neg H \ ; \ \neg SH \to H^+ \ ; \ CH \to H^+ \ .$$

Proof. The statements about H alone do not require any knowledge of forcing. $MA(\aleph_1) \to \neg H$ is Theorem III.3.43 and $\neg SH \to H$ is Lemma III.2.18 (where X is a Suslin line and $\mathbb{P} = \mathbb{Q} = \mathbb{O}(X)$). $CH \to H$ is Theorem III.8.19.

The statements about H^+ really do involve forcing, and, as we have just explained, can be made rigorous by using either the ctm approach or the "forcing over V" approach. Note that $H^+ \to H$ is immediate from the fact that ccc forcing preserves cardinals.

For $\neg SH \to H^+$, let T be a well-pruned Suslin tree. Let $\mathbb{P}$ be T, viewed as a forcing poset. So, $\mathbb{P}$ adds a path through T in the generic extension (Exercise IV.7.29). Let $\mathbb{Q}$ be the poset that makes T into a special Aronszajn tree (see the proof of Theorem III.5.19). Let G be $\mathbb{P} \times \mathbb{Q}$–generic over M. Applying Lemma V.1.1: in $M[G]$ we have a path C through T (which meets every level), and T is a countable union of antichains, $T = \bigcup_n A_n$. Since C has order type $(\omega_1)^M$ and each $|C \cap A_n| \leq 1$, C is countable in $M[G]$, so $(\omega_1)^M < (\omega_1)^{M[G]}$.

For $CH \to H^+$, let $\mathbb{P}, \mathbb{Q}$ be the $\mathbb{Q}_0, \mathbb{Q}_1$ from Exercise III.8.28; see the discussion following that exercise. We have an I of size $\aleph_1$ in the ground model such that after forcing with $\mathbb{Q}_0 \times \mathbb{Q}_1$, we get $I = \bigcup\{Z_m \cap Y_n : m, n \in \omega\}$ in the generic extension, where each $|Z_m \cap Y_n| \leq 1$; so, I becomes countable. □

It is consistent to have ccc posets $\mathbb{P}, \mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is Baire (so it preserves ω_1) but not ccc. We can even have $\mathbb{P} = \mathbb{Q}$ be a Suslin tree T. Although $T \times T$ is not ccc (Exercise III.5.48), it could be Baire:

Exercise V.2.30 *Assume $\diamondsuit$ and prove that there is a well-pruned Suslin tree T such that $T \times T$ is Baire.*

Hint. Modify Exercise III.7.10. Make sure that $p{\uparrow} \circledast q{\uparrow}$ is Suslin whenever $p, q \in T$ and $p \perp q$. □

The question of whether $\mathbb{P} \times \mathbb{Q}$ is ccc is related to whether $\mathbb{Q}$ remains ccc in the $\mathbb{P}$ extension:

Exercise V.2.31 *For ccc posets $\mathbb{P}, \mathbb{Q}$, the following are equivalent:*

1. *$\mathbb{P} \times \mathbb{Q}$ is ccc.* 2. *$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mathbb{Q}}$ is ccc.* 3. *$\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \check{\mathbb{P}}$ is ccc.*

Hint. For (1) $\to$ (2): If $\neg(2)$, then there is a $p \in \mathbb{P}$ and an $\mathring{f}$ such that p forces that $\mathring{f} : \omega_1 \to \mathbb{Q}$ and $\mathring{f}$ enumerates an antichain. Choose $p_\xi \leq p$ and $q_\xi \in \mathbb{Q}$ such that $p_\xi \Vdash \mathring{f}(\xi) = q_\xi$. Then $\{(p_\xi, q_\xi) : \xi < \omega_1\}$ is an antichain in $\mathbb{P} \times \mathbb{Q}$.

Assume (2) $\wedge \neg(1)$. Let $\{(p_\xi, q_\xi) : \xi < \omega_1\}$ be an antichain. Let $\mathring{D}$ be the $\mathbb{P}$–name $\{\langle \check{\xi}, p_\xi \rangle : \xi < \omega_1\}$. Get $\mathring{f}$ such that $\mathbb{1} \Vdash \mathring{f} : \mathring{D} \to \mathbb{Q}$ and $p_\xi \Vdash \mathring{f}(\xi) = q_\xi$. By (2), $\mathbb{1}$ forces $\mathring{D}$ to be countable, which leads to a contradiction. □

This exercise is a special case of Lemma V.3.9 and Exercise V.3.10, which have more complicated proofs.

V.3 General Two-Step Iteration

We now consider the situation where $\mathbb{P} \in M$ and $\mathbb{Q} \in M[G]$ and G is $\mathbb{P}$–generic over M and H is $\mathbb{Q}$–generic over $M[G]$. We build $M[G][H]$ in one step by producing another poset $\mathbb{R} \in M$ and a filter K that is $\mathbb{R}$–generic over M such that $M[G][H] = M[K]$. Section V.1 discussed the special case where $\mathbb{Q} \in M$; then $\mathbb{R}$ was $\mathbb{P} \times \mathbb{Q}$. If $\mathbb{Q} \notin M$ but we want $\mathbb{R}$ to be in M, then we work in M and build $\mathbb{R}$ out of $\mathbb{P}$ and a name for $\mathbb{Q}$.

First, we observe that formally, a forcing poset is really a triple $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$ (see Definition III.3.1), so we spell out more specifically how we name these three objects:

Definition V.3.1 *If $\mathbb{P}$ is a forcing poset, then a* $\mathbb{P}$–name for a forcing poset *is a triple of $\mathbb{P}$–names, $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$, such that $\mathring{\mathbb{1}}_{\mathbb{Q}} \in \mathrm{dom}(\mathring{\mathbb{Q}})$ and*

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \left[\mathring{\mathbb{1}}_{\mathbb{Q}} \in \mathring{\mathbb{Q}} \wedge \mathring{\leq}_{\mathbb{Q}} \text{ is a pre-order of } \mathring{\mathbb{Q}} \text{ with largest element } \mathring{\mathbb{1}}_{\mathbb{Q}}\right] .$$

We often write $\mathring{\mathbb{Q}}$ for $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$ and $\mathring{\leq}$ for $\mathring{\leq}_{\mathbb{Q}}$ and $\mathring{\mathbb{1}}$ for $\mathring{\mathbb{1}}_{\mathbb{Q}}$.

For example, say in M we wish to name the poset $[\omega]^\omega$ ordered by $\subseteq^*$ (as in Example III.3.5) — but we're naming the $[\omega]^\omega$ of $M[G]$, not of M. We could (in M) let $\mathring{\mathbb{Q}}$ be the set of all $\langle \vartheta, \mathbb{1}_{\mathbb{P}} \rangle$ such that ϑ is a nice $\mathbb{P}$–name for a subset of ω and $\mathbb{1}_{\mathbb{P}} \Vdash |\vartheta| = \aleph_0$. Then, let $\mathring{\mathbb{1}}_{\mathbb{Q}}$ be $\check{\omega}$, and let $\mathring{\leq}_{\mathbb{Q}}$ be some name for the $\subseteq^*$ relation on $\mathring{\mathbb{Q}}$.

It might seem more natural to define a $\mathbb{P}$–name for a poset to be just some name τ that is forced by some p to be a triple satisfying Definition III.3.1. It turns out that spelling out the three names $\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}}$ explicitly makes it easier to define the iterated forcing poset (see Definition V.3.3). First, we observe that Definition V.3.1 does not cause us to lose any generality in representing posets from $M[G]$:

Lemma V.3.2 *Let M be a ctm for ZFC with $\mathbb{P} \in M$ and let G be $\mathbb{P}$–generic over M. In $M[G]$, let $(\mathbb{X}, \leq_{\mathbb{X}}, \mathbb{1}_{\mathbb{X}})$ be a forcing poset as per Definition III.3.1. Then, there is a $\mathbb{P}$–name for a forcing poset $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}}) \in M$ such that in $M[G]$, the poset $(\mathrm{val}(\mathring{\mathbb{Q}}, G), \mathrm{val}(\mathring{\leq}_{\mathbb{Q}}, G), \mathrm{val}(\mathring{\mathbb{1}}_{\mathbb{Q}}, G))$ is isomorphic to $(\mathbb{X}, \leq_{\mathbb{X}}, \mathbb{1}_{\mathbb{X}})$. Furthermore, we may take $\mathring{\mathbb{Q}}$ to be $\check{\alpha}$ for some ordinal α, and we may take $\mathring{\mathbb{1}}_{\mathbb{Q}}$ to be $\check{0}$, and we may take $\mathring{\leq}_{\mathbb{Q}}$ to be a nice name for a subset of $\alpha \times \alpha$.*

Proof. Working in $M[G]$: Let $\alpha = |\mathbb{X}|$. Then, using a bijection from $\mathbb{X}$ onto α that moves $\mathbb{1}_{\mathbb{X}}$ to 0, we may define a pre-order $\trianglelefteq$ of α such that the poset $(\alpha, \trianglelefteq, 0) \cong (\mathbb{X}, \leq_{\mathbb{X}}, \mathbb{1}_{\mathbb{X}})$. Let $\varphi(u, v, w)$ abbreviate the statement "$u \subseteq v \times v$ is a pre-order of the set v with largest element $w \in v$"; so $\varphi(\trianglelefteq, \alpha, 0)$ is true.

Fix $\tau \in M^{\mathbb{P}}$ with $\tau_G = \trianglelefteq$. Let $\mathring{\mathbb{Q}}$ be $\check{\alpha}$ and let $\mathring{\mathbb{1}}_{\mathbb{Q}}$ be $\check{0}$. We cannot let $\mathring{\leq}_{\mathbb{Q}}$ be τ. If we did, then we would have everything required by the lemma *except* for the problems that τ need not be a nice name and $\mathbb{1}_{\mathbb{P}}$ need not force

the statement $\varphi(\tau, \check{\alpha}, \check{0})$ (although $M[G] \models \varphi(\tau, \check{\alpha}, \check{0})$, this only implies that $\varphi(\tau, \check{\alpha}, \check{0})$ is forced by some $p \in G$).

To fix these problems, note that $\mathbb{1} \Vdash \exists y\, [\varphi(y, \check{\alpha}, \check{0}) \wedge [\varphi(\tau, \check{\alpha}, \check{0}) \to y = \tau]]$, since we can let y be the trivial order $\alpha \times \alpha$ when $\neg\varphi(\tau, \check{\alpha}, \check{0})$. Then, by the Maximal Principle (Theorem IV.7.1), we can fix a name σ such that $\mathbb{1} \Vdash [\varphi(\sigma, \check{\alpha}, \check{0}) \wedge [\varphi(\tau, \check{\alpha}, \check{0}) \to \sigma = \tau]]$. Then, applying Lemma IV.3.10, let $\mathring{\leq}_{\mathbb{Q}}$ be a nice name for a subset of $\alpha \times \alpha$ such that $\mathbb{1} \Vdash (\sigma \subseteq \alpha \times \alpha \to \sigma = \mathring{\leq}_{\mathbb{Q}})$. Observe that $\mathrm{val}(\mathring{\leq}_{\mathbb{Q}}, G) = \tau_G = \trianglelefteq$. □

The "Furthermore" part of this lemma will be used in Section V.4, but it is not needed for the general theory, which works directly from Definition V.3.1.

Definition V.3.3 *If $\mathbb{P}$ is a forcing poset and $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$ is a $\mathbb{P}$–name for a forcing poset, then the* product $\mathbb{P} * \mathring{\mathbb{Q}}$ *is the triple* $(\mathbb{R}, \leq, \mathbb{1})$, *where*

$$\mathbb{R} = \{(p, \mathring{q}) \in \mathbb{P} \times \mathrm{dom}(\mathring{\mathbb{Q}}) : p \Vdash \mathring{q} \in \mathring{\mathbb{Q}}\}; \qquad \mathbb{1} = (\mathbb{1}_{\mathbb{P}}, \mathring{\mathbb{1}}_{\mathbb{Q}});$$
$$(p_1, \mathring{q}_1) \leq (p_2, \mathring{q}_2) \text{ iff } p_1 \leq_{\mathbb{P}} p_2 \text{ and } p_1 \Vdash [\mathring{q}_1 \mathring{\leq}_{\mathbb{Q}} \mathring{q}_2].$$

We often write "$(p, \mathring{q}) \in \mathbb{P} * \mathring{\mathbb{Q}}$" for "$(p, \mathring{q}) \in \mathbb{R}$".
Define $i : \mathbb{P} \to \mathbb{P} * \mathring{\mathbb{Q}}$ by: $i(p) = (p, \mathring{\mathbb{1}}_{\mathbb{Q}})$.

As usual, in the ctm approach, it is understood that $\mathbb{P} \in M$ and $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$ is a triple of $\mathbb{P}$–names from M. Of course, $\mathbb{P} * \mathring{\mathbb{Q}}$ should really be written as something like $\mathbb{P} * (\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$. Note that the set $\mathbb{R}$ contains all $i(p) = (p, \mathring{\mathbb{1}}_{\mathbb{Q}})$, since $\mathbb{1}_{\mathbb{P}} \Vdash \mathring{\mathbb{1}}_{\mathbb{Q}} \in \mathring{\mathbb{Q}}$, but $\mathbb{R}$ need not be all of $\mathbb{P} \times \mathrm{dom}(\mathring{\mathbb{Q}})$ (although $\mathbb{R} = \mathbb{P} \times \mathrm{dom}(\mathring{\mathbb{Q}})$ when $\mathring{\mathbb{Q}} = \check{\alpha}$ and the "Furthermore" part of Lemma V.3.2 holds).

If $\mathbb{Q} \in M$, then $\mathbb{P} \times \mathbb{Q}$ is isomorphic to $\mathbb{P} * \check{\mathbb{Q}}$. In the $\times$ product, we have an obvious isomorphism between $\mathbb{P} \times \mathbb{Q}$ and $\mathbb{Q} \times \mathbb{P}$. In Definition V.3.3, we lose this symmetry, since $\mathbb{P}$ is in the ground model and $\mathring{\mathbb{Q}}$ names something from the generic extension and "$\mathring{\mathbb{Q}} * \mathbb{P}$" is meaningless.

We now need to prove some results like the ones in Section V.1 for products, saying that this $*$ product "works". The following is easy from the definitions:

Lemma V.3.4 *Using the notation of Definition V.3.3, with $p_0, p_1 \in \mathbb{P}$ and $\mathring{q}_0, \mathring{q}_1 \in \mathrm{dom}(\mathring{\mathbb{Q}})$:*

1. *$\mathbb{P} * \mathring{\mathbb{Q}}$ is a forcing poset.*
2. *$p_0 \leq p_1 \leftrightarrow i(p_0) \leq i(p_1)$.*
3. *$i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{P} * \mathring{\mathbb{Q}}}$.*
4. *$p_0 \perp p_1 \to (p_0, \mathring{q}_0) \perp (p_1, \mathring{q}_1)$ whenever $(p_0, \mathring{q}_0), (p_1, \mathring{q}_1) \in \mathbb{P} * \mathring{\mathbb{Q}}$*
5. *$p_0 \perp p_1 \leftrightarrow (p_0, \mathring{\mathbb{1}}) \perp (p_1, \mathring{q}_1)$ whenever $(p_1, \mathring{q}_1) \in \mathbb{P} * \mathring{\mathbb{Q}}$*
6. *$p_0 \perp p_1 \leftrightarrow i(p_0) \perp i(p_1)$.*
7. *i is a complete embedding.*

Proof. (7) follows from (3),(2),(6), and (5), which implies that p_1 is a reduction of $(p_1, \mathring{q}_1)$ to $\mathbb{P}$ (see Lemma III.3.72). □

Next, as in Section V.1, when K is $\mathbb{P}*\mathring{\mathbb{Q}}$–generic over M, we need to identify $M[K]$ with $M[G][H]$. Now, K isn't exactly $G \times H$, since K is a set of pairs $(p, \mathring{q})$; but H is a filter on $\mathring{\mathbb{Q}}_G$, so its elements will be of the form $\mathring{q}_G$.

Definition V.3.5 *Assume that Definition V.3.3 holds within the model M. Let G be $\mathbb{P}$–generic over M and let H be any subset of $\mathring{\mathbb{Q}}_G$. Then $G * H = \{(p, \mathring{q}) \in \mathbb{P} * \mathring{\mathbb{Q}} : p \in G \wedge \mathring{q}_G \in H\}$.*

Theorem V.3.6 *Assume that Definition V.3.3 holds within M. Let K be $\mathbb{P} * \mathring{\mathbb{Q}}$–generic over M. Define $G = i^{-1}(K)$ and let $H = \{\mathring{q}_G : \mathring{q} \in \mathrm{dom}(\mathring{\mathbb{Q}}) \wedge \exists p\ (p, \mathring{q}) \in K\}$. Then G is $\mathbb{P}$–generic over M and H is $\mathring{\mathbb{Q}}_G$–generic over $M[G]$ and $K = G * H$ and $M[K] = M[G][H]$.*

Proof. G is generic because i is a complete embedding. Before proving H to be generic, we remark that "$\mathring{\mathbb{Q}}_G$–generic" really means "$(\mathring{\mathbb{Q}}_G, \mathring{\leq}_G, \mathring{\mathbb{1}}_G)$–generic".

$H \subseteq \mathring{\mathbb{Q}}_G$ because $K \subseteq \mathbb{P} * \mathring{\mathbb{Q}}$, so $(p, \mathring{q}) \in K \rightarrow p \Vdash \mathring{q} \in \mathring{\mathbb{Q}}$, so $\mathring{q}_G \in \mathring{\mathbb{Q}}_G$ whenever $p \in G$. To prove that H is a filter on $\mathring{\mathbb{Q}}_G$, we leave clauses (1) and (2) of Definition III.3.10 to the reader, but we prove (3) (upward closed), which is somewhat harder: Assume that $\mathring{q}_G \in H$, where $\mathring{q} \in \mathrm{dom}(\mathring{\mathbb{Q}})$, and we have $p \in G$ and $(p, \mathring{q}) \in K$. Assume that $\mathring{q}_G \leq \mathring{q}'_G$ (where the $\leq$ is really $\mathrm{val}(\mathring{\leq}_{\mathbb{Q}}, G)$). Then there is a $p' \in G$ such that $p' \Vdash \mathring{q} \mathring{\leq} \mathring{q}'$. Then $(p', \mathring{\mathbb{1}}), (p, \mathring{q}) \in K$ implies that there is a $(p'', \mathring{q}'') \in K$ below both of them, so $p'' \leq p'$ and $p'' \leq p$ and $p'' \Vdash \mathring{q}'' \mathring{\leq} \mathring{q}$. So, also $p'' \Vdash \mathring{q} \mathring{\leq} \mathring{q}'$, so that $p'' \Vdash \mathring{q}'' \mathring{\leq} \mathring{q}'$. Then $(p'', \mathring{q}'') \leq (p', \mathring{q}')$, so $(p', \mathring{q}') \in K$, so $\mathring{q}'_G \in H$.

Now, the proof that H is $\mathring{\mathbb{Q}}_G$–generic over $M[G]$ is almost exactly the same as the proof of (1) $\rightarrow$ (2) of Theorem V.1.2.

For $K \subseteq G * H$: If $(p, \mathring{q}) \in K$ then $p \in G$ and $\mathring{q}_G \in H$, so $(p, \mathring{q}) \in G * H$. For $G * H \subseteq K$: Fix $(p, \mathring{q}) \in G * H$. Then $(p, \mathring{q}) \in \mathbb{P} * \mathring{\mathbb{Q}}$ and $p \in G$ and $\mathring{q}_G \in H$, so $(p, \mathring{\mathbb{1}}) \in K$ and $(p', \mathring{q}) \in K$ for some p'. Fix $(p'', \mathring{q}'') \in K$ such that $(p'', \mathring{q}'') \leq (p, \mathring{\mathbb{1}})$ and $(p'', \mathring{q}'') \leq (p', \mathring{q})$. Then $p'' \leq p$ and $p'' \Vdash \mathring{q}'' \mathring{\leq} \mathring{q}$, so $(p'', \mathring{q}'') \leq (p, \mathring{q})$, so $(p, \mathring{q}) \in K$.

Finally, $M[K] \subseteq M[G][H]$ because $K = G * H \in M[G][H]$, and $M[G] \subseteq M[K]$ because $G \in M[K]$, and then $M[G][H] \subseteq M[K]$ because $H \in M[K]$. Thus, $M[K] = M[G][H]$. □

There is a converse to Theorem V.3.6 which we don't plan to use, so we leave it as an exercise:

Exercise V.3.7 *Assume that Definition V.3.3 holds within M. Let G be $\mathbb{P}$–generic over M and let H be $\mathring{\mathbb{Q}}_G$–generic over $M[G]$ and define $K = G * H$. Then K is $\mathbb{P} * \mathring{\mathbb{Q}}$–generic over M.*

In the special case where $\mathbb{P}, \mathbb{Q}$ are both in M, this exercise is contained in Theorem V.1.2, but was also never used in our applications of products.

When iterating ccc forcings, it will be important to know that $\mathbb{P} * \mathring{\mathbb{Q}}$ is also ccc, so we prove that below. First:

Lemma V.3.8 *Let $\kappa > \omega$ be regular and let $\mathbb{P}$ be κ-cc. Fix a name $\mathring{S}$ and assume that $\mathbb{1} \Vdash [\mathring{S} \subseteq \kappa \wedge |\mathring{S}| < \kappa]$. Then for some $\beta < \kappa$, $\mathbb{1} \Vdash \mathring{S} \subseteq \beta$.*

Proof. Let $E = \{\alpha < \kappa : \exists p [p \Vdash \alpha = \sup(\mathring{S})]\}$. For each $\alpha \in E$, choose p_α such that $p_\alpha \Vdash \alpha = \sup(\mathring{S})$. Since $\{p_\alpha : \alpha \in E\}$ is an antichain, $|E| < \kappa$. Since κ is regular, fix $\beta < \kappa$ such that $E \subseteq \beta$.

Note that $\mathbb{1} \Vdash [\kappa \text{ is regular}]$ (Theorem IV.7.9), so $\mathbb{1} \Vdash \sup(\mathring{S}) < \kappa$. If $\mathbb{1} \nVdash \mathring{S} \subseteq \beta$, then fix $q \in \mathbb{P}$ such that $q \Vdash \mathring{S} \not\subseteq \beta$, so $q \Vdash \sup(\mathring{S}) \in \kappa \backslash \beta$, so $q \Vdash \exists x \in \check{\kappa}[x \geq \beta \wedge \sup(\mathring{S}) = x]$. Then, by Lemma IV.6.1, there is an $\alpha < \kappa$ and a $p \leq q$ such that $p \Vdash [\alpha \geq \beta \wedge \sup(\mathring{S}) = \alpha]$. So $\alpha \geq \beta$ and $p \Vdash \sup(\mathring{S}) = \alpha$, contradicting the fact that $E \subseteq \beta$. □

Lemma V.3.9 *Assume that $\kappa > \omega$ is regular, $\mathbb{P}$ is κ-cc, $\mathring{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset, and $\mathbb{1} \Vdash_{\mathbb{P}} \mathring{\mathbb{Q}}$ is κ-ccc. Then $\mathbb{P} * \mathring{\mathbb{Q}}$ is κ-cc.*

Proof. As usual, although we have stated this lemma and the previous one as though we are forcing over V, we plan to use them in the ctm approach, so really $\mathbb{P}, \kappa, \mathring{\mathbb{Q}} \in M$.

In M: if $\mathbb{P} * \mathring{\mathbb{Q}}$ is not κ-cc, then let $(p_\xi, \mathring{q}_\xi)$, for $\xi < \kappa$, be pairwise incompatible in $\mathbb{P} * \mathring{\mathbb{Q}}$. Let $\mathring{S}$ be the $\mathbb{P}$-name $\{\langle \check{\xi}, p_\xi \rangle : \xi < \kappa\}$. Clearly, $\mathbb{1} \Vdash \mathring{S} \subseteq \kappa$, and there is no $\beta < \kappa$ such that $\mathbb{1} \Vdash \mathring{S} \subseteq \beta$, so we shall obtain a contradiction by showing that $\mathbb{1} \Vdash [|\mathring{S}| < \kappa]$.

Let G be $\mathbb{P}$-generic over M. In $M[G]$, we need to show that $|\mathring{S}_G| < \kappa$. Since $\mathring{\mathbb{Q}}_G$ has the κ-cc, $|\mathring{S}_G| < \kappa$ will follow if we prove that $(\mathring{q}_\xi)_G \perp (\mathring{q}_\eta)_G$ whenever $\xi, \eta \in \mathring{S}_G$ and $\xi \neq \eta$. So, fix such ξ, η, and assume that $(\mathring{q}_\xi)_G, (\mathring{q}_\eta)_G$ have a common extension $\mathring{q}_G$, where $\mathring{q} \in \mathrm{dom}(\mathring{\mathbb{Q}})$. The order on $\mathring{\mathbb{Q}}_G$ is $\mathring{\leq}_G$, so we may fix $p \in G$ such that $p \Vdash \mathring{q} \in \mathring{\mathbb{Q}}$ and $p \Vdash \mathring{q} \mathring{\leq} \mathring{q}_\xi$ and $p \Vdash \mathring{q} \mathring{\leq} \mathring{q}_\eta$. Also, $p_\xi, p_\eta \in G$ (since $\xi, \eta \in \mathring{S}_G$) and G is a filter, so we may assume that $p \leq p_\xi$ and $p \leq p_\eta$. But now we have $(p, \mathring{q}) \in \mathbb{P} * \mathring{\mathbb{Q}}$ and $(p, \mathring{q}) \leq (p_\xi, \mathring{q}_\xi)$ and $(p, \mathring{q}) \leq (p_\eta, \mathring{q}_\eta)$, contradicting $(p_\xi, \mathring{q}_\xi) \perp (p_\eta, \mathring{q}_\eta)$. □

The next exercise states the converse to Lemma V.3.9, which we do not plan to use later. This gives us the generalization of (1) $\leftrightarrow$ (2) from Exercise V.2.31. There is no (3) here, since the poset $\mathbb{Q}$ is not in the ground model.

Exercise V.3.10 *For regular κ, prove that if $\mathbb{P} * \mathring{\mathbb{Q}}$ is κ-cc, then $\mathbb{P}$ is κ-cc and $\mathbb{1} \Vdash_{\mathbb{P}} \mathring{\mathbb{Q}}$ is κ-ccc.*

Now, consider how we might iterate this $*$ product through the transfinite. So, we have $\mathbb{P}, \mathring{\mathbb{Q}}, \mathring{\mathbb{X}}, \mathring{\mathbb{Y}}, \mathring{\mathbb{Z}}, \ldots$, where $\mathring{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset, $\mathring{\mathbb{X}}$ is a $\mathbb{P} * \mathring{\mathbb{Q}}$-name for a poset, $\mathring{\mathbb{Y}}$ is a $(\mathbb{P} * \mathring{\mathbb{Q}}) * \mathring{\mathbb{X}}$-name for a poset, and so forth. Then, we can form the poset $(((\mathbb{P} * \mathring{\mathbb{Q}}) * \mathring{\mathbb{X}}) * \mathring{\mathbb{Y}}) * \mathring{\mathbb{Z}}$, whose elements are of the form $((((p, \mathring{q}), \mathring{x}), \mathring{y}), \mathring{z})$. Before discussing what to do at a limit ordinal, we shall modify the notation to make it a bit less ugly. To do this, we shall replace

the $((((p,\mathring{q}),\mathring{x}),\mathring{y}),\mathring{z})$ by the 5–tuple $(p,\mathring{q},\mathring{x},\mathring{y},\mathring{z})$. Then, at stage α of the iteration, the elements are α–tuples, so that the poset resembles the infinite products used in some of the arguments in Section V.2. This suggests:

Definition V.3.11 *For any ordinal α, an α–stage* iterated forcing construction *is a pair of sequences of the form:*

$$\big(\langle(\mathbb{P}_\xi,\leq_\xi,\mathbb{1}_\xi):\xi\leq\alpha\rangle\ ,\ \langle(\mathring{\mathbb{Q}}_\xi,\mathring{\leq}_{\mathring{\mathbb{Q}}_\xi},\mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi}):\xi<\alpha\rangle\big)$$

satisfying the following:

1. *Each $(\mathbb{P}_\xi,\leq_\xi,\mathbb{1}_\xi)$ is a forcing poset.*
2. *Each $(\mathring{\mathbb{Q}}_\xi,\mathring{\leq}_{\mathring{\mathbb{Q}}_\xi},\mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi})$ is a $(\mathbb{P}_\xi,\leq_\xi,\mathbb{1}_\xi)$–name for a forcing poset, as per Definition V.3.1.*
3. *Each $p\in\mathbb{P}_\xi$ is a sequence of the form $\langle\mathring{q}_\mu:\mu<\xi\rangle$, where each $\mathring{q}_\mu\in\mathrm{dom}(\mathring{\mathbb{Q}}_\mu)$. We use $(p)_\mu$ to denote this $\mathring{q}_\mu$.*
4. *If $\xi<\eta$ and $p\in\mathbb{P}_\eta$ then $p{\restriction}\xi\in\mathbb{P}_\xi$.*
5. *If $\xi<\eta$ and $p\in\mathbb{P}_\xi$ and p' is the η–sequence such that $p'{\restriction}\xi=p$ and $(p')_\mu=\mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}$ whenever $\xi\leq\mu<\eta$, then $p'\in\mathbb{P}_\eta$; $i^\eta_\xi(p)$ denotes this p'.*
6. *$\mathbb{1}_\xi$ is the sequence $\langle\mathring{q}_\mu:\mu<\xi\rangle$, where each $\mathring{q}_\mu=\mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}$.*
7. *If $p=\langle\mathring{q}_\mu:\mu<\xi\rangle\in\mathbb{P}_\xi$ and $p'=\langle\mathring{q}'_\mu:\mu<\xi\rangle\in\mathbb{P}_\xi$, then $p\leq_\xi p'$ iff $p{\restriction}\mu\Vdash_{\mathbb{P}_\mu}[\mathring{q}_\mu\leq\mathring{q}'_\mu]$ for all $\mu<\xi$.*
8. *If $\xi+1\leq\alpha$, then $\mathbb{P}_{\xi+1}$ is the set of all $p^\frown\mathring{q}$ such that $p\in\mathbb{P}_\xi$ and $\mathring{q}\in\mathrm{dom}(\mathring{\mathbb{Q}}_\xi)$ and $p\Vdash_{\mathbb{P}_\xi}\mathring{q}\in\mathring{\mathbb{Q}}_\xi$.*

This may be viewed as a prescription for a recursive construction of forcing posets. At stage 0, $\mathbb{P}_0=\{\emptyset\}$ and $\mathbb{1}_0=\emptyset$ by (3), since $\emptyset$ is the only sequence of length 0. At successor stages, (8) says essentially that $\mathbb{P}_{\xi+1}$ is $\mathbb{P}_\xi*\mathring{\mathbb{Q}}_\xi$, except that we are identifying the pair $(p,\mathring{q})$ from Definition V.3.3 with the concatenation $p^\frown\mathring{q}$ (as defined in Definition I.7.24). With this identification, note that we are defining the $\mathbb{1}$ and the order on $\mathbb{P}_{\xi+1}$ exactly as in Definition V.3.3. That is, by (6), $\mathbb{1}_{\xi+1}=\mathbb{1}_\xi{}^\frown\mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi}$, and by (7) (applied to $\mathbb{P}_\xi$ and $\mathbb{P}_{\xi+1}$), $p^\frown\mathring{q}\leq(p')^\frown\mathring{q}'$ iff $p\leq p'$ and $p'\Vdash[\mathring{q}_\mu\leq\mathring{q}'_\mu]$.

In Section V.4, we shall use these iterations to build models of $MA+\neg CH$. Then, the $\mathring{\mathbb{Q}}_\xi$ will enumerate names for those posets relevant to the verification of MA in $M[G]$; the exact meaning of "relevant" here involves some complexities that we shall discuss later.

When η is a limit ordinal, Definition V.3.11 does not completely specify $\mathbb{P}_\eta$ from the earlier $\mathbb{P}_\xi$. Clause (4) puts an upper limit on what can appear in $\mathbb{P}_\eta$, and clause (5) says that $\mathbb{P}_\eta$ contains at least enough to make the map $i^\eta_\xi:\mathbb{P}_\xi\to\mathbb{P}_\eta$ well-defined; in finite support iterations (Definition V.3.13), $\mathbb{P}_\eta$ contains only this much. Using (4)(5) it is easy to check (see Lemmas III.3.65 and III.3.72) that

Lemma V.3.12 *With the notation of Definition V.3.11, $i^\eta_\xi : \mathbb{P}_\xi \to \mathbb{P}_\eta$ is a complete embedding whenever $\xi \leq \eta \leq \alpha$, and for each $p \in \mathbb{P}_\eta$, $p{\restriction}\xi$ is a reduction of p to $\mathbb{P}_\xi$.*

One way to complete the prescription of the forcing at limits is to bound, by some cardinal, how many times the various $(p)_\mu$ can differ from $\mathbb{1}$:

Definition V.3.13 *With the notation of Definition V.3.11, if p is a sequence of length ξ, then its* support, $\mathrm{supt}(p)$ *is $\{\mu < \xi : (p)_\mu \neq \mathring{\mathbb{1}}_{\mathbb{Q}_\mu}\}$. If κ is an infinite cardinal, then the iteration is $<\kappa$–*supported *iff: for all limit $\eta \leq \alpha$, $\mathbb{P}_\eta$ is the set of all sequences p of length η such that $|\mathrm{supt}(p)| < \kappa$ and $p{\restriction}\xi \in \mathbb{P}_\xi$ for all $\xi < \eta$. Then* finite support *iterations are $<\aleph_0$–supported, and* countable support *iterations are $<\aleph_1$–supported.*

When $\xi \leq \eta \leq \alpha$, the map i^η_ξ is 1-1, so we may think of $\mathbb{P}_\xi$ as a complete sub-order of $\mathbb{P}_\eta$. For a fixed α, we can make this "think of" literally true by replacing each $\mathbb{P}_\xi$ by $\widehat{\mathbb{P}}_\xi := i^\alpha_\xi(\mathbb{P}_\xi)$. Now, we have an increasing chain of posets. In the special case of finite supports, we take unions at limit ordinals η — that is, as indicated on page 315, $\widehat{\mathbb{P}}_\eta = \bigcup_{\xi<\eta} \widehat{\mathbb{P}}_\xi$. We can now use delta system argument on the finite supports to prove (Lemma V.3.17) that finite support iterations preserve the ccc. First some preliminaries.

Definition V.3.14 *In a forcing poset, $\not\perp(p_1, \ldots, p_n)$ means that there is some q such that $q \leq p_\ell$ for each ℓ.*

Thus, $\not\perp(p_1, p_2)$ iff $p_1 \not\perp p_2$, and $C \subseteq \mathbb{P}$ is centered iff $\not\perp(p_1, \ldots, p_n)$ for all $n \geq 2$ and all $p_1, \ldots, p_n \in C$.

Lemma V.3.15 *In a $<\kappa$–supported iteration, using the notation of Definition V.3.11, fix $p^1, \ldots, p^n \in \mathbb{P}_\alpha$, and suppose that we have a $\xi < \alpha$ such that $\mathrm{supt}(p^i) \cap \mathrm{supt}(p^j) \subseteq \xi$ whenever $i \neq j$. Then $\not\perp(p^1, \ldots, p^n)$ iff $\not\perp(p^1{\restriction}\xi, \ldots, p^n{\restriction}\xi)$.*

Proof. For the $\to$ direction, if $\tilde{p} \leq p^\ell$ for each ℓ then $\tilde{p}{\restriction}\xi \leq p^\ell{\restriction}\xi$ for each ℓ.

For the $\leftarrow$ direction, assume that $\not\perp(p^1{\restriction}\xi, \ldots, p^n{\restriction}\xi)$, and fix $\hat{p} \in \mathbb{P}_\xi$ such that $\hat{p} \leq p^i{\restriction}\xi$ for each i. Then, define $p \in \mathbb{P}_\alpha$ such that $p{\restriction}\xi = \hat{p}$ and, for $\xi \leq \mu < \alpha$, $(p)_\mu = \mathring{\mathbb{1}}$ unless $\mu \in \mathrm{supt}(p^i)$ for some (unique) i, in which case $(p)_\mu = (p^i)_\mu$. Then $p \leq p^i$ for each i. □

Note that we appeal to the definition of "$<\kappa$–supported" to show that the p we defined really is in $\mathbb{P}_\alpha$; this lemma could fail for the general iterations described in Definition V.3.11.

Lemma V.3.16 *In a finite support iteration, using the notation of Definition V.3.11, assume that α is a limit ordinal and fix $p^\mu \in \mathbb{P}_\alpha$ for $\mu < \omega_1$. Then for some $\xi < \alpha$ and some $I \in [\omega_1]^{\aleph_1}$: All $n \geq 2$ and all $\mu_1, \ldots, \mu_n \in I$ satisfy: $\not\perp(p^{\mu_1}, \ldots, p^{\mu_n})$ iff $\not\perp(p^{\mu_1}{\restriction}\xi, \ldots, p^{\mu_n}{\restriction}\xi)$.*

Proof. Applying the Delta System Lemma, fix $I \in [\omega_1]^{\aleph_1}$ such that $\{\mathrm{supt}(p^\mu) : \mu \in I\}$ forms a delta system with some root R. Since R is finite, fix ξ with $R \subseteq \xi < \alpha$. Now apply Lemma V.3.15. □

Lemma V.3.17 *In a finite support iteration, using the notation of Definition V.3.11, assume that for each $\xi < \alpha$, $\mathbb{1}_\xi \Vdash [\mathring{\mathbb{Q}}_\xi$ is ccc$]$. Then $\mathbb{P}_\alpha$ is ccc.*

Proof. Induct on α. For the successor step, use Lemma V.3.9 and the identification described above of $\mathbb{P}_{\alpha+1}$ with $\mathbb{P}_\alpha * \mathring{\mathbb{Q}}_\alpha$. If α is a limit, use Lemma V.3.16 with $n = 2$: If $\{p^\mu : \mu < \omega_1\}$ is an antichain in $\mathbb{P}_\alpha$, then there is a $\xi < \alpha$ and an $I \in [\omega_1]^{\aleph_1}$ such that $\{p^\mu \restriction \xi : \mu < \omega_1\}$ is an antichain in $\mathbb{P}_\xi$. □

Lemma V.3.16 with $n > 2$ will be used later to prove that other properties of posets, such as pre-caliber $\aleph_1$, are also preserved by finite support iterations.

V.4 Finite Support Iteration

Here, we focus on finite support iterations (Definition V.3.13) and their use in building models for $MA + \neg CH$ and related statements. A typical result is:

Theorem V.4.1 *Assume that $\kappa > \omega$ is a regular cardinal and $2^{<\kappa} = \kappa$. Then there is a ccc poset $\mathbb{P}$ of size κ such that $\mathbb{1} \Vdash_{\mathbb{P}} MA \wedge \mathfrak{c} = \kappa$.*

As usual, we shall actually work within our ctm M and produce our $\mathbb{P}$ in M. This $\mathbb{P}$ will be the $\mathbb{P}_\kappa$ of a κ–stage iterated forcing construction.

Recall that $MA + \mathfrak{c} = \kappa$ implies that $\mathfrak{p} = \mathfrak{c}$, and hence that κ is regular and $2^{<\kappa} = \kappa$ (Lemmas III.1.26 and III.3.22). Now, if we want "κ is regular and $2^{<\kappa} = \kappa$" to hold in some ccc extension of M, then it must already hold in M, so we are assuming this in M as a hypothesis in Theorem V.4.1.

As indicated on page 333, the $\mathring{\mathbb{Q}}_\xi$, for $\xi < \kappa$, will list all the names for posets relevant to verifying $MA(\kappa)$ in the generic extension. But there is a problem with this: MA says, "for *all* ccc posets, ...", and the collection of all ccc posets forms a proper class. To reduce this collection to a set of size κ, we use the following definition and lemma saying that only some very special posets need *t*o *b*e *c*onsidered:

Definition V.4.2 *Let* $\mathrm{tbc}(\alpha, \trianglelefteq)$ *abbreviate the statement that $\alpha \in ON \backslash \{0\}$ and $\trianglelefteq$ is a subset of $\alpha \times \alpha$ and $(\alpha, \trianglelefteq, 0)$ is a ccc forcing poset (so $\mathbb{1} = 0$).*

Lemma V.4.3 *For any cardinal θ, $MA(\theta)$ holds iff $MA_{\mathbb{Q}}(\theta)$ holds for all posets $\mathbb{Q}$ of the form $(\alpha, \trianglelefteq, 0)$, where* $\mathrm{tbc}(\alpha, \trianglelefteq)$ *and $\alpha \leq \theta$.*

Proof. For the non-trivial direction: We have $MA_{\mathbb{Q}}(\theta)$ for all posets $\mathbb{Q}$ with $|\mathbb{Q}| \leq \theta$, and this implies $MA(\theta)$ by Lemma III.3.51. □

Now, to get MA holding in $M[G]$, we need to consider the various possible $(\alpha, \trianglelefteq) \in M[G]$ satisfying $\mathrm{tbc}(\alpha, \trianglelefteq)$. But we are doing this while working in M, so, in M, we need to list all relevant $\mathbb{P}_\kappa$-*names* for ccc posets. The Names to be considered are given by:

Definition V.4.4 *For a forcing poset* $\mathbb{P}$, $\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P})$ *holds iff* $\mathring{\trianglelefteq}$ *is a nice* $\mathbb{P}$*-name for a subset of* $\alpha \times \alpha$ *and* $\mathbb{1} \Vdash_{\mathbb{P}} \mathrm{tbc}(\check{\alpha}, \mathring{\trianglelefteq})$.

When we use this definition in the proof of Theorem V.4.1, the $\mathbb{P}$ will be some $\mathbb{P}_\xi$, and we shall apply:

Lemma V.4.5 *Suppose that* $\mathbb{P}_1$ *is ccc and* $\mathbb{P}_0 \subseteq_{\mathsf{c}} \mathbb{P}_1$ *and* $\mathring{\trianglelefteq}$ *is a* $\mathbb{P}_0$*-name (and hence also a* $\mathbb{P}_1$*-name). Then* $\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P}_1) \to \mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P}_0)$.

Proof. As usual, we work within our ctm M. Assume that $\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P}_1)$ and $\neg\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P}_0)$ both hold in M. Then fix $p \in \mathbb{P}_0$ such that $p \Vdash_{\mathbb{P}_0} \neg\mathrm{tbc}(\check{\alpha}, \mathring{\trianglelefteq})$. Let G be $\mathbb{P}_1$-generic over M with $p \in G$, and let $\trianglelefteq$ be $\mathring{\trianglelefteq}_G$. Applying Lemmas IV.4.2 and IV.4.4, $G \cap \mathbb{P}_0$ is $\mathbb{P}_0$-generic over M and $M[G \cap \mathbb{P}_0] \subseteq M[G]$. Also, $\trianglelefteq$ is in $M[G \cap \mathbb{P}_0]$ and $\mathrm{tbc}(\alpha, \trianglelefteq)$ is true in $M[G]$ (using $\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P}_1)$) and false in $M[G \cap \mathbb{P}_0]$ (using $p \Vdash_{\mathbb{P}_0} \neg\mathrm{tbc}(\check{\alpha}, \mathring{\trianglelefteq})$), and this is contradictory by an easy absoluteness argument. □

The following lemma says that the specific form of names required by $\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P})$ does not cause us to lose any generality in representing posets from $M[G]$. The proof is almost exactly as for Lemma V.3.2, but replacing the $\varphi(\tau, \check{\alpha}, \check{0})$ in that proof with $\mathrm{tbc}(\check{\alpha}, \tau)$ for this proof.

Lemma V.4.6 *Let* M *be a ctm for ZFC. In* M*, let* $\mathbb{P}$ *be a ccc poset and let* θ *be a cardinal. Let* G *be* $\mathbb{P}$*-generic over* M*. In* $M[G]$*, let* $(\mathbb{X}, \leq_{\mathbb{X}}, \mathbb{1}_{\mathbb{X}})$ *be a ccc forcing poset with* $|\mathbb{X}| \leq \theta$.

Then, there is a name $\mathring{\trianglelefteq}$ *in* M *and an* $\alpha \leq \theta$ *such that* $\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P})$ *holds in* M*, and such that in* $M[G]$*, the poset* $(\alpha, \mathring{\trianglelefteq}_G, 0)$ *is isomorphic to* $(\mathbb{X}, \leq_{\mathbb{X}}, \mathbb{1}_{\mathbb{X}})$.

Now, working in M, we need to list the various relevant $\mathbb{P}_\kappa$-names $\mathring{\trianglelefteq}$ in a κ-sequence. But, we are doing this while we build $\mathbb{P}_\kappa$. To avoid a circularity, we shall really only list relevant $\mathbb{P}_\beta$-names for $\beta < \kappa$ *after* we have constructed $\mathbb{P}_\beta$. Since $\mathbb{P}_\beta$ is contained in $\mathbb{P}_\kappa$, every $\mathbb{P}_\beta$-name is also a $\mathbb{P}_\kappa$-name, and we shall refer to Lemmas V.4.5 and V.4.6 at the end of the proof to show that every relevant $\mathbb{P}_\kappa$-name really got listed. The details for the listing will be explained in the proof of Theorem V.4.1; this method is often referred to as the *standard bookkeeping*, and will occur several times in this section and in later sections during iterated forcing arguments.

We remark that in our bookkeeping, we can focus on the posets and ignore the dense sets. It is true that in verifying $MA_{\mathbb{Q}}(\theta)$, we must consider all possible families of θ dense subsets of $\mathbb{Q}$. But that will not be a problem, since when we force with $\mathbb{Q}$, the generic filter meets *all* the dense sets in the ground model.

Proof of Theorem V.4.1. As usual, we work within our ctm M and produce $\mathbb{P}$, which will be the $\mathbb{P}_\kappa$ of a finite support iterated forcing construction, using the terminology of Definitions V.3.11 and V.3.13. We shall have $|\mathrm{dom}(\mathring{\mathbb{Q}}_\xi)| < \kappa$ for each $\xi < \kappa$, and $|\mathrm{dom}(\mathring{\mathbb{Q}}_\xi)| > 1$ for cofinally many ξ. By the regularity of κ, this will imply that $|\mathbb{P}_\xi| < \kappa$ for each $\xi < \kappa$ and $|\mathbb{P}_\kappa| = \kappa$. We shall also have $\mathbb{1}_\xi \Vdash [\mathring{\mathbb{Q}}_\xi \text{ is ccc}]$, so that each $\mathbb{P}_\xi$ will be ccc by Lemma V.3.17.

Still in M: To handle the standard bookkeeping, fix some $f : \kappa \xrightarrow[\text{onto}]{1-1} \kappa \times \kappa$ so that $f(\xi) = (\zeta, \mu) \to \zeta \leq \xi$. For example, the bijection arising from the usual proof that $|\kappa \times \kappa| = \kappa$ (Theorem I.11.6) has this property.

Since the iteration has finite supports, the construction is determined at the successor stages, where $\mathring{\mathbb{Q}}_\xi$ is obtained as follows:

First, for each $\zeta < \kappa$, once we have decided on $\mathbb{P}_\zeta$, let the sequence $\langle (\alpha_\zeta^\mu, \mathring{\trianglelefteq}_\zeta^\mu) : \mu < \kappa \rangle$ list all pairs $(\alpha, \mathring{\trianglelefteq})$ such that $0 < \alpha < \kappa$ and $\mathring{\trianglelefteq}$ is a nice $\mathbb{P}_\zeta$–name for a subset of $\alpha \times \alpha$. This listing is possible because $2^{<\kappa} = \kappa$. Since we use only the one poset $\mathring{\mathbb{Q}}_\zeta$ in passing from $\mathbb{P}_\zeta$ to $\mathbb{P}_{\zeta+1}$, we should view our list as a promise to consider each $(\alpha_\zeta^\mu, \mathring{\trianglelefteq}_\zeta^\mu)$ *eventually.*

Now, fix $\xi < \kappa$. Let $f(\xi) = (\zeta, \mu)$. Since $\zeta \leq \xi$, the name $i_*(\mathring{\trianglelefteq}_\zeta^\mu)$ (see Definition IV.4.3) is also a nice $\mathbb{P}_\xi$–name, where i means i_ζ^ξ. If $\mathrm{Ntbc}(\alpha_\zeta^\mu, i_*(\mathring{\trianglelefteq}_\zeta^\mu), \mathbb{P}_\xi)$ holds, let $\mathring{\mathbb{Q}}_\xi$ name this poset; formally, $(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_{\mathring{\mathbb{Q}}_\xi}, \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi})$ is $(\check{\alpha}_\zeta^\mu, i_*(\mathring{\trianglelefteq}_\zeta^\mu), \check{0})$. If $\mathrm{Ntbc}(\alpha_\zeta^\mu, i_*(\mathring{\trianglelefteq}_\zeta^\mu), \mathbb{P}_\xi)$ is false, let $\mathring{\mathbb{Q}}_\xi$ name the one-element poset $\{0\}$ (with $\mathbb{1} = 0$); then $\mathbb{P}_{\xi+1} \cong \mathbb{P}_\xi$.

This completes our construction of $\mathbb{P} = \mathbb{P}_\kappa$. Let G be $\mathbb{P}$–generic over M. Since $\mathbb{P}$ is ccc in M, cardinals and cofinalities are preserved. Using $\kappa^{\aleph_0} = \kappa = |\mathbb{P}|$ in M, we may count nice names in the usual way to verify that $M[G] \models \mathfrak{c} \leq \kappa$ (see Lemma IV.3.11).

Now fix a cardinal $\theta < \kappa$ and we shall verify that $M[G] \models MA(\theta)$. Since $MA(\theta) \to \mathfrak{c} > \theta$, if we do this for all $\theta < \kappa$, then we shall have $M[G] \models \mathfrak{c} = \kappa$.

As remarked after Definition V.3.13, we don't really have a chain of posets, but we can make it into a chain by defining $\widehat{\mathbb{P}}_\xi := i_\xi^\kappa(\mathbb{P}_\xi)$. Now, we have an increasing chain of posets, ending in $\widehat{\mathbb{P}}_\kappa = \mathbb{P}_\kappa = \mathbb{P}$.

Applying Lemma V.4.3 in $M[G]$: to verify $MA(\theta)$, it is enough to fix a poset $\mathbb{Q}$ of the form $(\alpha, \trianglelefteq, 0)$, where $\mathrm{tbc}(\alpha, \trianglelefteq)$ and $\alpha \leq \theta$, and verify $MA_{\mathbb{Q}}(\theta)$. So, let $D^\nu \subseteq \alpha$, for $\nu < \theta$, be dense subsets in $\mathbb{Q}$. Then let $\mathring{D}^\nu$ be a nice $\mathbb{P}$–name for a subset of α such that $D^\nu = \mathring{D}^\nu_G$, and let $\mathring{\trianglelefteq}$ be a nice $\mathbb{P}$–name for a subset of $\alpha \times \alpha$ such that $\mathring{\trianglelefteq}_G = \trianglelefteq$ and such that $\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P})$ (using Lemma V.4.6). Since all these names together mention fewer that κ forcing conditions, we can fix $\zeta < \kappa$ such that these are all $\widehat{\mathbb{P}}_\zeta$–names. Then fix μ such that α is α_ζ^μ and $\mathring{\trianglelefteq}$ is $(i_\zeta^\kappa)_*(\mathring{\trianglelefteq}_\zeta^\mu)$, and then fix $\xi \geq \zeta$ such that $f(\xi) = (\zeta, \mu)$. By Lemma V.4.5, $\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \mathbb{P})$ implies $\mathrm{Ntbc}(\alpha, \mathring{\trianglelefteq}, \widehat{\mathbb{P}}_\xi)$, so that $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * (\check{\alpha}, i_*(\mathring{\trianglelefteq}_\zeta^\mu), \check{0})$, and hence $M[G \cap \mathbb{P}_{\xi+1}]$ contains a filter on $\mathbb{Q}$ that meets all the dense sets in $M[G \cap \mathbb{P}_\xi]$, and in particular all the D^ν.

A foundational point: Obviously, we used AC in listing the set of nice names at each stage ζ. But AC doesn't imply *global choice* – that is, it doesn't give us a defined function that assigns to every set a well-order of that set. However, in this argument, everything that we consider lies in $H(\kappa^+)$. So, before we begin, we can fix a well-order $\sqsubset$ of $H(\kappa^+)$. Then, at stage ζ, when we have to *choose* a listing $\langle(\alpha^\mu_\zeta, \mathring{\trianglelefteq}^\mu_\zeta) : \mu < \kappa\rangle$, we simply choose the $\sqsubset$–first such sequence. □

Corollary V.4.7 *Assume that there is a ctm for ZFC. Then we can find a ctm $M[G]$ for ZFC satisfying $MA + \mathfrak{c} = \aleph_{17}$.*

A notational point: The proof of Theorem V.4.1 had some ugly notation such as $\widehat{\mathbb{P}}_\zeta$ and $(i^\kappa_\zeta)_*(\mathring{\trianglelefteq}^\mu_\zeta)$. Proofs like this in the literature (and later in this book) just say briefly which $\mathring{\mathbb{Q}}_\xi$ are relevant to the proof; they would assume that the reader has seen the standard bookkeeping, as described here, for listing them all. Such proofs just say that we are building a chain of posets, where $\zeta < \xi \to \mathbb{P}_\zeta \subseteq_{\mathsf{c}} \mathbb{P}_\xi$ and $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathring{\mathbb{Q}}_\xi$, where $\mathring{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$–name for a poset; for finite support iterations, we just take unions at limit ordinals. Although $\mathbb{P}_\xi$ is not literally a subset of $\mathbb{P}_{\xi+1}$, these proofs assume that the reader can translate the "$\subseteq$" into an inclusion with an appropriate i. Then, we don't have to talk about i_*; if $\zeta \leq \xi$ then each $\mathbb{P}_\zeta$–name *is* a $\mathbb{P}_\xi$–name

We shall discuss two kinds of variations on the proof of Theorem V.4.1: First, if we let the $\mathring{\mathbb{Q}}_\xi$ range over *fewer* than all relevant ccc posets, we shall obtain models for various weakenings of MA. Second, by controlling the $\mathring{\mathbb{Q}}_\xi$ more carefully, we shall obtain models for some axioms strengthening MA (see Section V.6).

The weakenings of MA are of interest because they show that certain consequences of MA don't imply other consequences. For example, instead of iterating with general ccc posets, we can iterate forcing only with posets that have $\aleph_1$ as a pre-caliber (Definition III.3.31). This will produce a model of $\mathfrak{p} = \mathfrak{c} = \kappa$ instead of $MA + \mathfrak{c} = \kappa$. We can then arrange for our model to satisfy $\neg SH$ and $\mathsf{add}(\mathsf{null}) = \aleph_1$, whereas full MA implies both SH (assuming $\mathfrak{c} > \aleph_1$) and $\mathsf{add}(\mathsf{null}) = \mathfrak{c}$. The axiom $\mathfrak{p} = \mathfrak{c}$ is of interest because it does imply many of the consequences of MA (see Lemmas III.1.24, III.1.25, III.1.26, and Theorem III.3.61).

A property intermediate between ccc and pre-caliber $\aleph_1$ is *property K*:

Definition V.4.8 *A poset $\mathbb{P}$ has* property K (*or the* Knaster property) *iff whenever $E \in [\mathbb{P}]^{\aleph_1}$, there is a linked $L \in [E]^{\aleph_1}$.*

By iterating with property K posets, we can also build a model of $\mathfrak{p} = \mathfrak{c} = \kappa$ plus $\neg SH$; but, $\mathsf{add}(\mathsf{null})$ must be $\mathfrak{c}$ in this model (see Corollary V.4.14).

The key to the proof of Theorem V.4.1 was the fact that the ccc is an "'iterable" property; that is, if we do a finite support iteration of ccc posets,

then the iterated forcing will remain ccc (Lemma V.3.17). This iterability was proved with the help of Lemma V.3.9 (the ccc is preserved by taking the $*$ product).

We first verify a version of Lemma V.3.9 for pre-caliber $\aleph_1$ and property K:

Lemma V.4.9 *Assume that $\mathring{\mathbb{Q}}$ is a $\mathbb{P}$–name for a poset. Let $\psi(\mathbb{P})$ say* either *that $\mathbb{P}$ has pre-caliber $\aleph_1$* or *that $\mathbb{P}$ has property K. Then $\psi(\mathbb{P})$ plus $\mathbb{1} \Vdash_{\mathbb{P}} \psi(\mathring{\mathbb{Q}})$ implies $\psi(\mathbb{P} * \mathring{\mathbb{Q}})$.*

Proof. As usual, we work within our ctm M, so really $\mathbb{P}, \mathring{\mathbb{Q}} \in M$. We shall assume that $\psi(\mathbb{P})$ says that $\mathbb{P}$ has property K; the proof for pre-caliber $\aleph_1$ is almost the same.

In M: Let $(p_\xi, \mathring{q}_\xi)$, for $\xi < \omega_1$, be arbitrary elements of $\mathbb{P} * \mathring{\mathbb{Q}}$. We must produce an $L \in [\omega_1]^{\aleph_1}$ such that $\{(p_\xi, \mathring{q}_\xi) : \xi \in L\}$ is linked. As in the proof of Lemma V.3.9, let $\mathring{S}$ be the $\mathbb{P}$–name $\{\langle \check{\xi}, p_\xi \rangle : \xi < \omega_1\}$. Then $\mathbb{1} \Vdash \mathring{S} \subseteq \omega_1$. Along with $\mathring{S}$, we have a name $\mathring{F}$ that is forced to name the function $\xi \mapsto \mathring{q}_\xi$ on $\mathring{S}$; formally, $\mathring{F} = \{\langle \mathrm{op}(\check{\xi}, \mathring{q}_\xi), p_\xi \rangle : \xi < \omega_1\}$ (see Lemma IV.2.14). Note that there is no $\beta < \omega_1$ such that $\mathbb{1} \Vdash \mathring{S} \subseteq \beta$, so $\mathbb{1} \not\Vdash |\mathring{S}| < \aleph_1$ (by Lemma V.3.8). So, fix G such that G is $\mathbb{P}$–generic over M and $M[G] \models |\mathring{S}| = \aleph_1$.

In $M[G]$: We have a set $\mathring{S}_G$ of size $\aleph_1$ together with a function $\mathring{F}_G$ that maps $\mathring{S}_G$ into the poset $\mathring{\mathbb{Q}}_G$ (since $\mathring{F}_G(\xi) = (\mathring{q}_\xi)_G$); and, $\mathring{\mathbb{Q}}_G$ has property K. There is thus, in $M[G]$, a set $B \in [\mathring{S}_G]^{\aleph_1}$ so that $\{\mathring{F}_G(\xi) : \xi \in B\}$ is linked. Say $B = \mathring{B}_G$. Fix $p^* \in G$ such that p^* forces that $\mathring{B} \subseteq \mathring{S}$ and that $\{\mathring{F}(\xi) : \xi \in \mathring{B}\}$ is linked.

In M: Define $A = \{\xi < \omega_1 : \exists p\, [p \leq p^* \wedge p \leq p_\xi \wedge p \Vdash \xi \in \mathring{B}]\}$. A is uncountable in M because, referring to the same G, $B \subseteq A$. To prove this, work in $M[G]$ (note that B need not be in M). If $\xi \in B$, then there is some $p \in G$ such that $p \Vdash \xi \in \mathring{B}$. Since $p^* \in G$ and $p_\xi \in G$ (since $\xi \in \mathring{S}_G$) and G is a filter, we can choose p so that also $p \leq p^*$ and $p \leq p_\xi$; so, $\xi \in A$.

In M: For $\xi \in A$, choose $\hat{p}_\xi \in \mathbb{P}$ such that $\hat{p}_\xi \leq p^*$ and $\hat{p}_\xi \leq p_\xi$ and $\hat{p}_\xi \Vdash \xi \in \mathring{B}$. Since A is uncountable and $\mathbb{P}$ has property K, fix $L \in [A]^{\aleph_1}$ such that $\{\hat{p}_\xi : \xi \in L\}$ is linked.

Still in M: To prove that $\{(p_\xi, \mathring{q}_\xi) : \xi \in L\}$ is linked, fix $\xi, \eta \in L$, and then fix $\tilde{p}$ such that $\tilde{p} \leq \hat{p}_\xi$ and $\tilde{p} \leq \hat{p}_\eta$. Then $\tilde{p}$ forces both $\xi \in \mathring{B}$ and $\eta \in \mathring{B}$, so that $\tilde{p}$ forces $\mathring{q}_\xi \not\perp \mathring{q}_\eta$, so $\tilde{p}$ forces the existence of a common extension of $\mathring{q}_\xi, \mathring{q}_\eta$. Applying Lemma IV.6.1, there is a $p^\dagger \leq \tilde{p}$ and a $\mathring{q}^\dagger \in \mathrm{dom}(\mathring{\mathbb{Q}})$ such that $p^\dagger \Vdash \mathring{q}^\dagger \leq \mathring{q}_\xi$ and $p^\dagger \Vdash \mathring{q}^\dagger \leq \mathring{q}_\eta$. Then, by the definition of $\leq$ on $\mathbb{P} * \mathring{\mathbb{Q}}$, $(p^\dagger, \mathring{q}^\dagger) \leq (p_\xi, \mathring{q}_\xi)$ and $(p^\dagger, \mathring{q}^\dagger) \leq (p_\eta, \mathring{q}_\eta)$. □

We next need a version of Lemma V.3.17, saying that pre-caliber $\aleph_1$ and property K are preserved by finite support iterations:

Lemma V.4.10 *Let $\psi(\mathbb{P})$ be as in Lemma V.4.9. In a finite support iteration, using the notation of Definition V.3.11, assume that for each $\xi < \alpha$, $\mathbb{1}_\xi \Vdash \psi(\mathring{\mathbb{Q}}_\xi)$. Then $\psi(\mathbb{P}_\alpha)$ holds.*

Proof. Induct on α. For the successor step, use Lemma V.4.9. For limit α, use Lemma V.3.16. □

We can now easily modify the proof of Theorem V.4.1 to produce models satisfying appropriate weakenings of *MA*.

Definition V.4.11 *MAK denotes the assertion that $MA_{\mathbb{P}}(\kappa)$ holds for all $\kappa < \mathfrak{c}$ and all posets $\mathbb{P}$ with property K. MAP denotes the assertion that $MA_{\mathbb{P}}(\kappa)$ holds for all $\kappa < \mathfrak{c}$ and all posets $\mathbb{P}$ with $\aleph_1$ as a pre-caliber.*

Neither "*MAK*" nor "*MAP*" is standard terminology. Clearly $MA \to MAK$ and $MAK \to MAP$. Also, $MAP \to \mathfrak{p} = \mathfrak{c}$, since by (the easy direction of) Bell's Theorem III.3.61, $\mathfrak{p} = \mathfrak{c}$ follows from *MA* restricted to σ–centered posets.

For both properties, *MAK* and *MAP*, note that, as for *MA*, it is enough to verify them for posets of size less than $\mathfrak{c}$. So, applying the method of the proof of Theorem V.4.1, and making the obvious changes in tbc and Ntbc, we get:

Theorem V.4.12 *Assume that $\kappa > \omega$ is a regular cardinal and $2^{<\kappa} = \kappa$. Then there are a posets $\mathbb{P}_K$ and $\mathbb{P}_P$ of size κ such that: $\mathbb{1} \Vdash_{\mathbb{P}_K}$ MAK $\wedge\ \mathfrak{c} = \kappa$ and $\mathbb{1} \Vdash_{\mathbb{P}_P}$ MAP $\wedge\ \mathfrak{c} = \kappa$, and $\mathbb{P}_K$ has property K and $\mathbb{P}_P$ has pre-caliber $\aleph_1$.*

Note that we need to verify here that if $\xi < \kappa$ and $\mathring{\trianglelefteq}$ is a $\mathbb{P}_\xi$–name, then $(\mathbb{1} \Vdash_{\mathbb{P}_\kappa} \psi(\alpha, \mathring{\trianglelefteq})) \to (\mathbb{1} \Vdash_{\mathbb{P}_\xi} \psi(\alpha, \mathring{\trianglelefteq}))$, where ψ is as in Lemma V.4.9. We leave the details of this as an exercise for the reader; the proof uses the fact that the extension from $M[G \cap \mathbb{P}_\xi]$ to $M[G \cap \mathbb{P}_\kappa]$ is by a forcing that has property ψ.

To see that *MAK* does not imply *MA*, use:

Lemma V.4.13 *If T is a Suslin tree and $\mathbb{P}$ has property K, then $\mathbb{1} \Vdash_{\mathbb{P}} T$ is Suslin.*

Proof. Working in the ctm M where T is Suslin: If $\mathbb{1}$ does not force T to remain Suslin, then some p forces T to have either an uncountable antichain or an uncountable chain. Say it is an antichain. Then we get a name $\mathring{f}$ and a $p \in \mathbb{P}$ that forces that $\mathring{f} : \omega_1 \xrightarrow{1\text{–}1} T$ and that $\mathring{f}$ enumerates an antichain. For $\xi < \omega_1$, choose $q_\xi \leq p$ and $x_\xi \in T$ such that $q_\xi \Vdash \mathring{f}(\xi) = x_\xi$. Using property K, fix $L \in [\omega_1]^{\aleph_1}$ such that $\{q_\xi : \xi \in L\}$ is linked. Then for each $\xi, \eta \in L$, there is some $q \leq p_\xi, p_\eta$, and $q \Vdash x_\xi \perp x_\eta$. Since the tree order is in the ground model, $x_\xi \perp x_\eta$ is true. Then $\{x_\xi : \xi \in L\}$ is an antichain (in M), contradicting our assumption that T is Suslin. □

Corollary V.4.14 *Assume that there is a ctm for ZFC. Then we can find a ctm $M[G]$ for ZFC satisfying $\mathfrak{c} = \aleph_{17}$ plus $\neg$SH plus MAK. In particular, $M[G] \models \mathfrak{p} = \mathfrak{c}$ and $M[G] \models \mathrm{add}(\mathsf{null}) = \mathfrak{c}$.*

Proof. In Theorem V.4.12, start with $M \models V = L$. In particular, $M \models GCH$ and there is a Suslin tree in M, which then remains Suslin in $M[G]$. To see that $MAK \to \mathrm{add}(\mathsf{null}) = \mathfrak{c}$, use the next lemma: □

Lemma V.4.15 *The amoeba order used in the proof of MA* $\rightarrow$ $\mathrm{add}(\mathsf{null}) = \mathfrak{c}$ *(Lemma III.3.28) has property K.*

Proof. It actually has the stronger property that $\mathbb{P}$ is σ–linked; that is, $\mathbb{P}$ is a countable union of linked families. To see this, refer to the proof of Lemma III.3.28; each linked family is of the form $\{p \in \mathbb{P} : p \supseteq W \wedge \mu(p \backslash W) \leq \delta \wedge \mu(p) \leq \varepsilon - 3\delta\}$, where $W \in \mathcal{B}$ and δ is a positive rational. □

If we just iterate with pre-caliber $\aleph_1$ posets, then we can get $\mathrm{add}(\mathsf{null}) = \aleph_1$:

Corollary V.4.16 *Assume that there is a ctm for ZFC. Then we can find a ctm* $M[G]$ *for ZFC satisfying* $\mathfrak{c} = \aleph_{17}$ *plus* $\neg SH$ *plus* $\mathrm{cov}(\mathsf{null}) = \aleph_1$ *(and hence* $\mathrm{add}(\mathsf{null}) = \aleph_1$*) plus MAP. In particular,* $M[G] \models \mathfrak{p} = \mathfrak{c}$.

Proof. Use the $\mathbb{P}_P$ of Theorem V.4.12. We just have to show that if $M \models CH$ and $\mathbb{P}$ has pre-caliber $\aleph_1$ in M, then $\mathrm{cov}(\mathsf{null}) = \aleph_1$ holds in $M[G]$ — roughly, because every real in $M[G]$ lies in some null G_δ set that is coded in M.

To define the coding, use the notation from Definition V.2.20 and Lemma V.2.22. So, if $c \in \mathcal{P}(\mathbb{Q} \times \mathbb{Q})$, then U_c is an open subset of $\mathbb{R}$. If $c \in M$, then $(U_c)^M$ is not the same set as $(U_c)^{M[G]}$, but an easy absoluteness argument shows that $(U_c)^M = (U_c)^{M[G]} \cap M$ and $\mu(U_c)^M$ is the same real as $\mu(U_c)^{M[G]}$, where μ denotes Lebesgue measure; note that the terminology $\mu(U_c)^M$ implies that we work completely within M to compute the open set U_c and then compute its measure. Of course, we are assuming that $\mathbb{R}$ has been defined in some reasonable way (such as by Definition I.14.2), so that $\mathbb{R}^M = \mathbb{R} \cap M$.

Also, if $\vec{c} \subseteq \omega \times (\mathbb{Q} \times \mathbb{Q})$, let $B_{\vec{c}}$ denote the G_δ set $\bigcap_{n \in \omega} U_{c_n}$. Again observe that $\mu(B_{\vec{c}})^M = \mu(B_{\vec{c}})^{M[G]}$.

Now, working in M, list $\mathbb{R}$ as $\{r_\xi : \xi < \omega_1\}$, and, for $\alpha < \omega_1$, let $E_\alpha = \{r_\xi : \xi < \alpha\}$. Then E_α is countable, and hence a null set, so choose a $\vec{c}^{\,\alpha} \subseteq \omega \times (\mathbb{Q} \times \mathbb{Q})$ such that $E_\alpha \subseteq B_{\vec{c}^{\,\alpha}}$ and $\mu(B_{\vec{c}^{\,\alpha}}) = 0$.

Working in $M[G]$: We have the $\aleph_1$ null G_δ sets $B_{\vec{c}^{\,\alpha}}$ (now, evaluated in $M[G]$), and we establish $\mathrm{cov}(\mathsf{null}) = \aleph_1$ by showing that these contain all reals. If this fails, then fix $x \in \mathbb{R}$ such that $x \notin B_{\vec{c}^{\,\alpha}}$ for all α, and then fix $a \in \omega$ such that $-a \leq x \leq a$. The map $\alpha \mapsto \vec{c}^{\,\alpha}$ is really a function, which we shall denote by f; so $f : \omega_1 \rightarrow \mathcal{P}(\omega \times (\mathbb{Q} \times \mathbb{Q}))$. Now, fix a name $\mathring{x} \in M^{\mathbb{P}}$ such that $\mathring{x}_G = x$. Then, fix $p \in \mathbb{P}$ that forces all this: that is, "$\mathring{x}$ is a real number and $-a \leq \mathring{x} \leq a$ and $\forall \alpha < \omega_1\, [\mathring{x} \notin B_{f(\alpha)}]$".

Back in M: For each $\alpha < \omega_1$, $p \Vdash \mathring{x} \notin B_{\vec{c}^{\,\alpha}}$, so $p \Vdash \exists n \in \omega\, [\mathring{x} \notin U_{\vec{c}^{\,\alpha}_n}]$, so we may fix an $n(\alpha) \in \omega$ and a $p_\alpha \leq p$ such that $p_\alpha \Vdash \mathring{x} \notin U_{\vec{c}^{\,\alpha}_{n(\alpha)}}$. Then fix $J \in [\omega_1]^{\aleph_1}$ such that $\{p_\alpha : \alpha \in J\}$ is centered. Note that $\mathbb{R} = \bigcup\{E_\alpha : \alpha \in J\}$ and each $E_\alpha \subseteq U_{\vec{c}^{\,\alpha}_{n(\alpha)}}$, so $[-a, a] \subseteq \bigcup\{U_{\vec{c}^{\,\alpha}_{n(\alpha)}} : \alpha \in J\}$. Since $[-a, a]$ is compact, we can fix a finite $S \subset J$ such that $[-a, a] \subseteq \bigcup\{U_{\vec{c}^{\,\alpha}_{n(\alpha)}} : \alpha \in S\}$. This last statement is absolute, so it holds in every generic extension, so $\mathbb{1} \Vdash [-a, a] \subseteq \bigcup\{U_{\vec{c}^{\,\alpha}_{n(\alpha)}} : \alpha \in S\}$. Fix q such that $q \leq p_\alpha$ for all $\alpha \in S$. Then $q \Vdash \mathring{x} \notin U_{\vec{c}^{\,\alpha}_{n(\alpha)}}$ for all $\alpha \in S$, so that $q \Vdash \mathring{x} \notin [-a, a]$, contradicting $p \Vdash -a \leq \mathring{x} \leq a$. □

Note that non(null) = $\mathfrak{c}$ does follow from $\mathfrak{p} = \mathfrak{c}$, using Lemma III.1.25 and Exercise III.1.8, since $\mathfrak{p} \leq$ add(meager) $\leq$ cov(meager) $\leq$ non(null).

The proof of Corollary V.4.16 may be summarized by saying that in this $M[G]$, there are no reals that are *random* over M. This is related to the random real forcing in Definition IV.7.33. In the literature (e.g., [55]), this is usually discussed using the Cantor set 2^ω rather than $\mathbb{R}$, and one introduces a coding for all Borel sets, but to maintain the terminology of the proof of Corollary V.4.16:

Definition V.4.17 *Let M be a ctm for ZFC and fix $x \in \mathbb{R}$. Then x is* random over M *iff $x \notin B_{\vec{c}}$ (evaluated in V) for all $\vec{c} \subseteq \omega \times (\mathbb{Q} \times \mathbb{Q})$ such that $\vec{c} \in M$ and $\mu(B_{\vec{c}}) = 0$.*

The notion of "random" involves models, but not forcing. Since there are only countably many $B_{\vec{c}}$ to consider, almost every real is random over M; none of these are in M (otherwise, $\{x\}$ would be a null G_δ coded in M). One can apply elementary probability theory to see that x has the properties you would expect if it were "chosen at random". For example,

Exercise V.4.18 *Let $x \in [0,1]$ be random over M, and let $0.d_0d_1d_2\cdots$ be the decimal expansion of x, so $x = \sum_n d_n \cdot 10^{-n-1}$ where $0 \leq d_n \leq 9$. Then for each $a < 10$, $\{n \in \omega : d_n = a\}$ has asymptotic density $1/10$.*

Note that in the proof of this exercise, you only need to consider $B_{\vec{c}}$ for some decidable sets $\vec{c} \subseteq \omega \times (\mathbb{Q} \times \mathbb{Q})$ — that is, x need only be a *computably random real.* Such reals are related to recursion theory; see, e.g., [24].

The next two exercises relate random reals to forcing:

Exercise V.4.19 *Let M be a ctm for ZFC. Working in M and using Definition IV.7.32, let $\mathbb{P} = \mathbb{MB}([0,1], \mu)$, where μ is Lebesgue measure.*

Let G be $\mathbb{P}$-generic over M. In $M[G]$, there is a unique $x = x(G) \in [0,1]$ such that $x \in [a,b]$ whenever $a, b \in \mathbb{Q}$ and $0 \leq a < b \leq 1$ and $[\,[a,b]\,] \in G$. Furthermore this x is random over M.

Here, $[\,[a,b]\,]$ denotes the equivalence class of the measurable set $[a,b]$. Note that even in the simple case of rational intervals, one should really, to make the terminology 100% rigorous, spell out whether the interval is evaluated in M (as when we ask whether $[\,[a,b]\,] \in G$) or in $M[G]$ (as when we ask whether $x \in [a,b]$ — i.e., $a \leq x \leq b$).

Exercise V.4.20 *Use the notation of Exercise V.4.19, and now fix an arbitrary real $y \in [0,1]$ that is random over M. Then there is a unique filter $G \subseteq \mathbb{P}$ such that G is $\mathbb{P}$-generic over M and $y = x(G)$.*

Although $MA + \neg CH$ implies SH, it does not refute Suslin or Aronszajn trees on cardinals larger than ω_1, as can be seen by:

Exercise V.4.21 *Let $\mathbb{P}$ be a ccc poset, let κ be regular, and let T be a κ–tree. Then show:*

1. *If T is an ω_1–Aronszajn tree and no subtree of T is a Suslin tree then $\mathbb{1} \Vdash T$ is Aronszajn.*
2. *If T is a κ–Aronszajn tree and $\kappa > \omega_1$ then $\mathbb{1} \Vdash T$ is Aronszajn.*
3. *If T is a κ–Suslin tree and $\kappa > \omega_1$ then $\mathbb{1} \Vdash T$ is Suslin.*

In particular, as claimed on page 205, it is consistent with $MA + \neg CH$ to have ω_2–Aronszajn trees and even ω_2–Suslin trees; $\mathfrak{c}$ can be $\aleph_2$ or larger.

Hint. For (1), if p forces $\mathring{C}$ to be a path through T, let S be the subtree $\{x \in T : \exists q \leq p\, [q \Vdash x \in \mathring{C}]\}$. For (2), apply Exercise III.6.32.

For (3), first observe that if T is a κ–Suslin tree where $\kappa > \omega_1$, then every subset of T of size κ contains an uncountable chain. □

Note that (2) and (3) are false when $\kappa = \omega_1$ and T is Suslin (setting $\mathbb{P} = T$). The Proper Forcing Axiom (*PFA*), which implies $MA \wedge \mathfrak{c} = \aleph_2$, also implies that there are no $\aleph_2$–Aronszajn trees; see Exercise V.7.7.

Obviously, the $\aleph_{17}$ in Corollaries V.4.7, V.4.14, and V.4.16 can be replaced by other regular cardinals, including inaccessible cardinals. Starting with κ strongly inaccessible in Theorem V.4.1, we get:

Corollary V.4.22 *Statements (1 – 4) of Lemma IV.3.17 are equivalent to*

5. *There is a ctm $M \models ZFC + MA + \mathfrak{c}$ is weakly inaccessible.*

The continuum can even be weakly Mahlo (defined in Exercise III.6.30). This can be proved with the aid of Exercise IV.7.56:

Exercise V.4.23 *If κ is strongly Mahlo in M, then there is a ccc extension $M[G]$ in which $\kappa = \mathfrak{c}$ and κ is weakly Mahlo and MA holds.*

By Exercise IV.7.58, one cannot make $\diamondsuit$ false using a ccc poset of size $\aleph_1$. One can use ccc forcing to get a model of *GCH* plus $\neg\diamondsuit$ and even *SH*; see Devlin and Johnsbråten [21]. They show that for an appropriate $M \models GCH$, there is a ccc extension satisfying $GCH + SH$. Their poset has size $\aleph_2$ in M and, unlike the poset that forces *MA*, doesn't add any reals. A Suslin tree is a simple example of a ccc poset of size $\aleph_1$ that doesn't add reals (see Lemmas IV.6.6 and IV.6.9 and Exercise IV.6.11), and their poset is an iteration of Suslin forcings. It is not a finite support iteration, however, since non-trivial finite support iterations always add reals:

Exercise V.4.24 *In M, construct a finite support iteration, $\langle \mathbb{P}_\xi : \xi \leq \omega \rangle$, as in Definitions V.3.11 and V.3.13, and assume that each $\mathring{\mathbb{Q}}_n$ is forced by $\mathbb{1}$ to have an antichain of size λ (where $\lambda \geq 2$ is a cardinal in M). If G is $\mathbb{P}_\omega$–generic over M, then prove that $M[G]$ contains a function $f : \omega \xrightarrow{\text{onto}} \lambda$ that is not in M.*

Hint. For each n, we have a maximal antichain A_n in $\mathbb{Q}_n$ of size at least λ, and A_n can be partitioned into disjoint non-empty subsets A_n^α for $\alpha < \lambda$. The $\mathbb{Q}_n$-generic filter we add meets exactly one of these A_n^α; let $f(n)$ be that α. This argument has to be made rigorous by an appropriate use of names. □

Note that an antichain of size 2 exists trivially in any atomless poset — and then we have an $f \in 2^\omega$ such that $f \in M[G]\backslash M$. If all the $\mathring{\mathbb{Q}}_n$ are forced to be non-ccc, then we can take $\lambda = \omega_1$, showing that this iteration collapses ω_1. A special case of this is iterating countably closed posets, which, if they are atomless, are not ccc. If all the $\mathring{\mathbb{Q}}_n$ are forced to be countably closed, then by induction, each $M[G \cap \mathbb{P}_n]$ has no new ω-sequences of ordinals, so that ω_1 doesn't get collapsed until we get to the $\mathbb{P}_\omega$-extension.

Section V.5 describes *countable* support iterations. Using these iterations with all the $\mathring{\mathbb{Q}}_n$ countably closed we can sometime make $\mathbb{P}_\omega$ countably closed as well, so then it doesn't add ω-sequences.

Exercise V.4.25 *Show that the f obtained in Exercise V.4.24 is actually $\mathrm{Fn}(\omega, \lambda)$-generic over M. That is, if we define the filter $G \subseteq \mathrm{Fn}(\omega, \lambda)$ to be $\{p : p \subseteq f\}$, then G is generic over M in the usual sense.*

Even the weakened versions of MA in obtained in Corollaries V.4.14 and V.4.16 imply that $\mathfrak{p} = \mathfrak{c}$ and hence that every non-principal ultrafilter on ω has character $\mathfrak{c}$ (see Definition III.1.32 and Lemma III.1.33). It is consistent with $\mathfrak{c}$ being arbitrarily large to have $\chi(\mathcal{U}) = \aleph_1$ for some non-principal $\mathcal{U}$ on ω. Before proving this, we remark that the Cohen model will not work here. Specifically, if we add κ Cohen reals, then Lemma V.2.22 shows that $\mathrm{cov}(\mathsf{meager}) \geq \kappa$ in the generic extension. It is also true that $\chi(\mathcal{U}) \geq \kappa$ must hold in the extension. We can prove this either by modifying the proof of Lemma V.2.22 or by quoting the following purely mathematical fact:

Lemma V.4.26 *Let $\mathcal{U}$ be any non-principal ultrafilter on ω. Then $\chi(\mathcal{U}) \geq \mathrm{cov}(\mathsf{meager})$ and $\chi(\mathcal{U}) \geq \mathrm{cov}(\mathsf{null})$.*

Proof. Let $\kappa = \chi(\mathcal{U})$, and then fix $\mathcal{E} \subseteq \mathcal{U}$ such that $|\mathcal{E}| = \kappa$ and $\mathcal{E}$ generates $\mathcal{U}$ in the sense of Definition III.1.31. For each $E \subseteq \omega$, let $E{\uparrow} = \{X \in \mathbb{P}(\omega) : E \subseteq X\}$ and let $E{\downarrow} = \{X \in \mathbb{P}(\omega) : E \supseteq X\}$. We may assume that $\mathcal{E}$ is closed under finite intersections, so that $\mathcal{U} = \bigcup\{E{\uparrow} : E \in \mathcal{E}\}$, and hence, since $\mathcal{U}$ is an ultrafilter, $\mathcal{P}(\omega) = \bigcup\{E{\uparrow} \cup (\omega\backslash E){\downarrow} : E \in \mathcal{E}\}$.

Give the space 2^ω its usual product topology (homeomorphic to the Cantor set) and its usual product measure (as on page 300). Then, identify $\mathcal{P}(\omega)$ with 2^ω by identifying sets with their characteristic functions. Note that the sets $E{\uparrow} \cup (\omega\backslash E){\downarrow}$ are closed in 2^ω, and are both null and meager whenever E is infinite. Since $\mathcal{U}$ is non-principal, we have $\mathcal{P}(\omega)$ expressed as a union of κ compact sets that are both null and meager.

Now, we defined the ideals meager and null on $\mathbb{R}$, not 2^ω, so to transfer the covering to $\mathbb{R}$, let $B : 2^\omega \xrightarrow{\text{onto}} [0,1]$ be the binary expansion map; $B(f) =$

$\sum_n f(n)\cdot 2^{-n-1}$. Although B is not 1-1 (since dyadic rationals have two binary expansions), B maps meager sets to meager sets and null sets to (Lebsegue) null sets. So, to cover $\mathbb{R}$ by κ compact sets that are both null and meager, take all integer translates of the sets $B(E\uparrow \cup (\omega\backslash E)\downarrow)$. □

Now, to obtain $\chi(\mathcal{U}) = \aleph_1$ in a model where $\mathfrak{c} = \kappa > \aleph_1$ we start with $\mathfrak{c} = \kappa$ in M and build $\mathcal{U}$ in $M[G]$ by iterating forcing ω_1 times, not κ times:

Lemma V.4.27 *In M, let $\kappa = \mathfrak{c}$. Then in M, there is a ccc poset $\mathbb{P}$ of size κ such that whenever G is $\mathbb{P}$-generic over M, it is true in $M[G]$ that $\mathfrak{c} = \kappa$ and that there is a non-principal ultrafilter $\mathcal{U}$ on ω of character $\aleph_1$.*

Proof. We present this proof as it might appear in the literature, leaving it up to the reader to sort out the ctms and to spell out appeals to the Maximal Principle, etc.; see the remarks following Corollary V.4.7. The bookkeeping is pretty simple here, since we are not trying to consider all possible posets.

Iterate forcing ω_1 times with finite supports. So, $\mathbb{P} = \mathbb{P}_{\omega_1}$ and $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathring{\mathbb{Q}}_\xi$, where $\mathring{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$-name for a ccc poset of size κ. Let $\mathbb{P}_0 = \{\mathbb{1}\}$. Along with $\mathbb{P}_\xi$, construct a $\mathbb{P}_\xi$-name $\mathring{\mathcal{U}}_\xi$ for a non-principal ultrafilter in the $\mathbb{P}_\xi$ extension; make sure it is forced by $\mathbb{1}$ that $\xi < \eta \to \mathring{\mathcal{U}}_\xi \subseteq \mathring{\mathcal{U}}_\eta$. Then $\mathring{\mathcal{U}}_0$ is $\check{\mathcal{U}}$ for some ground-model ultrafilter $\mathcal{U}$. Given $\mathring{\mathcal{U}}_\xi$, let $\mathring{\mathbb{Q}}_\xi$ be a $\mathbb{P}_\xi$-name for the poset used in the proof of $\mathfrak{m} \leq \mathfrak{p}$ (Lemma III.3.22), which forces an infinite set $\mathring{K}_\xi$ that is $\subseteq^*$ all elements of $\mathring{\mathcal{U}}_\xi$; so $\mathring{K}_\xi$ is a $\mathbb{P}_{\xi+1}$-name. Let $\mathring{\mathcal{U}}_{\xi+1}$ name some ultrafilter extending $\mathring{\mathcal{U}}_\xi \cup \{\mathring{K}_\xi\}$.

For limit $\eta < \omega_1$, $\mathbb{P}_\eta = \bigcup_{\xi<\eta} \mathbb{P}_\xi$, and we let $\mathring{\mathcal{U}}_\eta$ be some $\mathbb{P}_\eta$-name for an ultrafilter extending $\bigcup_{\xi<\eta} \mathring{\mathcal{U}}_\xi$. For $\eta = \omega_1$, every real in the P_{ω_1}-extension already occurs in the P_ξ-extension for some $\xi < \omega_1$, so $\bigcup_{\xi<\omega_1} \mathring{\mathcal{U}}_\xi$ already names an ultrafilter, which is forced by $\mathbb{1}$ to have character $\aleph_1$, since it is generated by the $\mathring{K}_\xi$ plus the cofinite sets.

In the ground model, we have $|\mathbb{P}| = \mathfrak{c} = \kappa$ and $\mathbb{P}$ is ccc, which implies that $\mathfrak{c} = \kappa$ in $M[G]$ also. □

A modification of Exercise V.4.25 shows that one can't do this by a finite support iteration of length larger than ω_1:

Exercise V.4.28 *Use the terminology of Exercises V.4.24 and V.4.25, but now build a chain of ccc posets, $\langle \mathbb{P}_\xi : \xi \leq \kappa\rangle$, where κ is regular, and assume only that each $\mathring{\mathbb{Q}}_\xi$ is forced by $\mathbb{1}$ to be atomless. If G is $\mathbb{P}_\kappa$-generic over M, then in $M[G]$, $\mathrm{cov}(\mathsf{meager}) \geq \kappa$ and hence every non-principal ultrafilter on ω has character at least κ.*

Hint. This is a little like Lemma V.2.22, but now using Exercise V.4.25 (with $\lambda = 2$) instead of Lemma V.2.21. That is, cofinally in the iteration you are adding reals that are Cohen generic over the earlier models, and any (code for a) family of $< \kappa$ dense open sets occurs in some $M[G \cap \mathbb{P}_\xi]$, where $\xi < \kappa$. □

This raises the issue: How do we get a model in which $\mathrm{cov}(\mathsf{null}) = \kappa > \aleph_1$ and $\mathrm{cov}(\mathsf{meager}) = \aleph_1$? The other way around was easy: To get $\mathrm{cov}(\mathsf{meager}) = \kappa > \aleph_1$ and $\mathrm{cov}(\mathsf{null}) = \aleph_1$, we iterate κ times to make $\mathrm{cov}(\mathsf{meager}) = \kappa$, using posets with pre-caliber $\aleph_1$, since these don't make $\mathrm{cov}(\mathsf{null})$ big (see Corollary V.4.16). We could even just iterate adding Cohen reals (see Exercise V.2.23). But if we iterate κ times with any ccc poset that makes $\mathrm{cov}(\mathsf{null})$ large (or with any ccc poset at all), then we get $\mathrm{cov}(\mathsf{meager}) \geq \kappa$ as well.

To get $\mathrm{cov}(\mathsf{null}) = \kappa > \aleph_1$ and $\mathrm{cov}(\mathsf{meager}) = \aleph_1$, we can use random real forcing. Specifically, let $\mathbb{P}_\alpha = (\mathbb{MB}(2^\alpha))^M = (\mathbb{MB}(2^\alpha, \mu))^M$, where μ is the standard product measure; see Definitions IV.7.32 and IV.7.33. Assuming in M that *GCH* holds and $\kappa > \omega$ is regular, the $\mathbb{P}_\kappa$ extension $M[G]$ will satisfy $\mathfrak{c} = \kappa$ and $\mathfrak{d} = \aleph_1$ (see Lemma IV.7.34). This $M[G]$ will also satisfy $\mathrm{cov}(\mathsf{meager}) = \aleph_1$. One can prove this directly, or quote

Lemma V.4.29 $\mathrm{cov}(\mathsf{meager}) \leq \mathfrak{d}$.

Proof. List the rationals as $\{a_n : n \in \omega\}$. For each $f \in \omega^\omega$, let E_f be the set of reals x such that $|x - a_n| \geq 2^{-f(n)}$ for all but finitely many n. Note that E_f is an F_σ set and is of first category. Also, $f \leq^* g \to E_f \subseteq E_g$. If $\mathcal{D}$ is a dominating family, then $\mathbb{R} = \bigcup\{E_f : f \in \omega^\omega\} = \bigcup\{E_g : g \in \mathcal{D}\}$. □

Now, to prove that $\mathrm{cov}(\mathsf{null})$ is large in the random real extension, we can view this extension as a kind of iterated forcing, but it is not a finite support iteration (see also Exercise V.4.34), so having $\mathrm{cov}(\mathsf{meager}) = \aleph_1$ does not contradict Exercise V.4.28. Roughly, $\mathrm{cov}(\mathsf{null})$ is large in the random real model by Exercises V.4.19 and V.4.20, saying that random reals are not in any null sets coded in the ground model. But, we must also consider null sets coded in $M[G]$. To do this, we use the idea behind the proof that $\mathfrak{d} = \mathrm{cov}(\mathsf{meager}) = \kappa$ in the Cohen extension, using $\mathrm{Fn}(\kappa, 2)$ (Lemmas V.2.19 and V.2.22). The main tool here was the observation that for $W \subseteq I$, $\mathrm{Fn}(I, J) \cong \mathrm{Fn}(W, J) \times \mathrm{Fn}(I \backslash W, J)$, so that we may view the extension by $\mathrm{Fn}(I, J)$ as an iteration, extending by $\mathrm{Fn}(W, J)$ and then by $\mathrm{Fn}(I \backslash W, J)$ (Lemma V.2.18).

Switching to measure algebras, we can identify 2^I with $2^W \times 2^{I \backslash W}$, but, this does not give us a product of the forcing posets; that is $\mathbb{MB}(2^I)$ is not the same poset as $\mathbb{MB}(2^W) \times \mathbb{MB}(2^{I \backslash W})$; see Exercise V.4.33. There is a product iteration lemma that is specific to forcing with measure algebras:

Exercise V.4.30 *Show that for $W \subseteq I$: $\mathbb{MB}(2^I) \cong \mathbb{MB}(2^W) * \mathring{\mathbb{Q}}$, where $\mathring{\mathbb{Q}}$ is a $\mathbb{MB}(2^W)$-name for the $\mathbb{MB}(2^{I \backslash W})$ in the $\mathbb{MB}(2^W)$-extension.*

Hint. Here, I, W are in the ground model M, and the asserted "$\cong$" (Definition IV.4.25) holds in M. Working in M, elements of $\mathbb{MB}(2^W) * \mathring{\mathbb{Q}}$ are, roughly, pairs $(S, \mathring{T})$, where S is a measurable subset of 2^W and $\mathring{T}$ is a name for a measurable subset of $2^{I \backslash W}$. Show how to associate this with a measurable subset of 2^I. Note that WLOG, $\mathring{T}$ names a G_δ set. □

Using this, together with the fact that measure algebra forcing adds random reals in the sense of Exercises V.4.19 and V.4.20, we get:

Exercise V.4.31 *In M, assume that $\kappa > \omega$ is regular and $2^{<\kappa} = \kappa$ and $\mathbb{P} = \mathbb{MB}(2^\kappa)$. Let G be $\mathbb{P}$-generic over M. Then* $\mathrm{non}(\mathsf{meager}) = \mathrm{cov}(\mathsf{null}) = \kappa = \mathfrak{c}$ *and* $\mathrm{non}(\mathsf{null}) = \mathrm{cov}(\mathsf{meager}) = \aleph_1$ *in $M[G]$.*

Section III.1 defined a number of cardinals that lie somewhere between $\aleph_1$ and $\mathfrak{c}$. Many more such cardinals are described in the literature, together with many consistency results about the values of these cardinals in models of $\neg CH$. We shall not attempt even to survey all such results here, but we shall point out that we have answered the specific questions mentioned in the Introduction on page 2. Converting to our current notation:

Proposition V.4.32 *Consider the two questions:*

Q1. Does $\mathrm{cov}(\mathsf{meager}) = \aleph_1$*?* *Q2. Does* $\mathrm{cov}(\mathsf{null}) = \aleph_1$*?*

Assuming $\mathrm{Con}(ZFC)$*, each of the $2 \cdot 2 = 4$ possible answers to Q1,Q2 is consistent with $ZFC + \mathfrak{c} = \aleph_{17}$.*

Proof. Start with $M \models GCH$. All four answers are obtained in some $M[G]$, where we force with some ccc poset of size $\kappa := \aleph_{17}$, and $\mathfrak{c} = \kappa$ in $M[G]$.

For "no", "no", let $M \models MA$, so $\mathrm{cov}(\mathsf{meager}) = \mathrm{cov}(\mathsf{null}) = \mathfrak{c}$.

For "no", "yes", use the Cohen model, forcing with $\mathrm{Fn}(\kappa, 2)$; by Exercise V.2.23, this satisfies $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c}$ and $\mathrm{cov}(\mathsf{null}) = \aleph_1$.

For "yes", "no", use the random real model; see Exercise V.4.31.

For "yes", "yes", first get $\mathfrak{c} = \kappa$ by using Cohen reals or random reals or the poset that forces MA. Then force over this model to get an ultrafilter on ω of character $\aleph_1$ (see Lemmas V.4.26 and V.4.27) □

We remark that although we have proved some general results about forcing with measure algebras, such as Lemma IV.7.34, all concrete examples just used measure algebras of the form $\mathbb{MB}(2^\kappa)$. This is not an accident. By a well-known theorem of Maharam [64], every generic extension obtained using some $\mathbb{MB}(X, \mu)$ is equivalent to one using some $\mathbb{MB}(2^\kappa)$.

Two more exercises on measure algebras:

Exercise V.4.33 *Forcing with a measure algebra never adds an $f \in 2^\omega$ that is Cohen generic (i.e., $\mathrm{Fn}(\omega, 2)$-generic) over the ground model, in the sense of Exercise V.4.25. But, forcing with $\mathbb{MB}(2^\omega) \times \mathbb{MB}(2^\omega)$ does add a Cohen generic f.*

Hint. View $2 = \{0, 1\}$ as the two-element abelian group, and 2^ω as the product group. Forcing with $\mathbb{MB}(2^\omega) \times \mathbb{MB}(2^\omega)$ adds a pair $(g_0, g_1) \in 2^\omega \times 2^\omega$, and each of g_0, g_1 is random over M. Let $f = g_0 + g_1$. To prove that f is Cohen generic over M, let U be a dense open subset of 2^ω that is coded in M; we must show that every forcing condition has an extension that forces f to lie in U. To do

this, use the fact from measure theory that if $X, Y \subseteq 2^\omega$ are measurable sets with positive measure, then $X + Y$ has non-empty interior. □

Compare this to Exercise V.4.30 and Theorem V.1.2. Let G denote the $\mathbb{MB}(2^\omega)$–generic filter. Then forcing with $(\mathbb{MB}(2^\omega) \times \mathbb{MB}(2^\omega))^M$ yields an extension of $M[G]$ obtained by forcing with the poset $(\mathbb{MB}(2^\omega))^M$, which, in $M[G]$, is not a measure algebra, whereas forcing with $(\mathbb{MB}(2^\omega \times 2^\omega))^M$ yields an extension of $M[G]$ obtained by forcing with the poset $(\mathbb{MB}(2^\omega))^{M[G]}$.

Exercise V.4.34 *Let $\mathbb{P}_\xi = \mathbb{MB}(2^\xi)$. Show how to view $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ as an α–stage iterated forcing construction in the sense of Definition V.3.11; but this is* not *a finite support iteration.*

Hint. The successor stages are essentially trivial, with $\mathring{\mathbb{Q}}_\xi$ naming a three-element poset. For limit η, use the fact that the (equivalence classes of) closed sets are dense in $\mathbb{MB}(2^\eta)$, and if p is a closed subset of 2^η, then $p = \bigcap_{\xi<\eta} p_\xi$, where p_ξ is the projection of p to 2^ξ. □

Next, two exercises on Suslin trees: Say $\diamondsuit$ holds in the ground model M, so there are lots of Suslin trees. If $M[G]$ is a ccc extension satisfying $MA + \neg CH$, what happens to all those Suslin trees in $M[G]$?

Say T is a Suslin tree in M. Since cardinals are preserved, T remains an ω_1–tree (Definition III.5.7). If T is Aronszajn in $M[G]$, then it must also be special (Theorem III.5.19), but it could also fail to be Aronszajn, since it may gain an uncountable chain in $M[G]$.

If you fix a specific Suslin tree in M and do the proof of Theorem V.4.1, then you have your choice. In that proof, $\mathbb{P}_1$ is essentially a ground model poset. We may let $\mathbb{P}_1$ be a well-pruned subtree of T, so that the forcing adds a path through T, making it non-Aronszajn. But we may also let $\mathbb{P}_1$ be the specializing order from Theorem III.5.19. Once T becomes special (a countable union of antichains), then it can never acquire an uncountable chain, so it remains a special Aronszajn tree in $M[G]$.

But if you ask about all the trees together, then the question becomes somewhat tricker.

Exercise V.4.35 *In M, let $\kappa \geq \aleph_2$ be a regular cardinal with $2^{<\kappa} = \kappa$. Show that one can modify the proof of Theorem V.4.1, obtaining $MA + \mathfrak{c} = \kappa$ in $M[G]$, so that all Aronszajn trees of M become special Aronszajn trees in $M[G]$.*

Hint. Let $\mathbb{P}_1$ be the finite support product of the specializing orders of all (up to isomorphism) Aronszajn trees (from M). Then $\mathbb{P}_1$ has the ccc (Exercise III.5.47 may help in proving this). Also, $|\mathbb{P}_1| = 2^{\aleph_1} \leq \kappa$, so that $2^{<\kappa} = \kappa$ continues to hold in the $\mathbb{P}_1$ extension, so one can now proceed to force MA. □

But you cannot in general make all the Suslin trees from M *non*-Aronszajn:

Exercise V.4.36 *Assume that $M \models \diamondsuit$. Prove that there are two Suslin trees, $S, T \in M$, such that for every forcing extension $M[G]$ that preserves ω_1, at least one of S, T is Aronszajn in $M[G]$.*

Hint. Using $\diamondsuit$, construct Suslin trees S, T, such each node in $\mathcal{L}_\alpha(T)$ encodes an order preserving map from S_α (Definition III.5.1) into the rationals, $\mathbb{Q}$, so that if there is a path through T in $M[G]$, then $M[G]$ will contain an order preserving map from S into $\mathbb{Q}$. □

Exercise V.4.37 *Let M be a ctm for ZFC + GCH. Obtain three cardinal-preserving generic extensions of M, all of which satisfy $ZFC + MA + 2^{\aleph_0} = \aleph_2$, but satisfying, respectively $2^{\aleph_2} = \aleph_3$, $2^{\aleph_2} = \aleph_6$, and $2^{\aleph_2} = \aleph_8$.*

Exercise V.4.38 *Lemma V.4.5 would be false if we replaced the $\rightarrow$ by an $\leftrightarrow$.*

Hint. A poset that is ccc in a model can cease to be ccc in a ccc extension of that model. □

There is a notion of *normal chain* that generalizes the finite support iterations we have used in this section:

Definition V.4.39 *For an ordinal α, a* normal α–chain *of posets is a sequence of posets $\langle \mathbb{P}_\xi : \xi < \alpha \rangle$ such that $\mathbb{P}_\xi \subseteq_c \mathbb{P}_\eta$ whenever $\xi \leq \eta$ and $\mathbb{P}_\eta = \bigcup_{\xi<\eta} \mathbb{P}_\xi$ whenever $\eta < \alpha$ is a limit ordinal.*

The finite support iteration, as in Definitions V.3.11 and V.3.13, does form a special kind of normal chain if we identify $\mathbb{P}_\xi$ with a sub-order of $\mathbb{P}_\eta$ as we explained after Definition V.3.13.

Note that Definition V.4.39 can be understood without any knowledge of forcing. This could have been placed in Section III.3 on Martin's Axiom, except that the motivation for studying these chains might not have been clear.

One can verify directly from Definition V.4.39 that certain properties of the normal chain are preserved at limit ordinals. The most basic property is *being* a normal chain:

Exercise V.4.40 *Let $\langle \mathbb{P}_\xi : \xi < \gamma \rangle$ be a normal γ–chain of posets, where γ is a limit ordinal. Let $\mathbb{P}_\gamma = \bigcup_{\xi<\gamma} \mathbb{P}_\xi$. Then $\langle \mathbb{P}_\xi : \xi \leq \gamma \rangle$ is a normal $(\gamma+1)$–chain.*

Less trivially, the ccc is preserved. In fact, we can prove the following stronger fact, which is the natural generalization of Lemma V.3.16. It says, roughly, that any pattern of incompatibilities in $\mathbb{P}_\alpha$ can be pulled back to some earlier $\mathbb{P}_\xi$.

Exercise V.4.41 *Let $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ be a normal chain of posets, where α is a limit ordinal. Fix $p_\mu \in \mathbb{P}_\alpha$ for $\mu < \omega_1$. Then for some $\xi < \alpha$ and some $I \in [\omega_1]^{\omega_1}$, there are $q_\mu \in \mathbb{P}_\xi$ for $\mu \in I$ such that for all $n \geq 2$ and all $\mu_1, \ldots, \mu_n \in I$:*

$$\not\perp(q_{\mu_1}, \ldots, q_{\mu_n}) \rightarrow \not\perp(p_{\mu_1}, \ldots, p_{\mu_n}) \ .$$

Hint. We need a replacement for the finite supports used in the delta system argument for Lemma V.3.16. For each $p \in \mathbb{P}_\alpha$, the least ξ such that $p \in \mathbb{P}_\xi$ is either 0 or a successor ordinal (since this is a normal chain). Define the "height" $\mathrm{ht}(p)$ to be ξ if $p \in \mathbb{P}_{\xi+1} \backslash \mathbb{P}_\xi$, and let $\mathrm{ht}(p) = -1$ if $p \in \mathbb{P}_0$.

When $\mathrm{ht}(p) = \xi \geq 0$, apply $\mathbb{P}_\xi \subseteq_{\mathsf{c}} \mathbb{P}_{\xi+1}$ to choose some $p^* \in \mathbb{P}_\xi$ that is a reduction of p to $\mathbb{P}_\xi$ (see Definition III.3.71 and Lemma III.3.72). Then $\forall q \in \mathbb{P}_\xi\, [q \perp p \to q \perp p^*]$ and $\mathrm{ht}(p^*) < \mathrm{ht}(p)$. Hence, for all p, q:

$$\mathrm{ht}(q) < \mathrm{ht}(p) \ \wedge\ q \perp p \ \ \to \ \ q \perp p^* \ . \tag{$\dagger$}$$

Every decreasing sequence of ordinals is finite. So, for each μ: define $m = m_\mu \in \omega$ and p^i_μ so that $p^0_\mu = p_\mu$, $p^m_\mu \in \mathbb{P}_0$, and, for $i < m$, $p^i_\mu \notin \mathbb{P}_0$ and $p^{i+1}_\mu = (p^i_\mu)^*$. So, $\mathrm{ht}(p^0_\mu) > \mathrm{ht}(p^1_\mu) > \cdots > \mathrm{ht}(p^m_\mu) = -1$. Now, applying the Delta System Lemma, get an $I \in [\omega_1]^{\aleph_1}$ and a $k \leq m < \omega$ and a $\xi < \alpha$ so that:

1. $m = m_\mu$ for all $\mu \in I$.
2. $\mathrm{ht}(p^k_\mu) < \xi$ for all $\mu \in I$.
3. Whenever $i, j < k$ and $\mu, \nu \in I$ and $\mu \neq \nu$: $\mathrm{ht}(p^i_\mu) \neq \mathrm{ht}(p^j_\nu)$.

Let $q_\mu = p^k_\mu$. To verify that these q_μ work is a little like Lemma V.3.15, but a bit more tedious. □

From this, the following is essentially obvious:

Lemma V.4.42 *Let $\psi(\mathbb{P})$ say* either *that $\mathbb{P}$ has pre-caliber $\aleph_1$* or *that $\mathbb{P}$ has property K* or *that $\mathbb{P}$ is ccc. Let $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ be a normal chain of posets, where α is a limit ordinal. If $\psi(\mathbb{P}_\xi)$ holds for all $\xi < \alpha$, then $\psi(\mathbb{P}_\alpha)$ holds also.*

We remark that the definition of "normal chain" requires that $\mathbb{P}_\eta = \bigcup_{\xi<\eta} \mathbb{P}_\xi$ for *every* $\eta \leq \alpha$, not just for $\eta = \alpha$. This was used in the proof of Exercise V.4.41 to define $\mathrm{ht}(p)$. If we dropped this requirement, then the following exercise would contradict Lemma V.4.42 (where $\alpha = \omega_1$):

Exercise V.4.43 *There are posets $\mathbb{P}_\xi$ for $\xi \leq \omega_1$ such that $\mathbb{P}_\xi \subseteq_{\mathsf{c}} \mathbb{P}_\eta$ whenever $\xi \leq \eta \leq \omega_1$ and $\mathbb{P}_{\omega_1} = \bigcup_{\xi<\omega_1} \mathbb{P}_\xi$, and $\mathbb{P}_\xi$ is countable (and hence ccc) for all $\xi < \omega_1$, and $\mathbb{P}_{\omega_1}$ is not ccc.*

Hint. If T is a well-pruned ω_1–tree, then $T_\xi \subseteq_{\mathsf{c}} T_\eta$ whenever $\xi \leq \eta$ and ξ is a *successor* ordinal. □

The proof of Exercise V.4.41 was definitely a bit more tedious than the corresponding Lemma V.3.16. So, after having done all this work, we can ask whether it gives us any new applications to forcing. The answer is "no", since, as Exercise V.4.47 shows, every normal chain is "forcing equivalent" to a finite support iteration as in Definitions V.3.11 and V.3.13.

Note that a normal chain need not be *isomorphic* to a finite support iteration because of the nature of the reductions. In the proof of Exercise V.4.41, we used $\mathbb{P}_\xi \subseteq_{\mathsf{c}} \mathbb{P}_{\xi+1}$ to *choose*, for $p \in \mathbb{P}_{\xi+1} \backslash \mathbb{P}_\xi$, some reduction of p^* of p to $\mathbb{P}_\xi$.

There is no "natural" choice, and we certainly cannot claim that $p^* \geq p$ (consider the case that $\mathbb{P}_\xi$ is dense in $\mathbb{P}_{\xi+1}$). But in finite support iterations, our p is a sequence of length $\xi+1$ and p^* is just $p\restriction\xi$, so we do have $p^* \geq p$ (identifying $\mathbb{P}_\xi$ with a sub-order of $\mathbb{P}_{\xi+1}$ as usual). This greatly simplifies the argument, and we didn't even mention the notion of reduction explicitly in the proof of Lemma V.3.16.

Before considering equivalence between chains, just consider two posets. One can show that if $\mathbb{P} \subseteq_c \mathbb{R}$, then there is a $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$ so that $\mathbb{P} * \mathring{\mathbb{Q}} \approxeq \mathbb{R}$ (see Definition IV.4.25). More generally, if we start with a complete embedding $i : \mathbb{P} \to \mathbb{R}$ we shall construct a $\mathring{\mathbb{Q}}$ such that $\mathbb{P} * \mathring{\mathbb{Q}} \approxeq \mathbb{R}$. First, we verify

Lemma V.4.44 *Whenever $i : \mathbb{P} \to \mathbb{R}$ is a complete embedding and G is a filter on $\mathbb{P}$, define $\mathbb{R}/G = \{r \in \mathbb{R} : \forall p \in G\, [r \not\perp i(p)]\}$. Of course, this depends on i as well. Observe that $\mathbb{R}/G$ is an upward closed subset of $\mathbb{R}$ and $i(G) \subseteq \mathbb{R}/G$.*

Now, assume that $\mathbb{P}, \mathbb{R}, i \in M$ and let $\mathring{G}$ be the standard name for the $\mathbb{P}$–generic filter. For any $p \in \mathbb{P}$ and $r \in \mathbb{R}$, $p \Vdash_{\mathbb{P}} r \in \mathbb{R}/\mathring{G}$ iff p is a reduction of r to $\mathbb{P}$.

Proof. $p \in \mathbb{P}$ is a reduction of r to $\mathbb{P}$ iff $\forall p_1 \in \mathbb{P}\, [i(p_1) \perp r \to p_1 \perp p]$ (see Definition III.3.71). Also, $p \Vdash_{\mathbb{P}} r \in \mathbb{R}/\mathring{G}$ iff $p \Vdash \forall p_1 \in \mathbb{P}\, [r \perp i(p_1) \to p_1 \notin \mathring{G}]$ iff $\forall p_1 \in \mathbb{P}\, [i(p_1) \perp r \to p \Vdash p_1 \notin \mathring{G}]$. Then, note that $p \Vdash p_1 \notin \mathring{G}$ iff $p_1 \perp p$. □

Now, let $\mathrm{red}(p, r)$ abbreviate the statement that p is a reduction of r to $\mathbb{P}$. Then, in M, we can define a name $\mathring{\mathbb{Q}}$ that is a $\mathbb{P}$–name for $\mathbb{R}/\mathring{G}$; so as a name, $\mathring{\mathbb{Q}} = \{\langle \check{r}, p\rangle : r \in \mathbb{R} \wedge p \in \mathbb{P} \wedge \mathrm{red}(p, r)\}$. Then in $M[G]$, we have $\mathbb{Q} := \mathring{\mathbb{Q}}_G = \{r \in \mathbb{R} : \exists p \in G\, [p \Vdash r \in \mathbb{R}/\mathring{G}]\} = \mathbb{R}/\mathring{G}$ by the Truth Lemma. To make this into a $\mathbb{P}$–name for a forcing poset, as per Definition V.3.1, we let $\mathring{\mathbb{1}}_{\mathbb{Q}}$ name the $\mathbb{1}$ of $\mathbb{R}$ (that is $\check{\mathbb{1}}_{\mathbb{R}}$), and $\mathring{\leq}_{\mathbb{Q}}$ name the $\leq$ of $\mathbb{R}$.

This may seem strange, since $\mathbb{Q}$ is actually a subset of $\mathbb{R}$, and $\mathrm{dom}(\mathring{\mathbb{Q}}) = \mathrm{dom}(\check{\mathbb{R}}) = \{\check{r} : r \in \mathbb{R}\}$, but since $\mathbb{Q}$ is usually a proper subset, its forcing properties can be very different from $\mathbb{R}$. Consider two simple special cases:

Suppose that $\mathbb{P} = \{\mathbb{1}_{\mathbb{R}}\}$ with i the identity map. Then $\mathring{\mathbb{Q}} = \{\langle \check{r}, \mathbb{1}\rangle : r \in \mathbb{R}\}$, so here $\mathbb{Q} = \mathbb{R}$, and we are saying that forcing with $\mathbb{R}$ is equivalent to forcing with $\{\mathbb{1}\}$ (which adds nothing) followed by forcing with $\mathbb{R}$.

Suppose that $\mathbb{P} = \mathbb{R}$ with i the identity map. Then, in $M[G]$, $\mathring{\mathbb{Q}}_G = \mathbb{R}/G = \{r \in \mathbb{R} : \forall p \in G\, [r \not\perp p]\} = G$ (using $\mathbb{P} = \mathbb{R}$; if $r \notin G$ then $r \perp p$ for some $p \in G$, since G is generic and $\{p \in \mathbb{P} : p \leq r \vee p \perp r\}$ is dense). So, forcing with $\mathbb{R}$ is equivalent to forcing with $\mathbb{R}$, followed by forcing with the filter G viewed as a poset (which adds nothing, since all elements of G are compatible in G).

Now, consider the general case. $\mathbb{P} * \mathring{\mathbb{Q}}$ is rarely isomorphic to $\mathbb{R}$, but

Lemma V.4.45 *With the above notation, $\mathbb{P} * \mathring{\mathbb{Q}} \approxeq \mathbb{R}$.*

Proof. First observe, by applying Definition V.3.3:

$$\mathbb{P} * \mathring{\mathbb{Q}} = \{(p, \check{r}) \in \mathbb{P} \times \mathrm{dom}(\check{\mathbb{R}}) : p \Vdash \check{r} \in \mathring{\mathbb{Q}}\} \cong \{(p, r) \in \mathbb{P} \times \mathbb{R} : \mathrm{red}(p, r)\} \quad .$$

Here, the "$\cong$" refers to the obvious bijection from $\mathbb{P} \times \mathrm{dom}(\check{\mathbb{R}})$ onto $\mathbb{P} \times \mathbb{R}$ Applying this bijection, the $\mathbb{1}$ of $\mathbb{P} * \mathring{\mathbb{Q}}$ is just $(\mathbb{1}_{\mathbb{P}}, \mathbb{1}_{\mathbb{R}})$, and, since the order on $\mathbb{Q}$ was just the order on $\mathbb{R}$ restricted to $\mathbb{Q}$, we have $(p_1, r_1) \leq (p_2, r_2)$ iff $p_1 \leq_{\mathbb{P}} p_2$ and $r_1 \leq_{\mathbb{R}} r_2$.

Now, let $j : \mathbb{R} \to \mathbb{A}$ be the boolean completion of $\mathbb{R}$; then certainly j is a dense embedding We prove that $\mathbb{P} * \mathring{\mathbb{Q}} \cong \mathbb{R}$ by producing a dense embedding $k : \mathbb{P} * \mathring{\mathbb{Q}} \to \mathbb{A}$. Recall that the forcing poset is actually $\mathbb{A}\backslash\{\mathbb{0}\}$, so that $\mathrm{ran}(k)$ must be contained in $\mathbb{A}\backslash\{\mathbb{0}\}$. Define $k(p, r) = j(r) \wedge j(i(p))$. Note that $k(p, r) \neq \mathbb{0}$ because $\mathrm{red}(p, r) \to r \not\perp i(p) \to j(r) \wedge j(i(p)) \neq \mathbb{0}$, and $\mathrm{ran}(k)$ is dense in $\mathbb{A}\backslash\{\mathbb{0}\}$ because $k(p, r) \leq j(r)$. To verify that k is a dense embedding, one must verify (1)(2)(3) of Definition III.3.65. Fix $(p_1, r_1), (p_2, r_2) \in \mathbb{P} \times \mathbb{R}$ with $\mathrm{red}(p_1, r_1)$ and $\mathrm{red}(p_2, r_2)$. So, one must verify

1. $k(\mathbb{1}, \mathbb{1}) = \mathbb{1}$.
2. $(p_1, r_1) \leq (p_2, r_2) \to k(p_1, r_1) \leq k(p_2, r_2)$.
3. $(p_1, r_1) \not\perp (p_2, r_2) \leftrightarrow k(p_1, r_1) \not\perp k(p_2, r_2)$.

Now, (1) and (2) are easy from the definitions, as is the $\to$ direction of (3). For the $\leftarrow$ direction, start with $j(r_1) \wedge j(i(p_1)) \wedge j(r_2) \wedge j(i(p_2)) \neq \mathbb{0}$. Then, in $\mathbb{R}$, $\not\perp(r_1, r_2, i(p_1), i(p_2))$ (see the proof of Lemma III.3.66), so get $r_3 \leq r_1, r_2, i(p_1), i(p_2)$. Now, note the following general facts about reductions:

$\mathrm{red}(p, r) \leftrightarrow \forall p_1 \in \mathbb{P}\,[p_1 \not\perp p \to i(p_1) \not\perp r] \leftrightarrow \forall p_1 \in \mathbb{P}\,[p_1 \leq p \to i(p_1) \not\perp r]$
$\mathrm{red}(p, r) \to i(p) \not\perp r$
$\mathrm{red}(p, r) \wedge p_1 \leq p \to \mathrm{red}(p_1, r)$
$\mathrm{red}(p, r) \wedge r \leq i(p_1) \to \exists p_2\,[p_2 \leq p_1 \wedge p_2 \leq p \wedge \mathrm{red}(p_2, r)]$
$r_3 \leq i(p_1) \wedge r_3 \leq i(p_2) \to \exists p_3\,[p_3 \leq p_1 \wedge p_3 \leq p_2 \wedge \mathrm{red}(p_3, r_3)]$

The last statement is what is needed in this argument. □

In this proof, we have two natural embeddings from $\mathbb{P}$ into $\mathbb{A}$. First, via $\mathbb{R}$, we have $p \mapsto j(i(p))$. Second, via $\mathbb{P} * \mathring{\mathbb{Q}}$, we have $p \mapsto k(p, \mathbb{1}) = j(\mathbb{1}) \wedge j(i(p))$. So, $j(i(p)) = k(p, \mathbb{1})$, and we may say that our equivalence between $\mathbb{P} * \mathring{\mathbb{Q}}$ and $\mathbb{R}$ *fixes* $\mathbb{P}$ in the obvious sense.

In practice, the usual way of constructing a normal chain is via some finite support iterated forcing construction, as in Definitions V.3.11 and V.3.13. One can now prove a result saying that *every* normal chain is equivalent to one constructed in that particular way. Of course, we must say what "equivalent" means. One way, motivated by what we have just done for pairs of posets, is:

Definition V.4.46 *Let $\langle \mathbb{P}_\xi : \xi < \alpha \rangle$ and $\langle \mathbb{X}_\xi : \xi < \alpha \rangle$ be two normal α-chains of posets. Then the two chains are* strongly forcing equivalent *iff there is a third normal chain $\langle \mathbb{A}_\xi : \xi < \alpha \rangle$, together with dense embeddings $i_\xi : \mathbb{P}_\xi \to \mathbb{A}_\xi$ and $j_\xi : \mathbb{X}_\xi \to \mathbb{A}_\xi$ such that $i_\xi = i_\eta \restriction \mathbb{P}_\xi$ and $j_\xi = j_\eta \restriction \mathbb{X}_\xi$ whenever $\xi < \eta < \alpha$.*

One can always take the $\mathbb{A}_\xi$, for successor ξ, to be complete boolean algebras. The unions, for limit ξ, cannot in general be complete.

Exercise V.4.47 *Let $\langle \mathbb{X}_\xi : \xi \leq \alpha \rangle$ be a normal $(\alpha+1)$-chain of posets. Obtain a finite support iterated forcing construction, as in Definitions V.3.11 and V.3.13, such that $\langle \mathbb{X}_\xi : \xi \leq \alpha \rangle$ is strongly forcing equivalent to the normal chain $\langle \widehat{\mathbb{P}}_\xi : \xi \leq \alpha \rangle$ obtained from the $\mathbb{P}_\xi$ as described after Definition V.3.13.*

As the above few results indicate, it is sometime useful to deal with complete boolean algebras, even though most of the "natural" forcing posets are not boolean algebras. Another example of this is given by the following three exercises. The first one is really more about relative constructibilty (Definition II.6.29) than about forcing.

Exercise V.4.48 *Let H be $\mathbb{Q}$-generic over M, and fix $A \in M[H]$. Then there is a smallest transitive N with the properties that $N \models ZF$ and $M \cup \{A\} \subseteq N$. Let $M(\!|A|\!)$ denote this N. If $A \subseteq I$ with $I \in M$, then $M(\!|A|\!) \models AC$. Also, $M(\!|H|\!) = M[H]$.*

One cannot prove in general that $M(\!|A|\!) \models AC$; this can fail with $A = (\mathcal{P}(\omega))^{M[H]}$; see Section IV.8. In the literature, one always writes $M[A]$ rather than $M(\!|A|\!)$, although the general method of obtaining it differs from the $M[G]$ construction (Definition IV.2.7). However, if we force with complete boolean algebras and $A \subseteq I \in M$, then this $M(\!|A|\!)$ really is an $M[G]$:

Exercise V.4.49 *Assume in M that $\mathbb{Q}$ is a complete boolean algebra, and let H be $\mathbb{Q}$-generic over M. Fix $A \in M[H]$ such that $A \subseteq I$ with $I \in M$. Then for some $\mathbb{P} \in M$: $\mathbb{P}$ is complete sub-algebra of $\mathbb{Q}$ in M, and $M(\!|A|\!) = M[H \cap \mathbb{P}]$.*

Hint. $A = \mathring{A}_H$, where $\mathring{A} = \{\langle \check{i}, q_i \rangle : i \in I\}$; $\mathbb{P}$ is generated by $\{q_i : i \in I\}$. □

This exercise can fail for an arbitrary $\mathbb{Q}$:

Exercise V.4.50 *Assume in M that $\mathbb{Q} = \mathrm{Fn}(\omega, \kappa)$, where $\kappa \geq \mathfrak{c}$, and let H be $\mathbb{Q}$-generic over M. Then there is a real number $A \in M[H]$ such that A is random over M (see Exercises V.4.19 and Exercises V.4.20); $A \subseteq I \in M$, where I is the set of rationals (see Definition I.14.2), but there is no poset $\mathbb{P} \in M$ such that $\mathbb{P} \subseteq_c \mathbb{Q}$ and $M(\!|A|\!) = M[H \cap \mathbb{P}]$.*

Hint. Get $A \in M[H]$ by meeting $\mathfrak{c}^M$ (which is countable) dense sets in the random real poset. Now, say $\mathbb{P} \in M$ and $\mathbb{P} \subseteq_c \mathbb{Q}$; let $G = H \cap \mathbb{P}$, and let $F = \bigcup G$; so F is a function with $\mathrm{dom}(F) \subseteq \omega$ and $\mathrm{ran}(F) \subseteq \kappa$. If $M[G] = M(\!|A|\!)$, then there is a countable $S \in M$ with $\mathrm{ran}(F) \subseteq S$. But then $M[G] = M(\!|A|\!)$ is obtained by forcing with a subposet of $\mathrm{Fn}(\omega, S)$, which is countable in M. You now get a contradiction by comparing Lemma IV.7.35 and Lemma IV.7.34; see also Exercise III.3.70 □

Exercise IV.7.55 says that "small forcings don't kill inaccessible cardinals". One might expect a related result saying that "small forcings don't kill MA". The situation is rather complicated here. Say $MA(\kappa) \wedge \kappa > \aleph_1$ holds in the

ground model M. If we force with $\mathrm{Fn}(\omega_1, 2)$, then even $MA(\aleph_1)$ fails miserably in $M[G]$, since many of the basic cardinals ($\mathfrak{p}$, add(null), etc.) that "should" be $\geq \kappa$ are actually $\aleph_1$; see Exercise V.4.51.

By Roitman [76], even countable posets kill MA. If $\mathbb{P} = \mathrm{Fn}(\omega, 2)$, then in $M[G]$, there are ccc posets $\mathbb{X}_0, \mathbb{X}_1$ such that $\mathbb{X}_0 \times \mathbb{X}_1$ is not ccc (see Exercise V.4.52); in particular, $MA(\aleph_1)$ is false by Theorem III.3.43. However, [76] also shows that countable posets do preserve $MA(\kappa)$ for σ–centered posets; that is, $\mathfrak{p} > \kappa$ holds in $M[G]$, see Exercise V.4.53 below.

Exercise V.4.51 *Let $M[G]$ be $\mathbb{P}$–generic over M, where $\mathbb{P} = \mathrm{Fn}(\omega_1^M, 2)$. Then in $M[G]$: $\mathfrak{p} = \mathfrak{b} = \mathrm{cov}(\mathsf{null}) = \mathrm{add}(\mathsf{null}) = \mathrm{non}(\mathsf{meager}) = \mathrm{add}(\mathsf{meager}) = \aleph_1$, and there exists a Luzin set.*

Hint. See Exercise V.2.23; this part of the exercise doesn't need the assumption that *CH* holds in M. □

Exercise V.4.52 *Let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$, and let G be $\mathbb{P}$–generic over M. Then in $M[G]$, there are ccc posets $\mathbb{X}_0, \mathbb{X}_1$ such that $\mathbb{X}_0 \times \mathbb{X}_1$ is not ccc*

Hint. This is a little like Theorem III.8.19, which produced such $\mathbb{X}_0, \mathbb{X}_1$ from *CH*. In $M[G]$, say $h : [\omega_1]^2 \to 2$. For $\mu = 0, 1$, let $\mathbb{X}_\mu$ be the set of all finite $x \subseteq \omega_1$ such that h has constant value μ on $[x]^2$. Then $\mathbb{X}_0 \times \mathbb{X}_1$ is never ccc, so you just have to define h so that $\mathbb{X}_0$ and $\mathbb{X}_1$ are ccc.

In M, fix $\psi_\alpha : \alpha \xrightarrow{1\text{–}1} \omega$ for $\alpha < \omega_1$ such that $\mathrm{ran}(\psi_\alpha) \cap \mathrm{ran}(\psi_\beta)$ is finite whenever $\alpha \neq \beta$. Then, in $M[G]$, let $F = \bigcup G : \omega \to 2$. When $\alpha < \beta$, define $h\{\alpha, \beta\} = F(\psi_\beta(\alpha))$.

To prove ccc: Suppose, in $M[G]$, that A is an uncountable antichain in $\mathbb{X}_\mu$. By Exercise IV.3.18, we may shrink A and assume that $A \in M$. Then, working in M and shrinking again, we may assume that A forms a delta system and that the root is empty, and that $|x| = n$ for all $x \in A$, and that $A = \{x_\xi : \xi < \omega_1\}$, where $\xi < \eta \to \max(x_\xi) < \min(x_\eta)$. You can now get a contradiction by showing that $x_\omega \not\perp x_\ell$ for some $\ell < \omega$. □

Roitman [76] also showed that adding one random real creates such $\mathbb{X}_0, \mathbb{X}_1$ and thereby destroys $MA(\aleph_1)$. Somewhat later, Shelah [82] showed that adding one Cohen real adds a Suslin tree, while Laver [61] showed that adding any number of random reals over a model of $MA(\aleph_1)$ does not add Suslin trees.

Exercise V.4.53 *Assume in M that $\mathfrak{p} > \kappa$ and $\mathbb{P}$ is countable in M. Then $\mathfrak{p} > \kappa$ holds in $M[G]$.*

Hint. WLOG, $\mathbb{P} = 2^{<\omega}$ (see the hint for Exercise III.3.70); then $\bigcup G \in {}^{\omega}2$. Working in M, assume we have $\mathbb{P}$–names τ_α for $\alpha < \kappa$ such that $\mathbb{1} \Vdash \tau_\alpha \subseteq \omega$ and $\mathbb{1} \Vdash |\tau_{\alpha_0} \cap \cdots \cap \tau_{\alpha_n}| = \aleph_0$ for each $\alpha_0, \ldots, \alpha_n$. We describe a σ–centered poset $\mathbb{Q}$ such that applying $MA_{\mathbb{Q}}(\kappa)$ in M (see Bell's Theorem III.3.61) shows

that $\{(\tau_\alpha)_G : \alpha < \kappa\}$ has a pseudo-intersection in $M[G]$. WLOG, for all $\alpha < \kappa$ and $j \in \omega$, there is a $\beta < \kappa$ such that $\mathbb{1} \Vdash \tau_\beta = \tau_\alpha \cup j$.

Working in M, let $\mathbb{Q}$ be the set of all q such that q is a function and for some $n = n(q) \in \omega$: $\mathrm{dom}(q) = {}^n2$ and for each $p \in {}^n2$, $q(p)$ is a pair $(a, s) = (a(q,p), s(q,p))$ such that $a \in [\omega]^{<\omega}$ and $s \in [\kappa]^{<\omega}$ and $p \Vdash_{\mathbb{P}} a \subseteq \tau_\alpha$ for all $\alpha \in s$. Define $q' \leq q$ iff $n(q') \geq n(q)$ and for each $p \in {}^{n'}2$: $a(q',p) \supseteq a(q, p{\restriction}n(q))$ and $s(q',p) \supseteq s(q, p{\restriction}n(q))$.

$\mathbb{Q}$ is σ–centered because every set $C \subseteq \mathbb{Q}$ is centered if the $n(q)$ are the same n for all $q \in C$ and the $a(q,p) = a(p)$ are the same for all $q \in C$ and $p \in {}^n2$.

Suppose that $H \in M$ is any filter on $\mathbb{Q}$. Working in $M[G]$, we define $F = \bigcup G : \omega \to 2$, and then let $A = \bigcup\{a(q, F{\restriction}n) : q \in H \wedge n = n(q)\} \subseteq \omega$.

For each $p \in \mathbb{P}$ and $k \in \omega$, $\{q : \exists p' \leq p\, [|a(q,p')| \geq k]\}$ is dense in $\mathbb{Q}$. If $H \in M$ meets all these dense sets, then the sets $\{p \in \mathbb{P} : \exists q \in H\, [|a(q,p')| \geq k]\}$ are dense in $\mathbb{P}$, so that A will be infinite.

Meeting more dense sets (using the last WLOG above) guarantees that A is a pseudo-intersection for $\{(\tau_\alpha)_G : \alpha < \kappa\}$. □

Note that these last two exercises give another way of getting results with the flavor of Corollaries V.4.14 and V.4.16, showing that some weakenings of *MA* don't imply full *MA*.

V.5 Other Iterations

Here we consider more general iterated forcing constructions, focusing mainly on countable support iterations (see Definitions V.3.13 and V.3.11). Consider the analogy:

$$\frac{\text{finite support}}{\text{countable support}} = \frac{\mathrm{Fn}(\kappa, 2)}{\mathrm{Fn}_{\aleph_1}(\kappa, 2)} = \frac{MA}{????}$$

Say, in the ground model, *GCH* holds and $\kappa \geq \aleph_2$ is regular. Then the posets $\mathrm{Fn}(\kappa, 2)$ and $\mathrm{Fn}_{\aleph_1}(\kappa, 2)$ both preserve cardinals, but $\mathrm{Fn}(\kappa, 2)$ makes $2^{\aleph_0} = \kappa$, while $\mathrm{Fn}_{\aleph_1}(\kappa, 2)$ makes $2^{\aleph_1} = \kappa$ but is countably closed, so *CH* continues to hold. Although we discussed these matters in Chapter IV before mentioning iterations, in hindsight both $\mathrm{Fn}(\kappa, 2)$ and $\mathrm{Fn}_{\aleph_1}(\kappa, 2)$ may be viewed as iterations. Let $\mathbb{Q}$ be the three-element poset $\mathrm{Fn}(1, 2)$, consisting of $\mathbb{1}$ plus two incompatible elements below $\mathbb{1}$. Let $\mathbb{P}_\alpha$ arise from an α–stage iterated forcing construction, as in Definition V.3.11, where all the names $\mathring{\mathbb{Q}}_\xi$ are $\check{\mathbb{Q}}$. If we use finite supports, then $\mathbb{P}_\alpha$ is isomorphic to $\mathrm{Fn}(\alpha, 2)$, while if we use countable supports, then $\mathbb{P}_\alpha$ is isomorphic to $\mathrm{Fn}_{\aleph_1}(\alpha, 2)$.

Theorem V.4.1 says that we may replace $\mathrm{Fn}(\kappa, 2)$ by a more sophisticated finite support iteration and get *MA* to hold in the generic extension along with $2^{\aleph_0} = \kappa$; this gives all the nice combinatorics described in Section III.3. This theorem was proved by letting the $\mathring{\mathbb{Q}}_\xi$ enumerate all "relevant" ccc posets.

Our analogy now suggests that we might replace "ccc" by "$\aleph_2$-cc + countably closed" and use a countable support iteration to obtain, in the generic extension, *CH* plus $2^{\aleph_1} = \kappa$ plus an axiom such as MA^*:

Definition V.5.1 *$MA^*(\kappa)$ is the statement that $MA_{\mathbb{P}}(\kappa)$ holds for all posets $\mathbb{P}$ that are $\aleph_2$-cc and countably closed.*

MA^* is the statement $\forall \kappa < 2^{\aleph_1}\ MA^*(\kappa)$.

To dispense with some trivialities,

Lemma V.5.2 *$2^{\aleph_1} = \aleph_2 \rightarrow MA^*$ and $\neg CH \rightarrow MA^*$.*

Proof. For the first statement, an easy modification of the Generic Filter Existence Lemma III.3.14 shows that $MA_{\mathbb{P}}(\aleph_1)$ holds for all countably closed posets. For the second statement, a tree argument shows that every atomless countably closed poset has an antichain of size $\mathfrak{c}$. □

So, we are interested in models of $MA^* + CH + 2^{\aleph_1} \geq \aleph_3$. Unfortunately, this is inconsistent, by an example of Shelah and Stanley [86]. But, we shall describe a weakened version of MA^*, due to Baumgartner, that is consistent with *CH* and $2^{\aleph_1}$ being arbitrarily large. For further information on this subject, see the survey of Tall [96], as well as [4, 80, 86, 87].

We begin by looking at what goes wrong with using the "obvious" modification of the proof of Theorem V.4.1 to construct models of $MA^* + CH + 2^{\aleph_1} \geq \aleph_3$. We would need to prove, assuming *CH*, that the properties of being $\aleph_2$-cc and being countably closed are preserved by countable support iterations. Unfortunately, both can fail without further assumptions. We consider first being countably closed. It is easy to check directly from the definitions that:

Lemma V.5.3 *Using the notation of Definition V.3.3, if $\mathbb{P}$ is countably closed and $\mathbb{1}$ forces $\mathring{\mathbb{Q}}$ to be countably closed then $\mathbb{P} * \mathring{\mathbb{Q}}$ is countably closed.*

Because of this, there is no problem for finite iterations, but there is a problem as soon as we get to stage ω:

Example V.5.4 *There is an ω-stage countable support iterated forcing construction, using the notation of Definitions V.3.11 and V.3.13, where $\mathbb{1}$ forces $\mathring{\mathbb{Q}}_n$ to be countably closed for each n, so that each $\mathbb{P}_n$ is countably closed, but $\mathbb{P}_\omega$ is not countably closed, and in fact collapses ω_1.*

Proof. In the ground model, let $\mathbb{T} = \omega^{<\omega_1} = \{t \in \mathrm{Fn}_{\aleph_1}(\omega_1, \omega) : \mathrm{dom}(t) \in \omega_1\}$, which is countably closed and a dense subposet of $\mathrm{Fn}_{\aleph_1}(\omega_1, \omega)$. For each $n \in \omega$, we shall have $\mathrm{dom}(\mathring{\mathbb{Q}}_n) \subseteq \{\check{t} : t \in \mathbb{T}\}$, and $\mathring{\mathbb{Q}}_n$ will be forced by $\mathbb{1}_{\mathbb{P}_n}$ to be a dense subposet of $\mathbb{T}$. Hence, by Lemma V.5.3 and induction on $n \in \omega$, each $\mathbb{P}_n$ will be countably closed, and hence $\mathbb{T}$ will still be the $\mathrm{Fn}_{\aleph_1}(\omega_1, \omega)$ of the $\mathbb{P}_n$-extension, so that $\mathbb{1}$ will force $\mathring{\mathbb{Q}}_n$ to be countably closed.

Applying Definition V.3.11, $\mathbb{P}_\omega$ is the set of all $\vec{q} = \langle \check{q}_n : n < \omega \rangle \in \mathbb{T}^\omega$ such that $\vec{q}{\restriction}n \Vdash \check{q}_n \in \mathring{\mathbb{Q}}_n$ for all n.

Since $\mathring{\mathbb{Q}}_n$ will name a subposet of $\mathbb{T}$, we can view $\mathring{\mathbb{Q}}_n$ as adding a function $f_n : \omega_1 \to \omega$ in the usual way; let $\mathring{f}_n$ be a name for this function in the $\mathbb{P}_{n+1}$ forcing language.

Let $\mathring{\mathbb{Q}}_0$ be $\check{\mathbb{T}}$. Let $\mathring{\mathbb{Q}}_{n+1}$ name the poset consisting of $\mathbb{1} = \emptyset$ plus all those $t \in \mathbb{T}$ such that $\mathrm{dom}(t) = \gamma + k$, where γ is a limit ordinal and k is $\mathring{f}_n(\gamma + \omega)$. So, if $\vec{q} \in \mathbb{P}_\omega$ and $q_{n+1} \neq \mathbb{1}$, then $\mathrm{dom}(q_{n+1}) = \gamma + k$ and $\vec{q}{\restriction}(n+1) \Vdash \check{q}_{n+1} \in \mathring{\mathbb{Q}}_{n+1}$, which means that q_n must force $\mathring{f}_n(\gamma + \omega)$ to be k, so that $\mathrm{dom}(q_n) > \gamma + \omega > \mathrm{dom}(q_{n+1})$. Since there is no infinite decreasing sequence of ordinals, $\vec{q} \in \mathbb{P}_\omega$ implies that $\{n : q_n \neq \mathbb{1}\}$ is finite.

But now, $\mathbb{P}_\omega$ is the same as a finite support iteration, so that Exercise V.4.24 applies to show that ω_1 is collapsed. □

This example also indicates what we need to do to ensure that the iterations are countably closed. We might expect a countable support iteration of posets dense in $\mathbb{T}$ to result in a poset dense in the product $\mathbb{T}^\omega$, which is countably closed; and the analogous result for finite support iterations does work (see Exercise V.5.22). The problem in Example V.5.4 is that the prescription $\mathrm{dom}(\mathring{\mathbb{Q}}_n) \subseteq \{\check{t} : t \in \mathbb{T}\}$, although fine for naming $\mathring{\mathbb{Q}}_n$ as a subset of $\mathbb{T}$, is not enough to give us all the names that "should" be in $\mathbb{P}_\omega$. Consider a sequence $\vec{t} \in \mathbb{T}^\omega$, where all $\mathrm{dom}(t_n) = \omega$. Then, as the above proof shows, no extension of $\vec{t}$ lies in $\mathbb{P}_\omega$. But working in the generic extension, where each name $\mathring{f}_n$ has value f_n, we could say that $\vec{t}$ "should" have an extension, $\vec{q}$, where $q_0 = t_0$ and each q_{n+1} extends t_{n+1} and has length $\omega + f_n(\omega + \omega)$. Now, in the ground model, we don't yet have a value for f_n, but by the Maximal Principal, there is a $\mathbb{P}_{n+1}$-name $\mathring{q}_{n+1}$ that is forced by $\mathbb{1}$ to be this q_{n+1}. We could let $\mathring{q}_0 = \check{t}_0$. Then, this sequence $\langle \mathring{q}_n : n < \omega \rangle$ "should" be in $\mathbb{P}_\omega$ and extend $\vec{t}$, except that the names $\mathring{q}_{n+1}$ are not in $\mathrm{dom}(\mathring{\mathbb{Q}}_n)$. We fix this problem by expanding the name $\mathring{\mathbb{Q}}_n$ to a *full name*, as described in the next definition and lemma:

Definition V.5.5 *For any poset $\mathbb{P}$, a $\mathbb{P}$-name τ is* full *iff for all $\mathbb{P}$-names σ and all $p \in \mathbb{P}$, if $p \Vdash \sigma \in \tau$ then there is a $\sigma' \in \mathrm{dom}(\tau)$ such that $p \Vdash \sigma' = \sigma$ and $\langle \sigma', p \rangle \in \tau$, so that $p \Vdash \sigma' \in \tau$.*

Lemma V.5.6 *Let τ be any $\mathbb{P}$-name. Then there is a full $\mathbb{P}$-name $\tilde{\tau}$ such that $\mathbb{1} \Vdash \tilde{\tau} = \tau$.*

Proof. Of course, we prove this in the ground model, M. Let $\pi = \bigcup \mathrm{dom}(\tau)$. Then, as in the proof of the Union Axiom in $M[G]$ (see Lemma IV.2.15), $\mathbb{1} \Vdash \bigcup \tau \subseteq \pi$, so that $\mathbb{1} \Vdash \forall x \in \tau\, [x \subseteq \pi]$. Let E be the set of all nice names for subsets of π, and let $\tilde{\tau} = \{\langle \sigma', p \rangle \in E \times \mathbb{P} : p \Vdash \sigma' \in \tau\}$. Then $\mathbb{1} \Vdash \tilde{\tau} \subseteq \tau$ is clear from the basic definitions (without using the definition of E).

Now, say $p \Vdash \sigma \in \tau$. Then, $p \Vdash \sigma \subseteq \pi$, so by Lemma IV.3.10, there is a $\sigma' \in E$ such that $p \Vdash \sigma' = \sigma$, so that $\langle \sigma', p \rangle \in \tilde{\tau}$ and $p \Vdash \sigma' \in \tilde{\tau}$. It follows from this that $\mathbb{1} \Vdash \tilde{\tau} = \tau$ and that $\tilde{\tau}$ is full. □

We did not mention full names earlier because in most arguments, they just give us a very ugly way of naming something simple. For example, the name $\check{2} = \{\langle \check{1}, \mathbb{1}\rangle, \langle \check{0}, \mathbb{1}\rangle\}$ is the natural way of naming 2, whereas

Exercise V.5.7 *If $\mathbb{P}$ is atomless and τ is a full $\mathbb{P}$–name for 2, then $|\tau| \geq 2^{\aleph_0}$.*

Hint. Whenever $A_0 \cup A_1$ is a maximal antichain in $\mathbb{P}$ with $A_0 \cap A_1 = \emptyset$, consider a σ in Definition V.5.5 naming the $i \in 2$ such that $G \cap A_i \neq \emptyset$. □

But, full names are useful in some iterated forcing arguments because of:

Lemma V.5.8 *Form a countable support α–stage iterated forcing construction using the notation of Definitions V.3.11 and V.3.13, and assume that for each $\xi < \alpha$, $\mathring{\mathbb{Q}}_\xi$ is a full name and $\mathbb{1}$ forces $\mathring{\mathbb{Q}}_\xi$ to be countably closed. Then $\mathbb{P}_\alpha$ is countably closed.*

Proof. Fix $p^n \in \mathbb{P}_\alpha$ for $n < \omega$ with $p^0 \geq p^1 \geq p^2 \geq \cdots$. We must produce a $p^\omega \in \mathbb{P}_\alpha$ such that $p^\omega \leq p^n$ for all $n < \omega$.

Say p^n is the sequence $\langle \mathring{q}^n_\mu : \mu < \alpha\rangle$. Our p^ω will be $\langle \mathring{q}^\omega_\mu : \mu < \alpha\rangle$. We shall obtain the $\mathring{q}^\omega_\mu$ by recursion on μ, checking inductively that for all $\xi \leq \alpha$:

$$(1_\xi)\ \ p^\omega{\restriction}\xi \in \mathbb{P}_\xi \qquad \text{and} \qquad (2_\xi)\ \ \forall n \in \omega\, [p^\omega{\restriction}\xi \leq p^n{\restriction}\xi]\ .$$

Then, setting $\xi = \alpha$, we shall be done. We shall also make sure that $q^\omega_\mu = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}$ whenever $\mu \notin \bigcup_n \operatorname{supt}(p^n)$. Hence, the supports remain countable, which make it easy to see that (1)(2) are preserved at limit ξ.

For the successor step in the recursion, fix $\xi < \alpha$, and assume that we have defined $p^\omega{\restriction}\xi$ so that $(1_\xi)(2_\xi)$ hold. We need to define $\mathring{q}^\omega_\xi$, which will determine $p^\omega{\restriction}(\xi+1)$. If $\xi \notin \bigcup_n \operatorname{supt}(p^n)$, set $q^\omega_\mu = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}$. If $\xi \in \bigcup_n \operatorname{supt}(p^n)$: For each n, $p^{n+1} \leq p^n$ plus the definition of $\leq$ on $\mathbb{P}_\alpha$ implies that $p^{n+1}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} [\mathring{q}^{n+1}_\xi \leq \mathring{q}^n_\xi]$ and hence $p^\omega{\restriction}\xi \Vdash_{\mathbb{P}_\xi} [\mathring{q}^{n+1}_\xi \leq \mathring{q}^n_\xi]$. Since $\mathring{\mathbb{Q}}_\xi$ is forced to be countably closed, the Maximal Principal gives us a $\mathbb{P}_\xi$–name $\mathring{q}^\omega_\xi$ such that $p^\omega{\restriction}\xi \Vdash_{\mathbb{P}_\xi} [\mathring{q}^\omega_\xi \leq \mathring{q}^n_\xi]$ for each n, and we may assume that $\mathring{q}^\omega_\xi \in \operatorname{dom}(\mathring{\mathbb{Q}}_\xi)$ because the name $\mathring{\mathbb{Q}}_\xi$ is full. But now we have $\mathring{q}^\omega_\xi$ and $p^\omega{\restriction}(\xi+1)$ such that $(1_{\xi+1})(2_{\xi+1})$ hold. □

In view of Lemma V.5.6, we can always do our iterated forcing using full names for the $\mathring{\mathbb{Q}}_\xi$. But preservation of the $\aleph_2$-cc fails completely. Fix any cardinal κ. Assuming *CH*, we can get an ω–stage countable support iteration where each $\mathring{\mathbb{Q}}_n$ is a full name that is forced by $\mathbb{1}$ to be countably closed and $\aleph_2$-cc but $\mathbb{P}_\omega$ is not κ-cc; see Exercise V.5.23. Note that each $\mathbb{P}_n$ will be $\aleph_2$-cc by Lemma V.3.9.

Baumgartner proved that each $\mathbb{P}_\alpha$ is $\aleph_2$-cc if we assume that that the $\mathbb{Q}_\xi$ satisfy the properties given by Definitions V.5.9 and V.5.11 below, and he obtained an axiom now called *BA* (Definition V.5.12). This *BA* is a weakening of MA^* (Definition V.5.1) and it is consistent with *CH* along with $2^{\aleph_1}$ being arbitrarily large.

Definition V.5.9 *A poset $\mathbb{Q}$ is $\aleph_1$–linked iff the set $\mathbb{Q}$ is a union of $\aleph_1$ linked sub-families. $\mathbb{Q}$ is $\aleph_1$–centered iff $\mathbb{Q}$ is a union of $\aleph_1$ centered sub-families.*

Clearly, "$\aleph_1$–centered" implies "$\aleph_1$–linked", and "$\aleph_1$–linked" implies the $\aleph_2$-cc. These properties are the natural generalizations of "σ–linked" and "σ–centered", which occur naturally in discussions of *MA* and weakenings thereof; see Definition III.3.23 and page 185. *BA* is an axiom about $\aleph_1$–linked posets.

Exercise V.5.10 *Let $\mathbb{P}_\kappa = \mathrm{Fn}_{\aleph_1}(\kappa, 2)$, which we may view as a κ–stage countable support iteration of $\mathrm{Fn}(1,2)$. Assuming CH, show that $\mathbb{P}_\kappa$ is $\aleph_1$–linked iff $\mathbb{P}_\kappa$ is $\aleph_1$–centered iff $\kappa \leq 2^{\aleph_1}$.*

Hint. See Lemma III.2.11 and Exercise III.2.13 for the finite support analog of these statements. Note that "linked" = "centered" for $\mathrm{Fn}_\lambda(I, J)$. □

Definition V.5.11 *The poset $\mathbb{Q}$ is* well-met *iff for all $q_1, q_2 \in \mathbb{Q}$ satisfying $q_1 \not\perp q_2$, there is a* meet*, called $q_1 \wedge q_2$, such that $q_1 \wedge q_2 \leq q_1$ and $q_1 \wedge q_2 \leq q_2$ and $q_1 \wedge q_2 \geq q$ for all q such that $q \leq q_1$ and $q \leq q_2$.*

If $\mathbb{Q} = \mathbb{B}\backslash\{0\}$, where $\mathbb{B}$ is a boolean algebra, then $q_1 \wedge q_2$ is the usual boolean meet. In a general poset, where $\leq$ may be a pre-order but not a partial order, the value of $q_1 \wedge q_2$ is not uniquely determined by Definition V.5.11, but we can always choose a value by the Axiom of Choice.

Definition V.5.12 *Baumgartner's Axiom BA is the statement that $MA_{\mathbb{P}}(\kappa)$ holds for all $\kappa < 2^{\aleph_1}$ and all countably closed well-met $\aleph_1$–linked posets $\mathbb{P}$. BACH is the statement $BA \wedge CH$.*

Note (Lemma V.5.2) that *BA* is trivially true under $\neg CH$ and *BACH* follows from $CH + 2^{\aleph_1} = \aleph_2$, so we are interested in *BACH* under $2^{\aleph_1} \geq \aleph_3$. The assumption of "well-met" may seem trivial, since every poset $\mathbb{P}$ may be densely embedded into a well-met poset $\mathbb{A}$ (for example, its boolean completion). This $\mathbb{A}$ will have the same chain condition as $\mathbb{P}$ has, but $\mathbb{A}$ may fail to be countably closed even if $\mathbb{P}$ is countably closed. For example, the poset of non-zero elements of a complete atomless boolean algebra is never countably closed. A related fact is that the obvious iteration lemma for preservation of "well-met" is false for the posets that we wish to consider:

Exercise V.5.13 *Assume that $\mathbb{P}$ is an atomless countably closed poset, with $\mathring{q}_0, \mathring{q}_1 \in \mathrm{dom}(\mathring{\mathbb{Q}})$. Assume that $\mathring{\mathbb{Q}}$ is a full name and that $\mathbb{1}_{\mathbb{P}}$ forces that $\mathring{q}_0, \mathring{q}_1$ are incompatible elements of $\mathring{\mathbb{Q}}$. Show that the iteration $\mathbb{P} * \mathring{\mathbb{Q}}$ is not well-met.*

Hint. First, since $\mathbb{P}$ is countably closed, find an antichain $\{p_n : n \in \omega\}$ such that the boolean supremum $\bigvee_n p_n$ is not realized in $\mathbb{P}$. Now, consider the pairs $(\mathbb{1}, \mathring{q}_0)$ and $(\mathbb{1}, \mathring{q}_2)$, where each $p_n \Vdash \mathring{q}_2 = \mathring{q}_0$, and $p \Vdash \mathring{q}_2 = \mathring{q}_1$ whenever $p \perp p_n$ for all n. □

But, "well-met" helps us prove the preservation of the $\aleph_2$-cc:

Lemma V.5.14 *Use the hypotheses of Lemma V.5.8, but assume also that CH holds and that for all $\xi < \alpha$, $\mathbb{1}$ forces $\mathring{\mathbb{Q}}_\xi$ to be well-met and $\aleph_1$-linked. Then $\mathbb{P}_\alpha$ has the $\aleph_2$-cc.*

Proof. For each $\xi < \alpha$: $\mathring{\mathbb{Q}}_\xi$ is forced to be $\aleph_1$–linked, so fix a $\mathbb{P}_\xi$–name $\mathring{L}_\xi$ such that $\mathbb{1}$ forces "$\mathring{L}_\xi : \mathring{\mathbb{Q}}_\xi \to \omega_1$, and each $\mathring{L}_\xi^{-1}\{\gamma\}$ is a linked family, and $L_\xi(\mathbb{1}) = 0$." Note that since the forcing is countably closed, the ω_1 is the same in all the models.

For each $p \in \mathbb{P}_\alpha$, choose ${}^*p \leq p$ such that $\mathrm{supt}({}^*p) \supseteq \mathrm{supt}(p)$ and for all $\xi < \alpha$, there is a $\gamma = \gamma(p,\xi) < \omega_1$ such that $({}^*p)\restriction\xi \Vdash \mathring{L}_\xi((p)_\xi) = \gamma$. *Remarks.* The notation $(p)_\xi$ is from Definition V.3.11(3). For $\xi \notin \mathrm{supt}(p)$, we may take $\gamma(p,\xi) = 0$. There is such a ${}^*p$ because the forcing is countably closed and $\mathrm{supt}(p)$ is countable.

For $p \in \mathbb{P}_\alpha$, let ${}^0p = p$ and ${}^{n+1}p = {}^*({}^np)$. Let $\Sigma(p) = \bigcup_{n\in\omega} \mathrm{supt}({}^np)$.

Fix $A \in [\mathbb{P}_\alpha]^{\aleph_2}$; we shall prove that A cannot be an antichain. Since each $\Sigma(p)$ is a countable set, *CH* plus the Delta System Lemma III.6.15 (applied with $\kappa = \aleph_2$ and $\lambda = \aleph_1$) implies that we may fix a $B \in [A]^{\aleph_2}$ such that $\{\Sigma(p) : p \in B\}$ forms a delta system with some root $R \in [\alpha]^{\leq\aleph_0}$. So, if $x, y \in B$ with $x \neq y$ then $\Sigma(x) \cap \Sigma(y) = R$. Now, fix $x, y \in B$ with $x \neq y$ such that $\gamma({}^nx,\xi) = \gamma({}^ny,\xi)$ for all $n \in \omega$ and $\xi \in R$.

We shall prove that $x \not\perp y$ by producing a p such that $p \leq {}^nx$ and $p \leq {}^ny$ for all n; so in particular $p \leq x$ and $p \leq y$.

Our p will be $\langle \mathring{q}_\xi : \xi < \alpha\rangle$. We shall define $\mathring{q}_\xi$ by recursion; so we assume inductively that $p\restriction\xi \leq ({}^nx)\restriction\xi$ and $p\restriction\xi \leq ({}^ny)\restriction\xi$ for all n, and we define $\mathring{q}_\xi$ so that for all n: $p\restriction\xi \Vdash \mathring{q}_\xi \leq ({}^nx)_\xi$ and $p\restriction\xi \Vdash \mathring{q}_\xi \leq ({}^ny)_\xi$, and hence $p\restriction(\xi+1) \leq ({}^nx)\restriction(\xi+1)$ and $p\restriction(\xi+1) \leq ({}^ny)\restriction(\xi+1)$.

If $\xi \notin \Sigma(x) \cup \Sigma(y)$, let $\mathring{q}_\xi = \mathring{\mathbb{1}}_\xi$; this will guarantee that $\mathrm{supt}(p)$ will be countable. If $\xi \in (\Sigma(x) \cup \Sigma(y)) \setminus R$, then we get $\mathring{q}_\xi$ just by using the fact that $\mathring{\mathbb{Q}}_\xi$ is forced to be countably closed and the name $\mathring{\mathbb{Q}}_\xi$ is full.

Now, say $\xi \in R$: Let ${}^n\gamma = \gamma({}^nx,\xi) = \gamma({}^ny,\xi)$. Since $p\restriction\xi$ forces that $\mathring{L}_\xi(({}^nx)_\xi) = \mathring{L}_\xi(({}^ny)_\xi) = {}^n\gamma$ for all n, $p\restriction\xi$ also forces $\forall n\,[({}^ny)_\xi \wedge ({}^ny)_\xi \in \mathring{\mathbb{Q}}_\xi]$ and $\forall n\,[({}^{n+1}y)_\xi \wedge ({}^{n+1}y)_\xi \leq ({}^ny)_\xi \wedge ({}^ny)_\xi]$. Then, since $\mathring{\mathbb{Q}}_\xi$ is forced to be countably closed and the name $\mathring{\mathbb{Q}}_\xi$ is full, we may choose $\mathring{q}_\xi$ such that $p\restriction\xi \Vdash \forall n\,[\mathring{q}_\xi \leq ({}^ny)_\xi \wedge ({}^ny)_\xi]$. □

Theorem V.5.15 *Assume that $\kappa > \aleph_1$ is a regular cardinal and $2^{<\kappa} = \kappa$ and $\forall\lambda < \kappa\,[\lambda^{\aleph_0} < \kappa]$. Then there is a countably closed $\aleph_2$-cc poset $\mathbb{P}$ of size κ such that $\mathbb{1} \Vdash_{\mathbb{P}}$ BACH $\wedge\ 2^{\aleph_1} = \kappa$.*

Proof. In view of Lemma V.5.14, the proof is very similar to the proof of Theorem V.4.1, which yielded a poset that forced $MA \wedge \mathfrak{c} = \kappa$. So, we leave most of the details as an exercise, but we elaborate on one slightly tricky point:

Working in the ground model, our $\mathbb{P}$ will be the $\mathbb{P}_\kappa$ of a countable support iterated forcing construction. We use the standard bookkeeping to make sure

that all relevant names for posets get considered, so we have to verify that it is enough to consider $2^{<\kappa} = \kappa$ such names.

First, in analogy with Lemmas V.4.3 and III.3.51, we need to verify, as a theorem of $ZFC + CH$, that $BACH$ follows from the statement that $MA_{\mathbb{Q}}(\lambda)$ holds for all $\lambda < 2^{\aleph_1}$ and all countably closed well-met $\aleph_1$–linked posets $\mathbb{Q}$ *such that* $|\mathbb{Q}| \leq \lambda^{\aleph_0}$. To do this, we can modify the proof of Lemma III.3.51, showing that for $\mathbb{Q}$ of arbitrary size, we may replace $\mathbb{Q}$ by an elementary submodel $\mathbb{Q}'$. Since we want $\mathbb{Q}'$ to be countably closed, we need to assume that whenever $q_0 \geq q_1 \geq \cdots$ are all elements of $\mathbb{Q}'$, there is a $q \in \mathbb{Q}'$ such that $q \leq q_n$ for all n. This requires a countable closure operation, applying Lemma I.13.23, so we get a $\mathbb{Q}'$ of size $\leq \lambda^{\aleph_0}$ rather than $\leq \lambda$.

In the iteration, we assume inductively that each $|\mathbb{P}_\xi| \leq \kappa$. At stage ζ, we list, in a κ–sequence, all relevant nice $\mathbb{P}_\zeta$–names for posets of the form $(\alpha, \leqslant, 0)$ where $\alpha < \kappa$. We may restrict ourselves to $\alpha < \kappa$ using $\forall \lambda < \kappa\, [\lambda^{\aleph_0} < \kappa]$; then, since $\mathbb{P}_\xi$ has the $\aleph_2$–cc, there are no more than $2^{<\kappa} = \kappa$ such nice names.

Now, the iteration requires that we use *full* names, not nice names, and these full names are objects of size κ. So, we just list the κ nice names, but, when we do the iteration, we obtain the $\mathring{\mathbb{Q}}_\xi$ by replacing each nice name by the corresponding full name obtained as in Lemma V.5.6. The fact that $|\mathbb{P}_\xi| \leq \kappa$ and $\mathbb{P}_\xi$ has the $\aleph_2$–cc is used in verifying that $|\mathrm{dom}(\mathring{\mathbb{Q}}_\xi)| \leq \kappa$, so that $|\mathbb{P}_{\xi+1}| \leq \kappa$. □

Actually, Baumgartner [4] shows that one may delete the hypothesis that $\forall \lambda < \kappa\, [\lambda^{\aleph_0} < \kappa]$, so, for example, $BACH + 2^{\aleph_1} = \aleph_{\omega+1}$ is consistent.

Next, we consider two applications of $BACH$. For more, see Tall [96].

First, we verify the analog of the fact that, by Lemmas III.3.22 and III.1.24, MA implies that $\mathfrak{p} = \mathfrak{b} = \mathfrak{d} = \mathfrak{c}$. This analog works because the proof that $MA \to \mathfrak{p} = \mathfrak{c}$ used MA with a σ–centered poset (Lemma III.3.26). Of course, we need to define the analogous cardinals $\mathfrak{p}_{\omega_1}, \mathfrak{b}_{\omega_1}, \mathfrak{d}_{\omega_1}$ (replacing ω by ω_1), but then the proof that $\mathfrak{p}_{\omega_1} = \mathfrak{b}_{\omega_1} = \mathfrak{d}_{\omega_1} = 2^{\aleph_1}$ uses $BACH$ with an $\aleph_1$–centered poset.

Definition V.5.16 *Fix a regular cardinal θ. To define $\mathfrak{b}_\theta$ and $\mathfrak{d}_\theta$, replace ω by θ in Definition III.1.11 and define $f \leq^* g$ iff $|\{\xi < \theta : g(x) > f(x)\}| < \theta$.*

A family $\mathcal{E} \subseteq \mathcal{P}(\theta)$ has the $S\theta IP$ iff $|\bigcap \mathcal{F}| = \theta$ for all $\mathcal{F} \in [\mathcal{E}]^{<\theta}$. Then $\mathfrak{p}_\theta$ is the least size of an $\mathcal{E} \subseteq \mathcal{P}(\theta)$ such that $\mathcal{E}$ has the $S\theta IP$ and there is no $K \in [\theta]^\theta$ with $K \subseteq^* Z$ for all $Z \in \mathcal{E}$; here $K \subseteq^* Z$ iff $|K \backslash Z| < \theta$.

Exercise V.5.17 $\theta^+ \leq \mathfrak{p}_\theta \leq \mathfrak{b}_\theta \leq \mathfrak{d}_\theta \leq 2^\theta$.

Hint. You need to modify the proof of $\mathfrak{p} \leq \mathfrak{b}$ slightly, since the given proof, via Lemma III.1.23, would seem to require $2^{<\theta} = \theta$. □

Exercise V.5.18 *$BACH$ implies that $\mathfrak{p}_{\omega_1} = \mathfrak{b}_{\omega_1} = \mathfrak{d}_{\omega_1} = 2^{\aleph_1}$.*

You need *BA* here; you cannot do this just by cardinal arithmetic. For example:

Exercise V.5.19 *Each of the following statements:*

$$1.\ \mathfrak{p}_{\omega_1} = \mathfrak{b}_{\omega_1} = \mathfrak{d}_{\omega_1} = \aleph_2 \qquad 2.\ \mathfrak{p}_{\omega_1} = \mathfrak{b}_{\omega_1} = \aleph_2 \wedge \mathfrak{d}_{\omega_1} = \aleph_{85}$$

is consistent with CH plus $2^{\aleph_1} = \aleph_{85} = \aleph_{85}^{\leq \aleph_{85}}$ *plus* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ *for all* $\alpha \geq 85$.

Hint. For (2), start with *GCH* and use $\mathrm{Fn}_{\aleph_1}(\omega_{85}, 2)$; see Lemmas V.2.19 and IV.4.10 for the Cohen real analog.

Now, we got $\mathfrak{p} = \mathfrak{b} = \mathfrak{d} = \aleph_1$ using the random real model (see Lemma IV.7.36), but this could also be done by iterating the dominating functions order (dfo) ω_1 times; see Exercise IV.4.11.

There is no known analog of random reals on ω_1. So, for (1), start with the correct cardinal arithmetic in the ground model. Then, iterate the analog of the dfo ω_2 times with countable supports to produce, in the generic extension, a dominating family in $\omega_1^{\omega_1}$ ordered by $\leq^*$ in type ω_2. □

This suggested proof for (1) is similar to the proof of Lemma V.4.27, which obtained models where $\mathfrak{c}$ is large and there is a non-principal ultrafilter $\mathcal{U}$ on ω of character $\aleph_1$. The analogous argument for ultrafilters on ω_1 does not seem to work to settle Question III.1.34, which asks for a uniform ultrafilter on ω_1 of character $\aleph_2 < 2^{\aleph_1}$. The problem is that for the $\mathrm{S}\aleph_1\mathrm{IP}$ to apply, the $\mathcal{U}_\xi$ in the analog of Lemma V.4.27 would have to be countably complete uniform ultrafilters on ω_1; but there are no such ultrafilters by Ulam's Theorem III.6.10.

Next, we show that $BACH + 2^{\aleph_1} > \aleph_2$ implies *KH* (that is, there is an ω_1–Kurepa tree). This may be more surprising than $\mathfrak{p}_{\omega_1} = 2^{\aleph_1}$, since an ω–Kurepa tree exists trivially in *ZFC*, so there is no analogous *MA* result. But it turns out that it is easy to force a Kurepa tree with countable forcing conditions, although one must be a little careful to ensure that the poset is well-met and $\aleph_1$–linked:

Exercise V.5.20 *Assume BACH and fix* $\kappa < 2^{\aleph_1}$. *Then there is an* ω_1*–tree* T *with at least* κ *paths through* T.

Hint. Force a Kurepa family (Definition III.5.39) instead. So, get $\mathcal{F} \subseteq \mathcal{P}(\omega_1)$ such that $|\mathcal{F}| \geq \kappa$ and $|\mathcal{F}{\restriction}\xi| \leq \aleph_0$ for all $\xi < \omega_1$.

Each $p \in \mathbb{P}$ is a pair (α_p, F_p) such that $\alpha_p < \omega_1$ and $\emptyset \in F_p \subset \mathcal{P}(\omega_1)$ and $|F_p| \leq \aleph_0$. Define $q \leq p$ iff $\alpha_q \geq \alpha_p$ and $F_q \supseteq F_p$ and $F_q{\restriction}\alpha_p = F_p{\restriction}\alpha_p$. Then $\mathbb{1} = (0, \{\emptyset\})$. Observe that whenever $\alpha_q \geq \alpha_p$, $q \not\perp p$ iff $F_q{\restriction}\alpha_p = F_p$; then $p \wedge q = (\alpha_q, F_p \cup F_q)$. Let $\mathcal{F} = \bigcup\{F_p : p \in G\}$, where G is a filter on $\mathbb{P}$.

To ensure that all $\mathcal{F}{\restriction}\xi$ are countable, meet the dense sets $\{p : \alpha_p \geq \xi\}$ for each $\xi < \omega_1$.

To ensure that $|\mathcal{F}| \geq \kappa$, observe that $\{p : \exists Y \in \mathcal{F}_p\, [Y =^* X]\}$ is dense whenever $X \subseteq \omega_1$; here, $Y =^* X$ holds iff $(X \backslash Y) \cup (Y \backslash X)$ is countable. □

We really do need *BA* here, since one can modify the proof of Theorem V.2.25 to show:

Exercise V.5.21 *Assume that in the ground model, GCH holds and there is a strongly inaccessible cardinal. Then some generic extension satisfies CH and* $2^{\aleph_1} = \aleph_{85}$ *and* $\neg KH$.

One can show that the pathology of Example V.5.4 cannot happen with finite support iterations:

Exercise V.5.22 *Let* $(\mathbb{R}, \leq, \mathbb{1})$ *denote some forcing poset in the ground model. Form a* finite *support* α*-stage iterated forcing construction exactly as in Definitions V.3.11 and V.3.13, but assume always that each* $\mathring{\mathbb{Q}}_\xi$ *is be forced by* $\mathbb{1}$ *to be a dense subposet of* $\mathbb{R}$. *Then* $\mathbb{P}_\alpha \approxeq \prod^{\text{fin}}_{\xi<\alpha} \mathbb{R}_\xi$, *where each* $\mathbb{R}_\xi = \mathbb{R}$ (*see Definitions III.3.40 and IV.4.25*).

The following exercise demonstrates a problem with the preservation of chain condition at limits in countable support iteration:

Exercise V.5.23 *Fix a cardinal* κ. *Show that there is a countable support* ω*-stage iterated forcing construction using the notation of Definitions V.3.11 and V.3.13 so that* $\mathbb{P}_\omega$ *does not have the* κ*-cc, and all the* $\mathring{\mathbb{Q}}_n$ *are full names that are forced by* $\mathbb{1}$ *to be countably closed and* $(2^{\aleph_0})^+$*-cc.*

Hint. $\mathbb{P}_0 = \{\mathbb{1}\}$. Let $\mathring{\mathbb{Q}}_0 \cong \mathbb{P}_1 \cong \text{Fn}_{\aleph_1}([\kappa]^2, \omega)$. Then $\mathbb{P}_1$ adds a partition $f : [\kappa]^2 \to \omega$.

Working in the $\mathbb{P}_1$ extension, let $\mathbb{Q}_{n+1} = \{q \in [\kappa]^{\leq \aleph_0} : f\text{“}([q]^2) \subseteq \omega \backslash \{n\}\}$. Then $\prod_{n\in\omega} \mathbb{Q}_{n+1}$ has an antichain of size κ (see Lemma III.8.22).

Back in the ground model, we have the names $\mathring{\mathbb{Q}}_{n+1}$ constructed from the name $\mathring{f}$, and the iteration $\mathbb{P}_\omega$ has an antichain of size κ. However, each $\mathbb{P}_{n+1}$ is $(2^{\aleph_0})^+$-cc by a delta system argument, so that $\mathbb{1}$ forces $\mathring{\mathbb{Q}}_n$ to be $(2^{\aleph_0})^+$-cc (see also Exercise V.3.10). □

We conclude this chapter with some examples of supports other than countable and finite. First, the natural generalization of Exercise V.5.19:

Exercise V.5.24 *In the ground model, assume GCH, and let* θ, κ *be regular cardinals with* $\aleph_0 < \theta < \kappa$. *Find generic extensions satisfying each of:*

$$1.\ \mathfrak{p}_\theta = \mathfrak{b}_\theta = \mathfrak{d}_\theta = \theta^+ \qquad 2.\ \mathfrak{p}_\theta = \mathfrak{b}_\theta = \theta^+ \wedge \mathfrak{d}_\theta = \kappa \ .$$

Both extensions should preserve cardinals and cofinalities and satisfy $2^\theta = \kappa$ *along with GCH below* θ.

Hint. For (2), just use $\text{Fn}_\theta(\kappa, 2)$. For (1), first use $\text{Fn}_\theta(\kappa, 2)$ to get the correct cardinal arithmetic, and then iterate the appropriate dfo θ^+ times with supports of size $< \theta$. □

Easton forcing was presented formally using product of posets (Definition V.2.2). As we said on page 294, it may be viewed informally as an iteration, but then the iteration is *top-down*, dealing with the largest cardinals first; formally, a top-down iteration is nonsense if there are infinitely many cardinals.

It does make sense to proceed bottom-up, although, as we pointed out (page 294), this may collapse some cardinals. This bottom-up iteration is now known as *reverse Easton* forcing, and it does have some uses. In particular, it was used by McAloon [67] to show that the axiom $V = HOD$ (see Section II.8) does not imply *GCH*; actually, his paper (1971) does not explicitly mention iterated forcing, and the details of reverse Easton forcing were developed somewhat later by Silver. We first point out that this forcing will preserve cardinals if we use a suitably "sparse" Easton index function (Definition V.2.1).

Exercise V.5.25 *In the ground model, assume GCH, and let E be an Easton index function satisfying $\forall \kappa, \kappa' \in \mathrm{dom}(E)\,[\kappa < \kappa' \to E(\kappa) < \kappa']$. List $\mathrm{dom}(E)$ in increasing order as $\{\kappa_\xi : \xi < \alpha\}$, and define an α-stage iterated forcing construction as in Definitions V.3.11 and V.3.13. Assume now that each $\mathring{\mathbb{Q}}_\xi$ is forced by $\mathbb{1}$ to be the $\mathrm{Fn}_{\kappa_\xi}(E(\kappa_\xi), 2)$ of the $\mathbb{P}_\xi$ extension* (not *of the ground model*). *For the supports, follow $(*)$ of Definition V.2.2, and use only those p such that $|\mathrm{supt}(p) \cap \lambda| < \lambda$ for all regular λ. Then prove that the iterated forcing $\mathbb{P}_\alpha$ preserves cardinals and cofinalities and that $2^\kappa = E(\kappa)$ holds in the generic extension for all $\kappa \in \mathrm{dom}(E)$.*

Exercise V.5.26 *Assume $V = L$ in the ground model, and produce a generic extension satisfying $V = HOD$ plus $2^{\aleph_0} = \aleph_5$.*

Hint. The extension will satisfy $V = L[A]$, where A is a subset of the least α such that $\alpha = \aleph_\alpha$. This can be viewed as a reverse Easton extension, except that now E is not in the ground model. Start the iteration with $\mathbb{P}_1 \cong \mathrm{Fn}(\omega_5, 2)$. Then the $\mathbb{P}_1$ extension satisfies $V = L[A_1]$ with $A_1 \subseteq \omega_5$, along with $2^{\aleph_0} = \aleph_5$ and $A_1 \notin OD$. To make A_1 ordinal definable, make a further extension in which, for all $\xi < \omega_5$, $2^{\aleph_{\omega \cdot \xi + 1}}$ is $\aleph_{\omega \cdot \xi + 2}$ when $\xi \in A_1$ and $\aleph_{\omega \cdot \xi + 3}$ when $\xi \notin A_1$. You will now have $V = L[A_2]$ with $A_2 \subseteq \omega_{\omega_5}$ and $A_2 \notin OD$ and $A_1 = A_2 \cap \omega_5$. You can put A_2 into OD by a third extension. If you repeat this ω times, the process will close off. □

V.6 Independence from Martin's Axiom

Consider the axioms $ZFC + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. Not surprisingly, as we saw in Section IV.7, these do not settle questions about larger cardinals, such as the values of $2^{\aleph_2}, 2^{\aleph_3}, \ldots$ But also, as we saw in Section V.4, these do not settle many basic questions about sets of smaller cardinality, such as the existence of Suslin trees and the values of the various cardinals $\mathfrak{p}$, $\mathrm{cov}(\mathsf{null})$, etc. But, these questions are all settled if we strengthen the axioms to $ZFC + MA + 2^{\aleph_0} = \aleph_2$,

since now we know that *SH* is true and that all the cardinals discussed in Section III.1 have value $\mathfrak{c}$.

By Exercise V.4.37, the axioms $ZFC+MA+2^{\aleph_0} = \aleph_2$ don't tell you the value of $2^{\aleph_2}$. But more surprisingly, there is a collection of combinatorial statements about sets of size $\aleph_1$ that these axioms don't settle. Most well-known among these is the *Open Coloring Axiom* (*OCA*). We introduce this subject here; further information is in Moore [70] and Todorčević [98] and Veličković [100].

Actually, there are several axioms with the same flavor as *OCA* discussed in the literature. Here, we shall focus just on one of them, called the *Semi Open Coloring Axiom* (*SOCA*). We shall later mention some of the other axioms and give some references to the literature.

The *SOCA* was introduced by Abraham, Rubin, and Shelah [2], who also proved it to be independent of $ZFC + MA + 2^{\aleph_0} = \aleph_2$. It is a statement about partitions of $[E]^2$ (these are often called *colorings*; see page 243). Since a topology on E is relevant here, we shall view the partitions in terms of symmetric subsets of $E \times E$, which has its usual product topology:

Definition V.6.1 *For any set E: Let $E^\dagger = (E \times E) \setminus \{(x,x) : x \in E\}$. Fix $W \subseteq E^\dagger$ with $W = W^{-1}$. Then $T \subseteq E$ is W–*free *iff $T^\dagger \cap W = \emptyset$; T is W–*connected *iff $T^\dagger \subseteq W$.*

Definition V.6.2 *SOCA is the statement: Whenever E is an uncountable separable metric space and W is an open subset of $E^\dagger$ with $W = W^{-1}$, there is a $T \in [E]^{\aleph_1}$ that is either W–connected or W–free.*

Note that we would get an equivalent statement if we restricted *SOCA* to spaces E such that $|E| = \aleph_1$, since we could always replace E by an $\aleph_1$ size subset of E before applying *SOCA*. So, *SOCA* really is a statement just about objects of size $\aleph_1$. It is important that W be open here; if we weakened "open" to "F_σ" or "G_δ", the axiom would become inconsistent (Exercise V.6.24). Of course, if we replace "open" by "closed in $E^\dagger$", we get an equivalent statement.

In the literature, it is common to state *SOCA* and related axioms in terms of subsets of $[E]^2$, which is given the quotient topology inherited from $E^\dagger$ obtained by identifying each (x,y) with (y,x).

SOCA is often used in conjunction with $MA + 2^{\aleph_0} = \aleph_2$, but *SOCA* alone contradicts *CH*, as we shall see by looking at one of its consequences in Euclidean geometry. As usual in $\mathbb{R}^n$, $\|x\|$ denotes the length of x, and S^{n-1} denotes the unit sphere, $\{x \in \mathbb{R}^n : \|x\| = 1\}$.

Definition V.6.3 *Let $\rho : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ be the perpendicular retraction given by $\rho(x) = x/\|x\|$. For $T \subseteq \mathbb{R}^n$, let* $\mathrm{dir}(T) = \{\rho(x-y) : x, y \in T \wedge x \neq y\}$.

Think of each $v \in S^{n-1}$ as a "direction"; when $n = 2$, the circle S^1 is the dial on your compass. Then $\rho(y-x)$ is the direction from x to y, and $\mathrm{dir}(T)$ is the set of all directions between pairs of points of T.

Definition V.6.4 *$T \subseteq \mathbb{R}^n$ is* skinny *iff* $\mathrm{dir}(T)$ *is not dense in* S^{n-1}.

If T is dense in some open set, then it cannot be skinny, so

Lemma V.6.5 *If $T \subseteq \mathbb{R}^n$ is skinny then T is nowhere dense in $\mathbb{R}^n$.*

As usual, a set is *nowhere dense* iff its *closure* has empty interior. We remark that the closure of a skinny set is also skinny.

Definition V.6.6 *Fix $E \subseteq \mathbb{R}^n$. E is a* Luzin set *iff E is uncountable and no uncountable subset of E is nowhere dense in $\mathbb{R}^n$. E is a* weakly Luzin set *iff E is uncountable and no uncountable subset of E is skinny.*

By Lemma V.6.5, every Luzin set is a weakly Luzin set. When $n = 1$, T is skinny iff $|T| \leq 1$, so every uncountable set is weakly Luzin.

The existence of Luzin sets under *CH* was first proved by Luzin in 1914 using the Baire (1899) Category Theorem. This is by now an easy exercise in elementary logic texts (e.g. [57] Exercise I.15.10); one can also check:

Exercise V.6.7 *Assume CH. Show that if X is any separable complete metric space without isolated points, then there is a Luzin set $E \subseteq X$. Then, show that for all $n \geq 2$, there is a weakly Luzin $E \subseteq \mathbb{R}^n$ that is not a Luzin set.*

Hint. Note that the definition of "Luzin set" (unlike "weakly Luzin") makes sense in any topological space. You can construct E in ω_1 steps, using the fact that there are only $\aleph_1$ closed nowhere dense sets. To get the weakly Luzin set, let X be a Cantor subset of $\mathbb{R}^n$ such that no non-empty (relatively) open subset of X is skinny, and let E be Luzin in X. □

A Luzin set is consistent with $\neg CH$ (Exercise V.2.23), but under $MA(\aleph_1)$, there are no Luzin sets, since then every set in $[E]^{\aleph_1}$ is of first category, and hence a countable union of nowhere dense sets.

SOCA implies that there are no weakly Luzin sets in $\mathbb{R}^n$ when $n \geq 2$ (Lemma V.6.10), so in particular $SOCA \to \neg CH$. Also, weakly Luzin sets are consistent with $MA + \mathfrak{c} = \aleph_2$ (Theorem V.6.19), so *SOCA* does not follow from $MA + \mathfrak{c} = \aleph_2$. But *SOCA* is consistent with $MA + \mathfrak{c} = \aleph_2$ (Theorem V.6.17). See [58] for some applications of weakly Luzin sets in topology.

Before proving that *SOCA* refutes weakly Luzin sets, we define a somewhat stronger notion than "skinny".

Definition V.6.8 *For $u, v \in S^{n-1}$, let $\angle(u,v) = 2\arcsin(\|u - v\|/2) \in [0, \pi]$ denote the angle between the vectors $\overrightarrow{0u}$ and $\overrightarrow{0v}$. Then $T \subseteq \mathbb{R}^n$ is* ε–directed *iff for some $v \in S^{n-1}$, $\angle(u,v) \leq \varepsilon$ or $\angle(-u,v) \leq \varepsilon$ for all $u \in \mathrm{dir}(T)$.*

We remark that the closure of an ε–directed set is also ε–directed. Every $T \subseteq \mathbb{R}^n$ is trivially $\pi/2$–directed, and T is 0–directed iff T is contained in a straight line. The curve $y = \sin(x)$ is $\pi/4$–directed (take $v = (1,0)$).

When $n \geq 2$ and $\varepsilon < \pi/2$, every ε–directed set is skinny. When $n = 2$, every skinny set is ε–directed for some $\varepsilon < \pi/2$, but this is false when $n > 2$, a plane in $\mathbb{R}^n$ being a trivial counter-example; informally, "ε–directed" means "almost in a straight line". Returning to *SOCA*,

Lemma V.6.9 *Let E be an uncountable separable metric space with $E^\dagger = \bigcup_{i<\ell} W_i$, where $\ell \in \omega$ and each $W_i = W_i^{-1}$ is open in $E^\dagger$. Assuming SOCA, there is an uncountable $T \subseteq E$ such that T is W_i–connected for some i.*

Proof. The case $\ell = 2$ is obvious from *SOCA*. Now, induct on ℓ. □

Lemma V.6.10 *Assume SOCA. Let E be an uncountable subset of $\mathbb{R}^n$, and fix $\varepsilon > 0$. Then there is an uncountable $T \subseteq E$ that is ε–directed. In particular, when $n \geq 2$, there are no weakly Luzin sets in $\mathbb{R}^n$.*

Proof. By compactness of S^{n-1}, there is a finite ℓ and $\{v_i : i < \ell\} \subset S^{n-1}$ such that $\forall u \in S^{n-1}\, \exists i < \ell\, [\angle(u, v_i) < \varepsilon]$. Now, apply Lemma V.6.9 where

$$W_i = \{(x,y) \in (\mathbb{R}^n)^\dagger : \angle(\rho(x-y), v_i) \leq \varepsilon \vee \angle(\rho(y-x), v_i) \leq \varepsilon\} \ .$$

The "in particular" follows by letting ε be *any* number less than $\pi/2$. □

We now turn to proving that both *SOCA* and $\neg$*SOCA* are consistent with $MA + \mathfrak{c} = \aleph_2$. For both proofs, we may assume *GCH* in the ground model, and obtain $MA + \mathfrak{c} = \aleph_2$ in the usual way. To get *SOCA* to be true in the generic extension, we show that our iteration can also ensure each of the possible $\aleph_2$ instances of *SOCA*. To get *SOCA* to be false in the generic extension, we shall show that we may pick any weakly Luzin set E in the ground model and make sure that E remains weakly Luzin in the generic extension.

Definition V.6.11 *(E, W) is* big *iff E is a separable metric space, $|E| = \aleph_1$, $W = W^{-1}$ is an open subset of $E^\dagger$, and no uncountable subset of E is W–free.*

SOCA asserts that whenever (E, W) is big, there is an uncountable W–connected set. We may try to force one using the poset $\mathbb{C}_{E,W}$:

Definition V.6.12 *Fix any set E and any $W \subseteq E^\dagger$ with $W = W^{-1}$. Let $\mathbb{C}_{E,W} = \{p \in [E]^{<\omega} : \forall x, y \in p\, [x \neq y \to (x,y) \in W]\}$. Define $p \leq q$ iff $p \supseteq q$, so that $\mathbb{1} = \emptyset$.*

This really just restated Definition III.8.20 with slightly different notation. If $G \subseteq \mathbb{C}_{E,W}$ is a filter, then $\bigcup G$ is W–connected. Unfortunately, $\mathbb{C}_{E,W}$ need not be ccc, even when (E, W) is big. Consider the following example:

Exercise V.6.13 *For $x = (x_1, x_2) \in \mathbb{R}^2$, let $x^\curvearrowleft = (-x_2, x_1)$. Assume CH, and prove that there is a Luzin set $E \subseteq \mathbb{R}^2 \backslash \{0\}$ such that*

a. E is the graph of a bijection from $\mathrm{dom}(E)$ *onto* $\mathrm{ran}(E)$.

b. E is fixed under $90°$ rotation: $x \in E \leftrightarrow x^\curvearrowleft \in E$.

By (a), the slope of the line through two points of E is never 0 or ∞. Let W be the set of all $(x,y) \in E^\dagger$ such that the line through x, y has positive slope. Then W is clopen in $E^\dagger$, and all W–connected sets are $\pi/4$–directed (take $v = (\sqrt{2}/2, \sqrt{2}/2)$) and all W–free sets are $\pi/4$–directed, so all W–connected sets and all W–free sets are countable, since E is Luzin and hence weakly Luzin.

Let $\mathbb{C} = \mathbb{C}_{E,W}$. Since E is weakly Luzin, there is no antichain in $\mathbb{C}$ of the form $\{\{x_\alpha\} : \alpha < \omega_1\}$. But, $\mathbb{C}$ is not ccc: Let $A = \{\{x, x^\frown\} : x \in E \wedge (x, x^\frown) \in W\} \subset \mathbb{C}$. A is uncountable, since $(x, x^\frown) \notin W \leftrightarrow (x^\frown, x^{\frown\frown}) \in W$. A is an antichain because $\{x, x^\frown, y, y^\frown\} \notin \mathbb{C}$ unless $x = y$, since if the line through x, y has positive slope, then the line through $x^\frown, y^\frown$ has negative slope.

When we build our model of $MA + \mathfrak{c} = \aleph_2 + \neg SOCA$, we can leave this example intact, so that in the generic extension, E will remain weakly Luzin. Since $\mathbb{C}_{E,W}$ is not ccc, MA will not apply to it. Our model of $MA + \mathfrak{c} = \aleph_2 + SOCA$ will kill this example with the aid of:

Lemma V.6.14 *Assume CH, and let (E,W) be big. Then there is an uncountable $\widetilde{E} \subseteq E$ such that $\mathbb{C}_{\widetilde{E},\widetilde{W}}$ is ccc, where $\widetilde{W} = W \cap \widetilde{E}^\dagger$.*

In our iterated forcing, we shall apply this lemma in the $\mathbb{P}_\xi$ extensions (for $\xi < \omega_2$), where CH holds, to obtain $SOCA$ in the $\mathbb{P}_{\omega_2}$ extension, where $\mathfrak{c} = \aleph_2$.

We shall prove this lemma by using a club of elementary submodels, following the terminology in Definition III.8.14; then $\widetilde{E}$ can be any set of elements of differing heights. The ccc will be easy using:

Lemma V.6.15 *Assume the following:*

1. *CH holds.*
2. *$m \in \omega$; and (E, W_i) is big for each $i \leq m$; and $\mathcal{B}$ is a countable open base for the topology of E.*
3. *θ is a suitably large regular cardinal and $\langle M_\xi : \xi < \omega_1 \rangle$ is a nice chain of elementary submodels of $H(\theta)$ with $E, \mathcal{B} \in M_1$.*
4. *$x^i_\alpha \in E$ for $\alpha < \omega_1$ and $i \leq m$.*
5. *$\mathrm{ht}(x^i_\alpha) \neq \mathrm{ht}(x^j_\beta)$ unless $\alpha = \beta$ and $i = j$.*

Then there are $\alpha \neq \beta$ such that $(x^i_\alpha, x^i_\beta) \in W_i$ for all $i \leq m$.

Remarks. Actually, θ can be $\aleph_2$ here, but as usual in these arguments, the value of θ does not matter. When we prove Lemma V.6.14, all W_i will be W. Some remarks on the M_ξ: These are countable and *not* transitive; e.g., $E \in M_\xi$ and $E \not\subseteq M_\xi$. Also, every closed subset of E is in M_μ for some μ, since by CH, $H(\theta)$, and hence M_1, contains a map Φ from ω_1 onto the set of all closed subsets of E; then $\xi \in M_\mu \to \Phi(\xi) \in M_\mu$. For countable K: $K \in M_\xi \to K \subseteq M_\xi$ (Lemma III.7.12).

Proof of Lemma V.6.15. Induct on m. When $m = 0$, this is immediate from the fact that (E, W_0) is big. Now, assume the lemma for $m-1$, and we prove it for m. Let $\vec{x}_\alpha = \langle x^0_\alpha, \dots, x^m_\alpha\rangle \in E^{m+1}$. Let $\xi(\alpha, i) = \mathrm{ht}(x^i_\alpha)$. Thinning the ω_1–sequence and rearranging each $\vec{x}_\alpha$ if necessary, we may assume that $\xi(\alpha,0) < \xi(\alpha,1) < \cdots < \xi(\alpha,m)$ and that $\alpha < \beta \to \xi(\alpha,m) < \xi(\beta,0)$. Let $F = \mathrm{cl}\{\vec{x}_\alpha : \alpha < \omega_1\} \subseteq E^{m+1}$, and fix $\mu < \omega_1$ such that $F \in M_\mu$.

For $\alpha \geq \mu$: Let $K_\alpha = \{z \in E : \langle x^0_\alpha, \dots, x^{m-1}_\alpha, z\rangle \in F\}$. K_α is uncountable because $K_\alpha \in M_{\xi(\alpha,m)}$ but K_α contains the element $x^m_\alpha \notin M_{\xi(\alpha,m)}$. Since (E, W_m) is big, choose $u_\alpha, v_\alpha \in K_\alpha$ with $(u_\alpha, v_\alpha) \in W_m$, and then choose $U_m, V_m \in \mathcal{B}$ with $u_\alpha \in U_m$, $v_\alpha \in V_m$, $U_m \cap V_m = \emptyset$, and $U_m \times V_m \subseteq W_m$.

Of course, U_m, V_m depend on α, but there are only $\aleph_0$ possible choices, so fix an uncountable set $S \subseteq \{\alpha : \mu \leq \alpha < \omega_1\}$ such that the U_m, V_m are the same for all $\alpha \in S$. By the lemma for $m-1$, fix $\gamma, \delta \in S$ such that $\gamma \neq \delta$ and $(x^i_\gamma, x^i_\delta) \in W_i$ for all $i < m$. Now choose disjoint open neighborhoods U_i of x^i_γ and V_i of x^i_δ for $i < m$ so that $U_i \times V_i \subseteq W_i$. Note that the two open sets $\prod_{i\leq m} U_i$ and $\prod_{i\leq m} V_i$ both meet F, since $u_\gamma \in K_\gamma$ and $v_\delta \in K_\delta$, so $\langle x^0_\gamma, \dots, x^{m-1}_\gamma, u_\gamma\rangle \in F \cap \prod_{i\leq m} U_i$ and $\langle x^0_\delta, \dots, x^{m-1}_\delta, v_\delta\rangle \in F \cap \prod_{i\leq m} V_i$. We may then choose α, β such that $\vec{x}_\alpha \in \prod_{i\leq m} U_i$ and $\vec{x}_\beta \in \prod_{i\leq m} V_i$. But then $(x^i_\alpha, x^i_\beta) \in W_i$ for all i. □

Proof of Lemma V.6.14. Use the notation in Lemma V.6.15, and choose $\widetilde{E}$ such that $\mathrm{ht}(x) \neq \mathrm{ht}(y)$ whenever $x, y \in \widetilde{E}$ and $x \neq y$. Let $\mathbb{C} = \mathbb{C}_{\widetilde{E},\widetilde{W}}$. Fix $A \in [\mathbb{C}]^{\aleph_1}$; we prove that A cannot be an antichain.

Thinning A, we may assume that A forms a delta system with root r, and that $|p|$ is the same for all $p \in A$. Then, if each $|p| = |r| + m + 1$, we may assume that $A = \{p_\alpha : \alpha < \omega_1\}$, where $p_\alpha = r \cup \{x^i_\alpha : i \leq m\}$. Since these form a delta system, $x^i_\alpha \neq x^j_\beta$ and hence $\mathrm{ht}(x^i_\alpha) \neq \mathrm{ht}(x^j_\beta)$, unless $\alpha = \beta$ and $i = j$.

For each α: $p_\alpha \in \mathbb{C}$ implies that $(x^i_\alpha, x^j_\alpha) \in W$ for $i, j \leq m$ with $i \neq j$, so, since W is open, we may choose disjoint basic open $B^i \subseteq E$ for $i \leq m$ such that each $x^i_\alpha \in B^i$ and $B^i \times B^j \subseteq W$ whenever $i, j \leq m$ with $i \neq j$.

The B^i depend on α, but there are only $\aleph_0$ possible choices, so fix an uncountable $S \subseteq \omega_1$ such that the B^i are the same for all $\alpha \in S$. Applying Lemma V.6.15 to the x^i_α for $\alpha \in S$, with all $W_i = W$, fix $\alpha \neq \beta$ with $\alpha, \beta \in S$ such that $(x^i_\alpha, x^i_\beta) \in W$ for all i. But also, if $i, j \leq m$ with $i \neq j$, then $x^i_\alpha \in B^i$ and $x^j_\beta \in B^j$ implies that $(x^i_\alpha, x^j_\beta) \in W$. Thus $p_\alpha \cup p_\beta$ is in $\mathbb{C}$ and extends both p_α and p_β. □

A minor point. If $\mathring{G}$ names the $\mathbb{C}$–generic filter, then $\mathbb{1}$ forces that $\bigcup \mathring{G}$ is W–connected. But to force *SOCA*, we need $\mathbb{1}$ to force $\bigcup \mathring{G}$ to be uncountable. But this could fail. Consider this trivial example: $W = E^\dagger \setminus (\{e\} \times E \cup E \times \{e\})$, where $e \in E$. Then (E, W) is big because no three-element subset of E is W–free, and the poset $\mathbb{C} = \mathbb{C}_{E,W}$ is already ccc. But $\mathbb{C}$ contains all singletons; in particular, $\{e\} \in E$, and $\{e\} \Vdash \bigcup \mathring{G} = \{e\}$. So, we need:

Lemma V.6.16 *Assume that (E, W) is big and $\mathbb{C} = \mathbb{C}_{E,W}$ is ccc. Then there is a $p \in \mathbb{C}$ such that $p \Vdash |\bigcup \dot{G}| = \aleph_1$.*

Hence, if $\mathbb{Q}$ is the poset $p{\downarrow}$ from Definition III.3.33, then $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "there is an uncountable W-connected set".

Proof. If this fails, then $\mathbb{1} \Vdash |\bigcup \dot{G}| \leq \aleph_0$, so there is a name $\dot{f}$ such that $\mathbb{1} \Vdash \dot{f} : \omega \xrightarrow{\text{onto}} \bigcup \dot{G}$. If $J = \{e \in E : \exists p \in \mathbb{C} \exists n \in \omega [p \Vdash \dot{f}(n) = e]\}$, then $\mathbb{1} \Vdash \bigcup \dot{G} \subseteq J$, and J is countable (by the ccc). But this is a contradiction, since if $e \in E \backslash J$, then $\{e\} \in \mathbb{C}$ and $\{e\} \Vdash e \in \bigcup \dot{G}$. □

Theorem V.6.17 *Assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Then there is a ccc poset $\mathbb{P}$ of size $\aleph_2$ such that $\mathbb{1} \Vdash_{\mathbb{P}} MA \wedge \mathfrak{c} = \aleph_2 \wedge SOCA$.*

Proof. This is almost exactly the same as the proof of Theorem V.4.1, which obtained a model for $MA \wedge \mathfrak{c} = \kappa$, so we focus only on the differences. Here, $\kappa = \aleph_2$. As before $f : \omega_2 \xrightarrow[\text{onto}]{1\text{–}1} \omega_2 \times \omega_2$ satisfies $f(\xi) = (\zeta, \mu) \rightarrow \zeta \leq \xi$.

We must arrange for the standard bookkeeping to track all relevant metric spaces as well as all relevant ccc posets. We may assume that the points of the metric space are just the countable ordinals, so that always $E = \omega_1$ and $W \subseteq \omega_1 \times \omega_1$. The topology can be described by a countable base $\{B_n : n \in \omega\}$, where $B \subseteq \omega \times \omega_1$ and $B_n = \{\delta : (n, \delta) \in B\}$.

When we get to stage $\zeta < \omega_2$ in the construction, we need to list names for metric spaces as well as names for ccc posets. Let $\mathfrak{E}$ be the set of even ordinals (of form "limit + $2n$") less than ω_2 and let $\mathfrak{O} = \omega_2 \backslash \mathfrak{E}$ (the odd ordinals). Let the sequence $\langle (\alpha_\zeta^\mu, \mathring{\trianglelefteq}_\zeta^\mu) : \mu \in \mathfrak{E} \rangle$ list all pairs $(\alpha, \mathring{\trianglelefteq})$ such that $0 < \alpha < \omega_2$ and $\mathring{\trianglelefteq}$ is a nice $\mathbb{P}_\zeta$–name for a subset of $\alpha \times \alpha$. Let $\langle (\dot{B}_\zeta^\mu, \dot{W}_\zeta^\mu) : \mu \in \mathfrak{O} \rangle$ list all pairs $(\dot{B}, \dot{W})$ such that $\dot{B}$ is a nice $\mathbb{P}_\zeta$–name for a subset of $\omega \times \omega_1$ and $\dot{W}$ is a nice $\mathbb{P}_\zeta$–name for a subset of $\omega_1 \times \omega_1$.

Fix $\xi < \omega_2$, and let $f(\xi) = (\zeta, \mu)$. If $\mu \in \mathfrak{E}$, we obtain $\mathring{\mathbb{Q}}_\xi$ exactly as before. Now, assume that $\mu \in \mathfrak{O}$, and let $\dot{B} = \dot{B}_\zeta^\mu$ and $\dot{W} = \dot{W}_\zeta^\mu$. Then, let $\mathring{\mathbb{Q}}_\xi$ name the one-element poset $\{0\}$ (with $\mathbb{1} = 0$) *unless* $\mathbb{1} \Vdash_{\mathbb{P}_\xi}$ "$\{\dot{B}_n : n \in \omega\}$ is a base for a metric topology on ω_1 and $(\omega_1, \dot{W})$ is big with respect to that topology". In that case, let $\mathring{\mathbb{Q}}_\xi$ name the poset constructed as above, working in the $\mathbb{P}_\xi$–extension. That is, using (E, W) for $(\omega_1, \dot{W})$, we apply Lemma V.6.14 to obtain a suitable $(\widetilde{E}, \widetilde{W})$, and then use Lemma V.6.16 to extract a $\mathbb{Q}$ from $(\widetilde{E}, \widetilde{W})$. Lemma V.6.14 applies in the $\mathbb{P}_\xi$–extension because $|\mathbb{P}_\xi| < \aleph_2$, so $\mathbb{1} \Vdash_{\mathbb{P}_\xi} CH$.

Note that it is important that we obtain $\mathring{\mathbb{Q}}_\xi$ from (E, W) in the $\mathbb{P}_\xi$ extension, when we force with it, and not in the earlier $\mathbb{P}_\zeta$ extension, when we listed it, so that $\mathbb{1} \Vdash_{\mathbb{P}_\xi} \mathring{\mathbb{Q}}_\xi$ is ccc. If we build $\mathring{\mathbb{Q}}_\xi$ in the $\mathbb{P}_\zeta$ extension, then $\mathbb{1} \Vdash_{\mathbb{P}_\zeta} \mathring{\mathbb{Q}}_\xi$ is ccc, but this ccc property could be lost by the time we get to the $\mathbb{P}_\xi$ extension. □

Now, to build a model of $MA \wedge \mathfrak{c} = \aleph_2 \wedge \neg SOCA$, we can start with a weakly Luzin set E in the ground model and get E to remain weakly Luzin

in some generic extension satisfying $MA \wedge \mathfrak{c} = \aleph_2$. We shall actually prove a more general result that does not mention geometry. The definition of "weakly Luzin" implies:

Lemma V.6.18 *$E \subseteq \mathbb{R}^n$ is weakly Luzin iff (E, W) is big whenever $W = \{(x,y) \in E^\dagger : \rho(x-y) \in U \vee \rho(y-x) \in U\}$ and U is a non-empty open subset of S^{n-1}.*

Then, we shall prove:

Theorem V.6.19 *Assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ and E is a separable metric space of size $\aleph_1$. Then there is a ccc poset $\mathbb{P}$ of size $\aleph_2$ such that*

1. *$\mathbb{1} \Vdash_{\mathbb{P}} MA \wedge \mathfrak{c} = \aleph_2$, and*
2. *For* all *Ω: If (E, Ω) is big then $\mathbb{1} \Vdash_{\mathbb{P}}$ "(E, Ω) is big".*

In particular, if $E \subseteq \mathbb{R}^n$ is a weakly Luzin set, then it remains weakly Luzin in the generic extension, so that $\mathbb{1} \Vdash_{\mathbb{P}} \neg SOCA$.

Note that the "in particular" follows from Lemma V.6.18 and the observation that every non-empty $U \subseteq S^{n-1}$ in the generic extension contains a non-empty open subset from the ground model.

To prove the theorem, we shall iterate forcing in the usual way to make $MA \wedge \mathfrak{c} = \aleph_2$ true, as in the proof of Theorem V.4.1. But part (2) of Theorem V.6.19 will require two kinds of lemmas about our iterated forcing:

A. The "big sets remain big" is preserved at limits.
B. We don't make a big set non-big at a successor stage.

Item (A) is handled by the next two lemmas. These have nothing to do with the topology on E, so we state them as results on abstract partitions:

Lemma V.6.20 *Fix any $W \subseteq (\omega_1)^\dagger$ with $W = W^{-1}$. Let $\mathbb{P}$ be a ccc forcing poset. Then the following are equivalent:*

a. For some $p \in \mathbb{P}$: $p \Vdash \exists T \in [\omega_1]^{\aleph_1}$ [T is W-free]
b. For some $J \in [\omega_1]^{\aleph_1}$, there are $p_\mu \in \mathbb{P}$ for $\mu \in J$ such that for all $\mu, \nu \in J$ with $\mu \neq \nu$: $p_\mu \not\perp p_\nu \to (\mu, \nu) \notin W$.

Proof. For $(a) \to (b)$, say $\mathring{T}$ is a name and p forces "$\mathring{T} \in [\omega_1]^{\aleph_1}$ and $\mathring{T}$ is W-free". Let $J = \{\mu : \exists q \leq p\, [q \Vdash \mu \in \mathring{T}]\}$. Then, for $\mu \in J$, choose $p_\mu \leq p$ such that $p_\mu \Vdash \mu \in \mathring{T}$.

For $(b) \to (a)$, let $\mathring{T}$ be the name $\{\langle \check{\mu}, p_\mu \rangle : \mu \in J\}$. Then $\mathbb{1} \Vdash$ "$\mathring{T}$ is W-free". Also, some $p \in \mathbb{P}$ forces $|\mathring{T}| = \aleph_1$ (this is like the proof of Lemma V.6.16). □

Applying (b) and Lemma V.3.16:

Lemma V.6.21 *Fix $W \subseteq (\omega_1)^\dagger$ with $W = W^{-1}$. In a finite support ccc iteration, using the notation of Definition V.3.11, assume that α is a limit ordinal and assume that for all $\xi < \alpha$, $\mathbb{1} \Vdash_{\mathbb{P}_\xi} \neg\exists T \in [\omega_1]^{\aleph_1}$ [T is W-free]. Then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha} \neg\exists T \in [\omega_1]^{\aleph_1}$ [T is W-free].*

For item (B), the main lemma is the following. Roughly, we're saying that if some ccc poset $\mathbb{D}$ makes a big set non-big, then in our *MA* construction, we can force with a different ccc poset $\mathbb{Q}$ that does preserve bigness and that also forces $\mathbb{D}$ to become not ccc, so that *MA* does not apply to $\mathbb{D}$.

Lemma V.6.22 *Assume CH, let (E, W) be big, and let $\mathbb{D}$ be any forcing poset such that $d \Vdash_{\mathbb{D}}$ "(E, W) is not big" for some $d \in \mathbb{D}$. Then there is a ccc poset $\mathbb{Q}$ of size $\aleph_1$ such that:*

1. *$\mathbb{Q} \times \mathbb{D}$ is not ccc*
2. *For* all *Ω: If (E, Ω) is big then $\mathbb{1} \Vdash_{\mathbb{Q}}$ "(E, Ω) is big".*
3. *$\mathbb{1} \Vdash_{\mathbb{Q}}$ "$\check{\mathbb{D}}$ is not ccc".*

Proof. First observe that it is sufficient to obtain (1) and (2). Then, by Exercise V.2.31, (1) implies $\mathbb{1} \not\Vdash_{\mathbb{Q}}$ "$\check{\mathbb{D}}$ is ccc", so there is a $q \in \mathbb{Q}$ such that $q \Vdash_{\mathbb{Q}}$ "$\check{\mathbb{D}}$ is not ccc". Then, to get (3), replace $\mathbb{Q}$ by $q{\downarrow}$ (Definition III.3.33).

Fix a $\mathbb{D}$-name $\mathring{Z}$ such that $d \Vdash$ "$\mathring{Z} \subseteq E$ is uncountable and W-free". Let $\mathcal{B}$ be a countable open base for the topology of E. Fix θ and the M_ξ so that (3) of Lemma V.6.15 holds.

Recursively choose $d_\alpha \leq d$ and $x^0_\alpha, x^1_\alpha \in E$ for $\alpha < \omega_1$ so that $d_\alpha \Vdash x^0_\alpha, x^1_\alpha \in \mathring{Z}$ and such that $\mathrm{ht}(x^0_\alpha) < \mathrm{ht}(x^1_\alpha) < \mathrm{ht}(x^0_\beta)$ whenever $\alpha < \beta < \omega_1$. Let

$$\mathbb{Q} = \left\{ q \in [\omega_1]^{<\omega} : \forall \{\alpha, \beta\} \in [q]^2 \left[(x^0_\alpha, x^0_\beta) \in W \text{ or } (x^1_\alpha, x^1_\beta) \in W \right] \right\} \quad .$$

$\mathbb{Q}$ is ordered by reverse inclusion, with $\mathbb{1} = \emptyset$. Each $\{\alpha\} \in \mathbb{Q}$, and the pairs $(\{\alpha\}, d_\alpha) \in \mathbb{Q} \times \mathbb{D}$ are incompatible, so $\mathbb{Q} \times \mathbb{D}$ is not ccc.

Now, suppose that we have some big (E, Ω) and $q \Vdash_{\mathbb{Q}}$ "(E, Ω) is not big"; we shall derive a contradiction. Fix a $\mathbb{Q}$-name $\mathring{T}$ such that $q \Vdash$ "$\mathring{T} \subseteq E$ is uncountable and Ω-free". Then, recursively choose $q_\mu \leq q$ and $t_\mu \in E$ for $\mu < \omega_1$ so that $q_\mu \Vdash t_\mu \in \mathring{T}$ and such that $\mathrm{ht}(t_\mu) < \mathrm{ht}(t_\nu)$ whenever $\mu < \nu < \omega_1$. Our contradiction will use the observation:

$$\mu \neq \nu \to (t_\mu, t_\nu) \notin \Omega \text{ or } q_\mu \perp q_\nu \quad . \tag{$*$}$$

Thinning the sequence and extending q if necessary, we may assume that the q_μ form a delta system with root q; so $q_\mu = q \cup \{\alpha(0,\mu), \ldots, \alpha(c,\mu)\}$, with $\alpha(0,\mu) < \ldots < \alpha(c,\mu)$. We also assume that $\max(q) < \alpha(0,0)$ and $\mu < \nu \to \alpha(c,\mu) < \alpha(0,\nu)$. Since $q_\mu \in \mathbb{Q}$,

$$i \neq j \to (x^0_{\alpha(i,\mu)}, x^0_{\alpha(j,\mu)}) \in W \text{ or } (x^1_{\alpha(i,\mu)}, x^1_{\alpha(j,\mu)}) \in W$$

for each μ. Since W is open and E is second countable, we may thin again and assume that whenever some $(x^\ell_{\alpha(i,\mu)}, x^\ell_{\alpha(j,\mu)}) \in W$, there are basic open sets $A^\ell_{i,j}, B^\ell_{i,j}$ (independently of μ) such that $x^\ell_{\alpha(i,\mu)} \in A^\ell_{i,j}$ and $x^\ell_{\alpha(j,\mu)} \in B^\ell_{i,j}$ and $A^\ell_{i,j} \times B^\ell_{i,j} \subseteq W$. Then, for all μ, ν:

$$i \neq j \to (x^0_{\alpha(i,\mu)}, x^0_{\alpha(j,\nu)}) \in W \text{ or } (x^1_{\alpha(i,\mu)}, x^1_{\alpha(j,\nu)}) \in W \ .$$

Thus, if $q_\mu \perp q_\nu$ then the incompatibility must come from the same index i, so that $(*)$ becomes

$$\mu \neq \nu \to (t_\mu, t_\nu) \notin \Omega \ \vee \ \exists i \leq c \left[(x^0_{\alpha(i,\mu)}, x^0_{\alpha(i,\nu)}) \notin W \wedge (x^1_{\alpha(i,\mu)}, x^1_{\alpha(i,\nu)}) \notin W \right] .$$

This comes close to contradicting Lemma V.6.15. With an eye to satisfying hypothesis (5), we thin the sequence again and assume that $\mathrm{ht}(t_\mu) \neq \mathrm{ht}(x^\ell_{\alpha(i,\nu)})$ whenever $\mu \neq \nu$. It is still possible to have $\mathrm{ht}(t_\mu) = \mathrm{ht}(x^\ell_{\alpha(i,\mu)})$, but for each μ, $\mathrm{ht}(t_\mu) = \mathrm{ht}(x^\ell_{\alpha(i,\mu)})$ can hold for at most one pair (ℓ, i). Thinning once more, we can assume WLOG that this ℓ is always 1, so that $\mathrm{ht}(t_\mu) \neq \mathrm{ht}(x^0_{\alpha(i,\nu)})$ for all $\mu < \omega_1$ and all $i \leq c$. But now the $(c+2)$–tuples $(t_\mu, x^0_{\alpha(0,\mu)}, \ldots x^0_{\alpha(c,\mu)})$ (for $\mu < \omega_1$) contradict Lemma V.6.15, where $W_0 = \Omega$ and the other $W_i = W$.

We also need to show that $\mathbb{Q}$ is ccc. If this fails, then let $\{q_\mu : \mu < \omega_1\}$ be an antichain. Derive a contradiction as before, but replace $(*)$ by the stronger fact $\mu \neq \nu \to q_\mu \perp q_\nu$, and delete all mention of Ω, $\mathring{T}$, and the t_μ. □

The following trivial restatement of Lemma V.6.22 will be useful in the proof of Theorem V.6.19:

Lemma V.6.23 *Assume CH, and fix an uncountable separable metric space E. Call a forcing poset $\mathbb{Q}$* nice *iff $\mathbb{Q}$ is ccc and for all Ω such that (E, Ω) is big, $\mathbb{1} \Vdash_{\mathbb{Q}}$ "(E, Ω) is big". Let $\mathbb{D} = (\omega_1, \leqslant_{\mathbb{D}}, 0)$ be a forcing poset. Then there is a nice poset $\mathbb{Q}$ (which we shall call* niceify($\mathbb{D}$)*), of the form $(\omega_1, \leqslant, 0)$, such that either $\mathbb{D}$ is nice and $\mathbb{Q} = \mathbb{D}$, or $\mathbb{D}$ is not nice and $\mathbb{1} \Vdash_{\mathbb{Q}}$ "$\mathbb{D}$ is not ccc".*

Proof of Theorem V.6.19. This is very similar to the proofs of Theorem V.6.17 and Theorem V.4.1, so we focus only on the differences.

At stage ζ, we only list names for posets, so we have $\langle (\omega_1, \mathring{\leqslant}^\mu_\zeta) : \mu < \omega_2 \rangle$, exactly as in the proof of Theorem V.4.1 with $\kappa = \omega_2$, except that we have simplified the "α^μ_ζ" in that proof to just ω_1, since it is easy to see that $MA(\aleph_1)$ is equivalent to "$MA_{\mathbb{Q}}(\aleph_1)$ holds for all ccc posets $\mathbb{Q}$ of size exactly $\aleph_1$".

We need to prove by induction on ξ that each $\mathbb{P}_\xi$ is nice, in the terminology of Lemma V.6.23. In particular, the fact that $\mathbb{P}_{\omega_2}$ is nice will give us part (2) of our theorem. But observe that to conclude that all $\mathbb{P}_\xi$ are nice, it is sufficient to guarantee that for each $\xi < \alpha$, $\mathbb{1}_\xi \Vdash [\mathring{\mathbb{Q}}_\xi \text{ is ccc}]$ and $\mathbb{1}_\xi \Vdash [\mathring{\mathbb{Q}}_\xi \text{ is nice}]$. Then, all $\mathbb{P}_\xi$ will be ccc by Lemma V.3.17. Also, if $\mathbb{P}_\xi$ is nice then the niceness of $\mathbb{P}_\xi * \mathring{\mathbb{Q}}_\xi$ and hence of $\mathbb{P}_{\xi+1}$ is clear from the definition of "nice", and niceness cannot first fail at a limit ordinal by Lemma V.6.21.

Fix $\xi < \omega_2$, and let $f(\xi) = (\zeta, \mu)$. We need to determine $\mathring{\mathbb{Q}}_\xi$. As in the proof of Theorem V.4.1, if $\mathrm{Ntbc}(\omega_1, i_*(\mathring{\trianglelefteq}_\zeta^\mu), \mathbb{P}_\xi)$ is false, let $\mathring{\mathbb{Q}}_\xi$ name the one-element poset, so assume that $\mathrm{Ntbc}(\omega_1, i_*(\mathring{\trianglelefteq}_\zeta^\mu), \mathbb{P}_\xi)$ is true. Let $(\mathring{\mathbb{D}}_\xi, \mathring{\leq}_{\mathring{\mathbb{D}}_\xi}, \mathring{\mathbb{1}}_{\mathring{\mathbb{D}}_\xi})$ be $(\check{\omega}_1, i_*(\mathring{\trianglelefteq}_\zeta^\mu), \check{0})$. If we always let $\mathring{\mathbb{Q}}_\xi$ be $\mathring{\mathbb{D}}_\xi$, we would guarantee $MA \wedge \mathfrak{c} = \aleph_2$ in the generic extension, but we might lose niceness. Instead, let $\mathring{\mathbb{Q}}_\xi$ (i.e, $(\omega_1, \mathring{\leq}_{\mathring{\mathbb{Q}}_\xi}, 0)$) be chosen so that $(\omega_1, \mathring{\leq}_{\mathring{\mathbb{Q}}_\xi}, 0)$ is a $\mathbb{P}_\xi$–name for niceify$(\mathring{\mathbb{D}}_\xi)$ (so, we are applying Lemma V.6.23 within the $\mathbb{P}_\xi$–extension). □

We remark that our discussion of *SOCA* yields another proof from *CH* that there are two ccc posets whose product is not ccc (Theorem III.8.19). Let $E \subseteq \mathbb{R}^2 \backslash \{0\}$ be a Luzin set such that E is the graph of a bijection from dom(E) onto ran(E) (as in Exercise V.6.13). Let W_0 be the set of all $(x, y) \in E^\dagger$ such that the line through x, y has positive slope and let W_1 be the set of all $(x, y) \in E^\dagger$ such that the line through x, y has negative slope. Applying Lemma V.6.14 twice, we may shrink E and assume that both posets $\mathbb{C}_{E,W_0}$ and $\mathbb{C}_{E,W_1}$ are ccc (of course, then (b) of Exercise V.6.13 cannot be true). But $\{(\{e\}, \{e\}) : e \in E\}$ is an uncountable antichain in $\mathbb{C}_{E,W_0} \times \mathbb{C}_{E,W_1}$.

Exercise V.6.24 *Prove that there is a separable metric space E of size $\aleph_1$ and a $W \subseteq E^\dagger$ with $W = W^{-1}$ such that W is both an F_σ and a G_δ set and no $T \in [E]^{\aleph_1}$ is either W–connected or W–free.*

Hint. By the example of Sierpiński (Lemma III.8.10), $\omega_1 \not\to (\omega_1)^2_2$. Translating that to our current terminology, there is a $W \subseteq \omega_1 \times \omega_1$ with $W = W^{-1}$ such that no $T \in [\omega_1]^{\aleph_1}$ is either W–connected or W–free, so let $E = \omega_1$ with an appropriate topology.

Choose $a_\alpha, b_\alpha \in [\omega]^\omega$ such that $a_\alpha \perp a_\beta \leftrightarrow (\alpha, \beta) \in W \leftrightarrow b_\alpha \not\perp b_\beta$ whenever $\alpha \neq \beta$. Then, identify ω_1 with $\{(a_\alpha, b_\alpha) : \alpha < \omega_1\} \subset \mathcal{P}(\omega) \times \mathcal{P}(\omega)$, and identify $\mathcal{P}(\omega)$ with the Cantor set 2^ω.

In choosing the a_α, it might help to assume inductively that all sets of the form $a_\alpha \backslash (a_{\xi_0} \cup a_{\xi_1} \cup \cdots \cup a_{\xi_n})$ are infinite when $\alpha \notin \{\xi_0, \xi_1, \ldots, \xi_n\}$. □

We now comment briefly on several variants of the *SOCA*. Most well-known is the *Open Coloring Axiom* (*OCA*), but there are two (non-equivalent) versions of the *OCA*, namely $OCA_{[ARS]}$ (called *OCA* in Abraham, Rubin, and Shelah [2]) and $OCA_{[T]}$ (called *OCA* in Todorčević [98]). The terminology $OCA_{[ARS]}$ and $OCA_{[T]}$ is due to Moore [70], who showed that $OCA_{[ARS]} + OCA_{[T]}$ implies $\mathfrak{c} = \aleph_2$. Variations on the argument given above show that $MA + \mathfrak{c} = \aleph_2$ is consistent with $OCA_{[ARS]} + OCA_{[T]} + SOCA$, but $MA + \mathfrak{c} > \aleph_2 + SOCA$ is also consistent by [2].

Definition V.6.25 *$OCA_{[T]}$ is the assertion: Whenever E is a separable metric space and $W \subseteq E^\dagger$ is open with $W = W^{-1}$, then either:*

a. $E = \bigcup_{n \in \omega} T_n$ for some T_n, each of which is W–free or

b. There is a $T \in [E]^{\aleph_1}$ that is W–connected.

Definition V.6.26 *$OCA_{[ARS]}$ is the assertion: Whenever E is a separable metric space and $|E| = \aleph_1$ and $E^\dagger = \bigcup_{i<r} W_i$, where $r \in \omega$ and each $W_i = W_i^{-1}$ is open in $E^\dagger$, then $E = \bigcup_{n\in\omega} T_n$, where each T_n is W_{i_n}-connected for some $i_n < r$.*

Both of these above statements easily are seen to contradict *CH*, since *CH* implies that there is a counter-example to *SOCA* with a clopen W. The paper [2] discusses other variants on *OCA*, including something like $OCA_{[T]}$, except that they included the assumption that $|E| = \aleph_1$, which Todorčević [98] realized could be dropped, resulting in a stronger axiom. It is easy to see that the $|E| = \aleph_1$ cannot be dropped in $OCA_{[ARS]}$; for example, let $E = \mathbb{R}^2$, let $r = 2$, and let $W_i = \{(x,y) \in E^\dagger : \rho(x-y) \in U_i\}$, where each U_i is a non-empty open subset of S^1 with $U_i = -U_i$, U_i not dense in S^1, and $U_0 \cup U_1 = S^1$. Then every W_i-connected set is nowhere dense, so $\bigcup_{n\in\omega} T_n$ cannot equal E by the Baire Category Theorem.

What does the "Semi" in "Semi Open Coloring Axiom" modify? In [2], which defined *SOCA* and $OCA_{[ARS]}$, they were probably thinking of the coloring (or partition) of $E^\dagger$ in *SOCA* as "semi-open" because one color, W, is open, and the other color, $E^\dagger \backslash W$, is closed in $E^\dagger$. In their *OCA*, all the colors, W_i, are open, although they can overlap. However, if you think of $OCA_{[T]}$ as your *OCA*, then it and *SOCA* both involve just W and $E^\dagger \backslash W$, but *SOCA* is obviously a weakening of $OCA_{[T]}$, so you can think of *SOCA* as a semi-*OCA*.

Exercise V.6.27 *Suppose we replace "open" by "closed" in the statement of $OCA_{[ARS]}$. Then the axiom becomes inconsistent.*

Hint. Let $E = A \times A \subseteq \mathbb{R}^2$, where $A \in [\mathbb{R}]^{\aleph_1}$. For $(x,y) \in E^\dagger$, let $L(x,y)$ be the line through x, y. Put (x,y) in W_0 iff the slope of $L(x,y)$ is ≥ 0 or infinite, and in W_1 iff the slope is ≤ 0 or infinite. □

Exercise V.6.28 *Call a topological space κ-dense iff it and every non-empty open set has size κ. Now, assume that $\mathfrak{p} > \kappa \geq \aleph_0$, and let X, Y be two κ-dense separable metric spaces. Prove that X and Y are homeomorphic. Furthermore, if X, Y are dense in 2^ω, then the homeomorphism can be the restriction of a homeomorphism of 2^ω onto itself.*

Hint. Force a homeomorphism, and then note that the natural poset is σ-centered, so apply Bell's Theorem III.3.61.

Let $\mathcal{A}, \mathcal{B}$ be countable clopen bases for X, Y respectively; obtaining these just uses $\mathfrak{c} > \kappa$, since the continuous real-valued functions separate closed sets. Assume that $\mathcal{A}, \mathcal{B}$ are boolean sub-algebras of $\mathcal{P}(X), \mathcal{P}(Y)$ respectively.

Naive attempts to force with finite partial functions from X to Y won't even be ccc. Instead fix $\varphi : X \to \kappa$ and $\psi : Y \to \kappa$ such that for all $\alpha < \kappa$, $\varphi^{-1}\{\alpha\}$ is countable and dense in X and $\psi^{-1}\{\alpha\}$ is countable and dense in

Y. Now, let $p \in \mathbb{P}$ be a pair (σ_p, h_p), where $\sigma_p \in \mathrm{Fn}(X, Y)$ is a 1-1 function satisfying $\varphi(x) = \psi(y)$ for all $(x, y) \in \sigma_p$; and $h_p \in \mathrm{Fn}(\mathcal{A}, \mathcal{B})$ is an isomorphism from a finite sub-algebra of $\mathcal{A}$ onto a finite sub-algebra of $\mathcal{B}$; and, of course, if $(x, y) \in \sigma_p$ and $(U, V) \in h_p$, then $x \in U$ iff $y \in V$. □

Remarks: For $\kappa = \aleph_0$, this is a classical *ZFC* fact; X and Y are both homeomorphic to the rational numbers $\mathbb{Q}$. If $X, Y \subseteq \mathbb{R}$, and are countable and dense in $\mathbb{R}$, then (by Cantor) X and Y are actually *isomorphic* to $\mathbb{Q}$. Now, say X, Y are $\aleph_1$–dense subsets of $\mathbb{R}$. Then $\mathfrak{p} > \aleph_1$ implies that X, Y are homeomorphic, but the assertion that X, Y must be isomorphic is independent of $MA(\aleph_1)$. The assertion that all such X, Y *are* isomorphic is sometimes called Baumgartner's Axiom *BA* (completely unrelated to the *BA* of Definition V.5.12) because Baumgartner first proved the consistency of $MA + \mathfrak{c} = \aleph_2 + BA$ (see [5]) by methods which led to the results of this section. It is also consistent to have $MA + \mathfrak{c} = \aleph_2 + \neg BA$ (see [2]).

The next exercise describes another way to produce weakly Luzin sets that are not Luzin:

Exercise V.6.29 *A* Sierpiński set *is a subset $E \subseteq \mathbb{R}^n$ such that E is uncountable and no uncountable subset of E is a Lebesgue null set in $\mathbb{R}^n$. Prove that Sierpiński sets exist under CH and can't exist under $MA(\aleph_1)$. Then show in ZFC that every Sierpiński set is a weakly Luzin set and not a Luzin set.*

Exercise V.6.30 *Prove that Sierpiński sets exist in the random real model, formed by forcing with $\mathbb{MB}(2^{\kappa\times\omega}, \mu)$, where $\kappa \geq \aleph_1$ (see Definition IV.7.33).*

Hint. See Exercise V.2.23. □

Exercise V.6.31 *Prove the consistency of $MA + \mathfrak{c} = \aleph_2 + OCA_{[T]}$.*

Hint. This is like Theorem V.6.17, but strengthening the *SOCA* there to $OCA_{[T]}$ requires two important changes.

First, to force a particular instance of $OCA_{[T]}$ (assuming *CH*), you need to modify Lemma V.6.14. Now, the assumption is that E is not a countable union of W–free sets. Since W is open, the closure of a W–free set is W–free. Since there are only $\mathfrak{c} = \aleph_1$ closed sets, a modification of the Luzin set construction produces an $E' \in [E]^{\aleph_1}$ in which every W–free set is countable. One can now apply Lemma V.6.14 to E'.

Second, in $OCA_{[T]}$, E is an arbitrary separable metric space, so in the generic extension, $|E| \leq \mathfrak{c} = \aleph_2$. In the *SOCA* proof, $|E| = \aleph_1$, so any such E in the $\mathbb{P}_{\omega_2}$–extension occurred already in some $\mathbb{P}_\zeta$–extension for $\zeta < \omega_2$, and E gets handled by the standard bookkeeping. For $OCA_{[T]}$, you can capture all relevant E by assuming that the ground model satisfies $\diamondsuit(\omega_2)$ on the ω_1–limits; for example, you can assume $V = L$ in the ground model (see Exercise III.7.18). □

In Lemma V.6.22 (and hence in Theorem V.6.19), for a given E we were able to preserve *all* big (E,Ω); this argument works because in the elementary submodels, we only required $E \in M_1$, not $\Omega \in M_1$. It is natural to ask whether we could handle *all* E as well; that is, replace the "For *all* Ω: If (E,Ω) is big $\cdots\cdots$" in (2) of Lemma V.6.22 and Theorem V.6.19 by "For *all* $\widetilde{E},\Omega$: If $(\widetilde{E},\Omega)$ is big $\cdots\cdots$". This turns out to be impossible:

Exercise V.6.32 *Assume CH, and let $\mathbb{P}$ be any ccc poset such that $\mathbb{1} \Vdash_{\mathbb{P}} MA(\aleph_1)$. Then there are some E, W, with W clopen in $E^\dagger$ and (E,W) and $(E, E^\dagger \backslash W)$ both big, such that $\mathbb{1} \Vdash_{\mathbb{P}}$ "(E,W) is not big".*

A hint for this exercise will be outlined below. Perhaps this result is not surprising, since assuming *GCH*, there are $\aleph_2$ separable metric spaces E of size $\aleph_1$, but for a given E of size $\aleph_1$, there are only $\aleph_1$ possible open sets Ω. One can handle $\aleph_1$ metric spaces:

Exercise V.6.33 *Say E_ξ for $\xi < \omega_1$ are separable metric spaces of size $\aleph_1$. Then in (2) of Lemma V.6.22 and Theorem V.6.19, one can replace the "For* all Ω*: If (E,Ω) is big $\cdots$" by "For* all $\xi < \omega_1$ *and all Ω: If (E_ξ,Ω) is big $\cdots$"*

Hint. Just make sure that each $E_\xi \in M_{\xi+1}$. □

As a start on Exercise V.6.32, observe:

Exercise V.6.34 *Let (E,W) be big with W clopen in $E^\dagger$, and let $\mathbb{P}$ be a ccc poset such that $\mathbb{P} \times \mathbb{C}_{E,W}$ is ccc and $\mathbb{1} \Vdash_{\mathbb{P}} MA(\aleph_1)$. Then $\mathbb{1} \Vdash_{\mathbb{P}}$ "$(E, E^\dagger \backslash W)$ is not big".*

Hint. In the $\mathbb{P}$-extension, $\mathbb{C}_{E,W}$ remains ccc, so *MA* applies to it. □

By Lemma V.6.14, it is easy to get *some* big (E,W) for which $\mathbb{C}_{E,W}$ is ccc, but then by Theorem V.6.19, a $\mathbb{P}$ that forces $MA(\aleph_1)$ may well have $\mathbb{P} \times \mathbb{C}_{E,W}$ not ccc. But then we can use an antichain in $\mathbb{P} \times \mathbb{C}_{E,W}$ to satisfy Exercise V.6.32 with some $(\widetilde{E},\widetilde{W})$, where $\widetilde{E}$ is contained in some E^m.

Definition V.6.35 *Suppose that $W \subseteq E^\dagger$ and $W = W^{-1}$. For $m \in \omega$, define $W^{(m)} = \{(\vec{x},\vec{y}) \in (E^m)^\dagger : \forall i < m\, [(x_i, y_i) \in W]\}$.*

Note that $W^{(m)} = (W^{(m)})^{-1}$, and $W^{(m)}$ is open whenever W is open. But, for a fairly trivial reason,

Remark V.6.36 *If $m \geq 2$ and $|E| > \aleph_0$ and $W \subseteq E^\dagger$, then $(E^m, W^{(m)})$ is not big.*

Proof. When $m = 2$: $((x,c),(x',c)) \in (E^2)^\dagger \setminus W^{(2)}$ whenever $x \neq x'$. Note that $(\vec{x},\vec{y}) \in (E^m)^\dagger \to \exists i\, [x_i \neq y_i]$, while $(\vec{x},\vec{y}) \in W^{(m)} \to \forall i\, [x_i \neq y_i]$. □

A less trivial question is whether various $(Z,\ W^{(m)} \cap Z)$ can be big where Z is *spaced*:

Definition V.6.37 *$Z \subseteq E^m$ is* spaced *iff for all $\vec{x}, \vec{y} \in Z$ and $i, j < m$: $x_i \neq y_j$ unless $\vec{x} = \vec{y}$ and $i = j$.*

Exercise V.6.38 *If W is clopen in $E^\dagger$ and $Z \subseteq E^m$ is spaced, then $W^{(m)} \cap Z^\dagger$ is clopen in $Z^\dagger$.*

Exercise V.6.39 *Assume that W is clopen in $E^\dagger$, where E is a separable metric space. Let $\mathbb{P}$ be a ccc poset such that $\mathbb{P} \times \mathbb{C}_{E,W}$ is not ccc. Then there is a spaced $Z \subseteq E^m$ and a $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ "$(Z,\ W^{(m)} \cap Z^\dagger)$ is not big".*

Hint. Applying the Delta System Lemma, get an antichain in $\mathbb{P} \times \mathbb{C}_{E,W}$ of the form $\{(p^\alpha, \{x_0^\alpha, \dots, x_{m-1}^\alpha\}) : \alpha < \omega_1\}$, where $x_i^\alpha \neq x_j^\beta$ unless $\alpha = \beta$ and $i = j$. Let $Z = \{\vec{x}^\alpha : \alpha < \omega_1\}$. Note that when $\alpha \neq \beta$, either $p^\alpha \perp p^\beta$ or $(\vec{x}^\alpha, \vec{x}^\beta) \notin W^{(m)}$. Let $\mathring{T}$ be a name for the subset of Z that contains $\vec{x}^\alpha$ iff $p^\alpha \in \mathring{G}$. Then $\mathbb{1} \Vdash$ "$\mathring{T}$ is $W^{(m)}$–free" and some p forces "$\mathring{T}$ is uncountable". □

This, together with Exercise V.6.34, would yield Exercise V.6.32 *if* we can arrange for all such $(Z, Z^\dagger \cap W^{(m)})$, together with $(E, E^\dagger \setminus W)$, to be big in the ground model. We can do this using our Luzin set examples.

Say E is a Luzin set in $\mathbb{R}$. Can E^2 be Luzin in $\mathbb{R}^2$? Trivially, no, since $\{(x,x) : x \in E\}$ and $\{(x,c) : x \in E\}$ are nowhere dense in $\mathbb{R}^2$. But, avoiding such trivialities, we do get a non-trivial notion related to the entangled sets described in [2].

Definition V.6.40 *For any topological space X, a subset $E \subseteq X$ is a* super Luzin set *iff E is uncountable and for all m, no uncountable spaced subset of E^m is nowhere dense in X^m.*

Equivalently, we could say that all uncountable spaced subsets of E^m are Luzin sets in X^m. Super Luzin sets are Luzin sets, since every subset of X^1 is spaced.

Exercise V.6.41 *Assume CH, and let X be a separable complete metric space without isolated points. Then there is a super Luzin set $E \subseteq X$.*

Hint. Start with a nice chain of elementary submodels, $\langle M_\xi : \xi < \omega_1 \rangle$. Let $E = \{x_\xi : \xi < \omega_1\}$, where $\mathrm{ht}(x_\xi) = \xi$ and $x_\xi \notin F$ for all $F \in M_\xi$ such that F is a first category subset of X. To prove that this works, you need to use the "Fubini Theorem for category"; that is, if $F \subseteq X^{m+1}$ is first category in X^{m+1}, then $\{y \in X : F_y$ is not first category in $X^m\}$ is first category in X.

You can rephrase this proof into a combinatorial argument that would have been understood by Luzin in 1914. □

Exercise V.6.42 *Let $E \subseteq \mathbb{R}^2$ be a super Luzin set, and assume also that E is the graph of a bijection from $\mathrm{dom}(E)$ onto $\mathrm{ran}(E)$; this is easily obtained from Exercise V.6.41. Let W_+ be the set of all $(x,y) \in E^\dagger$ such that the line through x, y has positive slope and let W_- be the set of all $(x,y) \in E^\dagger$ such that*

the line through x, y has negative slope. Then $\mathbb{C}_{E,W_+}$ and $\mathbb{C}_{E,W_-}$ are both ccc, and $(Z, Z^\dagger \cap W_+^{(m)})$, $(Z, Z^\dagger \cap W_-^{(m)})$, $(Z, Z^\dagger \setminus W_+^{(m)})$, $(Z, Z^\dagger \setminus W_-^{(m)})$ are all big whenever $Z \subseteq E^m$ is spaced.

Exercise V.6.43 *Complete the proof of Exercise V.6.32.*

Hint. The natural proof, applying Exercise V.6.39, would just produce some $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ "$(\widetilde{E}, \widetilde{W})$ is not big", where $\widetilde{E}$ is some spaced Z and $\widetilde{W}$ is of the form $Z^\dagger \cap W_\pm^{(m)}$ or $Z^\dagger \setminus W_\pm^{(m)}$. But also, the set of all such p is dense in $\mathbb{P}$, so you can get a (countable) maximal antichain of such p and take a countable disjoint union of spaces to get $\mathbb{1} \Vdash_{\mathbb{P}}$ "$(\widetilde{E}, \widetilde{W})$ is not big". □

V.7 Proper Forcing

Although a full discussion of the Proper Forcing Axiom (*PFA*) is beyond the scope of this book, we shall conclude with some brief remarks on the subject. For more information, see [1, 4, 5, 19, 42, 49, 81, 84, 98, 101].

There are two different currents in axiomatic set theory. On the one hand, there is a zoo of independence results that seem to say that "everything is independent of everything". On the other hand, there are attempts to simplify the chaos by introducing general axioms that settle large numbers of questions.

For an axiom to be useful, it should be *easily stated* and *natural* and have *many interesting consequences.* We do not demand any philosophical justification for the axiom to be "true" in any platonic sense, and the same set-theorist might well work on competing axioms.

One such axiom is certainly the *GCH*, which greatly simplifies cardinal arithmetic. But *GCH* does not settle many interesting questions, such as *SH* or *KH*. So, a better assumption along these lines is the Axiom of Constructibility ($V = L$) since this implies $\neg SH$ and KH.

The notions of "*easily stated*" and "*natural*" have evolved over the years. The statement of $V = L$ would have looked very mysterious to Suslin, but since the development of model theory, changing the $\mathcal{P}$ to a $\mathcal{D}$ to get the $L(\alpha)$ instead of the $R(\alpha)$ (see page 134) seems like a very natural thing to consider.

In the other direction, the axiom $\neg CH$ is also easily stated, but it leaves many basic questions unanswered, such as the values of the various cardinals discussed in Section III.1. A better simplifying assumption is $MA + \neg CH$, which implies that all these cardinals equal $\mathfrak{c}$. In addition, $MA + \neg CH$ implies *SH*, which isn't even a question about cardinals.

We have already seen that $MA + \neg CH$ isn't perfect, since there are many questions about sets of size $\leq \mathfrak{c}$ that $MA + \neg CH$ doesn't answer. Most notably is the size of $\mathfrak{c}$ itself. You could cure that by using $MA + \mathfrak{c} = \aleph_2$ as your axiom. But still, $MA + \mathfrak{c} = \aleph_2$ doesn't settle *KH*, which is a statement about $\aleph_1$ and $\aleph_2$. Also, there are a number of combinatorial properties, such as *SOCA*, that

only involve sets of size $\aleph_1$ and are not settled by $MA + \mathfrak{c} = \aleph_2$ (see Section V.6). These questions are settled by *PFA*, which is a strengthening of $MA + \mathfrak{c} = \aleph_2$. *PFA* implies *SOCA* along with $OCA_{[T]}$ and $OCA_{[ARS]}$ (Definitions V.6.25 and V.6.26). Also $PFA \to \neg KH$. Furthermore, *PFA* implies that there are no ω_2–Aronszajn trees (Exercise V.7.7); this statement is independent of $MA + \mathfrak{c} = \aleph_2$ (Exercise V.4.21).

So, there is no doubt that *PFA* improves $MA + \mathfrak{c} = \aleph_2$ in the sense of having many interesting consequences. But, it is not quite so natural and easily stated. Unlike *MA*, it is hard to understand what *PFA* says without first understanding much of Section V.3. It is even harder to prove that *PFA* is consistent.

PFA is obtained by replacing "ccc" by "proper" in the statement of $MA(\aleph_1)$.

Definition V.7.1 *A poset* $\mathbb{P}$ *is* proper *iff for all* κ *and all stationary* $\mathcal{S} \subseteq [\kappa]^{\leq \aleph_0}$ *(Definition III.6.24),* $\mathbb{1} \Vdash_{\mathbb{P}}$ *"*$\mathcal{S}$ *is stationary". PFA is the statement that* $MA_{\mathbb{P}}(\aleph_1)$ *holds for all proper* $\mathbb{P}$.

The following is clear from Exercises IV.7.59 and IV.7.61:

Lemma V.7.2 *All ccc posets and all countably closed posets are proper. Thus,* $PFA \to MA(\aleph_1)$.

Also note:

Lemma V.7.3 *If* $\mathbb{P}$ *is proper and* κ *is arbitrary and* $p \Vdash_{\mathbb{P}} \mathring{A} \in [\kappa]^{\leq \aleph_0}$, *then there is a* $B \in [\kappa]^{\leq \aleph_0}$ *and a* $q \leq p$ *such that* $q \Vdash \mathring{A} \subseteq B$. *In particular,* $\mathbb{P}$ *preserves* ω_1.

Proof. If not, let $\mathcal{S} = [\kappa]^{\leq \aleph_0}$, which is obviously stationary (and club) in $[\kappa]^{\leq \aleph_0}$. But p forces $\mathcal{S}$ to be disjoint from the club $\{X \in [\kappa]^{\leq \aleph_0} : \mathring{A} \subseteq X\}$ in the $\mathbb{P}$ extension.

For the "In particular", let $\mathring{A}$ denote the ω_1 of the ground model. □

That is, every countable set A of ordinals in the extension is covered by one in the ground model. Note that for countably closed posets, this is obvious (A is *in* the ground model), and for ccc posets this is clear from Lemma IV.3.5.

One may be tempted to improve *PFA* by weakening "proper" to "preserves ω_1", or maybe just the conclusion of Lemma V.7.3, but this is inconsistent. In fact, by Jensen's example in Exercise V.2.28, $MA_{\mathbb{P}}(\aleph_1)$ must be false for some Baire poset $\mathbb{P}$.

Unlike $MA(\aleph_1)$, which is consistent both with $\mathfrak{c} = \aleph_2$ and with $\mathfrak{c} > \aleph_2$, *PFA* implies that $\mathfrak{c} = \aleph_2$; see [70] or Theorem 1.8 of [101].

Note that $MA_{\mathbb{P}}(\aleph_1)$ for countably closed $\mathbb{P}$ is uninteresting, since it is provable in *ZFC* (see the proof of Lemma V.5.2). Nevertheless, countably closed forcings play an important role in deriving results from *PFA* because of the following remark, which is clear from the definition of "proper".

Lemma V.7.4 *If* $\mathbb{P}$ *is proper and* $\mathbb{1} \Vdash_{\mathbb{P}}$ *"*$\mathring{\mathbb{Q}}$ *is proper", then* $\mathbb{P} * \mathring{\mathbb{Q}}$ *is proper.*

The proofs from *PFA* of *SOCA*, $OCA_{[T]}$, $OCA_{[ARS]}$, $\neg KH$, and the non-existence of ω_2–Aronszajn trees, all use posets of the form $\mathbb{P} * \mathring{\mathbb{Q}}$, where $\mathbb{1} \Vdash_{\mathbb{P}}$ "$\mathring{\mathbb{Q}}$ is ccc" and $\mathbb{P}$ is the countably closed poset $\mathrm{Fn}_{\omega_1}(\omega_1, \mathfrak{c})$. For the proof of *SOCA*, see Theorem V.7.5 below.

Of course, as with other definitions that use forcing to define combinatorial notions, one needs to refer to the "forcing over V" approach to make the definition of "proper" rigorous. Subsection V.7.2 describes a purely combinatorial equivalent of "proper" that does not mention forcing at all. With this equivalent definition, in theory *PFA*, like *MA*, could be used by people who are not familiar with forcing. But, unlike with *MA*, it is hard to get very far with *PFA* without knowing forcing, because many proper posets are of the form $\mathbb{P} * \mathring{\mathbb{Q}}$.

Subsection V.7.1 below lists a few applications where the properness of the poset used is clear from what we have already done, and Subsection V.7.3 gives an application where the combinatorial equivalent from Subsection V.7.2 is helpful. Subsections V.7.1 and V.7.2 can be read in either order.

V.7.1 Some Applications

We saw in Section V.6 that there are a number of statements that can be proved to be independent from $MA + \mathfrak{c} = \aleph_2$ by iterated ccc forcing. *PFA* decides most of these one way or another. As an example of this:

Theorem V.7.5 *PFA implies SOCA.*

Proof. Fix an uncountable separable metric space E and an open $W \subseteq E \times E$ with $W = W^{-1}$. Assume that no uncountable subset of E is W–free. We shall produce a $T \in [E]^{\aleph_1}$ that is W–connected. We may assume WLOG that $|E| = \aleph_1$; otherwise replace E by an $\aleph_1$ size subset of E. Let $E = \{e_\xi : \xi < \omega_1\}$.

Now, Lemmas V.6.14 and V.6.16 construct a ccc poset $\mathbb{Q}$ that forces an uncountable W–connected set. Unfortunately, this construction assumes *CH*. So, we make *CH* true before applying these lemmas.

Let $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\omega_1, \mathfrak{c})$. Then $\mathbb{P}$ adds a generic map from ω_1 onto $\mathfrak{c}$, so *CH* becomes true in the $\mathbb{P}$ extension. Furthermore, $\mathbb{P}$ is countably closed. Hence, it not only preserves ω_1; it also preserves the fact that no uncountable subset of E is W–free (this is like Exercise IV.7.46). So, in the $\mathbb{P}$ extension, there is a ccc poset $\mathbb{Q}$ that forces an uncountable W–connected set.

From the point of view of the ground model, we have a $\mathbb{P}$–name $\mathring{\mathbb{Q}}$ that is forced by $\mathbb{1}_{\mathbb{P}}$ to be a ccc poset. We also have a $\mathbb{P} * \mathring{\mathbb{Q}}$–name $\mathring{T}$ that is forced by $\mathbb{1}_{\mathbb{P} * \mathring{\mathbb{Q}}}$ to be an uncountable W–connected subset of E. Let $D_\alpha = \{r \in \mathbb{P} * \mathring{\mathbb{Q}} : \exists \eta > \alpha\, [r \Vdash e_\eta \in \mathring{T}]\}$. Then D_α is dense whenever $\alpha < \omega_1$ because $\mathring{T}$ is forced to be uncountable. Since $\mathbb{P} * \mathring{\mathbb{Q}}$ is proper (Lemma V.7.4), fix a filter $G \subseteq \mathbb{P} * \mathring{\mathbb{Q}}$ such that $G \cap D_\alpha \neq \emptyset$ for each α. Then $I := \{\eta < \omega_1 : \exists r \in G\, [r \Vdash e_\eta \in \mathring{T}]\}$ is uncountable. Furthermore, $\{e_\eta : \eta \in I\}$ is W–connected. To prove this: Fix $\xi, \eta \in I$ with $\xi \neq \eta$. Then, since G is a filter, the definition of I implies that

there is an $r \in G$ such that $r \Vdash e_\xi \in \dot{T}$ and $r \Vdash e_\eta \in \dot{T}$. Since $\mathbb{1}$ forces $\dot{T}$ to be W–connected, $r \Vdash (e_\xi, e_\eta) \in W$; but this is a statement in the ground model, so $(e_\xi, e_\eta) \in W$ is simply true. □

This form of proof from *PFA* is known as the "collapse the continuum trick"; you force *CH* to be true, and then use *CH* in the extension to build a ccc poset. This trick is also used to prove the other results already mentioned above from *PFA*. We remark that we could have just as well used $\mathrm{Fn}_{\omega_1}(\omega_1, 2)$ instead of $\mathrm{Fn}_{\omega_1}(\omega_1, \mathfrak{c})$ to do the collapsing; see Exercise IV.7.19.

Exercise V.7.6 *PFA implies* $OCA_{[T]}$.

Hint. Use the "collapse the continuum trick", and modify the hint to Exercise V.6.31; but the second part of that hint is not relevant here. □

Exercise V.7.7 *PFA implies that there are no* $\aleph_2$*–Aronszajn trees.*

Hint. Suppose that T were such a tree.

Use the "collapse the continuum trick", so apply *PFA* to $\mathbb{P} * \mathring{\mathbb{Q}}$, where $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\omega_1, \omega_2)$ and $\mathring{\mathbb{Q}}$ is a $\mathbb{P}$–name for a ccc poset. Let λ denote the ordinal ω_2 (which will become a non-cardinal after we force with $\mathbb{P}$).

To obtain $\mathring{\mathbb{Q}}$, work in the $\mathbb{P}$–extension: Fix a club $C \subset \lambda$ of order type ω_1. Let $T{\restriction}C$ denote $\{x \in T : \mathrm{height}(x) \in C\}$, which is a tree with the induced order. $T{\restriction}C$ has height ω_1 and no uncountable paths (this is like Lemma V.2.26). Then let $\mathbb{Q}$ be a ccc poset that forces $T{\restriction}C$ to be special (that is, a countable union of antichains; see Theorem III.5.19).

Back in the real universe, meet $\aleph_1$ dense sets in $\mathbb{P} * \mathring{\mathbb{Q}}$ to obtain a limit $\gamma < \lambda$ and a club $C_0 \subseteq \gamma$ of order type ω_1 such that $T{\restriction}C_0$ is special. But this contradicts the fact that elements of $\mathcal{L}_\gamma(T)$ define paths through $T{\restriction}C_0$. □

V.7.2 A Combinatorial Equivalent

We begin with a warmup exercise on clubs:

Lemma V.7.8 *Fix* $A \subseteq B$, *and define* $\rho : [B]^{\leq \aleph_0} \xrightarrow{\text{onto}} [A]^{\leq \aleph_0}$ *by* $\rho(y) = y \cap A$. *Then:*

1. *If* $\mathcal{C} \subseteq [A]^{\leq \aleph_0}$ *is club then* $\rho^{-1}(\mathcal{C})$ *is club in* $[B]^{\leq \aleph_0}$.
2. *If* $\mathcal{C} \subseteq [B]^{\leq \aleph_0}$ *is club then* ρ"$\mathcal{C}$ *contains a club subset of* $[A]^{\leq \aleph_0}$.
3. *If* $\mathcal{S} \subseteq [A]^{\leq \aleph_0}$ *is stationary then* $\rho^{-1}(\mathcal{S})$ *is stationary in* $[B]^{\leq \aleph_0}$.
4. *If* $\mathcal{S} \subseteq [B]^{\leq \aleph_0}$ *is stationary then* ρ"$\mathcal{S}$ *is stationary in* $[A]^{\leq \aleph_0}$.

Proof. Item (1) is clear, and (3)(4) are clear from (2)(1). Regarding (2): Note that we cannot say that ρ"$\mathcal{C}$ *is* a club; for example, with $A = \omega_1$ and $B = \omega_2$, we might have each $n \cup \{\omega_1 + n\} \in \mathcal{C}$ but no $x \in \mathcal{C}$ such that $x \cap \omega_1 = \omega$.

To prove (2): First, for each finite $s \subseteq A$ choose $f(s) \supseteq s$ with $f(s) \in \mathcal{C}$ so that $s \subseteq t \to f(s) \subseteq f(t)$. This is easily done by choosing f first on the one–element sets, then on the two–element sets, and so forth. Let $\mathcal{C}_0 = \{x \in [A]^{\leq \aleph_0} : \forall s \in [x]^{<\aleph_0}\, [f(s) \cap A \subseteq x]\}$. Then $\mathcal{C}_0$ is club. If $x \in \mathcal{C}_0$ and $y = \bigcup\{f(s) : [x]^{<\aleph_0}\}$, then $y \in \mathcal{C}$ and $x = y \cap A$. So, $\mathcal{C}_0 \subseteq \rho\text{“}\mathcal{C}$. □

Generalizing Definition III.3.58 (of predense):

Definition V.7.9 *For $E \subseteq \mathbb{P}$: $p \perp E$ means $\forall q \in E\, [p \perp q]$, and E is* predense below p *(in symbols, $p \leq \bigvee E$) iff $\neg\exists q \leq p\, [q \perp E]$.*

So E is predense iff E is predense below $\mathbb{1}$. A remark on the notation: We do not define $\bigvee E$ in an arbitrary poset, and this “$p \leq \bigvee E$” is just a convenient shorthand. It is motivated by the fact that in the case that $\mathbb{P}$ is a dense suborder of a $\mathbb{B}\backslash\{\mathbb{0}\}$, where $\mathbb{B}$ is a complete boolean algebra, then $\bigvee E$ denotes the supremum of E, and E is predense below p iff $\bigvee E \geq p$.

Lemma V.7.10 *$p \leq \bigvee E$ iff $p \Vdash E \cap \mathring{G} \neq \emptyset$.*

Proof. Use the fact that $q \perp E$ iff $q \Vdash E \cap \mathring{G} = \emptyset$. □

Definition V.7.11 *If θ is an uncountable regular cardinal and $(\mathbb{P}, \leq, \mathbb{1}) \in M \preccurlyeq H(\theta)$, then $p \in \mathbb{P}$ is* $(M, \mathbb{P})$–generic *iff for all dense $D \subseteq \mathbb{P}$ such that $D \in M$: $p \leq \bigvee(D \cap M)$.*

This definition actually makes sense for any set M, but in practice it is used only with elementary submodels.

Exercise V.7.12 *In Definition V.7.11, show that we get an equivalent notion if the “all dense $D \subseteq \mathbb{P}$” is replaced by any one of “all dense open $D \subseteq \mathbb{P}$” or “all predense $D \subseteq \mathbb{P}$” or “all maximal antichains $D \subseteq \mathbb{P}$”.*

Hint. See the proof of Theorem III.3.60. Note that every dense set D contains a maximal antichain $A \subseteq D$, and if $D \in M \preccurlyeq H(\theta)$, then we can choose $A \in M$ also. □

Example V.7.13 *If $\mathbb{P} \in M \preccurlyeq H(\theta)$ and $\mathbb{P}$ is ccc, then* every $p \in \mathbb{P}$ *is $(M, \mathbb{P})$–generic.*

Proof. If $A \subseteq \mathbb{P}$ is a maximal antichain and $A \in M$, then $A \subseteq M$ because A is countable (Lemma III.7.12), so $A \cap M = A$, which is predense. □

Exercise V.7.14 *If $\mathbb{P} \in M \preccurlyeq H(\theta)$ and $\mathbb{P}$ is countably closed and $p \in M$ and M is countable, then there is a $q \leq p$ that is $(M, \mathbb{P})$–generic. Also, if $\mathbb{P}$ is atomless, then this q cannot be in M.*

Hint. Use $|M| = \aleph_0$ to let $\{D_n : n \in \omega\}$ list all the dense subsets that are in M. Then, apply $M \prec H(\theta)$ to get $p = p_0 \geq p_1 \geq \cdots$, where each $p_{n+1} \in D_n \cap M$. Then, get $q \in \mathbb{P}$ such that $q \leq p_n$ for all n. For the "Also", if $q \in M$, then there is a maximal antichain $A \in M$ such that $q{\downarrow} \cap A$ is uncountable. □

Example V.7.15 *If $M \prec H(\theta)$, where M is countable and $\theta \geq \aleph_2$ and $\mathbb{P} = \mathrm{Fn}(\omega, \omega_1)$, then $\mathbb{P} \in M$ and* no *q is $(M, \mathbb{P})$-generic.*

Proof. Fix $n \in \omega \backslash \mathrm{dom}(q)$, and consider $A = \{\, \{(n, \xi)\} : \xi < \omega_1 \,\}$. □

One more example, to illustrate the fact that for a given $\mathbb{P}$, $(M, \mathbb{P})$-generic elements may exist for some M and not for others:

Exercise V.7.16 *Let $S \subseteq \omega_1$ be stationary, and let $\mathbb{P}$ be Jensen's poset for shooting a club through S; see Exercise V.2.28. Fix $M \prec H(\theta)$, where M is countable and $S \in M$, so $\mathbb{P} \in M$ also. Let $\gamma = \sup(M \cap \omega_1)$. If $\gamma \in S$, then for all $p \in \mathbb{P} \cap M$, there is a $q \leq p$ that is $(M, \mathbb{P})$-generic. If $\gamma \notin S$, then* no *q is $(M, \mathbb{P})$-generic.*

The combinatorial equivalent of "proper" is given by:

Theorem V.7.17 *$\mathbb{P}$ is proper iff for all regular θ with $\mathbb{P} \in H(\theta)$, there is a club $\mathcal{C} \subseteq [H(\theta)]^{\leq \aleph_0}$ such that for all $M \in \mathcal{C}$: $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}) \in M$ and $M \prec H(\theta)$ and for all $p \in \mathbb{P} \cap M$, there is a $q \leq p$ that is $(M, \mathbb{P})$-generic.*

This equivalent makes no mention of forcing. Observe that by Example V.7.13 and Exercise V.7.14, this equivalent condition is satisfied by all ccc posets and all countably closed posets.

Lemma V.7.18 *In V, assume that θ is regular and uncountable and $p \in \mathbb{P}$ and $\mathbb{P} \in H(\theta)$. Then in any generic extension $V[G]$, there is a club $\mathcal{C} \subseteq [(H(\theta))^V]^{\leq \aleph_0}$ such that for all $M \in \mathcal{C}$ and $p \in \mathbb{P}$:*

1. *$M \prec (H(\theta))^V$ and $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}) \in M$.*
2. *For all $D \in M$, if D is a dense subset of $\mathbb{P}$ then $D \cap M \cap G \neq \emptyset$.*
3. If *$M \in V$ and $p \in G$, then there is a $q \leq p$ that is $(M, \mathbb{P})$-generic.*

Proof. First observe that working in $V[G]$, we must be careful to distinguish between $(H(\theta))^V$ (that is, the $H(\theta)$ from the ground model) and the $H(\theta)$ in the generic extension. Also, if $\mathbb{P}$ adds countable sets, then in $V[G]$ there may be countable $M \prec (H(\theta))^V$ that are not in V.

Working in $V[G]$, we obtain $\mathcal{C}$ satisfying (1)(2) by a simple closure argument. For (3): If $M \in V$, then (2) is a statement in V about the generic filter $\mathring{G}$, so there is a $q \in G$ with $q \leq p$ that forces (2). So for all dense $D \subseteq \mathbb{P}$ such that $D \in M$, $q \Vdash (D \cap M) \cap \mathring{G} \neq \emptyset$, so by Lemma V.7.10, $q \leq \bigvee(D \cap M)$. □

Regarding (3), we remark that for some $\mathbb{P}$ (see Exercise V.7.15), there are no $(M,\mathbb{P})$–generic q and M is never in V; then (3) is true vacuously.

It may be instructive to compare this lemma with Exercise V.7.16. In the generic extension $V[G]$, there is a club $C_0 \subset S$. Besides the obvious club provided by Jensen's argument, we have the club provided by the $\mathcal{C}$ of Lemma V.7.18, since if $K = \{M \cap \omega_1 : M \in \mathcal{C}\}$, then $K \subseteq S$ by Exercise V.7.16 and and K contains a club by Lemma V.7.8.

Lemma V.7.19 *Assume that* $M \in [H(\theta)]^{\leq \aleph_0}$ *and* $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}), \kappa, p \in M$ *and* $M \preccurlyeq H(\theta)$. *Let* $q \leq p$ *be* $(M,\mathbb{P})$*-generic, and let* τ *be a* $\mathbb{P}$*-name in* M *such that* $p \Vdash \tau \in [\kappa]^{\leq \aleph_0}$. *Then* $q \Vdash \tau \subseteq M \cap \kappa$.

Proof. Fix $\mathring{f} \in M$ such that $p \Vdash \mathring{f} \in \kappa^\omega \wedge \tau \subseteq \operatorname{ran}\mathring{f}$. Let $D_n = \{r \in \mathbb{P} : r \perp p \vee \exists \alpha\, [r \Vdash \mathring{f}(n) = \alpha]\}$. Then D_n is dense and $D_n \in M$, so $q \leq \bigvee(D_n \cap M)$, so $q \leq \bigvee\{r \leq p : \exists \alpha \in M\, [r \Vdash \mathring{f}(n) = \alpha]\}$. But then $q \Vdash \mathring{f}(n) \in M$. □

Proof of Theorem V.7.17. Note that $\{M \in [H(\theta)]^{\leq \aleph_0} : (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}) \in M \wedge M \preccurlyeq H(\theta)\}$ is always club by Exercise III.6.26.

For $\rightarrow$: If the stated condition fails, then by the Pressing Down Lemma (Exercise III.6.27), there is a stationary $\mathcal{S} \subseteq [H(\theta)]^{\leq \aleph_0}$ and a fixed $p \in \mathbb{P}$ such that for all $M \in \mathcal{S}$: $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}) \in M$ and $M \preccurlyeq H(\theta)$ and $p \in M$ and there is no $q \leq p$ that is $(M,\mathbb{P})$–generic. Then $p \Vdash_{\mathbb{P}}$ "$\mathcal{S}$ is not stationary", since Lemma V.7.18 produces a club $\mathcal{C} \subseteq [(H(\theta))^V]^{\leq \aleph_0}$ disjoint from $\mathcal{S}$ in $V[G]$ whenever $p \in G$. But then $\mathbb{P}$ is not proper; let $\kappa = |H(\theta)|$ in Definition V.7.1.

For $\leftarrow$: If $\mathbb{P}$ is not proper, fix κ and a stationary $\mathcal{S} \subseteq [\kappa]^{\leq \aleph_0}$ and a $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ "$\mathcal{S}$ is not stationary". Let $\mathring{\mathcal{K}}$ be a name such that p forces $\mathring{\mathcal{K}}$ to be a club disjoint from $\mathcal{S}$. Fix θ so that $[\kappa]^{\leq \aleph_0}, \mathcal{S}, \mathring{\mathcal{K}} \in H(\theta)$. Let $\widehat{\mathcal{S}} = \{M \in [H(\theta)]^{\leq \aleph_0} : (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}) \in M \wedge M \preccurlyeq H(\theta) \wedge \kappa \in M \wedge \mathring{\mathcal{K}} \in M \wedge M \cap \kappa \in \mathcal{S}\}$. Then $\widehat{\mathcal{S}}$ is stationary (Lemma V.7.8). We shall show that for $M \in \widehat{\mathcal{S}}$, there is no $(M,\mathbb{P})$–generic $q \leq p$, so that the stated condition fails for $\mathbb{P}$:

Fix $M \in \widehat{\mathcal{S}}$, and suppose that $q \leq p$ is $(M,\mathbb{P})$–generic. Choose finite $x_n \in M$ with $x_0 \subseteq x_1 \subseteq \cdots$ and $\bigcup_n x_n = M \cap \kappa$. Then, since p forces $\mathring{\mathcal{K}}$ to be unbounded, choose names $\tau_n \in M$ for $n \in \omega$ such that $p \Vdash \tau_n \in \mathring{\mathcal{K}}$ and $p \Vdash x_n \cup \tau_n \subseteq \tau_{n+1}$ for each n. By Lemma V.7.19, $q \Vdash \tau_n \subseteq M \cap \kappa$ for each n. Since q also forces $\mathring{\mathcal{K}}$ to be club, $q \Vdash M \cap \kappa \in \mathring{\mathcal{K}}$, contradicting $M \cap \kappa \in \mathcal{S}$ and $q \Vdash \mathring{\mathcal{K}} \cap \mathcal{S} = \emptyset$. □

V.7.3 Another Application

This application uses Theorem V.7.17 to establish that a poset is proper. Our application involves the notion of *club guessing* introduced by Shelah in the 1990s; for more on this and references to the literature, see [46].

Although $\diamondsuit$ easily implies *CH* (Lemma III.7.2), club guessing is an extreme weakening of $\diamondsuit$ that is consistent with $MA + \neg CH$.

Definition V.7.20 *Let* Lim *be the set of countable limit ordinals. A* ladder system *is a sequence* $\vec{C} = \langle C_\gamma : \gamma \in \mathrm{Lim}\rangle$ *such that each* C_γ *is a cofinal subset of* γ *of order type* ω.

The ladder system is club guessing *iff for all clubs* $C \subseteq \omega_1$, *there is a* $\gamma \in C$ *such that* $C_\gamma \subseteq C$.

Lemma V.7.21 $\diamondsuit$ *implies that there is a club guessing ladder system.*

Proof. Let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$ sequence. For $\gamma \in \mathrm{Lim}$, choose $C_\gamma \subseteq A_\gamma$ whenever $\sup(A_\gamma) = \gamma$; C_γ is arbitrary when $\sup(A_\gamma) < \gamma$. □

By Exercise IV.7.56, every club in a ccc extension contains a club in the ground model. It follows that

Lemma V.7.22 *If* $\vec{C}$ *is a club guessing ladder system and* $\mathbb{P}$ *is ccc, then* $\mathbb{1}_\mathbb{P}$ *forces that* $\vec{C}$ *remains a club guessing ladder system.*

In particular, the existence of club guessing ladder systems is consistent with $MA + \mathfrak{c} = \aleph_2$ and with $MA + \mathfrak{c} = \aleph_{17}$, etc. Also note, as in Exercise IV.7.46:

Exercise V.7.23 *If* $\vec{C}$ *is a club guessing ladder system and* $\mathbb{P}$ *is countably closed, then* $\mathbb{1}_\mathbb{P}$ *forces that* $\vec{C}$ *remains a club guessing ladder system.*

It follows that one cannot kill such ladder systems by forcing with posets of the form $\mathbb{P} * \mathring{\mathbb{Q}}$, where $\mathbb{P}$ is countably closed and $\mathring{\mathbb{Q}}$ is forced to be ccc. However, such ladder systems are refuted by *PFA*. This is true even if we weaken the "$C_\gamma \subseteq C$" in Definition V.7.20 to "$C_\gamma \subseteq^* C$" (i.e., $|C_\gamma \backslash C| < \aleph_0$):

Theorem V.7.24 *Assume PFA, and let* $\vec{C}$ *be a ladder system. Then there is a club* $C \subseteq \omega_1$ *such that* $C_\gamma \backslash C$ *is infinite for all* $\gamma \in \mathrm{Lim}$.

Proof. Let $\mathbb{P}$ be the set of all countable closed $p \subset \omega_1$ such that $\sup(p) \in \mathrm{Lim} \cup \{0\}$ and such that $C_\gamma \backslash p$ is infinite for all $\gamma \in \mathrm{Lim}$. Order $\mathbb{P}$ by end extension: $q \leq p$ iff $q \supseteq p$ and $(q \backslash p) \cap \sup(p) = \emptyset$. Then $\mathbb{1} = \emptyset$.

For $\alpha < \omega_1$, let $D_\alpha = \{p \in \mathbb{P} : \sup(p) > \alpha\}$. Then D_α is easily seen to be dense. If G is a filter meeting all D_α, we may let $C = \bigcup G$.

So, we are done if we can prove that $\mathbb{P}$ is proper. Fix a regular θ and a countable $M \prec H(\theta)$ with $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{1}) \in M$ and a $p \in \mathbb{P} \cap M$. We shall find a $q \leq p$ that is $(M, \mathbb{P})$–generic.

Let $\{D_n : n \in \omega\}$ list all the dense $D \subseteq \mathbb{P}$ such that $D \in M$. So, we need to find a $q \leq p$ such that $q \leq \bigvee(D_n \cap M)$ for each n. In fact, we shall get $q \in (D_n \cap M)\downarrow$ for each n, which is a stronger statement.

Outside of M, let $\gamma = M \cap \omega_1$ and fix $\alpha_n \nearrow \gamma$. Recursively choose $p_n \in M$ such that $p = p_0 \geq p_1 \geq \cdots$ and $\sup(p_{n+1}) > \alpha_n$ and $p_{n+1} \in D_n$. Note that in passing from p_n to p_{n+1}, we stay in M (since $D_n \in M$ and $M \prec H(\theta)$), but

the sequence $\langle \alpha_n : n \in \omega \rangle$ is outside of M. We can simply let $q = \bigcup_n p_n \cup \{\gamma\}$ *except* for the problem that q need not be in $\mathbb{P}$, since $C_\gamma \backslash q$ could be finite (note that there is no problem with limit ordinals other than this γ). To avoid this problem, before choosing p_{n+1}, first choose $\hat{p}_n \in M$ such that $\hat{p}_n = p_n \cup \{\delta_n\} < p_n$, where $\delta_n > \sup(p_n)$ and there is at least one ordinal in C_γ between $\sup(p_n)$ and δ_n; then choose $p_{n+1} \leq \hat{p}_n$. □

Posets like this, where we can get $q \in (D_n \cap M)\downarrow$ rather than just $q \leq \bigvee(D_n \cap M)$, are called *totally proper*; this is equivalent to being proper and not adding ω–sequences; see, e.g., [27], for more on these and references to the literature. Note that Exercise V.7.14 really showed that countably closed posets are totally proper.

Exercise V.7.25 *Improve Theorem V.7.24 to get C so that $C \cap C_\gamma$ is finite for all $\gamma \in$ Lim.*

Hint. Modify $\mathbb{P}$ to require that $p \cap C_\gamma$ be finite for all $\gamma \in$ Lim. But now, in constructing q, you have to make sure that $q \cap C_\gamma$ is finite. To do this, let $\mathcal{D} \subseteq [H(\theta)]^{\leq \aleph_0}$ be a club of elementary submodels so that each $M \in \mathcal{D}$ is also the union of an increasing ω^2–chain of elementary submodels of the form $\langle N_\mu : \mu < \omega^2 \rangle$. Then, for $M \in \mathcal{D}$, we build our $(M, \mathbb{P})$–generic q as before, except that now we make sure that all $p_n \cap C_\gamma = p \cap C_\gamma$. Given p_n: First fix $\mu < \omega^2$ such that $C_\gamma \cap (N_{\mu+1} \backslash N_\mu) = \emptyset$. Then choose $\delta_n \in (N_{\mu+1} \backslash N_\mu)$, and then choose $p_{n+1} \leq \hat{p}_n$ in $N_{\mu+1}$. □

V.7.4 Concluding Remarks

Regarding the consistency of *PFA*: Obviously, this is proved by iterated forcing. We shall indicate this argument *very* briefly. See Jech [49], Theorem 31.21, for an exposition of the proof with most of the details given and references to the literature. Other references are [1, 4, 19, 42, 81, 84].

The first step is to show that "proper" is preserved by the iteration. One cannot use finite support iteration here, since we are iterating posets that are not all ccc, so that a finite support iteration will collapse ω_1 (see Exercise V.4.24), and hence not be proper. However, one can show that a countable support iteration of proper posets is proper, assuming that full names are used.

A more serious difficulty in the proof is the lack of a bound on the size of the posets we need to consider. Recall (Lemma III.3.51) that to verify $MA(\aleph_1)$, it is enough to consider posets of size $\leq \aleph_1$. Then, in proving the consistency of $MA(\aleph_1)$ (Theorem V.4.1), the standard bookkeeping lets us list all names for posets of size $\leq \aleph_1$ in an ω_2–sequence to get $MA(\aleph_1) + \mathfrak{c} = \aleph_2$ in $M[G]$. But, there is no analogous result for proper forcing. In fact, any application of the "collapse the continuum trick" (as in our proof of *PFA* $\rightarrow$ *SOCA* in Theorem V.7.5) requires a poset of size at least $\aleph_2$.

So, one cannot simply start with *GCH* in the ground model and iterate proper forcings ω_2 times. Also note that some cardinals must be collapsed. Since the forcing is proper, $\aleph_1$ is preserved. By *CH*, there is an ω_2–Aronszajn tree T in the ground model that is also special (the union of $\aleph_1$ antichains; see Exercise III.5.44, Hint 1). But *PFA* implies that there are no such trees, and the only way that T could be destroyed is if $\aleph_2$ gets collapsed.

In fact, the only way known to obtain a model of *PFA* is to start in the ground model with a supercompact cardinal κ. This is a large cardinal (much larger than measurable, which is much larger than strongly Mahlo, which is much larger than strongly inaccessible). This κ becomes the $\aleph_2$ of the generic extension. For more on such large cardinals, see Kanamori [51].

The exact consistency strength of *PFA* is not known, although it is known to be more than a mere measurable cardinal. Viale and Weiß [102] show that if you start with an inaccessible κ in the ground model and make *PFA* true in the "standard way" as indicated above (so κ is the $\aleph_2$ of the extension), then κ must be supercompact in the ground model.

There is a strengthening of *PFA* called *MM* (*Martin's Maximum*), which weakens "proper" to the assertion that $\mathbb{P}$ just preserves stationary subsets of ω_1. This *MM* was proved consistent by Foreman, Magidor, and Shelah [34] (again, using a supercompact cardinal). They also give some consequences of *MM* that do not follow from *PFA*.

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Indices

The General Index lists terms in alphabetic order. Symbolic expressions have no natural order, and are listed in the Index of Symbols in order of first occurrence in this book.

Symbolic expressions are often listed in context, with commonly used parameters. For example, we list "$\mathcal{P}(x)$" rather than just the power set function "$\mathcal{P}$", and we list "$[B]^{\theta}$" (the family of subsets of B of size θ) rather than just something like "$[\cdots]^{\cdots}$".

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This book is designed for readers who know elementary mathematical logic and axiomatic set theory, and who want to learn more about set theory.

The primary focus of the book is on the independence proofs. Most famous among these is the independence of the Continuum Hypothesis (CH); that is, there are models of the axioms of set theory (ZFC) in which CH is true, and other models in which CH is false. More generally, cardinal exponentiation on the regular cardinals can consistently be anything not contradicting the classical theorems of Cantor and König.

The basic methods for the independence proofs are the notion of constructibility, introduced by Gödel, and the method of forcing, introduced by Cohen. This book describes these methods in detail, verifies the basic independence results for cardinal exponentiation, and also applies these methods to prove the independence of various mathematical questions in measure theory and general topology.

Before the chapters on forcing, there is a fairly long chapter on "infinitary combinatorics". This consists of just mathematical theorems (not independence results), but it stresses the areas of mathematics where set-theoretic topics (such as cardinal arithmetic) are relevant.

There is, in fact, an interplay between infinitary combinatorics and independence proofs. Infinitary combinatorics suggests many set-theoretic questions that turn out to be independent of ZFC, but it also provides the basic tools used in forcing arguments. In particular, Martin's Axiom, which is one of the topics under infinitary combinatorics, introduces many of the basic ingredients of forcing.

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