Springer Monographs in Mathematics

Thomas Jech

Set Theory

The Third Millennium Edition,
revised and expanded

Springer

Thomas Jech
e-mail: jech@math.cas.cz

The original edition was published in 1978 by Academic Press, Inc. in the series *Pure and Applied Mathematics*, vol. 79, with the ISBN 0-12-381950-4

The second edition was published in 1997 by Springer-Verlag under the same title in the series *Perspectives in Mathematical Logic*

Catalog-in-Publication Data applied for
Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Jech, Thomas J.:
Set theory / Thomas Jech. - 3rd Millennium ed, rev. and expanded. - Berlin ;
Heidelberg ; New York ; Hong Kong ; London ; Milan ; Paris ; Tokyo :
Springer, 2002
(Springer monographs in mathematics)
ISBN 3-540-44085-2

Mathematics Subject Classification (2000): 03Exx

ISSN 1439-7382

Corrected 4th printing 2006
ISBN-10 3-540-44085-2 Springer-Verlag Berlin Heidelberg New York
ISBN-13 978-3-540-44085-7 Springer-Verlag Berlin Heidelberg New York

ISBN 3-540-63048-1 2nd ed. Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable for prosecution under the German Copyright Law.

Springer is a part of Springer Science+Business Media
springer.com
© Springer-Verlag Berlin Heidelberg 1997, 2003
Printed in Germany

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting by the author using a Springer TEX macro package
Production: LE-TEX Jelonek, Schmidt & Vöckler GbR, Leipzig
Cover design: Erich Kirchner, Heidelberg

Printed on acid-free paper 46/3100YL - 5 4 3 2 1 0

For Paula, Pavel, and Susanna

Preface

When I wrote the first edition in the 1970s my goal was to present the state of the art of a century old discipline that had recently undergone a revolutionary transformation. After the book was reprinted in 1997 I started contemplating a revised edition. It has soon become clear to me that in order to describe the present day set theory I would have to write a more or less new book.

As a result this edition differs substantially from the 1978 book. The major difference is that the three major areas (forcing, large cardinals and descriptive set theory) are no longer treated as separate subjects. The progress in past quarter century has blurred the distinction between these areas: forcing has become an indispensable tool of every set theorist, while descriptive set theory has practically evolved into the study of $L(\boldsymbol{R})$ under large cardinal assumptions. Moreover, the theory of inner models has emerged as a major part of the large cardinal theory.

The book has three parts. The first part contains material that every student of set theory should learn and all results contain a detailed proof. In the second part I present the topics and techniques that I believe every set theorist should master; most proofs are included, even if some are sketchy. For the third part I selected various results that in my opinion reflect the state of the art of set theory at the turn of the millennium.

I wish to express my gratitude to the following institutions that made their facilities available to me while I was writing the book: Mathematical Institute of the Czech Academy of Sciences, The Center for Theoretical Study in Prague, CRM in Barcelona, and the Rockefeller Foundation's Bellagio Center. I am also grateful to numerous set theorists who I consulted on various subjects, and particularly to those who made invaluable comments on preliminary versions of the manuscript. My special thanks are to Miroslav Repický who converted the handwritten manuscript to LaTeX. He also compiled the three indexes that the reader will find extremely helpful.

Finally, and above all, I would like to thank my wife for her patience and encouragement during the writing of this book.

Prague, May 2002 *Thomas Jech*

Table of Contents

Part I. Basic Set Theory

1. Axioms of Set Theory 3
Axioms of Zermelo-Fraenkel. Why Axiomatic Set Theory? Language of Set Theory, Formulas. Classes. Extensionality. Pairing. Separation Schema. Union. Power Set. Infinity. Replacement Schema. Exercises. Historical Notes.

2. Ordinal Numbers 17
Linear and Partial Ordering. Well-Ordering. Ordinal Numbers. Induction and Recursion. Ordinal Arithmetic. Well-Founded Relations. Exercises. Historical Notes.

3. Cardinal Numbers 27
Cardinality. Alephs. The Canonical Well-Ordering of $\alpha \times \alpha$. Cofinality. Exercises. Historical Notes.

4. Real Numbers 37
The Cardinality of the Continuum. The Ordering of $\boldsymbol{R}$. Suslin's Problem. The Topology of the Real Line. Borel Sets. Lebesgue Measure. The Baire Space. Polish Spaces. Exercises. Historical Notes.

5. The Axiom of Choice and Cardinal Arithmetic 47
The Axiom of Choice. Using the Axiom of Choice in Mathematics. The Countable Axiom of Choice. Cardinal Arithmetic. Infinite Sums and Products. The Continuum Function. Cardinal Exponentiation. The Singular Cardinal Hypothesis. Exercises. Historical Notes.

6. The Axiom of Regularity 63
The Cumulative Hierarchy of Sets. $\in$-Induction. Well-Founded Relations. The Bernays-Gödel Axiomatic Set Theory. Exercises. Historical Notes.

7. Filters, Ultrafilters and Boolean Algebras 73
Filters and Ultrafilters. Ultrafilters on ω. κ-Complete Filters and Ideals. Boolean Algebras. Ideals and Filters on Boolean Algebras. Complete Boolean Algebras. Complete and Regular Subalgebras. Saturation. Distributivity of Complete Boolean Algebras. Exercises. Historical Notes.

8. Stationary Sets 91
Closed Unbounded Sets. Mahlo Cardinals. Normal Filters. Silver's Theorem. A Hierarchy of Stationary Sets. The Closed Unbounded Filter on $P_\kappa(\lambda)$. Exercises. Historical Notes.

9. Combinatorial Set Theory 107
Partition Properties. Weakly Compact Cardinals. Trees. Almost Disjoint Sets and Functions. The Tree Property and Weakly Compact Cardinals. Ramsey Cardinals. Exercises. Historical Notes.

10. Measurable Cardinals 125
The Measure Problem. Measurable and Real-Valued Measurable Cardinals. Measurable Cardinals. Normal Measures. Strongly Compact and Supercompact Cardinals. Exercises. Historical Notes.

11. Borel and Analytic Sets 139
Borel Sets. Analytic Sets. The Suslin Operation $\mathcal{A}$. The Hierarchy of Projective Sets. Lebesgue Measure. The Property of Baire. Analytic Sets: Measure, Category, and the Perfect Set Property. Exercises. Historical Notes.

12. Models of Set Theory 155
Review of Model Theory. Gödel's Theorems. Direct Limits of Models. Reduced Products and Ultraproducts. Models of Set Theory and Relativization. Relative Consistency. Transitive Models and Δ_0 Formulas. Consistency of the Axiom of Regularity. Inaccessibility of Inaccessible Cardinals. Reflection Principle. Exercises. Historical Notes.

Part II. Advanced Set Theory

13. Constructible Sets 175
The Hierarchy of Constructible Sets. Gödel Operations. Inner Models of ZF. The Lévy Hierarchy. Absoluteness of Constructibility. Consistency of the Axiom of Choice. Consistency of the Generalized Continuum Hypothesis. Relative Constructibility. Ordinal-Definable Sets. More on Inner Models. Exercises. Historical Notes.

14. Forcing 201
Forcing Conditions and Generic Sets. Separative Quotients and Complete Boolean Algebras. Boolean-Valued Models. The Boolean-Valued Model V^B. The Forcing Relation. The Forcing Theorem and the Generic Model Theorem. Consistency Proofs. Independence of the Continuum Hypothesis. Independence of the Axiom of Choice. Exercises. Historical Notes.

15. Applications of Forcing 225
Cohen Reals. Adding Subsets of Regular Cardinals. The κ-Chain Condition. Distributivity. Product Forcing. Easton's Theorem. Forcing with a Class of Conditions. The Lévy Collapse. Suslin Trees. Random Reals. Forcing with Perfect Trees. More on Generic Extensions. Symmetric Submodels of Generic Models. Exercises. Historical Notes.

16. Iterated Forcing and Martin's Axiom 267
Two-Step Iteration. Iteration with Finite Support. Martin's Axiom. Independence of Suslin's Hypothesis. More Applications of Martin's Axiom. Iterated Forcing. Exercises. Historical Notes.

17. Large Cardinals 285
Ultrapowers and Elementary Embeddings. Weak Compactness. Indescribability. Partitions and Models. Exercises. Historical Notes.

18. Large Cardinals and L 311
Silver Indiscernibles. Models with Indiscernibles. Proof of Silver's Theorem and $0^\sharp$. Elementary Embeddings of L. Jensen's Covering Theorem. Exercises. Historical Notes.

19. Iterated Ultrapowers and $L[U]$ 339
The Model $L[U]$. Iterated Ultrapowers. Representation of Iterated Ultrapowers. Uniqueness of the Model $L[D]$. Indiscernibles for $L[D]$. General Iterations. The Mitchell Order. The Models $L[\mathcal{U}]$. Exercises. Historical Notes.

20. Very Large Cardinals 365
Strongly Compact Cardinals. Supercompact Cardinals. Beyond Supercompactness. Extenders and Strong Cardinals. Exercises. Historical Notes.

21. Large Cardinals and Forcing 389
Mild Extensions. Kunen-Paris Forcing. Silver's Forcing. Prikry Forcing. Measurability of $\aleph_1$ in ZF. Exercises. Historical Notes.

22. Saturated Ideals 409
Real-Valued Measurable Cardinals. Generic Ultrapowers. Precipitous Ideals. Saturated Ideals. Consistency Strength of Precipitousness. Exercises. Historical Notes.

23. The Nonstationary Ideal 441
Some Combinatorial Principles. Stationary Sets in Generic Extensions. Precipitousness of the Nonstationary Ideal. Saturation of the Nonstationary Ideal. Reflection. Exercises. Historical Notes.

24. The Singular Cardinal Problem 457
The Galvin-Hajnal Theorem. Ordinal Functions and Scales. The pcf Theory. The Structure of pcf. Transitive Generators and Localization. Shelah's Bound on $2^{\aleph_\omega}$. Exercises. Historical Notes.

25. Descriptive Set Theory 479
The Hierarchy of Projective Sets. Π^1_1 Sets. Trees, Well-Founded Relations and κ-Suslin Sets. Σ^1_2 Sets. Projective Sets and Constructibility. Scales and Uniformization. Σ^1_2 Well-Orderings and Σ^1_2 Well-Founded Relations. Borel Codes. Exercises. Historical Notes.

26. The Real Line 511
Random and Cohen reals. Solovay Sets of Reals. The Lévy Collapse. Solovay's Theorem. Lebesgue Measurability of $\mathbf{\Sigma}^1_2$ Sets. Ramsey Sets of Reals and Mathias Forcing. Measure and Category. Exercises. Historical Notes.

Part III. Selected Topics

27. Combinatorial Principles in L 545
The Fine Structure Theory. The Principle $\square_\kappa$. The Jensen Hierarchy. Projecta, Standard Codes and Standard Parameters. Diamond Principles. Trees in L. Canonical Functions on ω_1. Exercises. Historical Notes.

28. More Applications of Forcing 557
A Nonconstructible Δ^1_3 Real. Namba Forcing. A Cohen Real Adds a Suslin Tree. Consistency of Borel's Conjecture. κ^+-Aronszajn Trees. Exercises. Historical Notes.

29. More Combinatorial Set Theory 573
Ramsey Theory. Gaps in ω^ω. The Open Coloring Axiom. Almost Disjoint Subsets of ω_1. Functions from ω_1 into ω. Exercises. Historical Notes.

30. Complete Boolean Algebras 585
Measure Algebras. Cohen Algebras. Suslin Algebras. Simple Algebras. Infinite Games on Boolean Algebras. Exercises. Historical Notes.

31. Proper Forcing 601
Definition and Examples. Iteration of Proper Forcing. The Proper Forcing Axiom. Applications of PFA. Exercises. Historical Notes.

32. More Descriptive Set Theory 615
$\mathbf{\Pi}^1_1$ Equivalence Relations. $\mathbf{\Sigma}^1_1$ Equivalence Relations. Constructible Reals and Perfect Sets. Projective Sets and Large Cardinals. Universally Baire sets. Exercises. Historical Notes.

33. Determinacy 627
Determinacy and Choice. Some Consequences of AD. AD and Large Cardinals. Projective Determinacy. Consistency of AD. Exercises. Historical Notes.

34. Supercompact Cardinals and the Real Line 647
Woodin Cardinals. Semiproper Forcing. The Model $L(\boldsymbol{R})$. Stationary Tower Forcing. Weakly Homogeneous Trees. Exercises. Historical Notes.

35. Inner Models for Large Cardinals 659
The Core Model. The Covering Theorem for K. The Covering Theorem for $L[U]$. The Core Model for Sequences of Measures. Up to a Strong Cardinal. Inner Models for Woodin Cardinals. Exercises. Historical Notes.

36. Forcing and Large Cardinals 669
Violating GCH at a Measurable Cardinal. The Singular Cardinal Problem. Violating SCH at $\aleph_\omega$. Radin Forcing. Stationary Tower Forcing. Exercises. Historical Notes.

37. Martin's Maximum 681
RCS iteration of semiproper forcing. Consistency of MM. Applications of MM. Reflection Principles. Forcing Axioms. Exercises. Historical Notes.

38. More on Stationary Sets 695
The Nonstationary Ideal on $\aleph_1$. Saturation and Precipitousness. Reflection. Stationary Sets in $P_\kappa(\lambda)$. Mutually Stationary Sets. Weak Squares. Exercises. Historical Notes.

Bibliography 707

Notation 733

Name Index 743

Index 749

Part I

Basic Set Theory

1. Axioms of Set Theory

Axioms of Zermelo-Fraenkel

1.1. ***Axiom of Extensionality.*** *If X and Y have the same elements, then $X = Y$.*

1.2. ***Axiom of Pairing.*** *For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.*

1.3. ***Axiom Schema of Separation.*** *If P is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P.*

1.4. ***Axiom of Union.*** *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*

1.5. ***Axiom of Power Set.*** *For any X there exists a set $Y = P(X)$, the set of all subsets of X.*

1.6. ***Axiom of Infinity.*** *There exists an infinite set.*

1.7. ***Axiom Schema of Replacement.*** *If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.*

1.8. ***Axiom of Regularity.*** *Every nonempty set has an $\in$-minimal element.*

1.9. ***Axiom of Choice.*** *Every family of nonempty sets has a choice function.*

The theory with axioms 1.1–1.8 is the Zermelo-Fraenkel axiomatic set theory ZF; ZFC denotes the theory ZF with the Axiom of Choice.

Why Axiomatic Set Theory?

Intuitively, a set is a collection of all elements that satisfy a certain given property. In other words, we might be tempted to postulate the following rule of formation for sets.

1.10. ***Axiom Schema of Comprehension* (*false*).** *If P is a property, then there exists a set $Y = \{x : P(x)\}$.*

This principle, however, is false:

1.11. Russell's Paradox. Consider the set S whose elements are all those (and only those) sets that are not members of themselves: $S = \{X : X \notin X\}$. Question: Does S belong to S? If S belongs to S, then S is not a member of itself, and so $S \notin S$. On the other hand, if $S \notin S$, then S belongs to S. In either case, we have a contradiction.

Thus we must conclude that

$$\{X : X \notin X\}$$

is not a set, and we must revise the intuitive notion of a set.

The safe way to eliminate paradoxes of this type is to abandon the Schema of Comprehension and keep its weak version, the *Schema of Separation*:

If P is a property, then for any X there exists a set $Y = \{x \in X : P(x)\}$.

Once we give up the full Comprehension Schema, Russell's Paradox is no longer a threat; moreover, it provides this useful information: The set of all sets does not exist. (Otherwise, apply the Separation Schema to the property $x \notin x$.)

In other words, it is the concept of the set of all sets that is paradoxical, not the idea of comprehension itself.

Replacing full Comprehension by Separation presents us with a new problem. The Separation Axioms are too weak to develop set theory with its usual operations and constructions. Notably, these axioms are not sufficient to prove that, e.g., the union $X \cup Y$ of two sets exists, or to define the notion of a real number.

Thus we have to add further construction principles that postulate the existence of sets obtained from other sets by means of certain operations.

The axioms of ZFC are generally accepted as a correct formalization of those principles that mathematicians apply when dealing with sets.

Language of Set Theory, Formulas

The Axiom Schema of Separation as formulated above uses the vague notion of a *property*. To give the axioms a precise form, we develop axiomatic set theory in the framework of the first order predicate calculus. Apart from the equality predicate $=$, the language of set theory consists of the binary predicate $\in$, the *membership relation*.

The *formulas* of set theory are built up from the *atomic formulas*

$$x \in y, \qquad x = y$$

by means of *connectives*

$$\varphi \wedge \psi, \qquad \varphi \vee \psi, \qquad \neg\varphi, \qquad \varphi \to \psi, \qquad \varphi \leftrightarrow \psi$$

(conjunction, disjunction, negation, implication, equivalence), and *quantifiers*

$$\forall x\, \varphi, \qquad \exists x\, \varphi.$$

In practice, we shall use in formulas other symbols, namely defined predicates, operations, and constants, and even use formulas informally; but it will be tacitly understood that each such formula can be written in a form that only involves $\in$ and $=$ as nonlogical symbols.

Concerning formulas with free variables, we adopt the notational convention that all free variables of a formula

$$\varphi(u_1, \ldots, u_n)$$

are among $u_1, \ldots, u_n$ (possibly some u_i are not free, or even do not occur, in φ). A formula without free variables is called a *sentence*.

Classes

Although we work in ZFC which, unlike alternative axiomatic set theories, has only one type of object, namely sets, we introduce the informal notion of a *class*. We do this for practical reasons: It is easier to manipulate classes than formulas.

If $\varphi(x, p_1, \ldots, p_n)$ is a formula, we call

$$C = \{x : \varphi(x, p_1, \ldots, p_n)\}$$

a *class*. Members of the class C are all those sets x that satisfy $\varphi(x, p_1, \ldots, p_n)$:

$$x \in C \quad \text{if and only if} \quad \varphi(x, p_1, \ldots, p_n).$$

We say that C is *definable from* $p_1, \ldots, p_n$; if $\varphi(x)$ has no parameters p_i then the class C is *definable*.

Two classes are considered equal if they have the same elements: If

$$C = \{x : \varphi(x, p_1, \ldots, p_n)\}, \qquad D = \{x : \psi(x, q_1, \ldots, q_m)\},$$

then $C = D$ if and only if for all x

$$\varphi(x, p_1, \ldots, p_n) \leftrightarrow \psi(x, q_1, \ldots, q_m).$$

The *universal class*, or *universe*, is the class of all sets:

$$V = \{x : x = x\}.$$

We define *inclusion* of classes (C is a *subclass* of D)

$$C \subset D \text{ if and only if for all } x,\ x \in C \text{ implies } x \in D,$$

and the following operations on classes:

$$\begin{aligned} C \cap D &= \{x : x \in C \text{ and } x \in D\}, \\ C \cup D &= \{x : x \in C \text{ or } x \in D\}, \\ C - D &= \{x : x \in C \text{ and } x \notin D\}, \\ \bigcup C &= \{x : x \in S \text{ for some } S \in C\} = \bigcup\{S : S \in C\}. \end{aligned}$$

Every set can be considered a class. If S is a set, consider the formula $x \in S$ and the class

$$\{x : x \in S\}.$$

That the set S is uniquely determined by its elements follows from the Axiom of Extensionality.

A class that is not a set is a *proper class*.

Extensionality

If X and Y have the same elements, then $X = Y$:

$$\forall u\,(u \in X \leftrightarrow u \in Y) \rightarrow X = Y.$$

The converse, namely, if $X = Y$ then $u \in X \leftrightarrow u \in Y$, is an axiom of predicate calculus. Thus we have

$$X = Y \quad \text{if and only if} \quad \forall u\,(u \in X \leftrightarrow u \in Y).$$

The axiom expresses the basic idea of a set: A set is determined by its elements.

Pairing

For any a and b there exists a set $\{a, b\}$ that contains exactly a and b:

$$\forall a\,\forall b\,\exists c\,\forall x\,(x \in c \leftrightarrow x = a \vee x = b).$$

By Extensionality, the set c is unique, and we can define the *pair*

$$\{a, b\} = \text{the unique } c \text{ such that } \forall x\,(x \in c \leftrightarrow x = a \vee x = b).$$

The *singleton* $\{a\}$ is the set

$$\{a\} = \{a, a\}.$$

Since $\{a, b\} = \{b, a\}$, we further define an *ordered pair*

$$(a, b)$$

so as to satisfy the following condition:

(1.1) $\qquad (a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

For the formal definition of an ordered pair, we take

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

We leave the verification of (1.1) to the reader (Exercise 1.1).

We further define ordered triples, quadruples, etc., as follows:

$$\begin{aligned}(a, b, c) &= ((a, b), c),\\ (a, b, c, d) &= ((a, b, c), d),\\ &\vdots\\ (a_1, \ldots, a_{n+1}) &= ((a_1, \ldots, a_n), a_{n+1}).\end{aligned}$$

It follows that two ordered n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are equal if and only if $a_1 = b_1$, …, $a_n = b_n$.

Separation Schema

Let $\varphi(u, p)$ be a formula. For any X and p, there exists a set $Y = \{u \in X : \varphi(u, p)\}$:

(1.2) $\qquad \forall X\,\forall p\,\exists Y\,\forall u\,(u \in Y \leftrightarrow u \in X \wedge \varphi(u, p)).$

For each formula $\varphi(u, p)$, the formula (1.2) is an Axiom (of Separation). The set Y in (1.2) is unique by Extensionality.

Note that a more general version of Separation Axioms can be proved using ordered n-tuples: Let $\psi(u, p_1, \ldots, p_n)$ be a formula. Then

(1.3) $\qquad \forall X\,\forall p_1\,\ldots\,\forall p_n\,\exists Y\,\forall u\,(u \in Y \leftrightarrow u \in X \wedge \psi(u, p_1, \ldots, p_n)).$

Simply let $\varphi(u, p)$ be the formula

$$\exists p_1, \ldots \exists p_n \, (p = (p_1, \ldots, p_n) \text{ and } \psi(u, p_1, \ldots, p_n))$$

and then, given X and $p_1, \ldots, p_n$, let

$$Y = \{u \in X : \varphi(u, (p_1, \ldots, p_n))\}.$$

We can give the Separation Axioms the following form: Consider the class $C = \{u : \varphi(u, p_1, \ldots, p_n)\}$; then by (1.3)

$$\forall X \, \exists Y \, (C \cap X = Y).$$

Thus the intersection of a class C with any set is a set; or, we can say even more informally

a subclass of a set is a set.

One consequence of the Separation Axioms is that the intersection and the difference of two sets is a set, and so we can define the operations

$$X \cap Y = \{u \in X : u \in Y\} \quad \text{and} \quad X - Y = \{u \in X : u \notin Y\}.$$

Similarly, it follows that the empty class

$$\emptyset = \{u : u \neq u\}$$

is a set—the *empty set*; this, of course, only under the assumption that at least one set X exists (because $\emptyset \subset X$):

$$\exists X \, (X = X). \tag{1.4}$$

We have not included (1.4) among the axioms, because it follows from the Axiom of Infinity.

Two sets X, Y are called *disjoint* if $X \cap Y = \emptyset$.

If C is a nonempty class of sets, we let

$$\bigcap C = \bigcap \{X : X \in C\} = \{u : u \in X \text{ for every } X \in C\}.$$

Note that $\bigcap C$ is a set (it is a subset of any $X \in C$). Also, $X \cap Y = \bigcap \{X, Y\}$.

Another consequence of the Separation Axioms is that the universal class V is a proper class; otherwise,

$$S = \{x \in V : x \notin x\}$$

would be a set.

Union

For any X there exists a set $Y = \bigcup X$:

$$\forall X\, \exists Y\, \forall u\, (u \in Y \leftrightarrow \exists z\, (z \in X \wedge u \in z)). \tag{1.5}$$

Let us introduce the abbreviations

$$(\exists z \in X)\, \varphi \quad \text{for} \quad \exists z\, (z \in X \wedge \varphi),$$

and

$$(\forall z \in X)\, \varphi \quad \text{for} \quad \forall z\, (z \in X \to \varphi).$$

By (1.5), for every X there is a unique set

$$Y = \{u : (\exists z \in X)\, u \in z\} = \bigcup \{z : z \in X\} = \bigcup X,$$

the *union* of X.

Now we can define

$$X \cup Y = \bigcup \{X, Y\}, \qquad X \cup Y \cup Z = (X \cup Y) \cup Z, \qquad \text{etc.},$$

and also

$$\{a, b, c\} = \{a, b\} \cup \{c\},$$

and in general

$$\{a_1, \ldots, a_n\} = \{a_1\} \cup \ldots \cup \{a_n\}.$$

We also let

$$X \triangle Y = (X - Y) \cup (Y - X),$$

the *symmetric difference* of X and Y.

Power Set

For any X there exists a set $Y = P(X)$:

$$\forall X\, \exists Y\, \forall u\, (u \in Y \leftrightarrow u \subset X).$$

A set U is a *subset* of X, $U \subset X$, if

$$\forall z\, (z \in U \to z \in X).$$

If $U \subset X$ and $U \neq X$, then U is a *proper subset* of X.

The set of all subsets of X,

$$P(X) = \{u : u \subset X\},$$

is called the *power set* of X.

Using the Power Set Axiom we can define other basic notions of set theory.

The *product* of X and Y is the set of all pairs (x, y) such that $x \in X$ and $y \in Y$:

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}. \tag{1.6}$$

The notation $\{(x, y) : \dots\}$ in (1.6) is justified because

$$\{(x, y) : \varphi(x, y)\} = \{u : \exists x\, \exists y\, (u = (x, y) \text{ and } \varphi(x, y))\}.$$

The product $X \times Y$ is a set because

$$X \times Y \subset PP(X \cup Y).$$

Further, we define

$$X \times Y \times Z = (X \times Y) \times Z,$$

and in general

$$X_1 \times \dots \times X_{n+1} = (X_1 \times \dots \times X_n) \times X_{n+1}.$$

Thus

$$X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) : x_1 \in X_1 \wedge \dots \wedge x_n \in X_n\}.$$

We also let

$$X^n = \underbrace{X \times \dots \times X}_{n \text{ times}}.$$

An *n-ary relation* R is a set of n-tuples. R is a relation *on* X if $R \subset X^n$. It is customary to write $R(x_1, \dots, x_n)$ instead of

$$(x_1, \dots, x_n) \in R,$$

and in case that R is binary, then we also use

$$x\ R\ y$$

for $(x, y) \in R$.

If R is a binary relation, then the *domain* of R is the set

$$\mathrm{dom}(R) = \{u : \exists v\, (u, v) \in R\},$$

and the *range* of R is the set

$$\mathrm{ran}(R) = \{v : \exists u\, (u, v) \in R\}.$$

Note that $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ are sets because

$$\mathrm{dom}(R) \subset \bigcup\bigcup R, \qquad \mathrm{ran}(R) \subset \bigcup\bigcup R.$$

The *field* of a relation R is the set $\mathrm{field}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R)$.

In general, we call a class R an n-ary relation if all its elements are n-tuples; in other words, if

$$R \subset V^n = \text{the class of all } n\text{-tuples},$$

where C^n (and $C \times D$) is defined in the obvious way.

A binary relation f is a *function* if $(x, y) \in f$ and $(x, z) \in f$ implies $y = z$. The unique y such that $(x, y) \in f$ is the *value* of f at x; we use the standard notation

$$y = f(x)$$

or its variations $f : x \mapsto y$, $y = f_x$, etc. for $(x, y) \in f$.

f is a function *on* X if $X = \mathrm{dom}(f)$. If $\mathrm{dom}(f) = X^n$, then f is an *n-ary function* on X.

f is a function *from* X to Y,

$$f : X \to Y,$$

if $\mathrm{dom}(f) = X$ and $\mathrm{ran}(f) \subset Y$. The set of all functions from X to Y is denoted by Y^X. Note that Y^X is a set:

$$Y^X \subset P(X \times Y).$$

If $Y = \mathrm{ran}(f)$, then f is a function *onto* Y. A function f is *one-to-one* if

$$f(x) = f(y) \quad \text{implies} \quad x = y.$$

An *n-ary operation on* X is a function $f : X^n \to X$.

The *restriction* of a function f to a set X (usually a subset of $\mathrm{dom}(f)$) is the function

$$f{\restriction}X = \{(x, y) \in f : x \in X\}.$$

A function g is an *extension* of a function f if $g \supset f$, i.e., $\mathrm{dom}(f) \subset \mathrm{dom}(g)$ and $g(x) = f(x)$ for all $x \in \mathrm{dom}(f)$.

If f and g are functions such that $\mathrm{ran}(g) \subset \mathrm{dom}(f)$, then the *composition* of f and g is the function $f \circ g$ with domain $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$ such that $(f \circ g)(x) = f(g(x))$ for all $x \in \mathrm{dom}(g)$.

We denote the *image* of X by f either $f\text{“}X$ or $f(X)$:

$$f\text{“}X = f(X) = \{y : (\exists x \in X)\, y = f(x)\},$$

and the *inverse image* by

$$f_{-1}(X) = \{x : f(x) \in X\}.$$

If f is one-to-one, then f^{-1} denotes the *inverse* of f:

$$f^{-1}(x) = y \quad \text{if and only if} \quad x = f(y).$$

The previous definitions can also be applied to classes instead of sets. A class F is a function if it is a relation such that $(x, y) \in F$ and $(x, z) \in F$

implies $y = z$. For example, $F``C$ or $F(C)$ denotes the image of the class C by the function F.

It should be noted that a function is often called a *mapping* or a *correspondence* (and similarly, a set is called a *family* or a *collection*).

An *equivalence relation* on a set X is a binary relation $\equiv$ which is *reflexive*, *symmetric*, and *transitive*: For all $x, y, z \in X$,

$$x \equiv x,$$
$$x \equiv y \text{ implies } y \equiv x,$$
$$\text{if } x \equiv y \text{ and } y \equiv z \text{ then } x \equiv z.$$

A family of sets is *disjoint* if any two of its members are disjoint. A *partition* of a set X is a disjoint family P of nonempty sets such that

$$X = \bigcup\{Y : Y \in P\}.$$

Let $\equiv$ be an equivalence relation on X. For every $x \in X$, let

$$[x] = \{y \in X : y \equiv x\}$$

(the *equivalence class* of x). The set

$$X/\equiv = \{[x] : x \in X\}$$

is a partition of X (the *quotient* of X by $\equiv$). Conversely, each partition P of X defines an equivalence relation on X:

$$x \equiv y \quad \text{if and only if} \quad (\exists Y \in P)(x \in Y \text{ and } y \in Y).$$

If an equivalence relation is a class, then its equivalence classes may be proper classes. In Chapter 6 we shall introduce a trick that enables us to handle equivalence classes as if they were sets.

Infinity

There exists an infinite set.

To give a precise formulation of the Axiom of Infinity, we have to define first the notion of finiteness. The most obvious definition of finiteness uses the notion of a natural number, which is as yet undefined. We shall define natural numbers (as finite ordinals) in Chapter 2 and give only a quick treatment of natural numbers and finiteness in the exercises below.

In principle, it is possible to give a definition of finiteness that does not mention numbers, but such definitions necessarily look artificial.

We therefore formulate the Axiom of Infinity differently:

$$\exists S\,(\emptyset \in S \wedge (\forall x \in S)x \cup \{x\} \in S).$$

We call a set S with the above property *inductive*. Thus we have:

Axiom of Infinity. *There exists an inductive set.*

The axiom provides for the existence of infinite sets. In Chapter 2 we show that an inductive set is infinite (and that an inductive set exists if there exists an infinite set).

We shall introduce natural numbers and finite sets in Chapter 2, as a part of the introduction of ordinal numbers. In Exercises 1.3–1.9 we show an alternative approach.

Replacement Schema

If a class F is a function, then for every set X, $F(X)$ is a set.

For each formula $\varphi(x, y, p)$, the formula (1.7) is an Axiom (of Replacement):

$$(1.7)\quad \forall x\,\forall y\,\forall z\,(\varphi(x,y,p) \wedge \varphi(x,z,p) \to y = z) \\ \to \forall X\,\exists Y\,\forall y\,(y \in Y \leftrightarrow (\exists x \in X)\,\varphi(x,y,p)).$$

As in the case of Separation Axioms, we can prove the version of Replacement Axioms with several parameters: Replace p by $p_1, \ldots, p_n$.

If $F = \{(x, y) : \varphi(x, y, p)\}$, then the premise of (1.7) says that F is a function, and we get the formulation above. We can also formulate the axioms in the following ways:

If a class F is a function and $\mathrm{dom}(F)$ *is a set, then* $\mathrm{ran}(F)$ *is a set.*
If a class F is a function, then $\forall X\,\exists f\,(F \restriction X = f)$.

The remaining two axioms, Choice and Regularity, will by introduced in Chapters 5 and 6.

Exercises

1.1. Verify (1.1).

1.2. There is no set X such that $P(X) \subset X$.

Let

$$\boldsymbol{N} = \bigcap\{X : X \text{ is inductive}\}.$$

$\boldsymbol{N}$ is the smallest inductive set. Let us use the following notation:

$$0 = \emptyset, \qquad 1 = \{0\}, \qquad 2 = \{0, 1\}, \qquad 3 = \{0, 1, 2\}, \qquad \ldots.$$

If $n \in \boldsymbol{N}$, let $n + 1 = n \cup \{n\}$. Let us define $<$ (on $\boldsymbol{N}$) by $n < m$ if and only if $n \in m$.

A set T is *transitive* if $x \in T$ implies $x \subset T$.

1.3. If X is inductive, then the set $\{x \in X : x \subset X\}$ is inductive. Hence $\boldsymbol{N}$ is transitive, and for each n, $n = \{m \in \boldsymbol{N} : m < n\}$.

1.4. If X is inductive, then the set $\{x \in X : x \text{ is transitive}\}$ is inductive. Hence every $n \in \boldsymbol{N}$ is transitive.

1.5. If X is inductive, then the set $\{x \in X : x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n + 1$ for each $n \in \boldsymbol{N}$.

1.6. If X is inductive, then $\{x \in X : x$ is transitive and every nonempty $z \subset x$ has an $\in$-minimal element$\}$ is inductive (t is *$\in$-minimal* in z if there is no $s \in z$ such that $s \in t$).

1.7. Every nonempty $X \subset \boldsymbol{N}$ has an $\in$-minimal element.
[Pick $n \in X$ and look at $X \cap n$.]

1.8. If X is inductive then so is $\{x \in X : x = \emptyset$ or $x = y \cup \{y\}$ for some $y\}$. Hence each $n \neq 0$ is $m + 1$ for some m.

1.9 (Induction). Let A be a subset of $\boldsymbol{N}$ such that $0 \in A$, and if $n \in A$ then $n + 1 \in A$. Then $A = \boldsymbol{N}$.

A set X *has n elements* (where $n \in \boldsymbol{N}$) if there is a one-to-one mapping of n onto X. A set is *finite* if it has n elements for some $n \in \boldsymbol{N}$, and *infinite* if it is not finite.

A set S is *T-finite* if every nonempty $X \subset P(S)$ has a $\subset$-maximal element, i.e., $u \in X$ such that there is no $v \in X$ with $u \subset v$ and $u \neq v$. S is T-*infinite* if it is not T-finite. (T is for Tarski.)

1.10. Each $n \in \boldsymbol{N}$ is T-finite.

1.11. $\boldsymbol{N}$ is T-infinite; the set $\boldsymbol{N} \subset P(\boldsymbol{N})$ has no $\subset$-maximal element.

1.12. Every finite set is T-finite.

1.13. Every infinite set is T-infinite.
[If S is infinite, consider $X = \{u \subset S : u \text{ is finite}\}$.]

1.14. The Separation Axioms follow from the Replacement Schema.
[Given φ, let $F = \{(x, x) : \varphi(x)\}$. Then $\{x \in X : \varphi(x)\} = F(X)$, for every X.]

1.15. Instead of Union, Power Set, and Replacement Axioms consider the following weaker versions:

(1.8) $\forall X\, \exists Y \bigcup X \subset Y$, i.e., $\forall X\, \exists Y\, (\forall x \in X)(\forall u \in x)\, u \in Y$,

(1.9) $\forall X\, \exists Y\, P(X) \subset Y$, i.e., $\forall X\, \exists Y\, \forall u\, (u \subset X \to u \in Y)$,

(1.10) If a class F is a function, then $\forall X\, \exists Y\, F(X) \subset Y$.

Then axioms 1.4, 1.5, and 1.7 can be proved from (1.8), (1.9), and (1.10), using the Separation Schema (1.3).

Historical Notes

Set theory was invented by Georg Cantor. The first attempt to consider infinite sets is attributed to Bolzano (who introduced the term *Menge*). It was however Cantor who realized the significance of one-to-one functions between sets and introduced the notion of cardinality of a set. Cantor originated the theory of cardinal and ordinal numbers as well as the investigations of the topology of the real line. Much of the development in the first four chapters follows Cantor's work. The main reference to Cantor's work is his collected works, Cantor [1932]. Another source of references to the early research in set theory is Hausdorff's book [1914].

Cantor started his investigations in [1874], where he proved that the set of all real numbers is uncountable, while the set of all algebraic reals is countable. In [1878] he gave the first formulation of the celebrated Continuum Hypothesis.

The axioms for set theory (except Replacement and Regularity) are due to Zermelo [1908]. The Replacement Schema is due to Fraenkel [1922a] and Skolem (see [1970], pp. 137–152).

Exercises 1.12 and 1.13: Tarski [1925a].

2. Ordinal Numbers

In this chapter we introduce ordinal numbers and prove the Transfinite Recursion Theorem.

Linear and Partial Ordering

Definition 2.1. A binary relation $<$ on a set P is a *partial ordering* of P if:

(i) $p \not< p$ for any $p \in P$;
(ii) if $p < q$ and $q < r$, then $p < r$.

$(P, <)$ is called a *partially ordered set*. A partial ordering $<$ of P is a *linear ordering* if moreover

(iii) $p < q$ or $p = q$ or $q < p$ for all $p, q \in P$.

If $<$ is a partial (linear) ordering, then the relation $\leq$ (where $p \leq q$ if either $p < q$ or $p = q$) is also called a partial (linear) ordering (and $<$ is sometimes called a *strict* ordering).

Definition 2.2. If $(P, <)$ is a partially ordered set, X is a nonempty subset of P, and $a \in P$, then:

a is a *maximal* element of X if $a \in X$ and $(\forall x \in X)\, a \not< x$;
a is a *minimal* element of X if $a \in X$ and $(\forall x \in X)\, x \not< a$;
a is the *greatest* element of X if $a \in X$ and $(\forall x \in X)\, x \leq a$;
a is the *least* element of X if $a \in X$ and $(\forall x \in X)\, a \leq x$;
a is an *upper bound* of X if $(\forall x \in X)\, x \leq a$;
a is a *lower bound* of X if $(\forall x \in X)\, a \leq x$;
a is the *supremum* of X if a is the least upper bound of X;
a is the *infimum* of X if a is the greatest lower bound of X.

The supremum (infimum) of X (if it exists) is denoted $\sup X$ ($\inf X$). Note that if X is linearly ordered by $<$, then a maximal element of X is its greatest element (similarly for a minimal element).

If $(P, <)$ and $(Q, <)$ are partially ordered sets and $f : P \to Q$, then f is *order-preserving* if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then an order-preserving function is also called *increasing*.

A one-to-one function of P onto Q is an *isomorphism* of P and Q if both f and f^{-1} are order-preserving; $(P, <)$ is then *isomorphic* to $(Q, <)$. An isomorphism of P onto itself is an *automorphism* of $(P, <)$.

Well-Ordering

Definition 2.3. A linear ordering $<$ of a set P is a *well-ordering* if every nonempty subset of P has a least element.

The concept of well-ordering is of fundamental importance. It is shown below that well-ordered sets can be compared by their lengths; ordinal numbers will be introduced as order-types of well-ordered sets.

Lemma 2.4. *If $(W, <)$ is a well-ordered set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$.*

Proof. Assume that the set $X = \{x \in W : f(x) < x\}$ is nonempty and let z be the least element of X. If $w = f(z)$, then $f(w) < w$, a contradiction. □

Corollary 2.5. *The only automorphism of a well-ordered set is the identity.*

Proof. By Lemma 2.4, $f(x) \geq x$ for all x, and $f^{-1}(x) \geq x$ for all x. □

Corollary 2.6. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique.* □

If W is a well-ordered set and $u \in W$, then $\{x \in W : x < u\}$ is an *initial segment* of W (given by u).

Lemma 2.7. *No well-ordered set is isomorphic to an initial segment of itself.*

Proof. If $\text{ran}(f) = \{x : x < u\}$, then $f(u) < u$, contrary to Lemma 2.4. □

Theorem 2.8. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds*:

(i) *W_1 is isomorphic to W_2;*
(ii) *W_1 is isomorphic to an initial segment of W_2;*
(iii) *W_2 is isomorphic to an initial segment of W_1.*

Proof. For $u \in W_i$, $(i = 1, 2)$, let $W_i(u)$ denote the initial segment of W_i given by u. Let

$$f = \{(x, y) \in W_1 \times W_2 : W_1(x) \text{ is isomorphic to } W_2(y)\}.$$

Using Lemma 2.7, it is easy to see that f is a one-to-one function. If h is an isomorphism between $W_1(x)$ and $W_2(y)$, and $x' < x$, then $W_1(x')$ and $W_2(h(x'))$ are isomorphic. It follows that f is order-preserving.

If $\mathrm{dom}(f) = W_1$ and $\mathrm{ran}(f) = W_2$, then case (i) holds.

If $y_1 < y_2$ and $y_2 \in \mathrm{ran}(f)$, then $y_1 \in \mathrm{ran}(f)$. Thus if $\mathrm{ran}(f) \neq W_2$ and y_0 is the least element of $W_2 - \mathrm{ran}(f)$, we have $\mathrm{ran}(f) = W_2(y_0)$. Necessarily, $\mathrm{dom}(f) = W_1$, for otherwise we would have $(x_0, y_0) \in f$, where $x_0 =$ the least element of $W_1 - \mathrm{dom}(f)$. Thus case (ii) holds.

Similarly, if $\mathrm{dom}(f) \neq W_1$, then case (iii) holds.

In view of Lemma 2.7, the three cases are mutually exclusive. □

If W_1 and W_2 are isomorphic, we say that they have the same *order-type*. Informally, an ordinal number is the order-type of a well-ordered set.

We shall now give a formal definition of ordinal numbers.

Ordinal Numbers

The idea is to define ordinal numbers so that

$$\alpha < \beta \quad \text{if and only if} \quad \alpha \in \beta, \qquad \text{and} \quad \alpha = \{\beta : \beta < \alpha\}.$$

Definition 2.9. A set T is *transitive* if every element of T is a subset of T. (Equivalently, $\bigcup T \subset T$, or $T \subset P(T)$.)

Definition 2.10. A set is an *ordinal number* (an *ordinal*) if it is transitive and well-ordered by $\in$.

We shall denote ordinals by lowercase Greek letters α, β, γ, ... The class of all ordinals is denoted by *Ord*.

We define

$$\alpha < \beta \quad \text{if and only if} \quad \alpha \in \beta.$$

Lemma 2.11.

(i) $0 = \emptyset$ *is an ordinal.*
(ii) *If* α *is an ordinal and* $\beta \in \alpha$, *then* β *is an ordinal.*
(iii) *If* $\alpha \neq \beta$ *are ordinals and* $\alpha \subset \beta$, *then* $\alpha \in \beta$.
(iv) *If* α, β *are ordinals, then either* $\alpha \subset \beta$ *or* $\beta \subset \alpha$.

Proof. (i), (ii) by definition.

(iii) If $\alpha \subset \beta$, let γ be the least element of the set $\beta - \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ. Thus $\alpha = \{\xi \in \beta : \xi < \gamma\} = \gamma$, and so $\alpha \in \beta$.

(iv) Clearly, $\alpha \cap \beta$ is an ordinal, $\alpha \cap \beta = \gamma$. We have $\gamma = \alpha$ or $\gamma = \beta$, for otherwise $\gamma \in \alpha$, and $\gamma \in \beta$, by (iii). Then $\gamma \in \gamma$, which contradicts the definition of an ordinal (namely that $\in$ is a *strict* ordering of α). □

Using Lemma 2.11 one gets the following facts about ordinal numbers (the proofs are routine):

(2.1) $<$ is a linear ordering of the class *Ord*.

(2.2) For each α, $\alpha = \{\beta : \beta < \alpha\}$.

(2.3) If C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$.

(2.4) If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal, and $\bigcup X = \sup X$.

(2.5) For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$.

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$ (the *successor* of α). In view of (2.4), the class *Ord* is a proper class; otherwise, consider $\sup Ord + 1$.

We can now prove that the above definition of ordinals provides us with order-types of well-ordered sets.

Theorem 2.12. *Every well-ordered set is isomorphic to a unique ordinal number.*

Proof. The uniqueness follows from Lemma 2.7. Given a well-ordered set W, we find an isomorphic ordinal as follows: Define $F(x) = \alpha$ if α is isomorphic to the initial segment of W given by x. If such an α exists, then it is unique. By the Replacement Axioms, $F(W)$ is a set. For each $x \in W$, such an α exists (otherwise consider the least x for which such an α does not exist). If γ is the least $\gamma \notin F(W)$, then $F(W) = \gamma$ and we have an isomorphism of W onto γ. □

If $\alpha = \beta + 1$, then α is a *successor ordinal*. If α is not a successor ordinal, then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$; α is called a *limit ordinal*. We also consider 0 a limit ordinal and define $\sup \emptyset = 0$.

The existence of limit ordinals other than 0 follows from the Axiom of Infinity; see Exercise 2.3.

Definition 2.13 (Natural Numbers). We denote the least nonzero limit ordinal ω (or $\boldsymbol{N}$). The ordinals less than ω (elements of $\boldsymbol{N}$) are called *finite ordinals*, or *natural numbers*. Specifically,

$$0 = \emptyset, \qquad 1 = 0 + 1, \qquad 2 = 1 + 1, \qquad 3 = 2 + 1, \qquad \text{etc.}$$

A set X is *finite* if there is a one-to-one mapping of X onto some $n \in \boldsymbol{N}$. X is *infinite* if it is not finite.

We use letters n, m, l, k, j, i (most of the time) to denote natural numbers.

Induction and Recursion

Theorem 2.14 (Transfinite Induction). *Let C be a class of ordinals and assume that*:

(i) $0 \in C$;
(ii) *if* $\alpha \in C$, *then* $\alpha + 1 \in C$;
(iii) *if* α *is a nonzero limit ordinal and* $\beta \in C$ *for all* $\beta < \alpha$, *then* $\alpha \in C$.

Then C is the class of all ordinals.

Proof. Otherwise, let α be the least ordinal $\alpha \notin C$ and apply (i), (ii), or (iii). □

A function whose domain is the set $\boldsymbol{N}$ is called an (*infinite*) *sequence* (A *sequence in* X is a function $f : \boldsymbol{N} \to X$.) The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

or variants thereof. A *finite sequence* is a function s such $\mathrm{dom}(s) = \{i : i < n\}$ for some $n \in \boldsymbol{N}$; then s is a *sequence of length* n.

A *transfinite sequence* is a function whose domain is an ordinal:

$$\langle a_\xi : \xi < \alpha \rangle.$$

It is also called an α-*sequence* or a *sequence of length* α. We also say that a sequence $\langle a_\xi : \xi < \alpha \rangle$ is an *enumeration* of its range $\{a_\xi : \xi < \alpha\}$. If s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cup \{(\alpha, x)\}.$$

Sometimes we shall call a "sequence"

$$\langle a_\alpha : \alpha \in Ord \rangle$$

a function (a proper class) on *Ord*.

"Definition by transfinite recursion" usually takes the following form: Given a function G (on the class of transfinite sequences), then for every θ there exists a unique θ-sequence

$$\langle a_\alpha : \alpha < \theta \rangle$$

such that

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

for every $\alpha < \theta$.

We shall give a general version of this theorem, so that we can also construct sequences $\langle a_\alpha : \alpha \in Ord \rangle$.

Theorem 2.15 (Transfinite Recursion). *Let G be a function (on V), then* (2.6) *below defines a unique function F on Ord such that*

$$F(\alpha) = G(F{\restriction}\alpha)$$

for each α.

In other words, if we let $a_\alpha = F(\alpha)$, then for each α,

$$a_\alpha = G(\langle a_\xi : \xi < \alpha\rangle).$$

(Note that we tacitly use Replacement: $F{\restriction}\alpha$ is a set for each α.)

Corollary 2.16. *Let X be a set and θ an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ such that $\mathrm{ran}(G) \subset X$ there exists a unique θ-sequence $\langle a_\alpha : \alpha < \theta\rangle$ in X such that $a_\alpha = G(\langle a_\xi : \xi < \alpha\rangle)$ for every $\alpha < \theta$.* □

Proof. Let

(2.6) $F(\alpha) = x \leftrightarrow$ there is a sequence $\langle a_\xi : \xi < \alpha\rangle$ such that:
(i) $(\forall \xi < \alpha)\, a_\xi = G(\langle a_\eta : \eta < \xi\rangle)$;
(ii) $x = G(\langle a_\xi : \xi < \alpha\rangle)$.

For every α, if there is an α-sequence that satisfies (i), then such a sequence is unique: If $\langle a_\xi : \xi < \alpha\rangle$ and $\langle b_\xi : \xi < \alpha\rangle$ are two α-sequences satisfying (i), one shows $a_\xi = b_\xi$ by induction on ξ. Thus $F(\alpha)$ is determined uniquely by (ii), and therefore F is a function. It follows, again by induction, that for each α there is an α-sequence that satisfies (i) (at limit steps, we use Replacement to get the α-sequence as the union of all the ξ-sequences, $\xi < \alpha$). Thus F is defined for all $\alpha \in Ord$. It obviously satisfies

$$F(\alpha) = G(F{\restriction}\alpha).$$

If F' is any function on Ord that satisfies

$$F'(\alpha) = G(F'{\restriction}\alpha)$$

then it follows by induction that $F'(\alpha) = F(\alpha)$ for all α. □

Definition 2.17. Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi : \xi < \alpha\rangle$ be a *nondecreasing* sequence of ordinals (i.e., $\xi < \eta$ implies $\gamma_\xi \le \gamma_\eta$). We define the *limit* of the sequence by

$$\lim_{\xi\to\alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}.$$

A sequence of ordinals $\langle \gamma_\alpha : \alpha \in Ord\rangle$ is *normal* if it is increasing and *continuous*, i.e., for every limit α, $\gamma_\alpha = \lim_{\xi\to\alpha} \gamma_\xi$.

Ordinal Arithmetic

We shall now define addition, multiplication and exponentiation of ordinal numbers, using Transfinite Recursion.

Definition 2.18 (Addition). For all ordinal numbers α

(i) $\alpha + 0 = \alpha$,
(ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all β,
(iii) $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limit $\beta > 0$.

Definition 2.19 (Multiplication). For all ordinal numbers α

(i) $\alpha \cdot 0 = 0$,
(ii) $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ for all β,
(iii) $\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi$ for all limit $\beta > 0$.

Definition 2.20 (Exponentiation). For all ordinal numbers α

(i) $\alpha^0 = 1$,
(ii) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ for all β,
(iii) $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$ for all limit $\beta > 0$.

As defined, the operations $\alpha + \beta$, $\alpha \cdot \beta$ and α^β are normal functions in the second variable β. Their properties can be proved by transfinite induction. For instance, $+$ and $\cdot$ are associative:

Lemma 2.21. *For all ordinals α, β and γ,*

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,
(ii) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

Proof. By induction on γ. □

Neither $+$ nor $\cdot$ are commutative:

$$1 + \omega = \omega \neq \omega + 1, \qquad 2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega.$$

Ordinal sums and products can be also defined geometrically, as can sums and products of arbitrary linear orders:

Definition 2.22. Let $(A, <_A)$ and $(B, <_B)$ be disjoint linearly ordered sets. The *sum* of these linear orders is the set $A \cup B$ with the ordering defined as follows: $x < y$ if and only if

(i) $x, y \in A$ and $x <_A y$, or
(ii) $x, y \in B$ and $x <_B y$, or
(iii) $x \in A$ and $y \in B$.

Definition 2.23. Let $(A, <)$ and $(B, <)$ be linearly ordered sets. The *product* of these linear orders is the set $A \times B$ with the ordering defined by

$$(a_1, b_1) < (a_2, b_2) \text{ if and only if either } b_1 < b_2 \text{ or } (b_1 = b_2 \text{ and } a_1 < a_2).$$

Lemma 2.24. *For all ordinals α and β, $\alpha + \beta$ and $\alpha \cdot \beta$ are, respectively, isomorphic to the sum and to the product of α and β.*

Proof. By induction on β. □

Ordinal sums and products have some properties of ordinary addition and multiplication of integers. For instance:

Lemma 2.25.

(i) *If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$.*
(ii) *If $\alpha < \beta$ then there exists a unique δ such that $\alpha + \delta = \beta$.*
(iii) *If $\beta < \gamma$ and $\alpha > 0$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.*
(iv) *If $\alpha > 0$ and γ is arbitrary, then there exist a unique β and a unique $\rho < \alpha$ such that $\gamma = \alpha \cdot \beta + \rho$.*
(v) *If $\beta < \gamma$ and $\alpha > 1$, then $\alpha^\beta < \alpha^\gamma$.*

Proof. (i), (iii) and (v) are proved by induction on γ.
(ii) Let δ be the order-type of the set $\{\xi : \alpha \leq \xi < \beta\}$; δ is unique by (i).
(iv) Let β be the greatest ordinal such that $\alpha \cdot \beta \leq \gamma$. □

For more, see Exercises 2.10 and 2.11.

Theorem 2.26 (Cantor's Normal Form Theorem). *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n,$$

where $n \geq 1$, $\alpha \geq \beta_1 > \ldots > \beta_n$, and $k_1, \ldots, k_n$ are nonzero natural numbers.

Proof. By induction on α. For $\alpha = 1$ we have $1 = \omega^0 \cdot 1$; for arbitrary $\alpha > 0$ let β be the greatest ordinal such that $\omega^\beta \leq \alpha$. By Lemma 2.25(iv) there exists a unique δ and a unique $\rho < \omega^\beta$ such that $\alpha = \omega^\beta \cdot \delta + \rho$; this δ must necessarily be finite. The uniqueness of the normal form is proved by induction. □

In the normal form it is possible to have $\alpha = \omega^\alpha$; see Exercise 2.12. The least ordinal with this property is called ε_0.

Well-Founded Relations

Now we shall define an important generalization of well-ordered sets.

A binary relation E on a set P is *well-founded* if every nonempty $X \subset P$ has an E-*minimal* element, that is $a \in X$ such that there is no $x \in X$ with $x\ E\ a$.

Clearly, a well-ordering of P is a well-founded relation.

Given a well-founded relation E on a set P, we can define the *height* of E, and assign to each $x \in P$ an ordinal number, the *rank* of x in E.

Theorem 2.27. *If E is a well-founded relation on P, then there exists a unique function ρ from P into the ordinals such that for all $x \in P$,*

$$\rho(x) = \sup\{\rho(y) + 1 : y\ E\ x\}. \tag{2.7}$$

The range of ρ is an initial segment of the ordinals, thus an ordinal number. This ordinal is called the *height* of E.

Proof. We shall define a function ρ satisfying (2.7) and then prove its uniqueness. By induction, let

$$P_0 = \emptyset, \qquad P_{\alpha+1} = \{x \in P : \forall y\,(y\ E\ x \to y \in P_\alpha)\},$$
$$P_\alpha = \bigcup_{\xi<\alpha} P_\xi \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Let θ be the least ordinal such that $P_{\theta+1} = P_\theta$ (such θ exists by Replacement). First, it should be easy to see that $P_\alpha \subset P_{\alpha+1}$ for each α (by induction). Thus $P_0 \subset P_1 \subset \ldots \subset P_\theta$. We claim that $P_\theta = P$. Otherwise, let a be an E-minimal element of $P - P_\theta$. It follows that each $x\ E\ a$ is in P_θ, and so $a \in P_{\theta+1}$, a contradiction. Now we define $\rho(x)$ as the least α such that $x \in P_{\alpha+1}$. It is obvious that if $x\ E\ y$, then $\rho(x) < \rho(y)$, and (2.7) is easily verified. The ordinal θ is the height of E.

The uniqueness of ρ is established as follows: Let ρ' be another function satisfying (2.7) and consider an E-minimal element of the set $\{x \in P : \rho(x) \neq \rho'(x)\}$. □

Exercises

2.1. The relation "$(P, <)$ is isomorphic to $(Q, <)$" is an equivalence relation (on the class of all partially ordered sets).

2.2. α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$, for every β.

2.3. If a set X is inductive, then $X \cap \mathit{Ord}$ is inductive. The set $\boldsymbol{N} = \bigcap\{X : X \text{ is inductive}\}$ is the least limit ordinal $\neq 0$.

2.4. (Without the Axiom of Infinity). Let ω = least limit $\alpha \neq 0$ if it exists, $\omega = Ord$ otherwise. Prove that the following statements are equivalent:

(i) There exists an inductive set.
(ii) There exists an infinite set.
(iii) ω is a set.

[For (ii) $\rightarrow$ (iii), apply Replacement to the set of all finite subsets of X.]

2.5. If W is a well-ordered set, then there exists no sequence $\langle a_n : n \in \boldsymbol{N} \rangle$ in W such that $a_0 > a_1 > a_2 > \ldots$.

2.6. There are arbitrarily large limit ordinals; i.e., $\forall \alpha \, \exists \beta > \alpha \, (\beta \text{ is a limit})$.
[Consider $\lim_{n \to \omega} \alpha_n$, where $\alpha_{n+1} = \alpha_n + 1$.]

2.7. Every normal sequence $\langle \gamma_\alpha : \alpha \in Ord \rangle$ has arbitrarily large *fixed points*, i.e., α such that $\gamma_\alpha = \alpha$.
[Let $\alpha_{n+1} = \gamma_{\alpha_n}$, and $\alpha = \lim_{n \to \omega} \alpha_n$.]

2.8. For all α, β and γ,

(i) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$,
(ii) $\alpha^{\beta + \gamma} = \alpha^\beta \cdot \alpha^\gamma$,
(iii) $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.

2.9. (i) Show that $(\omega + 1) \cdot 2 \neq \omega \cdot 2 + 1 \cdot 2$.
(ii) Show that $(\omega \cdot 2)^2 \neq \omega^2 \cdot 2^2$.

2.10. If $\alpha < \beta$ then $\alpha + \gamma \leq \beta + \gamma$, $\alpha \cdot \gamma \leq \beta \cdot \gamma$, and $\alpha^\gamma \leq \beta^\gamma$,

2.11. Find α, β, γ such that

(i) $\alpha < \beta$ and $\alpha + \gamma = \beta + \gamma$,
(ii) $\alpha < \beta$ and $\alpha \cdot \gamma = \beta \cdot \gamma$,
(iii) $\alpha < \beta$ and $\alpha^\gamma = \beta^\gamma$.

2.12. Let $\varepsilon_0 = \lim_{n \to \omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$ for all n. Show that ε_0 is the least ordinal ε such that $\omega^\varepsilon = \varepsilon$.

A limit ordinal $\gamma > 0$ is called *indecomposable* if there exist no $\alpha < \gamma$ and $\beta < \gamma$ such that $\alpha + \beta = \gamma$.

2.13. A limit ordinal $\gamma > 0$ is indecomposable if and only if $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some α.

2.14. If E is a well-founded relation on P, then there is no sequence $\langle a_n : n \in \boldsymbol{N} \rangle$ in P such that $a_1 \; E \; a_0$, $a_2 \; E \; a_1$, $a_3 \; E \; a_2$, ...

2.15 (Well-Founded Recursion). Let E be a well-founded relation on a set P, and let G be a function. Then there exists a function F such that for all $x \in P$, $F(x) = G(x, F \restriction \{y \in P : y \; E \; x\})$.

Historical Notes

The theory of well-ordered sets was developed by Cantor, who also introduced transfinite induction. The idea of identifying an ordinal number with the set of smaller ordinals is due to Zermelo and von Neumann.

3. Cardinal Numbers

Cardinality

Two sets X, Y have the same *cardinality* (cardinal number, cardinal),

$$|X| = |Y|, \tag{3.1}$$

if there exists a one-to-one mapping of X onto Y.

The relation (3.1) is an equivalence relation. We assume that we can assign to each set X its *cardinal number* $|X|$ so that two sets are assigned the same cardinal just in case they satisfy condition (3.1). Cardinal numbers can be defined either using the Axiom of Regularity (via equivalence classes of (3.1)), or using the Axiom of Choice. In this chapter we define cardinal numbers of well-orderable sets; as it follows from the Axiom of Choice that every set can be well-ordered, this defines cardinals in ZFC.

We recall that a set X is *finite* if $|X| = |n|$ for some $n \in \boldsymbol{N}$; then X is said to *have n elements*. Clearly, $|n| = |m|$ if and only if $n = m$, and so we define *finite cardinals* as natural numbers, i.e., $|n| = n$ for all $n \in \boldsymbol{N}$.

The ordering of cardinal numbers is defined as follows:

$$|X| \leq |Y| \tag{3.2}$$

if there exists a one-to-one mapping of X into Y. We also define the strict ordering $|X| < |Y|$ to mean that $|X| \leq |Y|$ while $|X| \neq |Y|$. The relation $\leq$ in (3.2) is clearly transitive. Theorem 3.2 below shows that it is indeed a partial ordering, and it follows from the Axiom of Choice that the ordering is linear—any two sets are comparable in this ordering.

The concept of cardinality is central to the study of infinite sets. The following theorem tells us that this concept is not trivial:

Theorem 3.1 (Cantor). *For every set X, $|X| < |P(X)|$.*

Proof. Let f be a function from X into $P(X)$. The set

$$Y = \{x \in X : x \notin f(x)\}$$

is not in the range of f: If $z \in X$ were such that $f(z) = Y$, then $z \in Y$ if and only if $z \notin Y$, a contradiction. Thus f is not a function of X onto $P(X)$. Hence $|P(X)| \neq |X|$.

The function $f(x) = \{x\}$ is a one-to-one function of X into $P(X)$ and so $|X| \leq |P(X)|$. It follows that $|X| < |P(X)|$. □

In view of the following theorem, $<$ is a partial ordering of cardinal numbers.

Theorem 3.2 (Cantor-Bernstein). *If* $|A| \leq |B|$ *and* $|B| \leq |A|$, *then* $|A| = |B|$.

Proof. If $f_1 : A \to B$ and $f_2 : B \to A$ are one-to-one, then if we let $B' = f_2(B)$ and $A_1 = f_2(f_1(A))$, we have $A_1 \subset B' \subset A$ and $|A_1| = |A|$. Thus we may assume that $A_1 \subset B \subset A$ and that f is a one-to-one function of A onto A_1; we will show that $|A| = |B|$.

We define (by induction) for all $n \in \boldsymbol{N}$:

$$\begin{aligned} A_0 &= A, \qquad & A_{n+1} &= f(A_n), \\ B_0 &= B, \qquad & B_{n+1} &= f(B_n). \end{aligned}$$

Let g be the function on A defined as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise.} \end{cases}$$

Then g is a one-to-one mapping of A onto B, as the reader will easily verify. Thus $|A| = |B|$. □

The arithmetic operations on cardinals are defined as follows:

$$\begin{aligned} (3.3)\quad & \kappa + \lambda = |A \cup B| && \text{where } |A| = \kappa,\ |B| = \lambda, \text{ and } A,\ B \text{ are disjoint}, \\ & \kappa \cdot \lambda = |A \times B| && \text{where } |A| = \kappa,\ |B| = \lambda, \\ & \kappa^{\lambda} = |A^B| && \text{where } |A| = \kappa,\ |B| = \lambda. \end{aligned}$$

Naturally, the definitions in (3.3) are meaningful only if they are independent of the choice of A and B. Thus one has to check that, e.g., if $|A| = |A'|$ and $|B| = |B'|$, then $|A \times B| = |A' \times B'|$.

Lemma 3.3. *If* $|A| = \kappa$, *then* $|P(A)| = 2^{\kappa}$.

Proof. For every $X \subset A$, let χ_X be the function

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \in A - X. \end{cases}$$

The mapping $f : X \to \chi_X$ is a one-to-one correspondence between $P(A)$ and $\{0,1\}^A$. □

Thus Cantor's Theorem 3.1 can be formulated as follows:

$$\kappa < 2^\kappa \quad \text{for every cardinal } \kappa.$$

A few simple facts about cardinal arithmetic:

(3.4) $+$ and $\cdot$ are associative, commutative and distributive.

(3.5) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

(3.6) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.

(3.7) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

(3.8) If $\kappa \leq \lambda$, then $\kappa^\mu \leq \lambda^\mu$.

(3.9) If $0 < \lambda \leq \mu$, then $\kappa^\lambda \leq \kappa^\mu$.

(3.10) $\kappa^0 = 1$; $1^\kappa = 1$; $0^\kappa = 0$ if $\kappa > 0$.

To prove (3.4)–(3.10), one has only to find the appropriate one-to-one functions.

Alephs

An ordinal α is called a *cardinal number* (a cardinal) if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$. We shall use κ, λ, μ, ... to denote cardinal numbers.

If W is a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$. Thus we let

$$|W| = \text{the least ordinal such that } |W| = |\alpha|.$$

Clearly, $|W|$ is a cardinal number.

Every natural number is a cardinal (a *finite cardinal*); and if S is a finite set, then $|S| = n$ for some n.

The ordinal ω is the least infinite cardinal. Note that all infinite cardinals are limit ordinals. The infinite ordinal numbers that are cardinals are called *alephs*.

Lemma 3.4.

(i) *For every α there is a cardinal number greater than α.*
(ii) *If X is a set of cardinals, then $\sup X$ is a cardinal.*

For every α, let α^+ be the least cardinal number greater than α, the *cardinal successor* of α.

Proof. (i) For any set X, let

(3.11) $$h(X) = \text{the least } \alpha \text{ such that there is no one-to-one function of } \alpha \text{ into } X.$$

There is only a set of possible well-orderings of subsets of X. Hence there is only a set of ordinals for which a one-to-one function of α into X exists. Thus $h(X)$ exists.

If α is an ordinal, then $|\alpha| < |h(\alpha)|$ by (3.11). That proves (i).

(ii) Let $\alpha = \sup X$. If f is a one-to-one mapping of α onto some $\beta < \alpha$, let $\kappa \in X$ be such that $\beta < \kappa \leq \alpha$. Then $|\kappa| = |\{f(\xi) : \xi < \kappa\}| \leq \beta$, a contradiction. Thus α is a cardinal. □

Using Lemma 3.4, we define the increasing enumeration of all alephs. We usually use $\aleph_\alpha$ when referring to the cardinal number, and ω_α to denote the order-type:

$$\aleph_0 = \omega_0 = \omega, \qquad \aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+,$$
$$\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Sets whose cardinality is $\aleph_0$ are called *countable*; a set is *at most countable* if it is either finite or countable. Infinite sets that are not countable are *uncountable*.

A cardinal $\aleph_{\alpha+1}$ is a *successor cardinal*. A cardinal $\aleph_\alpha$ whose index is a limit ordinal is a *limit cardinal*.

Addition and multiplication of alephs is a trivial matter, due to the following fact:

Theorem 3.5. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

To prove Theorem 3.5 we use a pairing function for ordinal numbers:

The Canonical Well-Ordering of $\alpha \times \alpha$

We define a well-ordering of the class $Ord \times Ord$ of ordinal pairs. Under this well-ordering, each $\alpha \times \alpha$ is an initial segment of Ord^2; the induced well-ordering of α^2 is called the *canonical well-ordering* of α^2. Moreover, the well-ordered class Ord^2 is isomorphic to the class Ord, and we have a one-to-one function Γ of Ord^2 onto Ord. For many α's the order-type of $\alpha \times \alpha$ is α; in particular for those α that are alephs.

We define:

$$(3.12) \qquad (\alpha, \beta) < (\gamma, \delta) \leftrightarrow \text{either } \max\{\alpha, \beta\} < \max\{\gamma, \delta\},$$
$$\text{or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\} \text{ and } \alpha < \gamma,$$
$$\text{or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\},\ \alpha = \gamma \text{ and } \beta < \delta.$$

The relation $<$ defined in (3.12) is a linear ordering of the class $Ord \times Ord$. Moreover, if $X \subset Ord \times Ord$ is nonempty, then X has a least element. Also, for each α, $\alpha \times \alpha$ is the initial segment given by $(0, \alpha)$. If we let

$$\Gamma(\alpha, \beta) = \text{the order-type of the set } \{(\xi, \eta) : (\xi, \eta) < (\alpha, \beta)\},$$

then Γ is a one-to-one mapping of Ord^2 onto Ord, and

$$(\alpha,\beta) < (\gamma,\delta) \quad \text{if and only if} \quad \Gamma(\alpha,\beta) < \Gamma(\gamma,\delta). \tag{3.13}$$

Note that $\Gamma(\omega\times\omega) = \omega$ and since $\gamma(\alpha) = \Gamma(\alpha\times\alpha)$ is an increasing function of α, we have $\gamma(\alpha) \geq \alpha$ for every α. However, $\gamma(\alpha)$ is also continuous, and so $\Gamma(\alpha \times \alpha) = \alpha$ for arbitrarily large α.

Proof of Theorem 3.5. Consider the canonical one-to-one mapping Γ of $Ord\times Ord$ onto Ord. We shall show that $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$. This is true for $\alpha = 0$. Thus let α be the least ordinal such that $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$. Let $\beta,\gamma < \omega_\alpha$ be such that $\Gamma(\beta,\gamma) = \omega_\alpha$. Pick $\delta < \omega_\alpha$ such that $\delta > \beta$ and $\delta > \gamma$. Since $\delta\times\delta$ is an initial segment of $Ord \times Ord$ in the canonical well-ordering and contains (β,γ), we have $\Gamma(\delta\times\delta) \supset \omega_\alpha$, and so $|\delta\times\delta| \geq \aleph_\alpha$. However, $|\delta\times\delta| = |\delta|\cdot|\delta|$, and by the minimality of α, $|\delta|\cdot|\delta| = |\delta| < \aleph_\alpha$. A contradiction. □

As a corollary we have

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha,\aleph_\beta\}. \tag{3.14}$$

Exponentiation of cardinals will be dealt with in Chapter 5. Without the Axiom of Choice, one cannot prove that $2^{\aleph_\alpha}$ is an aleph (or that $P(\omega_\alpha)$ can be well-ordered), and there is very little one can prove about $2^{\aleph_\alpha}$ or $\aleph_\alpha^{\aleph_\beta}$.

Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $\langle\alpha_\xi : \xi < \beta\rangle$, β a limit ordinal, is *cofinal* in α if $\lim_{\xi\to\beta}\alpha_\xi = \alpha$. Similarly, $A \subset \alpha$ is *cofinal* in α if $\sup A = \alpha$. If α is an infinite limit ordinal, the *cofinality* of α is

$$\begin{aligned}\operatorname{cf}\alpha = {} & \text{the least limit ordinal } \beta \text{ such that there is an increasing} \\ & \beta\text{-sequence } \langle\alpha_\xi : \xi < \beta\rangle \text{ with } \lim_{\xi\to\beta}\alpha_\xi = \alpha.\end{aligned}$$

Obviously, $\operatorname{cf}\alpha$ is a limit ordinal, and $\operatorname{cf}\alpha \leq \alpha$. Examples: $\operatorname{cf}(\omega+\omega) = \operatorname{cf}\aleph_\omega = \omega$.

Lemma 3.6. $\operatorname{cf}(\operatorname{cf}\alpha) = \operatorname{cf}\alpha$.

Proof. If $\langle\alpha_\xi : \xi < \beta\rangle$ is cofinal in α and $\langle\xi(\nu) : \nu < \gamma\rangle$ is cofinal in β, then $\langle\alpha_{\xi(\nu)} : \nu < \gamma\rangle$ is cofinal in α. □

Two useful facts about cofinality:

Lemma 3.7. *Let $\alpha > 0$ be a limit ordinal.*

(i) *If $A \subset \alpha$ and $\sup A = \alpha$, then the order-type of A is at least $\operatorname{cf}\alpha$.*

(ii) *If* $\beta_0 \leq \beta_1 \leq \ldots \leq \beta_\xi \leq \ldots$, $\xi < \gamma$, *is a nondecreasing* γ*-sequence of ordinals in* α *and* $\lim_{\xi\to\gamma} \beta_\xi = \alpha$, *then* $\operatorname{cf}\gamma = \operatorname{cf}\alpha$.

Proof. (i) The order-type of A is the length of the increasing enumeration of A which is an increasing sequence with limit α.

(ii) If $\gamma = \lim_{\nu\to\operatorname{cf}\gamma} \xi(\nu)$, then $\alpha = \lim_{\nu\to\operatorname{cf}\gamma} \beta_{\xi(\nu)}$, and the nondecreasing sequence $\langle \beta_{\xi(\nu)} : \nu < \operatorname{cf}\gamma\rangle$ has an increasing subsequence of length $\leq \operatorname{cf}\gamma$, with the same limit. Thus $\operatorname{cf}\alpha \leq \operatorname{cf}\gamma$.

To show that $\operatorname{cf}\gamma \leq \operatorname{cf}\alpha$, let $\alpha = \lim_{\nu\to\operatorname{cf}\alpha} \alpha_\nu$. For each $\nu < \operatorname{cf}\alpha$, let $\xi(\nu)$ be the least ξ greater than all $\xi(\iota)$, $\iota < \nu$, such that $\beta_\xi > \alpha_\nu$. Since $\lim_{\nu\to\operatorname{cf}\alpha} \beta_{\xi(\nu)} = \alpha$, it follows that $\lim_{\nu\to\operatorname{cf}\alpha} \xi(\nu) = \gamma$, and so $\operatorname{cf}\gamma \leq \operatorname{cf}\alpha$. □

An infinite cardinal $\aleph_\alpha$ is *regular* if $\operatorname{cf}\omega_\alpha = \omega_\alpha$. It is *singular* if $\operatorname{cf}\omega_\alpha < \omega_\alpha$.

Lemma 3.8. *For every limit ordinal* α, $\operatorname{cf}\alpha$ *is a regular cardinal.*

Proof. It is easy to see that if α is not a cardinal, then using a mapping of $|\alpha|$ onto α, one can construct a cofinal sequence in α of length $\leq |\alpha|$, and therefore $\operatorname{cf}\alpha < \alpha$.

Since $\operatorname{cf}(\operatorname{cf}\alpha) = \operatorname{cf}\alpha$, it follows that $\operatorname{cf}\alpha$ is a cardinal and is regular. □

Let κ be a limit ordinal. A subset $X \subset \kappa$ is *bounded* if $\sup X < \kappa$, and *unbounded* if $\sup X = \kappa$.

Lemma 3.9. *Let* κ *be an aleph.*

(i) *If* $X \subset \kappa$ *and* $|X| < \operatorname{cf}\kappa$ *then* X *is bounded.*
(ii) *If* $\lambda < \operatorname{cf}\kappa$ *and* $f : \lambda \to \kappa$ *then the range of* f *is bounded.*

It follows from (i) that every unbounded subset of a regular cardinal has cardinality κ.

Proof. (i) Lemma 3.7(i).

(ii) If $X = \operatorname{ran}(f)$ then $|X| \leq \lambda$, and use (i). □

There are arbitrarily large singular cardinals. For each α, $\aleph_{\alpha+\omega}$ is a singular cardinal of cofinality ω.

Using the Axiom of Choice, we shall show in Chapter 5 that every $\aleph_{\alpha+1}$ is regular. (The Axiom of Choice is necessary.)

Lemma 3.10. *An infinite cardinal* κ *is singular if and only if there exists a cardinal* $\lambda < \kappa$ *and a family* $\{S_\xi : \xi < \lambda\}$ *of subsets of* κ *such that* $|S_\xi| < \kappa$ *for each* $\xi < \lambda$, *and* $\kappa = \bigcup_{\xi<\lambda} S_\xi$. *The least cardinal* λ *that satisfies the condition is* $\operatorname{cf}\kappa$.

Proof. If κ is singular, then there is an increasing sequence $\langle \alpha_\xi : \xi < \operatorname{cf}\kappa\rangle$ with $\lim_\xi \alpha_\xi = \kappa$. Let $\lambda = \operatorname{cf}\kappa$, and $S_\xi = \alpha_\xi$ for all $\xi < \lambda$.

If the condition holds, let $\lambda < \kappa$ be the least cardinal for which there is a family $\{S_\xi : \xi < \lambda\}$ such that $\kappa = \bigcup_{\xi<\lambda} S_\xi$ and $|S_\xi| < \kappa$ for each $\xi < \lambda$. For

every $\xi < \lambda$, let β_ξ be the order-type of $\bigcup_{\nu<\xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda \rangle$ is nondecreasing, and by the minimality of λ, $\beta_\xi < \kappa$ for all $\xi < \lambda$. We shall show that $\lim_\xi \beta_\xi = \kappa$, thus proving that $\operatorname{cf} \kappa \le \lambda$.

Let $\beta = \lim_{\xi\to\lambda} \beta_\xi$. There is a one-to-one mapping f of $\kappa = \bigcup_{\xi<\lambda} S_\xi$ into $\lambda \times \beta$: If $\alpha \in \kappa$, let $f(\alpha) = (\xi, \gamma)$, where ξ is the least ξ such that $\alpha \in S_\xi$ and γ is the order-type of $S_\xi \cap \alpha$. Since $\lambda < \kappa$ and $|\lambda \times \beta| = \lambda \cdot |\beta|$, it follows that $\beta = \kappa$. □

One cannot prove without the Axiom of Choice that ω_1 is not a countable union of countable sets. Compare this with Exercise 3.13

The only cardinal inequality we have proved so far is Cantor's Theorem $\kappa < 2^\kappa$. It follows that $\kappa < \lambda^\kappa$ for every $\lambda > 1$, and in particular $\kappa < \kappa^\kappa$ (for $\kappa \ne 1$). The following theorem gives a better inequality. This and other cardinal inequalities will also follow from König's Theorem 5.10, to be proved in Chapter 5.

Theorem 3.11. *If κ is an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf}\kappa}$.*

Proof. Let F be a collection of κ functions from $\operatorname{cf}\kappa$ to κ: $F = \{f_\alpha : \alpha < \kappa\}$. It is enough to find $f : \operatorname{cf}\kappa \to \kappa$ that is different from all the f_α. Let $\kappa = \lim_{\xi\to\operatorname{cf}\kappa} \alpha_\xi$. For $\xi < \operatorname{cf}\kappa$, let

$$f(\xi) = \text{least } \gamma \text{ such that } \gamma \ne f_\alpha(\xi) \text{ for all } \alpha < \alpha_\xi.$$

Such γ exists since $|\{f_\alpha(\xi) : \alpha < \alpha_\xi\}| \le |\alpha_\xi| < \kappa$. Obviously, $f \ne f_\alpha$ for all $\alpha < \kappa$. □

Consequently, $\kappa^\lambda > \kappa$ whenever $\lambda \ge \operatorname{cf}\kappa$.

An uncountable cardinal κ is *weakly inaccessible* if it is a limit cardinal and is regular. There will be more about inaccessible cardinals later, but let me mention at this point that existence of (weakly) inaccessible cardinals is not provable in ZFC.

To get an idea of the size of an inaccessible cardinal, note that if $\aleph_\alpha > \aleph_0$ is limit and regular, then $\aleph_\alpha = \operatorname{cf}\aleph_\alpha = \operatorname{cf}\alpha \le \alpha$, and so $\aleph_\alpha = \alpha$.

Since the sequence of alephs is a normal sequence, it has arbitrarily large fixed points; the problem is whether some of them are regular cardinals. For instance, the least fixed point $\aleph_\alpha = \alpha$ has cofinality ω:

$$\kappa = \lim\langle \omega, \omega_\omega, \omega_{\omega_\omega}, \ldots \rangle = \lim_{n\to\omega} \kappa_n$$
$$\text{where } \kappa_0 = \omega,\ \kappa_{n+1} = \omega_{\kappa_n}.$$

Exercises

3.1. (i) A subset of a finite set is finite.
(ii) The union of a finite set of finite sets is finite.
(iii) The power set of a finite set is finite.
(iv) The image of a finite set (under a mapping) is finite.

3.2. (i) A subset of a countable set is at most countable.
(ii) The union of a finite set of countable sets is countable.
(iii) The image of a countable set (under a mapping) is at most countable.

3.3. $\boldsymbol{N} \times \boldsymbol{N}$ is countable.
[$f(m, n) = 2^m(2n + 1) - 1$.]

3.4. (i) The set of all finite sequences in $\boldsymbol{N}$ is countable.
(ii) The set of all finite subsets of a countable set is countable.

3.5. Show that $\Gamma(\alpha \times \alpha) \leq \omega^\alpha$.

3.6. There is a well-ordering of the class of all finite sequences of ordinals such that for each α, the set of all finite sequences in ω_α is an initial segment and its order-type is ω_α.

We say that a set B is a *projection* of a set A if there is a mapping of A onto B. Note that B is a projection of A if and only if there is a partition P of A such that $|P| = |B|$. If $|A| \geq |B| > 0$, then B is a projection of A. Conversely, using the Axiom of Choice, one shows that if B is a projection of A, then $|A| \geq |B|$. This, however, cannot be proved without the Axiom of Choice.

3.7. If B is a projection of ω_α, then $|B| \leq \aleph_\alpha$.

3.8. The set of all finite subsets of ω_α has cardinality $\aleph_\alpha$.
[The set is a projection of the set of finite sequences.]

3.9. If B is a projection of A, then $|P(B)| \leq |P(A)|$.
[Consider $g(X) = f_{-1}(X)$, where f maps A onto B.]

3.10. $\omega_{\alpha+1}$ is a projection of $P(\omega_\alpha)$.
[Use $|\omega_\alpha \times \omega_\alpha| = \omega_\alpha$ and project $P(\omega_\alpha \times \omega_\alpha)$: If $R \subset \omega_\alpha \times \omega_\alpha$ is a well-ordering, let $f(R)$ be its order-type.]

3.11. $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$.
[Use Exercises 3.10 and 3.9.]

3.12. If $\aleph_\alpha$ is an uncountable limit cardinal, then $\operatorname{cf} \omega_\alpha = \operatorname{cf} \alpha$; ω_α is the limit of a cofinal sequence $\langle \omega_\xi : \xi < \operatorname{cf} \alpha \rangle$ of cardinals.

3.13 (ZF). Show that ω_2 is not a countable union of countable sets.
[Assume that $\omega_2 = \bigcup_{n<\omega} S_n$ with S_n countable and let α_n be the order-type of S_n. Then $\alpha = \sup_n \alpha_n \leq \omega_1$ and there is a mapping of $\omega \times \alpha$ onto ω_2.]

A set S is *Dedekind-finite* (D-finite) if there is no one-to-one mapping of S onto a proper subset of S. Every finite set is D-finite. Using the Axiom of Choice, one proves that every infinite set is D-infinite, and so D-finiteness is the same as finiteness. Without the Axiom of Choice, however, one cannot prove that every D-finite set is finite.

The set $\boldsymbol{N}$ of all natural numbers is D-infinite and hence every S such that $|S| \geq \aleph_0$, is D-infinite.

3.14. S is D-infinite if and only if S has a countable subset.
[If S is D-infinite, let $f : S \to X \subset S$ be one-to-one. Let $x_0 \in S - X$ and $x_{n+1} = f(x_n)$. Then $S \supset \{x_n : n < \omega\}$.]

3.15. (i) If A and B are D-finite, then $A \cup B$ and $A \times B$ are D-finite.
(ii) The set of all finite one-to-one sequences in a D-finite set is D-finite.
(iii) The union of a disjoint D-finite family of D-finite sets is D-finite.

On the other hand, one cannot prove without the Axiom of Choice that a projection, power set, or the set of all finite subsets of a D-finite set is D-finite, or that the union of a D-finite family of D-finite sets is D-finite.

3.16. If A is an infinite set, then $PP(A)$ is D-infinite.
[Consider the set $\{\{X \subset A : |X| = n\} : n < \omega\}$.]

Historical Notes

Cardinal numbers and alephs were introduced by Cantor. The proof of the Cantor-Bernstein Theorem is Bernstein's; see Borel [1898], p. 103. (There is an earlier proof by Dedekind.) The first proof of $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ appeared in Hessenberg [1906], p. 593. Regularity of cardinals was investigated by Hausdorff, who also raised the question of existence of regular limit cardinals. D-finiteness was formulated by Dedekind.

4. Real Numbers

The set of all real numbers $\boldsymbol{R}$ (the *real line* or the *continuum*) is the unique ordered field in which every nonempty bounded set has a least upper bound. The proof of the following theorem marks the beginning of Cantor's theory of sets.

Theorem 4.1 (Cantor). *The set of all real numbers is uncountable.*

Proof. Let us assume that the set $\boldsymbol{R}$ of all reals is countable, and let c_0, $c_1, \ldots, c_n, \ldots, n \in \boldsymbol{N}$, be an enumeration of $\boldsymbol{R}$. We shall find a real number different from each c_n.

Let $a_0 = c_0$ and $b_0 = c_{k_0}$ where k_0 is the least k such that $a_0 < c_k$. For each n, let $a_{n+1} = c_{i_n}$ where i_n is the least i such that $a_n < c_i < b_n$, and $b_{n+1} = c_{k_n}$ where k_n is the least k such that $a_{n+1} < c_k < b_n$. If we let $a = \sup\{a_n : n \in \boldsymbol{N}\}$, then $a \neq c_k$ for all k. □

The Cardinality of the Continuum

Let $\mathfrak{c}$ denote the cardinality of $\boldsymbol{R}$. As the set $\boldsymbol{Q}$ of all rational numbers is dense in $\boldsymbol{R}$, every real number r is equal to $\sup\{q \in \boldsymbol{Q} : q < r\}$ and because $\boldsymbol{Q}$ is countable, it follows that $\mathfrak{c} \leq |P(\boldsymbol{Q})| = 2^{\aleph_0}$.

Let $\boldsymbol{C}$ (the *Cantor set*) be the set of all reals of the form $\sum_{n=1}^{\infty} a_n/3^n$, where each $a_n = 0$ or 2. $\boldsymbol{C}$ is obtained by removing from the closed interval $[0, 1]$, the open intervals $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{9}, \frac{2}{9})$, $(\frac{7}{9}, \frac{8}{9})$, etc. (the middle-third intervals). $\boldsymbol{C}$ is in a one-to-one correspondence with the set of all ω-sequences of 0's and 2's and so $|\boldsymbol{C}| = 2^{\aleph_0}$.

Therefore $\mathfrak{c} \geq 2^{\aleph_0}$, and so by the Cantor-Bernstein Theorem we have

$$\mathfrak{c} = 2^{\aleph_0}. \tag{4.1}$$

By Cantor's Theorem 4.1 (or by Theorem 3.1), $\mathfrak{c} > \aleph_0$. Cantor conjectured that every set of reals is either at most countable or has cardinality of the continuum. In ZFC, every infinite cardinal is an aleph, and so $2^{\aleph_0} \geq \aleph_1$. Cantor's conjecture then becomes the statement

$$2^{\aleph_0} = \aleph_1$$

known as the *Continuum Hypothesis* (CH).

Among sets of cardinality $\mathfrak{c}$ are the set of all sequences of natural numbers, the set of all sequences of real numbers, the set of all complex numbers. This is because $\aleph_0^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, $2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$.

Cantor's proof of Theorem 4.1 yielded more than uncountability of $\boldsymbol{R}$; it showed that the set of all transcendental numbers has cardinality $\mathfrak{c}$ (cf. Exercise 4.5).

The Ordering of $\boldsymbol{R}$

A linear ordering $(P, <)$ is *complete* if every nonempty bounded subset of P has a least upper bound. We stated above that $\boldsymbol{R}$ is the unique complete ordered field. We shall generally disregard the field properties of $\boldsymbol{R}$ and will concern ourselves more with the order properties.

One consequence of being a complete ordered field is that $\boldsymbol{R}$ contains the set $\boldsymbol{Q}$ of all rational numbers as a dense subset. The set $\boldsymbol{Q}$ is countable and its ordering is dense.

Definition 4.2. A linear ordering $(P, <)$ is *dense* if for all $a < b$ there exists a c such that $a < c < b$.

A set $D \subset P$ is a *dense subset* if for all $a < b$ in P there exists a $d \in D$ such that $a < d < b$.

The following theorem proves the uniqueness of the ordered set $(\boldsymbol{R}, <)$. We say that an ordered set is *unbounded* if it has neither a least nor a greatest element.

Theorem 4.3 (Cantor).

(i) *Any two countable unbounded dense linearly ordered sets are isomorphic.*

(ii) $(\boldsymbol{R}, <)$ *is the unique complete linear ordering that has a countable dense subset isomorphic to* $(\boldsymbol{Q}, <)$.

Proof. (i) Let $P_1 = \{a_n : n \in \boldsymbol{N}\}$ and let $P_2 = \{b_n : n \in \boldsymbol{N}\}$ be two such linearly ordered sets. We construct an isomorphism $f : P_1 \to P_2$ in the following way: We first define $f(a_0)$, then $f^{-1}(b_0)$, then $f(a_1)$, then $f^{-1}(b_1)$, etc., so as to keep f order-preserving. For example, to define $f(a_n)$, if it is not yet defined, we let $f(a_n) = b_k$ where k is the least index such that f remains order-preserving (such a k always exists because f has been defined for only finitely many $a \in P_1$, and because P_2 is dense and unbounded).

(ii) To prove the uniqueness of $\boldsymbol{R}$, let C and C' be two complete dense unbounded linearly ordered sets, let P and P' be dense in C and C', respectively, and let f be an isomorphism of P onto P'. Then f can be extended (uniquely) to an isomorphism f^* of C and C': For $x \in C$, let $f^*(x) = \sup\{f(p) : p \in P \text{ and } p \leq x\}$. □

The existence of $(\boldsymbol{R}, <)$ is proved by means of *Dedekind cuts* in $(\boldsymbol{Q}, <)$. The following theorem is a general version of this construction:

Theorem 4.4. *Let $(P, <)$ be a dense unbounded linearly ordered set. Then there is a complete unbounded linearly ordered set $(C, \prec)$ such that*:

(i) $P \subset C$, *and* $<$ *and* $\prec$ *agree on* P;
(ii) P *is dense in* C.

Proof. A *Dedekind cut* in P is a pair (A, B) of disjoint nonempty subsets of P such that

(i) $A \cup B = P$;
(ii) $a < b$ for any $a \in A$ and $b \in B$;
(iii) A does not have a greatest element.

Let C be the set of all Dedekind cuts in P and let $(A_1, B_1) \preccurlyeq (A_2, B_2)$ if $A_1 \subset A_2$ (and $B_1 \supset B_2$). The set C is complete: If $\{(A_i, B_i) : i \in I\}$ is a nonempty bounded subset of C, then $(\bigcup_{i\in I} A_i, \bigcap_{i\in I} B_i)$ is its supremum. For $p \in P$, let

$$A_p = \{x \in P : x < p\}, \qquad B_p = \{x \in P : x \geq p\}.$$

Then $P' = \{(A_p, B_p) : p \in P\}$ is isomorphic to P and is dense in C. □

Suslin's Problem

The real line is, up to isomorphism, the unique linearly ordered set that is dense, unbounded, complete and contains a countable dense subset.

Since $\boldsymbol{Q}$ is dense in $\boldsymbol{R}$, every nonempty open interval of $\boldsymbol{R}$ contains a rational number. Hence if S is a disjoint collection of open intervals, S is at most countable. (Let $\langle r_n : n \in \boldsymbol{N}\rangle$ be an enumeration of the rationals. To each $J \in S$ assign $r_n \in J$ with the least possible index n.)

Let P be a dense linearly ordered set. If every disjoint collection of open intervals in P is at most countable, then we say that P satisfies the *countable chain condition*.

Suslin's Problem. *Let P be a complete dense unbounded linearly ordered set that satisfies the countable chain condition. Is P isomorphic to the real line?*

This question cannot be decided in ZFC; we shall return to the problem in Chapter 9.

The Topology of the Real Line

The real line is a metric space with the metric $d(a,b) = |a-b|$. Its metric topology coincides with the order topology of $(\boldsymbol{R}, <)$. Since $\boldsymbol{Q}$ is a dense set in $\boldsymbol{R}$ and since every Cauchy sequence of real numbers converges, $\boldsymbol{R}$ is a separable complete metric space. (A metric space is *separable* if it has a countable dense set; it is *complete* if every Cauchy sequence converges.)

Open sets are unions of open intervals, and in fact, every open set is the union of open intervals with rational endpoints. This implies that the number of all open sets in $\boldsymbol{R}$ is the continuum and so is the number of all closed sets in $\boldsymbol{R}$ (Exercise 4.6).

Every open interval has cardinality $\mathfrak{c}$, therefore every nonempty open set has cardinality $\mathfrak{c}$. We show below that every uncountable closed set has cardinality $\mathfrak{c}$. Proving this was Cantor's first step in the search for the proof of the Continuum Hypothesis. In Chapter 11 we show that CH holds for Borel and analytic sets as well.

A nonempty closed set is *perfect* if it has no isolated points. Theorems 4.5 and 4.6 below show that every uncountable closed set contains a perfect set.

Theorem 4.5. *Every perfect set has cardinality* $\mathfrak{c}$.

Proof. Given a perfect set P, we want to find a one-to-one function F from $\{0,1\}^\omega$ into P. Let S be the set of all finite sequences of 0's and 1's. By induction on the length of $s \in S$ one can find closed intervals I_s such that for each n and all $s \in S$ of length n,

(i) $I_s \cap P$ is perfect,
(ii) the diameter of I_s is $\leq 1/n$,
(iii) $I_{s^\frown 0} \subset I_s$, $I_{s^\frown 1} \subset I_s$ and $I_{s^\frown 0} \cap I_{s^\frown 1} = \emptyset$.

For each $f \in \{0,1\}^\omega$, the set $P \cap \bigcap_{n=0}^{\infty} I_{f\restriction n}$ has exactly one element, and we let $F(f)$ to be this element of P. □

The same proof gives a more general result: Every perfect set in a separable complete metric space contains a closed copy of the Cantor set (Exercise 4.19).

Theorem 4.6 (Cantor-Bendixson). *If F is an uncountable closed set, then $F = P \cup S$, where P is perfect and S is at most countable.*

Corollary 4.7. *If F is a closed set, then either $|F| \leq \aleph_0$ or $|F| = 2^{\aleph_0}$.* □

Proof. For every $A \subset \boldsymbol{R}$, let

$$A' = \text{the set of all limit points of } A$$

It is easy to see that A' is closed, and if A is closed then $A' \subset A$. Thus we let

$$F_0 = F, \qquad F_{\alpha+1} = F'_\alpha,$$
$$F_\alpha = \bigcap_{\gamma<\alpha} F_\gamma \quad \text{if } \alpha > 0 \text{ is a limit ordinal.}$$

Since $F_0 \supset F_1 \supset \ldots \supset F_\alpha \supset \ldots$, there exists an ordinal θ such that $F_\alpha = F_\theta$ for all $\alpha \geq \theta$. (In fact, the least θ with this property must be countable, by the argument below.) We let $P = F_\theta$.

If P is nonempty, then $P' = P$ and so it is perfect. Thus the proof is completed by showing that $F - P$ is at most countable.

Let $\langle J_k : k \in \boldsymbol{N} \rangle$ be an enumeration of rational intervals. We have $F - P = \bigcup_{\alpha<\theta}(F_\alpha - F'_\alpha)$; hence if $a \in F - P$, then there is a unique α such that a is an isolated point of F_α. We let $k(a)$ be the least k such that a is the only point of F_α in the interval J_k. Note that if $\alpha \leq \beta$, $b \neq a$ and $b \in F_\beta - F'_\beta$, then $b \notin J_{k(a)}$, and hence $k(b) \neq k(a)$. Thus the correspondence $a \mapsto k(a)$ is one-to-one, and it follows that $F - P$ is at most countable. □

A set of reals is called *nowhere dense* if its closure has empty interior. The following theorem shows that $\boldsymbol{R}$ is not the union of countably many nowhere dense sets ($\boldsymbol{R}$ is not of *the first category*).

Theorem 4.8 (The Baire Category Theorem). *If D_0, D_1, …, D_n, …, $n \in \boldsymbol{N}$, are dense open sets of reals, then the intersection $D = \bigcap_{n=0}^{\infty} D_n$ is dense in $\boldsymbol{R}$.*

Proof. We show that D intersects every nonempty open interval I. First note that for each n, $D_0 \cap \ldots \cap D_n$ is dense and open. Let $\langle J_k : k \in \boldsymbol{N} \rangle$ be an enumeration of rational intervals. Let $I_0 = I$, and let, for each n, $I_{n+1} = J_k = (q_k, r_k)$, where k is the least k such that the closed interval $[q_k, r_k]$ is included in $I_n \cap D_n$. Then $a \in D \cap I$, where $a = \lim_{k\to\infty} q_k$. □

Borel Sets

Definition 4.9. An *algebra of sets* is a collection $\mathcal{S}$ of subsets of a given set S such that

(4.2)
(i) $S \in \mathcal{S}$,
(ii) if $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ then $X \cup Y \in \mathcal{S}$,
(iii) if $X \in \mathcal{S}$ then $S - X \in \mathcal{S}$.

(Note that $\mathcal{S}$ is also closed under intersections.)

A *σ-algebra* is additionally closed under countable unions (and intersections):

(iv) If $X_n \in \mathcal{S}$ for all n, then $\bigcup_{n=0}^{\infty} X_n \in \mathcal{S}$.

For any collection $\mathcal{X}$ of subsets of S there is a smallest algebra (σ-algebra) $\mathcal{S}$ such that $\mathcal{S} \supset \mathcal{X}$; namely the intersection of all algebras (σ-algebras) $\mathcal{S}$ of subsets of S for which $\mathcal{X} \subset \mathcal{S}$.

Definition 4.10. A set of reals B is *Borel* if it belongs to the smallest σ-algebra $\mathcal{B}$ of sets of reals that contains all open sets.

In Chapter 11 we investigate Borel sets in more detail. In particular, we shall classify Borel sets by defining a hierarchy of ω_1 levels. For that we need however a weak version of the Axiom of Choice that is not provable in ZF alone. At this point we mention the lowest level of the hierarchy (beyond open sets and closed sets): The intersections of countably many open sets are called G_δ *sets*, and the unions of countably many closed sets are called F_σ *sets*.

Lebesgue Measure

We assume that the reader is familiar with the basic theory of Lebesgue measure. As we shall return to the subject in Chapter 11 we do not define the concept of measure at this point. We also caution the reader that some of the basic theorems on Lebesgue measure require the Countable Axiom of Choice (to be discussed in Chapter 5).

Lebesgue measurable sets form a σ-algebra and contain all open intervals (the measure of an interval is its length). Thus all Borel sets are Lebesgue measurable.

The Baire Space

The *Baire space* is the space $\mathcal{N} = \omega^\omega$ of all infinite sequences of natural numbers, $\langle a_n : n \in \boldsymbol{N}\rangle$, with the following topology: For every finite sequence $s = \langle a_k : k < n\rangle$, let

$$O(s) = \{f \in \mathcal{N} : s \subset f\} = \{\langle c_k : k \in \boldsymbol{N}\rangle : (\forall k < n)\, c_k = a_k\}. \tag{4.3}$$

The sets (4.3) form a basis for the topology of $\mathcal{N}$. Note that each $O(s)$ is also closed.

The Baire space is separable and is metrizable: consider the metric $d(f, g) = 1/2^{n+1}$ where n is the least number such that $f(n) \neq g(n)$. The countable set of all eventually constant sequences is dense in $\mathcal{N}$. This separable metric space is complete, as every Cauchy sequence converges.

Every infinite sequence $\langle a_n : n \in \boldsymbol{N}\rangle$ of positive integers defines a *continued fraction* $1/(a_0 + 1/(a_1 + 1/(a_2 + \ldots)))$, an irrational number between 0 and 1. Conversely, every irrational number in the interval $(0, 1)$ can be so represented, and the one-to-one correspondence is a homeomorphism. It follows that the Baire space is homeomorphic to the space of all irrational numbers.

For various reasons, modern descriptive set theory uses the Baire space rather than the real line. Often the functions in ω^ω are called reals.

Clearly, the space $\mathcal{N}$ satisfies the Baire Category Theorem; the proof is similar to the proof of Theorem 4.8 above. The Cantor-Bendixson Theorem holds as well. For completeness we give a description of perfect sets in $\mathcal{N}$.

Let *Seq* denote the set of all finite sequences of natural numbers. A (*sequential*) *tree* is a set $T \subset Seq$ that satisfies

$$\text{if } t \in T \text{ and } s = t\restriction n \text{ for some } n, \text{ then } s \in T. \tag{4.4}$$

If $T \subset Seq$ is a tree, let $[T]$ be the set of all *infinite paths* through T:

$$[T] = \{f \in \mathcal{N} : f\restriction n \in T \text{ for all } n \in \boldsymbol{N}\}. \tag{4.5}$$

The set $[T]$ is a closed set in the Baire space: Let $f \in \mathcal{N}$ be such that $f \notin [T]$. Then there is $n \in \boldsymbol{N}$ such that $f\restriction n = s$ is not in T. In other words, the open set $O(s) = \{g \in \mathcal{N} : g \supset s\}$, a neighborhood of f, is disjoint from $[T]$. Hence $[T]$ is closed.

Conversely, if F is a closed set in $\mathcal{N}$, then the set

$$T_F = \{s \in Seq : s \subset f \text{ for some } f \in F\} \tag{4.6}$$

is a tree, and it is easy to verify that $[T_F] = F$: If $f \in \mathcal{N}$ is such that $f\restriction n \in T$ for all $n \in \boldsymbol{N}$, then for each n there is some $g \in F$ such that $g\restriction n = f\restriction n$; and since F is closed, it follows that $f \in F$.

If f is an isolated point of a closed set F in $\mathcal{N}$, then there is $n \in \mathcal{N}$ such that there is no $g \in F$, $g \neq f$, such that $g\restriction n = f\restriction n$. Thus the following definition:

A nonempty sequential tree T is *perfect* if for every $t \in T$ there exist $s_1 \supset t$ and $s_2 \supset t$, both in T, that are *incomparable*, i.e., neither $s_1 \supset s_2$ nor $s_2 \supset s_1$.

Lemma 4.11. *A closed set $F \subset \mathcal{N}$ is perfect if and only if the tree T_F is a perfect tree.* □

The Cantor-Bendixson analysis for closed sets in the Baire space is carried out as follows: For each tree $T \subset Seq$, we let

$$T' = \{t \in T : \text{there exist incomparable } s_1 \supset t \text{ and } s_2 \supset t \text{ in } T\}. \tag{4.7}$$

(Thus T is perfect if and only if $\emptyset \neq T = T'$.)

The set $[T] - [T']$ is at most countable: For each $f \in [T]$ such that $f \notin [T']$, let $s_f = f\restriction n$ where n is the least number such that $f\restriction n \notin T'$. If $f, g \in [T] - [T']$, then $s_f \neq s_g$, by (4.7). Hence the mapping $f \mapsto s_f$ is one-to-one and $[T] - [T']$ is at most countable.

Now we let

$$\begin{aligned} &T_0 = T, \qquad T_{\alpha+1} = T'_\alpha, \\ &T_\alpha = \bigcap_{\beta<\alpha} T_\beta \quad \text{if } \alpha > 0 \text{ is a limit ordinal.} \end{aligned} \tag{4.8}$$

Since $T_0 \supset T_1 \supset \ldots \supset T_\alpha \supset \ldots$, and T_0 is at most countable, there is an ordinal $\theta < \omega_1$ such that $T_{\theta+1} = T_\theta$. If $T_\theta \neq \emptyset$, then it is perfect.

Now it is easy to see that $\left[\bigcap_{\beta<\alpha} T_\beta\right] = \bigcap_{\beta<\alpha}[T_\beta]$, and so

$$[T] - [T_\theta] = \bigcup_{\alpha<\theta} ([T_\alpha] - [T'_\alpha]); \tag{4.9}$$

hence (4.9) is at most countable. Thus if $[T]$ is an uncountable closed set in $\mathcal{N}$, the sets $[T_\theta]$ and $[T] - [T_\theta]$ constitute the decomposition of $[T]$ into a perfect and an at most countable set.

In modern descriptive set theory one often speaks about the *Lebesgue measure* on $\mathcal{N}$. This measure is the extension of the product measure m on Borel sets in the Baire space induced by the probability measure on $\boldsymbol{N}$ that gives the singleton $\{n\}$ measure $1/2^{n+1}$. Thus for every sequence $s \in Seq$ of length $n \geq 1$ we have

$$m(O(s)) = \prod_{k=0}^{n-1} 1/2^{s(k)+1}. \tag{4.10}$$

Polish Spaces

Definition 4.12. A *Polish space* is a topological space that is homeomorphic to a separable complete metric space.

Examples of Polish spaces include $\boldsymbol{R}$, $\mathcal{N}$, the Cantor space, the unit interval $[0,1]$, the unit circle T, the Hilbert cube $[0,1]^\omega$, etc.

Every Polish space is a continuous image of the Baire space. In Chapter 11 we prove a somewhat more general statement.

Exercises

4.1. The set of all continuous functions $f : \boldsymbol{R} \to \boldsymbol{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^{\mathfrak{c}}$).

[A continuous function on $\boldsymbol{R}$ is determined by its values at rational points.]

4.2. There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets.

[For every sequence $a = \langle a_n : n \in \boldsymbol{N}\rangle$ of natural numbers consider the order-type

$$\tau_a = a_0 + \xi + a_1 + \xi + a_2 + \dots$$

where ξ is the order-type of the integers. Show that if $a \neq b$, then $\tau_a \neq \tau_b$.]

A real number is *algebraic* if it is a root of a polynomial whose coefficients are integers. Otherwise, it is *transcendental*.

4.3. The set of all algebraic reals is countable.

4.4. If S is a countable set of reals, then $|\boldsymbol{R} - S| = \mathfrak{c}$.

[Use $\boldsymbol{R} \times \boldsymbol{R}$ rather than $\boldsymbol{R}$ (because $|\boldsymbol{R} \times \boldsymbol{R}| = 2^{\aleph_0}$).]

4.5. (i) The set of all irrational numbers has cardinality $\mathfrak{c}$.
(ii) The set of all transcendental numbers has cardinality $\mathfrak{c}$.

4.6. The set of all open sets of reals has cardinality $\mathfrak{c}$.

4.7. The Cantor set is perfect.

4.8. If P is a perfect set and (a, b) is an open interval such that $P \cap (a, b) \neq \emptyset$, then $|P \cap (a, b)| = \mathfrak{c}$.

4.9. If $P_2 \not\subset P_1$ are perfect sets, then $|P_2 - P_1| = \mathfrak{c}$.
[Use Exercise 4.8.]

If A is a set of reals, a real number a is called a *condensation point* of A if every neighborhood of a contains uncountably many elements of A. Let A^* denote the set of all condensation points of A.

4.10. If P is perfect then $P^* = P$.
[Use Exercise 4.8.]

4.11. If F is closed and $P \subset F$ is perfect, then $P \subset F^*$.
[$P = P^* \subset F^*$.]

4.12. If F is an uncountable closed set and P is the perfect set constructed in Theorem 4.6, then $F^* \subset P$; thus $F^* = P$.
[Every $a \in F^*$ is a limit point of P since $|F - P| \leq \aleph_0$.]

4.13. If F is an uncountable closed set, then $F = F^* \cup (F - F^*)$ is the unique partition of F into a perfect set and an at most countable set.
[Use Exercise 4.9.]

4.14. $\boldsymbol{Q}$ is not the intersection of a countable collection of open sets.
[Use the Baire Category Theorem.]

4.15. If B is Borel and f is a continuous function then $f_{-1}(B)$ is Borel.

4.16. Let $f : \boldsymbol{R} \to \boldsymbol{R}$. Show that the set of all x at which f is continuous is a G_δ set.

4.17. (i) $\mathcal{N} \times \mathcal{N}$ is homeomorphic to $\mathcal{N}$.
(ii) $\mathcal{N}^\omega$ is homeomorphic to $\mathcal{N}$.

4.18. The tree T_F in (4.6) has no maximal node, i.e., $s \in T$ such that there is no $t \in T$ with $s \subset t$. The map $F \mapsto T_F$ is a one-to-one correspondence between closed sets in $\mathcal{N}$ and sequential trees without maximal nodes.

4.19. Every perfect Polish space has a closed subset homeomorphic to the Cantor space.

4.20. Every Polish space is homeomorphic to a G_δ subspace of the Hilbert cube.
[Let $\{x_n : n \in \boldsymbol{N}\}$ be a dense set, and define $f(x) = \langle d(x, x_n) : n \in \boldsymbol{N} \rangle$.]

Historical Notes

Theorems 4.1, 4.3 and 4.5 are due to Cantor. The construction of real numbers by completion of the rationals is due to Dedekind [1872].

Suslin's Problem: Suslin [1920].

Theorem 4.6: Cantor, Bendixson [1883].

Theorem 4.8: Baire [1899].

Exercise 4.5: Cantor.

5. The Axiom of Choice and Cardinal Arithmetic

The Axiom of Choice

***Axiom of Choice* (AC).** *Every family of nonempty sets has a choice function.*

If S is a family of sets and $\emptyset \notin S$, then a *choice function* for S is a function f on S such that

$$f(X) \in X \tag{5.1}$$

for every $X \in S$.

The Axiom of Choice postulates that for every S such that $\emptyset \notin S$ there exists a function f on S that satisfies (5.1).

The Axiom of Choice differs from other axioms of ZF by postulating the existence of a set (i.e., a choice function) without defining it (unlike, for instance, the Axiom of Pairing or the Axiom of Power Set). Thus it is often interesting to know whether a mathematical statement can be proved without using the Axiom of Choice. It turns out that the Axiom of Choice is independent of the other axioms of set theory and that many mathematical theorems are unprovable in ZF without AC.

In some trivial cases, the existence of a choice function can be proved outright in ZF:

(i) when every $X \in S$ is a singleton $X = \{x\}$;
(ii) when S is finite; the existence of a choice function for S is proved by induction on the size of S;
(iii) when every $X \in S$ is a finite set of real numbers; let $f(X) =$ the least element of X.

On the other hand, one cannot prove existence of a choice function (in ZF) just from the assumption that the sets in S are finite; even when every $X \in S$ has just two elements (e.g., sets of reals), we cannot necessarily prove that S has a choice function.

Using the Axiom of Choice, one proves that every set can be well-ordered, and therefore every infinite set has cardinality equal to some $\aleph_\alpha$. In particular,

any two sets have comparable cardinals, and the ordering

$$|X| \leq |Y|$$

is a well-ordering of the class of all cardinals.

Theorem 5.1 (Zermelo's Well-Ordering Theorem). *Every set can be well-ordered.*

Proof. Let A be a set. To well-order A, it suffices to construct a transfinite one-to-one sequence $\langle a_\alpha : \alpha < \theta \rangle$ that enumerates A. That we can do by induction, using a choice function f for the family S of all nonempty subsets of A. We let for every α

$$a_\alpha = f(A - \{a_\xi : \xi < \alpha\})$$

if $A - \{a_\xi : \xi < \alpha\}$ is nonempty. Let θ be the least ordinal such that $A = \{a_\xi : \xi < \theta\}$. Clearly, $\langle a_\alpha : \alpha < \theta \rangle$ enumerates A. □

In fact, Zermelo's Theorem 5.1 is equivalent to the Axiom of Choice: If every set can be well-ordered, then every family S of nonempty sets has a choice function. To see this, well-order $\bigcup S$ and let $f(X)$ be the least element of X for every $X \in S$.

Of particular importance is the fact that the set of all real numbers can be well-ordered. It follows that $2^{\aleph_0}$ is an aleph and so $2^{\aleph_0} \geq \aleph_1$.

The existence of a well-ordering of $\boldsymbol{R}$ yields some interesting counterexamples. Well known is Vitali's construction of a nonmeasurable set (Exercise 10.1); another example is an uncountable set of reals without a perfect subset (Exercise 5.1).

If every set can be well-ordered, then every infinite set has a countable subset: Well-order the set and take the first ω elements. Thus every infinite set is Dedekind-infinite, and so finiteness and Dedekind finiteness coincide.

Dealing with cardinalities of sets is much easier when we have the Axiom of Choice. In the first place, any two sets have comparable cardinals. Another consequence is:

(5.2) *if f maps A onto B then $|B| \leq |A|$.*

To show (5.2), we have to find a one-to-one function from B to A. This is done by choosing one element from $f_{-1}(\{b\})$ for each $b \in B$.

Another consequence of the Axiom of Choice is:

(5.3) *The union of a countable family of countable sets is countable.*

(By the way, this often used fact cannot be proved in ZF alone.) To prove (5.3) let A_n be a countable set for each $n \in \boldsymbol{N}$. For each n, let us *choose* an

enumeration $\langle a_{n,k} : k \in \boldsymbol{N}\rangle$ of A_n. That gives us a projection of $\boldsymbol{N} \times \boldsymbol{N}$ onto $\bigcup_{n=0}^{\infty} A_n$:

$$(n, k) \mapsto a_{n,k}.$$

Thus $\bigcup_{n=0}^{\infty} A_n$ is countable.

In a similar fashion, one can prove a more general statement.

Lemma 5.2. $|\bigcup S| \leq |S| \cdot \sup\{|X| : X \in S\}$.

Proof. Let $\kappa = |S|$ and $\lambda = \sup\{|X| : X \in S\}$. We have $S = \{X_\alpha : \alpha < \kappa\}$ and for each $\alpha < \kappa$, we choose an enumeration $X_\alpha = \{a_{\alpha,\beta} : \beta < \lambda_\alpha\}$, where $\lambda_\alpha \leq \lambda$. Again we have a projection

$$(\alpha, \beta) \mapsto a_{\alpha,\beta}$$

of $\kappa \times \lambda$ onto $\bigcup S$, and so $|\bigcup S| \leq \kappa \cdot \lambda$. □

In particular, the union of $\aleph_\alpha$ sets, each of cardinality $\aleph_\alpha$, has cardinality $\aleph_\alpha$.

Corollary 5.3. *Every $\aleph_{\alpha+1}$ is a regular cardinal.*

Proof. This is because otherwise $\omega_{\alpha+1}$ would be the union of at most $\aleph_\alpha$ sets of cardinality at most $\aleph_\alpha$. □

Using the Axiom of Choice in Mathematics

In algebra and point set topology, one often uses the following version of the Axiom of Choice. We recall that if $(P, <)$ is a partially ordered set, then $a \in P$ is called *maximal* in P if there is no $x \in P$ such that $a < x$. If X is a nonempty subset of P, then $c \in P$ is an *upper bound* of X if $x \leq c$ for every $x \in X$.

We say that a nonempty $C \subset P$ is a *chain* in P if C is linearly ordered by $<$.

Theorem 5.4 (Zorn's Lemma). *If $(P, <)$ is a nonempty partially ordered set such that every chain in P has an upper bound, then P has a maximal element.*

Proof. We construct (using a choice function for nonempty subsets of P), a chain in P that leads to a maximal element of P. We let, by induction,

$$a_\alpha = \text{an element of } P \text{ such that } a_\alpha > a_\xi \text{ for every } \xi < \alpha \text{ if there is one.}$$

Clearly, if $\alpha > 0$ is a limit ordinal, then $C_\alpha = \{a_\xi : \xi < \alpha\}$ is a chain in P and a_α exists by the assumption. Eventually, there is θ such that there is no $a_{\theta+1} \in P$, $a_{\theta+1} > a_\theta$. Thus a_θ is a maximal element of P. □

Like Zermelo's Theorem 5.1, Zorn's Lemma 5.4 is equivalent to the Axiom of Choice (in ZF); see Exercise 5.5.

There are numerous examples of proofs using Zorn's Lemma. To mention only of few:

Every vector space has a basis.
Every field has a unique algebraic closure.
The Hahn-Banach Extension Theorem.
Tikhonov's Product Theorem for compact spaces.

The Countable Axiom of Choice

Many important consequences of the Axiom of Choice, particularly many concerning the real numbers, can be proved from a weaker version of the Axiom of Choice.

The Countable Axiom of Choice. *Every countable family of nonempty sets has a choice function.*

For instance, the countable AC implies that the union of countably many countable sets is countable. In particular, the real line is not a countable union of countable sets. Similarly, it follows that $\aleph_1$ is a regular cardinal. On the other hand, the countable AC does not imply that the set of all reals can be well-ordered.

Several basic theorems about Borel sets and Lebesgue measure use the countable AC; for instance, one needs it to show that the union of countably many F_σ sets is F_σ. In modern descriptive set theory one often works without the Axiom of Choice and uses the countable AC instead. In some instances, descriptive set theorists use a somewhat stronger principle (that follows from AC):

***The Principle of Dependent Choices* (DC).** *If E is a binary relation on a nonempty set A, and if for every $a \in A$ there exists $b \in A$ such that $b\ E\ a$, then there is a sequence $a_0, a_1, \ldots, a_n, \ldots$ in A such that*

$$a_{n+1}\ E\ a_n \quad \text{for all } n \in \boldsymbol{N}. \tag{5.4}$$

The Principle of Dependent Choices is stronger than the Countable Axiom of Choice; see Exercise 5.7.

As an application of DC we have the following characterization of well-founded relations and well-orderings:

Lemma 5.5.

(i) *A linear ordering $<$ of a set P is a well-ordering of P if and only if there is no infinite descending sequence*

$$a_0 > a_1 > \ldots > a_n > \ldots$$

in A.

(ii) *A relation E on P is well-founded if and only if there is no infinite sequence $\langle a_n : n \in \boldsymbol{N}\rangle$ in P such that*

$$a_{n+1}\ E\ a_n \quad \textit{for all } n \in \boldsymbol{N}. \tag{5.5}$$

Proof. Note that (i) is a special case of (ii) since a well-ordering is a well-founded linear ordering.

If $a_0, a_1, \ldots, a_n, \ldots$ is a sequence that satisfies (5.5), then the set $\{a_n : n \in \boldsymbol{N}\}$ has no E-minimal element and hence E is not well-founded.

Conversely, if E is not well-founded, then there is a nonempty set $A \subset P$ with no E-minimal element. Using the Principle of Dependent Choices we construct a sequence $a_0, a_1, \ldots, a_n, \ldots$ that satisfies (5.5). □

Cardinal Arithmetic

In the presence of the Axiom of Choice, every set can be well-ordered and so every infinite set has the cardinality of some $\aleph_\alpha$. Thus addition and multiplication of infinite cardinal numbers is simple: If κ and λ are infinite cardinals then

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

The exponentiation of cardinals is more interesting. The rest of Chapter 5 is devoted to the operations 2^κ and κ^λ, for infinite cardinals κ and λ.

Lemma 5.6. *If $2 \le \kappa \le \lambda$ and λ is infinite, then $\kappa^\lambda = 2^\lambda$.*

Proof.

$$2^\lambda \le \kappa^\lambda \le (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\lambda. \tag{5.6}$$ □

If κ and λ are infinite cardinals and $\lambda < \kappa$ then the evaluation of κ^λ is more complicated. First, if $2^\lambda \ge \kappa$ then we have $\kappa^\lambda = 2^\lambda$ (because $\kappa^\lambda \le (2^\lambda)^\lambda = 2^\lambda$), but if $2^\lambda < \kappa$ then (because $\kappa^\lambda \le \kappa^\kappa = 2^\kappa$) we can only conclude

$$\kappa \le \kappa^\lambda \le 2^\kappa. \tag{5.7}$$

Not much more can be claimed at this point, except that by Theorem 3.11 in Chapter 3 ($\kappa^{\operatorname{cf}\kappa} > \kappa$) we have

$$\kappa < \kappa^\lambda \quad \text{if } \lambda \ge \operatorname{cf}\kappa. \tag{5.8}$$

If λ is a cardinal and $|A| \ge \lambda$, let

$$[A]^\lambda = \{X \subset A : |X| = \lambda\}. \tag{5.9}$$

Lemma 5.7. *If $|A| = \kappa \ge \lambda$, then the set $[A]^\lambda$ has cardinality κ^λ.*

Proof. On the one hand, every $f : \lambda \to A$ is a subset of $\lambda \times A$, and $|f| = \lambda$. Thus $\kappa^\lambda \leq |[\lambda \times A]|^\lambda = |[A]^\lambda|$. On the other hand, we construct a one-to-one function $F : [A]^\lambda \to A^\lambda$ as follows: If $X \subset A$ and $|X| = \lambda$, let $F(X)$ be some function f on λ whose range is X. Clearly, F is one-to-one. □

If λ is a limit cardinal, let

$$\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu \text{ is a cardinal and } \mu < \lambda\}. \tag{5.10}$$

For the sake of completeness, we also define $\kappa^{<\lambda^+} = \kappa^\lambda$ for infinite successor cardinals λ^+.

If κ is an infinite cardinal and $|A| \geq \kappa$, let

$$[A]^{<\kappa} = P_\kappa(A) = \{X \subset A : |X| < \kappa\}. \tag{5.11}$$

It follows from Lemma 5.7 and Lemma 5.8 below that the cardinality of $P_\kappa(A)$ is $|A|^{<\kappa}$.

Infinite Sums and Products

Let $\{\kappa_i : i \in I\}$ be an indexed set of cardinal numbers. We define

$$\sum_{i \in I} \kappa_i = \Big|\bigcup_{i \in I} X_i\Big|, \tag{5.12}$$

where $\{X_i : i \in I\}$ is a disjoint family of sets such that $|X_i| = \kappa_i$ for each $i \in I$.

This definition does not depend on the choice of $\{X_i\}_i$; this follows from the Axiom of Choice (see Exercise 5.9).

Note that if κ and λ are cardinals and $\kappa_i = \kappa$ for each $i < \lambda$, then

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \kappa.$$

In general, we have the following

Lemma 5.8. *If λ is an infinite cardinal and $\kappa_i > 0$ for each $i < \lambda$, then*

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i. \tag{5.13}$$

Proof. Let $\kappa = \sup_{i<\lambda} \kappa_i$ and $\sigma = \sum_{i<\lambda} \kappa_i$. On the one hand, since $\kappa_i \leq \kappa$ for all i, we have $\sum_{i<\lambda} \kappa \leq \lambda \cdot \kappa$. On the other hand, since $\kappa_i \geq 1$ for all i, we have $\lambda = \sum_{i<\lambda} 1 \leq \sigma$, and since $\sigma \geq \kappa_i$ for all i, we have $\sigma \geq \sup_{i<\lambda} \kappa_i = \kappa$. Therefore $\sigma \geq \lambda \cdot \kappa$. □

In particular, if $\lambda \le \sup_{i<\lambda} \kappa_i$, we have

$$\sum_{i<\lambda} \kappa_i = \sup_{i<\lambda} \kappa_i.$$

Thus we can characterize singular cardinals as follows: An infinite cardinal κ is singular just in case

$$\kappa = \sum_{i<\lambda} \kappa_i$$

where $\lambda < \kappa$ and for each i, $\kappa_i < \kappa$.

An infinite product of cardinals is defined using infinite products of sets. If $\{X_i : i \in I\}$ is a family of sets, then the *product* is defined as follows:

$$\prod_{i\in I} X_i = \{f : f \text{ is a function on } I \text{ and } f(i) \in X_i \text{ for each } i \in I\}. \tag{5.14}$$

Note that if some X_i is empty, then the product is empty. If all the X_i are nonempty, then AC implies that the product is nonempty.

If $\{\kappa_i : i \in I\}$ is a family of cardinal numbers, we define

$$\prod_{i\in I} \kappa_i = \Big|\prod_{i\in I} X_i\Big|, \tag{5.15}$$

where $\{X_i : i \in I\}$ is a family of sets such that $|X_i| = \kappa_i$ for each $i \in I$. (We abuse the notation by using $\prod$ both for the product of sets and for the product of cardinals.)

Again, it follows from AC that the definition does not depend on the choice of the sets X_i (Exercise 5.10).

If $\kappa_i = \kappa$ for each $i \in I$, and $|I| = \lambda$, then $\prod_{i\in I} \kappa_i = \kappa^\lambda$. Also, infinite sums and products satisfy some of the rules satisfied by finite sums and products. For instance, $\prod_i \kappa_i^\lambda = (\prod_i \kappa_i)^\lambda$, or $\prod_i \kappa^{\lambda_i} = \kappa^{\sum_i \lambda_i}$. Or if I is a disjoint union $I = \bigcup_{j\in J} A_j$, then

$$\prod_{i\in I} \kappa_i = \prod_{j\in J} \Big(\prod_{i\in A_j} \kappa_i\Big). \tag{5.16}$$

If $\kappa_i \ge 2$ for each $i \in I$, then

$$\sum_{i\in I} \kappa_i \le \prod_{i\in I} \kappa_i. \tag{5.17}$$

(The assumption $\kappa_i \ge 2$ is necessary: $1+1 > 1\cdot 1$.) If I is finite, then (5.17) is certainly true; thus assume that I is infinite. Since $\prod_{i\in I} \kappa_i \ge \prod_{i\in I} 2 = 2^{|I|} > |I|$, it suffices to show that $\sum_i \kappa_i \le |I| \cdot \prod_i \kappa_i$. If $\{X_i : i \in I\}$ is a disjoint family, we assign to each $x \in \bigcup_i X_i$ a pair (i, f) such that $x \in X_i$, $f \in \prod_i X_i$ and $f(i) = x$. Thus we have (5.17).

Infinite product of cardinals can be evaluated using the following lemma:

Lemma 5.9. *If λ is an infinite cardinal and $\langle \kappa_i : i < \lambda \rangle$ is a nondecreasing sequence of nonzero cardinals, then*

$$\prod_{i<\lambda} \kappa_i = (\sup_i \kappa_i)^\lambda.$$

Proof. Let $\kappa = \sup_i \kappa_i$. Since $\kappa_i \le \kappa$ for each $i < \lambda$, we have

$$\prod_{i<\lambda} \kappa_i \le \prod_{i<\lambda} \kappa = \kappa^\lambda.$$

To prove that $\kappa^\lambda \le \prod_{i<\lambda} \kappa_i$, we consider a partition of λ into λ disjoint sets A_j, each of cardinality λ:

$$\lambda = \bigcup_{j<\lambda} A_j. \tag{5.18}$$

(To get a partition (5.18), we can, e.g., use the canonical pairing function $\Gamma : \lambda \times \lambda \to \lambda$ and let $A_j = \Gamma(\lambda \times \{j\})$.) Since a product of nonzero cardinals is greater than or equal to each factor, we have $\prod_{i\in A_j} \kappa_i \ge \sup_{i\in A_j} \kappa_i = \kappa$, for each $j < \lambda$. Thus, by (5.16),

$$\prod_{i<\lambda} \kappa_i = \prod_{j<\lambda} \Big(\prod_{i\in A_j} \kappa_i \Big) \ge \prod_{j<\lambda} \kappa = \kappa^\lambda. \qquad \square$$

The strict inequalities in cardinal arithmetic that we proved in Chapter 3 can be obtained as special cases of the following general theorem.

Theorem 5.10 (König). *If $\kappa_i < \lambda_i$ for every $i \in I$, then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Proof. We shall show that $\sum_i \kappa_i \not\ge \prod_i \lambda_i$. Let T_i, $i \in I$, be such that $|T_i| = \lambda_i$ for each $i \in I$. It suffices to show that if Z_i, $i \in I$, are subsets of $T = \prod_{i\in I} T_i$, and $|Z_i| \le \kappa_i$ for each $i \in I$, then $\bigcup_{i\in I} Z_i \ne T$.

For every $i \in I$, let S_i be the projection of Z_i into the ith coordinate:

$$S_i = \{f(i) : f \in Z_i\}.$$

Since $|Z_i| < |T_i|$, we have $S_i \subset T_i$ and $S_i \ne T_i$. Now let $f \in T$ be a function such that $f(i) \notin S_i$ for every $i \in I$. Obviously, f does not belong to any Z_i, $i \in I$, and so $\bigcup_{i\in I} Z_i \ne T$. $\square$

Corollary 5.11. *$\kappa < 2^\kappa$ for every κ.*

Proof. $\underbrace{1+1+\dots}_{\kappa \text{ times}} < \underbrace{2\cdot 2\cdot \dots}_{\kappa \text{ times}}$. $\square$

Corollary 5.12. $\operatorname{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$.

Proof. It suffices to show that if $\kappa_i < 2^{\aleph_\alpha}$ for $i < \omega_\alpha$, then $\sum_{i<\omega_\alpha} \kappa_i < 2^{\aleph_\alpha}$. Let $\lambda_i = 2^{\aleph_\alpha}$.

$$\sum_{i<\omega_\alpha} \kappa_i < \prod_{i<\omega_\alpha} \lambda_i = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}. \qquad \square$$

Corollary 5.13. $\operatorname{cf}(\aleph_\alpha^{\aleph_\beta}) > \aleph_\beta$.

Proof. We show that if $\kappa_i < \aleph_\alpha^{\aleph_\beta}$ for $i < \omega_\beta$, then $\sum_{i<\omega_\beta} \kappa_i < \aleph_\alpha^{\aleph_\beta}$. Let $\lambda_i = \aleph_\alpha^{\aleph_\beta}$.

$$\sum_{i<\omega_\beta} \kappa_i < \prod_{i<\omega_\beta} \lambda_i = (\aleph_\alpha^{\aleph_\beta})^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}. \qquad \square$$

Corollary 5.14. *$\kappa^{\operatorname{cf}\kappa} > \kappa$ for every infinite cardinal κ.*

Proof. Let $\kappa_i < \kappa$, $i < \operatorname{cf}\kappa$, be such that $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$. Then

$$\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i < \prod_{i<\operatorname{cf}\kappa} \kappa = \kappa^{\operatorname{cf}\kappa}. \qquad \square$$

The Continuum Function

Cantor's Theorem 3.1 states that $2^{\aleph_\alpha} > \aleph_\alpha$, and therefore $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$, for all α. The *Generalized Continuum Hypothesis* (GCH) is the statement

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

for all α. GCH is independent of the axioms of ZFC. Under the assumption of GCH, cardinal exponentiation is evaluated as follows:

Theorem 5.15. *If* GCH *holds and κ and λ are infinite cardinals then*:

(i) *If $\kappa \leq \lambda$, then $\kappa^\lambda = \lambda^+$.*
(ii) *If $\operatorname{cf}\kappa \leq \lambda < \kappa$, then $\kappa^\lambda = \kappa^+$.*
(iii) *If $\lambda < \operatorname{cf}\kappa$, then $\kappa^\lambda = \kappa$.*

Proof. (i) Lemma 5.6.
(ii) This follows from (5.7) and (5.8).
(iii) By Lemma 3.9(ii), the set κ^λ is the union of the sets α^λ, $\alpha < \kappa$, and $|\alpha^\lambda| \leq 2^{|\alpha|\cdot\lambda} = (|\alpha| \cdot \lambda)^+ \leq \kappa$. $\square$

The *beth function* is defined by induction:

$$\beth_0 = \aleph_0, \qquad \beth_{\alpha+1} = 2^{\beth_\alpha},$$
$$\beth_\alpha = \sup\{\beth_\beta : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Thus GCH is equivalent to the statement $\beth_\alpha = \aleph_\alpha$ for all α.

We shall now investigate the general behavior of the *continuum function* 2^κ, without assuming GCH.

Theorem 5.16.

(i) *If* $\kappa < \lambda$ *then* $2^\kappa \le 2^\lambda$.
(ii) $\operatorname{cf} 2^\kappa > \kappa$.
(iii) *If* κ *is a limit cardinal then* $2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa}$.

Proof. (ii) By Corollary 5.12,
(iii) Let $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$, where $\kappa_i < \kappa$ for each i. We have

$$2^\kappa = 2^{\sum_i \kappa_i} = \prod_i 2^{\kappa_i} \le \prod_i 2^{<\kappa} = (2^{<\kappa})^{\operatorname{cf}\kappa} \le (2^\kappa)^{\operatorname{cf}\kappa} \le 2^\kappa. \qquad \square$$

For regular cardinals, the only conditions Theorem 5.16 places on the continuum function are $2^\kappa > \kappa$ and $2^\kappa \le 2^\lambda$ if $\kappa < \lambda$. We shall see that these are the only restrictions on 2^κ for regular κ that are provable in ZFC.

Corollary 5.17. *If* κ *is a singular cardinal and if the continuum function is eventually constant below* κ, *with value* λ, *then* $2^\kappa = \lambda$.

Proof. If κ is a singular cardinal that satisfies the assumption of the theorem, then there is μ such that $\operatorname{cf}\kappa \le \mu < \kappa$ and that $2^{<\kappa} = \lambda = 2^\mu$. Thus

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa} = (2^\mu)^{\operatorname{cf}\kappa} = 2^\mu. \qquad \square$$

The *gimel function* is the function

$$\beth(\kappa) = \kappa^{\operatorname{cf}\kappa}. \tag{5.19}$$

If κ is a limit cardinal and if the continuum function below κ is not eventually constant, then the cardinal $\lambda = 2^{<\kappa}$ is a limit of a nondecreasing sequence

$$\lambda = 2^{<\kappa} = \lim_{\alpha\to\kappa} 2^{|\alpha|}$$

of length κ. By Lemma 3.7(ii), we have

$$\operatorname{cf}\lambda = \operatorname{cf}\kappa.$$

Using Theorem 5.16(iii), we get

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\lambda}. \tag{5.20}$$

If κ is a regular cardinal, then $\kappa = \operatorname{cf}\kappa$; and since $2^\kappa = \kappa^\kappa$, we have

$$2^\kappa = \kappa^{\operatorname{cf}\kappa}. \tag{5.21}$$

Thus (5.20) and (5.21) show that the continuum function can be defined in terms of the gimel function:

Corollary 5.18.

(i) *If* κ *is a successor cardinal, then* $2^\kappa = \beth(\kappa)$.
(ii) *If* κ *is a limit cardinal and if the continuum function below* κ *is eventually constant, then* $2^\kappa = 2^{<\kappa} \cdot \beth(\kappa)$.
(iii) *If* κ *is a limit cardinal and if the continuum function below* κ *is not eventually constant, then* $2^\kappa = \beth(2^{<\kappa})$. $\square$

Cardinal Exponentiation

We shall now investigate the function κ^λ for infinite cardinal numbers κ and λ.

We start with the following observation: If κ is a regular cardinal and $\lambda < \kappa$, then every function $f : \lambda \to \kappa$ is bounded (i.e., $\sup\{f(\xi) : \xi < \lambda\} < \kappa$). Thus

$$\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda.$$

and so

$$\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda.$$

In particular, if κ is a successor cardinal, we obtain the *Hausdorff formula*

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}. \tag{5.22}$$

(Note that (5.22) holds for all α and β.)

In general, we can compute κ^λ using the following lemma. If κ is a limit cardinal, we use the notation $\lim_{\alpha\to\kappa} \alpha^\lambda$ to abbreviate $\sup\{\mu^\lambda : \mu \text{ is a cardinal and } \mu < \kappa\}$.

Lemma 5.19. *If κ is a limit cardinal, and $\lambda \geq \operatorname{cf}\kappa$, then*

$$\kappa^\lambda = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa}.$$

Proof. Let $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$, where $\kappa_i < \kappa$ for each i. We have $\kappa^\lambda \leq (\prod_{i<\operatorname{cf}\kappa} \kappa_i)^\lambda = \prod_i \kappa_i^\lambda \leq \prod_i (\lim_{\alpha\to\kappa} \alpha^\lambda) = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa} \leq (\kappa^\lambda)^{\operatorname{cf}\kappa} = \kappa^\lambda$. □

Theorem 5.20. *Let λ be an infinite cardinal. Then for all infinite cardinals κ, the value of κ^λ is computed as follows, by induction on κ:*

(i) *If $\kappa \leq \lambda$ then $\kappa^\lambda = 2^\lambda$.*
(ii) *If there exists some $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.*
(iii) *If $\kappa > \lambda$ and if $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then:*
  (a) *if $\operatorname{cf}\kappa > \lambda$ then $\kappa^\lambda = \kappa$,*
  (b) *if $\operatorname{cf}\kappa \leq \lambda$ then $\kappa^\lambda = \kappa^{\operatorname{cf}\kappa}$.*

Proof. (i) Lemma 5.6

(ii) $\mu^\lambda \leq \kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda$.

(iii) If κ is a successor cardinal, we use the Hausdorff formula. If κ is a limit cardinal, we have $\lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$. If $\operatorname{cf}\kappa > \lambda$ then every $f : \lambda \to \kappa$ is bounded and we have $\kappa^\lambda = \lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$. If $\operatorname{cf}\kappa \leq \lambda$ then by Lemma 5.19, $\kappa^\lambda = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa} = \kappa^{\operatorname{cf}\kappa}$. □

Theorem 5.20 shows that all cardinal exponentiation can be defined in terms of the gimel function:

Corollary 5.21. *For every κ and λ, the value of κ^λ is either 2^λ, or κ, or $\beth(\mu)$ for some μ such that* $\operatorname{cf}\mu \leq \lambda < \mu$.

Proof. If $\kappa^\lambda > 2^\lambda \cdot \kappa$, let μ be the least cardinal such that $\mu^\lambda = \kappa^\lambda$, and by Theorem 5.20 (for μ and λ), $\mu^\lambda = \mu^{\operatorname{cf}\mu}$. □

In the Exercises, we list some properties of the gimel function.

A cardinal κ is a *strong limit* cardinal if

$$2^\lambda < \kappa \quad \text{for every } \lambda < \kappa.$$

Obviously, every strong limit cardinal is a limit cardinal. If the GCH holds, then every limit cardinal is a strong limit.

It is easy to see that if κ is a strong limit cardinal, then

$$\lambda^\nu < \kappa \quad \text{for all } \lambda, \nu < \kappa.$$

An example of a strong limit cardinal is $\aleph_0$. Actually, the strong limit cardinals form a proper class: If α is an arbitrary cardinal, then the cardinal

$$\kappa = \sup\{\alpha, 2^\alpha, 2^{2^\alpha}, \dots\}$$

(of cofinality ω) is a strong limit cardinal.

Another fact worth mentioning is:

(5.23) *If κ is a strong limit cardinal, then $2^\kappa = \kappa^{\operatorname{cf}\kappa}$.*

We recall that κ is weakly inaccessible if it is uncountable, regular, and limit. We say that a cardinal κ is *inaccessible* (strongly) if $\kappa > \aleph_0$, κ is regular, and κ is strong limit.

Every inaccessible cardinal is weakly inaccessible. If the GCH holds, then every weakly inaccessible cardinal κ is inaccessible.

The inaccessible cardinals owe their name to the fact that they cannot be obtained from smaller cardinals by the usual set-theoretical operations.

If κ is inaccessible and $|X| < \kappa$, then $|P(X)| < \kappa$. If $|S| < \kappa$ and if $|X| < \kappa$ for every $X \in S$, then $|\bigcup S| < \kappa$.

In fact, $\aleph_0$ has this property too. Thus we can say that in a sense an inaccessible cardinal is to smaller cardinals what $\aleph_0$ is to finite cardinals. This is one of the main themes of the theory of large cardinals.

The Singular Cardinal Hypothesis

The *Singular Cardinal Hypothesis* (SCH) is the statement: For every singular cardinal κ, if $2^{\operatorname{cf}\kappa} < \kappa$, then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.

Obviously, the Singular Cardinals Hypothesis follows from GCH. If $2^{\operatorname{cf}\kappa} \geq \kappa$ then $\kappa^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa}$. If $2^{\operatorname{cf}\kappa} < \kappa$, then κ^+ is the least possible value of $\kappa^{\operatorname{cf}\kappa}$.

We shall prove later in the book that if SCH fails then a large cardinal axiom holds. In fact, the failure of SCH is equiconsistent with the existence of a certain large cardinal.

Under the assumption of SCH, cardinal exponentiation is determined by the continuum function on regular cardinals:

Theorem 5.22. *Assume that* SCH *holds.*

(i) *If κ is a singular cardinal then*
  (a) $2^\kappa = 2^{<\kappa}$ *if the continuum function is eventually constant below* κ,
  (b) $2^\kappa = (2^{<\kappa})^+$ *otherwise.*

(ii) *If κ and λ are infinite cardinals, then*:
  (a) *If* $\kappa \leq 2^\lambda$ *then* $\kappa^\lambda = 2^\lambda$.
  (b) *If* $2^\lambda < \kappa$ *and* $\lambda < \operatorname{cf}\kappa$ *then* $\kappa^\lambda = \kappa$.
  (c) *If* $2^\lambda < \kappa$ *and* $\operatorname{cf}\kappa \leq \lambda$ *then* $\kappa^\lambda = \kappa^+$.

Proof. (i) If κ is a singular cardinal, then by Theorem 5.16, 2^κ is either λ or $\lambda^{\operatorname{cf}\kappa}$ where $\lambda = 2^{<\kappa}$. The latter occurs if 2^α is not eventually constant below κ. Then $\operatorname{cf}\lambda = \operatorname{cf}\kappa$, and since $2^{\operatorname{cf}\kappa} < 2^{<\kappa} = \lambda$, we have $\lambda^{\operatorname{cf}\lambda} = \lambda^+$ by the Singular Cardinals Hypothesis.

(ii) We proceed by induction on κ, for a fixed λ. Let $\kappa > 2^\lambda$. If κ is a successor cardinal, $\kappa = \nu^+$, then $\nu^\lambda \leq \kappa$ (by the induction hypothesis), and $\kappa^\lambda = (\nu^+)^\lambda = \nu^+ \cdot \nu^\lambda = \kappa$, by the Hausdorff formula.

If κ is a limit cardinal, then $\nu^\lambda < \kappa$ for all $\nu < \kappa$. By Theorem 5.20, $\kappa^\lambda = \kappa$ if $\lambda < \operatorname{cf}\kappa$, and $\kappa^\lambda = \kappa^{\operatorname{cf}\kappa}$ if $\lambda \geq \operatorname{cf}\kappa$, In the latter case, $2^{\operatorname{cf}\kappa} \leq 2^\lambda < \kappa$, and by the Singular Cardinals Hypothesis, $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. □

Exercises

5.1. There exists a set of reals of cardinality $2^{\aleph_0}$ without a perfect subset.
[Let $\langle P_\alpha : \alpha < 2^{\aleph_0}\rangle$ be an enumeration of all perfect sets of reals. Construct disjoint $A = \{a_\alpha : \alpha < 2^{\aleph_0}\}$ and $B = \{b_\alpha : \alpha < 2^{\aleph_0}\}$ as follows: Pick a_α such that $a_\alpha \notin \{a_\xi : \xi < \alpha\} \cup \{b_\xi : \xi < \alpha\}$, and b_α such that $b_\alpha \in P_\alpha - \{a_\xi : \xi \leq \alpha\}$. Then A is the set.]

5.2. If X is an infinite set and S is the set of all finite subsets of X, then $|S| = |X|$.
[Use $|X| = \aleph_\alpha$.]

5.3. Let $(P, <)$ be a linear ordering and let κ be a cardinal. If every initial segment of P has cardinality $< \kappa$, then $|P| \leq \kappa$.

5.4. If A can be well-ordered then $P(A)$ can be linearly ordered.
[Let $X < Y$ if the least element of $X \triangle Y$ belongs to X.]

5.5. Prove the Axiom of Choice from Zorn's Lemma.
[Let S be a family of nonempty sets. To find a choice function on S, let $P = \{f :$ f is a choice function on some $Z \subset S\}$, and apply Zorn's Lemma to the partially ordered set $(P, \subset)$.]

5.6. The countable AC implies that every infinite set has a countable subset.

[If A is infinite, let $A_n = \{s : s \text{ is a one-to-one sequence in } A \text{ of length } n\}$ for each n. Use a choice function for $S = \{A_n : n \in \boldsymbol{N}\}$ to obtain a countable subset of A.]

5.7. Use DC to prove the countable AC.

[Given $S = \{A_n : n \in \boldsymbol{N}\}$, consider the set A of all choice functions on some $S_n = \{A_i : i \le n\}$, with the binary relation $\supset$.]

5.8 (The Milner-Rado Paradox). For every ordinal $\alpha < \kappa^+$ there are sets $X_n \subset \alpha$ $(n \in \boldsymbol{N})$ such that $\alpha = \bigcup_n X_n$, and for each n the order-type of X_n is $\le \kappa^n$.

[By induction on α, choosing a sequence cofinal in α.]

5.9. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are two disjoint families such that $|X_i| = |Y_i|$ for each $i \in I$, then $|\bigcup_{i\in I} X_i| = |\bigcup_{i\in I} Y_i|$.

[Use AC.]

5.10. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are such that $|X_i| = |Y_i|$ for each $i \in I$, then $|\prod_{i\in I} X_i| = |\prod_{i\in I} Y_i|$.

[Use AC.]

5.11. $\prod_{0<n<\omega} n = 2^{\aleph_0}$.

5.12. $\prod_{n<\omega} \aleph_n = \aleph_\omega^{\aleph_0}$.

5.13. $\prod_{\alpha<\omega+\omega} \aleph_\alpha = \aleph_{\omega+\omega}^{\aleph_0}$.

5.14. If GCH holds then

(i) $2^{<\kappa} = \kappa$ for all κ, and
(ii) $\kappa^{<\kappa} = \kappa$ for all regular κ.

5.15. If β is such that $2^{\aleph_\alpha} = \aleph_{\alpha+\beta}$ for every α, then $\beta < \omega$.

[Let $\beta \ge \omega$. Let α be least such that $\alpha + \beta > \beta$. We have $0 < \alpha \le \beta$, and α is limit. Let $\kappa = \aleph_{\alpha+\alpha}$; since $\operatorname{cf}\kappa = \operatorname{cf}\alpha \le \alpha < \kappa$, κ is singular. For each $\xi < \alpha$, $\xi + \beta = \beta$, and so $2^{\aleph_{\alpha+\xi}} = \aleph_{\alpha+\xi+\beta} = \aleph_{\alpha+\beta}$. By Corollary 5.17, $2^\kappa = \aleph_{\alpha+\beta}$, a contradiction, since $\aleph_{\alpha+\beta} < \aleph_{\alpha+\alpha+\beta}$.]

5.16. $\prod_{\alpha<\omega_1+\omega} \aleph_\alpha = \aleph_{\omega_1+\omega}^{\aleph_1}$.

[$\aleph_{\omega_1+\omega}^{\aleph_1} \le (\prod_{n=0}^{\infty} \aleph_{\omega_1+n})^{\aleph_1} = \prod_n \aleph_{\omega_1+n}^{\aleph_1} = \prod_n (\aleph_{\omega_1}^{\aleph_1} \cdot \aleph_{\omega_1+n}) = \aleph_{\omega_1}^{\aleph_1} \cdot \prod_n \aleph_{\omega_1+n} = \prod_{\alpha<\omega_1+\omega} \aleph_\alpha$.]

5.17. If κ is a limit cardinal and $\lambda < \operatorname{cf}\kappa$, then $\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda$.

5.18. $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.

5.19. If $\alpha < \omega_1$, then $\aleph_\alpha^{\aleph_1} = \aleph_\alpha^{\aleph_0} \cdot 2^{\aleph_1}$.

5.20. If $\alpha < \omega_2$, then $\aleph_\alpha^{\aleph_2} = \aleph_\alpha^{\aleph_1} \cdot 2^{\aleph_2}$.

5.21. If κ is regular and limit, then $\kappa^{<\kappa} = 2^{<\kappa}$. If κ is regular and strong limit then $\kappa^{<\kappa} = \kappa$.

5.22. If κ is singular and is not strong limit, then $\kappa^{<\kappa} = 2^{<\kappa} > \kappa$.

5.23. If κ is singular and strong limit, then $2^{<\kappa} = \kappa$ and $\kappa^{<\kappa} = \kappa^{\operatorname{cf}\kappa}$.

5.24. If $2^{\aleph_0} > \aleph_\omega$, then $\aleph_\omega^{\aleph_0} = 2^{\aleph_0}$.

5.25. If $2^{\aleph_1} = \aleph_2$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$, then $\aleph_{\omega_1}^{\aleph_1} = \aleph_\omega^{\aleph_0}$.

5.26. If $2^{\aleph_0} \geq \aleph_{\omega_1}$, then $\beth(\aleph_\omega) = 2^{\aleph_0}$ and $\beth(\aleph_{\omega_1}) = 2^{\aleph_1}$.

5.27. If $2^{\aleph_1} = \aleph_2$, then $\aleph_\omega^{\aleph_0} \neq \aleph_{\omega_1}$.

5.28. If κ is a singular cardinal and if $\kappa < \beth(\lambda)$ for some $\lambda < \kappa$ such that $\operatorname{cf}\kappa \leq \operatorname{cf}\lambda$ then $\beth(\kappa) \leq \beth(\lambda)$.

5.29. If κ is a singular cardinal such that $2^{\operatorname{cf}\kappa} < \kappa \leq \lambda^{\operatorname{cf}\kappa}$ for some $\lambda < \kappa$, then $\beth(\kappa) = \beth(\lambda)$ where λ is the least λ such that $\kappa \leq \lambda^{\operatorname{cf}\kappa}$.

Historical Notes

The Axiom of Choice was formulated by Zermelo, who used it to prove the Well-Ordering Theorem in [1904]. Zorn's Lemma is as in Zorn [1935]; for a related principle, see Kuratowski [1922]. (Hausdorff in [1914], pp. 140–141, proved that every partially ordered set has a maximal linearly ordered subset.) The Principle of Dependent Choices was formulated by Bernays in [1942].

König's Theorem 5.10 appeared in J. König [1905]. Corollary 5.17 was found independently by Bukovský [1965] and Hechler. The discovery that cardinal exponentiation is determined by the gimel function was made by Bukovský; cf. [1965]. The inductive computation of κ^λ in Theorem 5.20 is as in Jech [1973a].

The Hausdorff formula (5.22): Hausdorff [1904].

Inaccessible cardinals were introduced in the paper by Sierpiński and Tarski [1930]; see Tarski [1938] for more details.

Exercise 5.1: Felix Bernstein.

Exercise 5.8: Milner and Rado [1965].

Exercise 5.15: L. Patai.

Exercise 5.17: Tarski [1925b].

Exercises 5.28–5.29: Jech [1973a].

6. The Axiom of Regularity

The Axiom of Regularity states that the relation $\in$ on any family of sets is well-founded:

Axiom of Regularity. *Every nonempty set has an $\in$-minimal element*:

$$\forall S\,(S \neq \emptyset \to (\exists x \in S)\, S \cap x = \emptyset).$$

As a consequence, there is no infinite sequence

$$x_0 \ni x_1 \ni x_2 \ni \ldots.$$

(Consider the set $S = \{x_0, x_1, x_2, \ldots\}$ and apply the axiom.) In particular, there is no set x such that

$$x \in x$$

and there are no "cycles"

$$x_0 \in x_1 \in \ldots \in x_n \in x_0.$$

Thus the Axiom of Regularity postulates that sets of certain type do no exist. This restriction on the universe of sets is not contradictory (i.e., the axiom is consistent with the other axioms) and is irrelevant for the development of ordinal and cardinal numbers, natural and real numbers, and in fact of all ordinary mathematics. However, it is extremely useful in the metamathematics of set theory, in construction of models. In particular, all sets can be assigned ranks and can be arranged in a cumulative hierarchy.

We recall that a set T is *transitive* if $x \in T$ implies $x \subset T$.

Lemma 6.1. *For every set S there exists a transitive set $T \supset S$.*

Proof. We define by induction

$$S_0 = S, \qquad S_{n+1} = \bigcup S_n$$

and

$$T = \bigcup_{n=0}^{\infty} S_n. \tag{6.1}$$

Clearly, T is transitive and $T \supset S$. □

Since every transitive set must satisfy $\bigcup T \subset T$, it follows that the set in (6.1) is the smallest transitive $T \supset S$; it is called *transitive closure* of S:

$$\mathrm{TC}(S) = \bigcap\{T : T \supset S \text{ and } T \text{ is transitive}\}.$$

Lemma 6.2. *Every nonempty class C has an $\in$-minimal element.*

Proof. Let $S \in C$ be arbitrary. If $S \cap C = \emptyset$, then S is a minimal element of C; if $S \cap C \neq \emptyset$, we let $X = T \cap C$ where $T = \mathrm{TC}(S)$. X is a nonempty set and by the Axiom of Regularity, there is $x \in X$ such that $x \cap X = \emptyset$. It follows that $x \cap C = \emptyset$; otherwise, if $y \in x$ and $y \in C$, then $y \in T$ since T is transitive, and so $y \in x \cap T \cap C = x \cap X$. Hence x is a minimal element of C. □

The Cumulative Hierarchy of Sets

We define, by transfinite induction,

$$V_0 = \emptyset, \qquad V_{\alpha+1} = P(V_\alpha),$$
$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}$$

The sets V_α have the following properties (by induction):

(i) Each V_α is transitive.
(ii) If $\alpha < \beta$, then $V_\alpha \subset V_\beta$.
(iii) $\alpha \subset V_\alpha$.

The Axiom of Regularity implies that every set is in some V_α:

Lemma 6.3. *For every x there is α such that $x \in V_\alpha$:*

$$\bigcup_{\alpha \in Ord} V_\alpha = V. \tag{6.2}$$

Proof. Let C be the class of all x that are not in any V_α. If C is nonempty, then C has an $\in$-minimal element x. That is, $x \in C$, and $z \in \bigcup_\alpha V_\alpha$ for every $z \in x$. Hence $x \subset \bigcup_{\alpha \in Ord} V_\alpha$. By Replacement, there exists an ordinal γ such that $x \subset \bigcup_{\alpha<\gamma} V_\alpha$. Hence $x \subset V_\gamma$ and so $x \in V_{\gamma+1}$. Thus C is empty and we have (6.2). □

Since every x is in some V_α, we may define the *rank* of x:

$$\mathrm{rank}(x) = \text{the least } \alpha \text{ such that } x \in V_{\alpha+1}. \tag{6.3}$$

Thus each V_α is the collection of all sets of rank less than α, and we have

(i) If $x \in y$, then $\mathrm{rank}(x) < \mathrm{rank}(y)$.

(ii) $\operatorname{rank}(\alpha) = \alpha$.

One of the uses of the rank function is a definition of equivalence classes for equivalence relations on a proper class. The basic trick is the following:

Given a class C, let

$$\hat{C} = \{x \in C : (\forall z \in C)\ \operatorname{rank} x \leq \operatorname{rank} z\}. \tag{6.4}$$

$\hat{C}$ is always a set, and if C is nonempty, then $\hat{C}$ is nonempty. Moreover, (6.4) can be applied uniformly.

Thus, for example, if $\equiv$ is an equivalence on a proper class C, we apply (6.4) to each equivalence class of $\equiv$, and define

$$[x] = \{y \in C : y \equiv x \text{ and } \forall z \in C\,(z \equiv x \to \operatorname{rank} y \leq \operatorname{rank} z)\}$$

and

$$C/{\equiv} = \{[x] : x \in C\}.$$

In particular, this trick enables us to define isomorphism types for a given isomorphism. For instance, one can define order-types of linearly ordered sets, or cardinal numbers (even without AC).

We use the same argument to prove the following.

Collection Principle.

$$\forall X\, \exists Y\, (\forall u \in X)[\exists v\, \varphi(u, v, p) \to (\exists v \in Y)\, \varphi(u, v, p)] \tag{6.5}$$

(*p is a parameter*).

The Collection Principle is a schema of formulas. We can formulate it as follows:

Given a "collection of classes" C_u, $u \in X$ (X is a set), then there is a set Y such that for every $u \in X$,

$$\text{if } C_u \neq \emptyset, \quad \text{then } C_u \cap Y \neq \emptyset.$$

To prove (6.5), we let

$$Y = \bigcup_{u \in X} \hat{C}_u$$

where $C_u = \{v : \varphi(u, v, p)\}$, i.e.,

$$v \in Y \leftrightarrow (\exists u \in X)(\varphi(u, v, p) \text{ and } \forall z\,(\varphi(u, z, p) \to \operatorname{rank} v \leq \operatorname{rank} z)).$$

That Y is a set follows from the Replacement Schema.

Note that the Collection Principle implies the Replacement Schema: Given a function F, then for every set X we let Y be a set such that

$$(\forall u \in X)(\exists v \in Y)\, F(u) = v.$$

Then

$$F{\restriction}X = F \cap (X \times Y)$$

is a set by the Separation Schema.

∈-Induction

The method of transfinite induction can be extended to an arbitrary transitive class (instead of *Ord*), both for the proof and for the definition by induction:

Theorem 6.4 (∈-Induction). *Let T be a transitive class, let Φ be a property. Assume that*

(i) $\Phi(\emptyset)$;
(ii) *if $x \in T$ and $\Phi(z)$ holds for every $z \in x$, then $\Phi(x)$.*

Then every $x \in T$ has property Φ.

Proof. Let C be the class of all $x \in T$ that do not have the property Φ. If C is nonempty, then it has an $\in$-minimal element x; apply (i) or (ii). □

Theorem 6.5 (∈-Recursion). *Let T be a transitive class and let G be a function (defined for all x). Then there is a function F on T such that*

$$F(x) = G(F{\restriction}x) \tag{6.6}$$

for every $x \in T$.

Moreover, F is the unique function that satisfies (6.6).

Proof. We let, for every $x \in T$,

$$\begin{aligned} F(x) = y \leftrightarrow {} & \text{there exists a function } f \text{ such that} \\ & \operatorname{dom}(f) \text{ is a transitive subset of } T \text{ and:} \\ & \text{(i)}\ (\forall z \in \operatorname{dom}(f))\, f(z) = G(f{\restriction}z), \\ & \text{(ii)}\ f(x) = y. \end{aligned}$$

That F is a (unique) function on T satisfying (6.6) is proved by $\in$-induction. □

Corollary 6.6. *Let A be a class. There is a unique class B such that*

$$B = \{x \in A : x \subset B\}. \tag{6.7}$$

Proof. Let

$$F(x) = \begin{cases} 1 & \text{if } x \in A \text{ and } F(z) = 1 \text{ for all } z \in x, \\ 0 & \text{otherwise.} \end{cases}$$

Let $B = \{x : F(x) = 1\}$. The uniqueness of B is proved by $\in$-induction. □

We say that each $x \in B$ is *hereditarily* in A.

One consequence of the Axiom of Regularity is that the universe does not admit nontrivial $\in$-automorphisms. More generally:

Theorem 6.7. *Let T_1, T_2 be transitive classes and let π be an $\in$-isomorphism of T_1 onto T_2; i.e., π is one-to-one and*

$$u \in v \leftrightarrow \pi u \in \pi v. \tag{6.8}$$

Then $T_1 = T_2$ and $\pi u = u$ for every $u \in T_1$.

Proof. We show, by $\in$-induction, that $\pi x = x$ for every $x \in T_1$. Assume that $\pi z = z$ for each $z \in x$ and let $y = \pi x$.

We have $x \subset y$ because if $z \in x$, then $z = \pi z \in \pi x = y$.

We also have $y \subset x$: Let $t \in y$. Since $y \subset T_2$, there is $z \in T_1$ such that $\pi z = t$. Since $\pi z \in y$, we have $z \in x$, and so $t = \pi z = z$. Thus $t \in x$.

Therefore $\pi x = x$ for all $x \in T_1$, and $T_2 = T_1$. □

Well-Founded Relations

The notion of well-founded relations that was introduced in Chapter 2 can be generalized to relations on proper classes, and one can extend the method of induction to well-founded relations.

Let E be a binary relation on a class P. For each $x \in P$, we let

$$\mathrm{ext}_E(x) = \{z \in P : z \; E \; x\}$$

the *extension* of x.

Definition 6.8. A relation E on P is well-founded, if:

(6.9) (i) every nonempty set $x \subset P$ has an E-minimal element;
(ii) $\mathrm{ext}_E(x)$ is a set, for every $x \in P$.

(Condition (ii) is vacuous if P is a set.) Note that the relation $\in$ is well-founded on any class, by the Axiom of Regularity.

Lemma 6.9. *If E is a well-founded relation on P, then every nonempty class $C \subset P$ has an E-minimal element.*

Proof. We follow the proof of Lemma 6.2; we are looking for $x \in C$ such that $\mathrm{ext}_E(x) \cap C = \emptyset$. Let $S \in C$ be arbitrary and assume that $\mathrm{ext}_E(S) \cap C \neq \emptyset$. We let $X = T \cap C$ where

$$T = \bigcup_{n=0}^{\infty} S_n$$

and

$$S_0 = \mathrm{ext}_E S, \qquad S_{n+1} = \bigcup\{\mathrm{ext}_E(z) : z \in S_n\}.$$

As in Lemma 6.2, it follows that an E-minimal element x of X is E-minimal in C. □

Theorem 6.10 (Well-Founded Induction). *Let E be a well-founded relation on P. Let Φ be a property. Assume that*:

(i) *every E-minimal element x has property Φ;*
(ii) *if $x \in P$ and if $\Phi(z)$ holds for every z such that $z\ E\ x$, then $\Phi(x)$.*

Then every $x \in P$ has property Φ.

Proof. A modification of the proof of Theorem 6.4. □

Theorem 6.11 (Well-Founded Recursion). *Let E be a well-founded relation on P. Let G be a function (on $V \times V$). Then there is a unique function F on P such that*

$$F(x) = G(x, F\restriction \mathrm{ext}_E(x)) \tag{6.10}$$

for every $x \in P$.

Proof. A modification of the proof of Theorem 6.5. □

(Note that if $F(x) = G(F\restriction \mathrm{ext}(x))$ for some G, then $F(x) = F(y)$ whenever $\mathrm{ext}(x) = \mathrm{ext}(y)$; in particular, $F(x)$ is the same for all minimal elements.)

Example 6.12 (The Rank Function). We define, by induction, for all $x \in P$:

$$\rho(x) = \sup\{\rho(z) + 1 : z\ E\ x\}$$

(compare with (2.7)). The range of ρ is either an ordinal or the class *Ord*. For all $x, y \in P$,

$$x\ E\ y \to \rho(x) < \rho(y).$$ □

Example 6.13 (The Transitive Collapse). By induction, let

$$\pi(x) = \{\pi(z) : z\ E\ x\}$$

for every $x \in P$. The range of π is a transitive class, and for all $x, y \in P$,

$$x\ E\ y \to \pi(x) \in \pi(y).$$ □

The transitive collapse of a well-founded relation is not necessarily a one-to-one function. It is one-to-one if E satisfies an additional condition, extensionality.

Definition 6.14. A well-founded relation E on a class P is *extensional* if

$$\mathrm{ext}_E(X) \neq \mathrm{ext}_E(Y) \tag{6.11}$$

whenever X and Y are distinct elements of P.

A class M is *extensional* if the relation $\in$ on M is extensional, i.e., if for any distinct X and $Y \in M$, $X \cap M \neq Y \cap M$.

The following theorem shows that the transitive collapse of an extensional well-founded relation is one-to-one, and that every extensional class is $\in$-isomorphic to a transitive class.

Theorem 6.15 (Mostowski's Collapsing Theorem).

(i) *If E is a well-founded and extensional relation on a class P, then there is a transitive class M and an isomorphism π between (P, E) and $(M, \in)$. The transitive class M and the isomorphism π are unique.*

(ii) *In particular, every extensional class P is isomorphic to a transitive class M. The transitive class M and the isomorphism π are unique.*

(iii) *In case* (ii), *if $T \subset P$ is transitive, then $\pi x = x$ for every $x \in T$.*

Proof. Since (ii) is a special case of (i) ($E = \in$ in case (ii)), we shall prove the existence of an isomorphism in the general case.

Since E is a well-founded relation, we can define π by well-founded induction (Theorem 6.11), i.e., $\pi(x)$ can be defined in terms of the $\pi(z)$'s, where $z\ E\ x$. We let, for each $x \in P$

$$\pi(x) = \{\pi(z) : z\ E\ x\}. \tag{6.12}$$

In particular, in the case $E = \in$, (6.12) becomes

$$\pi(x) = \{\pi(z) : z \in x \cap P\}. \tag{6.13}$$

The function π maps P onto a class $M = \pi(P)$, and it is immediate from the definition (6.12) that M is transitive.

We use the extensionality of E to show that π is one-to-one. Let $z \in M$ be of least rank such that $z = \pi(x) = \pi(y)$ for some $x \neq y$. Then $\mathrm{ext}_E(x) \neq \mathrm{ext}_E(y)$ and there is, e.g., some $u \in \mathrm{ext}_E(x)$ such that $u \notin \mathrm{ext}_E(y)$. Let $t = \pi(u)$. Since $t \in z = \pi(y)$, there is $v \in \mathrm{ext}_E(y)$ such that $t = \pi(v)$. Thus we have $t = \pi(u) = \pi(v)$, $u \neq v$, and t is of lesser rank than z (since $t \in z$). A contradiction.

Now it follows easily that

$$x\ E\ y \leftrightarrow \pi(x) \in \pi(y). \tag{6.14}$$

If $x\ E\ y$, then $\pi(x) \in \pi(y)$ by definition (6.12). On the other hand, if $\pi(x) \in \pi(y)$, then by (6.12), $\pi(x) = \pi(z)$ for some $z\ E\ y$. Since π is one-to-one, we have $x = z$ and so $x\ E\ y$.

The uniqueness of the isomorphism π, and the transitive class $M = \pi(P)$, follows from Theorem 6.7. If π_1 and π_2 are two isomorphisms of P and M_1, M_2, respectively, then $\pi_2\pi_1^{-1}$ is an isomorphism between M_1 and M_2, and therefore the identity mapping. Hence $\pi_1 = \pi_2$.

It remains to prove (iii). If $T \subset P$ is transitive, then we first observe that $x \subset P$ for every $x \in T$ and so $x \cap P = x$, and we have

$$\pi(x) = \{\pi(z) : z \in x\}$$

for all $x \in T$. It follows easily by $\in$-induction that $\pi(x) = x$ for all $x \in T$. $\square$

The Bernays-Gödel Axiomatic Set Theory

There is an alternative axiomatization of set theory. We consider two types of objects: *sets* (for which we use lower case letters) and *classes* (denoted by capital letters).

A. 1. Extensionality: $\forall u\,(u \in X \leftrightarrow y \in Y) \to X = Y$.
   2. Every set is a class.
   3. If $X \in Y$, then X is a set.
   4. Pairing: For any sets x and y there is a set $\{x, y\}$.

B. Comprehension:

$$\forall X_1 \ldots \forall X_n \exists Y\ Y = \{x : \varphi(x, X_1, \ldots, X_n)\}$$

where φ is a formula in which only set variables are quantified.

C. 1. Infinity: There is an infinite set.
   2. Union: For every set x the set $\bigcup x$ exists.
   3. Power Set: For every set x the power set $P(x)$ of x exists.
   4. Replacement: If a class F is a function and x is a set, then $\{F(z) : z \in x\}$ is a set.

D. Regularity.

E. Choice: There is a function F such that $F(x) \in x$ for every nonempty set x.

Let BG denote the axiomatic theory A–D and let BGC denote BG + Choice.

If a set-theoretical statement is provable in ZF (ZFC), then it is provable in BG (BGC).

On the other hand, a theorem of Shoenfield (using proof-theoretic methods) states that if a sentence involving only set variables is provable in BG, then it is provable in ZF. This result can be extended to BGC/ZFC using the method of forcing.

Exercises

6.1. $\operatorname{rank}(x) = \sup\{\operatorname{rank}(z) + 1 : z \in x\}$.

6.2. $|V_\omega| = \aleph_0$, $|V_{\omega+\alpha}| = \beth_\alpha$.

6.3. If κ is inaccessible, then $|V_\kappa| = \kappa$.

6.4. If x and y have rank $\leq \alpha$ then $\{x, y\}$, $\langle x, y\rangle$, $x \cup y$, $\bigcup x$, $P(x)$, and x^y have rank $< \alpha + \omega$

6.5. The sets $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ are in $V_{\omega+\omega}$.

6.6. Let B be the class of all x that are hereditarily in the class A. Show that

(i) $x \in B$ if and only if $\operatorname{TC}(x) \subset A$,
(ii) B is the largest transitive class $B \subset A$.

Historical Notes

The Axiom of Regularity was introduced by von Neumann in [1925], although a similar principle had been considered previously by Skolem (see [1970], pp. 137–152). The concept of rank appears first in Mirimanov [1917]. The transitive collapse is defined in Mostowski [1949]. Induction on well-founded relations (Theorems 6.10, 6.11) was formulated by Montague in [1955].

The axiomatic system BG was introduced by Bernays in [1937]. Shoenfield's result was published in [1954].

For more references on the history of axioms of set theory consult Fraenkel *et al.* [1973].

7. Filters, Ultrafilters and Boolean Algebras

Filters and Ultrafilters

Filters and ideals play an important role in several mathematical disciplines (algebra, topology, logic, measure theory). In this chapter we introduce the notion of filter (and ideal) on a given set. The notion of ideal extrapolates the notion of small sets: Given an ideal I on S, a set $X \subset S$ is considered small if it belongs to I.

Definition 7.1. A *filter* on a nonempty set S is a collection F of subsets of S such that

(7.1) (i) $S \in F$ and $\emptyset \notin F$,
(ii) if $X \in F$ and $Y \in F$, then $X \cap Y \in F$,
(iii) If $X, Y \subset S$, $X \in F$, and $X \subset Y$, then $Y \in F$.

An *ideal* on a nonempty set S is a collection I of subsets of S such that:

(7.2) (i) $\emptyset \in I$ and $S \notin I$,
(ii) if $X \in I$ and $Y \in I$, then $X \cup Y \in I$,
(iii) if $X, Y \subset S$, $X \in I$, and $Y \subset X$, then $Y \in I$.

If F is a filter on S, then the set $I = \{S - X : X \in F\}$ is an ideal on S; and conversely, if I is an ideal, then $F = \{S - X : X \in I\}$ is a filter. If this is the case we say that F and I are *dual* to each other.

Examples. 1. A *trivial* filter: $F = \{S\}$.

2. A *principal* filter. Let X_0 be a nonempty subset of S. The filter $F = \{X \subset S : X \supset X_0\}$ is a *principal* filter. Note that every filter on a finite set is principal.

The dual notions are a *trivial* ideal and a *principal* ideal.

3. The *Fréchet filter*. Let S be an infinite set, and let I be the ideal of all finite subsets of S. The dual filter $F = \{X \subset S : S - X \text{ is finite}\}$ is called the Fréchet filter on S. Note that the Fréchet filter is not principal.

4. Let A be an infinite set and let $S = [A]^{<\omega}$ be the set of all finite subsets of A. For each $P \in S$, let $\hat{P} = \{Q \in S : P \subset Q\}$. Let F be the set of all $X \subset S$ such that $X \supset \hat{P}$ for some $P \in S$. Then F is a nonprincipal filter on S.

5. A set $A \subset \boldsymbol{N}$ has *density* 0 if $\lim_{n\to\infty} |A \cap n|/n = 0$. The set of all A of density 0 is an ideal on $\boldsymbol{N}$.

A family G of sets has the *finite intersection property* if every finite $H = \{X_1, \ldots, X_n\} \subset G$ has a nonempty intersection $X_1 \cap \ldots \cap X_n \neq \emptyset$. Every filter has the finite intersection property.

Lemma 7.2.

(i) *If $\mathcal{F}$ is a nonempty family of filters on S, then $\bigcap \mathcal{F}$ is a filter on S.*
(ii) *If $\mathcal{C}$ is a $\subset$-chain of filters on S, then $\bigcup \mathcal{C}$ is a filter on S.*
(iii) *If $G \subset P(S)$ has the finite intersection property, then there is a filter F on S such that $G \subset F$.*

Proof. (i) and (ii) are easy to verify.

(iii) Let F be the set of all $X \subset S$ such that there is a finite $H = \{X_1, \ldots, X_n\} \subset G$ with $X_1 \cap \ldots \cap X_n \subset X$. Then F is a filter and $F \supset G$. □

Since every filter $F \supset G$ must contain all finite intersections of sets in G, it follows that the filter F constructed in the proof of Lemma 7.2(iii) is the smallest filter on S that extends G:

$$F = \bigcap \{D : D \text{ is a filter on } S \text{ and } G \subset D\}.$$

We say that the filter F is *generated* by G.

Definition 7.3. A filter U on a set S is an *ultrafilter* if

(7.3) for every $X \subset S$, either $X \in U$ or $S - X \in U$.

The dual notion is a *prime ideal*: For every $X \subset S$, either $X \in I$ or $S - X \in I$. Note that $I = P(S) - U$.

A filter F on S is *maximal* if there is no filter F' on S such that $F \subset F'$ and $F \neq F'$.

Lemma 7.4. *A filter F on S is an ultrafilter if and only if it is maximal.*

Proof. (a) An ultrafilter U is clearly a maximal filter: Assume that $U \subset F$ and $X \in F - U$. Then $S - X \in U$, and so both $S - X \in F$ and $X \in F$, a contradiction.

(b) Let F be a filter that is not an ultrafilter. We will show that F is not maximal. Let $Y \subset S$ be such that neither Y nor $S - Y$ is in F. Consider the family $G = F \cup \{Y\}$; we claim that G has the finite intersection property. If $X \in F$, then $X \cap Y \neq \emptyset$, for otherwise we would have $S - Y \supset X$ and $S - Y \in F$. Thus, if $X_1, \ldots, X_n \in F$, we have $X_1 \cap \ldots \cap X_n \in F$ and so $Y \cap X_1 \cap \ldots \cap X_n \neq \emptyset$. Hence G has the finite intersection property, and by Lemma 7.2(iii) there is a filter $F' \supset G$. Since $Y \in F' - F$, F is not maximal. □

Theorem 7.5 (Tarski). *Every filter can be extended to an ultrafilter.*

Proof. Let F_0 be a filter on S. Let P be the set of all filters F on S such that $F \supset F_0$ and consider the partially ordered set $(P, \subset)$. If $\mathcal{C}$ is a chain in P, then by Lemma 7.2(ii), $\bigcup \mathcal{C}$ is a filter and hence an upper bound of $\mathcal{C}$ in P. By Zorn's Lemma there exists a maximal element U in P. This U is an ultrafilter by Lemma 7.4. □

For every $a \in S$, the principal filter $\{X \subset S : a \in X\}$ is an ultrafilter. If S is finite, then every ultrafilter on S is principal.

If S is infinite, then there is a nonprincipal ultrafilter on S: If U extends the Fréchet filter, then U is nonprincipal.

The proof of Theorem 7.5 uses the Axiom of Choice. We shall see later that the existence of nonprincipal ultrafilters cannot be proved without AC.

If S is an infinite set of cardinality κ, then because every ultrafilter on S is a subset of $P(S)$, there are at most 2^{2^κ} ultrafilters on S. The next theorem shows that the number of ultrafilters on κ is exactly 2^{2^κ}. To get a slightly stronger result, let us call an ultrafilter D on κ *uniform* if $|X| = \kappa$ for all $X \in D$.

Theorem 7.6 (Pospíšil). *For every infinite cardinal κ, there exist 2^{2^κ} uniform ultrafilters on κ.*

We prove first the following lemma. Let us call a family $\mathcal{A}$ of subsets of κ *independent* if for any distinct sets $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ in $\mathcal{A}$, the intersection

$$X_1 \cap \ldots \cap X_n \cap (\kappa - Y_1) \cap \ldots \cap (\kappa - Y_m) \tag{7.4}$$

has cardinality κ.

Lemma 7.7. *There exists an independent family of subsets of κ of cardinality 2^κ.*

Proof. Let us consider the set P of all pairs $(F, \mathcal{F})$ where F is a finite subset of κ and $\mathcal{F}$ is a finite set of finite subsets of κ. Since $|P| = \kappa$, it suffices to find an independent family $\mathcal{A}$ of subsets of P, of size 2^κ.

For each $u \subset \kappa$, let

$$X_u = \{(F, \mathcal{F}) \in P : F \cap u \in \mathcal{F}\}$$

and let $\mathcal{A} = \{X_u : u \subset \kappa\}$. If u and v are distinct subsets of κ, then $X_u \neq X_v$: For example, if $\alpha \in u$ but $\alpha \notin v$, then let $F = \{\alpha\}$, $\mathcal{F} = \{F\}$, and $(F, \mathcal{F}) \in X_u$ while $(F, \mathcal{F}) \notin X_v$. Hence $|\mathcal{A}| = 2^\kappa$.

To show that $\mathcal{A}$ is independent, let $u_1, \ldots, u_n, v_1, \ldots, v_m$ be distinct subsets of κ. For each $i \leq n$ and each $j \leq m$, let $\alpha_{i,j}$ be some element of κ such that either $\alpha_{i,j} \in u_i - v_j$ or $\alpha_{i,j} \in v_j - u_i$. Now let F be any finite

subset of κ such that $F \supset \{\alpha_{i,j} : i \leq n,\ j \leq m\}$ (note that there are κ many such finite sets). Clearly, we have $F \cap u_i \neq F \cap v_j$ for any $i \leq n$ and $j \leq m$. Thus if we let $\mathcal{F} = \{F \cap u_i : i \leq n\}$, we have $(F, \mathcal{F}) \in X_{u_i}$ for all $i \leq n$ and $(F, \mathcal{F}) \notin X_{v_j}$ for all $j \leq m$. Consequently, the intersection

$$X_{u_1} \cap \ldots \cap X_{u_n} \cap (P - X_{v_1}) \cap \ldots \cap (P - X_{v_m})$$

has cardinality κ. □

Proof of Theorem 7.6. Let $\mathcal{A}$ be an independent family of subsets of κ. For every function $f : \mathcal{A} \to \{0, 1\}$, consider this family of subsets of κ:

$$G_f = \{X : |\kappa - X| < \kappa\} \cup \{X : f(X) = 1\} \cup \{\kappa - X : f(X) = 0\}. \tag{7.5}$$

By (7.4), the family G_f has the finite intersection property, and so there exists an ultrafilter D_f such that $D_f \supset G_f$. If follows from (7.5) that D_f is uniform. If $f \neq g$, then for some $X \in \mathcal{A}$, $f(X) \neq g(X)$; e.g., $f(X) = 1$ and $g(X) = 0$ and then $X \in D_f$, while $\kappa - X \in D_g$. Thus we obtain 2^{2^κ} distinct uniform ultrafilters on κ. □

Ultrafilters on ω

We present two properties of ultrafilters on ω that are frequently used in set-theoretic topology.

Let D be a nonprincipal ultrafilter on ω. D is called a *p-point* if for every partition $\{A_n : n \in \omega\}$ of ω into $\aleph_0$ pieces such that $A_n \notin D$ for all n, there exists $X \in D$ such that $X \cap A_n$ is finite, for all $n \in \omega$.

First we notice that it is easy to find a nonprincipal ultrafilter that is not a p-point: Let $\{A_n : n \in \omega\}$ be any partition of ω into $\aleph_0$ infinite pieces, and let F be the following filter on ω:

(7.6) $X \in F$ if and only if except for finitely many n, $X \cap A_n$ contains all but finitely many elements of A_n.

If D is any ultrafilter extending F, then D is not a p-point.

Theorem 7.8 below shows that existence of p-points follows from the Continuum Hypothesis. By a result of Shelah there exists a model of ZFC in which there are no p-points.

A nonprincipal ultrafilter D on ω is a *Ramsey* ultrafilter if for every partition $\{A_n : n \in \omega\}$ of ω into $\aleph_0$ pieces such that $A_n \notin D$ for all n, there exists $X \in D$ such that $X \cap A_n$ has one element for all $n \in \omega$.

Every Ramsey ultrafilter is a p-point.

Theorem 7.8. *If* $2^{\aleph_0} = \aleph_1$, *then a Ramsey ultrafilter exists.*

Proof. Let $\mathcal{A}_\alpha$, $\alpha < \omega_1$, enumerate all partitions of ω and let us construct an ω_1-sequence of infinite subsets of ω as follows: Given X_α, let $X_{\alpha+1} \subset X_\alpha$ be such that either $X_{\alpha+1} \subset A$ for some $A \in \mathcal{A}_\alpha$, or that $|X_{\alpha+1} \cap A| \leq 1$ for all $A \in \mathcal{A}_\alpha$. If α is a limit ordinal, let X_α be such that $X_\alpha - X_\beta$ is finite for all $\beta < \alpha$. (Such a set X_α exists because α is countable.) Then $D = \{X : X \supset X_\alpha \text{ for some } \alpha < \omega_1\}$ is a Ramsey ultrafilter. □

κ-Complete Filters and Ideals

A filter F on S is *countably complete* (*σ-complete*) if whenever $\{X_n : n \in \boldsymbol{N}\}$ is a countable family of subsets of S and $X_n \in F$ for every n, then

$$\bigcap_{n=0}^{\infty} X_n \in F. \tag{7.7}$$

A *countably complete* ideal (a *σ-ideal*) is such that if $X_n \in I$ for every n, then

$$\bigcup_{n=0}^{\infty} X_n \in I.$$

More generally, if κ is a regular uncountable cardinal, and F is a filter on S, then F is called *κ-complete* if F is closed under intersection of less than κ sets, i.e., if whenever $\{X_\alpha : \alpha < \gamma\}$ is a family of subsets of S, $\gamma < \kappa$, and $X_\alpha \in F$ for every $\alpha < \gamma$, then

$$\bigcap_{\alpha<\gamma} X_\alpha \in F. \tag{7.8}$$

The dual notion is a κ-complete ideal.

An example of a κ-complete ideal is $I = \{X \subset S : |X| < \kappa\}$, on any set S such that $|S| \geq \kappa$.

A σ-complete filter is the same as an $\aleph_1$-complete filter.

There is no nonprincipal σ-complete filter on a countable set S. If S is uncountable, then

$$\{X \subset S : |X| \leq \aleph_0\}$$

is a σ-ideal on S.

Similarly, if $\kappa > \omega$ is regular and $|S| \geq \kappa$, then

$$\{X \subset S : |X| < \kappa\}$$

is the smallest κ-complete ideal on S containing all singletons $\{a\}$.

The question whether a nonprincipal *ultrafilter* on a set can be σ-complete gives rise to deep investigations of the foundations of set theory. In particular, if such ultrafilters exist, then there exist large cardinals (inaccessible, etc.).

Boolean Algebras

An algebra of sets (see Definition 4.9) is a collection of subsets of a given nonempty set that is closed under unions, intersections and complements. These properties of algebras of sets are abstracted in the notion of Boolean algebra:

Definition 7.9. A *Boolean algebra* is a set B with at least two elements, 0 and 1, endowed with binary operations $+$ and $\cdot$ and a unary operation $-$.

The Boolean operations satisfy the following axioms:

$$
\begin{aligned}
(7.9)\quad & u+v=v+u, && u\cdot v=v\cdot u, && \text{(commutativity)}\\
& u+(v+w)=(u+v)+w, && u\cdot(v\cdot w)=(u\cdot v)\cdot w, && \text{(associativity)}\\
& u\cdot(v+w)=u\cdot v+u\cdot w, && u+(v\cdot w)=(u+v)\cdot(u+w), && \text{(distributivity)}\\
& u\cdot(u+v)=u, && u+(u\cdot v)=u, && \text{(absorption)}\\
& u+(-u)=1, && u\cdot(-u)=0. && \text{(complementation)}
\end{aligned}
$$

An algebra of sets $\mathcal{S}$, with $\bigcup\mathcal{S}=S$, is a Boolean algebra, with Boolean operations $X\cup Y$, $X\cap Y$ and $S-X$, and with $\emptyset$ and S being 0 and 1. If follows from Stone's Representation Theorem below that every Boolean algebra is isomorphic to an algebra of sets.

From the axioms (7.9) one can derive additional Boolean algebraic rules that correspond to rules for the set operations $\cup$, $\cap$ and $-$. Among others, we have

$$u+u=u,\quad u\cdot u=u,\quad u+0=u,\quad u\cdot 0=0,\quad u+1=1,\quad u\cdot 1=u$$

and the De Morgan laws

$$-(u+v)=-u\cdot -v,\qquad -(u\cdot v)=-u+-v.$$

Two elements $u,v\in B$ are *disjoint* if $u\cdot v=0$. Let us define

$$u-v=u\cdot(-v),$$

and

$$(7.10)\qquad u\le v \quad\text{if and only if}\quad u-v=0.$$

It is easy to see that $\le$ is a partial ordering of B and that

$$u\le v \quad\text{if and only if}\quad u+v=v \quad\text{if and only if}\quad u\cdot v=u.$$

Moreover, 1 is the greatest element of B and 0 is the least element. Also, for any $u,v\in B$, $u+v$ is the least upper bound of $\{u,v\}$ and $u\cdot v$ is the greatest

lower bound of $\{u, v\}$. Since $-u$ is the unique v such that $u + v = 1$ and $u \cdot v = 0$, it follows that all Boolean-algebraic operations can be defined in terms of the partial ordering of B.

We shall now give an example showing the relation between Boolean algebras and logic:

Let $\mathcal{L}$ be a first order language and let S be the set of all sentences of $\mathcal{L}$. We consider the equivalence relation $\vdash \varphi \leftrightarrow \psi$ on S. The set B of all equivalence classes $[\varphi]$ is a Boolean algebra under the following operations:

$$\begin{aligned} [\varphi] + [\psi] &= [\varphi \vee \psi], & 0 &= [\varphi \wedge \neg\varphi], \\ [\varphi] \cdot [\psi] &= [\varphi \wedge \psi], & 1 &= [\varphi \vee \neg\varphi]. \\ -[\varphi] &= [\neg\varphi], \end{aligned}$$

This algebra is called the *Lindenbaum algebra*.

A subset A of a Boolean algebra B is a *subalgebra* if it contains 0 and 1 and is closed under the Boolean operations:

(7.11) (i) $0 \in A$, $1 \in A$;
(ii) if $u, v \in A$, then $u + v \in A$, $u \cdot v \in A$, $-u \in A$.

If $X \subset B$, then there is a smallest subalgebra A of B that contains X; A can be described either as $\bigcap\{A : X \subset A \subset B$ and A is a subalgebra$\}$, or as the set of all Boolean combinations in B of elements of X. The subalgebra A is *generated* by X. If X is infinite, then $|A| = |X|$. See Exercises 7.18–7.20.

If B is a Boolean algebra, let $B^+ = B - \{0\}$ denote the set of all nonzero elements of B. If $a \in B^+$, the set $B\restriction a = \{u \in B : u \leq a\}$ with the partial order inherited from B, is a Boolean algebra; its $+$ and $\cdot$ are the same as in B, and the complement of u is $a - u$. An element $a \in B$ is called an *atom* if it is a minimal element of B^+; equivalently, if there is no x such that $0 < x < a$. A Boolean algebra is *atomic* if for every $u \in B^+$ there is an atom $a \leq u$; B is *atomless* if it has no atoms.

Let B and C be two Boolean algebras. A mapping $h : B \to C$ is a *homomorphism* if it preserves the operations:

(7.12) (i) $h(0) = 0$, $h(1) = 1$,
(ii) $h(u + v) = h(u) + h(v)$, $h(u \cdot v) = h(u) \cdot h(v)$, $h(-u) = -h(u)$.

Note that the range of a homomorphism is a subalgebra of C and that $h(u) \leq h(v)$ whenever $u \leq v$. A one-to-one homomorphism of B onto C is called an *isomorphism*. An *embedding* of B in C is an isomorphism of B onto a subalgebra of C. Note that if $h : B \to C$ is a one-to-one mapping such that $u \leq v$ if and only if $h(u) \leq h(v)$, then h is an isomorphism. An isomorphism of a Boolean algebra onto itself is called an *automorphism*.

Ideals and Filters on Boolean Algebras

The definition of filter (and ideal) given earlier in this chapter generalizes to arbitrary Boolean algebras. Let B be a Boolean algebra. An *ideal* on B is a subset I of B such that:

(7.13) (i) $0 \in I$, $1 \notin I$;
(ii) if $u \in I$ and $v \in I$, then $u + v \in I$;
(iii) if $u, v \in B$, $u \in I$ and $v \leq u$, then $v \in I$.

A *filter* on B is a subset F of B such that:

(7.14) (i) $1 \in F$, $0 \notin F$;
(ii) if $u \in F$ and $v \in F$, then $u \cdot v \in F$;
(iii) if $u, v \in B$, $u \in F$ and $u \leq v$, then $v \in F$.

The *trivial* ideal is the ideal $\{0\}$; an ideal is *principal* if $I = \{u \in B : u \leq u_0\}$ for some $u_0 \neq 1$. Similarly for filters.

A subset G of $B - \{0\}$ has the *finite intersection property* if for every finite $\{u_1, \ldots, u_n\} \subset G$, $u_1 \cdot \ldots \cdot u_n \neq 0$. Every $G \subset B$ that has the finite intersection property generates a filter on B; this and the other two clauses of Lemma 7.2 hold also for Boolean algebras.

There is a relation between ideals and homomorphisms. If $h : B \to C$ is a homomorphism, then

$$I = \{u \in B : h(u) = 0\} \tag{7.15}$$

is an ideal on B (the *kernel* of the homomorphism). On the other hand, let I be an ideal on B. Let us consider the following equivalence relation on B:

$$u \sim v \quad \text{if and only if} \quad u \triangle v \in I \tag{7.16}$$

where

$$u \triangle v = (u - v) + (v - u).$$

Let C be the set of all equivalence classes, $C = B/\!\sim$, and endow C with the following operations:

$$\begin{aligned} [u] + [v] &= [u + v], & 0 &= [0], \\ [u] \cdot [v] &= [u \cdot v], & 1 &= [1]. \\ -[u] &= [-u], \end{aligned} \tag{7.17}$$

Then C is a Boolean algebra, the *quotient* of B mod I, and is a homomorphic image of B under the homomorphism

$$h(u) = [u]. \tag{7.18}$$

The quotient algebra is denoted B/I.

An ideal I on B is a *prime ideal* if

(7.19) for every $u \in B$, either $u \in I$ or $-u \in I$.

The dual of a prime ideal is an *ultrafilter*.

Lemma 7.4 holds in general: An ideal is a prime ideal (and a filter is an ultrafilter) if and only if it is maximal. Also, an ideal I on B is prime if and only if the quotient of B mod I is the trivial algebra $\{0, 1\}$.

Tarski's Theorem 7.5 easily generalizes to Boolean algebras:

Theorem 7.10 (The Prime Ideal Theorem). *Every ideal on B can be extended to a prime ideal.* □

The proof of the Prime Ideal Theorem uses the Axiom of Choice. It is known that the theorem cannot be proved without using the Axiom of Choice. However, it is also known that the Prime Ideal Theorem is weaker than the Axiom of Choice.

Theorem 7.11 (Stone's Representation Theorem). *Every Boolean algebra is isomorphic to an algebra of sets.*

Proof. Let B be a Boolean algebra. We let

(7.20) $$S = \{p : p \text{ is an ultrafilter on } B\}.$$

For every $u \in B$, let X_u be the set of all $p \in S$ such that $u \in p$. Let

(7.21) $$\mathcal{S} = \{X_u : u \in B\}.$$

Let us consider the mapping $\pi(u) = X_u$ from B onto $\mathcal{S}$. Clearly, $\pi(1) = S$ and $\pi(0) = \emptyset$. It follows from the definition of ultrafilter that

$$\pi(u \cdot v) = \pi(u) \cap \pi(v), \qquad \pi(u + v) = \pi(u) \cup \pi(v), \qquad \pi(-u) = S - \pi(u).$$

Thus π is a homomorphism of B onto the algebra of sets $\mathcal{S}$. It remains to show that π is one-to-one.

If $u \neq v$, then using the Prime Ideal Theorem, one can find an ultrafilter p on B containing one of these two elements but not the other. Thus π is an isomorphism. □

Complete Boolean Algebras

The partial ordering $\leq$ of a Boolean algebra can be used to define infinitary operations on B, generalizing $+$ and $\cdot$. Let us recall that $u + v = \sup\{u, v\}$

and $u \cdot v = \inf\{u, v\}$ in the partial ordering of B. Thus for any nonempty $X \subset B$, we define

$$\sum\{u : u \in X\} = \sup X \quad \text{and} \quad \prod\{u : u \in X\} = \inf X, \tag{7.22}$$

provided that the least upper bound (the greatest lower bound) exists. We also define $\sum \emptyset = 0$ and $\prod \emptyset = 1$.

If the infinitary sum and product is defined for all $X \subset B$, the Boolean algebra is called *complete*. Similarly, we call B *κ-complete* (where κ is a regular uncountable cardinal) if sums and products exist for all X of cardinality $< \kappa$. An $\aleph_1$-complete Boolean algebra is called *σ-complete* or *countably complete*.

An algebra of sets $\mathcal{S}$ is *κ-complete* if it is closed under unions and intersections of $< \kappa$ sets. A κ-complete algebra of sets is a κ-complete Boolean algebra and for every $X \subset \mathcal{S}$ such that $|X| < \kappa$, $\sum X = \bigcup X$.

An ideal I on a κ-complete Boolean algebra is *κ-complete* if

$$\sum\{u : u \in X\} \in I$$

whenever $X \subset I$ and $|X| < \kappa$. A *κ-complete filter* is the dual notion.

If I is a κ-complete ideal on a κ-complete Boolean algebra B, then B/I is κ-complete, and

$$\sum\{[u] : u \in X\} = [\sum\{u : u \in X\}]$$

for every $X \subset B$, $|X| < \kappa$. Similarly for products.

An $\aleph_1$-complete ideal is called a *σ-ideal*.

There are two important examples of σ-ideals on the Boolean algebra of all Borel sets of reals: the σ-ideal of Borel sets of Lebesgue measure 0, and the σ-ideal of meager Borel sets. (Exercises 7.14 and 7.15.)

Let A be a subalgebra of a Boolean algebra B. A is a *dense* subalgebra of B if for every $u \in B^+$ there is a $v \in A^+$ such that $v \leq u$.

A *completion* of a Boolean algebra B is a complete Boolean algebra C such that B is a dense subalgebra of C.

Lemma 7.12. *The completion of a Boolean algebra B is unique up to isomorphism.*

Proof. Let C and D be completions of B. We define an isomorphism $\pi : C \to D$ by

$$\pi(c) = \sum\nolimits^{D}\{u \in B : u \leq c\}. \tag{7.23}$$

To verify that π is an isomorphism, one uses the fact that B is a dense subalgebra of both C and D. For example, to show that $\pi(c) \neq 0$ whenever $c \neq 0$: There is $u \in B$ such that $0 < u \leq c$, and we have $0 < u \leq \pi(c)$. □

Theorem 7.13. *Every Boolean algebra has a completion.*

Proof. We use a construction similar to the method of Dedekind cuts. Let A be a Boolean algebra. Let us call a set $U \subset A^+$ a *cut* if

$$p \leq q \text{ and } q \in U \text{ implies } p \in U. \tag{7.24}$$

For every $p \in A^+$, let U_p denote the cut $\{x : x \leq p\}$.

A cut U is *regular* if

$$\text{whenever } p \notin U, \text{ then there exists } q \leq p \text{ such that } U_q \cap U = \emptyset. \tag{7.25}$$

Note that every U_p is regular, and that every cut includes some U_p.

We let B be the set of all regular cuts in A^+. We claim that B, under the partial ordering by inclusion, is a complete Boolean algebra. Note that the intersection of any collection of regular cuts is a regular cut, and hence each cut U is included in a least regular cut $\overline{U}$. In fact,

$$\overline{U} = \{p : (\forall q \leq p)\, U \cap U_q \neq \emptyset\}.$$

Thus for $u, v \in B$ we have

$$u \cdot v = u \cap v, \qquad u + v = \overline{u \cup v}.$$

The complement of $u \in B$ is the regular cut

$$-u = \{p : U_p \cap u = \emptyset\}.$$

And, of course, $\emptyset$ and A^+ are the zero and the unit of B. It is not difficult to verify that B is a complete Boolean algebra, and we leave the verification to the reader.

Furthermore, for all $p, q \in A^+$ we have $U_p + U_q = U_{p+q}$, $U_p \cdot U_q = U_{p \cdot q}$ and $-U_p = U_{-p}$. Thus A embeds in B as a dense subalgebra. □

Complete and Regular Subalgebras

Let B be a complete Boolean algebra. A subalgebra A of B is a *complete subalgebra* if $\sum X \in A$ and $\prod X \in A$ for all $X \subset A$. (Caution: A subalgebra A of B that is itself complete is not necessarily a complete subalgebra of B.) Similarly, a *complete homomorphism* is a homomorphism h of B into C such that for all $X \subset B$,

$$h(\textstyle\sum X) = \sum h(X), \qquad h(\prod X) = \prod h(X). \tag{7.26}$$

A *complete embedding* is an embedding that satisfies (7.26). Note that every isomorphism is complete.

Since the intersection of any collection of complete subalgebras of B is a complete subalgebra, every $X \subset B$ is included in a smallest complete subalgebra of B. This algebra is called the complete subalgebra of B *completely generated* by X.

Definition 7.14. A set $W \subset B^+$ is an *antichain* in a Boolean algebra B if $u \cdot v = 0$ for all distinct $u, v \in W$.

If W is an antichain and if $\sum W = u$ then we say that W is a *partition* of u. A partition of 1 is just a *partition*, or a *maximal antichain*.

If B is a Boolean algebra and A is a subalgebra of B then an antichain in A that is maximal in A need not be maximal in B. If every maximal antichain in A is also maximal in B, then A is called a *regular subalgebra* of B.

If A is a complete subalgebra of a complete Boolean algebra B then A is a regular subalgebra of B. Also, if A is a dense subalgebra of B then A is a regular subalgebra. See also Exercise 7.31.

Saturation

Let κ be an infinite cardinal. A Boolean algebra B is *κ-saturated* if there is no partition W of B such that $|W| = \kappa$, and

(7.27) $\qquad \mathrm{sat}(B) = \text{the least } \kappa \text{ such that } B \text{ is } \kappa\text{-saturated.}$

B is also said to satisfy the *κ-chain condition*; this is because if B is complete, B is κ-saturated if and only if there exists no descending κ-sequence $u_0 > u_1 > \ldots > u_\alpha > \ldots$, $\alpha < \kappa$, of elements of B. The $\aleph_1$-chain condition is called the *countable chain condition* (c.c.c.).

Theorem 7.15. *If B is an infinite complete Boolean algebra, then* $\mathrm{sat}(B)$ *is a regular uncountable cardinal.*

Proof. Let $\kappa = \mathrm{sat}(B)$. It is clear that κ is uncountable. Let us assume that κ is singular; we shall obtain a contradiction by constructing a partition of size κ.

For $u \in B$, $u \neq 0$, let $\mathrm{sat}(u)$ denote $\mathrm{sat}(B_u)$. Let us call $u \in B$ *stable* if $\mathrm{sat}(v) = \mathrm{sat}(u)$ for every nonzero $v \leq u$. The set S of stable elements is dense in B; otherwise, there would be a descending sequence $u_0 > u_1 > u_2 > \ldots$ with decreasing cardinals $\mathrm{sat}(u_0) > \mathrm{sat}(u_1) > \ldots$. Let T be a maximal set of pairwise disjoint elements of S. Thus T is a partition of B, and $|T| < \kappa$.

First we show that $\sup\{\mathrm{sat}(u) : u \in T\} = \kappa$. For every regular $\lambda < \kappa$ such that $\lambda > |T|$, consider a partition W of B of size λ. Then at least one $u \in T$ is partitioned by W into λ pieces.

Thus we consider two cases:

Case I. There is $u \in T$ such that $\mathrm{sat}(u) = \kappa$. Since $\mathrm{cf}\,\kappa < \kappa$, there is a partition W of u of size $\mathrm{cf}\,\kappa$: $W = \{u_\alpha : \alpha < \mathrm{cf}\,\kappa\}$. Let κ_α, $\alpha < \mathrm{cf}\,\kappa$, be an increasing sequence with limit κ. For each α, $\mathrm{sat}(u_\alpha) = \mathrm{sat}(u) = \kappa$ and so let W_α be a partition of u_α of size κ_α. Then $\bigcup_{\alpha < \mathrm{cf}\,\kappa} W_\alpha$ is a partition of u of size κ.

Case II. For all $u \in T$, $\mathrm{sat}(u) < \kappa$, but $\sup\{\mathrm{sat}(u) : u \in T\} = \kappa$. Again, let $\kappa_\alpha \to \kappa$, $\alpha < \mathrm{cf}\,\kappa$. For each $\alpha < \mathrm{cf}\,\kappa$ (by induction), we find $u_\alpha \in T$, distinct from all u_β, $\beta < \alpha$, which admits a partition W_α of size κ_α. Then $\bigcup_{\alpha<\mathrm{cf}\,\kappa} W_\alpha$ is an antichain in B of size κ. □

Distributivity of Complete Boolean Algebras

The following distributive law holds for every complete Boolean algebra:

$$\sum_{i\in I} u_{0,i} \cdot \sum_{u\in J} u_{1,j} = \sum_{(i,j)\in I\times J} u_{0,i}\cdot u_{1,j}.$$

To formulate a general distributive law, let κ be a cardinal, and let us call B *κ-distributive* if

$$\prod_{\alpha<\kappa}\sum_{i\in I_\alpha} u_{\alpha,i} = \sum_{f\in\prod_{\alpha<\kappa} I_\alpha}\prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}. \tag{7.28}$$

(Every complete algebra of sets satisfies (7.28).) We shall see later that distributivity plays an important role in generic models. For now, let us give two equivalent formulations of κ-distributivity.

If W and Z are partitions of B, then W is a *refinement* of Z if for every $w \in W$ there is $z \in Z$ such that $w \le z$. A set $D \subset B$ is *open dense* if it is dense in B and $0 \ne u \le v \in D$ implies $u \in D$.

Lemma 7.16. *The following are equivalent, for any complete Boolean algebra B:*

(i) *B is κ-distributive.*
(ii) *The intersection of κ open dense subsets of B is open dense.*
(iii) *Every collection of κ partitions of B has a common refinement.*

Proof. (i) → (ii). Let D_α, $\alpha < \kappa$, be open dense, $D = \bigcap_{\alpha<\kappa} D_\alpha$. D is certainly open; thus let $u \ne 0$. If we let $\{u_{\alpha,i} : i \in I_\alpha\} = \{u \cdot v : v \in D_\alpha\}$, then $\sum_i u_{\alpha,i} = u$ for every α and the left-hand side of (7.28) is u. For each $f \in \prod_\alpha I_\alpha$, let $u_f = \prod_\alpha u_{\alpha,f(\alpha)}$; clearly, each nonzero u_f is in D. However, $\sum_f u_f = u$, by (7.28), and so some u_f is nonzero.

(ii) → (iii). Let W_α, $\alpha < \kappa$ be partions of B. For each α, let $D_\alpha = \{u : u \le v \text{ for some } v \in W_\alpha\}$; each D_α is open dense. Let $D = \bigcap_{\alpha<\kappa} D_\alpha$, and let W be a maximal set of pairwise disjoint elements of D. Since D is dense, W is a partition of B, and clearly, W is a refinement of each W_α.

(iii) → (i). Let $\{u_{\alpha,i} : \alpha < \kappa,\ i \in I_\alpha\}$ be a collection of elements of B. First we show that the right-hand side of (7.28) is always $\le$ the left-hand side. For each $f \in \prod_{\alpha<\kappa} I_\alpha$, let $u_f = \prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}$; we have $u_f \le u_{\alpha,f(\alpha)}$ and so $u_f \le \sum_{i\in I_\alpha} u_{\alpha,i}$ for each α. Thus, for each α,

$$\sum_f u_f \le \sum_i u_{\alpha,i}$$

and so

$$\sum_f \prod_\alpha u_{\alpha,f(\alpha)} = \sum_f u_f \leq \prod_\alpha \sum_i u_{\alpha,i}.$$

To prove (7.28), assume that (iii) holds, and let $u = \prod_\alpha \sum_i u_{\alpha,i}$; we want to show that $\sum_f \prod_\alpha u_{\alpha,f(\alpha)} = u$. Without loss of generality, we can assume that $u = 1$ (otherwise we argue in the algebra $B\restriction u$). For each α, let us replace $\{u_{\alpha,i} : i \in I_\alpha\}$ by pairwise disjoint $\{v_{\alpha,i} : i \in I_\alpha\} = W_\alpha$ such that $v_{\alpha,i} \leq u_{\alpha,i}$ and $\sum_i v_{\alpha,i} = \sum_i u_{\alpha,i}$ (some of the $v_{\alpha,i}$ may be 0). Clearly $\sum_f \prod_\alpha v_{\alpha,f(\alpha)} \leq \sum_f \prod_\alpha u_{\alpha,f(\alpha)}$. Each W_α is a partition of B and so there is a partition W that is a refinement of each W_α. Now for each $w \in W$ there exists f such that $w \leq \prod_\alpha v_{\alpha,f(\alpha)}$, and so $\sum_f \prod_\alpha v_{\alpha,f(\alpha)} = 1$. □

Exercises

7.1. If F is a filter and $X \in F$, then $P(X) \cap F$ is a filter on X.

7.2. The filter in Example 4 is generated by the sets $\{a\}^\wedge$, $a \in A$.

7.3. If U is an ultrafilter and $X \cup Y \in U$, then either $X \in U$ or $Y \in U$.

7.4. Let U be an ultrafilter on S. Then the set of all $X \subset S \times S$ such that $\{a \in S : \{b \in S : (a,b) \in X\} \in U\} \in U$ is an ultrafilter on $S \times S$.

7.5. Let U be an ultrafilter on S and let $f : S \to T$. Then the set $f_*(U) = \{X \subset T : f_{-1}(X) \in U\}$ is an ultrafilter on T.

7.6. Let U be an ultrafilter on $\boldsymbol{N}$ and let $\langle a_n \rangle_{n=0}^\infty$ be a bounded sequence of real numbers. Prove that there exists a unique *U-limit* $a = \lim_U a_n$ such that for every $\varepsilon > 0$, $\{n : |a_n - a| < \varepsilon\} \in U$.

7.7. A nonprincipal ultrafilter D on ω is a p-point if and only if it satisfies the following: If $A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots$ is a decreasing sequence of elements of D, then there exists $X \in D$ such that for each n, $X - A_n$ is finite.

7.8. If $(P, <)$ is a countable linearly ordered set and if D is a p-point on P, then there exists $X \in D$ such that the order-type of X is either ω or ω^*. (X has order-type ω^* if and only if $X = \{x_n\}_{n=0}^\infty$ and $x_0 > x_1 \ldots > x_n > \ldots$.)

7.9. An ultrafilter D on ω is Ramsey if and only if every function $f : \omega \to \omega$ is either one-to-one on a set in D, or constant on a set in D.

If D and E are ultrafilters on ω, then $D \leq E$ means that for some function $f : \omega \to \omega$, $D = f_*(E)$ (the Rudin-Keisler ordering, see Exercise 7.5).

$D \equiv E$ means that there is a one-to-one function of ω onto ω such that $E = f_*(D)$.

7.10. If $D = f_*(D)$, then $\{n : f(n) = n\} \in D$.

[Let $X = \{n : f(n) < n\}$, $Y = \{n : f(n) > n\}$. For each $n \in X$, let $l(n)$ be the length of the maximal sequence such that $n > f(n) > f(f(n)) > \ldots$. Let $X_0 = \{n \in X : l(n) \text{ is even}\}$ and $X_1 = \{n \in X : l(n) \text{ is odd}\}$. Neither X_0 nor X_1 can be in D since, e.g., $X_0 \cap f_{-1}(X_0) = \emptyset$. The set Y is handled similarly,

except that it remains to show that the set Z of all n such that the sequence $n < f(n) < f^2(n) < f^3(n) < \dots$ is infinite cannot be in D. For $x, y \in Z$ let $x \equiv y$ if $f^k(x) = f^m(y)$ for some k and m. For each $x \in Z$, let a_x be a fixed representative of the class $\{y : y \equiv x\}$; let $l(x)$ be the least $k + m$ such that $f^k(x) = f^m(a_x)$. Let $Z_0 = \{x \in Z : l(x) \text{ is even}\}$ and $Z_1 = \{x \in Z : l(x) \text{ is odd}\}$. Clearly $f_{-1}(Z_1) \cap Z = Z_0$.]

7.11. If $D \leq E$ and $E \leq D$, then $D \equiv E$.
[Use Exercise 7.10.]

Thus $\leq$ is a partial ordering of ultrafilters on ω. A nonprincipal ultrafilter D is *minimal* if there is no nonprincipal E such that $E \leq D$ and $E \not\equiv D$.

7.12. An ultrafilter D on ω is minimal if and only if it is Ramsey.
[If D is Ramsey and $E = f_*(D)$ is nonprincipal, then f is unbounded mod D, hence one-to-one mod D and consequently, $E \equiv D$. If D is minimal and f is unbounded mod D, then $D \leq f_*(D)$ and hence $D = g_*(f_*(D))$ for some g. It follows, by Exercise 7.10, that f is one-to-one mod D.]

7.13. If ω_α is singular, then there is no nonprincipal ω_α-complete ideal on ω_α.

7.14. The set of all sets $X \subset \boldsymbol{R}$ that have Lebesgue measure 0 is a σ-ideal.

A set $X \subset \boldsymbol{R}$ is *meager* if it is the union of a countable collection of nowhere dense sets.

7.15. The set of all meager sets $X \subset \boldsymbol{R}$ is a σ-ideal.
[By the Baire Category Theorem, $\boldsymbol{R}$ is not meager.]

7.16. Let κ be a regular uncountable cardinal, let $|A| \geq \kappa$ and let $S = P_\kappa(A)$. Let F be the set of all $X \subset S$ such that $X \supset \hat{P}$ for some $P \in S$, where $\hat{P} = \{Q \in S : P \subset Q\}$. Then F is a κ-complete filter on S.

7.17. Let B be a Boolean algebra and define

$$u \oplus v = (u - v) + (v - u).$$

Then B with operations $\oplus$ and $\cdot$ is a ring (with zero 0 and unit 1).

7.18. Every element of the subalgebra generated by X is equal to $u_1 + \dots + u_n$ where each u_s is of the form $u_s = \pm x_1 \cdot \pm x_2 \cdot \dots \cdot \pm x_k$ with $x_i \in X$.

7.19. If A is a subalgebra of B and $u \in B$, then the subalgebra generated by $A \cup \{u\}$ is equal to $\{a \cdot u + (b - u) : a, b \in A\}$.

7.20. A finitely generated Boolean algebra is finite. If A has k generators, then $|A| \leq 2^{2^k}$.

7.21. Every finite Boolean algebra is atomic. If $A = \{a_1, \dots, a_n\}$ are the atoms of B, then B is isomorphic to the field of sets $P(A)$. Hence B has 2^n elements.

7.22. Any two countable atomless Boolean algebras are isomorphic.

7.23. $B \restriction a$ is isomorphic to B/I where I is the principal ideal $\{u : u \leq -a\}$.

7.24. Let A be a subalgebra of a Boolean algebra B and let $u \in B - A$. Then there exist ultrafilters F, G on B such that $u \in F$, $u \notin G$, and $F \cap A = G \cap A$.

7.25. Let B be an infinite Boolean algebra, $|B| = \kappa$. There are at least κ ultrafilters on B.

[Assume otherwise. For each pair $(F, G) \in S \times S$ pick $u \in F - G$, and let these u's generate a subalgebra A. Since $|A| \leq |S| < \kappa$, let $u \in B - A$. Use Exercise 7.24 to get a contradiction.]

7.26. For B to be complete it is sufficient that all the sums $\sum X$ exist.

[$\prod X = \sum\{u : u \leq x \text{ for all } x \in X\}$.]

7.27. Let B be a complete Boolean algebra.

(i) Verify the distributive laws:

$$a \cdot \sum\{u : u \in X\} = \sum\{a \cdot u : u \in X\},$$
$$a + \prod\{u : u \in X\} = \prod\{a + u : u \in X\}.$$

(ii) Verify the De Morgan laws:

$$-\sum\{u : u \in X\} = \prod\{-u : u \in X\},$$
$$-\prod\{u : u \in X\} = \sum\{-u : u \in X\}.$$

7.28. Let A and B be σ-complete Boolean algebras. If A is isomorphic to $B\restriction b$ and B is isomorphic to $A\restriction a$, then A and B are isomorphic.

[Follow the proof of the Cantor-Bernstein Theorem.]

7.29. Let A be a subalgebra of a Boolean algebra B, let $u \in B$ and let $A(u)$ be the algebra generated by $A \cup \{u\}$. If h is a homomorphism from A into a complete Boolean algebra C then h extends to a homomorphism from $A(u)$ into C.

[Let $v \in C$ be such that $\sum\{h(a) : a \in A, a \leq u\} \leq v \leq \sum\{h(b) : b \in A, u \leq b\}$. Define $h(a \cdot u + b \cdot (-u)) = h(a) \cdot v + h(b) \cdot (-v)$.]

7.30 (Sikorski's Extension Theorem). Let A be a subalgebra of a Boolean algebra B and let h be a homomorphism from A into a complete Boolean algebra C. Then h can be extended to a homomorphism from B into C.

[Use Exercise 7.29 and Zorn's Lemma.]

7.31. If B is a Boolean algebra and A is a regular subalgebra of B then the inclusion mapping extends to a (unique) complete embedding of the completion of A into the completion of B.

[Use Sikorski's Extension Theorem.]

7.32. If B is an infinite complete Boolean algebra, then $|B|^{\aleph_0} = |B|$.

[First consider the case when $|B\restriction a| = |B|$ for all $a \neq 0$: There is a partition W such that $|W| = \aleph_0$, and $|B| = \prod\{|B\restriction a| : a \in W\} = |B|^{\aleph_0}$. In general, call $a \neq 0$ *stable* if $|B\restriction x| = |B\restriction a|$ for all $x \leq a$, $x \neq 0$. The set of all stable $a \in B$ is dense, and $|B\restriction a| = 2$ or $|B\restriction a|^{\aleph_0} = |B\restriction a|$ if a is stable. Let W be a partition of B such that each $a \in W$ is stable; we have $|B| = \prod\{|B\restriction a| : a \in W\}$ and the theorem follows.]

7.33. If B is a κ-complete, κ-saturated Boolean algebra, then B is complete.

[It suffices to show that $\sum X$ exists for every *open* X (i.e., $u \leq v \in X$ implies $u \in X$). If $X \subset B$ is open, show that $\sum X = \sum W$ where W is a maximal subset of X that is an antichain.]

Historical Notes

The notion of filter is, according to Kuratowski's book [1966], due to H. Cartan. Theorem 7.5 was first proved by Tarski in [1930].

Theorem 7.6 is due to Pospíšil [1937]; the present proof uses independent sets (Lemma 7.7); cf. Fichtenholz and Kantorovich [1935] ($\kappa = \omega$) and Hausdorff [1936b].

W. Rudin [1956] proved that p-points exist if $2^{\aleph_0} = \aleph_1$, a recent result of Shelah shows that existence of p-points is unprovable in ZFC. Galvin showed that $2^{\aleph_0} = \aleph_1$ implies the existence of Ramsey ultrafilters.

Facts about Boolean algebras can be found in Handbook of Boolean algebras [1989] which also contains an extensive bibliography. The Representation Theorem for Boolean algebras as well as the existence of the completion (Theorems 7.11 and 7.13) are due to Stone [1936]. Theorem 7.15 on saturation was proved by Erdős and Tarski [1943].

Exercise 7.8: Booth [1970/71].

Exercise 7.10: Frolík [1968], M. E. Rudin [1971].

The Rudin-Keisler equivalence was first studied by W. Rudin in [1956]; the study of the Rudin-Keisler ordering was initiated by M. E. Rudin [1966].

Exercise 7.25: Makinson [1969].

Exercises 7.29 and 7.30: Sikorski [1964].

Exercise 7.32: Pierce [1958]. The assumption can be weakened to "σ-complete," see Comfort and Hager [1972].

8. Stationary Sets

In this chapter we develop the theory of closed unbounded and stationary subsets of a regular uncountable cardinal, and its generalizations.

Closed Unbounded Sets

If X is a set of ordinals and $\alpha > 0$ is a limit ordinal then α is a *limit point* of X if $\sup(X \cap \alpha) = \alpha$.

Definition 8.1. Let κ be a regular uncountable cardinal. A set $C \subset \kappa$ is a *closed unbounded* subset of κ if C is unbounded in κ and if it contains all its limit points less than κ.

A set $S \subset \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded subset C of κ.

An unbounded set $C \subset \kappa$ is closed if and only if for every sequence $\alpha_0 < \alpha_1 < \ldots < \alpha_\xi < \ldots$ $(\xi < \gamma)$ of elements of C, of length $\gamma < \kappa$, we have $\lim_{\xi \to \gamma} \alpha_\xi \in C$.

Lemma 8.2. *If C and D are closed unbounded, then $C \cap D$ is closed unbounded.*

Proof. It is immediate that $C \cap D$ is closed. To show that $C \cap D$ is unbounded, let $\alpha < \kappa$. Since C is unbounded, there exists an $\alpha_1 > \alpha$ with $\alpha_1 \in C$. Similarly there exists an $\alpha_2 > \alpha_1$ with $\alpha_2 \in D$. In this fashion, we construct an increasing sequence

$$\alpha < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \ldots \tag{8.1}$$

such that $\alpha_1, \alpha_3, \alpha_5, \ldots \in C$, $\alpha_2, \alpha_4, \alpha_6, \ldots \in D$. If we let β be the limit of the sequence (8.1), then $\beta < \kappa$, and $\beta \in C$ and $\beta \in D$. □

The collection of all closed unbounded subsets of κ has the finite intersection property. The filter generated by the closed unbounded sets consists of all $X \subset \kappa$ that contain a closed unbounded subset. We call this filter the *closed unbounded filter* on κ.

The set of all limit ordinals $\alpha < \kappa$ is closed unbounded in κ. If A is an unbounded subset of κ, then the set of all limit points $\alpha < \kappa$ of A is closed unbounded.

A function $f : \kappa \to \kappa$ is *normal* if it is increasing and continuous ($f(\alpha) = \lim_{\xi\to\alpha} f(\xi)$ for every nonzero limit $\alpha < \kappa$). The range of a normal function is a closed unbounded set. Conversely, if C is closed unbounded, there is a unique normal function that enumerates C.

The closed unbounded filter on κ is κ-complete:

Theorem 8.3. *The intersection of fewer than κ closed unbounded subsets of κ is closed unbounded.*

Proof. We prove, by induction on $\gamma < \kappa$, that the intersection of a sequence $\langle C_\alpha : \alpha < \gamma\rangle$ of closed unbounded subsets of κ is closed unbounded. The induction step works at successor ordinals because of Lemma 8.2. If γ is a limit ordinal, we assume that the assertion is true for every $\alpha < \gamma$; then we can replace each C_α by $\bigcap_{\xi\le\alpha} C_\xi$ and obtain a decreasing sequence with the same intersection. Thus assume that

$$C_0 \supset C_1 \supset \ldots \supset C_\alpha \supset \ldots \qquad (\alpha < \gamma)$$

are closed unbounded, and let $C = \bigcap_{\alpha<\gamma} C_\alpha$.

It is easy to see that C is closed. To show that C is unbounded, let $\alpha < \kappa$. We construct a γ-sequence

$$\beta_0 < \beta_1 < \ldots \beta_\xi < \ldots \qquad (\xi < \gamma) \tag{8.2}$$

as follows: We let $\beta_0 \in C_0$ be such that $\beta_0 > \alpha$, and for each $\xi < \gamma$, let $\beta_\xi \in C_\xi$ be such that $\beta_\xi > \sup\{\beta_\nu : \nu < \xi\}$. Since κ is regular and $\gamma < \kappa$, such a sequence (8.2) exists and its limit β is less than κ. For each $\eta < \gamma$, β is the limit of a sequence $\langle\beta_\xi : \eta \le \xi < \gamma\rangle$ in C_η, and so $\beta \in C_\eta$. Hence $\beta \in C$. □

Let $\langle X_\alpha : \alpha < \kappa\rangle$ be a sequence of subsets of κ. The *diagonal intersection* of X_α, $\alpha < \kappa$, is defined as follows:

$$\triangle_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}. \tag{8.3}$$

Note that $\triangle X_\alpha = \triangle Y_\alpha$ where $Y_\alpha = \{\xi \in X_\alpha : \xi > \alpha\}$. Note also that $\triangle X_\alpha = \bigcap_\alpha (X_\alpha \cup \{\xi : \xi \le \alpha\})$.

Lemma 8.4. *The diagonal intersection of a κ-sequence of closed unbounded sets is closed unbounded.*

Proof. Let $\langle C_\alpha : \alpha < \kappa\rangle$ be a sequence of closed unbounded sets. It is clear from the definition that if we replace each C_α by $\bigcap_{\xi\le\alpha} C_\xi$, the diagonal

intersection is the same. In view of Theorem 8.3 we may thus assume that

$$C_0 \supset C_1 \supset \ldots \supset C_\alpha \supset \ldots \qquad (\alpha < \kappa).$$

Let $C = \triangle_{\alpha<\kappa} C_\alpha$. To show that C is closed, let α be a limit point of C. We want to show that $\alpha \in C$, or that $\alpha \in C_\xi$ for all $\xi < \alpha$. If $\xi < \alpha$, let $X = \{\nu \in C : \xi < \nu < \alpha\}$. Every $\nu \in X$ is in C_ξ, by (8.3). Hence $X \subset C_\xi$ and $\alpha = \sup X \in C_\xi$. Therefore $\alpha \in C$ and C is closed.

To show that C is unbounded, let $\alpha < \kappa$. We construct a sequence $\langle \beta_n : n < \omega \rangle$ as follows: Let $\beta_0 > \alpha$ be such that $\beta_0 \in C_0$, and for each n, let $\beta_{n+1} > \beta_n$ be such that $\beta_{n+1} \in C_{\beta_n}$. Let us show that $\beta = \lim_n \beta_n$ is in C: If $\xi < \beta$, let us show that $\beta \in C_\xi$. Since $\xi < \beta$, there is an n such that $\xi < \beta_n$. Each β_k, $k > n$, belongs to C_{β_n} and so $\beta \in C_{\beta_n}$. Therefore $\beta \in C_\xi$. Thus $\beta \in C$, and C is unbounded. □

Corollary 8.5. *The closed unbounded filter on κ is closed under diagonal intersections.* □

The dual of the closed unbounded filter is the ideal of nonstationary sets, the *nonstationary ideal* I_{NS}. I_{NS} is κ-complete and is closed under *diagonal unions*:

$$\sum_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcup_{\alpha<\xi} X_\alpha\}.$$

The quotient algebra $B = P(\kappa)/I_{\mathrm{NS}}$ is a κ-complete Boolean algebra, where the Boolean operations $\sum_{\alpha<\gamma}$ and $\prod_{\alpha<\gamma}$ for $\gamma < \kappa$ are induced by $\bigcup_{\alpha<\gamma}$ and $\bigcap_{\alpha<\gamma}$. As a consequence of Lemma 8.4, B is κ^+-complete: If $\{X_\alpha : \alpha < \kappa\}$ is a collection of subsets of κ then the equivalence classes of $\triangle_{\alpha<\kappa} X_\alpha$ and $\sum_{\alpha<\kappa} X_\alpha$ are, respectively, the greatest lower bound and the least upper bound of the equivalence classes $[X_\alpha]$ in B. It also follows that if $\langle X_\alpha : \alpha < \kappa \rangle$ and $\langle Y_\alpha : \alpha < \kappa \rangle$ are two enumerations of the same collection, then $\triangle_{\alpha<\kappa} X_\alpha$ and $\triangle_{\alpha<\kappa} Y_\alpha$ differ only by a nonstationary set.

Definition 8.6. An ordinal function f on a set S is *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in S$, $\alpha > 0$.

Theorem 8.7 (Fodor). *If f is a regressive function on a stationary set $S \subset \kappa$, then there is a stationary set $T \subset S$ and some $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$.*

Proof. Let us assume that for each $\gamma < \kappa$, the set $\{\alpha \in S : f(\alpha) = \gamma\}$ is nonstationary, and choose a closed unbounded set C_γ such that $f(\alpha) \neq \gamma$ for each $\alpha \in S \cap C_\gamma$. Let $C = \triangle_{\gamma<\kappa} C_\gamma$. The set $S \cap C$ is stationary and if $\alpha \in S \cap C$, we have $f(\alpha) \neq \gamma$ for every $\gamma < \alpha$; in other words, $f(\alpha) \geq \alpha$. This is a contradiction. □

For a regular uncountable cardinal κ and a regular $\lambda < \kappa$, let

$$E^\kappa_\lambda = \{\alpha < \kappa : \operatorname{cf}\alpha = \lambda\}. \tag{8.4}$$

It is easy to see that each E^κ_λ is a stationary subset of κ.

The closed unbounded filter on κ is not an ultrafilter. This is because there is a stationary subset of κ whose complement is stationary. If $\kappa > \omega_1$, this is clear: The sets E^κ_ω and $E^\kappa_{\omega_1}$ are disjoint. If $\kappa = \omega_1$, the decomposition of ω_1 into disjoint stationary sets uses the Axiom of Choice.

The use of AC is necessary: It is consistent (relative to large cardinals) that the closed unbounded filter on ω_1 is an ultrafilter.

In Theorem 8.10 below we show that every stationary subset of κ is the union of κ disjoint stationary sets. In the following lemma we prove a weaker result that illustrates a typical use of Fodor's Theorem.

Lemma 8.8. *Every stationary subset of E^κ_ω is the union of κ disjoint stationary sets.*

Proof. Let $W \subset \{\alpha < \kappa : \operatorname{cf}\alpha = \omega\}$ be stationary. For every $\alpha \in W$, we choose an increasing sequence $\langle a^\alpha_n : n \in \boldsymbol{N}\rangle$ such that $\lim_n a^\alpha_n = \alpha$. First we show that there is an n such that for all $\eta < \kappa$, the set

$$\{\alpha \in W : a^\alpha_n \geq \eta\} \tag{8.5}$$

is stationary. Otherwise there is η_n and a closed unbounded set C_n such that $a^\alpha_n < \eta_n$ for all $\alpha \in C_n \cap W$, for every n. If we let η be the supremum of the η_n and C the intersection of the C_n, we have $a^\alpha_n < \eta$ for all n and all $\alpha \in C \cap W$. This is a contradiction. Now let n be such that (8.5) is stationary for every $\eta < \kappa$. Let f be the following function on W: $f(\alpha) = a^\alpha_n$. The function f is regressive; and so for every $\eta < \kappa$, we find by Fodor's Theorem a stationary subset S_η of (8.5) and $\gamma_\eta \geq \eta$ such that $f(\alpha) = \gamma_\eta$ on S_η. If $\gamma_\eta \neq \gamma_{\eta'}$, then $S_\eta \cap S_{\eta'} = \emptyset$, and since κ is regular, we have $|\{S_\eta : \eta < \kappa\}| = |\{\gamma_\eta : \eta < \kappa\}| = \kappa$. □

The proof easily generalizes to the case when $\lambda > \omega$: Every stationary subset of E^κ_λ the union of κ stationary sets. From that it follows that every stationary subset W of the set $\{\alpha < \kappa : \operatorname{cf}\alpha < \alpha\}$ admits such a decomposition: By Fodor's Theorem, there exists some $\lambda < \kappa$ such that $W \cap E^\kappa_\lambda$ is stationary. The remaining case in Theorem 8.10 is when the set $\{\alpha < \kappa : \alpha$ is a regular cardinal$\}$ is stationary and the following lemma plays the key role.

Lemma 8.9. *Let S be a stationary subset of κ and assume that every $\alpha \in S$ is a regular uncountable cardinal. Then the set $T = \{\alpha \in S : S \cap \alpha$ is not a stationary subset of $\alpha\}$ is stationary.*

Proof. We prove that T intersects every closed unbounded subset of κ. Let C be closed unbounded. The set C' of all limit points of C is also closed

unbounded, and hence $S \cap C' \neq \emptyset$. Let α be the least element of $S \cap C'$. Since α is regular and a limit point of C, $C \cap \alpha$ is a closed unbounded subset of α, and so is $C' \cap \alpha$. As α is the least element of $S \cap C'$, $C' \cap \alpha$ is disjoint from $S \cap \alpha$ and so $S \cap \alpha$ is a nonstationary subset of α. Hence $\alpha \in T \cap C$. □

Theorem 8.10 (Solovay). *Let κ be a regular uncountable cardinal. Then every stationary subset of κ is the disjoint union of κ stationary subsets.*

Proof. We follow the proof of Lemma 8.8 as much as possible. Let A be a stationary subset of κ. By Lemma 8.8, by the subsequent discussion and by Lemma 8.9, we may assume that the set W of all $\alpha \in A$ such that α is a regular cardinal and $A \cap \alpha$ is not stationary, is stationary. There exists for each $\alpha \in W$ a continuous increasing sequence $\langle a_\xi^\alpha : \xi < \alpha \rangle$ such that $a_\xi^\alpha \notin W$, for all α and ξ, and $\alpha = \lim_{\xi \to \alpha} a_\xi^\alpha$.

First we show that there is ξ such that for all $\eta < \kappa$, the set

$$\{\alpha \in W : a_\xi^\alpha \geq \eta\} \tag{8.6}$$

is stationary. Otherwise, there is for each ξ some $\eta(\xi)$ and a closed unbounded set C_ξ such that $a_\xi^\alpha < \eta(\xi)$ for all $\alpha \in C_\xi \cap W$ if a_ξ^α is defined. Let C be the diagonal intersection of the C_ξ. Thus if $\alpha \in C \cap W$, then $a_\xi^\alpha < \eta(\xi)$ for all $\xi < \alpha$. Now let D be the closed unbounded set of all $\gamma \in C$ such that $\eta(\xi) < \gamma$ for all $\xi < \gamma$. Since W is stationary, $W \cap D$ is also stationary; let $\gamma < \alpha$ be two ordinals in $W \cap D$. Now if $\xi < \gamma$, then $a_\xi^\alpha < \eta(\xi) < \gamma$ and it follows that $a_\gamma^\alpha = \gamma$. This is a contradiction since $\gamma \in W$ and $a_\gamma^\alpha \notin W$.

Once we have found ξ such that (8.6) is stationary for all $\eta < \kappa$, we proceed as in Lemma 8.8. Let f be the function on W defined by $f(\alpha) = a_\xi^\alpha$. The function f is regressive; and so for every $\eta < \kappa$, we find by Fodor's Theorem a stationary subset S_η of (8.6) and $\gamma_\eta \geq \eta$ such that $f(\alpha) = \gamma_\eta$ on S_η. If $\gamma_\eta \neq \gamma_{\eta'}$, then $S_\eta \cap S_{\eta'} = \emptyset$; and since κ is regular, we have $|\{S_\eta : \eta < \kappa\}| = |\{\gamma_\eta : \eta < \kappa\}| = \kappa$. □

Mahlo Cardinals

Let κ be an inaccessible cardinal. The set of all cardinals below κ is a closed unbounded subset of κ, and so is the set of its limit points, the set of all limit cardinals. In fact, the set of all strong limit cardinals below κ is closed unbounded.

If κ is the least inaccessible cardinal, then all strong limit cardinals below κ are singular, and so the set of all singular strong limit cardinals below κ is closed unbounded. If κ is the αth inaccessible, where $\alpha < \kappa$, then still the set of all regular cardinals below κ is nonstationary.

An inaccessible cardinal κ is called a *Mahlo cardinal* if the set of all regular cardinals below κ is stationary.

(Then the set of all inaccessibles below κ is stationary, and κ is the κth inaccessible cardinal.)

Similarly, we define a *weakly Mahlo* cardinal as a cardinal κ that is weakly inaccessible and the set of all regular cardinals bellow κ is stationary (then the set of all weakly inaccessibles is stationary in κ).

Normal Filters

Let F be a filter on a cardinal κ; F is *normal* if it is closed under diagonal intersections:

$$\text{if } X_\alpha \in F \text{ for all } \alpha < \kappa, \quad \text{then } \triangle_{\alpha<\kappa} X_\alpha \in F. \tag{8.7}$$

An ideal I on κ is *normal* if the dual filter is normal.

The closed unbounded filter is κ-complete and normal, and contains all complements of bounded sets. It is the smallest such filter on κ:

Lemma 8.11. *If κ is regular and uncountable and if F is a normal filter on κ that contains all final segments $\{\alpha : \alpha_0 < \alpha < \kappa\}$, then F contains all closed unbounded sets.*

Proof. First we note that the set C_0 of all limit ordinals is in F: C_0 is the diagonal intersection of the sets $X_\alpha = \{\xi : \alpha + 1 < \xi < \kappa\}$. Now let C be a closed unbounded set, and let $C = \{a_\alpha : \alpha < \kappa\}$ be its increasing enumeration. We let $X_\alpha = \{\xi : a_\alpha < \xi < \kappa\}$. Then $C \supset C_0 \cap \triangle_{\alpha<\kappa} X_\alpha$. □

Silver's Theorem

We shall now apply the techniques using ultrafilters and stationary sets to prove the following theorems.

Theorem 8.12 (Silver). *Let κ be a singular cardinal such that $\operatorname{cf}\kappa > \omega$. If $2^\alpha = \alpha^+$ for all cardinals $\alpha < \kappa$, then $2^\kappa = \kappa^+$.*

Theorem 8.13 (Silver). *If the Singular Cardinals Hypothesis holds for all singular cardinals of cofinality ω, then it holds for all singular cardinals.*

The proofs of both theorems use the following lemma:

Lemma 8.14. *Let κ be a singular cardinal, let $\operatorname{cf}\kappa > \omega$, and assume that $\lambda^{\operatorname{cf}\kappa} < \kappa$ for all $\lambda < \kappa$. If $\langle \kappa_\alpha : \alpha < \operatorname{cf}\kappa\rangle$ is a normal sequence of cardinals such that $\lim \kappa_\alpha = \kappa$, and if the set $\{\alpha < \operatorname{cf}\kappa : \kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+\}$ is stationary in $\operatorname{cf}\kappa$, then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.*

If GCH holds below κ then the assumptions of Lemma 8.14 are satisfied, and $2^\kappa = \kappa^{\operatorname{cf}\kappa}$. Thus Theorem 8.12 follows from Lemma 8.14.

Proof of Theorem 8.13. We prove by induction on the cofinality of κ that $2^{\operatorname{cf}\kappa} < \kappa$ implies $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. The assumption of the theorem is that this holds for each κ of cofinality ω. Thus let κ be of uncountable cofinality and let $2^{\operatorname{cf}\kappa} < \kappa$. Using the induction hypothesis and the proof of Theorem 5.22(ii) one verifies, by induction on λ, that $\lambda^{\operatorname{cf}\kappa} < \kappa$ for all $\lambda < \kappa$.

Let $\langle \kappa_\alpha : \alpha < \operatorname{cf}\kappa\rangle$ be any normal sequence of cardinals such that $\lim \kappa_\alpha = \kappa$. The set $S = \{\alpha < \operatorname{cf}\kappa : \operatorname{cf}\kappa_\alpha = \omega \text{ and } 2^{\aleph_0} < \kappa_\alpha\}$ is clearly stationary in $\operatorname{cf}\kappa$, and for every $\alpha \in S$, $\kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+$ by the assumption. Hence $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. □

We now proceed toward a proof of Lemma 8.14. To simplify the notation, we shall consider the special case when

$$\kappa = \aleph_{\omega_1}.$$

The general case is proved in a similar way.

Let f and g be two functions on ω_1. We say that f and g are *almost disjoint* if there is $\alpha_0 < \omega_1$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \alpha_0$. A family F of functions on ω_1 is an *almost disjoint family* if any two distinct $f, g \in F$ are almost disjoint.

Lemma 8.14 follows from

Lemma 8.15. *Assume that* $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ *for all* $\alpha < \omega_1$. *Let* F *be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha,$$

such that the set

$$\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_{\alpha+1}\} \tag{8.8}$$

is stationary. Then $|F| \leq \aleph_{\omega_1+1}$.

[In the general case, we consider almost disjoint functions on $\operatorname{cf}\kappa$.]

Proof of Lemma 8.14 *from Lemma* 8.15. We assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ and that $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha} = \aleph_{\alpha+1}$ for a stationary set of α's; we want to show that $\aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1+1}$. For every $h : \omega_1 \to \aleph_{\omega_1}$, we let $f_h = \langle h_\alpha : \alpha < \omega_1\rangle$, where $\operatorname{dom} h_\alpha = \omega_1$ and

$$h_\alpha(\xi) = \begin{cases} h(\xi) & \text{if } h(\xi) < \aleph_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

and let $F = \{f_h : h \in {\aleph_{\omega_1}}^{\omega_1}\}$. If $h \neq g$, then f_h and f_g are almost disjoint. Moreover,

$$F \subset \prod_{\alpha<\omega_1} {\aleph_\alpha}^{\omega_1}.$$

Since for a stationary set of α's, $\aleph_\alpha^{\aleph_1} = \aleph_{\alpha+1}$ (namely for all α such that $\aleph_\alpha > 2^{\aleph_1}$ and $\aleph_\alpha^{\aleph_0} = \aleph_{\alpha+1}$), we have $|F| \leq \aleph_{\omega_1+1}$, and so $|{\aleph_{\omega_1}}^{\omega_1}| = \aleph_{\omega_1+1}$.

[In the general case of Lemma 8.14 we have to show that

$$\{\alpha < \operatorname{cf}\kappa : \kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+\}$$

is stationary. Note that the set

$$C = \{\alpha : \alpha \text{ is a limit ordinal and } (\forall \lambda < \kappa_\alpha)\, \lambda^{\operatorname{cf}\kappa} < \kappa_\alpha\}$$

is closed unbounded in $\operatorname{cf}\kappa$; if $\alpha \in C$, then $\operatorname{cf}\kappa_\alpha < \operatorname{cf}\kappa$ and we have $\kappa_\alpha^{\operatorname{cf}\kappa} = \kappa_\alpha^{\operatorname{cf}\alpha}$.] □

The first step in the proof of Lemma 8.15 is

Lemma 8.16. *Assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Let F be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha$$

such that the set

$$\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_\alpha\} \tag{8.9}$$

is stationary. Then $|F| \leq \aleph_{\omega_1}$.

(The assumption (8.8) is replaced by (8.9) and the bound for $|F|$ is $\aleph_{\omega_1}$ rather than $\aleph_{\omega_1+1}$.)

Proof. We may as well assume that each A_α is a set of ordinals and that $A_\alpha \subset \omega_\alpha$ for all α in some stationary subset of $\aleph_1$. Let

$$S_0 = \{\alpha < \omega_1 : \alpha \text{ is a limit ordinal and } A_\alpha \subset \omega_\alpha\}.$$

Thus if $f \in F$, then $f(\alpha) < \omega_\alpha$ for all $\alpha \in S_0$. Given $f \in F$, we can find for each $\alpha > 0$ in S_0 some $\beta < \alpha$ such that $f(\alpha) < \omega_\beta$; call this $\beta = g(\alpha)$. The function g is regressive on S, and by Fodor's Theorem there is a stationary $S \subset S_0$ such that g is constant on S. In other words, the function f is bounded on S, by some $\omega_\gamma < \omega_{\omega_1}$.

We assign to each f a pair $(S, f\restriction S)$ where $S \subset S_0$ is a stationary set and $f\restriction S$ is a bounded function. For any S, if $f\restriction S = g\restriction S$, then $f = g$ since any two distinct functions in F are almost disjoint. Thus the correspondence

$$f \mapsto (S, f\restriction S)$$

is one-to-one.

For a given S, the number of bounded functions on S is at most

$$\sum_{\gamma<\omega_1} \aleph_\gamma^{|S|} = \sup_{\gamma<\omega_1} \aleph_\gamma^{\aleph_1} = \aleph_{\omega_1}.$$

Since $|P(\omega_1)| = 2^{\aleph_1} < \aleph_{\omega_1}$, the number of pairs $(S, f\restriction S)$ is at most $\aleph_{\omega_1}$. Hence $|F| \leq \aleph_{\omega_1}$. □

Proof of Lemma 8.15. Let U be an ultrafilter on ω_1 that extends the closed unbounded filter. Every $S \in U$ is stationary.

We may assume that each A_α is a subset of $\omega_{\alpha+1}$. For every $f, g \in F$, let

$$f < g \text{ if and only if } \{\alpha < \omega_1 : f(\alpha) < g(\alpha)\} \in U. \tag{8.10}$$

Since U is a filter, the relation $f < g$ is transitive. Since U is an ultrafilter, and $\{\alpha : f(\alpha) = g(\alpha)\} \notin U$ for distinict $f, g \in F$, the relation $f < g$ is a linear ordering of F. For every $f \in F$, let $F_f = \{g \in F :$ for some stationary set T, $g(\alpha) < f(\alpha)$ for all $\alpha \in T\}$. By Lemma 8.16, $|F_f| \leq \aleph_{\omega_1}$. If $g < f$, then $g \in F_f$, and so $|\{g \in F : g < f\}| \leq \aleph_{\omega_1}$. It follows that $|F| \leq \aleph_{\omega_1+1}$. □

A Hierarchy of Stationary Sets

If α is a limit ordinal of uncountable cofinality, it still makes sense to talk about closed unbounded and stationary subsets of α. Since $\operatorname{cf}\alpha > \omega$, Lemma 8.2 holds, and the closed unbounded sets generate a filter on α. The closed unbounded filter is $\operatorname{cf}\alpha$-complete. A set $S \subset \alpha$ is stationary if and only if for some (or for any) normal function $f : \operatorname{cf}\alpha \to \alpha$, $f_{-1}(S)$ is a stationary subset of $\operatorname{cf}\alpha$.

Let κ be a regular uncountable cardinal, and let us consider the following operation (the *Mahlo operation*) on stationary sets:

Definition 8.17. If $S \subset \kappa$ is stationary, the *trace* of S is the set

$$\operatorname{Tr}(S) = \{\alpha < \kappa : \operatorname{cf}\alpha > \omega \text{ and } S \cap \alpha \text{ is stationary}\}.$$

The Mahlo operation is invariant under equivalence mod I_{NS} and can thus be considered as an operation on the Boolean algebra $P(\kappa)/I_{\mathrm{NS}}$ (see Exercise 8.11).

In the context of closed unbounded and stationary sets we use the phrase *for almost all* $\alpha \in S$ to mean that the set of all contrary $\alpha \in S$ is nonstationary.

Definition 8.18. Let S and T be stationary subsets of κ.

$$S < T \quad \text{if and only if} \quad S \cap \alpha \text{ is stationary for almost all } \alpha \in T.$$

(It is implicit in the definition that almost all $\alpha \in T$ have uncountable cofinality.)

As an example, if $\lambda < \mu$ are regular, then $E^\kappa_\lambda < E^\kappa_\mu$. The following properties are easily verified:

Lemma 8.19.

(i) $A < \operatorname{Tr}(A)$,

(ii) *if* $A < B$ *and* $B < C$ *then* $A < C$,
(iii) *if* $A < B$, $A \simeq A' \bmod I_{\rm NS}$ *and* $B \simeq B' \bmod I_{\rm NS}$ *then* $A' < B'$. □

Thus $<$ is a transitive relation on $P(\kappa)/I_{\rm NS}$. The next theorem shows that it is a well-founded partial ordering:

Theorem 8.20 (Jech). *The relation* $<$ *is well-founded.*

Proof. Assume to the contrary that there exist stationary sets such that $A_1 > A_2 > A_3 \ldots$. Therefore there exist closed unbounded sets C_n such that $A_n \cap C_n \subset {\rm Tr}(A_{n+1})$ for $n = 1, 2, 3, \ldots$. For each n, let

$$B_n = A_n \cap C_n \cap {\rm Lim}(C_{n+1}) \cap {\rm Lim}({\rm Lim}(C_{n+2})) \cap \ldots$$

where ${\rm Lim}(C)$ is the set of all limit points of C.

Each B_n is stationary, and for every n, $B_n \subset {\rm Tr}(B_{n+1})$. Let $\alpha_n = \min B_n$. Since $B_{n+1} \cap \alpha_n$ is stationary, we have $\alpha_{n+1} < \alpha_n$ and therefore, a decreasing sequence $\alpha_1 > \alpha_2 > \ldots$. A contradiction. □

The rank of a stationary set $A \subset \kappa$ in the well-founded relation $<$ is called the *order* of the set A, and the height of $<$ is the *order* of the cardinal κ:

$$o(A) = \sup\{o(X) + 1 : X < A\},$$
$$o(\kappa) = \sup\{o(A) + 1 : A \subset \kappa \text{ is stationary}\}.$$

We also define $o(\aleph_0) = 0$, and $o(\alpha) = o({\rm cf}\,\alpha)$ for every limit ordinal α. Note that $o(E^\kappa_\omega) = 0$, $o(E^\kappa_{\omega_1}) = 1$, $o(\aleph_1) = 1$, $o(\aleph_2) = 2$, etc. See Exercises 8.13 and 8.14.

The Closed Unbounded Filter on $P_\kappa(\lambda)$

We shall now consider a generalization of closed unbounded and stationary sets, to the space $P_\kappa(\lambda)$. This generalization replaces $(\kappa, <)$ with the structure $(P_\kappa(\lambda), \subset)$.

Let κ be a regular uncountable cardinal and let A be a set of cardinality at least κ.

Definition 8.21. A set $X \subset P_\kappa(A)$ is *unbounded* if for every $x \in P_\kappa(A)$ there exists a $y \supset x$ such that $y \in X$.

A set $X \subset P_\kappa(A)$ is *closed* if for any chain $x_0 \subset x_1 \subset \ldots \subset x_\xi \subset \ldots$, $\xi < \alpha$, of sets in X, with $\alpha < \kappa$, the union $\bigcup_{\xi<\alpha} x_\xi$ is in X.

A set $C \subset P_\kappa(A)$ is *closed unbounded* if it is closed and unbounded.

A set $S \subset P_\kappa(A)$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded $C \subset P_\kappa(A)$.

The *closed unbounded filter* on $P_\kappa(A)$ is the filter generated by the closed unbounded sets.

When $|A| = |B|$ then $P_\kappa(A)$ and $P_\kappa(B)$ are isomorphic, with closed unbounded and stationary sets corresponding to closed unbounded and stationary sets, and so it often suffices to consider such sets in $P_\kappa(\lambda)$ where λ is a cardinal $\geq \kappa$.

When $|A| = \kappa$, then the set $\kappa \subset P_\kappa(\kappa)$ is closed unbounded, and the closed unbounded filter on κ is the restriction to κ of the closed unbounded filter on $P_\kappa(\kappa)$.

Theorem 8.22 (Jech). *The closed unbounded filter on $P_\kappa(A)$ is κ-complete.*

Proof. This is a generalization of Theorem 8.3. First we proceed as in Lemma 8.2 and show that if C and D are closed unbounded then $C \cap D$ is closed unbounded. Both proofs have straightforward generalizations from $(\kappa, <)$ to $(P_\kappa(A), \subset)$. □

Fodor's Theorem also generalizes to $P_\kappa(A)$; with regressive functions replaced by choice functions. The *diagonal intersection* of subsets of $P_\kappa(A)$ is defined as follows

$$\triangle_{a\in A} X_a = \{x \in P_\kappa(A) : x \in \bigcap_{a\in x} X_a\}.$$

Lemma 8.23. *If $\{C_a : a \in A\}$ is a collection of closed unbounded subsets of $P_\kappa(A)$ then its diagonal intersection is closed unbounded.*

Proof. Let $C = \triangle_{a\in A} C_a$. First we show that C is closed. Let $x_0 \subset x_1 \subset \ldots \subset x_\xi \subset \ldots$, $\xi < \alpha$, be a chain in C, with $\alpha < \kappa$, and let x be its union. To show that $x \in C$, let $a \in x$ and let us show that $x \in C_a$. There is some $\eta < \alpha$ such that $a \in x_\xi$ for all $\xi \geq \eta$; hence $x_\xi \in C_a$ for all $\xi \geq \eta$, and so $x \in C_a$.

Now we show that C is unbounded. Let $x_0 \in P_\kappa(A)$, we shall find an $x \in C$ such that $x \supset x_0$. By induction, we find $x_0 \subset x_1 \subset \ldots \subset x_n \subset \ldots$, $n \in \boldsymbol{N}$, such that $x_{n+1} \in \bigcap_{a\in x_n} C_a$; this is possible because each $\bigcap_{a\in x_n} C_a$ is closed unbounded. Then we let $x = \bigcup_{n=0}^{\infty} x_n$ and show that $x \in C_a$ for all $a \in x$. But if $a \in x$ then $a \in x_k$ for some k, and then $x_n \in C_a$ for all $n \geq k+1$. Hence $x \in C_a$. □

Theorem 8.24 (Jech). *If f is a function on a stationary set $S \subset P_\kappa(\lambda)$ and if $f(x) \in x$ for every nonempty $x \in S$, then there exist a stationary set $T \subset S$ and some $a \in A$ such that $f(x) = a$ for all $a \in T$.*

Proof. The proof uses Lemma 8.23 and generalizes the proof of Theorem 8.7. □

Let us call a set $D \subset P_\kappa(A)$ *directed* if for all x and y in D there is a $z \in D$ such that $x \cup y \subset z$.

Lemma 8.25. *If C is a closed subset of $P_\kappa(A)$ then for every directed set $D \subset C$ with $|D| < \kappa$, $\bigcup D \in C$.*

Proof. By induction on $|D|$. Let $|D| = \gamma$, $D = \{x_\alpha : \alpha < \gamma\}$, and assume the lemma holds for every directed set of cardinality $< \gamma$. By induction on $\alpha < \gamma$, let D_α be a smallest directed subset of D such that $x_\alpha \in D_\alpha$ and $D_\alpha \supset \bigcup_{\beta<\alpha} D_\beta$. Letting $y_\alpha = \bigcup D_\alpha$, we have $y_\alpha \in C$ for all $\alpha < \gamma$, and $y_\beta \subset y_\alpha$ if $\beta < \alpha$. It follows that $\bigcup D = \bigcup_{\alpha<\gamma} y_\alpha \in C$. □

Consider a function $f : [A]^{<\omega} \to P_\kappa(A)$; a set $x \in P_\kappa(A)$ is a *closure point* of f if $f(e) \subset x$ whenever $e \subset x$. The set C_f of all closure points $x \in P_\kappa(A)$ is a closed unbounded set. Moreover, the sets C_f generate the closed unbounded filter:

Lemma 8.26. *For every closed unbounded set C in $P_\kappa(A)$ there exists a function $f : [A]^{<\omega} \to P_\kappa(A)$ such that $C_f \subset C$.*

Proof. By induction on $|e|$ we find for each $e \in [A]^{<\omega}$ an infinite set $f(e) \in C$ such that $e \subset f(e)$ and that $f(e_1) \subset f(e_2)$ whenever $e_1 \subset e_2$. We will show that $C_f \subset C$. Let x be a closure point of f. As $x = \bigcup\{f(e) : e \in [x]^{<\omega}\}$ is the union of a directed subset of C (of cardinality $< \kappa$), by Lemma 8.25 we have $x \in C$. □

Let $A \subset B$ (and $|A| \geq \kappa$). For $X \in P_\kappa(B)$, the *projection* of X to A is the set

$$X{\restriction}A = \{x \cap A : x \in X\}.$$

For $Y \in P_\kappa(A)$, the *lifting* of Y to B is the set

$$Y^B = \{x \in P_\kappa(B) : x \cap A \in Y\}.$$

Theorem 8.27 (Menas). *Let $A \subset B$.*

(i) *If S is stationary in $P_\kappa(B)$, then $S{\restriction}A$ is stationary in $P_\kappa(A)$.*
(ii) *If S is stationary in $P_\kappa(A)$, then S^B is stationary in $P_\kappa(B)$.*

Proof. (i) holds because if C is a closed unbounded set in $P_\kappa(A)$, then C^B is closed unbounded in $P_\kappa(B)$. For (ii), it suffices to prove that if C is closed unbounded in $P_\kappa(B)$, then $C{\restriction}A$ contains a closed unbounded set.

If $C \subset P_\kappa(B)$ is closed unbounded, then by Lemma 8.26, $C \supset C_f$ for some $f : [B]^{<\omega} \to P_\kappa(B)$. Let $g : [A]^{<\omega} \to P_\kappa(A)$ be the following function: For $e \in [A]^{<\omega}$, let x be the smallest closure point of f such that $x \supset e$, and let $g(e) = x \cap A$. Then $C_f{\restriction}A = C_g$ (where C_f is defined in $P_\kappa(B)$ and C_g in $P_\kappa(A)$), and we have $C_g \subset C{\restriction}A$. □

When $\kappa = \omega_1$, Lemma 8.26 can be improved to give the following basis theorem for $[A]^\omega = \{x \subset A : |x| = \aleph_0\}$. An *operation* on A is a function $F : [A]^{<\omega} \to A$. A set x is *closed under* F if $f(e) \in x$ for all $e \in [x]^{<\omega}$.

Theorem 8.28 (Kueker). *For every closed unbounded set $C \subset [A]^\omega$ there is an operation F on A such that $C \supset C_F = \{x \in [A]^\omega : x \text{ is closed under } F\}$.*

Proof. We may assume that $A = \lambda$ is an infinite cardinal, and let C be a closed unbounded subset of $[\lambda]^\omega$. As in the proof of Lemma 8.26 there exists a function $f : [\lambda]^{<\omega} \to C$ such that $e \subset f(e)$ and $f(e_1) \subset f(e_2)$ if $e_1 \subset e_2$. As each $f(e)$ is countable, there exist functions f_k, $k \in \boldsymbol{N}$, such that $f(e) = \{f_k(e) : k \in \boldsymbol{N}\}$ for all e. Let $n \mapsto (k_n, m_n)$ be a pairing function.

Now we define an operation F on λ as follows: Let $F(\{\alpha\}) = \alpha + 1$, and if $\alpha_1 < \ldots < \alpha_n$, let $F(\{\alpha_1, \ldots, \alpha_n\}) = f_{k_n}(\{\alpha_1, \ldots, \alpha_{m_n}\})$. It is enough to show that if $x \in [\lambda]^\omega$ is closed under F then x is a closure point of f, and so $C_F \subset C_f \subset C$.

Let x be closed under F, let $k \in \boldsymbol{N}$ and let $e \in [x]^{<\omega}$; we want to show that $f_k(e) \in x$. If $e = \{\alpha_1, \ldots, \alpha_m\}$ with $\alpha_1 < \ldots < \alpha_m$, let $n \geq m$ be such that $k = k_n$ and $m = m_n$. As x does not have a greatest element (because $F(\{\alpha\}) = \alpha + 1$), there are $\alpha_{m+1}, \ldots, \alpha_n \in x$ such that $f_k(\{\alpha_1, \ldots, \alpha_m\}) = F(\{\alpha_1, \ldots, \alpha_n\}) \in x$. □

Theorem 8.28 does not generalize outright to $P_\kappa(A)$ for $\kappa > \omega_1$ (see Exercise 8.18); we shall return to the subject in Part III.

Exercises

8.1. The set of all fixed points (i.e., $f(\alpha) = \alpha$) of a normal function is closed unbounded.

8.2. If $f : \kappa \to \kappa$, then the set of all $\alpha < \kappa$ such that $f(\xi) < \alpha$ for all $\xi < \alpha$ is closed unbounded.

8.3. If S is stationary and C is closed unbounded, then $S \cap C$ is stationary.

8.4. If $X \subset \kappa$ is nonstationary, then there exists a regressive function f on X such that $\{\alpha : f(\alpha) \leq \gamma\}$ is bounded, for every $\gamma < \kappa$.

[Let $C \cap X = \emptyset$, and let $f(\alpha) = \sup(C \cap \alpha)$.]

8.5. For every stationary $S \subset \omega_1$ and every $\alpha < \omega_1$ there is a closed set of ordinals A of length α such that $A \subset S$.

[By induction on α: $\forall\gamma\ \exists$closed $A \subset S$ of length α such that $\gamma < \min A$. The nontrivial step: If true for a limit α, find a closed $A \subset S$ of length α such that $\sup A \in S$. Let A_ξ, $\xi < \omega_1$, be closed subsets of S, of length α, such that $\lambda_\xi = \sup \bigcup_{\nu<\xi} A_\nu < \min A_\xi$. There is ξ such that $\lambda_\xi \in S$. Let $\xi = \lim_n \xi_n$. Pick initial segments $B_{\xi_n} \subset A_{\xi_n}$ of length $\alpha_n + 1$ where $\lim_n \alpha_n = \alpha$. Let $A = \bigcup_{n=0}^\infty B_{\xi_n}$.]

Exercise 8.5 does not generalize to closed sets of uncountable length. It is not provable in ZFC that given $X \subset \omega_2$, either X or $\omega_2 - X$ contains a closed set of length ω_1. On the other hand, this statement is consistent, relative to large cardinals.

8.6. Let κ be the least inaccessible cardinal such that κ is the κth inaccessible cardinal. Then κ is not Mahlo.

[Use $f(\lambda) = \alpha$ where λ is the αth inaccessible.]

8.7. If κ is a limit (weakly inaccessible, weakly Mahlo) cardinal and the set of all strong limit cardinals below κ is unbounded in κ, then κ is a strong limit (inaccessible, Mahlo) cardinal.

8.8. A κ-complete ideal I on κ is normal if and only if for every $S_0 \notin I$ and any regressive f on S_0 there is $S \subset S_0$, $S \notin I$, such that f is constant on S.

[One direction is like Fodor's Theorem. For the other direction, let $X_\alpha \in F$ for each $\alpha < \kappa$. If $\triangle X_\alpha \notin F$, let $S_0 = \kappa - \triangle X_\alpha$ and let $f(\alpha) =$ some $\xi < \alpha$ such that $\alpha \notin X_\xi$. If $f(\alpha) = \gamma$ for all $\alpha \in S$, then $X_\gamma \cap S = \emptyset$, a contradiction.]

8.9. There is no normal nonprincipal filter on ω.

[Use the regressive function $f(n+1) = n$.]

8.10. If κ is singular, then there is no normal ideal on κ that contains all bounded subsets of κ.

8.11. (i) If $S \subset T$ then $\mathrm{Tr}(S) \subset \mathrm{Tr}(T)$,
(ii) $\mathrm{Tr}(S \cup T) = \mathrm{Tr}(S) \cup \mathrm{Tr}(T)$,
(iii) $\mathrm{Tr}(\mathrm{Tr}(S)) \subset \mathrm{Tr}(S)$,
(iv) if $S \simeq T \bmod I_{\mathrm{NS}}$ then $\mathrm{Tr}(S) \simeq \mathrm{Tr}(T) \bmod I_{\mathrm{NS}}$.

8.12. Show that $\mathrm{Tr}(E^\kappa_\lambda) = \{\alpha < \kappa : \mathrm{cf}\,\alpha \geq \lambda^+\}$.

8.13. If $\lambda < \kappa$ is the αth regular cardinal cardinal, then $o(E^\kappa_\lambda) = \alpha$.

8.14. $o(\kappa) \geq \kappa$ if and only if κ is weakly inaccessible; $o(\kappa) \geq \kappa + 1$ if and only if κ is weakly Mahlo.

8.15. For each $a \in P_\kappa(A)$, the set $\{x \in P_\kappa(A) : x \supset a\}$ is closed unbounded.

A κ-complete filter F on $P_\kappa(A)$ is *normal* if for every $a \in A$, $\{x \in P_\kappa(A) : a \in x\} \in F$, and if F is closed under diagonal intersections. A set $X \subset P_\kappa(A)$ is *F-positive* if its complement is not in F.

8.16. Let F be a normal κ-complete filter on $P_\kappa(A)$. If g is a function on an F-positive set such that $g(x) \in [x]^{<\omega}$ for all x, then g is constant on an F-positive set.

8.17. If F is a normal κ-complete filter on $P_\kappa(A)$ then F contains all closed unbounded sets.

[Use Lemma 8.26 and Exercise 8.16.]

8.18. If $\kappa > \omega_1$ then the set $\{x \in P_\kappa(A) : |x| \geq \aleph_1\}$ is closed unbounded.

Contrast this with the fact that for every $F : [A]^{<\omega} \to A$ there exists a countable x closed under A.

8.19. The set $\{x \in P_\kappa(\lambda) : x \cap \kappa \in \kappa\}$ is closed unbounded.

Historical Notes

The definition of stationary set is due to Bloch [1953], and the fundamental theorem (Theorem 8.7) was proved by Fodor [1956]. (A precursor of Fodor's Theorem appeared in Aleksandrov-Urysohn [1929].) The concept of stationary sets is implicit in Mahlo [1911].

Theorem 8.10 was proved by Solovay [1971] using the technique of saturated ideals.

Mahlo cardinals are named after P. Mahlo, who in 1911–1913 investigated what is now called weakly Mahlo cardinals. Theorems 8.12 and 8.13 are due to Silver [1975]. Silver's proof uses generic ultrapowers; the elementary proof given here is as in Baumgartner-Prikry [1976, 1977]. Lemma 8.16: Erdös, Hajnal, and Milner [1968].

Definition 8.18 and Theorem 8.20 are due to Jech [1984]. The generalization of closed unbounded and stationary sets (Definition 8.21 and Theorems 8.22 and 8.24) was given by Jech [1971b] and [1972/73]; Kueker [1972, 1977] also formulated these concepts for $\kappa = \omega_1$ and proved Theorem 8.28. Theorem 8.27 is due to Menas [1974/75].

Exercise 8.5: Friedman [1974].

Exercise 8.17: Carr [1982].

9. Combinatorial Set Theory

In this chapter we discuss topics in infinitary combinatorics such as trees and partition properties.

Partition Properties

Let us consider the following argument (the *pigeonhole principle*): If seven pigeons occupy three pigeonholes, then at least one pigenhole is occupied by three pigeons. More generally: If an infinite set is partitioned into finitely many pieces, then at least one piece is infinite.

Recall that a *partition* of a set S is a pairwise disjoint family $P = \{X_i : i \in I\}$ such that $\bigcup_{i\in I} X_i = S$. With the partition P we can associate a function $F : S \to I$ such that $F(x) = F(y)$ if and only if x and y are in the same $X \in P$. Conversely, any function $F : S \to I$ determines a partition of S. (We shall sometimes say that F is a partition of S.)

For any set A and any natural number $n > 0$,

$$[A]^n = \{X \subset A : |X| = n\} \tag{9.1}$$

is the set of all subsets of A that have exactly n elements. It is sometimes convenient, when A is a set of ordinals, to identify $[A]^n$ with the set of all sequences $\langle \alpha_1, \ldots, \alpha_n\rangle$ in A such that $\alpha_1 < \ldots < \alpha_n$. We shall consider partitions of sets $[A]^n$ for various infinite sets A and natural numbers n. Our starting point is the theorem of Ramsey dealing with finite partitions of $[\omega]^n$.

If $\{X_i : i \in I\}$ is a partition of $[A]^n$, then a set $H \subset A$ is *homogeneous* for the partition if for some i, $[H]^n$ is included in X_i; that is, if all the n-element subsets of H are in the same piece of the partition.

Theorem 9.1 (Ramsey). *Let n and k be natural numbers. Every partition $\{X_1, \ldots, X_k\}$ of $[\omega]^n$ into k pieces has an infinite homogeneous set.*

Equivalently, for every $F : [\omega]^n \to \{1, \ldots, k\}$ there exists an infinite $H \subset \omega$ such that F is constant on $[H]^n$.

Proof. By induction on n. If $n = 1$, the theorem is trivial, so we assume that it holds for n and prove for $n + 1$. Let F be a function from $[\omega]^{n+1}$ into

$\{1,\dots,k\}$. For each $a \in \omega$, let F_a be the function on $[\omega - \{a\}]^n$ defined as follows:

$$F_a(X) = F(\{a\} \cup X).$$

By the induction hypothesis, there exists for each $a \in \omega$ and each infinite $S \subset \omega$ an infinite set $H_a^S \subset S - \{a\}$ such that F_a is constant on $[H_a^S]^n$. We construct an infinite sequence $\langle a_i : i = 0, 1, 2, \dots\rangle$: We let $S_0 = \omega$ and $a_0 = 0$, and

$$S_{i+1} = H_{a_i}^{S_i}, \qquad a_{i+1} = \text{the least element of } S_{i+1} \text{ greater than } a_i.$$

It is clear that for each $i \in \omega$, the function F_{a_i} is constant on $[\{a_m : m > i\}]^n$; let $G(a_i)$ be its value. Now there is an infinite subset $H \subset \{a_i : i \in \omega\}$ such that G is constant on H. It follows that F is constant on $[H]^{n+1}$; this is because for $x_1 < \dots < x_{n+1}$ in H we have $F(\{x_1,\dots,x_{n+1}\}) = F_{x_1}(\{x_2,\dots,x_{n+1}\})$. □

The following lemma explains the terminology introduced in Chapter 7 where Ramsey ultrafilters were defined:

Lemma 9.2. *Let D be a nonprincipal ultrafilter on ω. D is Ramsey if and only if for all natural numbers n and k, every partition $F : [\omega]^n \to \{1,\dots,k\}$ has a homogeneous set $H \in D$.*

Proof. First assume that D has the partition property stated in the lemma. Let $\mathcal{A}$ be a partition of ω such that $A \notin D$ for all $A \in \mathcal{A}$; we shall find $X \in D$ such that $|X \cap A| \leq 1$ for all $A \in \mathcal{A}$. Let $F : [\omega]^2 \to \{0,1\}$ be as follows: $F(x,y) = 1$ if x and y are in different members of $\mathcal{A}$. If $H \in D$ is homogeneous for F, then clearly H has at most one element common with each $A \in \mathcal{A}$.

Now let us assume that D is a Ramsey ultrafilter. We shall first prove that D has the following property:

(9.2) if $X_0 \supset X_1 \supset X_2 \supset \dots$ are sets in D, then there is a sequence $a_0 < a_1 < a_2 < \dots$ such that $\{a_n\}_{n=0}^\infty \in D$, $a_0 \in X_0$ and $a_{n+1} \in X_{a_n}$ for all n.

Thus let $X_0 \supset X_1 \supset \dots$ be sets in D. Since D is a p-point, there exists $Y \in D$ such that each $Y - X_n$ is finite. Let us define a sequence $y_0 < y_1 < \dots$ in Y as follows:

$$\begin{aligned}
y_0 &= \text{the least } y_0 \in Y \text{ such that } \{y \in Y : y > y_0\} \subset X_0,\\
y_1 &= \text{the least } y_1 \in Y \text{ such that } y_1 > y_0 \text{ and } \{y \in Y : y > y_1\} \subset X_{y_0},\\
&\dots\\
y_n &= \text{the least } y_n \in Y \text{ such that } y_n > y_{n-1} \text{ and } \{y \in Y : y > y_n\} \subset X_{y_{n-1}}.
\end{aligned}$$

For each n, let $A_n = \{y \in Y : y_n < y \leq y_{n+1}\}$. Since D is Ramsey, there exists a set $\{z_n\}_{n=0}^\infty \in D$ such that $z_n \in A_n$ for all n.

We observe that for each n, $z_{n+2} \in X_{z_n}$: Since $z_{n+2} > y_{n+2}$, we have $z_{n+2} \in X_{y_{n+1}}$, and since $y_{n+1} \geq z_n$, we have $X_{y_{n+1}} \subset X_{z_n}$ and hence $z_{n+2} \in X_{z_n}$.

Thus if we let $a_n = z_{2n}$ and $b_n = z_{2n+1}$, for all n, then either $\{a_n\}_{n=0}^{\infty} \in D$ or $\{b_n\}_{n=0}^{\infty} \in D$; and in either case we get a sequence that satisfies (9.2).

Now we use the property (9.2) to prove the partition property; we proceed by induction on n and follow closely the proof of Ramsey's Theorem. Let F be a function from $[\omega]^{n+1}$ into $\{1, \ldots, k\}$. For each $a \in \omega$, let F_a be the function on $[\omega - \{a\}]^n$ defined by $F_a(x) = F(x \cup \{a\})$. By the induction hypothesis, there exists for each $a \in \omega$ a set $H_a \in D$ such that F_a is constant on $[H_a]^n$. There exists $X \in D$ such that the constant value of F_a is the same for all $a \in X$; say $F_a(x) = r$ for all $a \in X$ and all $x \in [H_a]^n$.

For each n, let $X_n = X \cap H_0 \cap H_1 \cap \ldots \cap H_n$. By (9.2) there exists a sequence $a_0 < a_1 < a_2 < \ldots$ such that $a_0 \in X_0$ and $a_{n+1} \in X_{a_n}$ for each n, and that $\{a_n\}_{n=0}^{\infty} \in D$. Let $H = \{a_n\}_{n=0}^{\infty}$. It is clear that for each $i \in \omega$, $a_i \in X$ and $\{a_m : m > i\} \subset H_{a_i}$. Hence $F_{a_i}(x) = r$ for all $x \in [\{a_m : m > i\}]^n$, and it follows that F is constant on $[H]^{n+1}$. □

To facilitate our investigation of generalizations of Ramsey's Theorem, we shall now introduce the *arrow notation*. Let κ and λ be infinite cardinal numbers, let n be a natural number and let m be a (finite or infinite) cardinal. The symbol

$$\kappa \to (\lambda)^n_m \tag{9.3}$$

(read: κ *arrows* λ) denotes the following *partition property*: Every partition of $[\kappa]^n$ into m pieces has a homogeneous set of size λ. In other words, every $F : [\kappa]^n \to m$ is constant on $[H]^n$ for some $H \subset \kappa$ such that $|H| = \lambda$. Using the arrow notation, Ramsey's Theorem is expressed as follows:

$$\aleph_0 \to (\aleph_0)^n_k \qquad (n, k \in \omega). \tag{9.4}$$

The subscript m in (9.3) is usually deleted when $m = 2$, and so

$$\kappa \to (\lambda)^n$$

is the same as $\kappa \to (\lambda)^n_2$.

The relation $\kappa \to (\lambda)^n_m$ remains true if κ is made larger or if λ or m are made smaller. A moment's reflection is sufficient to see that the relation also remains true when n is made smaller.

Obviously, the relation (9.3) makes sense only if $\kappa \geq \lambda$ and $\kappa > m$; if $m = \kappa$, then it is clearly false. Thus we always assume $2 \leq m < \kappa$ and $\lambda \leq \kappa$. If $n = 1$, then (9.3) holds just in case either $\kappa > \lambda$, or $\kappa = \lambda$ and $\operatorname{cf} \kappa > m$. We shall concentrate on the nontrivial case: $n \geq 2$.

We start with two negative partition relations.

Lemma 9.3. *For all* κ,

$$2^\kappa \not\to (\omega)^2_\kappa.$$

In other words, there is a partition of 2^κ *into* κ *pieces that does not have an infinite homogeneous set.*

Proof. In fact, we find a partition that has no homogeneous set of size 3. Let $S = \{0,1\}^\kappa$ and let $F : [S]^2 \to \kappa$ be defined by $F(\{f,g\}) =$ the least $\alpha < \kappa$ such that $f(\alpha) \neq g(\alpha)$. If f, g, h are distinct elements of S, it is impossible to have $F(\{f,g\}) = F(\{f,h\}) = F(\{g,h\})$. □

Lemma 9.4. *For every* κ,

$$2^\kappa \not\to (\kappa^+)^2.$$

(Thus the obvious generalization of Ramsey's Theorem, namely $\aleph_1 \to (\aleph_1)^2_2$, is false.)

To construct a partition of $[2^\kappa]^2$ that violates the partition property, let us consider the linearly ordered set $(P,<)$ where $P = \{0,1\}^\kappa$, and $f < g$ if and only if $f(\alpha) < g(\alpha)$ where α is the least α such that $f(\alpha) \neq g(\alpha)$ (the lexicographic ordering of P).

Lemma 9.5. *The lexicographically ordered set* $\{0,1\}^\kappa$ *has no increasing or decreasing* κ^+*-sequence.*

Proof. Assume that $W = \{f_\alpha : \alpha < \kappa^+\} \subset \{0,1\}^\kappa$ is such that $f_\alpha < f_\beta$ whenever $\alpha < \beta$ (the decreasing case is similar). Let $\gamma \leq \kappa$ be the least γ such that the set $\{f_\alpha \restriction \gamma : \alpha < \kappa^+\}$ has size κ^+, and let $Z \subset W$ be such that $|Z| = \kappa^+$ and $f \restriction \gamma \neq g \restriction \gamma$ for $f, g \in Z$. We may as well assume that $Z = W$, so let us do so.

For each $\alpha < \kappa^+$, let ξ_α be such that $f_\alpha \restriction \xi_\alpha = f_{\alpha+1} \restriction \xi_\alpha$ and $f_\alpha(\xi_\alpha) = 0$, $f_{\alpha+1}(\xi_\alpha) = 1$; clearly $\xi_\alpha < \gamma$. Hence there exists $\xi < \gamma$ such that $\xi = \xi_\alpha$ for κ^+ elements f_α of W. However, if $\xi = \xi_\alpha = \xi_\beta$ and $f_\alpha \restriction \xi = f_\beta \restriction \xi$, then $f_\beta < f_{\alpha+1}$ and $f_\alpha < f_{\beta+1}$; hence $f_\alpha = f_\beta$. Thus the set $\{f_\alpha \restriction \xi : \alpha < \kappa^+\}$ has size κ^+, contrary to the minimality assumption on γ. □

Proof of Lemma 9.4. Let $2^\kappa = \lambda$ and let $\{f_\alpha : \alpha < \lambda\}$ be an enumeration of the set $P = \{0,1\}^\kappa$. Let $\prec$ be a linear ordering of λ induced by the lexicographic ordering of P: $\alpha \prec \beta$ if $f_\alpha < f_\beta$.

Now we define a partition $F : [\lambda]^2 \to \{0,1\}$ by letting $F(\{\alpha,\beta\}) = 1$ when the ordering $\prec$ of $\{\alpha,\beta\}$ agrees with the natural ordering; and letting $F(\{\alpha,\beta\}) = 0$ otherwise. If $H \subset \lambda$ is a homogeneous set of order-type κ^+, then $\{f_\alpha : \alpha \in H\}$ constitutes an increasing or decreasing κ^+-sequence in $(P,<)$; a contradiction. □

By Lemma 9.4, the relation $\kappa \to (\aleph_1)^2$ is false if $\kappa \leq 2^{\aleph_0}$. On the other hand, if $\kappa > 2^{\aleph_0}$, then $\kappa \to (\aleph_1)^2$ is true, as follows from this more general theorem:

Theorem 9.6 (Erdős-Rado).

$$\beth_n^+ \to (\aleph_1)^{n+1}_{\aleph_0}.$$

In particular, $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$.

Proof. We shall first prove the case $n = 1$ since the induction step parallels closely this case. Thus let $\kappa = (2^{\aleph_0})^+$ and let $F : [\kappa]^2 \to \omega$ be a partition of $[\kappa]^2$ into $\aleph_0$ pieces. We want to find a homogeneous $H \subset \kappa$ of size $\aleph_1$.

For each $a \in \kappa$, let F_a be a function on $\kappa - \{a\}$ defined by $F_a(x) = F(\{a, x\})$. We shall first prove the following claim: There exists a set $A \subset \kappa$ such that $|A| = 2^{\aleph_0}$ and such that for every countable $C \subset A$ and every $u \in \kappa - C$ there exists $v \in A - C$ such that F_v agrees with F_u on C.

To prove the claim, we construct an ω_1-sequence $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$, $\alpha < \omega_1$, of subsets of κ, each of size $2^{\aleph_0}$, as follows: Let A_0 be arbitrary, and for each limit α, let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Given A_α, there exists a set $A_{\alpha+1} \supset A_\alpha$ of size $2^{\aleph_0}$ such that for each countable $C \subset A_\alpha$ and every $u \in \kappa - C$ there exists $v \in A_{\alpha+1} - C$ such that F_v agrees with F_u on C (because the number of such functions is $\leq 2^{\aleph_0}$). Then we let $A = \bigcup_{\alpha<\omega_1} A_\alpha$, and clearly A has the required property.

Next we choose some $a \in \kappa - A$, and construct a sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in A as follows: Let x_0 be arbitrary, and given $\{x_\beta : \beta < \alpha\} = C$, let x_α be some $v \in A - C$ such that F_v agrees with F_a on C. Let $X = \{x_\alpha : \alpha < \omega_1\}$.

Now we consider the function $G : X \to \omega$ defined by $G(x) = F_a(x)$. It is clear that if $\alpha < \beta$, then $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha) = F_a(x_\alpha) = G(x_\alpha)$. Since the range of G is countable, there exists $H \subset X$ of size $\aleph_1$ such that G is constant on H. It follows that F is constant on $[H]^2$.

Thus we have proved the theorem for $n = 1$. The general case is proved by induction. Let us assume that $\beth_{n-1}^+ \to (\aleph_1)^n_{\aleph_0}$ and let $F : [\kappa]^{n+1} \to \omega$, where $\kappa = \beth_n^+$. For each $a \in \kappa$, let $F_a : [\kappa - \{a\}]^n \to \omega$ be defined by $F_a(x) = F(x \cup \{a\})$. As in the case $n = 1$, there exists a set $A \subset \kappa$ of size $\beth_n$ such that for every $C \subset A$ of size $|C| \leq \beth_{n-1}$ and every $u \in \kappa - C$ there exists $v \in A - C$ such that F_v agrees with F_u on $[C]^n$.

Next we choose $a \in \kappa - A$ and construct a set $X = \{x_\alpha : \alpha < \beth_{n-1}^+\} \subset A$ such that for each α, F_{x_α} agrees with F_a on $[\{x_\beta : \beta < \alpha\}]^n$.

Then we consider $G : [X]^n \to \omega$ where $G(x) = F_a(x)$. As before, if $\alpha_1 < \ldots < \alpha_{n+1}$, then $F(\{x_{\alpha_1}, \ldots, x_{\alpha_{n+1}}\}) = G(\{x_{\alpha_1}, \ldots, x_{\alpha_n}\})$. By the induction hypothesis, there exists $H \subset X$ of size $\aleph_1$ such that G is constant on $[H]^n$. It follows that F is constant on $[H]^{n+1}$. □

Erdős and Rado proved that for each n, the partition property $\beth_n^+ \to (\aleph_1)^{n+1}_{\aleph_0}$ is best possible. The property also generalizes easily to larger cardinals.

A natural generalization of the partition property (9.3) is when we allow λ to be a limit ordinal, not just a cardinal. Let κ, n and m be as in (9.3) and

let $\alpha > 0$ be a limit ordinal. The symbol

$$\kappa \to (\alpha)^n_m \tag{9.5}$$

stands for: For every $F : [\kappa]^n \to m$ there exists an $H \subset \kappa$ of order-type α such that F is constant on $[H]^n$.

There are various results about the partition relation (9.5). For instance, Baumgartner and Hajnal proved in [1973] that $\aleph_1 \to (\alpha)^2$ for all $\alpha < \omega_1$. The analogous case for $\aleph_2$ is different: If $2^{\aleph_0} = \aleph_1$, then $\aleph_2 \to (\omega_1)^2$ (by Erdős-Rado), but it is consistent (with $2^{\aleph_0} = \aleph_1$) that $\aleph_2 \not\to (\omega_1 + \omega)^2$.

Among other generalizations of (9.3), we mention the following:

$$\kappa \to (\alpha, \beta)^n \tag{9.6}$$

means that for every $F : [\kappa]^n \to \{0, 1\}$, either there exists an $H_1 \subset \kappa$ of order-type α such that $F = 0$ on $[H_1]^n$ or there exists and $H_2 \subset \kappa$ of order-type β such that $F = 1$ on $[H_2]^n$.

Theorem 9.7 (Dushnik-Miller). *For every infinite cardinal κ,*

$$\kappa \to (\kappa, \omega)^2.$$

Proof. Let $\{A, B\}$ be a partition of $[\kappa]^2$. For every $x \in \kappa$, let $B_x = \{y \in \kappa : x < y \text{ and } \{x, y\} \in B\}$. First let us assume that in every set $X \subset \kappa$ of cardinality κ there exists an $x \in X$ such that $|B_x \cap X| = \kappa$. In this case, we construct an infinite H with $[H]^2 \subset B$ as follows:

Let $X_0 = \kappa$ and $x_0 \in X_0$ such that $|B_{x_0} \cap X_0| = \kappa$. For each n, let $X_{n+1} = B_{x_n} \cap X_n$ and let $x_{n+1} \in X_{n+1}$ be such that $x_{n+1} > x_n$ and $|B_{x_{n+1}} \cap X_{n+1}| = \kappa$. Then let $H = \{x_n\}_{n=0}^{\infty}$; it is clear that $[H]^2 \subset B$.

Thus let us assume, for the rest of the proof, that there exists a set $S \subset \kappa$ of cardinality κ such that

$$\text{for every } x \in S,\ |B_x \cap S| < \kappa. \tag{9.7}$$

If κ is regular, then we construct (by induction) an increasing κ-sequence $\langle x_\alpha : \alpha < \kappa \rangle$ in S such that $\{x_\alpha, x_\beta\} \in A$ for all $\alpha < \beta$; this is possible by (9.7).

Thus let us assume that κ is singular, let $\lambda = \operatorname{cf} \kappa$ and let $\langle \kappa_\xi : \xi < \lambda \rangle$ be an increasing sequence of regular cardinals $> \lambda$ with limit κ. Furthermore, we assume that there is no infinite H with $[H]^2 \subset B$, and that $\kappa_\xi \to (\kappa_\xi, \omega)^2$ holds for every $\xi < \lambda$. We shall find a set $H \subset \kappa$ of cardinality κ such that $[H]^2 \subset A$.

Let $\{S_\xi : \xi < \lambda\}$ be a partition of S into disjoint sets such that $S_\xi = \kappa_\xi$. It follows from our assumptions that there exist sets $K_\xi \subset S_\xi$, $|K_\xi| = \kappa_\xi$, such that $[K_\xi]^2 \subset A$.

For every $x \in K_\xi$ there exists, by (9.7), some $\alpha < \lambda$ such that $|B_x \cap S| < \kappa_\alpha$; since $\lambda < \kappa_\xi$, there exists an $\alpha(\xi)$ such that the set $Z_\xi = \{x \in K_\xi : |B_x \cap S| < \kappa_{\alpha(\xi)}\}$ has cardinality κ_ξ.

Let $\langle \xi_\nu : \nu < \lambda \rangle$ be an increasing sequence of ordinals $< \lambda$ such that if $\nu_1 < \nu_2$ then $\alpha(\xi_{\nu_1}) < \xi_{\nu_2}$. We define, by induction on ν,

$$H_\nu = Z_{\xi(\nu)} - \bigcup\{B_x : x \in \bigcup_{\eta<\nu} Z_{\xi(\eta)}\}.$$

Clearly, $|H_\nu| = \kappa_{\xi(\nu)}$, and $[H_\nu]^2 \subset A$.

Finally, we let $H = \bigcup_{\nu<\lambda} H_\nu$. It follows from the construction of H that $[H]^2 \subset A$. □

Weakly Compact Cardinals

In the positive results given by the Erdős-Rado Theorem, the size of the homogeneous set is smaller than the size of the set being partitioned. A natural question arises, whether the relation $\kappa \to (\kappa)^2$ can hold for cardinals other than $\kappa = \omega$.

Definition 9.8. A cardinal κ is *weakly compact* if it is uncountable and satisfies the partition property $\kappa \to (\kappa)^2$.

The reason for the name "weakly compact" is that these cardinals satisfy a certain compactness theorem for infinitary languages; we shall investigate weakly compact cardinals further in Part II.

Lemma 9.9. *Every weakly compact cardinal is inaccessible.*

Proof. Let κ be a weakly compact cardinal. To show that κ is regular, let us assume that κ is the disjoint union $\bigcup\{A_\gamma : \gamma < \lambda\}$ such that $\lambda < \kappa$ and $|A_\gamma| < \kappa$ for each $\gamma < \lambda$. We define a partition $F : [\kappa]^2 \to \{0, 1\}$ as follows: $F(\{\alpha, \beta\}) = 0$ just in case α and β are in the same A_γ. Obviously, this partition does not have a homogeneous set $H \subset \kappa$ of size κ.

That κ is a strong limit cardinal follows from Lemma 9.4: If $\kappa \leq 2^\lambda$ for some $\lambda < \kappa$, then because $2^\lambda \not\to (\lambda^+)^2$, we have $\kappa \not\to (\lambda^+)^2$ and hence $\kappa \not\to (\kappa)^2$. □

We shall prove in Chapter 17 that every weakly compact cardinal κ is the κth inaccessible cardinal.

Trees

Many problems in combinatorial set theory can be formulated as problems about trees.

In this chapter we discuss Suslin's Problem as well as the use of trees in partition calculus and large cardinals.

Definition 9.10. A *tree* is a partially ordered set $(T, <)$ with the property that for each $x \in T$, the set $\{y : y < x\}$ of all predecessors of x is well-ordered by $<$.

The αth *level* of T consists of all $x \in T$ such that $\{y : y < x\}$ has order-type α. The height of T is the least α such that the αth level of T is empty; in other words, it is the height of the well-founded relation $<$:

$$\begin{aligned} o(x) &= \text{the order-type of } \{y : y < x\}, \\ \alpha\text{th level} &= \{x : o(x) = \alpha\}, \\ \text{height}(T) &= \sup\{o(x) + 1 : x \in T\}. \end{aligned} \tag{9.8}$$

A *branch* in T is a maximal linearly ordered subset of T. The *length* of a branch b is the order-type of b. An α-*branch* is a branch of length α.

We shall now turn our attention to Suslin's Problem introduced in Chapter 4. In Lemma 9.14 below we show that the problem can be restated as a question about the existence of certain trees of height ω_1.

Suslin's Problem asks whether the real line is the only complete dense unbounded linearly ordered set that satisfies the countable chain condition. An equivalent question is whether every dense linear ordering that satisfies the countable chain condition is *separable*, i.e., has a countable dense subset.

Definition 9.11. A *Suslin line* is a dense linearly ordered set that satisfies the countable chain condition and is not separable.

Thus Suslin's Problem asks whether a Suslin line exists. We shall show that the existence of a Suslin line is equivalent to the existence of a Suslin tree.

Let T be a tree. An *antichain* in T is a set $A \subset T$ such that any two distinct elements x, y of A are incomparable, i.e., neither $x < y$ nor $y < x$.

Definition 9.12. A tree T is a *Suslin tree* if

(i) the height of T is ω_1;
(ii) every branch in T is at most countable;
(iii) every antichain in T is at most countable.

For the formulation of Suslin's Problem in terms of trees it is useful to consider Suslin trees that are called *normal*.

Let α be an ordinal number, $\alpha \le \omega_1$. A *normal* α-*tree* is a tree T with the following properties:

(9.9)
(i) $\text{height}(T) = \alpha$;
(ii) T has a unique least point (the root);
(iii) each level of T is at most countable;
(iv) if x is not maximal in T, then there are infinitely many $y > x$ at the next level (immediate successors of x);

(v) for each $x \in T$ there is some $y > x$ at each higher level less than α;
(vi) if $\beta < \alpha$ is a limit ordinal and x, y are both at level β and if $\{z : z < x\} = \{z : z < y\}$, then $x = y$.

See Exercise 9.6 for a representation of normal trees.

Lemma 9.13. *If there exists a Suslin tree then there exists a normal Suslin tree.*

Proof. Let T be a Suslin tree. T has height ω_1, and each level of T is countable. We first discard all points $x \in T$ such that $T_x = \{y \in T : y \geq x\}$ is at most countable, and let $T_1 = \{x \in T : T_x \text{ is uncountable}\}$. Note that if $x \in T_1$ and $\alpha > o(x)$, then $|T_y| = \aleph_1$ for some $y > x$ at level α. Hence T_1 satisfies condition (v). Next, we satisfy property (vi): For every chain $C = \{z : z < y\}$ in T_1 of limit length we add an extra node a_C and stipulate that $z < a_C$ for all $z \in C$, and $a_C < x$ for every x such that $x > z$ for all $z \in C$. The resulting tree T_2 satisfies (iii), (v) and (vi). For each $x \in T_2$ there are uncountably many branching points $z > x$, i.e., points that have at least two immediate successors (because there is no uncountable chain and T_2 satisfies (v)). The tree $T_3 = \{\text{the branching points of } T_2\}$ satisfies (iii), (v) and (vi) and each $x \in T_3$ is a branching point. To get property (iv), let T_4 consists of all $z \in T_3$ at limit levels of T_3. The tree T_4 satisfies (i), (iii), (iv), and (v); and then $T_5 \subset T_4$ satisfying (ii) as well is easily obtained. □

Lemma 9.14. *There exists a Suslin line if and only if there exists a Suslin tree.*

Proof. (a) Let S be a Suslin line. We shall construct a Suslin tree. The tree will consist of closed (nondegenerate) intervals on the Suslin line S. The partial ordering of T is by inverse inclusion: If $I, J \in T$, then $I \leq J$ if and only if $I \supset J$.

The collection T of intervals is constructed by induction on $\alpha < \omega_1$. We let $I_0 = [a_0, b_0]$ be arbitrary (such that $a_0 < b_0$). Having constructed I_β, $\beta < \alpha$, we consider the countable set $C = \{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}$ of endpoints of the intervals I_β, $\beta < \alpha$. Since S is a Suslin line, C is not dense in S and so there exists an interval $[a, b]$ disjoint from C; we pick some such $[a_\alpha, b_\alpha] = I_\alpha$. The set $T = \{I_\alpha : \alpha < \omega_1\}$ is uncountable and partially ordered by $\supset$. If $\alpha < \beta$, then either $I_\alpha \supset I_\beta$ or I_α is disjoint from I_β. It follows that for each α, $\{I \in T : I \supset I_\alpha\}$ is well-ordered by $\supset$ and thus T is a tree.

We shall show that T has no uncountable branches and no uncountable antichains. Then it is immediate that the height of T is at most ω_1; and since every level is an antichain and T is uncountable, we have $\text{height}(T) = \omega_1$.

If $I, J \in T$ are incomparable, then they are disjoint intervals of S; and since S satisfies the countable chain condition, every antichain in T is at most countable. To show that T has no uncountable branch, we note first that if

b is a branch of length ω_1, then the left endpoints of the intervals $I \in B$ form an increasing sequence $\{x_\alpha : \alpha < \omega_1\}$ of points of S. It is clear that the intervals $(x_\alpha, x_{\alpha+1})$, $\alpha < \omega_1$, form a disjoint uncountable collection of open intervals in S, contrary to the assumption that S satisfies the countable chain condition.

(b) Let T be a normal Suslin tree. The line S will consist of all branches of T (which are all countable). Each $x \in T$ has countably many immediate successors, and we order these successors as rational numbers. Then we order the elements of S lexicographically: If α is the least level where two branches $a, b \in S$ differ, then α is a successor ordinal and the points $a_\alpha \in a$ and $b_\alpha \in b$ are both successors of the same point at level $\alpha - 1$; we let $a < b$ or $b < a$ according to whether $a_\alpha < b_\alpha$ or $b_\alpha < a_\alpha$.

It is easy to see that S is linearly ordered and dense. If (a, b) is an open interval in S, then one can find $x \in T$ such that $I_x \subset (a, b)$, where I_x is the interval $I_x = \{c \in S : x \in c\}$. And if I_x and I_y are disjoint, then x and y are incomparable points of T. Thus every disjoint collection of open intervals of S must be at most countable, and so S satisfies the countable chain condition.

The line S is not separable: If C is a countable set of branches of T, let α be a countable ordinal bigger than the length of any branches $b \in C$. Then if x is any point at level α, the interval I_x does not contain any $b \in C$, and so C is not dense in S. □

Lemma 9.14 reduces Suslin's Problem to a purely combinatorial problem. In Part II we shall return to it and show that the problem is independent of the axioms of set theory.

We now turn our attention to the following problem, related to Suslin trees.

Definition 9.15. An *Aronszajn tree* is a tree of height ω_1 all of whose levels are at most countable and which has no uncountable branches.

Theorem 9.16 (Aronszajn). *There exists an Aronszajn tree.*

Proof. We construct a tree T whose elements are some bounded increasing transfinite sequences of rational numbers. If $x, y \in T$ are two such sequences, then we let $x \leq y$ just in case y extends x, i.e., $x \subset y$. Also, if $y \in T$ and x is an initial segment of y, then we let $x \in T$; thus the αth level of T will consist of all those $x \in T$ whose length is α.

It is clear that an uncountable branch in T would yield an increasing ω_1-sequence of rational numbers, which is impossible. Thus T will be an Aronszajn tree, provided we arrange that T has $\aleph_1$ levels, all of them at most countable. We construct T by induction on levels. For each $\alpha < \omega_1$ we construct a set U_α of increasing α-sequences of rationals; U_α will be the αth level of T. We construct the U_α so that for each α, $|U_\alpha| \leq \aleph_0$, and that:

(9.10) For each $\beta < \alpha$, each $x \in U_\beta$ and each $q > \sup x$ there is $y \in U_\alpha$ such that $x \subset y$ and $q \geq \sup y$.

Condition (9.10) enables us to continue the construction at limit steps.

To start, we let $U_0 = \{\emptyset\}$. The successor steps of the construction are also fairly easy. Given U_α, we let $U_{\alpha+1}$ be the set of all extensions $x^\frown r$ of sequences in U_α such that $r > \sup x$. It is clear that since U_α satisfies condition (9.10), $U_{\alpha+1}$ satisfies it also (for $\alpha + 1$), and it is equally clear that $U_{\alpha+1}$ is at most countable.

Thus let α be a limit ordinal ($\alpha < \omega_1$) and assume that we have constructed all levels U_γ, $\gamma < \alpha$, of T below α, and that all the U_γ satisfy (9.10); we shall construct U_α. The points $x \in T$ below level α form a (normal) tree T_α of length α. We claim that T_α has the following property:

(9.11) For each $x \in T_\alpha$ and each $q > \sup x$ there is an increasing α-sequence of rationals y such that $x \subset y$ and $q \geq \sup y$ and that $y{\restriction}\beta \in T_\alpha$ for all $\beta < \alpha$.

The last condition means that $\{y{\restriction}\beta : \beta < \alpha\}$ is a branch in T_α. To prove the claim, we let α_n, $n = 0, 1, \ldots$, be an increasing sequence of ordinals such that $x \in U_{\alpha_0}$ and $\lim_n \alpha_n = \alpha$, and let $\{q_n\}_{n=0}^\infty$ be an increasing sequence of rational numbers such that $q_0 > \sup x$ and $\lim_n q_n \leq q$. Using repeatedly condition (9.10), for all α_n ($n = 0, 1, \ldots$), we can construct a sequence $y_0 \subset y_1 \subset \ldots \subset y_n \ldots$ such that $y_0 = x$, $y_n \in U_{\alpha_n}$, and $\sup y_n \leq q_n$ for each n. Then we let $y = \bigcup_{n=0}^\infty y_n$; clearly, y satisfies (9.11).

Now we construct U_α as follows: For each $x \in T_\alpha$ and each rational number q such that $q > \sup x$, we choose a branch y in T_α that satisfies (9.11), and let U_α consists of all these $y : \alpha \to \boldsymbol{Q}$. The set U_α so constructed is countable and satisfies condition (9.10).

Then $T = \bigcup_{\alpha<\omega_1} U_\alpha$ is an Aronszajn tree. □

The Aronszajn tree constructed in Theorem 9.16 has the property that there exists a function $f : T \to \boldsymbol{R}$ such that $f(x) < f(y)$ whenever $x < y$ (Exercise 9.8). With a little more care, one can construct T so that there is a function $f : T \to \boldsymbol{Q}$ such that $f(x) < f(y)$ if $x < y$. Such trees are called *special Aronszajn trees*. In Part II we'll show that it is consistent that all Aronszajn trees are special.

Almost Disjoint Sets and Functions

In combinatorial set theory one often consider families of sets that are as much different as possible; a typical example is an almost disjoint family of infinite sets—any two intersect in a finite set. Here we present a sample of results and problems of this kind.

Definition 9.17. A collection of finite sets Z is called a Δ-*system* if there exists a finite set S such that $X \cap Y = S$ for any two distinct sets $X, Y \in Z$.

The following theorem is often referred to as the Δ-*Lemma*:

Theorem 9.18 (Shanin). *Let W be an uncountable collection of finite sets. Then there exists an uncountable $Z \subset W$ that is a Δ-system.*

Proof. Since W is uncountable, it is clear that uncountably many $X \in W$ have the same size; thus we may assume that for some n, $|X| = n$ for all $X \in W$. We prove the theorem by induction on n. If $n = 1$, the theorem is trivial. Thus assume that the theorem holds for n, and let W be such that $|X| = n + 1$ for all $X \in W$.

If there is some element a that belongs to uncountably many $X \in W$, we apply the induction hypothesis to the collection $\{X - \{a\} : X \in W$ and $a \in X\}$, and obtain $Z \subset W$ with the required properties.

Otherwise, each a belongs to at most countably many $X \in W$, and we construct a disjoint collection $Z = \{X_\alpha : \alpha < \omega_1\}$ as follows, by induction on α. Given X_ξ, $\xi < \alpha$, we find $X = X_\alpha \in W$ that is disjoint from all X_ξ, $\xi < \alpha$. □

For an alternative proof, using Fodor's Theorem, see Exercise 9.10. Theorem 9.18 generalizes to greater cardinals, under the assumption of GCH:

Theorem 9.19. *Assume $\kappa^{<\kappa} = \kappa$. Let W be a collection of sets of cardinality less than κ such that $|W| = \kappa^+$. Then there exist a collection $Z \subset W$ such that $|Z| = \kappa^+$ and a set A such that $X \cap Y = A$ for any two distinct elements $X, Y \in Z$.* □

Definition 9.20. If X and Y are infinite subsets of ω then X and Y are *almost disjoint* if $X \cap Y$ is finite.

Let κ be a regular cardinal. If $X \cap Y$ are subsets of κ of cardinality κ then X and Y are *almost disjoint* if $|X \cap Y| < \kappa$.

An *almost disjoint family* of sets is a family of pairwise almost disjoint sets.

Lemma 9.21. *There exists an almost disjoint family of $2^{\aleph_0}$ subsets of ω.*

Proof. Let S be the set of all finite 0–1 sequences: $S = \bigcup_{n=0}^{\infty} \{0,1\}^n$. For every $f : \omega \to \{0,1\}$, let $A_f \subset S$ be the set $A_f = \{s \in S : s \subset f\} = \{f \restriction n : n \in \omega\}$. Clearly, $A_f \cap A_g$ is finite if $f \neq g$; thus $\{A_f : f \in \{0,1\}^\omega\}$ is a family of $2^{\aleph_0}$ almost disjoint subsets of the countable set S, and the lemma follows. □

A generalization from ω to arbitrary regular κ is not provable in ZFC (although under GCH the generalization is straightforward; see Exercise 9.11). Without assuming the GCH, the best one can do is to find an almost disjoint family of κ^+ subsets of κ; this follows from Lemma 9.23 below.

Definition 9.22. Let κ be a regular cardinal. Two functions f and g on κ are *almost disjoint* if $|\{\alpha : f(\alpha) = g(\alpha)\}| < \kappa$.

Lemma 9.23. *For every regular cardinal κ, there exists an almost disjoint family of κ^+ functions from κ to κ.*

Proof. It suffices to show that given κ almost disjoint functions $\{f_\nu : \nu < \kappa\}$, then there exists $f : \kappa \to \kappa$ almost disjoint from all f_ν, $\nu < \kappa$; this we do by diagonalization: Let $f(\alpha) \neq f_\nu(\alpha)$ for all $\nu < \alpha$. □

Let us consider the special case when $\kappa = \omega_1$.

Definition 9.24. A tree $(T, <)$ is a *Kurepa tree* if:

(i) height$(T) = \omega_1$;
(ii) each level of T is at most countable;
(iii) T has at least $\aleph_2$ uncountable branches.

If T is a Kurepa tree, then the family of all ω_1-branches is an almost disjoint family of uncountable subsets of T. In fact, since the levels of T are countable, we can identify the ω_1-branches with the functions from ω_1 into ω and get the following result: *There exists an almost disjoint family of $\aleph_2$ functions $f : \omega_1 \to \omega$.*

Lemma 9.25. *A Kurepa tree exists if and only if there exists a family $\mathcal{F}$ of subsets of ω_1 such that*:

(9.12) (i) $|\mathcal{F}| \geq \aleph_2$;
(ii) *for each* $\alpha < \omega_1$, $|\{X \cap \alpha : X \in \mathcal{F}\}| \leq \aleph_0$.

Proof. (a) Let $(T, <_T)$ be a Kurepa tree. Since T has size $\aleph_1$, we may assume that $T = \omega_1$, and moreover that $\alpha < \beta$ whenever $\alpha <_T \beta$. If we let $\mathcal{F}$ be the family of all ω_1-branches of T, then $\mathcal{F}$ satisfies (9.12).

(b) Let $\mathcal{F}$ be a family of subsets of ω_1 such that (9.12) holds. For each $X \in \mathcal{F}$, let f_X be the functions on ω_1 defined by

$$f_X(\alpha) = X \cap \alpha \qquad (\alpha < \omega_1).$$

For each $\alpha < \omega_1$, let $U_\alpha = \{f_X \restriction \alpha : X \in \mathcal{F}\}$ and let $T = \bigcup_{\alpha<\omega_1} U_\alpha$. Then $(T, \subset)$ is a tree, the U_α are the levels of T and the functions f_X correspond to branches of T. By (9.12)(ii), every U_α is countable, and it follows that T is a Kurepa tree. □

The existence of a Kurepa tree is independent of the axioms of set theory. In fact, the nonexistence of Kurepa trees is equiconsistent with an inaccessible cardinal.

The Tree Property and Weakly Compact Cardinals

Generalizing the concept of Aronszajn tree to cardinals $> \omega_1$ we say that a regular uncountable cardinal κ has the *tree property* if every tree of height κ whose levels have cardinality $< \kappa$ has a branch of cardinality κ.

Lemma 9.26.

(i) *If κ is weakly compact, then κ has the tree property.*
(ii) *If κ is inaccessible and has the tree property, then κ is weakly compact, and in fact $\kappa \to (\kappa)^2_m$ for every $m < \kappa$.*

Proof. (i) Let κ be weakly compact and let $(T, <_T)$ be a tree of height κ such that each level of T has size $< \kappa$. Since κ is inaccessible, $|T| = \kappa$ and we may assume that $T = \kappa$. We extend the partial ordering $<_T$ of κ to a linear ordering $\prec$: If $\alpha <_T \beta$, then we let $\alpha \prec \beta$; if α and β are incomparable and if ξ is the first level where the predecessors α_ξ, β_ξ of α and β are distinct, we let $\alpha \prec \beta$ if and only if $\alpha_\xi < \beta_\xi$.

Let $F : [\kappa]^2 \to \{0, 1\}$ be the partition defined by $F(\{\alpha, \beta\}) = 1$ if and only if $\prec$ agrees with $<$ on $\{\alpha, \beta\}$. By weak compactness, let $H \subset \kappa$ be homogeneous for F, $|H| = \kappa$.

We now consider the set $B \subset \kappa$ of all $x \in \kappa$ such that $\{\alpha \in H : x <_T \alpha\}$ has size κ. Since every level has size $< \kappa$, it is clear that at each level there is at least one x in B. Thus if we show that any two elements of B are $<_T$-comparable, we shall have proved that B is a branch in T of size κ.

Thus assume that x, y are incomparable elements of B; let $x \prec y$. Since both x and y have κ successors in H, there exist $\alpha < \beta < \gamma$ in H such that $x <_T \alpha$, $y <_T \beta$, and $x <_T \gamma$. By the definition of $\prec$, we have $\alpha \prec \beta$ and $\gamma \prec \beta$. Thus $F(\{\alpha, \beta\}) = 1$ and $F(\{\gamma, \beta\}) = 0$, contrary to the homogeneity of H.

(ii) Let κ be an inaccessible cardinal with the tree property, and let $F : [\kappa]^2 \to I$ be a partition such that $|I| < \kappa$. We shall find a homogeneous $H \subset \kappa$ of size κ.

We construct a tree $(T, \subset)$ whose elements are some functions $t : \gamma \to I$, $\gamma < \kappa$. We construct T by induction: At step $\alpha < \kappa$, we put into T one more element t, calling it t_α. Let $t_0 = \emptyset$. Having constructed $t_0, \ldots, t_\beta, \ldots, \beta < \alpha$, let us construct t_α as follows, by induction on ξ. Having constructed $t_\alpha{\restriction}\xi$, we look first whether $t_\alpha{\restriction}\xi$ is among the t_β, $\beta < \alpha$ (note that for $\xi = 0$ we have $t_\alpha{\restriction}0 = t_0$). If not, then we consider t_α constructed: $t_\alpha = t_\alpha{\restriction}\xi$. If $t_\alpha{\restriction}\xi = t_\beta$ for some $\beta < \alpha$, then we let $t_\alpha(\xi) = i$ where $i = F(\{\beta, \alpha\})$.

$(T, \subset)$ is a tree of size κ; and since κ is inaccessible, each level of T has size $< \kappa$ and the height of T is κ. It follows from the construction that if $t_\beta \subset t_\alpha$, then $\beta < \alpha$ and $F(\{\beta, \alpha\}) = t_\alpha(\text{length}(t_\beta))$. By the assumption, T has a branch B of size κ. If we now let, for each $i \in I$,

$$H_i = \{\alpha : t_\alpha \in B \text{ and } t_\alpha^\frown i \in B\}, \tag{9.13}$$

then each H_i is homogeneous for the partition F, and at least one H_i has size κ. □

It should be mentioned that an argument similar to the one above, only more complicated, shows that if κ is inaccessible and has the tree property, then $\kappa \to (\kappa)^n_m$ for all $n \in \omega$, $m < \kappa$.

Ramsey Cardinals

Let us consider one more generalization of Ramsey's Theorem. Let κ be an infinite cardinal, let α be an infinite limit ordinal, $\alpha \leq \kappa$, and let m be a cardinal, $2 \leq m < \kappa$. The symbol

$$\kappa \to (\alpha)^{<\omega}_m \tag{9.14}$$

denotes the property that for every partition F of the set $[\kappa]^{<\omega} = \bigcup_{n=0}^{\infty} [\kappa]^n$ into m pieces, there exists a set $H \subset \kappa$ of order-type α such that for each $n \in \omega$, F is constant on $[H]^n$. (Again, the subscript m is deleted when $m = 2$.)

It is not difficult to see that the partition property $\omega \to (\omega)^{<\omega}$ is false (see Exercise 9.13).

A cardinal κ is a *Ramsey cardinal* if $\kappa \to (\kappa)^{<\omega}$. Clearly, every Ramsey cardinal is weakly compact. We shall investigate Ramsey cardinals and property (9.14) in general in Part II.

Exercises

9.1. (i) Every infinite partially ordered set either has an infinite chain or has an infinite set of mutually incomparable elements.
(ii) Every infinite linearly ordered set either has an infinite increasing sequence of elements or has an infinite decreasing sequence of elements.
[Use Ramsey's Theorem.]

For each κ, let $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

9.2. For every κ, $(\exp_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa$. In particular, we have $(2^\kappa)^+ \to (\kappa^+)^2$.

9.3. $\omega_1 \to (\omega_1, \omega + 1)^2$.
[Let $\{A, B\}$ be a partition of $[\omega_1]^2$. For every limit ordinal α let K_α be a maximal subset of α such that $[K_\alpha \cup \{\alpha\}]^2 \subset B$. If K_α is finite for each α, use Fodor's Theorem to find a stationary set S such that all K_α, $\alpha \in S$, are the same. Then $[S]^2 \subset A$.]

If A is an infinite set of ordinals and α an ordinal, let $[A]^\alpha$ denote the set of all increasing α-sequences in A. The symbol

$$\kappa \to (\lambda)^\alpha$$

stands for: For every partition $F : [\kappa]^\alpha \to \{0, 1\}$ of $[\kappa]^\alpha$ into two pieces, there exists a set H of order-type λ such that F is constant on $[H]^\alpha$.

9.4. For all infinite cardinals κ, $\kappa \not\to (\omega)^\omega$.

[For $s, t \in [\kappa]^\omega$ let $s \equiv t$ if and only if $\{n : s(n) \neq t(n)\}$ is finite. Pick a representative in each equivalence class. Let $F(s) = 0$ if s differs from the representative of its class at an even number of places; let $F(s) = 1$ otherwise. F has no infinite homogeneous set.]

9.5 (König's Lemma). If T is a tree of height ω such that each level of T is finite, then T has an infinite branch.

[To construct a branch $\{x_0, x_1, \ldots, x_n, \ldots\}$ in T, pick x_0 at level 0 such that $\{y : y > x_0\}$ is infinite. Then pick x_1, x_2, ... similarly.]

9.6. If T is a normal α-tree, then T is isomorphic to a tree $\overline{T}$ whose elements are β-sequences ($\beta < \alpha$), ordered by extension; if $t \subset s$ and $s \in \overline{T}$, then $t \in \overline{T}$, and the βth level of $\overline{T}$ is the set $\{t \in \overline{T} : \operatorname{dom} t = \beta\}$.

9.7. If T is a normal ω_1-tree and if T has uncountable branch, then T has an uncountable antichain.

[For each x in the branch B pick a successor z_x of x such that $z_x \notin B$. Let $A = \{z_x : x \in B\}$.]

9.8. Show that if T is the tree in Theorem 9.16 then there exists some $f : T \to \boldsymbol{R}$ such that $f(x) < f(y)$ whenever $x < y$.

9.9. An Aronszajn tree is special if and only if T is the union of ω antichains.

[If $T = \bigcup_{n=0}^{\infty} A_n$, where each A_n is an antichain, define $\pi : T \to \boldsymbol{Q}$ by induction on n, constructing $\pi \restriction A_n$ at stage n, so that the range of π remains finite.]

9.10. Prove Theorem 9.18 using Fodor's Theorem.

[Let $W = \{X_\alpha : \alpha < \omega_1\}$ with $X_\alpha \subset \omega_1$. For each α, let $f(\alpha) = X_\alpha \cap \alpha$. By Fodor's Theorem, f is constant on a stationary set S; by induction construct a Δ-system $W \subset \{X_\alpha : \alpha \in S\}$.]

9.11. If $2^{<\kappa} = \kappa$, then there exists an almost disjoint family of 2^κ subsets of κ.

[As in Lemma 9.21, let $S = \bigcup_{\alpha<\kappa} \{0,1\}^\alpha$; $|S| = \kappa$.]

9.12. Given a family $\mathcal{F}$ of $\aleph_2$ almost disjoint functions $f : \omega_1 \to \omega$, there exists a collection $\mathcal{S}$ of $\aleph_2$ pairwise disjoint *stationary* subsets of ω_1.

[Each $f \in \mathcal{F}$ is constant on a stationary set S_f with value n_f. There is $\mathcal{G} \subset \mathcal{F}$ of size $\aleph_2$ such that n_f is the same for all $f \in \mathcal{G}$. Let $\mathcal{S} = \{S_f : f \in \mathcal{G}\}$.]

9.13. $\omega \not\to (\omega)^{<\omega}$.

[For $x \in [\omega]^{<\omega}$, let $F(x) = 1$ if $|x| \in x$, and $F(x) = 0$ otherwise. If $H \subset \omega$ is infinite, pick $n \in H$ and show that F is not constant on $[H]^n$.]

Historical Notes

Theorem 9.1 is due to Ramsey [1929/30]. Ramsey ultrafilters are investigated in Booth [1970/71]. The theory of partition relations has been developed by Erdős, who has written a number of papers on the subject, some coauthored by Rado, Hajnal, and others. The arrow notation is introduced in Erdős and Rado [1956]. Other major comprehensive articles on partition relations are Erdős, Hajnal, and Rado [1965] and Erdős and Hajnal [1971].

Theorem 9.6 appears in Erdős and Rado [1956]. Lemma 9.4 is due to Sierpiński [1933]. Theorem 9.7 is in Dushnik-Miller [1941].

Weakly compact cardinals (as in Definition 9.8 as well as the tree property) were introduced by Erdős and Tarski in [1961].

The equivalence of Suslin's Problem with the tree formulation (Lemma 9.14) is due to Kurepa [1935]; this paper also presents Aronszajn's construction and Kurepa trees, with Lemma 9.25.

Theorems 9.18 and 9.19: Shanin [1946] and Erdős-Rado [1960].

Ramsey cardinals were first studied by Erdős and Hajnal in [1962].

Exercise 9.2: Erdős-Rado [1956], Exercises 9.4 and 9.13: Erdős-Rado [1952].

Exercise 9.5: D. König [1927].

Exercise 9.9: Galvin.

10. Measurable Cardinals

The theory of large cardinals owes its origin to the basic problem of measure theory, the Measure Problem of H. Lebesgue.

The Measure Problem

Let S be an infinite set. A (*nontrivial σ-additive probabilistic*) *measure* on S is a real-valued function μ on $P(S)$ such that:

(10.1)
(i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$;
(ii) if $X \subset Y$, then $\mu(X) \leq \mu(Y)$;
(iii) $\mu(\{a\}) = 0$ for all $a \in S$ (nontriviality);
(iv) if X_n, $n = 0, 1, 2, \ldots$, are pairwise disjoint, then

$$\mu\Big(\bigcup_{n=0}^{\infty} X_n\Big) = \sum_{n=0}^{\infty} \mu(X_n) \qquad (\sigma\text{-additivity}).$$

It follows from (ii) that $\mu(X)$, the *measure of* X, is nonnegative for every $X \subset S$; in a special case of (iv) we get $\mu(X \cup Y) = \mu(X) + \mu(Y)$ whenever $X \cap Y = \emptyset$ (finite additivity).

More generally, let $\mathcal{A}$ be a σ-complete algebra of sets. A measure on $\mathcal{A}$ is a real-valued function μ on $\mathcal{A}$ satisfying (i)–(iv). Thus a measure on S is a measure on $P(S)$.

An example of a measure on a σ-complete algebra of sets is the Lebesgue measure on the algebra of all Lebesgue measurable subsets of the unit interval $[0, 1]$. The Lebesgue measure has, in addition to (i)–(iv), the following property:

(10.2) If X is congruent by translation to a measurable set Y, then X is measurable and $\mu(X) = \mu(Y)$.

It is well known that there exist sets of reals that are not Lebesgue measurable, and in fact that there is no measure on $[0, 1]$ with the property (10.2) (*translation invariant measure*); see Exercise 10.1.

The natural question to ask is whether the Lebesgue measure can be extended to some measure (not translation invariant) such that all subsets

of $[0,1]$ are measurable, or whether there exists any measure on $[0,1]$. Or, whether there exists a measure on some set S.

The investigation of this problem has lead to important discoveries in set theory, opening up a new field, the theory of large cardinal numbers, which has far-reaching consequences both in pure set theory and in descriptive set theory.

A measure μ on S is *two-valued* if $\mu(X)$ is either 0 or 1 for all $X \subset S$. If μ is a two-valued measure on S, let

$$U = \{X \subset S : \mu(X) = 1\}. \tag{10.3}$$

It is easy to verify that U is an ultrafilter on S. (For instance, if $X \in U$ and $Y \in U$, then $X \cap Y \in U$. If $\mu(X) = \mu(Y) = 1$, then $X = (X-Y) \cup (X \cap Y)$ and $Y = (Y-X) \cup (X \cap Y)$. If $\mu(X \cap Y)$ were not 1, then $\mu(X-Y) = \mu(Y-X) = 1$, and we would have $\mu(X \cup Y) = 2$.)

Next we note that the ultrafilter U is σ-complete. This is so because μ is σ-additive, and an ultrafilter U on S is σ-complete if and only if there is no partition of S into countably many disjoint parts $S = \bigcup_{n=0}^{\infty} X_n$ such that $X_n \notin U$, for all n.

Thus if μ is a two-valued measure on S, U is a σ-complete ultrafilter on S. Conversely, if U is a σ-complete ultrafilter on S, then the following function is a two-valued measure on S:

$$\mu(X) = \begin{cases} 1 & \text{if } X \in U, \\ 0 & \text{if } X \notin U. \end{cases} \tag{10.4}$$

Let μ be a measure on S. A set $A \subset S$ is an *atom* of μ if $\mu(A) > 0$ and if for every $X \subset A$, we have either $\mu(X) = 0$ or $\mu(X) = \mu(A)$.

If μ has an atom A, then

$$U = \{X \subset S : \mu(X \cap A) = \mu(A)\} \tag{10.5}$$

is again a σ-complete ultrafilter on S.

A measure μ on S is *atomless* if it has no atoms. Then every set $X \subset S$ of positive measure can be split into two disjoint sets of positive measure: $X = Y \cup Z$, and $\mu(Y) > 0$, $\mu(Z) > 0$.

We shall eventually prove various strong consequences of the existence of a nontrivial σ-additive measure and establish the relationship between the Measure Problem and large cardinals. Our starting point is the following theorem which shows that if a measure exists, then there exists at least a weakly inaccessible cardinal.

Theorem 10.1 (Ulam). *If there is a σ-additive nontrivial measure on S, then either there exists a two-valued measure on S and $|S|$ is greater than or equal to the least inaccessible cardinal, or there exists an atomless measure on $2^{\aleph_0}$ and $2^{\aleph_0}$ is greater than or equal to the least weakly inaccessible cardinal.*

Theorem 10.1 will be proved in a sequence of lemmas, which will also provide additional information on the Measure Problem and introduce basic notions and methods of the theory of large cardinals. First we make the following observation. Let κ be the least cardinal that carries a nontrivial σ-additive two-valued measure. Clearly, κ is uncountable and is also the least cardinal that has a nonprincipal countably complete ultrafilter. And we observe that such an ultrafilter is in fact κ-complete:

Lemma 10.2. *Let κ be the least cardinal with the property that there is a nonprincipal σ-complete ultrafilter on κ, and let U be such an ultrafilter. Then U is κ-complete.*

Proof. Let U be a σ-complete ultrafilter on κ, and let us assume that U is not κ-complete. Then there exists a partition $\{X_\alpha : \alpha < \gamma\}$ of κ such that $\gamma < \kappa$, and $X_\alpha \notin U$ for all $\alpha < \gamma$. We shall now use this partition to construct a nonprincipal σ-complete ultrafilter on γ, thus contradicting the choice of κ as the least cardinal that carries such an ultrafilter.

Let f be the mapping of κ onto γ defined as follows:

$$f(x) = \alpha \quad \text{if and only if} \quad x \in X_\alpha \qquad (x \in \kappa).$$

The mapping f induces a σ-complete ultrafilter on γ: we define $D \subset P(\gamma)$ by

$$Z \in D \quad \text{if and only if} \quad f_{-1}(Z) \in U. \tag{10.6}$$

The ultrafilter D is nonprincipal: Assume that $\{\alpha\} \in D$ for some $\alpha < \gamma$. Then $X_\alpha \in U$, contrary to our assumption on X_α. Thus γ carries a σ-complete nonprincipal ultrafilter. □

Measurable and Real-Valued Measurable Cardinals

We are now ready to define the central notion of this chapter.

Definition 10.3. An uncountable cardinal κ is *measurable* if there exists a κ-complete nonprincipal ultrafilter U on κ.

By Lemma 10.2, the least cardinal that carries a nontrivial two-valued σ-additive measure is measurable. Note that if U is a κ-complete nonprincipal ultrafilter on κ, then every set $X \in U$ has cardinality κ because every set of smaller size is the union of fewer than κ singletons. For similar reasons, κ is a regular cardinal because if κ is singular, then it is the union of fewer than κ small sets. The next lemma gives a first link of the Measure Problem with large cardinals.

Lemma 10.4. *Every measurable cardinal is inaccessible.*

Proof. We have just given an argument why a measurable cardinal is regular. Let us show that measurable cardinals are strong limit cardinals. Let κ be measurable, and let us assume that there exists $\lambda < \kappa$ such that $2^\lambda \geq \kappa$; we shall reach a contradiction.

Let S be a set of functions $f : \lambda \to \{0, 1\}$ such that $|S| = \kappa$, and let U be a κ-complete nonprincipal ultrafilter on S. For each $\alpha < \lambda$, let X_α be that one of the two sets $\{f \in S : f(\alpha) = 0\}$, $\{f \in S : f(\alpha) = 1\}$ which is in U, and let ε_α be 0 or 1 accordingly. Since U is κ-complete, the set $X = \bigcap_{\alpha<\lambda} X_\alpha$ is in U. However, X has at most one element, namely the function f that has the values $f(\alpha) = \varepsilon_\alpha$. A contradiction. □

Let us now turn our attention to measures that are not necessarily two-valued. Let μ be a nontrivial σ-additive measure on a set S. In analogy with (10.3) we consider the ideal of all *null sets*:

$$I_\mu = \{X \subset S : \mu(X) = 0\}. \tag{10.7}$$

I_μ is a nonprincipal σ-complete ideal on S. Moreover, it has these properties:

(10.8) (i) $\{x\} \in I$ for every $x \in S$;
(ii) every family of pairwise disjoint sets $X \subset S$ that are not in I is at most countable.

To see that (ii) holds, note that if W is a disjoint family of set of positive measure, then for each integer $n > 0$, there are only finitely many sets $X \in W$ of measure $\geq 1/n$.

A σ-complete nonprincipal ideal I on S is called *σ-saturated* if it satisfies (10.8).

The following lemma is an analog of Lemma 10.2:

Lemma 10.5.

(i) *Let κ be the least cardinal that carries a nontrivial σ-additive measure and let μ be such a measure on κ. Then the ideal I_μ of null sets is κ-complete.*
(ii) *Let κ be the least cardinal with the property that there is a σ-complete σ-saturated ideal on κ, and let I be such an ideal. Then I is κ-complete.*

Proof. (i) Let us assume that I_μ is not κ-complete. There exists a collection of null sets $\{X_\alpha : \alpha < \gamma\}$ such that $\gamma < \kappa$ and that their union X has positive measure. We may assume without loss of generality that the sets X_α, $\alpha < \gamma$, are pairwise disjoint; let $m = \mu(X)$.

Let f be the following mapping of X onto γ:

$$f(x) = \alpha \quad \text{if and only if} \quad x \in X_\alpha \qquad (x \in X).$$

The mapping f induces a measure ν on γ:

$$\nu(Z) = \frac{1}{m} \cdot \mu(f_{-1}(Z)). \tag{10.9}$$

The measure ν is σ-additive and is nontrivial since $\nu(\{\alpha\}) = \mu(X_\alpha) = 0$ for each $\alpha \in \gamma$. This contradicts the choice of κ as the least cardinal that carries a measure.

(ii) The proof is similar. We define an ideal J on γ by: $Z \in J$ if and only if $f_{-1}(Z) \in I$. The induced ideal J is σ-complete and σ-saturated. □

Let $\{r_i : i \in I\}$ be a collection of nonnegative real numbers. We define

$$\sum_{i\in I} r_i = \sup\Big\{\sum_{i\in E} r_i : E \text{ is a finite subset of } I\Big\}. \tag{10.10}$$

Note that if the sum (10.10) is not ∞, then at most countably many r_i are not equal to 0.

Let κ be an uncountable cardinal. A measure μ on S is called *κ-additive* if for every $\gamma < \kappa$ and for every disjoint collection X_α, $\alpha < \gamma$, of subsets of S,

$$\mu\Big(\bigcup_{\alpha<\gamma} X_\alpha\Big) = \sum_{\alpha<\gamma} \mu(X_\alpha). \tag{10.11}$$

If μ is a κ-additive measure, then the ideal I_μ of null sets is κ-complete. The converse is also true and we get a better analog of Lemma 10.2 for real-valued measures:

Lemma 10.6. *Let μ be a measure on S, and let I_μ be the ideal of null sets. If I_μ is κ-complete, then μ is κ-additive.*

Proof. Let $\gamma < \kappa$, and let X_α, $\alpha < \gamma$, be disjoint subsets of S. Since the X_α are disjoint, at most countably many of them have positive measure. Thus let us write

$$\{X_\alpha : \alpha < \gamma\} = \{Y_n : n = 0, 1, 2, \ldots\} \cup \{Z_\alpha : \alpha < \gamma\},$$

where each Z_α has measure 0. Then we have

$$\mu\Big(\bigcup_{\alpha<\gamma} X_\alpha\Big) = \mu\Big(\bigcup_{n=0}^{\infty} Y_n\Big) + \mu\Big(\bigcup_{\alpha<\gamma} Z_\alpha\Big).$$

Now first μ is σ-additive, and we have

$$\mu\Big(\bigcup_{n=0}^{\infty} Y_n\Big) = \sum_{n=0}^{\infty} \mu(Y_n),$$

and secondly I_μ is κ-complete and

$$\mu\Big(\bigcup_{\alpha<\gamma} Z_\alpha\Big) = 0 = \sum_{\alpha<\gamma} \mu(Z_\alpha).$$

Thus $\mu(\bigcup_\alpha X_\alpha) = \sum_\alpha \mu(X_\alpha)$. □

Corollary 10.7. *Let κ be the least cardinal that carries a nontrivial σ-additive measure and let μ be such a measure. Then μ is κ-additive.* □

Definition 10.8. An uncountable cardinal κ is *real-valued measurable* if there exists a nontrivial κ-additive measure μ on κ.

By Corollary 10.7, the least cardinal that carries a nontrivial σ-additive measure is real-valued measurable. We shall show that if a real-valued measurable cardinal κ is not measurable, then $\kappa \leq 2^{\aleph_0}$. Note that if μ is a nontrivial κ additive measure on κ, then every set of size $< \kappa$ has measure 0, and moreover κ cannot be the union of fewer than κ sets of size $< \kappa$. Thus a real-valued measurable cardinal is regular. We shall show that it is weakly inaccessible.

We shall first prove the first claim made in the preceding paragraph.

Lemma 10.9.

(i) *If there exists an atomless nontrivial σ-additive measure, then there exists a nontrivial σ-additive measure on some $\kappa \leq 2^{\aleph_0}$.*

(ii) *If I is a σ-complete σ-saturated ideal on S, then either there exists $Z \subset S$, such that $I \restriction Z = \{X \subset Z : X \in I\}$ is a prime ideal, or there exists a σ-complete σ-saturated ideal on some $\kappa \leq 2^{\aleph_0}$.*

Proof. (i) Let μ be such a measure on S. We construct a tree T of subsets of S, partially ordered by reverse inclusion. The 0th level of T is $\{S\}$. Each level of T consists of pairwise disjoint subsets of S of positive measure. Each $X \in T$ has two immediate successors: We choose two sets Y, Z of positive measure such that $Y \cup Z = X$ and $Y \cap Z = \emptyset$. If α is a limit ordinal, then the αth level consists of all intersections $X = \bigcap_{\xi<\alpha} X_\xi$ such that each X_ξ is on the ξth level of T and such that X has positive measure.

We observe that every branch of T has countable length: If $\{X_\xi : \xi < \alpha\}$ is a branch in T, then the set $\{Y_\xi : \xi < \alpha\}$, where $Y_\xi = X_\xi - X_{\xi+1}$, is a disjoint collection of sets of positive measure. Consequently, T has height at most ω_1. Similarly, each level of T is at most countable, and it follows that T has at most $2^{\aleph_0}$ branches.

Let $\{b_\alpha : \alpha < \kappa\}$, $\kappa \leq 2^{\aleph_0}$, be an enumeration of all branches $b = \{X_\xi : \xi < \gamma\}$ such that $\bigcap_{\xi<\gamma} X_\xi$ is nonempty; for each $\alpha < \kappa$, let $Z_\alpha = \bigcap\{X : X \in b_\alpha\}$. The collection $\{Z_\alpha : \alpha < \kappa\}$ is a partition of S into κ sets of measure 0.

We induce a measure ν on κ as follows: Let f be the mapping of S onto κ defined by

$$f(x) = \alpha \quad \text{if and only if} \quad x \in Z_\alpha \qquad (x \in S),$$

and let

$$\nu(Z) = \mu(f_{-1}(Z))$$

for all $Z \subset \kappa$. It follows that ν is a nontrivial σ-additive measure on κ.

(ii) The proof is similar. We define a tree T as above and then induce an ideal J on κ by letting $Z \in J$ if and only if $f_{-1}(Z) \in I$. □

The proof of Lemma 10.9 shows that if μ is atomless, then there is a partition of S into at most $2^{\aleph_0}$ null sets; in other words, μ is not $(2^{\aleph_0})^+$-additive. Hence if κ carries an atomless κ-additive measure, then $\kappa \le 2^{\aleph_0}$ and we have:

Corollary 10.10. *If κ is a real-valued measurable cardinal, then either κ is measurable or $\kappa \le 2^{\aleph_0}$.*

More generally, if κ carries a κ-complete σ-saturated ideal, then either κ is measurable or $\kappa \le 2^{\aleph_0}$. □

The measure ν obtained in Lemma 10.9(i) is atomless; this follows from the fact that $\kappa \le 2^{\aleph_0}$ and Lemma 10.4. If there exists an atomless σ-additive measure, then there is one on some $\kappa \le 2^{\aleph_0}$. Clearly, such a measure can be extended to a measure on $2^{\aleph_0}$: For $X \subset 2^{\aleph_0}$, we let $\mu(X) = \mu(X \cap \kappa)$. Thus we conclude that there exists an atomless σ-additive measure on the set $\boldsymbol{R}$ of all reals. It turns out that using the same assumption, we can obtain a σ-additive measure on $\boldsymbol{R}$ that extends Lebesgue measure. This can be done by a slight modification of the proof of Lemma 10.9:

Using Exercise 10.3, we construct for each finite 0–1 sequence s, a set $X_s \subset S$ such that $X_\emptyset = S$, and for every $s \in Seq$, $X_{s^\frown 0} \cup X_{s^\frown 1} = X_s$, $X_{s^\frown 0} \cap X_{s^\frown 1} = \emptyset$, and $\mu(X_{s^\frown 0}) = \mu(X_{s^\frown 1}) = \frac{1}{2} \cdot \mu(X_{s^\frown 0})$. Then we define a measure ν_1 on 2^ω by

$$\nu_1(Z) = \mu(\bigcup\{X_f : f \in Z\}),$$

where $X_f = \bigcap_{n=0}^\infty X_{f\restriction n}$ for each $f \in 2^\omega$. Using the mapping $F : 2^\omega \to [0,1]$ defined by

$$F(f) = \sum_{n=0}^{\infty} f(n)/2^{n+1}$$

we obtain a nontrivial σ-additive measure ν on $[0,1]$. This measure agrees with the Lebesgue measure on all intervals $[k/2^n, (k+1)/2^n]$, and hence on all Borel sets. Every set of Lebesgue measure 0 is included in a Borel (in fact, G_δ) set of Lebesgue measure 0 and hence has ν-measure 0. Every Lebesgue measurable set X can be written as $X = (B - N_1) \cup N_2$, where N_1 and N_2 have Lebesgue measure 0, and hence the Lebesgue measure of X is equal to $\nu(X)$. Thus ν agrees with the Lebesgue measure on all Lebesgue measurable subsets of $[0,1]$.

We shall now show that a real-valued measurable cardinal is weakly inaccessible. The proof is by a combinatorial argument, using matrices of sets.

Definition 10.11. An *Ulam matrix* (more precisely, an Ulam $(\aleph_1, \aleph_0)$-matrix) is a collection $\{A_{\alpha,n} : \alpha < \omega_1,\ n < \omega\}$ of subsets of ω_1 such that:

(10.12) (i) if $\alpha \ne \beta$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for every $n < \omega$;
(ii) for each α, the set $\omega_1 - \bigcup_{n=0}^\infty A_{\alpha,n}$ is at most countable.

An Ulam matrix has $\aleph_1$ rows and $\aleph_0$ columns. Each column consists of pairwise disjoint sets, and the union of each row contains all but countably many elements of ω_1.

Lemma 10.12. *An Ulam matrix exists.*

Proof. For each $\xi < \omega_1$, let f_ξ be a function on ω such that $\xi \subset \mathrm{ran}(f_\xi)$. Let us define $A_{\alpha,n}$ for $\alpha < \omega_1$ and $n < \omega$ by

$$\xi \in A_{\alpha,n} \quad \text{if and only if} \quad f_\xi(n) = \alpha. \tag{10.13}$$

If $n < \omega$, then for each $\xi \in \omega_1$ there is only one α such that $\xi \in A_{\alpha,n}$, namely $\alpha = f_\xi(n)$; and we have property (i) of (10.12). If $\alpha < \omega_1$, then for each $\xi > \alpha$ there is an n such that $f_\xi(n) = \alpha$ and hence $(\omega_1 - \bigcup_{n=0}^{\infty} A_{\alpha,n}) \subset \alpha + 1$; that verifies property (ii). □

Using an Ulam matrix, we can show that there is no measure on ω_1:

Lemma 10.13. *There is no nontrivial σ-additive measure on ω_1. More generally, there is no σ-complete σ-saturated ideal on ω_1.*

Proof. Let $\{A_{\alpha,n} : \alpha < \omega_1,\ n < \omega\}$ be an Ulam matrix. Assuming that we have a measure on ω_1, there is for each α some $n = n_\alpha$ such that $A_{\alpha,n}$ has positive measure (because of (10.12)(ii)). Hence there exist an uncountable set $W \subset \omega_1$ and some $n < \omega$ such that $n_\alpha = n$ for all $\alpha \in W$. Then $\{A_{\alpha,n} : \alpha \in W\}$ is an uncountable, pairwise disjoint (by (10.12)(i)) family of sets of positive measure; a contradiction. □

A straightforward generalization of Lemmas 10.12 and 10.13 gives the result mentioned above:

Lemma 10.14. *If $\kappa = \lambda^+$, then there is no κ-complete σ-saturated ideal on κ.*

Proof. For each $\xi < \lambda^+$, we let f_ξ be a function on λ such that $\xi \subset \mathrm{ran}(f_\xi)$, and let

$$\xi \in A_{\alpha,\eta} \quad \text{if and only if} \quad f_\xi(\eta) = \alpha.$$

Then $\{A_{\alpha,\eta} : \alpha < \lambda^+,\ \eta < \lambda\}$ is an Ulam (λ^+, λ)-matrix, that is a collection of subsets of λ^+ such that:

$$\begin{aligned} &\text{(i)}\ \ A_{\alpha,\eta} \cap A_{\beta,\eta} = \emptyset \text{ whenever } \alpha \neq \beta < \lambda^+, \text{ and } \eta < \lambda; \\ &\text{(ii)}\ \ |\lambda^+ - \textstyle\bigcup_{\eta<\lambda} A_{\alpha,\eta}| \leq \lambda \text{ for each } \alpha < \lambda^+. \end{aligned} \tag{10.14}$$

The proof of Lemma 10.13 generalizes to show that there is no κ-complete σ-saturated ideal on κ. □

Corollary 10.15. *Every real-valued measurable cardinal is weakly inaccessible.* □

Lemma 10.14 completes the proof of Theorem 10.1: If there is a σ-additive nontrivial measure on S, then either the measure has an atom A and we can construct a two-valued measure on S via a σ-complete nonprincipal ultrafilter on A, and then $|S| \geq$ the least measurable cardinal, which is inaccessible; or the measure on S is atomless and we construct, as in Lemma 10.9, an atomless measure on $2^{\aleph_0}$, and then $2^{\aleph_0} \geq$ the least real-valued measurable cardinal, which is weakly inaccessible. □

Prior to Ulam's work, Banach and Kuratowski proved that if the Continuum Hypothesis holds then there exists no σ-additive measure on $\boldsymbol{R}$. We present their proof below; in fact, Lemma 10.16 gives a slightly more general result.

If f and g are functions from ω to ω, let $f < g$ mean that $f(n) < g(n)$ for all but finitely many $n \in \omega$. A κ-sequence of functions $\langle f_\alpha : \alpha < \kappa \rangle$ is called a *κ-scale* if $f_\alpha < f_\beta$ whenever $\alpha < \beta$, and if for every $g : \omega \to \omega$ there exists an α such that $g < f_\alpha$.

Lemma 10.16. *If there exists a κ-scale, then κ is not a real-valued measurable cardinal.*

Proof. Let f_α, $\alpha < \kappa$, be a κ-scale. We define an $(\aleph_0, \aleph_0)$-matrix of subsets of κ as follows: For $n, k < \omega$, let

$$\alpha \in A_{n,k} \quad \text{if and only if} \quad f_\alpha(n) = k \qquad (\alpha \in \kappa). \tag{10.15}$$

Since for each n and each α there is k such that $\alpha \in A_{n,k}$, we have

$$\bigcup_{k=0}^{\infty} A_{n,k} = \kappa$$

for every $n = 0, 1, 2, \ldots$.

Now assume that μ is a nontrivial κ-additive measure on κ. For each n, let k_n be such that

$$\mu(A_{n,0} \cup A_{n,1} \cup \ldots \cup A_{n,k_n}) \geq 1 - (1/2^{n+2}),$$

and let $B_n = A_{n,0} \cup \ldots \cup A_{n,k_n}$. If we let $B = \bigcap_{n=0}^{\infty} B_n$, then we clearly have $\mu(B) \geq 1/2$.

Let $g : \omega \to \omega$ be the function $g(n) = k_n$. If $\alpha \in B$, then by the definition of B and by (10.15), we have

$$f_\alpha(n) \leq g(n)$$

for all $n = 0, 1, 2, \ldots$; hence $g \not< f_\alpha$. However, since B has positive measure, B has size κ, and therefore we have $g \not< f_\alpha$ for cofinally many $\alpha < \kappa$. This contradicts the assumption that the f_α form a scale. □

Corollary 10.17. *If there is a measure on $2^{\aleph_0}$, then $2^{\aleph_0} > \aleph_1$.*

Proof. If $2^{\aleph_0} = \aleph_1$, then there exists an ω_1-scale; a scale $\langle f_\alpha : \alpha < \omega_1 \rangle$ is constructed by transfinite induction to ω_1:

Let $\{g_\alpha : \alpha < \omega_1\}$ enumerate all functions from ω to ω. At stage α, we construct, by diagonalization, a function f_α such that for all $\beta < \alpha$, $f_\alpha > f_\beta$ and $f_\alpha > g_\beta$. Then $\langle f_\alpha : \alpha < \omega_1 \rangle$ is an ω_1-scale. □

Measurable Cardinals

By Lemma 10.4, every measurable cardinal is inaccessible. While we shall investigate measurable cardinals extensively in Part II, we now present a few basic results that establish the relationship of measurable cardinals and the large cardinals introduced in Chapter 9.

We recall that by Lemma 9.26, a cardinal κ is weakly compact if and only if it is inaccessible and has the tree property.

Lemma 10.18. *Every measurable cardinal is weakly compact.*

Proof. Let κ be a measurable cardinal. To show that κ is weakly compact, it suffices to prove the tree property. Let $(T, <)$ be a tree of height κ with levels of size $< \kappa$. We consider a nonprincipal κ-complete ultrafilter U on T. Let B be the set of all $x \in T$ such that the set of all successors of x is in U. It is clear that B is a branch in T and it is easy to verify that each level of T has one element in B; thus B is a branch of size κ. □

Normal Measures

In Chapter 8 we defined the notion of a normal κ-complete filter, namely a filter closed under diagonal intersections (8.7).

Thus we call a normal κ-complete nonprincipal ultrafilter a *normal measure* on κ. Note that by Exercise 8.8, a measure is normal if and only if every regressive function on a set of measure one is constant on a set of measure one.

Lemma 10.19. *If D is a normal measure on κ, then every set in D is stationary.*

Proof. By Lemma 8.11, every closed unbounded set is in D, and the lemma follows. □

Theorem 10.20 below shows that if κ is measurable cardinal then a normal measure exists.

Theorem 10.20. *Every measurable cardinal carries a normal measure. If U is a nonprincipal κ-complete ultrafilter on κ then there exists a function $f : \kappa \to \kappa$ such that $f_*(U) = \{X \subset \kappa : f_{-1}(X) \in U\}$ is a normal measure.*

Proof. Let U be a nonprincipal κ-complete ultrafilter on κ. For f and g in κ^κ, let

$$f \equiv g \quad \text{if and only if} \quad \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U.$$

It is easily seen that $\equiv$ is an equivalence relation on κ^κ. Let $[f]$ denote the equivalence class of $f \in \kappa^\kappa$. Furthermore, if we let

$$f < g \quad \text{if and only if} \quad \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in U,$$

then $<$ is a linear ordering of (the equivalence classes of) κ^κ.

There exists no infinite descending sequence $f_0 > f_1 > \ldots > f_n > \ldots$: Otherwise, let $X_n = \{\alpha : f_n(\alpha) > f_{n+1}(\alpha)\}$, and let $X = \bigcap_{n=0}^{\infty} X_n$. X is nonempty, and if $\alpha \in X$, we would have $f_0(\alpha) > f_1(\alpha) > \ldots > f_n(\alpha) > \ldots$, a contradiction.

Thus $<$ is a well-ordering of $\kappa^\kappa/\equiv$.

Now let $f : \kappa \to \kappa$ be the least function (in this well-ordering) with the property that for all $\gamma < \kappa$, $\{\alpha : f(\alpha) > \gamma\} \in U$. Such functions exist: for instance, the *diagonal function* $d(\alpha) = \alpha$ has this property.

Let $D = f_*(U) = \{X \subset \kappa : f_{-1}(X) \in U\}$. We claim that D is a normal measure.

It is easy to verify that D is a κ-complete ultrafilter. For every $\gamma < \kappa$, we have $f_{-1}(\{\gamma\}) \notin U$, and so $\{\gamma\} \notin D$, and so D is nonprincipal.

In order to show that D is normal, let h be a regressive function on a set $X \in D$. We shall show that h is constant on a set in D. Let g be the function defined by $g(\alpha) = h(f(\alpha))$. As $g(\alpha) < f(\alpha)$ for all $\alpha \in f_{-1}(X)$, we have $g < f$, and it follows by the minimality of f that g is constant on some $Y \in U$. Hence h is constant on $f(Y)$ and $f(Y) \in D$. □

As an application of normal measures we show that every measurable cardinal is a Mahlo cardinal, and improve Lemma 10.18 by showing that every measurable cardinal is a Ramsey cardinal.

Lemma 10.21. *Every measurable cardinal is a Mahlo cardinal.*

Proof. Let κ be a measurable cardinal. We shall show that the set of all inaccessible cardinals $\alpha < \kappa$ is stationary. As κ is strong limit, the set of all strong limit cardinals $\alpha < \kappa$ is closed unbounded, and it suffices to show that the set of all regular cardinals $\alpha < \kappa$ is stationary.

Let D be a normal measure on κ. We claim that $\{\alpha < \kappa : \alpha \text{ is regular}\} \in D$; this will complete the proof, since every set in D is stationary, by Lemma 10.19.

Toward a contradiction, assume that $\{\alpha : \operatorname{cf} \alpha < \alpha\} \in D$. By normality, there is some $\lambda < \kappa$ such that $E_\lambda = \{\alpha : \operatorname{cf} \alpha = \lambda\} \in D$. For each $\alpha \in E_\lambda$, let $\langle x_{\alpha,\xi} : \xi < \lambda \rangle$ be an increasing sequence with limit α. For each $\xi < \lambda$ there exist y_ξ and $A_\xi \in D$ such that $x_{\alpha,\xi} = y_\xi$ for all $\alpha \in A_\xi$. Let $A = \bigcap_{\xi<\lambda} A_\xi$. Then $A \in D$, but A contains only one element, namely $\lim_{\xi\to\lambda} y_\xi$; a contradiction. □

Theorem 10.22. *Let κ be a measurable cardinal, let D be a normal measure on κ, and let F be a partition of $[\kappa]^{<\omega}$ into less than κ pieces. Then there exists a set $H \in D$ homogeneous for F. Hence every measurable cardinal is a Ramsey cardinal.*

Proof. Let D be a normal measure on κ, and let F be a partition of $[\kappa]^{<\omega}$ into fewer than κ pieces. It suffices to show that for each $n = 1, 2, \ldots$, there is $H_n \in D$ such that F is constant on $[H_n]^n$; then $H = \bigcap_{n=1}^{\infty} H_n$ is homogeneous for F.

We prove, by induction on n, that every partition of $[\kappa]^n$ into fewer than κ pieces is constant on $[H]^n$ for some $H \in D$. The assertion is trivial for $n = 1$, so we assume that it is true for n and prove that it holds also for $n + 1$. Let $F : [\kappa]^{n+1} \to I$, where $|I| < \kappa$. For each $\alpha < \kappa$, we define F_α on $[\kappa - \{\alpha\}]^n$ by $F_\alpha(x) = F(\{\alpha\} \cup x)$.

By the induction hypothesis, there exists for each $\alpha < \kappa$ a set $X_\alpha \in D$ such that F_α is constant on $[X_\alpha]^n$; let i_a be its constant value. Let X be the diagonal intersection $X = \{\alpha < \kappa : \alpha \in \bigcap_{\gamma<\alpha} X_\gamma\}$. We have $X \in D$ since D is normal; also, if $\gamma < \alpha_1 < \ldots < \alpha_n$ are in X, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\gamma]^n$ and so $F(\{\gamma, \alpha_1, \ldots, \alpha_n\}) = F_\gamma(\{\alpha_1, \ldots, \alpha_n\}) = i_\gamma$. Now, there exist $i \in I$ and $H \subset X$ in D such that $i_\gamma = i$ for all $\gamma \in H$. It follows that $F(x) = i$ for all $x \in [H]^{n+1}$. □

Strongly Compact and Supercompact Cardinals

Among the various large cardinals that we shall investigate in more detail in Part II there are two that are immediate generalizations of measurable cardinals.

Definition 10.23. An uncountable cardinal κ is *strongly compact* if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

Clearly, every strongly compact cardinal is measurable.

Let A be a set of size at least κ, and let us consider the filter F on $P_\kappa(A)$ generated by the sets $\hat{P} = \{Q \in P_\kappa(A) : P \subset Q\}$. F is a κ-complete filter and if κ is strongly compact, F can be extended to a κ-complete ultrafilter U. A κ-complete ultrafilter U on $P_\kappa(A)$ that extends F is called a *fine measure*. In Part II we prove that if a fine measure on $P_\kappa(A)$ exists for every A, then κ is strongly compact.

A fine measure U on $P_{<\kappa}(A)$ is *normal* if whenever $f : P_\kappa(A) \to A$ is such that $f(P) \in P$ for all P in a set in U, then f is constant on a set in U. Equivalently, U is normal if it is closed under diagonal intersections $\triangle_{a\in A} X_a = \{x \in P_\kappa(A) : x \in \bigcap_{a\in x} X_a\}$.

Definition 10.24. An uncountable cardinal κ is *supercompact* if for every A such that $|A| \geq \kappa$ there exists a normal measure on $P_\kappa(A)$.

We return to the subject of strongly compact and supercompact cardinals in Part II.

Exercises

10.1 (Vitali). Let M be maximal (under $\subset$) subset of $[0,1]$ with the property that $x-y$ is not a rational number, for any pair of distinct $x, y \in M$. Show that M is not Lebesgue measurable.

[Consider the sets $M_q = \{x+q : x \in M\}$ where q is rational. They are pairwise disjoint and $[0,1] \subset \bigcup\{M_q : q \in \boldsymbol{Q} \cap [-1,1]\} \subset [-1,2]$.]

10.2. Prove directly that the measure ν defined in the proof of Lemma 10.9(i) is atomless.

[Assume that Z is an atom of ν, and let $Y = f_{-1}(Z)$. If $X \in T$ is such that $\mu(Y \cap X) \neq 0$ and if X_1, X_2 are the two immediate successors of X, then either $\mu(Y \cap X_1) = 0$ or $\mu(Y \cap X_2) = 0$. Prove by induction that on each level of T there is a unique X such that $\mu(Y \cap X) \neq 0$, and that these X's constitute a branch in T of length ω_1; a contradiction.]

10.3. If μ is an atomless measure on S, there exists $Z \subset S$ such that $\mu(Z) = 1/2$. More generally, given $Z_0 \subset S$, there exists $Z \subset Z_0$ such that $\mu(Z) = (1/2) \cdot \mu(Z_0)$.

[Construct a sequence $S = S_0 \supset S_1 \supset \ldots \supset S_\alpha \supset \ldots$, $\alpha < \omega_1$, such that $\mu(S_\alpha) \geq 1/2$, and if $\mu(S_\alpha) > 1/2$, then $1/2 \leq \mu(S_{\alpha+1}) < \mu(S_\alpha)$; if α is a limit ordinal, let $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$. There exists $\alpha < \omega_1$ such that $\mu(S_\alpha) = 1/2$.]

10.4. Let μ be a two-valued measure and U the ultrafilter of all sets of measure one. Then μ is κ-additive if and only if U is κ-complete.

10.5. A measure U on κ is normal if and only if the diagonal function $d(\alpha) = \alpha$ is the least function f with the property that for all $\gamma < \kappa$, $\{\alpha : f(\alpha) > \gamma\} \in U$.

10.6. Let D be a normal measure on κ and let $f : [\kappa]^{<\omega} \to \kappa$ be such that $f(x) = 0$ or $f(x) < \min x$ for all $x \in [\kappa]^{<\omega}$. Then there is $H \in D$ such that for each n, f is constant on $[H]^n$.

[By induction, as in Theorem 10.22. Given f on $[\kappa]^{n+1}$, let $f_\alpha(s) = f(\{\alpha\} \cup s)$ for $\alpha < \min s$; f_α is constant on $[X_\alpha]^n$ with value $\gamma_\alpha < \alpha$. Let X be the diagonal intersection of X_α, $\alpha < \kappa$, and let γ and $H \subset X$ be such that $H \in D$ and $\gamma_\alpha = \gamma$ for all $\alpha \in H$.]

10.7. If κ is measurable then there exists a normal measure on $P_\kappa(\kappa)$.

Historical Notes

The study of measurable cardinals originated around 1930 with the work of Banach, Kuratowski, Tarski, and Ulam. Ulam showed in [1930] that measurable cardinals are large, that the least measurable cardinal is at least as large as the least inaccessible cardinal.

The main result on measurable and real-valued measurable cardinals (Theorem 10.1) is due to Ulam [1930]. The fact that a measurable cardinal is inaccessible (Lemma 10.4) was discovered by Ulam and Tarski (cf. Ulam [1930]). Prior to Ulam,

Banach and Kuratowski proved in [1929] that if $2^{\aleph_0} = \aleph_1$, then there is no measure on the continuum; their proof is as in Lemma 10.16. Real-valued measurable cardinals were introduced by Banach in [1930].

Lemma 10.18: Erdős and Tarski [1943]. Hanf [1963/64a] proved that the least inaccessible cardinal is not measurable. That every measurable cardinal is a Ramsey cardinal was proved by Erdős and Hajnal [1962]; the stronger version (Theorem 10.22) is due to Rowbottom [1971].

Strongly compact cardinals were introduced by Keisler and Tarski in [1963/64]; supercompact cardinals were defined by Reinhardt and Solovay, cf. Solovay *et al.* [1978].

Exercise 10.1: Vitali [1905].

11. Borel and Analytic Sets

Descriptive set theory deals with sets of reals that are described in some simple way: sets that have a simple topological structure (e.g., continuous images of closed sets) or are definable in a simple way. The main theme is that questions that are difficult to answer if asked for arbitrary sets of reals, become much easier when asked for sets that have a simple description. An example of that is the Cantor-Bendixson Theorem (Theorem 4.6): Every closed set of reals is either at most countable or has size $2^{\aleph_0}$.

Since properties of definable sets can usually be established effectively, without use of the Axiom of Choice, we shall work in set theory ZF without the Axiom of Choice. When some statement depends on the Axiom of Choice, we shall explicitly say so. However, we shall assume a weak form of the Axiom of Choice. The reason is that in descriptive set theory one frequently considers unions and intersections of countably many sets of reals, and we shall often use facts like "the union of countably many countable sets is countable." Thus we shall work, throughout this chapter, in set theory ZF + the Countable Axiom of Choice.

In this chapter we develop the basic theory of Borel and analytic sets in Polish spaces. A Polish space is a topological space that is homeomorphic to a complete separable metric space (Definition 4.12).

A canonical example of a Polish space is the Baire space $\mathcal{N}$. The following lemma shows that every Polish space is a continuous image of $\mathcal{N}$:

Lemma 11.1. *Let X be a Polish space. Then there exists a continuous mapping from $\mathcal{N}$ onto X.*

Proof. Let X be a complete separable metric space; we construct a mapping f of $\mathcal{N}$ onto X as follows: It is easy to construct, by induction on the length of $s \in Seq$, a collection $\{C_s : s \in Seq\}$ of closed balls such that $C_\emptyset = X$ and

$$\begin{aligned} &\text{(i) } \operatorname{diameter}(C_s) \le 1/n \text{ where } n = \operatorname{length}(s), \\ &\text{(ii) } C_s \subset \bigcup_{k=0}^{\infty} C_{s^\frown k} \text{ (all } s \in Seq), \\ &\text{(iii) if } s \subset t \text{ then } \operatorname{center}(C_t) \in C_s. \end{aligned} \tag{11.1}$$

For each $a \in \mathcal{N}$, let $f(a)$ be the unique point in $\bigcap\{C_s : s \subset a\}$; it is easily checked that f is continuous and that $X = f(\mathcal{N})$. □

Borel Sets

Let X be a Polish space. A set $A \subset X$ is a *Borel set* if it belongs to the smallest σ-algebra of subsets of X containing all closed sets. We shall now give a more explicit description of Borel sets. For each $\alpha < \omega_1$, let us define the collections $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of subsets of X:

$$
\begin{aligned}
(11.2)\qquad \mathbf{\Sigma}^0_1 &= \text{the collection of all open sets;}\\
\mathbf{\Pi}^0_1 &= \text{the collection of all closed sets;}\\
\mathbf{\Sigma}^0_\alpha &= \text{the collection of all sets } A = \textstyle\bigcup_{n=0}^{\infty} A_n, \text{ where each } A_n\\
&\qquad \text{belongs to } \mathbf{\Pi}^0_\beta \text{ for some } \beta < \alpha;\\
\mathbf{\Pi}^0_\alpha &= \text{the collection of all complements of sets in } \mathbf{\Sigma}^0_\alpha\\
&= \text{the collection of all sets } A = \textstyle\bigcap_{n=0}^{\infty} A_n, \text{ where each } A_n\\
&\qquad \text{belongs to } \mathbf{\Sigma}^0_\beta \text{ for some } \beta < \alpha.
\end{aligned}
$$

It is clear (by induction on α) that the elements of each $\mathbf{\Sigma}^0_\alpha$ and each $\mathbf{\Pi}^0_\alpha$ are Borel sets. Since every open set is the union of countably many closed sets, we have $\mathbf{\Sigma}^0_1 \subset \mathbf{\Sigma}^0_2$, and consequently, if $\alpha < \beta$, then

$$\mathbf{\Sigma}^0_\alpha \subset \mathbf{\Sigma}^0_\beta, \qquad \mathbf{\Sigma}^0_\alpha \subset \mathbf{\Pi}^0_\beta, \qquad \mathbf{\Pi}^0_\alpha \subset \mathbf{\Pi}^0_\beta, \qquad \mathbf{\Pi}^0_\alpha \subset \mathbf{\Sigma}^0_\beta.$$

Hence

$$(11.3)\qquad \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$$

and it is easy to verify that the collection (11.3) is a σ-algebra (here we use the Countable Axiom of Choice). Hence every Borel set is in some $\mathbf{\Sigma}^0_\alpha$, $\alpha < \omega_1$.

Note that each $\mathbf{\Sigma}^0_\alpha$ (and each $\mathbf{\Pi}^0_\alpha$) is closed under finite unions, finite intersections, and inverse images by continuous functions (i.e., if $A \in \mathbf{\Sigma}^0_\alpha$ in Y, then $f_{-1}(A) \in \mathbf{\Sigma}^0_\alpha$ in X whenever $f : X \to Y$ is a continuous function).

If the Polish space X is countable, then of course every $A \in X$ is a Borel set, in fact an F_σ set. Uncountable Polish spaces are more interesting: Not all sets are Borel, and the collections $\mathbf{\Sigma}^0_\alpha$ form a hierarchy. We show below that for each α, $\mathbf{\Sigma}^0_\alpha \not\subset \mathbf{\Pi}^0_\alpha$, and hence $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Sigma}^0_{\alpha+1}$ for all $\alpha < \omega_1$.

While we prove the next lemma for the special case when X is the Baire space, the proof can be modified to prove the same result for any uncountable Polish space.

Lemma 11.2. *For each $\alpha \geq 1$ there exists a set $U \subset \mathcal{N}^2$ such that U is $\mathbf{\Sigma}^0_\alpha$ (in $\mathcal{N}^2$), and that for every $\mathbf{\Sigma}^0_\alpha$ set A in $\mathcal{N}$ there exists some $a \in \mathcal{N}$ such that*

$$(11.4)\qquad A = \{x : (x, a) \in U\}.$$

U is a *universal* $\mathbf{\Sigma}^0_\alpha$ set.

Proof. By induction on α. To construct a universal open set in $\mathcal{N}^2$, let $G_1, \ldots,$ $G_k, \ldots$ be an enumeration of all basic open sets in $\mathcal{N}$, and let $G_0 = \emptyset$. Let

$$(x,y) \in U \text{ if and only if } x \in G_{y(n)} \text{ for some } n. \tag{11.5}$$

Since $U = \bigcup_{n=0}^{\infty} H_n$ where each $H_n = \{(x,y) : x \in G_{y(n)}\}$ is an open set in $\mathcal{N}^2$, we see that U is open. Now if G is an open set in $\mathcal{N}$, we let $a \in \mathcal{N}$ be such that $G = \bigcup_{n=0}^{\infty} G_{a(n)}$; then $G = \{x : (x,a) \in U\}$.

Next let U be a universal $\mathbf{\Sigma}^0_\alpha$ set, and let us construct a universal $\mathbf{\Sigma}^0_{\alpha+1}$ set V. Let us consider some continuous mapping of $\mathcal{N}$ onto the product space $\mathcal{N}^\omega$; for each $a \in \mathcal{N}$ and each n, let $a_{(n)}$ be the nth coordinate of the image of a. [For instance, let us define $a_{(n)}$ as follows: $a_{(n)}(k) = a(\Gamma(n,k))$, where Γ is the canonical one-to-one pairing function $\Gamma : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$.] Now let

$$(x,y) \in V \text{ if and only if for some } n,\ (x, y_{(n)}) \notin U. \tag{11.6}$$

Since $V = \bigcup_{n=0}^{\infty} H_n$ where each $H_n = \{(x,y) : (x, y_{(n)}) \notin U\}$ is a $\mathbf{\Pi}^0_\alpha$ set, we see that V is $\mathbf{\Sigma}^0_{\alpha+1}$. If A is a $\mathbf{\Sigma}^0_{\alpha+1}$ set in $\mathcal{N}$, then $A = \bigcup_{n=0}^{\infty} A_n$ where each A_n is $\mathbf{\Pi}^0_\alpha$. For each n, let a_n be such that $\mathcal{N} - A_n = \{x : (x, a_n) \in U\}$, and let a be such that $a_{(n)} = a_n$ for all n. Then $A = \{x : (x,a) \in V\}$.

Finally, let α be a limit ordinal, and let U_β, $1 \le \beta \le \alpha$, be universal $\mathbf{\Sigma}^0_\beta$ sets. Let $1 \le \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ be an increasing sequence of ordinals such that $\lim_{n\to\infty} \alpha_n = \alpha$. Let

$$(x,y) \in U \text{ if and only if for some } n,\ (x, y_{(n)}) \notin U_{\alpha_n} \tag{11.7}$$

(where $a_{(n)}$ has the same meaning as above). The set U is $\mathbf{\Sigma}^0_\alpha$. If A is a $\mathbf{\Sigma}^0_\alpha$ set in $\mathcal{N}$ then $A = \bigcup_{n=0}^{\infty} A_n$ where each A_n is $\mathbf{\Pi}^0_{\alpha_n}$. For each n, let a_n be such that $\mathcal{N} - A_n = \{x : (x, a_n) \in U_{\alpha_n}\}$, and let a be such that $a_{(n)} = a_n$ for all n. Then $A = \{x : (x,a) \in U\}$. □

Corollary 11.3. *For every $\alpha \ge 1$, there is a set $A \subset \mathcal{N}$ that is $\mathbf{\Sigma}^0_\alpha$ but not $\mathbf{\Pi}^0_\alpha$.*

Proof. Let $U \subset \mathcal{N}^2$ be a universal $\mathbf{\Sigma}^0_\alpha$ set. Let us consider the set

$$A = \{x : (x,x) \in U\}. \tag{11.8}$$

Clearly, A is a $\mathbf{\Sigma}^0_\alpha$ set. If A were $\mathbf{\Pi}^0_\alpha$, then its complement would be $\mathbf{\Sigma}^0_\alpha$ and there would be some a such that

$$A = \{x : (x,a) \notin U\}.$$

But this contradicts (11.8): Simply let $x = a$. □

Analytic Sets

While the collection of Borel sets of reals is closed under Boolean operations, and countable unions and intersections, it is not closed under continuous images: As we shall learn presently, the image of a Borel set by a continuous function need not be a Borel set. We shall now investigate the continuous images of Borel sets.

Definition 11.4. A subset of A of a Polish space X is *analytic* if there exists a continuous function $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$.

Definition 11.5. The *projection* of a set $S \subset X \times Y$ (into X) is the set $P = \{x \in X : \exists y\, (x, y) \in S\}$.

The following lemma gives equivalent definitions of analytic sets.

Lemma 11.6. *The following are equivalent, for any set A in a Polish space X:*

(i) *A is the continuous image of $\mathcal{N}$.*
(ii) *A is the continuous image of a Borel set B (in some Polish space Y).*
(iii) *A is the projection of a Borel set in $X \times Y$, for some Polish space Y.*
(iv) *A is the projection of a closed set in $X \times \mathcal{N}$.*

Proof. We shall prove that every closed set (in any Polish space) is analytic and that every Borel set is the projection of a closed set in $X \times \mathcal{N}$. Then the lemma follows: Since the projection map $\pi : X \times Y \to X$ defined by $\pi(x, y) = x$ is continuous, it follows that every Borel set is analytic and that the continuous image of a Borel set is analytic. Conversely, if $A \subset X$ is an analytic set, $A = f(\mathcal{N})$, then A is the projection of the set $\{(f(x), x) : x \in \mathcal{N}\}$ which is a closed set in $X \times \mathcal{N}$.

In order to prove that every closed set is analytic, note that every closed set in a Polish space is itself a Polish space, and thus a continuous image of $\mathcal{N}$ by Lemma 11.1.

In order to prove that every Borel set in X is the projection of a closed set in $X \times \mathcal{N}$, it suffices to show that the family P of all subsets of X that are such projections contains all closed sets, all open sets, and is closed under countable unions and intersections.

Clearly, the family P contains all closed sets. Moreover, every open set is a countable union of closed sets; thus it suffices to show that P is closed under $\bigcup_{n=0}^{\infty}$ and $\bigcap_{n=0}^{\infty}$.

Recall the continuous mapping $a \mapsto \langle a_{(n)} : n \in \boldsymbol{N} \rangle$ of $\mathcal{N}$ onto $\mathcal{N}^\omega$ from Lemma 11.2, and also recall that the inverse image of a closed set under a continuous function is closed. Let A_n, $n < \omega$, be projections of closed sets in $X \times \mathcal{N}$; we shall show that $\bigcup_{n=0}^{\infty} A_n$ and $\bigcap_{n=0}^{\infty} A_n$ are projections of closed sets.

For each n, let $F_n \subset X \times \mathcal{N}$ be a closed set such that

$$A_n = \{x : \exists a\,(x, a) \in F_a\}.$$

Thus

$$\begin{aligned} x \in \bigcup_{n=0}^{\infty} A_n &\leftrightarrow \exists n\,\exists a\,(x, a) \in F_n \\ &\leftrightarrow \exists a\,\exists b\,(x, a) \in F_{b(0)} \\ &\leftrightarrow \exists c\,(x, c_{(0)}) \in F_{c_{(1)}(0)}, \end{aligned}$$

and

$$\begin{aligned} x \in \bigcap_{n=0}^{\infty} A_n &\leftrightarrow \forall n\,\exists a\,(x, a) \in F_n \\ &\leftrightarrow \exists c\,\forall n\,(x, c_{(n)}) \in F_n \\ &\leftrightarrow \exists c\,(x, c) \in \bigcap_{n=0}^{\infty} \{(x, c) : (x, c_{(n)}) \in F_n\}. \end{aligned}$$

Hence $\bigcup_{n=0}^{\infty} A_n$ is the projection of the closed set

$$\{(x, c) : (x, c_{(0)}) \in F_{c_{(1)}(0)}\}$$

and $\bigcap_{n=0}^{\infty} A_n$ is the projection of an intersection of closed sets. □

The Suslin Operation $\mathcal{A}$

For each $a \in \omega^\omega$, $a{\restriction}n$ is the finite sequence $\langle a_k : k < n\rangle$. For each $s \in Seq$, $O(s)$ is the basic open set $\{a \in \mathcal{N} : a{\restriction}n = s\}$ of the Baire space. $O(s)$ is both open and closed. For every set A in a Polish space, $\overline{A}$ denotes the closure of A.

Let $\{A_s : s \in Seq\}$ be a collection of sets indexed by elements of Seq. We define

$$\mathcal{A}\{A_s : s \in Seq\} = \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} A_{a{\restriction}n} \tag{11.9}$$

Note that if $\{B_s : s \in Seq\}$ is arbitrary, then

$$\bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} B_{a{\restriction}n} = \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} (B_{a{\restriction}0} \cap B_{a{\restriction}1} \cap \ldots \cap B_{a{\restriction}n})$$

and hence $\mathcal{A}\{B_s : s \in Seq\} = \mathcal{A}\{A_s : s \in Seq\}$ where the sets A_s are finite intersections of the sets B_s and satisfy the following condition:

$$\text{if } s \subset t, \text{ then } A_s \supset A_t. \tag{11.10}$$

Thus we shall restrict our use of $\mathcal{A}$ to families that satisfy condition (11.10). The operation $\mathcal{A}$ is called the *Suslin operation.*

Lemma 11.7. *A set A in a Polish space is analytic if and only if A is the result of the operation $\mathcal{A}$ applied to a family of closed sets.*

Proof. First we show that if F_s, $s \in Seq$, are closed sets in a Polish space X, then $A = \mathcal{A}\{F_s : s \in Seq\}$ is analytic. We have

$$\begin{aligned} x \in A &\leftrightarrow \exists a \in \mathcal{N}\, x \in \bigcap_{n=0}^{\infty} F_{a\restriction n} \\ &\leftrightarrow \exists a\, (x, a) \in \bigcap_{n=0}^{\infty} B_n \end{aligned}$$

where $B_n = \{(x, a) : x \in F_{a\restriction n}\}$. Clearly, each B_n is a Borel set in $X \times \mathcal{N}$ and hence A is analytic.

Conversely, let $A \subset X$ be analytic. There is a continuous function $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$. Notice that for every $a \in \mathcal{N}$,

$$\bigcap_{n=0}^{\infty} f(O(a\restriction n)) = \bigcap_{n=0}^{\infty} \overline{f(O(a\restriction n))} = \{f(a)\}. \tag{11.11}$$

Thus

$$A = f(\mathcal{N}) = \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} \overline{f(O(a\restriction n))},$$

and hence A is the result of the operation $\mathcal{A}$ applied to the closed sets $\overline{f(O(s))}$ (which satisfy the condition (11.10)). □

It follows from the preceding lemmas that the collection of all analytic sets in a Polish space is closed under countable unions and intersections, continuous images, and inverse images, and the Suslin operation (the last statement is proved like the first part of Lemma 11.7). It is however not the case that the complement of an analytic set is analytic (if X is an uncountable Polish space). In the next section we establish exactly that; we show that there exists an analytic set (in $\mathcal{N}$) whose complement is not analytic.

The Hierarchy of Projective Sets

For each $n \geq 1$, we define the collections $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ of subsets of a Polish space X as follows:

$$\begin{aligned} \mathbf{\Sigma}^1_1 &= \text{the collection of all analytic sets,} \\ \mathbf{\Pi}^1_1 &= \text{the complements of analytic sets,} \\ \mathbf{\Sigma}^1_{n+1} &= \text{the collection of the projections of all } \mathbf{\Pi}^1_n \text{ sets in } X \times \mathcal{N}, \\ \mathbf{\Pi}^1_n &= \text{the complements of the } \mathbf{\Sigma}^1_n \text{ sets in } X, \\ \mathbf{\Delta}^1_n &= \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n. \end{aligned} \tag{11.12}$$

The sets belonging to one of the collections $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ are called *projective sets*. It is easily seen that for every n, $\mathbf{\Delta}^1_n \subset \mathbf{\Sigma}^1_n \subset \mathbf{\Delta}^1_{n+1}$ and $\mathbf{\Delta}^1_n \subset \mathbf{\Pi}^1_n \subset \mathbf{\Delta}^1_{n+1}$.

We shall show that for each n there is a $\mathbf{\Sigma}^1_n$ set in $\mathcal{N}$ that is not $\mathbf{\Pi}^1_n$; thus the above inclusions are proper inclusions.

Lemma 11.8. *For each $n \geq 1$, there exists a universal $\mathbf{\Sigma}^1_n$ set in $\mathcal{N}^2$; i.e., a set $U \subset \mathcal{N}^2$ such that U is $\mathbf{\Sigma}^1_n$ and that for every $\mathbf{\Sigma}^1_n$ set A in $\mathcal{N}$ there exists some $v \in \mathcal{N}$ such that*

$$A = \{x : (x, v) \in U\}.$$

Proof. Let h be a homeomorphism of $\mathcal{N} \times \mathcal{N}$ onto $\mathcal{N}$. If $n = 1$, let V be a universal $\mathbf{\Sigma}^0_1$ set; if $n > 1$, let V be, by the induction hypothesis, a universal $\mathbf{\Sigma}^1_{n-1}$ set. Let

$$(x, y) \in U \quad \text{if and only if} \quad \exists a \in \mathcal{N}\, (h(x, a), y) \notin V. \tag{11.13}$$

Since the set $\{(x, y, a) : (h(x, a), y) \notin V\}$ is closed (if $n = 1$) or $\mathbf{\Pi}^1_{n-1}$ (if $n > 1$), U is $\mathbf{\Sigma}^1_n$.

If $A \subset \mathcal{N}$ is $\mathbf{\Sigma}^1_n$, there is a closed (or $\mathbf{\Pi}^1_{n-1}$) set B such that

$$x \in A \quad \text{if and only if} \quad \exists a \in \mathcal{N}\, (x, a) \in B. \tag{11.14}$$

The set $C = \mathcal{N} - h(B)$ is open (or $\mathbf{\Sigma}^1_{n-1}$) in $\mathcal{N}$ and since V is universal, there exists a v such that $C = \{u : (u, v) \in V\}$. Then by (11.13), we have

$$\begin{aligned} x \in A &\leftrightarrow (\exists a \in \mathcal{N})\, (x, a) \in B \leftrightarrow (\exists a \in \mathcal{N})\, h(x, a) \notin C \\ &\leftrightarrow (\exists a \in \mathcal{N})\, (h(x, a), v) \notin V \leftrightarrow (x, v) \in U. \end{aligned}$$

Hence U is a universal $\mathbf{\Sigma}^1_n$ set. □

Corollary 11.9. *For each $n \geq 1$, there is a set $A \subset \mathcal{N}$ that is $\mathbf{\Sigma}^1_n$ but not $\mathbf{\Pi}^1_n$.*

Proof. Let $U \subset \mathcal{N}^2$ be a universal $\mathbf{\Sigma}^1_n$ set and let

$$A = \{x : (x, x) \in U\}$$ □

The collection of all $\mathbf{\Delta}^1_1$ sets in a Polish space is a σ-algebra and contains all Borel sets. It turns out that $\mathbf{\Delta}^1_1$ is exactly the collection of all Borel sets.

Theorem 11.10 (Suslin). *Every analytic set whose complement is also analytic is a Borel set. Thus $\mathbf{\Delta}^1_1$ is the collection of all Borel sets.*

Let X be a Polish space and let A and B be two disjoint analytic sets in X. We say that A and B are *separated* by a Borel set if there exists a Borel set D such that $A \subset D$ and $B \subset X - D$.

Lemma 11.11. *Any two disjoint analytic sets are separated by a Borel set.*

This lemma is often called "the $\mathbf{\Sigma}^1_1$-Separation Principle." It clearly implies Suslin's Theorem since if A is an analytic set such that $B = X - A$ is also analytic, A and B are separated by a Borel set D and we clearly have $D = A$.

Proof. First we make the following observation: If $A = \bigcup_{n=0}^{\infty} A_n$ and $B = \bigcup_{m=0}^{\infty} B_m$ are such that for all n and m, A_n and B_m are separated, then A and B are separated. This is proved as follows: For each n and each m, let $D_{n,m}$ be a Borel set such that $A_n \subset D_{n,m} \subset X - B_m$. Then A and B are separated by the Borel set $D = \bigcup_{n=0}^{\infty} \bigcap_{m=0}^{\infty} D_{n,m}$.

Let A and B be two disjoint analytic sets in X. Let f and g be continuous functions such that $A = f(\mathcal{N})$ and $B = g(\mathcal{N})$. For each $s \in Seq$, let $A_s = f(O(s))$ and $B_s = g(O(s))$; the sets A_s and B_s are all analytic sets. For each s we have $A_s = \bigcup_{n=0}^{\infty} A_{s^\frown n}$ and $B_s = \bigcup_{m=0}^{\infty} B_{s^\frown m}$. If $a \in \omega^\omega$, then

$$\{f(a)\} = \bigcap_{n=0}^{\infty} f(O(a\restriction n)) = \bigcap_{n=0}^{\infty} A_{a\restriction n},$$

and similarly for the sets B_s.

Let $a, b \in \omega^\omega$ be arbitrary. Since $f(\mathcal{N})$ and $g(\mathcal{N})$ are disjoint, we have $f(a) \neq g(b)$. Let G_a and G_b be two disjoint open neighbourhoods of $f(a)$ and $g(b)$, respectively. By the continuity of f and g there exists some n such that $A_{a\restriction n} \subset G_a$ and $B_{b\restriction n} \subset G_b$. It follows that the sets $A_{a\restriction n}$ and $B_{b\restriction n}$ are separated by a Borel set.

We shall now show, by contradiction, that the sets A and B are separated by a Borel set. If A and B are not separated, then because $A = \bigcup_{n=0}^{\infty} A_{\langle n\rangle}$ and $B = \bigcup_{m=0}^{\infty} B_{\langle m\rangle}$, there exist n_0 and m_0 such that the sets $A_{\langle n_0\rangle}$ and $B_{\langle m_0\rangle}$ are not separated. Then similarly there exist n_1 and m_1 such that the sets $A_{\langle n_0,n_1\rangle}$ and $B_{\langle m_0,m_1\rangle}$ are not separated, and so on. In other words, there exist $a = \langle n_0, n_1, n_2, \ldots\rangle$ and $b = \langle m_0, m_1, m_2, \ldots\rangle$ such that for every k, the sets $A_{\langle n_0,\ldots,n_k\rangle}$ and $B_{\langle m_0,\ldots,m_k\rangle}$ are not separated. This is a contradiction since in the preceding paragraph we proved exactly the opposite: There is k such that $A_{a\restriction k}$ and $B_{b\restriction k}$ are separated. □

Lebesgue Measure

We shall now review basic properties of Lebesgue measure on the n-dimensional Euclidean space.

The standard way of defining Lebesgue measure is to define first the *outer measure* $\mu^*(X)$ of a set $X \subset \boldsymbol{R}^n$ as the infimum of all possible sums $\sum\{v(I_k) : k \in \boldsymbol{N}\}$ where $\{I_k : k \in \boldsymbol{N}\}$ is a collection of n-dimensional intervals such that $X \subset \bigcup_{k=0}^{\infty} I_k$, and $v(I)$ denotes the volume of I. For each X, $\mu^*(X) \geq 0$ and possibly $= \infty$. A set X is *null* if $\mu^*(X) = 0$.

A set $A \subset \boldsymbol{R}^n$ is *Lebesgue measurable* if for each $X \subset \boldsymbol{R}^n$,

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X - A).$$

For a measurable set A, we write $\mu(A)$ instead of $\mu^*(A)$ and call $\mu(A)$ the *Lebesgue measure of* A.

The standard development of the theory of Lebesgue measure gives the following facts:

(11.15)
(i) Every interval is Lebesgue measurable, and its measure is equal to its volume.
(ii) The Lebesgue measurable sets form a σ-algebra; hence every Borel set is measurable.
(iii) μ is σ-additive: If A_n, $n < \omega$, are pairwise disjoint and measurable, then
$$\mu\Big(\bigcup_{n=0}^{\infty} A_n\Big) = \sum_{n=0}^{\infty} \mu(A_n).$$
(iv) μ is σ-finite: If A is measurable, then there exist measurable sets A_n, $n < \omega$, such that $A = \bigcup_{n=0}^{\infty} A_n$, and $\mu(A_n) < \infty$ for each n.
(v) Every null set is measurable. The null sets form a σ-ideal and contain all singletons.
(vi) If A is measurable, then
$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ is compact}\}.$$
(vii) If A is measurable, then there is an F_σ set F and a G_δ set G such that $F \subset A \subset G$ and $G - F$ is null.

This last property gives this characterization of Lebesgue measurable sets: A set $A \subset \boldsymbol{R}^n$ is measurable if and only if there is a Borel set B such that the symmetric difference $A \triangle B = (A - B) \cup (B - A)$ is null.

One consequence of this is that if we denote by $\mathcal{B}$ the σ-algebra of Borel sets and by $\mathcal{M}$ the σ-algebra of measurable sets, and if I_μ is the ideal of all null sets, then $\mathcal{B}/I_\mu = \mathcal{M}/I_\mu$. The Boolean algebra $\mathcal{B}/I_\mu$ is σ-complete; and since a familiar argument shows that I_μ is (as an ideal in $\mathcal{M}$) σ-saturated, we conclude that $\mathcal{B}/I_\mu$ is a complete Boolean algebra. We shall return to this in Part II.

Assuming the Axiom of Choice one can show that there exists a set of reals that is not Lebesgue measurable. One such example is the Vitali set in Exercise 10.1. As another example there exists a set $X \subset \boldsymbol{R}^n$ such that neither X nor its complement has a perfect subset (see Exercise 5.1 for a construction of such a set). The set X is not measurable: Otherwise, e.g., $\mu(X) > 0$ and by (11.15)(vi) there is a closed $K \subset X$ such that $\mu(K) > 0$; thus K is uncountable and hence contains a perfect subset, a contradiction.

However, we shall show in Part II that it is consistent (with ZF + DC) that all sets or reals are Lebesgue measurable.

We conclude this review of Lebesgue measurability with two lemmas. One is the well-known Fubini Theorem, and we state it here, without proof, for the sake of completeness. The other lemma will be used in the proof of Theorem 11.18 below.

If A is a subset of the plane $\boldsymbol{R}^2$ and $x \in \boldsymbol{R}$, let A_x denote the set $\{y : (x, y) \in A\}$.

Lemma 11.12. *Let $A \subset \boldsymbol{R}^2$ be a measurable set. Then A is null if and only if for almost all x, A_x is null (i.e., the set $\{x : A_x$ is not null$\}$ is null).* □

Lemma 11.13. *For any set $X \subset \boldsymbol{R}^n$ there exists a measurable set $A \supset X$ with the property that whenever $Z \subset A - X$ is measurable, then Z is null.*

Proof. If $\mu^*(X) < \infty$, then because $\mu^*(X) = \inf\{\mu(A) : A$ is measurable and $A \supset X\}$, there is a measurable $A \supset X$ such that $\mu(A) = \mu^*(X)$; clearly such an A will do. If $\mu^*(X) = \infty$, there exist pairwise disjoint X_n such that $X = \bigcup_{n=0}^{\infty} X_n$ and that for each n, $\mu^*(X_n) < \infty$. Let $A_n \supset X_n$, $n < \omega$, be measurable sets such that $\mu(A_n) = \mu^*(X_n)$, and let $A = \bigcup_{n=0}^{\infty} A_n$. □

It should be mentioned that the main results of descriptive set theory on Lebesgue measure can be proved in a more general context, namely for reasonable σ-additive measures on Polish spaces. An example of such a measure is the product measure in the Cantor space $\{0, 1\}^\omega$.

The Property of Baire

In Chapter 4 we proved the Baire Category Theorem (Theorem 4.8): The intersection of countably many dense open sets of reals is nonempty. It is fairly easy to see that the proof works not only for the real line $\boldsymbol{R}$ but for any Polish space.

Let us consider a Polish space X. Let us call a set $A \subset X$ *nowhere dense* if the complement of A contains a dense open set. Note that A is nowhere dense just in case for every nonempty open set G, there is a nonempty open set $H \subset G$ such that $A \cap H = \emptyset$. A set A is nowhere dense if and only if its closure $\overline{A}$ is nowhere dense.

A set $A \subset X$ is *meager* (or of *first category*) if A is the union of countably many nowhere dense sets. A nonmeager set is called a set of *second category*.

The Baire Category Theorem states in effect that in a Polish space every nonempty open set is of second category.

The meager sets form a σ-ideal. Moreover, in case of $\boldsymbol{R}^n$, $\mathcal{N}$, or the Cantor space, every singleton $\{x\}$ is nowhere dense and so the ideal of meager sets contains all countable sets.

Definition 11.14. A set A has the *Baire property* if there exists an open set G such that $A \triangle G$ is meager.

Clearly, every meager set has the Baire property. Note that if G is open, then $\overline{G} - G$ is nowhere dense. Hence if $A \triangle G$ is meager then $(X - A) \triangle (X - \overline{G}) = A \triangle \overline{G}$ is meager, and it follows that the complement of a set with the Baire property also has the Baire property. It is also easy to see that the union of countably many sets with the Baire property has the Baire property and we have:

Lemma 11.15. *The sets having the Baire property form a σ-algebra; hence every Borel set has the Baire property.* □

If $\mathcal{B}$ denotes the σ-algebra of Borel sets, and if we denote by $\mathcal{C}$ the σ-algebra of sets with the Baire property, and if I is the σ-ideal of meager sets, we have $\mathcal{B}/I = \mathcal{C}/I$. Note that the algebra $\mathcal{B}/I$ is σ-saturated: Let O be a countable topology base for X. For each nonmeager set X with the Baire property there exists $G \in O$ such that $G - X$ is meager. Thus the set $D = \{[G] : G \in O\}$ of equivalence classes is a dense set in $\mathcal{B}/I$. Hence $\mathcal{B}/I$ is σ-saturated and is a complete Boolean algebra.

The Axiom of Choice implies that sets without the Baire property exist. For instance, the Vitali set (Exercise 10.1) is such, see Exercise 11.7.

If $X \subset \boldsymbol{R}^n$ is such that neither X nor its complement has a perfect subset, then X does not have the Baire property: Otherwise, e.g., X is of second category and hence X contains a G_δ subset G of second category. Now G is uncountable, and this is a contradiction since as we shall prove in Theorem 11.18, every uncountable Borel set (even analytic) has a perfect subset.

The following two lemmas are analogs of Lemmas 11.12 and 11.13. The first one, although not very difficult to prove, is again stated without proof.

Lemma 11.16. *Let $A \subset \boldsymbol{R}^2$ have the property of Baire. Then A is meager if and only if A_x is meager for all x except a meager set.* □

Lemma 11.17. *For any set S in a Polish space X, there exists a set $A \supset S$ that has the Baire property and such that whenever $Z \subset A - S$ has the Baire property, then Z is meager.*

Proof. Let us consider a fixed countable topology basis O for X. Let $S \subset X$. Let

$$D(S) = \{x \in X : \text{for every } U \in O \text{ such that } x \in U,\ U \cap S \text{ is not meager}\}.$$

Note that the complement of $D(S)$ is the union of open sets and hence open; thus $D(S)$ is closed.

The set $S - D(S)$ is the union of all $S \cap U$ where $U \in O$ and $S \cap U$ is meager; since O is countable, $X - D(S)$ is meager. Let

$$A = S \cup D(S).$$

Since $A = (S - D(S)) \cup D(S)$ is the union of a meager and a closed set, A has the Baire property.

Let $Z \subset A - S$ have the Baire property; we shall show that Z is meager. Otherwise there is $U \in O$ such that $U - Z$ is meager; hence $U \cap S$ is meager. Since $U \cap Z \neq \emptyset$ and $Z \subset D(S)$, there is $x \in U$ such that $x \in D(S)$, and hence $U \cap S$ is not meager, a contradiction. □

Although both "null" and "meager" mean in a sense "negligible," see Exercise 11.8 that shows that the real line can be decomposed into a null set and a meager set.

Analytic Sets: Measure, Category, and the Perfect Set Property

Theorem 11.18.

(i) *Every analytic set of reals is Lebesgue measurable.*
(ii) *Every analytic set has the Baire property.*
(iii) *Every uncountable analytic set contains a perfect subset.*

Corollary 11.19. *Every* $\mathbf{\Pi}^1_1$ *set of reals is Lebesgue measurable and has the Baire property.* □

Corollary 11.20. *Every analytic* (*and in particular every Borel*) *set is either at most countable or has cardinality* $\mathfrak{c}$. □

We prove (ii) and (iii) for an arbitrary Polish space. The proof of (i) is general enough to work for other measures (in Polish spaces) as well.

Proof. The proof of (i) and (ii) is exactly the same and uses either Lemma 11.13 or Lemma 11.17 (and basic facts on Lebesgue measure and the Baire property). We give the proof of (i) and leave (ii) to the reader.

Let A be an analytic set of reals (or a subset of $\boldsymbol{R}^n$). Let $f : \mathcal{N} \to \boldsymbol{R}$ be a continuous function such that $A = f(\mathcal{N})$. For each $s \in Seq$, let $A_s = f(O(s))$. We have

$$A = \mathcal{A}\{A_s : s \in Seq\} = \mathcal{A}\{\overline{A_s} : s \in Seq\}, \tag{11.16}$$

and for every $s \in Seq$,

$$A_s = \bigcup_{n=0}^{\infty} A_{s^\frown n}. \tag{11.17}$$

By Lemma 11.13, there exists for each $s \in Seq$ a measurable set $B_s \supset A_s$ such that every measurable $Z \subset B_s - A_s$ is null. Since $\overline{A_s}$ is measurable, we may actually find B_s such that $A_s \subset B_s \subset \overline{A_s}$.

Let $B = B_\emptyset$. Since B is measurable, it suffices to show that $B - A$ is a null set. Notice that because $A_s \subset B_s \subset \overline{A_s}$, and because (11.16) holds, we have

$$A = \mathcal{A}\{B_s : s \in Seq\}.$$

Thus

$$B - A = B - \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} B_{a\restriction n}.$$

We claim that

$$B - \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} B_{a\restriction n} \subset \bigcup_{s\in Seq} \Big(B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}\Big). \tag{11.18}$$

To prove (11.18), assume that $x \in B$ is such that x is not a member of the right-hand side. Then for every s, if $x \in B_s$, then $x \in B_{s^\frown k}$ for some k. Hence there is k_0 such that $x \in B_{\langle k_0\rangle}$, then there is k_1 such that $x \in B_{\langle k_0,k_1\rangle}$, etc. Let $a = \langle k_0, k_1, k_2, \ldots\rangle$; we have $x \in \bigcap_{n=0}^{\infty} B_{a\restriction n}$ and hence x is not a member of the left-hand side.

Thus we have

$$B - A \subset \bigcup_{s\in Seq} \Big(B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}\Big).$$

Since Seq is a countable set, it suffices to show that each $B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}$ is null. Let $s \in Seq$, and let $Z = B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}$. We have

$$Z = B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k} \subset B_s - \bigcup_{k=0}^{\infty} A_{s^\frown k} = B_s - A_s.$$

Now because $Z \subset B_s - A_s$ and because Z is measurable, Z must be null.

(iii) The proof is a variant of the Cantor-Bendixson argument for closed sets in the Baire space. Recall that every closed set F in $\mathcal{N}$ is of the form $F = [T] = \{a : \forall n\ a\restriction n \in T\}$, where T is a tree, $T \subset Seq$. For each tree $T \subset Seq$ and each $s \in Seq$, let T_s denote the tree $\{t \in T : t \subset s \text{ or } s \subset t\}$; note that $[T_s] = [T] \cap O(s)$.

Let A be an analytic set (in a Polish space X), and let f be a continuous function such that $A = f(\mathcal{N})$. For each tree $T \subset Seq$, we define

$$T' = \{s \in T : f([T_s]) \text{ is uncountable}\}.$$

For each $\alpha < \omega_1$, we define $T^{(\alpha)}$ as follows:

$$T^{(0)} = Seq, \qquad T^{(\alpha+1)} = (T^{(\alpha)})',$$
$$T^{(\alpha)} = \bigcap_{\beta<\alpha} T^{(\beta)} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Let $\alpha < \omega_1$ be the least ordinal such that $T^{(\alpha+1)} = T^{(\alpha)}$. If $T^{(\alpha)} = \emptyset$, then

$$A = \bigcup_{\beta<\alpha} \{f([T_s^{(\beta)}]) : s \in T^{(\beta)} - T^{(\beta+1)}\},$$

and hence A is countable. Thus if A is uncountable, $T^{(\alpha)}$ is nonempty and for every $s \in T^{(\alpha)}$, $f([T_s^{(\alpha)}])$ is uncountable. In this case, we shall find a perfect subset of A.

Let $s \in T^{(\alpha)}$ be arbitrary. Since $f([T_s^{(\alpha)}])$ has at least two elements, there exist $s_{\langle 0\rangle} \supset s$ and $s_{\langle 1\rangle} \supset s$ (in $T^{(\alpha)}$) such that $f([T_{s_{\langle 0\rangle}}^{(\alpha)}])$ and $f([T_{s_{\langle 1\rangle}}^{(\alpha)}])$ are disjoint. Then there are $s_{\langle 0,0\rangle} \supset s_{\langle 0\rangle}$ and $s_{\langle 0,1\rangle} \supset s_{\langle 0\rangle}$, and $s_{\langle 1,0\rangle} \supset s_{\langle 1\rangle}$, $s_{\langle 1,1\rangle} \supset s_{\langle 1\rangle}$ such that the four sets $f([T_{s_{\langle i,j\rangle}}^{(\alpha)}])$, $i, j = 0, 1$ are pairwise disjoint. In this fashion we construct $s_t \in T^{(\alpha)}$ for each finite 0–1 sequence t. These elements s_t generate a subtree $U = \{s : s \subset s_t \text{ for some } t\}$ of $T^{(\alpha)}$ such that (1) U is perfect, (2) every s has at most two immediate successors in U (hence $[U]$ is a compact set in $\mathcal{N}$), and (3) f is one-to-one on $[U]$.

Let P be the image of $[U]$ under the function f. Since $[U]$ is compact and f is continuous, P is also compact, and hence closed. Moreover, P has no isolated points because $[U]$ is perfect and f is continuous. Thus P is a perfect subset of A. □

Exercises

11.1. The operations $\bigcup_{n=0}^{\infty}$ and $\bigcap_{n=0}^{\infty}$ are special cases of the operation $\mathcal{A}$.

11.2. Let A_s, $s \in Seq$, be Borel sets satisfying (11.10) and the additional condition: For each $s \in Seq$ and all $n \neq m$, $A_{s^\frown n} \cap A_{s^\frown m} = \emptyset$. Then $\mathcal{A}\{A_s : s \in Seq\}$ is a Borel set.

[$\bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} A_{a\restriction n} = \bigcap_{n=0}^{\infty} \bigcup\{A_s : \text{length}(s) = n\}$.]

11.3. Let A_n, $n = 0, 1, 2, \ldots$, be pairwise disjoint analytic sets. Then there exist pairwise disjoint Borel sets D_n such that $A_n \subset D_n$ for all n.

[Modify the proof of Lemma 11.11.]

11.4. If A is a null set and $a_0 \geq a_1 \geq \ldots \geq a_n \geq \ldots$ is a sequence of positive numbers with $\lim_n a_n = 0$, then there exists a sequence G_n, $n = 0, 1, \ldots$, of finite unions of open intervals such that $A \subset \bigcup_{n=0}^{\infty} G_n$ and $\mu(G_n) < a_n$ for each n. Moreover, the intervals can be required to have rational endpoints.

[First find a sequence of open intervals I_k such that $A \subset \bigcup_{k=0}^{\infty} I_k$ and $\sum_{k=0}^{\infty} \mu(I_k) \leq a_0$.]

11.5. For every set A with the Baire property, there exist a G_δ set G and an F_σ set F such that $G \subset A \subset F$ and such that $F - G$ is meager.

[Note that every meager set is included in a meager F_σ set.]

11.6. For every set A with the Baire property, there exists a unique regular open set U such that $A \triangle U$ is meager.

[An open set U is *regular* if $U = \text{int}(\overline{U})$.]

11.7. The Vitali set M from Exercise 10.1 does not have the Baire property.

["Meager" and "Baire property" are invariant under translation. If M has the Baire property, then there is an interval (a, b) such that $(a, b) - M$ is meager. Then $(a, b) \cap M_q$ is meager for all rational $q \neq 0$, hence each $M \cap (a - q, b - q)$ is meager, hence M is meager, hence each M_q is meager; a contradiction since $\boldsymbol{R} = \bigcup_{q\in\boldsymbol{Q}} M_q$.]

11.8. There is a null set of reals whose complement is meager.

[Let $q_1, q_2, \ldots$ be an enumeration of the rationals. For each $n \geq 1$ and $k \geq 1$, let $I_{n,k}$ be the open interval with center q_n and length $1/(k \cdot 2^n)$. Let $D_k = \bigcup_{n=1}^{\infty} I_{n,k}$, and $A = \bigcap_{k=1}^{\infty} D_k$. Each D_k is open and dense, and $\mu(D_k) \leq 1/k$. Hence A is null and $\boldsymbol{R} - A$ is meager.]

Historical Notes

Borel sets were introduced by Borel in [1905]. Lebesgue in [1905] proved in effect Lemma 11.2. Suslin's discovery of an error in a proof in Lebesgue's article led to a construction of an analytic non-Borel set and introduction of the operation $\mathcal{A}$. The basic results on analytic sets as well as Theorem 11.10 appeared in Suslin's article [1917].

Projective sets were introduced by Luzin [1925] and [1927a], and Sierpiński [1925] and [1927]. The present notation ($\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$) appeared first in the paper [1959] of Addison who noticed the analogy between Luzin's hierarchy of projective sets and Kleene's hierarchy of analytic predicates [1955].

Lemma 11.8: Luzin [1930].

Lemma 11.11: Luzin [1927b].

For detailed treatment of Lebesgue measure, we refer the reader to Halmos' book [1950]; Lebesgue introduced his measure and integral in his thesis [1902]. Sets of first and second category were introduced by Baire [1899].

Lemmas 11.13 and 11.17: Marczewski [1930a].

Lemma 11.16: Kuratowski and Ulam [1932].

Theorem 11.18(i) (measurability of analytic sets) is due to Luzin [1917]. Theorem 11.18(ii) (Baire property) is due to Luzin and Sierpiński [1923] and Theorem 11.18(iii) (perfect subsets) is due to Suslin; cf. Luzin [1930]. The present proof of (i) and (ii) follows Marczewski [1930a]. Prior to Suslin (and following the Cantor-Bendixson Theorem for closed sets) Young proved in [1906] the perfect subset result for G_δ and F_σ sets; and Hausdorff [1916] and Aleksandrov [1916] proved the same for Borel sets.

12. Models of Set Theory

Modern set theory uses extensively construction of models to establish relative consistency of various axioms and conjectures. As the techniques often involve standard model-theoretic concepts, we assume familiarity with basic notions of models and satisfaction, submodels and embeddings, as well as Skolem functions, direct limit and ultraproducts. We shall review the basic notions, notation and terminology of model theory.

Review of Model Theory

A *language* is a set of symbols: relation symbols, function symbols, and constant symbols:

$$\mathcal{L} = \{P, \ldots, F, \ldots, c, \ldots\}.$$

Each P is assumed to be an n-placed relation for some integer $n \geq 1$; each F is an m-placed function symbol for some $m \geq 1$.

Terms and *formulas* of a language $\mathcal{L}$ are certain finite sequences of symbols of $\mathcal{L}$, and of *logical symbols* (identity symbol, parentheses, variables, connectives, and quantifiers). The set of all terms and the set of all formulas are defined by recursion. If the language is countable (i.e., if $|\mathcal{L}| \leq \aleph_0$), then we may identify the symbols of $\mathcal{L}$, as well as the logical symbols, with some hereditarily finite sets (elements of V_ω); then formulas are also hereditarily finite.

A *model* for a given language $\mathcal{L}$ is a pair $\mathfrak{A} = (A, \mathcal{I})$, where A is the universe of $\mathfrak{A}$ and $\mathcal{I}$ is the *interpretation* function which maps the symbols of $\mathcal{L}$ to appropriate relations, functions, and constants in A. A model for $\mathcal{L}$ is usually written in displayed form as

$$\mathfrak{A} = (A, P^{\mathfrak{A}}, \ldots, F^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots)$$

By recursion on length of terms and formulas one defines the *value* of a term

$$t^{\mathfrak{A}}[a_1, \ldots, a_n]$$

and *satisfaction*

$$\mathfrak{A} \vDash \varphi[a_1, \ldots, a_n]$$

where t is a term, φ is a formula, and $\langle a_1, \ldots, a_n \rangle$ is a finite sequence in A.

Two models $\mathfrak{A} = (A, P, \ldots, F, \ldots, c, \ldots)$ and $\mathfrak{A}' = (A', P', \ldots, F', \ldots, c', \ldots)$ are *isomorphic* if there is an *isomorphism* between $\mathfrak{A}$ and $\mathfrak{A}'$, that is a one-to-one function f of A onto A' such that

(i) $P(x_1, \ldots, x_n)$ if and only if $P'(f(x_1), \ldots, f(x_n))$,
(ii) $f(F(x_1, \ldots, x_n)) = F'(f(x_1), \ldots, f(x_n))$,
(iii) $f(c) = c'$,

for all relations, functions, and constants of $\mathfrak{A}$. If f is an isomorphism, then $f(t^{\mathfrak{A}}[a_1, \ldots, a_n]) = t^{\mathfrak{A}'}[f(a_1), \ldots, f(a_n)]$ for each term, and

$$\mathfrak{A} \vDash \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathfrak{A}' \vDash \varphi[f(a_1), \ldots, f(a_n)]$$

for each formula φ and all $a_1, \ldots, a_n \in A$.

A *submodel* of $\mathfrak{A}$ is a subset $B \subset A$ endowed with the relations $P^{\mathfrak{A}} \cap B^n$, ..., functions $F^{\mathfrak{A}} \restriction B^m$, ..., and constants $c^{\mathfrak{A}}$, ...; all $c^{\mathfrak{A}}$ belong to B, and B is closed under all $F^{\mathfrak{A}}$ (if $(x_1, \ldots, x_m) \in B^m$, then $F^{\mathfrak{A}}(x_1, \ldots, x_m) \in B$).

An *embedding* of $\mathfrak{B}$ into $\mathfrak{A}$ is an isomorphism between $\mathfrak{B}$ and a submodel $\mathfrak{B}' \subset \mathfrak{A}$.

A submodel $\mathfrak{B} \subset \mathfrak{A}$ is an *elementary submodel*

$$\mathfrak{B} \prec \mathfrak{A}$$

if for every formula φ, and every $a_1, \ldots, a_n \in B$,

$$\mathfrak{B} \vDash \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathfrak{A} \vDash \varphi[a_1, \ldots, a_n]. \tag{12.1}$$

Two models $\mathfrak{A}$, $\mathfrak{B}$ are *elementarily equivalent* if they satisfy the same sentences.

The key lemma in construction of elementary submodels is this: A subset $B \subset A$ forms an elementary submodel of $\mathfrak{A}$ if and only if for every formula $\varphi(u, x_1, \ldots, x_n)$, and every $a_1, \ldots, a_n \in B$,

(12.2) if $\exists a \in A$ such that $\mathfrak{A} \vDash \varphi[a, a_1, \ldots, a_n]$, then $\exists a \in B$ such that $\mathfrak{A} \vDash \varphi[a, a_1, \ldots, a_n]$.

A function $h : A^n \to A$ is a *Skolem function* for φ if

$$(\exists a \in A)\, \mathfrak{A} \vDash \varphi[a, a_1, \ldots, a_n] \quad \text{implies} \quad \mathfrak{A} \vDash \varphi[h(a_1, \ldots, a_n), a_1, \ldots, a_n]$$

for every a_1, ..., a_n. Using the Axiom of Choice, one can construct a Skolem function for every φ. If a subset $B \subset A$ is closed under (some) Skolem functions for all formulas, then B satisfies (12.2) and hence forms an elementary submodel of $\mathfrak{A}$.

Given a set of Skolem functions, one for each formula of $\mathcal{L}$, the closure of a set $X \subset A$ is a *Skolem hull* of X. It is clear that the Skolem hull of X is an elementary submodel of $\mathfrak{A}$, and has cardinality at most $|X| \cdot |\mathcal{L}| \cdot \aleph_0$. In particular, we have the following:

Theorem 12.1 (Löwenheim-Skolem). *Every infinite model for a countable language has a countable elementary submodel.* □

An *elementary embedding* is an embedding whose range is an elementary submodel.

A set $X \subset A$ is *definable over* $\mathfrak{A}$ if there exist a formula φ and some $a_1, \ldots, a_n \in A$ such that

$$X = \{x \in A : \mathfrak{A} \vDash \varphi[x, a_1, \ldots, a_n]\}.$$

We say that X is *definable in* $\mathfrak{A}$ *from* $a_1, \ldots, a_n$. If φ is a formula of x only, without parameters $a_1, \ldots, a_n$, then X is *definable* in $\mathfrak{A}$. An element $a \in A$ is *definable* (*from* $a_1, \ldots, a_n$) if the set $\{a\}$ is definable (from $a_1, \ldots, a_n$).

Gödel's Theorems

The cornerstone of modern logic are Gödel's theorems: the Completeness Theorem and two incompleteness theorems.

A set Σ of sentences of a language $\mathcal{L}$ is *consistent* if there is no formal proof of contradiction from Σ. The Completeness Theorem states that every consistent set of sentences has a model.

The First Incompleteness Theorem shows that no consistent (recursive) extension of Peano Arithmetic is complete: there exists a statement that is undecidable in the theory. In particular, if ZFC is consistent (as we believe), no additional axioms can prove or refute every sentence in the language of set theory.

The Second Incompleteness Theorem proves that sufficiently strong mathematical theories such as Peano Arithmetic or ZF (if consistent) cannot prove its own consistency. Gödel's Second Incompleteness Theorem implies that it is unprovable in ZF that there exists a model of ZF. This fact is significant for the theory of large cardinals, and we shall return to it later in this chapter.

Direct Limits of Models

An often used construction in model theory is the direct limit of a directed system of models. A *directed set* is a partially ordered set $(D, <)$ such that for every $i, j \in D$ there is a $k \in D$ such that $i \leq k$ and $j \leq k$.

First consider a system of models $\{\mathfrak{A}_i : i \in D\}$, indexed by a directed set D, such that for all $i, j \in D$, if $i < j$ then $\mathfrak{A}_i \prec \mathfrak{A}_j$. Let $\mathfrak{A} = \bigcup_{i \in D} \mathfrak{A}_i$; i.e., the universe of $\mathfrak{A}$ is the union of the universes of the $\mathfrak{A}_i$, $P^{\mathfrak{A}} = \bigcup_{i \in D} P^{\mathfrak{A}_i}$, etc. It is easily proved by induction on the complexity of formulas that $\mathfrak{A}_i \prec \mathfrak{A}$ for all i.

In general, we consider a *directed system* of models which consists of models $\{\mathfrak{A}_i : i \in D\}$ together with elementary embeddings $e_{i,j} : \mathfrak{A}_i \to \mathfrak{A}_j$ such that $e_{i,k} = e_{j,k} \circ e_{i,j}$ for all $i < j < k$.

Lemma 12.2. *If* $\{\mathfrak{A}_i, e_{i,j} : i, j \in D\}$ *is a directed system of models, there exists a model* $\mathfrak{A}$*, unique up to isomorphism, and elementary embeddings* $e_i : \mathfrak{A}_i \to \mathfrak{A}$ *such that* $\mathfrak{A} = \bigcup_{i \in D} e_i(\mathfrak{A}_i)$ *and that* $e_i = e_j \circ e_{i,j}$ *for all* $i < j$*.*

The model $\mathfrak{A}$ is called the *direct limit* of $\{\mathfrak{A}_i, e_{i,j}\}_{i,j \in D}$.

Proof. Consider the set S of all pairs (i, a) such that $i \in D$ and $a \in A_i$, and define an equivalence relation on S by

$$(i, a) \equiv (j, b) \leftrightarrow \exists k\, (i \leq k,\ j \leq k \text{ and } e_{i,k}(a) = e_{j,k}(b)).$$

Let $A = S/\equiv$ be the set of all equivalence classes, and let $e_i(a) = [(i, a)]$ for all $i \in D$ and $a \in A_i$. The rest is routine. □

In set theory, a frequent application of direct limits involves the case when D is an ordinal number (and $<$ is its well-ordering).

Reduced Products and Ultraproducts

An important method in model theory uses filters and ultrafilters. Let S be a nonempty set and let $\{\mathfrak{A}_x : x \in S\}$ be a system of models (for a language $\mathcal{L}$). Let F be a filter on S. Consider the set

$$A = \prod_{x \in S} A_x / =_F$$

where $=_F$ is the equivalence relation on $\prod_{x \in x} A_x$ defined as follows:

(12.3) $\qquad f =_F g \quad$ if and only if $\quad \{x \in S : f(x) = g(x)\} \in F.$

It follows easily that $=_F$ is an equivalence relation.

The model $\mathfrak{A}$ with universe A is obtained by interpreting the language as follows:

If $P(x_1, \ldots, x_n)$ is a predicate, let

(12.4) $\quad P^{\mathfrak{A}}([f_1], \ldots, [f_n])$ if and only if $\{x \in S : P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \in F.$

If $F(x_1, \ldots, x_n)$ is a function, let

(12.5) $\quad F^{\mathfrak{A}}([f_1], \ldots, [f_n]) = [f]$ where $f(x) = F^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))$ for all $x \in S$.

If c is a constant, let

(12.6) $\quad c^{\mathfrak{A}} = [f]$ where $f(x) = c^{\mathfrak{A}_x}$ for all $x \in S$.

(Note that (12.4) and (12.5) does not depend on the choice of representatives from the equivalence classes $[f_1], \ldots, [f_n]$).

The model $\mathfrak{A}$ is called a *reduced product* of $\{\mathfrak{A}_x : x \in S\}$ (by F).

Reduced products are particularly important in the case when the filter is an ultrafilter. If U is an ultrafilter on S then the reduced product defined in (12.3)–(12.6) is called the *ultraproduct* of $\{\mathfrak{A}_x : x \in S\}$ by U:

$$\mathfrak{A} = \mathrm{Ult}_U\{\mathfrak{A}_x : x \in S\}.$$

The importance of ultraproducts is due mainly to the following fundamental property.

Theorem 12.3 (Łoś). *Let U be an ultrafilter on S and let $\mathfrak{A}$ be the ultraproduct of $\{\mathfrak{A}_x : x \in S\}$ by U.*

(i) *If φ is a formula, then for every $f_1, \dots, f_n \in \prod_{x\in S} A_x$,*

$$\mathfrak{A} \vDash \varphi([f_1], \dots, [f_n]) \quad \textit{if and only if} \quad \{x \in S : \mathfrak{A}_x \vDash \varphi[f_1(x), \dots, f_n(x)]\} \in U.$$

(ii) *If σ is a sentence, then*

$$\mathfrak{A} \vDash \sigma \quad \textit{if and only if} \quad \{x \in S : \mathfrak{A}_x \vDash \sigma\} \in U.$$

Part (ii) is a consequence of (i). Note that by the theorem, the satisfaction of φ at $[f_1]$, …, $[f_n]$ does not depend on the choice of representatives f_1, …, f_n for the equivalence classes $[f_1]$, …, $[f_n]$. Thus we may further abuse the notation and write

$$\mathfrak{A} \vDash \varphi[f_1, \dots, f_n].$$

It will also be convenient to adopt a measure-theoretic terminology. If

$$\{x \in S : \mathfrak{A}_x \vDash \varphi[f_1(x), \dots, f_n(x)]\} \in U$$

we say that $\mathfrak{A}_x$ satisfies $\varphi(f_1(x), \dots, f_n(x))$ for *almost all* x, or that $\mathfrak{A}_x \vDash \varphi(f_1(x), \dots, f_n(x))$ holds *almost everywhere*. In this terminology, Łoś's Theorem states that $\varphi(f_1, \dots, f_n)$ holds in the ultraproduct if and only if for almost all x, $\varphi(f_1(x), \dots, f_n(x))$ holds in $\mathfrak{A}_x$.

Proof. We shall prove (i) by induction on the complexity of formulas. We shall prove that (i) holds for atomic formulas, and then prove the induction step for $\neg$, $\wedge$, and $\exists$.

Atomic formulas. First we consider the formula $u = v$, and we have

$$\begin{aligned} \mathfrak{A} \vDash [f] = [g] &\leftrightarrow [f] = [g] \\ &\leftrightarrow f =_U g \\ &\leftrightarrow \{x : f(x) = g(x)\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash f(x) = g(x)\} \in U. \end{aligned} \tag{12.7}$$

For a predicate $P(v_1, \dots, v_n)$ we have

$$\begin{aligned} \mathfrak{A} \vDash P([f_1], \dots, [f_n]) &\leftrightarrow P^{\mathfrak{A}}([f_1], \dots, [f_n]) \\ &\leftrightarrow \{x : P^{\mathfrak{A}_x}(f_1(x), \dots, f_n(x))\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash P(f_1(x), \dots, f_n(x))\} \in U. \end{aligned} \tag{12.8}$$

Both (12.7) and (12.8) remain true if variables are replaced by terms, and so (i) holds for all atomic formulas.

Logical connectives. First we assume that (i) holds for φ and show that it also holds for $\neg\varphi$ (here we use that $X \in U$ if and only if $S - X \notin U$).

$$\begin{aligned} \mathfrak{A} \vDash \neg\varphi[f] &\leftrightarrow \text{not } \mathfrak{A} \vDash \varphi[f] \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \varphi[f(x)]\} \notin U \\ &\leftrightarrow \{x : \mathfrak{A}_x \nvDash \varphi[f(x)]\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \neg\varphi[f(x)]\} \in U. \end{aligned}$$

Similarly, if (i) is true for φ and ψ, we have

$$\begin{aligned} \mathfrak{A} \vDash \varphi \wedge \psi &\leftrightarrow \mathfrak{A} \vDash \varphi \text{ and } \mathfrak{A} \vDash \psi \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \varphi\} \in U \text{ and } \{x : \mathfrak{A}_x \vDash \psi\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \varphi \wedge \psi\} \in U \end{aligned}$$

(The last equivalence uses this: $X \in U$ and $Y \in U$ if and only if $X \cap Y \in U$.)

Existential quantifier. We assume that (i) is true for $\varphi(u, v_1, \dots, v_n)$ and show that it remains true for the formula $\exists u\, \varphi$. Let us assume first that

$$\mathfrak{A} \vDash \exists u\, \varphi[f_1, \dots, f_n]. \tag{12.9}$$

Then there is $g \in \prod_{x \in S} A_x$ such that $\mathfrak{A} \vDash \varphi[g, f_1, \dots, f_n]$, and therefore

$$\{x : \mathfrak{A}_x \vDash \varphi[g(x), f_1(x), \dots, f_n(x)]\} \in U, \tag{12.10}$$

and it clearly follows that

$$\{x : \mathfrak{A}_x \vDash \exists u\, \varphi[u, f_1(x), \dots, f_n(x)]\} \in U. \tag{12.11}$$

Now let us assume that (12.11) holds. For each $x \in S$, let $u_x \in A_x$ be such that $\mathfrak{A}_x \vDash [u_x, f_1(x), \dots, f_n(x)]$ if such u_x exists, and arbitrary otherwise. If we define $g \in \prod_{x \in S} A_x$ by $g(x) = u_x$, then we have (12.10), and therefore

$$\mathfrak{A} \vDash \varphi[g, f_1, \dots, f_n].$$

Now (12.9) follows. □

Let us consider now the special case of ultraproducts, when each $\mathfrak{A}_x$ is the same model $\mathfrak{A}$. Then the ultraproduct is called an *ultrapower* of $\mathfrak{A}$; denoted $\mathrm{Ult}_U\,\mathfrak{A}$.

Corollary 12.4. *An ultrapower of a model $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{A}$.*

Proof. By Theorem 12.3(ii) we have $\mathrm{Ult}_U\,\mathfrak{A} \vDash \sigma$ if and only if $\{x : \mathfrak{A} \vDash \sigma\}$ is either S or empty, according to whether $\mathfrak{A} \vDash \sigma$ or not. □

We shall now show that a model $\mathfrak{A}$ is elementarily embeddable in its ultrapower. If U is an ultrafilter on S, we define the *canonical embedding* $j : \mathfrak{A} \to \mathrm{Ult}_U\,\mathfrak{A}$ as follows: For each $a \in A$, let c_a be the *constant function* with value a:

$$c_a(x) = a \qquad (\text{for every } x \in S), \tag{12.12}$$

and let

$$j(a) = [c_a]. \tag{12.13}$$

Corollary 12.5. *The canonical embedding $j : \mathfrak{A} \to \mathrm{Ult}_U\,\mathfrak{A}$ is an elementary embedding.*

Proof. Let $a \in A$. By Łoś's Theorem, $\mathrm{Ult}_U\,\mathfrak{A} \vDash \varphi[j(a)]$ if and only if $\mathrm{Ult}_U\,\mathfrak{A} \vDash \varphi[c_a]$ if and only if $\mathfrak{A} \vDash \varphi[a]$ for almost all x if and only if $\mathfrak{A} \vDash \varphi[a]$. □

Models of Set Theory and Relativization

The language of set theory consists of one binary predicate symbol $\in$, and so models of set theory are given by its universe M and a binary relation E on M that interprets $\in$.

We shall also consider models of set theory that are proper classes. However, due to Gödel's Second Incompleteness Theorem, we have to be careful how the generalization is formulated.

Definition 12.6. Let M be a class, E a binary relation on M and let $\varphi(x_1, \ldots, x_n)$ be a formula of the language of set theory. The *relativization* of φ to M, E is the formula

$$\varphi^{M,E}(x_1, \ldots, x_n) \tag{12.14}$$

defined inductively as follows:

$$\begin{aligned} (x \in y)^{M,E} &\leftrightarrow x\ E\ y \\ (x = y)^{M,E} &\leftrightarrow x = y \\ (\neg\varphi)^{M,E} &\leftrightarrow \neg\varphi^{M,E} \\ (\varphi \wedge \psi)^{M,E} &\leftrightarrow \varphi^{M,E} \wedge \psi^{M,E} \\ (\exists x\,\varphi)^{M,E} &\leftrightarrow (\exists x \in M)\,\varphi^{M,E} \end{aligned} \tag{12.15}$$

and similarly for the other connectives and $\forall$.

When E is $\in$, we write φ^M instead of $\varphi^{M,\in}$.

When using relativization $\varphi^{M,E}(x_1, \ldots, x_n)$ it is implicitly assumed that the variables $x_1, \ldots, x_n$ range over M. We shall often write

$$(M, E) \vDash \varphi(x_1, \ldots, x_n)$$

instead of (12.14) and say that the *model* (M, E) satisfies φ. We point out however that while this is a legitimate statement in every particular case of φ, the general satisfaction relation is formally undefinable in ZF.

Let *Form* denote the set of all formulas of the language $\{\in\}$. As with any actual (metamathematical) natural number we can associate the corresponding element of $\boldsymbol{N}$, we can similarly associate with any given formula of set theory the corresponding element of the set *Form*. To make the distinction, if φ is a formula, let $\ulcorner\varphi\urcorner$ denote the corresponding element of *Form*.

If M is a set and E is a binary relation on M and if $a_1, \ldots, a_n$ are elements of M, then

$$\varphi^{M,E}(a_1, \ldots, a_n) \leftrightarrow (M, E) \vDash \ulcorner\varphi\urcorner[a_1, \ldots, a_n] \tag{12.16}$$

as can easily be verified. Thus in the case when M is a set and φ a particular (metamathematical) formula, we shall not make a distinction between the two meanings of the symbol $\vDash$. We note however that the left-hand side of (12.16) (relativization) is *not* defined for $\varphi \in$ *Form*, and the right-hand side (satisfaction) is *not* defined if M is a proper class.

Below we sketch a proof of a theorem of Tarski, closely related to Gödel's Second Incompleteness Theorem. The theorem states that there is no set-theoretical property $T(x)$ such that if σ is a sentence that $T(\ulcorner\sigma\urcorner)$ holds if and only if σ holds.

Let us arithmetize the syntax and consider some fixed effective enumeration of all expressions by natural numbers (*Gödel numbering*). In particular, if σ is a sentence, then $\#\sigma$ is the Gödel number of σ, a natural number. We say that $T(x)$ is a *truth definition* if:

(12.17) (i) $\forall x\,(T(x) \to x \in \omega)$;
(ii) if σ is a sentence, then $\sigma \leftrightarrow T(\#\sigma)$.

Theorem 12.7 (Tarski). *A truth definition does not exist.*

Proof. Let us assume that there is a formula $T(x)$ satisfying (12.17). Let

$$\varphi_0, \quad \varphi_1, \quad \varphi_2, \quad \ldots$$

be an enumeration of all formulas with one free variable. Let $\psi(x)$ be the formula

$$x \in \omega \wedge \neg T(\#(\varphi_x(x))).$$

There is a natural number k such that ψ is φ_k. Let σ be the sentence $\psi(k)$. Then we have

$$\sigma \leftrightarrow \psi(k) \leftrightarrow \neg T(\#(\varphi_k(k))) \leftrightarrow \neg T(\#\sigma)$$

which contradicts (12.17). □

Relative Consistency

By Gödel's Second Incompleteness Theorem it is impossible to show the consistency of ZF (or related theories) by means limited to ZF alone.

Once we assume that ZF (or ZFC) is consistent, we may ask whether the theory remains consistent if we add an additional axiom A.

Let T be a mathematical theory (in our case, T is either ZF or ZFC), and let A be an additional axiom. We say that T + A is *consistent relative to* T (or that A is *consistent with* T) if the following implication holds:

$$\text{if T is consistent, then so is T + A.}$$

If both A and ¬A are consistent with T, we say that A is *independent of* T.

The question whether A is consistent with T is equivalent to the question whether the negation of A is provable in T (provided T is consistent); this is because T + A is consistent if and only if ¬A is not provable in T.

The way to show that an axiom A is consistent with ZF (ZFC) is to use models. For assume that we have a model M (possibly a proper class) of ZF such that $M \vDash \mathrm{A}$. (More precisely, the relativizations σ^M hold for all axioms σ of ZF, as well as A^M.) Then A is consistent with ZF: If it were not, then ¬A would be provable in ZF, and since M is a model of ZF, M would satisfy ¬A. However, $(\neg\mathrm{A})^M$ contradicts A^M.

Transitive Models and Δ_0 Formulas

If M is a transitive class then the model $(M, \in)$ is called a *transitive model*. We note that transitive models satisfy the Axiom of Extensionality (see Exercise 12.4) and that every well-founded extensional model is isomorphic to a transitive model (Theorem 6.15).

Definition 12.8. A formula of set theory is a Δ_0-formula if

(i) it has no quantifiers, or
(ii) it is $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\varphi \to \psi$ or $\varphi \leftrightarrow \psi$ where φ and ψ are Δ_0-formulas, or
(iii) it is $(\exists x \in y)\, \varphi$ or $(\forall x \in y)\, \varphi$ where φ is a Δ_0-formula.

Lemma 12.9. *If M is a transitive class and φ is a Δ_0-formula, then for all $x_1, \ldots, x_n$,*

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n). \tag{12.18}$$

If (12.18) holds, we say that the formula φ is *absolute* for the transitive model M.

Proof. If φ is an atomic formula, then (12.18) holds. If (12.18) holds for φ and ψ, then it holds for $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, and $\varphi \leftrightarrow \psi$.

Let φ be the formula $(\exists u \in x)\, \psi(u, x, \ldots)$ and assume that (12.18) is true for ψ. We show that (12.18) is true for φ (the proof for $\forall u \in x$ is similar).

If φ^M holds then we have $(\exists u\, (u \in x \wedge \psi))^M$, i.e., $(\exists u \in M)(u \in x \wedge \psi^M)$. Since $\psi^M \leftrightarrow \psi$, it follows that $(\exists u \in x)\, \psi$. Conversely, if $(\exists u \in x)\, \psi$, then since M is transitive, u belongs to M, and since $\psi(u, x, \ldots) \leftrightarrow \psi^M(u, x, \ldots)$, we have $\exists u\, (u \in M \wedge u \in x \wedge \psi^M)$ and so $((\exists u \in x)\, \psi)^M$. □

Lemma 12.10. *The following expressions can be written as Δ_0-formulas and thus are absolute for all transitive models.*

(i) $x = \{u, v\}$, $x = (u, v)$, *x is empty*, $x \subset y$, *x is transitive*, *x is an ordinal*, *x is a limit ordinal*, *x is a natural number*, $x = \omega$.
(ii) $Z = X \times Y$, $Z = X - Y$, $Z = X \cap Y$, $Z = \bigcup X$, $Z = \operatorname{dom} X$, $Z = \operatorname{ran} X$.
(iii) *X is a relation*, *f is a function*, $y = f(x)$, $g = f \restriction X$.

Proof.

(i) $x = \{u, v\} \leftrightarrow u \in x \wedge v \in x \wedge (\forall w \in x)(w = u \vee w = v)$.
$x = (u, v) \leftrightarrow (\exists w \in x)(\exists z \in x)(w = \{u\} \wedge z = \{u, v\})$
$\wedge\, (\forall w \in x)(w = \{u\} \vee w = \{u, v\})$.
x is empty $\leftrightarrow (\forall u \in x)\, u \neq u$.
$x \subset y \leftrightarrow (\forall u \in x)\, u \in y$.
x is transitive $\leftrightarrow (\forall u \in x)\, u \subset x$.
x is an ordinal $\leftrightarrow x$ is transitive $\wedge (\forall u \in x)(\forall v \in x)(u \in v \vee v \in u \vee u = v)$
$\wedge\, (\forall u \in x)(\forall v \in x)(\forall w \in x)(u \in v \in w \to u \in w)$.
x is a limit ordinal $\leftrightarrow x$ is an ordinal $\wedge\, (\forall u \in x)(\exists v \in x)\, u \in v$.
x is a natural number $\leftrightarrow x$ is an ordinal $\wedge$ (x is not a limit $\vee\, x = 0$)
$\wedge\, (\forall u \in x)(u = 0 \vee u$ is not a limit$)$.
$x = \omega \leftrightarrow x$ is a limit ordinal $\wedge\, x \neq 0 \wedge (\forall u \in x)\, x$ is a natural number.

(ii) $Z = X \times Y \leftrightarrow (\forall z \in Z)(\exists x \in X)(\exists y \in Y)\, z = (x, y)$
$\wedge\, (\forall x \in X)(\forall y \in Y)(\exists z \in Z)\, z = (x, y)$.
$Z = X - Y \leftrightarrow (\forall z \in Z)(z \in X \wedge z \notin Y) \wedge (\forall z \in X)(z \notin Y \to z \in Z)$.
$Z = X \cap Y$ ... similar.
$Z = \bigcup X \leftrightarrow (\forall z \in Z)(\exists x \in X)\, z \in x \wedge (\forall x \in X)(\forall z \in x)\, z \in Z$.
$Z = \operatorname{dom}(X) \leftrightarrow (\forall z \in Z)\, z \in \operatorname{dom} X \wedge (\forall z \in \operatorname{dom} X)\, z \in Z$,

and we show that:

(12.19) (a) $z \in \operatorname{dom} X$ is a Δ_0-formula;
(b) if φ is Δ_0, then $(\forall z \in \operatorname{dom} X)\, \varphi$ is Δ_0.

(a) $z \in \operatorname{dom} X \leftrightarrow (\exists x \in X)(\exists u \in X)(\exists v \in u)\, x = (z, v)$.
(b) $(\forall z \in \operatorname{dom} X)\, \varphi \leftrightarrow (\forall x \in X)(\forall u \in x)(\forall z, v \in u)(x = (z, v) \to \varphi)$.
An assertion similar to (12.19) holds for $\operatorname{ran}(X)$, and for $\exists$.

(iii) X is a relation $\leftrightarrow (\forall x \in X)(\exists u \in \operatorname{dom} X)(\exists v \in \operatorname{ran} X)\, x = (u, v)$.
f is a function $\leftrightarrow f$ is a relation $\wedge$
$$(\forall x \in \operatorname{dom} f)(\forall y, z \in \operatorname{ran} f)((x, y) \in f \wedge (x, z) \in f \rightarrow y = z)$$
where
$$(x, y) \in f \leftrightarrow (\exists u \in f)\, u = (x, y).$$
$g = f \restriction X \leftrightarrow g$ is a function $\wedge\, g \subset f \wedge (\forall x \in \operatorname{dom} g)\, x \in X$
$$\wedge\, (\forall x \in X)(x \in \operatorname{dom} f \rightarrow x \in \operatorname{dom} g). \quad \square$$

It should be emphasized that cardinal concepts are generally not absolute. In particular, the following expressions are known not to be absolute:

$$Y = P(X), \quad |Y| = |X|, \quad \alpha \text{ is a cardinal}, \quad \beta = \operatorname{cf}(\alpha), \quad \alpha \text{ is regular}.$$

Compare with Exercise 12.6.

Consistency of the Axiom of Regularity

As an application of the theory of transitive models we show that the Axiom of Regularity is consistent with the other axioms of ZF. In this section only we work in the theory ZF minus Regularity, i.e., axioms 1.1–1.7.

The cumulative hierarchy V_α is defined as in Chapter 6, and we denote (in the present section only) V not the universal class but the class $\bigcup_{\alpha \in Ord} V_\alpha$. We shall show that V is a transitive model of ZF. Thus the Axiom of Regularity is consistent relative to the theory 1.1–1.7.

Theorem 12.11. *In* ZF *minus Regularity,* σ^V *holds for every axiom* σ *of* ZF.

Proof. We use absoluteness of Δ_0-formulas and the fact that for every set x, if $x \subset V$, then $x \in V$.

Extensionality. The formula
$$((\forall u \in X)\, u \in Y \wedge (\forall u \in Y)\, u \in X) \rightarrow X = Y$$
is Δ_0.

Pairing. Given $a, b \in V$, let $c = \{a, b\}$. The set c is in V and since "$c = \{a, b\}$" is Δ_0 (see Lemma 12.10), the Pairing Axiom holds in V.

Separation. Let φ be a formula; we shall show that
$$V \vDash \forall X\, \forall p\, \exists Y\, \forall u\, (u \in Y \leftrightarrow u \in X \wedge \varphi(u, p)).$$
Given $X, p \in V$, we let $Y = \{u \in X : \varphi^V(u, p)\}$. Since $Y \subset X$ and $X \in V$, we have $Y \in V$, and
$$V \vDash \forall u\, (y \in Y \leftrightarrow u \in X \wedge \varphi(u, p)).$$

Union. Given $X \in V$, let $Y = \bigcup X$. The set Y is in V and since "$Y = \bigcup X$" is Δ_0, the Axiom of Union holds in V.

Power Set. Given $X \in V$, let $Y = P(X)$. The set Y is in V, and we claim that $V \vDash \forall u\, \varphi(u)$ where $\varphi(u)$ is the formula $u \in Y \leftrightarrow u \subset X$. Since $\varphi(u)$ is Δ_0 and because $\varphi(u)$ holds for all u, we have $\varphi^V(u)$ for all $u \in V$, as claimed.

Infinity. We want to show that

$$V \vDash \exists S\, (\emptyset \in S \wedge (\forall x \in S)\, x \cup \{x\} \in S). \tag{12.20}$$

The formula in (12.20) contains defined notions, $\{\ \}$, $\cup$, and $\emptyset$; and strictly speaking, we should first eliminate these symbols and use a formula in which they are replaced by their definitions, using only $\in$ and $=$. However, we have already proved that both pairing and union are the same in the universe as in V, and similarly one shows that $X \in V$ is empty if and only if (X is empty)^V. In other words,

$$\{a, b\}^V = \{a, b\}, \qquad \bigcup{}^V X = \bigcup X, \qquad \emptyset^V = \emptyset$$

where $\{a, b\}^V$, $\bigcup^V$, and $\emptyset^V$ denote pairing, union, and the empty set in the model V.

Since $\omega \in V$, we easily verify that (12.20) holds when $S = \omega$.

Replacement. Let φ be a formula; we shall show that

$$\begin{aligned} V \vDash \forall x\, \forall y\, \forall z\, (\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y = z) \\ \rightarrow \forall X\, \exists Y\, \forall y\, (y \in Y \leftrightarrow (\exists x \in X)\, \varphi(x, y, p)). \end{aligned}$$

Given $p \in V$, assume that $V \vDash \forall x\, \forall y\, \forall z\, (\dots)$. Thus

$$F = \{(x, y) \in V : \varphi^V(x, y, p)\}$$

is a function, and we let $Y = F(X)$. Since $Y \subset V$, we have $Y \in V$, and one verifies that for every $y \in V$,

$$V \vDash y \in Y \leftrightarrow (\exists x \in X)\, \varphi(x, y, p).$$

Regularity. We want to show that $V \vDash \forall S\, \varphi(S)$, where φ is the formula

$$S \neq \emptyset \rightarrow (\exists x \in S)\, S \cap x = \emptyset.$$

If $S \in V$ is nonempty, then let $x \in S$ be of least rank; then $S \cap x = \emptyset$. Hence $\varphi(S)$ is true for any S; moreover, $(S \cap x)^V = S \cap x$, and φ is Δ_0. Thus $V \vDash \forall S\, \varphi(S)$. □

Inaccessibility of Inaccessible Cardinals

Theorem 12.12. *The existence of inaccessible cardinals is not provable in* ZFC. *Moreover, it cannot be shown that the existence of inaccessible cardinals is consistent with* ZFC.

We shall prove the first assertion and invoke Gödel's Second Incompleteness Theorem to obtain the second part.

First we prove (in ZFC):

Lemma 12.13. *If κ is an inaccessible cardinal, then V_κ is a model of* ZFC.

Proof. The proof of all axioms of ZFC except Replacement is as in the proof of consistency of the Axiom of Regularity (see Exercises 12.7 and 12.8). To show that $V_\kappa \vDash$ Replacement, it is enough to show:

(12.21) If F is a function from some $X \in V_\kappa$ into V_κ, then $F \in V_\kappa$.

Since κ is inaccessible, we have $|V_\kappa| = \kappa$ and $|X| < \kappa$ for every $X \in V_\kappa$. If F is a function from $X \in V_\kappa$ into V_κ, then $|F(X)| \leq |X| < \kappa$ and (since κ is regular) $F(X) \subset V_\alpha$ for some $\alpha < \kappa$. It follows that $F \in V_\kappa$. □

Proof of Theorem 12.12. If κ is an inaccessible cardinal, then not only is V_κ a model of ZFC, but in addition

$$(\alpha \text{ is an ordinal})^{V_\kappa} \leftrightarrow \alpha \text{ is an ordinal.}$$
$$(\alpha \text{ is a cardinal})^{V_\kappa} \leftrightarrow \alpha \text{ is a cardinal.}$$
$$(\alpha \text{ is a regular cardinal})^{V_\kappa} \leftrightarrow \alpha \text{ is a regular cardinal.}$$
$$(\alpha \text{ is an inaccessible cardinal})^{V_\kappa} \leftrightarrow \alpha \text{ is an inaccessible cardinal.}$$

We leave the details to the reader.

In particular, if κ is inaccessible cardinal, then

$$V_\kappa \vDash \text{there is no inaccessible cardinal.}$$

Thus we have a model of ZFC + "there is no inaccessible cardinal" (if there is no inaccessible cardinal, we take the universe as the model). Hence it cannot be proved in ZFC that inaccessible cardinals exist.

To prove the second part, assume that it can be shown that the existence of inaccessible cardinals is consistent with ZFC; in other words, we assume

$$\text{if ZFC is consistent, then so is ZFC} + \text{I}$$

where I is the statement "there is an inaccessible cardinal."

We naturally assume that ZFC is consistent. Since I is consistent with ZFC, we conclude that ZFC + I is consistent. It is provable in ZFC + I that there is a model of ZFC (Lemma 12.13). Thus the sentence "ZFC is consistent" is provable in ZFC + I. However, we have assumed that "I is consistent with ZFC" is provable, and so "ZFC + I is consistent" is provable in ZFC + I. This contradicts Gödel's Second Incompleteness Theorem. □

The wording of the second part of Theorem 12.12 (and its proof) is somewhat vague; "it cannot be shown" means: It cannot be shown by methods formalizable in ZFC.

Reflection Principle

The theorem that we prove below is the analog of the Löwenheim-Skolem Theorem. While that theorem states that every model has a small elementary submodel, the Reflection Principle provides, for any *finite* number of formulas, a set M that is like an "elementary submodel" of the universe, with respect to the given formulas. The theorem is proved without the use of the Axiom of Choice, but using the Axiom of Choice, one can obtain countable model.

Theorem 12.14 (Reflection Principle).

(i) *Let $\varphi(x_1, \dots, x_n)$ be a formula. For each M_0 there exists a set $M \supset M_0$ such that*

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \tag{12.22}$$

for every $x_1, \dots, x_n \in M$. (We say that M reflects φ.)

(ii) *Moreover, there is a transitive $M \supset M_0$ that reflects φ; moreover, there is a limit ordinal α such that $M_0 \subset V_\alpha$ and V_α reflects φ.*

(iii) *Assuming the Axiom of Choice, there is an $M \supset M_0$ such that M reflects φ and $|M| \le |M_0| \cdot \aleph_0$. In particular, there is a countable M that reflects φ.*

Remarks. 1. We may require either that M be transitive or that $|M| \le |M_0| \cdot \aleph_0$ but not both.

2. The proof works for any finite number of formulas, not just one. Thus if $\varphi_1, \dots, \varphi_n$ are formulas, then there exists a set M that reflects each of $\varphi_1, \dots, \varphi_n$.

3. If σ is a true sentence, then the Reflection Principle yields a set M that is a model of σ; using the Axiom of Choice, one can get a countable transitive model of σ.

4. As a consequence of the Reflection Principle, and of Gödel's Second Incompleteness Theorem, it follows that the theory ZF is not finitely axiomatizable: Any finite number of theorems of ZF have a model (a set) by the Reflection Principle, while the existence of a model of ZF is not provable. (By the same argument, no consistent extension of ZF is finitely axiomatizable.)

The key step in the proof of Theorem 12.14 is the following lemma, which we prove first.

Lemma 12.15.

(i) *Let $\varphi(u_1, \dots, u_n, x)$ be a formula. For each set M_0 there exists a set $M \supset M_0$ such that*

$$\text{if } \exists x\, \varphi(u_1, \dots, u_n, x) \quad \text{then } (\exists x \in M)\, \varphi(u_1, \dots, u_n, x) \tag{12.23}$$

for every $u_1, \dots, u_n \in M$. Assuming the Axiom of Choice, there is $M' \supset M_0$ such that (12.23) *holds for M' and $|M'| \leq |M_0| \cdot \aleph_0$.*

(ii) *If $\varphi_1, \dots, \varphi_k$ are formulas, then for each M_0 there is an $M \supset M_0$ such that* (12.23) *holds for each $\varphi_1, \dots, \varphi_k$.*

Proof. We shall give a detailed proof of (i). An obvious modification of the proof gives (ii); we leave that to the reader.

Note that the operation $H(u_1, \dots, u_n)$ defined below plays the same role as Skolem functions in the Löwenheim-Skolem Theorem.

Let us recall the definition (6.4):

$$\hat{C} = \{x \in C : (\forall z \in C)\ \operatorname{rank} x \leq \operatorname{rank} z\}. \tag{12.24}$$

For every $u_1, \dots, u_n$, let

$$H(u_1, \dots, u_n) = \hat{C} \tag{12.25}$$

where

$$C = \{x : \varphi(u_1, \dots, u_n, x)\}. \tag{12.26}$$

Thus $H(u_1, \dots, u_n)$ is a set with the property

$$\text{if } \exists x\, \varphi(u_1, \dots, u_n, x), \quad \text{then } (\exists x \in H(u_1, \dots, u_n))\, \varphi(u_1, \dots, u_n, x). \tag{12.27}$$

We construct the set M by induction. We let $M = \bigcup_{i=0}^{\infty} M_i$ where for each $i \in \boldsymbol{N}$,

$$M_{i+1} = M_i \cup \bigcup \{H(u_1, \dots, u_n) : u_1, \dots, u_n \in M_i\}. \tag{12.28}$$

Now, if $u_1, \dots, u_n \in M$, then there is an $i \in \boldsymbol{N}$ such that $u_1, \dots, u_n \in M_i$ and if $\varphi(u_1, \dots, u_n, x)$ holds for some x, then it holds for some $x \in M_{i+1}$, by (12.27) and (12.28).

Assuming the Axiom of Choice, let F be a choice function on $P(M)$. For every $u_1, \dots, u_n \in M$, let $h(u_1, \dots, u_n) = F(H(u_1, \dots, u_n))$ (and let $h(u_1, \dots, u_n)$ remain undefined if $H(u_1, \dots, u_n)$ is empty). Let us define $M' = \bigcup_{i=0}^{\infty} M'_i$, where $M'_0 = M_0$ and for each $i \in \boldsymbol{N}$,

$$M'_{i+1} = M'_i \cup \{h(u_1, \dots, u_n) : u_1, \dots, u_n \in M'_i\}.$$

Condition (12.23) can be verified for M' in the same way as for M. Moreover, each M'_i has cardinality at most $|M_0| \cdot \aleph_0$, and so does M'. □

Proof of Theorem 12.14. Let $\varphi(x_1, \dots, x_n)$ be a formula. We may assume that the universal quantifier does not occur in φ ($\forall x \dots$ can be replaced by $\neg\exists x \neg \dots$). Let $\varphi_1, \dots, \varphi_k$ be all the subformulas of the formula φ.

Given a set M_0, there exists, by Lemma 12.15(ii), a set $M \supset M_0$, such that

$$\exists x\, \varphi_j(u, \dots, x) \to (\exists x \in M)\, \varphi_j(u, \dots, x), \qquad j = 1, \dots, k \tag{12.29}$$

for all $u, \dots \in M$. We claim that M reflects each φ_j, $j = 1, \dots, k$, and in particular M reflects φ. This is proved by induction on the complexity of φ_j.

It is easy to see that (every) M reflects atomic formulas, and that if M reflects formulas ψ and χ, then M reflects $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \to \chi$, and $\psi \leftrightarrow \chi$. Thus assume that M reflects $\varphi_j(u_1, \dots, u_m, x)$ and let us prove that M reflects $\exists x\, \varphi_j$.

If $u_1, \dots, u_m \in M$, then

$$\begin{aligned} M \vDash \exists x\, \varphi_j(u_1, \dots, u_m, x) &\leftrightarrow (\exists x \in M)\, \varphi_j^M(u_1, \dots, u_m, x) \\ &\leftrightarrow (\exists x \in M)\, \varphi_j(u_1, \dots, u_m, x) \\ &\leftrightarrow \exists x\, \varphi_j(u_1, \dots, u_m, x). \end{aligned}$$

The last equivalence holds by (12.29).

This proves part (i) of the theorem. Part (iii) is proved by taking M of size $\leq |M_0| \cdot \aleph_0$. To prove (ii), one has to modify the proof of Lemma 12.15 so that the set M used in (12.29) is transitive (or $M = V_\alpha$). This is done as follows: In (12.28), we replace M_{i+1} by its transitive closure (or by the least $V_\gamma \supset M_{i+1}$). Then M is transitive (or $M = V_\alpha$). □

Exercises

12.1. Let U be a principal ultrafilter on S, such that $\{a\} \in U$. Show that the ultraproduct $\mathrm{Ult}_U\{\mathfrak{A}_x : x \in S\}$ is isomorphic to $\mathfrak{A}_a$.

12.2. If U is a principal ultrafilter, then the canonical embedding j is an isomorphism between $\mathfrak{A}$ and $\mathrm{Ult}_U\, \mathfrak{A}$.

12.3. Let κ be a measurable cardinal and let U be an ultrafilter on κ. Let $(A, <^*)$ be the ultrapower of $(\kappa, <)$ by U, and let $j : \kappa \to A$ be the canonical embedding.

(i) $(A, <^*)$ is a linear ordering.
(ii) If U is σ-complete then $(A, <^*)$ is a well-ordering; $(A, <^*)$ is isomorphic, and can be identified with, $(\gamma, <)$, where γ is an ordinal.
(iii) If U is κ-complete then $j(\alpha) = \alpha$ for all $\alpha < \kappa$
(iv) If d is the diagonal function, $[d] \geq \kappa$. The measure U is normal if and only if $[d] = \kappa$.
[Compare with Exercise 10.5.]

12.4. A class M is extensional if and only if σ^M holds where σ is the Axiom of Extensionality.

12.5. The following can be written as Δ_0-formulas: x is an ordered pair, x is a partial (linear) ordering of y, x and y are disjoint, $z = x \cup y$, $y = x \cup \{x\}$, x is an inductive set, f is a one-to-one function of X into (onto) Y, f is an increasing ordinal function, f is a normal function.

12.6. Let M be a transitive class.

(i) If $M \vDash |X| \leq |Y|$, then $|X| \leq |Y|$.
(ii) If $\alpha \in M$ and if α is a cardinal, then $M \vDash \alpha$ is a cardinal.

[$|X| \leq |Y| \leftrightarrow \exists f\, \varphi(f, X, Y)$; α is a cardinal $\leftrightarrow \neg\exists f\, (\exists \beta \in \alpha)\, \psi(\alpha, \beta, f)$, where φ and ψ are Δ_0-formulas.]

12.7. If α is a limit ordinal, then V_α is a model of Extensionality, Pairing, Separation, Union, Power Set, and Regularity. If AC holds, then V_α is a model of AC.

12.8. If $\alpha > \omega$, then V_α is a model of Infinity.

12.9. V_ω, the set of all hereditarily finite sets, is a model of ZFC minus Infinity.

12.10. The existence of an infinite set is not provable in ZFC minus Infinity. Moreover, it cannot be shown that the existence of an infinite set is consistent with ZFC minus Infinity.

12.11. If κ is an inaccessible cardinal then $V_\kappa \vDash$ there is a countable model of ZFC.

[Since $\langle V_\kappa, \in\rangle$ is a model of ZFC, there is a countable model (by the Löwenheim-Skolem Theorem). Thus there is $E \subset \omega \times \omega$ such that $\mathfrak{A} = (\omega, E)$ is a model of ZFC. Verify that $V_\kappa \vDash (\mathfrak{A}$ is a countable model of ZFC).]

12.12. If κ is an inaccessible cardinal, then there is $\alpha < \kappa$ such that $\langle V_\alpha, \in\rangle \prec \langle V_\kappa, \in\rangle$ Moreover, the set $\{\alpha < \kappa : \langle V_\alpha, \in\rangle \prec \langle V_\kappa, \in\rangle\}$ is closed unbounded.

[Construct Skolem functions h for V_κ, and let $\alpha = \lim_n \alpha_n$, where $\alpha_{n+1} < \kappa$ is such that $h(V_{\alpha_n}) \subset V_{\alpha_{n+1}}$ for each h.]

For every infinite regular cardinal κ let H_κ be the set of all x such that $|\operatorname{TC}(x)| < \kappa$. The sets in H_ω are hereditarily finite sets. The sets in H_{ω_1} are *hereditarily countable* sets. Each H_κ is transitive and $H_\kappa \subset V_\kappa$.

12.13. If κ is a regular uncountable cardinal then H_κ is a model of ZFC minus the Power Set Axiom.

12.14. For every formula φ, there is a closed unbounded class C_φ of ordinals such that for each $\alpha \in C_\varphi$, V_α reflects φ.

[$C_{\varphi\wedge\psi} = C_\varphi \cap C_\psi$, $C_{\exists x\, \varphi} = C_\varphi \cap K_\varphi$, where K_φ is the closed unbounded class $\{\alpha \in \mathit{Ord} : \forall x_1, \ldots, x_n \in V_\alpha\, (\exists x\, \varphi(x, x_1, \ldots, x_n) \rightarrow (\exists x \in V_\alpha)\, \varphi(x, x_1, \ldots, x_n))\}$.]

12.15. Let M be a transitive class and let φ be a formula. For each $M_0 \subset M$ there exists a set $M_1 \supset M_0$ such that $M_1 \subset M$ and that $\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi^{M_1}(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in M_1$.

A transfinite sequence $\langle W_\alpha : \alpha \in \mathit{Ord}\rangle$ is called a *cumulative hierarchy* if $W_0 = \emptyset$ and

(12.30) (i) $W_\alpha \subset W_{\alpha+1} \subset P(W_\alpha)$,
(ii) if α is limit, then $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$.

Each W_α is transitive and $W_\alpha \subset V_\alpha$.

12.16. Let $\langle W_\alpha : \alpha \in \mathit{Ord}\rangle$ be a cumulative hierarchy, and let $W = \bigcup_{\alpha\in \mathit{Ord}} W_\alpha$. Let φ be a formula. Show that there are arbitrary large limit ordinals α such that $\varphi^W(x_1, \ldots, x_n) \leftrightarrow \varphi^{W_\alpha}(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in W_\alpha$.

Historical Notes

For concepts of model theory, the history of the subject and for model-theoretical terminology, I refer the reader to Chang and Keisler's book [1973].

Reduced products were first investigated by Łoś in [1955], who also proved Theorem 12.3 on ultraproducts.

For Tarski's Theorem 12.7, see Tarski [1939].

The impossibility of a consistency proof of the existence of inaccessible cardinals follows from Gödel's Theorem [1931]. An argument that more or less establishes the consistency of the Axiom of Regularity appeared in Skolem's work in 1923 (see Skolem [1970], pp. 137–152).

The study of transitive models of set theory originated with Gödel's work on constructible sets. The Reflection Principle was introduced by Montague; see [1961] and Lévy [1960b].

Exercise 12.12: Montague and Vaught [1959].

Exercise 12.14: Galvin.

Part II

Advanced Set Theory

13. Constructible Sets

Constructible sets were introduced by Gödel in his proof of consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis. The class L of all constructible sets (the *constructible universe*) is a transitive model of ZFC, and is the smallest transitive model of ZF that contains all ordinal numbers. In this chapter we study constructible sets and some related concepts.

The Hierarchy of Constructible Sets

Recall that a set X is *definable* over a model $(M, \in)$ (where M is a set) if there exist a formula $\varphi \in Form$ (the set of all formulas of the language $\{\in\}$) and some $a_1, \ldots, a_n \in M$ such that $X = \{x \in M : (M, \in) \vDash \varphi[x, a_1, \ldots, a_n]\}$. Let

$$\mathrm{def}(M) = \{X \subset M : X \text{ is definable over } (M, \in)\}.$$

Clearly, $M \in \mathrm{def}(M)$ and $M \subset \mathrm{def}(M) \subset P(M)$.

Definition 13.1. We define by transfinite induction

(i) $L_0 = \emptyset$, $L_{\alpha+1} = \mathrm{def}(L_\alpha)$,
(ii) $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ if α is a limit ordinal, and
(iii) $L = \bigcup_{\alpha \in Ord} L_\alpha$.

The (definable) class L is the class of *constructible* sets. The statement $V = L$, i.e., "every set is constructible," is the *Axiom of Constructibility*.

It follows from Definition 13.1 that $\langle L_\alpha : \alpha \in Ord\rangle$ is a *cumulative hierarchy* (see (12.30)); in particular, each L_α is transitive, $L_\alpha \subset L_\beta$ if $\alpha < \beta$, and L is a transitive class.

Lemma 13.2. *For every* α, $\alpha \subset L_\alpha$ (*and* $L_\alpha \cap Ord = \alpha$).

Proof. By induction on α. At stage $\alpha + 1$, we need to show that $\alpha \in L_{\alpha+1}$, or that α is a definable subset of L_α. Since $\alpha = \{x \in L_\alpha : x \text{ is an ordinal}\}$, and "$x$ is an ordinal" is a Δ_0 formula, we have $\alpha = \{x \in L_\alpha : L_\alpha \vDash x \text{ is an ordinal}\}$. □

Theorem 13.3. *L is a model of* ZF.

Proof. We show that σ^L holds for every axiom σ of ZF. Since L is a transitive class, every Δ_0 formula is absolute for L.

Extensionality. L is transitive and therefore extensional.

Pairing. Given $a, b \in L$, let $c = \{a, b\}$. Let α be such that $a \in L_\alpha$ and $b \in L_\alpha$. Since $\{a, b\}$ is definable over L_α, we have $c \in L_{\alpha+1}$, and since "$c = \{a, b\}$" is Δ_0, the Pairing Axiom holds in L.

Separation. Let φ be a formula. Given $X, p \in L$, we wish to show that the set $Y = \{u \in X : \varphi^L(u, p)\}$ is in L. By the Reflection Principle (applied to the cumulative hierarchy L_α, cf. Exercise 12.6), there exists an α such that $X, p \in L_\alpha$ and $Y = \{u \in X : \varphi^{L_\alpha}(u, p)\}$. Thus $Y = \{u \in L_\alpha : L_\alpha \vDash u \in X \wedge \varphi(u, p)\}$ and so $Y \in L$.

Union. Given $X \in L$, let $Y = \bigcup X$. As L is transitive, we have $Y \subset L$; let α be such that $X \in L_\alpha$ and $Y \subset L_\alpha$. Y is definable over L_α by the Δ_0 formula "$x \in \bigcup X$" and so $Y \in L$. Since "$Y = \bigcup X$" is Δ_0, the Axiom of Union holds in L.

Power Set. Given $X \in L$, let $Y = P(X) \cap L$. Let α be such that $Y \subset L_\alpha$. Y is definable over L_α by the Δ_0 formula "$x \subset X$" and so $Y \in L$. We claim that $Y = P^L(X)$, i.e., that "Y is the power set of X" holds in L. But "$x \in Y \leftrightarrow x \subset X$" is a Δ_0 formula true for every $x \in L$.

Infinity. We can repeat the proof from Theorem 12.11 as $\omega \in L$.

Replacement. The easiest way to verify these axioms is to refer to Exercise 1.15, specifically to (1.10). If a class F is a function in L then for every $X \in L$ there exists an α such that $\{F(x) : x \in X\} \subset L_\alpha$. Since $L_\alpha \in L$, this suffices.

Regularity. If $S \in L$ is nonempty, let $x \in S$ be such that $x \cap S = \emptyset$. Then $x \in L$ and the Δ_0 formula "$x \cap S = \emptyset$" holds in L. □

We will show that the model L satisfies both the Axiom of Choice and the Generalized Continuum Hypothesis, thus establishing the consistency of AC and GCH (relative to ZF). This will be done by showing that L is a model of the Axiom of Constructibility ($V = L$), and that $V = L$ implies both AC and GCH.

It is rather clear that $V = L$ implies AC: it is relatively straightforward to define a well-ordering of L (by transfinite induction, using some enumeration of the set *Form* of all formulas).

It may appear that L is trivially a model of "every set is constructible." However, to verify $V = L$ in L, we have to prove first that the property "x is constructible" is absolute for L, i.e., that for every $x \in L$ we have (x is constructible)^L. We shall do this by analyzing the complexity of the property "constructible." While this can be done working directly with the model-theoretic concepts involved, we prefer to use an alternative approach (also due to Gödel).

Gödel Operations

The Axiom Schema of Separation states that given a formula $\varphi(x)$, for every X there exists a set $Y = \{u \in X : \varphi(u)\}$. It turns out that for Δ_0 formulas, the construction of Y from X can be described by means of a finite number of elementary operations.

Theorem 13.4 (Gödel's Normal Form Theorem). *There exist operations $G_1, \ldots, G_{10}$ such that if $\varphi(u_1, \ldots, u_n)$ is a Δ_0 formula, then there is a composition G of $G_1, \ldots, G_{10}$ such that for all $X_1, \ldots, X_n$,*

(13.1)
$$G(X_1, \ldots, X_n) = \{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \varphi(u_1, \ldots, u_n)\}.$$

The operations $G_1, \ldots, G_{10}$ will be defined below. Compositions of $G_1, \ldots, G_{10}$ are called *Gödel operations*.

We call the following sentence an instance of Δ_0-*Separation*:

$$\forall p_1 \ldots \forall p_n \forall X \exists Y \forall u \, (u \in Y \leftrightarrow u \in X \wedge \varphi(u, p_1, \ldots, p_n)) \tag{13.2}$$

where φ is a Δ_0 formula. We say that a transitive class M satisfies Δ_0-*Separation* if for every Δ_0 formula φ, M satisfies (13.2).

A class C is *closed under an operation F* if $F(x_1, \ldots, x_n) \in C$ whenever $x_1, \ldots, x_n \in C$. If a class M is closed under the operations $G_1, \ldots, G_{10}$ then M is closed under all Gödel operations.

Corollary 13.5. *If M is a transitive class closed under Gödel operations then M satisfies Δ_0-Separation.*

Proof. Let $\varphi(u, p_1, \ldots, p_n)$ be a Δ_0 formula, and let $X, p_1, \ldots, p_n \in M$. Let

$$Y = \{u \in X : \varphi(u, p_1, \ldots, p_n)\}.$$

By Lemma 12.9 it suffices to show that $Y \in M$, in order that M satisfy (13.2). By Gödel's Normal Form Theorem, there is a Gödel operation G such that

$$G(X, \{p_1\}, \ldots, \{p_n\}) = \{(u, p_1, \ldots, p_n) : u \in X \wedge \varphi(u, p_1, \ldots, p_n)\}.$$

It follows that

$$\begin{aligned} Y &= \{u : \exists u_1 \ldots \exists u_n \, (u, u_1, \ldots, u_n) \in G(X, \{p_1\}, \ldots, \{p_n\})\} \\ &= \underbrace{\mathrm{dom} \ldots \mathrm{dom}}_{n \text{ times}} G(X, \{p_1\}, \ldots, \{p_n\}). \end{aligned}$$

Since both $\{x, y\}$ and $\mathrm{dom}(x)$ are Gödel operations (see below) and since M is closed under Gödel operations, we have $Y \in M$. □

Definition 13.6 (Gödel Operations).

$$\begin{aligned}
G_1(X,Y) &= \{X,Y\},\\
G_2(X,Y) &= X \times Y,\\
G_3(X,Y) &= \varepsilon(X,Y) = \{(u,v) : u \in X \wedge v \in Y \wedge u \in v\},\\
G_4(X,Y) &= X - Y,\\
G_5(X,Y) &= X \cap Y,\\
G_6(X) &= \bigcup X,\\
G_7(X) &= \mathrm{dom}(X),\\
G_8(X) &= \{(u,v) : (v,u) \in X\},\\
G_9(X) &= \{(u,v,w) : (u,w,v) \in X\},\\
G_{10}(X) &= \{(u,v,w) : (v,w,u) \in X\}.
\end{aligned}$$

Proof of Theorem 13.4. The theorem is proved by induction on the complexity of Δ_0 formulas. To simplify matters, we consider only formulas of this form:

(13.3)
(i) the only logical symbols in φ are $\neg$, $\wedge$, and restricted $\exists$;
(ii) $=$ does not occur;
(iii) the only occurrence of $\in$ is $u_i \in u_j$ where $i \neq j$;
(iv) the only occurrence of $\exists$ is

$$(\exists u_{m+1} \in u_i)\, \psi(u_1, \ldots, u_{m+1})$$

where $i \leq m$.

Every Δ_0 formula can be rewritten in this form: The use of logical symbols can be restricted to $\neg$, $\wedge$, and $\exists$; $x = y$ can be replaced by $(\forall u \in x)\, u \in y \wedge (\forall v \in y)\, v \in x$, $x \in x$ can be replaced by $(\exists u \in x)\, u = x$ and the bound variables in $\varphi(u_1, \ldots, u_n)$ can be renamed so that the variable with the highest index is quantified.

Note that we allow dummy variables, so that for instance $\varphi(u_1, \ldots, u_5) = u_3 \in u_2$ and $\varphi(u_1, \ldots, u_6) = u_3 \in u_2$ are considered separately.

Thus let $\varphi(u_1, \ldots, u_n)$ be a formula in the form (13.3) and let us assume that the theorem holds for all subformulas of φ.

Case I. $\varphi(u_1, \ldots, u_n)$ is an atomic formula $u_i \in u_j$ $(i \neq j)$. We prove this case by induction on n.

Case Ia. $n = 2$. Here we have

$$\{(u_1,u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_1 \in u_2\} = \varepsilon(X_1, X_2)$$

and

$$\{(u_1,u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_2 \in u_1\} = G_8(\varepsilon(X_2, X_1)).$$

Case Ib. $n > 2$ and $i, j \neq n$. By the induction hypothesis, there is a G such that

$$\{(u_1, \ldots, u_{n-1}) : u_1 \in X_1, \ldots, u_{n-1} \in X_{n-1} \wedge u_i \in u_j\} = G(X_1, \ldots, X_{n-1}).$$

Obviously

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \wedge u_i \in u_j\} = G(X_1, \ldots, X_{n-1}) \times X_n.$$

Case Ic. $n > 2$ and $i, j \neq n-1$. By the induction hypothesis (Case Ib) there is a G such that

$$\{(u_1, \ldots, u_{n-2}, u_n, u_{n-1}) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_i \in u_j\} = G(X_1, \ldots, X_n).$$

Noting that

$$(u_1, \ldots, u_{n-2}, u_n, u_{n-1}) = ((u_1, \ldots, u_{n-2}), u_n, u_{n-1})$$

we get

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_i \in u_j\} = G_9(G(X_1, \ldots, X_n)).$$

Case Id. $i = n-1$, $j = n$. By Ia, we have

$$\{(u_{n-1}, u_n) : u_{n-1} \in X_{n-1} \wedge u_n \in X_n \wedge u_{n-1} \in u_n\} = \varepsilon(X_{n-1}, X_n)$$

and so

$$\{((u_{n-1}, u_n), (u_1, \ldots, u_{n-2})) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_{n-1} \in u_n\} = \varepsilon(X_{n-1}, X_n) \times (X_1 \times \ldots \times X_{n-2}) = G(X_1, \ldots, X_n).$$

Now we note that

$$((u_{n-1}, u_n), (u_1, \ldots, u_{n-2})) = (u_{n-1}, u_n, (u_1, \ldots, u_{n-2}))$$

and

$$(u_1, \ldots, u_n) = ((u_1, \ldots, u_{n-2}), u_{n-1}, u_n)$$

and thus

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_{n-1} \in u_n\} = G_{10}(G(X_1, \ldots, X_n)).$$

Case Ie. $i = n$, $j = n-1$. Similar to Case Id.

Case II. $\varphi(u_1, \ldots, u_n)$ is a negation, $\neg\psi(u_1, \ldots, u_n)$. By the induction hypothesis, there is a G such that

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \psi(u_1, \ldots, u_n)\} = G(X_1, \ldots, X_n).$$

Clearly,

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \varphi(u_1, \ldots, u_n)\} = X_1 \times \ldots \times X_n - G(X_1, \ldots, X_n).$$

Case III. φ is a conjunction, $\psi_1 \wedge \psi_2$. By the induction hypothesis,

$$\{(u_1, \dots, u_n) : u_1 \in X_1, \dots, u_n \in X_n \text{ and } \psi_i(u_1, \dots, u_n)\} = G_{(i)}(X_1, \dots, X_n)$$

$(i = 1, 2)$. Hence

$$\{(u_1, \dots, u_n) : u_1 \in X_1, \dots, u_n \in X_n \text{ and } \varphi(u_1, \dots, u_n)\} \\ = G_{(1)}(X_1, \dots, X_n) \cap G_{(2)}(X_1, \dots, X_n).$$

Case IV. $\varphi(u_1, \dots, u_n)$ is the formula $(\exists u_{n+1} \in u_i)\, \psi(u_1, \dots, u_{n+1})$. Let $\chi(u_1, \dots, u_{n+1})$ be the formula $\psi(u_1, \dots, u_{n+1}) \wedge u_{n+1} \in u_i$. By the induction hypothesis (we consider χ less complex than φ), there is a G such that

$$\{(u_1, \dots, u_{n+1}) : u_1 \in X_1, \dots, u_{n+1} \in X_{n+1} \text{ and } \chi(u_1, \dots, u_{n+1})\} \\ = G(X_1, \dots, X_{n+1})$$

for all $X_1, \dots, X_{n+1}$. We claim that

$$\{(u_1, \dots, u_n) : u_1 \in X_1, \dots, u_n \in X_n \text{ and } \varphi(u_1, \dots, u_n)\} \\ = (X_1 \times \dots \times X_n) \cap \operatorname{dom}(G(X_1, \dots, X_n, \textstyle\bigcup X_i)). \tag{13.4}$$

Let us denote $u = (u_1, \dots, u_n)$ and $X = X_1 \times \dots \times X_n$. For all $u \in X$, we have

$$\begin{aligned} \varphi(u) &\leftrightarrow (\exists v \in u_i)\, \psi(u_i, v) \\ &\leftrightarrow \exists v\, (v \in u_i \wedge \psi(u, v) \wedge v \in \textstyle\bigcup X_i) \\ &\leftrightarrow u \in \operatorname{dom}\{(u, v) \in X \times \textstyle\bigcup X_i : \chi(u, v)\} \end{aligned}$$

and (13.4) follows. This completes the proof of Theorem 13.4. □

The following lemma shows that Gödel operations are absolute for transitive models.

Lemma 13.7. *If G is a Gödel operation then the property $Z = G(X_1, \dots, X_n)$ can be written as a Δ_0 formula.*

Proof. We show, by induction on the complexity of G (a composition of $G_1, \dots, G_{10}$):

(13.5)
(i) $u \in G(X, \dots)$ is Δ_0.
(ii) If φ is Δ_0, then so are $\forall u \in G(X, \dots)\, \varphi$ and $\exists u \in G(X, \dots)\, \varphi$.
(iii) $Z = G(X, \dots)$ is Δ_0.
(iv) If φ is Δ_0, then so is $\varphi(G(X, \dots))$.

We proved (iii) for most of the $G_1, \dots, G_{10}$ in Lemma 12.10; the rest of the G_i are handled similarly, e.g.,

$$\begin{aligned} &Z = G_8(X) \\ &\leftrightarrow (\forall z \in Z)(\exists x \in X)(\exists u \in \operatorname{ran} X)(\exists v \in \operatorname{dom} X)(x = (v, u) \wedge z = (u, v)) \\ &\quad \wedge (\forall x \in X)(\forall u \in \operatorname{ran} X)(\forall v \in \operatorname{dom} X)(\exists z \in Z)(x = (v, u) \to z = (u, v)). \end{aligned}$$

We shall prove (i) and (ii) only for a typical example and leave the full proof to the reader (see also (12.19)). In (i) consider the formula

$$u \in F(X, \ldots) \times G(X, \ldots).$$

This can be written as

$$\exists x \in F(X, \ldots)\, \exists y \in G(X, \ldots)\, u = (x, y).$$

In (ii), consider the formula

$$\forall u \in \{F(X, \ldots), G(X, \ldots)\}\, \varphi(u),$$

which can be written as

$$\varphi(F(X, \ldots)) \wedge \varphi(G(X, \ldots)).$$

(iii) follows from (i) and (ii):

$$Z = G(X, \ldots) \leftrightarrow (\forall u \in Z)\, u \in G(X, \ldots) \wedge \forall u \in G(X, \ldots)\, u \in Z.$$

To prove (iv), let φ be a Δ_0 formula. Then $G(X, \ldots)$ occurs in $\varphi(G(X, \ldots))$ in the form $u \in G(X, \ldots)$, $G(X, \ldots) \in u$, $Z = G(X, \ldots)$, $\forall u \in G(X, \ldots)$, or $\exists u \in G(X, \ldots)$. Since $G(X, \ldots) \in u$ can be replaced by $(\exists v \in u)\, v = G(X, \ldots)$, we use (i)–(iii) to show that $\varphi(G(X, \ldots))$ is a Δ_0 property. □

If φ is a formula then φ^M is a Δ_0 formula, and so by Theorem 13.4 there is a Gödel operation G such that for every transitive set M and all $a_1, \ldots, a_n$,

$$\begin{aligned}\{x \in M : M \vDash \varphi[x, a_1, \ldots, a_n]\} &= \{x \in M : \varphi^M(x, a_1, \ldots, a_n)\} \\ &= G(M, a_1, \ldots, a_n).\end{aligned}$$

The same argument, by induction on the complexity of φ, shows that for every $\varphi \in \mathit{Form}$, the set $\{x \in M : M \vDash \varphi[x, a_1, \ldots, a_n]\}$ is in the closure of $M \cup \{M\}$ under $G_1, \ldots, G_{10}$.

Conversely, if G is a composition of $G_1, \ldots, G_{10}$ then by Lemma 13.7 there is a Δ_0 formula φ such that for all M and all $a_1, \ldots, a_n$, if $X = G(M, a_1, \ldots, a_n)$ then $X = \{x : \varphi(M, x, a_1, \ldots, a_n)\}$. If, moreover, M is transitive and $X \subset M$, then $X = \{x \in M : M \vDash \psi[x, a_1, \ldots, a_n]\}$ (where ψ is an obvious modification of φ, e.g., replacing $\exists u \in M$ by $\exists u$). Thus we have the following description of $\mathrm{def}(M)$:

Corollary 13.8. *For every transitive set M,*

$$\mathrm{def}(M) = \mathrm{cl}(M \cup \{M\}) \cap P(M),$$

where cl *denotes the closure under $G_1, \ldots, G_{10}$.* □

Inner Models of ZF

An *inner model* of ZF is a transitive class that contains all ordinals and satisfies the axioms of ZF. The constructible universe L is an inner model of ZF, and as we show later in this chapter, L is the smallest inner model of ZF.

In Chapter 12 we proved that Δ_0 formulas are absolute for all transitive models, i.e., φ^M is equivalent to φ, for every transitive class M. One can extend the use of superscripts to concepts other than formulas, namely classes, operations and constants:

If C is a class $\{x : \varphi(x)\}$ then C^M denotes the class $\{x : \varphi^M(x)\}$. As an example, Ord^M is either Ord (if M contains all ordinals), or is the least ordinal not in M.

If F is an operation then F^M is the corresponding operation in M (if $x \in M$ then $F^M(x)$ is defined if M satisfies the statement that $F(x)$ exists). If $F^M(x) = F(x)$ for all x for which $F^M(x)$ is defined, we say that F is *absolute* for M. By Lemma 13.7, all Gödel operations are absolute for transitive models. As an example, $P^M(X) = P(X) \cap M$, and $V_\alpha^M = V_\alpha \cap M$ (Exercise 13.6).

Similarly, if c is a constant symbol then c^M, if it exists, is the corresponding constant in M. Thus $\emptyset^M = \emptyset$ (if $\emptyset \in M$), $\omega^M = \omega$ (if $\omega \in M$), etc.

The following theorem gives a necessary and sufficient condition for a transitive class to be an inner model of ZF:

Theorem 13.9. *A transitive class M is an inner model of* ZF *if and only if it is closed under Gödel operations and is* almost universal, *i.e., every subset $X \subset M$ is included in some $Y \in M$.*

Proof. As Gödel operations are absolute for transitive models, an inner model is necessarily closed under $G_1, \ldots, G_{10}$. If X is a subset of an inner model M, then $X \subset V_\alpha \cap M$ for some α, and $V_\alpha \cap M$ is in M because $\alpha \in M$ and $V_\alpha \cap M = V_\alpha^M$. Thus the condition is necessary.

Now let M be a transitive almost universal class that is closed under Gödel operations. Except for the Separation Schema, the verification of the axioms of ZF in M follows closely the proof of Theorem 13.3 (or of Theorem 12.11), but using almost universality. For example, if $X \in M$ then $P(X) \cap M$ is included in some $Y \in M$, verifying the weak version (1.9) of the Power Set Axiom. We leave the details to the reader.

Separation. We will show that for every $X \in M$ the set $Y = \{u \in X : \varphi^M(u)\}$ is in M. (For simplicity, we disregard the parameter in the formula φ.)

Let $\varphi(u_1, \ldots, u_n)$ be a formula with k quantifiers. We let $\bar{\varphi}(u_1, \ldots, u_n, Y_1, \ldots, Y_k)$ be the Δ_0 formula obtained by replacing each $\exists x$ (or $\forall x$) in φ by $\exists x \in Y_j$ (or $\forall x \in Y_j$) for $j = 1, \ldots, k$. We shall prove, by induction on k, that for every $\varphi(u_1, \ldots, u_n)$ with k quantifiers, for every $X \in M$ there exist

$Y_1, \ldots, Y_k \in M$ such that

$$\varphi^M(u_1, \ldots, u_n) \text{ if and only if } \bar{\varphi}(u_1, \ldots, u_n, Y_1, \ldots, Y_k)$$

for all $u_1, \ldots, u_n \in X$. Then it follows that $Y = \{u \in X : \bar{\varphi}(u, Y_1, \ldots, Y_k)\}$, and since M satisfies Δ_0-Separation (by Corollary 13.5), we have verified that $Y \in M$, completing the proof.

If $k = 0$ then $\bar{\varphi} = \varphi$. For the induction step, let $\varphi(u)$ be $\exists v\, \psi(u, v)$ where ψ has k quantifiers. Thus $\bar{\varphi}$ is $(\exists v \in Y_{k+1})\, \bar{\psi}(u, v, Y_1, \ldots, Y_k)$.

Let $X \in M$. We look for $Y_1, \ldots, Y_k, Y_{k+1} \in M$ such that for every $u \in X$,

$$(\exists v\, \psi(u, v))^M \quad \text{if and only if} \quad (\exists v \in Y_{k+1})\, \bar{\psi}(u, v, Y_1, \ldots, Y_k). \tag{13.6}$$

By the Collection Principle (6.5) (applied to the formula $v \in M \wedge \psi^M(u, v)$), there exists a set M_1 such that $X \subset M_1 \subset M$ and that for every $u \in X$,

$$(\exists v \in M)\, \psi^M(u, v) \quad \text{if and only if} \quad (\exists v \in M_1)\, \psi^M(u, v). \tag{13.7}$$

Since M is almost universal, there exists a set $Y \in M$ such that $M_1 \subset Y$. It follows from (13.7) that for every $u \in X$,

$$(\exists v \in M)\, \psi^M(u, v) \quad \text{if and only if} \quad (\exists v \in Y)\, \psi^M(u, v).$$

By the induction hypothesis, given $Y \in M$, there exist $Y_1, \ldots, Y_k \in M$ such that for all $u, v \in Y$,

$$\psi^M(u, v) \quad \text{if and only if} \quad \bar{\psi}(u, v, Y_1, \ldots, Y_k).$$

Thus we let $Y_{k+1} = Y$, and since $X \subset Y$, we have for all $u \in X$,

$$\begin{aligned} (\exists v\, \psi(u, v))^M \quad &\text{if and only if} \quad (\exists v \in M)\, \psi^M(u, v) \\ &\text{if and only if} \quad (\exists v \in Y)\, \psi^M(u, v) \\ &\text{if and only if} \quad (\exists v \in Y)\, \bar{\psi}(u, v, Y_1, \ldots, Y_k). \end{aligned}$$

□

The Lévy Hierarchy

Definable concepts can be classified by means of the following hierarchy of formulas, introduced by Azriel Lévy:

A formula is Σ_0 and Π_0 if its only quantifiers are bounded, i.e., a Δ_0 formula. Inductively, a formula is Σ_{n+1} if it is of the form $\exists x\, \varphi$ where φ is Π_n, and Π_{n+1} if its is of the form $\forall x\, \varphi$ where φ is Σ_n.

We say that a property (class, relation) is Σ_n (or Π_n) if it can be expressed by a Σ_n (or Π_n) formula. A function F is Σ_n (Π_n) if the relation $y = F(x)$ is Σ_n (Π_n).

This classification of definable concepts is not syntactical: To verify that a concept can be expressed in a certain way may need a proof (in ZF). To illustrate this, consider the proof of Lemma 13.10 bellow: To contract two like quantifiers into one uses an application of the Pairing Axiom.

Whenever we say that a property P is Σ_n we always mean P can be expressed by a Σ_n formula in ZF, unless we specifically state which axioms of ZF are assumed. Since every proof uses only finitely many axioms, every specific property requires a finite set Σ of axioms of ZF for its classification in the hierarchy. This finite set is implicit in the use of the defining formula. When M is a transitive model of Σ then the relativization P^M is unambiguous, namely the formula φ^M. We call such transitive models *adequate* for P. A property is Δ_n if it is both Σ_n and Π_n.

Lemma 13.10. *Let $n \geq 1$.*

(i) *If P, Q are Σ_n properties, then so are $\exists x\, P$, $P \wedge Q$, $P \vee Q$, $(\exists u \in x)\, P$, $(\forall u \in x)\, P$.*
(ii) *If P, Q are Π_n properties, then so are $\forall x\, P$, $P \wedge Q$, $P \vee Q$, $(\forall u \in x)\, P$, $(\exists u \in x)\, P$.*
(iii) *If P is Σ_n, then $\neg P$ is Π_n; if P is Π_n, then $\neg P$ is Σ_n.*
(iv) *If P is Π_n and Q is Σ_n, then $P \to Q$ is Σ_n; if P is Σ_n and Q is Π_n, then $P \to Q$ is Π_n*
(v) *If P and Q are Δ_n, then so are $\neg P$, $P \wedge Q$, $P \vee Q$, $P \to Q$, $P \leftrightarrow Q$, $(\forall u \in x)\, P$, $(\exists u \in x)\, P$.*
(vi) *If F is a Σ_n function, then $\mathrm{dom}(F)$ is a Σ_n class.*
(vii) *If F is a Σ_n function and $\mathrm{dom}(F)$ is Δ_n, then F is Δ_n.*
(viii) *If F and G are Σ_n functions, then so is $F \circ G$.*
(ix) *If F is a Σ_n function and if P is a Σ_n property, then $P(F(x))$ is Σ_n.*

Proof. Let us prove the lemma for $n = 1$. The general case follows easily by induction.

(i) Let

$$P(x, \ldots) \leftrightarrow \exists z\, \varphi(z, x, \ldots),$$
$$Q(x, \ldots) \leftrightarrow \exists u\, \psi(u, x, \ldots)$$

where φ and ψ are Δ_0 formulas. We have

$$\begin{aligned} \exists x\, P(x, \ldots) &\leftrightarrow \exists x\, \exists z\, \varphi(z, x, \ldots) \\ &\leftrightarrow \exists v\, \exists w \in v\, \exists x \in w\, \exists z \in w\, (v = (x, z) \wedge \varphi(z, x, \ldots)). \end{aligned} \tag{13.8}$$

The right-hand side of (13.8) is a Σ_1 formula. Furthermore,

$$P(x, \ldots) \wedge Q(x, \ldots) \leftrightarrow \exists z\, \exists u\, (\varphi(z, x, \ldots) \wedge \psi(u, x, \ldots)),$$
$$P(x, \ldots) \vee Q(x, \ldots) \leftrightarrow \exists z\, \exists u\, (\varphi(z, x, \ldots) \vee \psi(u, x, \ldots)),$$
$$(\exists u \in x)\, P(u, \ldots) \leftrightarrow \exists z\, \exists u\, (u \in x \wedge \varphi(z, u, \ldots)).$$

To show that $(\forall u \in x)\, P$ is a Σ_1 property, we use the Collection Principle:

$$\begin{aligned}(\forall u \in x)\, P(u, \ldots) &\leftrightarrow (\forall u \in x)\, \exists z\, \varphi(z, u, \ldots) \\ &\leftrightarrow \exists y\, (\forall u \in x)(\exists z \in y)\, \varphi(z, u, \ldots).\end{aligned}$$

(ii) follows from (i) and (iii).
(iii)

$$\begin{aligned}\neg\exists z\, \varphi(z, x, \ldots) &\leftrightarrow \forall z\, \neg\varphi(z, x, \ldots), \\ \neg\forall z\, \varphi(z, x, \ldots) &\leftrightarrow \exists z\, \neg\varphi(z, x, \ldots).\end{aligned}$$

(iv)

$$(P \to Q) \leftrightarrow (\neg P \vee Q).$$

(v) follows from (i)–(iv).
(vi)

$$x \in \mathrm{dom}(F) \leftrightarrow \exists y\, y = F(x).$$

(vii) Since F is a function, we have

$$y = F(x) \leftrightarrow x \in \mathrm{dom}(F) \wedge \forall z\, (z = F(x) \to y = z). \tag{13.9}$$

If $z = F(x)$ is Σ_n and $x \in \mathrm{dom}(F)$ is Π_n, then the right-hand side of (13.9) is Π_n.
(viii)

$$y = F(G(x)) \leftrightarrow \exists z\, (z = G(x) \wedge y = F(z)).$$

(ix)

$$P(F(x)) \leftrightarrow \exists y\, (y = F(x) \wedge P(y)).$$ □

Since Δ_0 properties are absolute for all transitive models, it is clear that Σ_1 properties are *upward absolute*: If $P(x)$ is Σ_1 and if M is a transitive model (adequate for P) then for all $x \in M$, $P^M(x)$ implies $P(x)$. Similarly, Π_1 properties are *downward absolute*, and consequently, Δ_1 properties are absolute for transitive models.

As an example of a Δ_1 property we show

Lemma 13.11. "*E is a well-founded relation on P*" *is a Δ_1 property.*

Proof. The following is a Π_1 formula: E is a relation on P and $\forall X\, \varphi(E, P, X)$, where $\varphi(E, P, X)$ is the formula

$$\emptyset \neq X \subset P \to (\exists a \in X)\, a \text{ is } E\text{-minimal in } X.$$

(Both "E is a relation on P" and $\varphi(E, P, X)$ are Δ_0 formulas.)

On the other hand, E is well-founded if and only if there exists a function f from P into *Ord* such that $f(x) < f(y)$ whenever $x\ E\ y$. Thus we have an equivalent Σ_1 formula: E is a relation on P and $\exists f\, (f$ is a function $\wedge\ (\forall u \in \mathrm{ran}(f))\, u$ is an ordinal $\wedge\ (\forall x, y \in P)(x\ E\ y \to f(x) < f(y)))$. □

Other examples of Δ_1 concepts are given in the Exercises.

Lemma 13.12. *Let $n \geq 1$, let G be a Σ_n function (on V), and let F be defined by induction*:

$$F(\alpha) = G(F\restriction\alpha).$$

Then F is a Σ_n function on Ord.

Proof. Since *Ord* is a Σ_0 class, it is enough to verify that the following expression is Σ_n:

$$\begin{aligned}(13.10)\quad y = F(\alpha) \text{ if and only if } \exists f\,(f \text{ is a function} \wedge \operatorname{dom}(f) = \alpha \\ \wedge\, (\forall \xi < \alpha)\, f(\xi) = G(f\restriction\xi) \wedge y = G(f)).\end{aligned}$$

All the properties and operations in (13.10) are Σ_0 and G is Σ_n, and hence $y = F(\alpha)$ is Σ_n. □

The power set operation $P(X)$ is obviously Π_1; since it is not absolute as we shall see in Chapter 14, it is not Σ_1. Similarly, cardinal concepts are Π_1 but not Σ_1:

Lemma 13.13. "*α is a cardinal*," "*α is a regular cardinal*," *and* "*α is a limit cardinal*" *are* Π_1.

Proof. (a) $\neg\exists f\,(f$ is a function and $\operatorname{dom}(f) \in \alpha$ and $\operatorname{ran}(f) = \alpha)$.
(b) $\alpha > 0$ is a limit ordinal and
$$\neg\exists f\,(f \text{ is a function and } \operatorname{dom}(f) \in \alpha \text{ and } \bigcup \operatorname{ran}(f) = \alpha).$$
(c) $(\forall \beta < \alpha)(\exists \gamma < \alpha)(\beta < \gamma$ and γ is a cardinal). □

Consequently, if M is an inner model of ZF, then every cardinal (regular cardinal, limit cardinal) is a cardinal (regular cardinal, limit cardinal) in M, and if $|X|^M = |Y|^M$ then $|X| = |Y|$.

In Chapter 12 we pointed out that the satisfaction relation $(V, \in) \vDash \varphi[a_1, \ldots, a_n]$ (for $\varphi \in \mathit{Form}$) is not formalizable in ZF; this follows from Theorem 12.7. For any particular n, the satisfaction relation $\vDash_n$ restricted to Σ_n formulas is formalizable: For $n = 0$, we can use the absoluteness of Δ_0 formulas for transitive models,

$$\vDash_0 \varphi[a_1, \ldots, a_k] \quad \text{if and only if}$$
$$\varphi \in \mathit{Form},\ \varphi \text{ is } \Delta_0, \text{ and } \exists M\,(M \text{ is transitive and } (M, \in) \vDash \varphi[a_1, \ldots, a_k]);$$

then inductively

$$\vDash_{n+1} (\exists x\, \varphi)[a_1, \ldots, a_k] \quad \text{if and only if}$$
$$\varphi \in \mathit{Form},\ \varphi \text{ is } \Pi_n, \text{ and } \exists a \neg \vDash_n (\neg\varphi)[a, a_1, \ldots, a_k].$$

Similarly, we can define $\vDash_n^M$ for any particular n and any transitive class M. Even more generally, we can define $\vDash_n^{(M,\in)}$ for any class M (transitive or not).

If $M \subset N$, we say that $(M, \in)$ is a Σ_n-*elementary submodel* of $(N, \in)$,

$$(M, \in) \prec_{\Sigma_n} (N, \in),$$

if for every Σ_n formula $\varphi \in \mathit{Form}$ and all $a_1, \ldots, a_k \in M$, $\vDash_n^M \varphi[a_1, \ldots, a_k] \leftrightarrow \vDash_n^N \varphi[a_1, \ldots, a_k]$.

Absoluteness of Constructibility

We prove in this section that the property "x is constructible" is absolute for inner models of ZF.

Lemma 13.14. *The function $\alpha \mapsto L_\alpha$ is Δ_1.*

Proof. The function L_α is defined by transfinite induction and so by Lemma 13.12 it suffices to show that the induction step is Σ_1. In view of Corollary 13.8 it suffices to verify that

$$Y = \mathrm{cl}(M) \tag{13.11}$$

(where cl denotes closure under Gödel operations) is Σ_1. But (13.11) is equivalent to

$$\begin{aligned}\exists W\, [&W \text{ is a function} \wedge \mathrm{dom}(W) = \omega \wedge Y = \bigcup \mathrm{ran}(W) \wedge W(0) = M \\ &\wedge (\forall n \in \mathrm{dom}(W))(W(n+1) = W(n) \cup \{G_i(x, y) : x \in W(n),\ y \in W(n), \\ &\qquad\qquad i = 1, \ldots, 10\})].\end{aligned}$$

□

Corollary 13.15. *The property "x is constructible" is absolute for inner models of* ZF.

Proof. Let M be an inner model of ZF. Since $M \supset \mathit{Ord}$, we have for all $x \in M$

$$(x \text{ is constructible})^M \leftrightarrow \exists \alpha \in M\, x \in L_\alpha^M \leftrightarrow \exists \alpha\, x \in L_\alpha \leftrightarrow x \text{ is constructible.}$$

□

As an immediate consequence we have.

Theorem 13.16 (Gödel).

(i) *L satisfies the Axiom of Constructibility ($V = L$).*
(ii) *L is the smallest inner model of* ZF.

Proof. (i) For every $x \in L$, $(x \text{ is constructible})^L$ if and only if x is constructible, and hence "every set is constructible" holds in L.

(ii) If M is an inner model then L^M (the class of all constructible sets in M) is L and so $L \subset M$. □

A detailed analysis of absoluteness of L_α for transitive models reveals that the following concept of adequacy suffices: Let us call a transitive set M *adequate* if

(13.12) (i) M is closed under $G_1, \ldots, G_{10}$,
(ii) for all $U \in M$, $\{G_i(x,y) : x, y \in U \text{ and } i = 1, \ldots, 10\} \in M$,
(iii) if $\alpha \in M$ then $\langle L_\beta : \beta < \alpha \rangle \in M$.

It follows that the Δ_1 function $\alpha \mapsto L_\alpha$ is absolute for every adequate transitive set M. Also, we can verify that for every limit ordinal δ, the transitive set L_δ is adequate. Moreover, adequacy can by formulated as follows: There is a sentence σ such that for every transitive set M, M is adequate if and only if $(M, \in) \vDash \sigma$. Therefore there exists a sentence σ (which is Π_2) such that for every transitive set M

(13.13) $(M, \in) \vDash \sigma$ if and only if $M = L_\delta$ for some limit ordinal δ.

This leads to the following:

Lemma 13.17 (Gödel's Condensation Lemma). *For every limit ordinal δ, if $M \prec (L_\delta, \in)$ then the transitive collapse of M is L_γ for some $\gamma \leq \delta$.* □

We wish to make two remarks at this point. First, it is enough to assume only $M \prec_{\Sigma_1} L_\delta$ for the Condensation Lemma to hold (as the sentence σ in (13.13)) is Π_2. Secondly, the careful analysis of the definition of L_α makes it possible to find a Π_2 sentence σ such that (13.13) holds even for (infinite) successor ordinals δ. Thus Gödel's Condensation Lemma holds for all infinite ordinals δ, a fact that is useful in some applications of L.

Consistency of the Axiom of Choice

Theorem 13.18 (Gödel). *There exists a well-ordering of the class L. Thus $V = L$ implies the Axiom of Choice.*

Combining Theorems 13.16 and 13.18, we conclude that the Axiom of Choice holds in the model L, and so it is consistent with ZF.

Proof. We will show that L has a definable well-ordering.

By induction, we construct for each α a well-ordering $<_\alpha$ of L_α. We do it in such a way that if $\alpha < \beta$, then $<_\beta$ is an *end-extension* of $<_\alpha$, i.e.,

(13.14) (i) if $x <_\alpha y$, then $x <_\beta y$;
(ii) if $x \in L_\alpha$ and $y \in L_\beta - L_\alpha$, then $x <_\beta y$.

Notice that (13.14) implies that if $x \in y \in L_\alpha$, then $x <_\alpha y$.

First let us assume that α is a limit ordinal and that we have constructed $<_\beta$ for all $\beta < \alpha$ and that if $\beta_1 < \beta_2 < \alpha$, then $<_{\beta_2}$ is an end-extension of $<_{\beta_1}$. In this case we simply let

$$<_\alpha = \bigcup_{\beta<\alpha} <_\beta,$$

i.e., if $x, y \in L_\alpha$, we let

$$x <_\alpha y \quad \text{if and only if} \quad (\exists \beta < \alpha)\, x <_\beta y.$$

Thus assume that we have defined $<_\alpha$ and let us construct $<_{\alpha+1}$, a well-ordering of $L_{\alpha+1}$. We recall the definition of $L_{\alpha+1}$:

$$L_{\alpha+1} = P(L_\alpha) \cap \mathrm{cl}(L_\alpha \cup \{L_\alpha\}) = P(L_\alpha) \cap \bigcup_{n=0}^{\infty} W_n^\alpha,$$

where

$$W_0^\alpha = L_\alpha \cup \{L_\alpha\},$$
$$W_{n+1}^\alpha = \{G_i(X, Y) : X, Y \in W_n^\alpha,\ i = 1, \ldots, 10\}.$$

The idea of the construction of $<_{\alpha+1}$ is now as follows: First we take the elements of L_α, then L_α, then the remaining elements of W_1^α, then the remaining elements of W_2^α, etc. To order the elements of W_{n+1}^α, we use the already defined well-ordering of W_n^α since every $x \in W_{n+1}^\alpha$ is equal to $G_i(u, v)$ for some $i = 1, \ldots, 10$ and some $u, v \in W_n^\alpha$. We let

(13.15)

(i) $<_{\alpha+1}^0$ is the well-ordering of $L_\alpha \cup \{L_\alpha\}$ that extends $<_\alpha$ and such that L_α is the last element.

(ii) $<_{\alpha+1}^{n+1}$ is the following well-ordering of W_{n+1}^α:
$x <_{\alpha+1}^{n+1} y$ if and only if either: $x <_{\alpha+1}^n y$,
or: $x \in W_n^\alpha$ and $y \notin W_n^\alpha$,
or: $x \notin W_n^\alpha$ and $y \notin W_n^\alpha$ and

(a) the least i such that $\exists u, v \in W_n^\alpha\, (x = G_i(u, v)) <$ the least j such that $\exists s, t \in W_n^\alpha\, (x = G_i(s, t))$, or

(b) the least $i =$ the least j and
[the $<_{\alpha+1}^n$-least $u \in W_n^\alpha$ such that $\exists v \in W_n^\alpha\, (x = G_i(u, v))$]
$<_{\alpha+1}^n$ [the $<_{\alpha+1}^n$-least $s \in W_n^\alpha$ such that $\exists t \in W_n^\alpha\, (x = G_i(s, t))$], or

(c) the least $i =$ the least j and the least $u =$ the least s and
[the $<_{\alpha+1}^n$-least $v \in W_n^\alpha$ such that $x = G_i(u, v)$] $<_{\alpha+1}^n$
[the $<_{\alpha+1}^n$-least $t \in W_n^\alpha$ such that $x = G_i(u, t)$].

Now we let

$$<_{\alpha+1} = \bigcup_{n=0}^{\infty} <_{\alpha+1}^n \cap (P(L_\alpha) \times P(L_\alpha)), \tag{13.16}$$

and it is clear that $<_{\alpha+1}$ is an end-extension of $<_\alpha$ and is a well-ordering of $L_{\alpha+1}$.

Having defined $<_\alpha$ for all α, we let

$$x <_L y \quad \text{if and only if} \quad \exists\alpha\, x <_\alpha y.$$

The relation $<_L$ is a well-ordering of L. □

We call $<_L$ the *canonical well-ordering* of L.

The proof of Theorem 13.18 gives additional information about the complexity of the canonical well-ordering of L.

Lemma 13.19. *The relation $<_L$ is Σ_1 and moreover, for every limit ordinal δ and every $y \in L_\delta$, $x <_L y$ if and only if $x \in L_\delta$ and $(L_\delta, \in) \vDash x <_L y$.*

Proof. It suffices to prove that the function, $\alpha \mapsto <_\alpha$ which assigns to each α the canonical well-ordering of L_α is Σ_1.

The function $\alpha \mapsto <_\alpha$ is defined by induction and thus it suffices to show that the induction step is Σ_1. In fact, $<_{\alpha+1}$ is defined by induction from $<_\alpha$ (see (13.15) and (13.16)). It suffices to verify that $<_{\alpha+1}$ is obtained from $<_\alpha$ by means of a Δ_1 operation (similar to the way in which $L_{\alpha+1}$ is obtained from L_α by $L_{\alpha+1} = \mathrm{def}(L_\alpha)$). The operation that yields $<_{\alpha+1}$ when applied to $<_\alpha$ is described in detail in (13.15). It can be written in a Σ_1 fashion in very much the same way as (13.11). The only potential difficulty might be the use of the words "the $<$-least," and that can be overcome as follows: For example, in (13.15)(ii)(c)

$$\text{the } <^n_{\alpha+1}\text{-least } v \in W^\alpha_n \text{ such that } x = G_i(u, v)$$
$$<^n_{\alpha+1} \text{ the } <^n_{\alpha+1}\text{-least } t \in W^\alpha_n \text{ such that } y = G_i(u, t)$$

can be written as

$$(\exists v \in W^\alpha_n)[x = G_i(u, v) \wedge (\forall t \in W^\alpha_n)(y = G_i(u, t) \to v <^n_{\alpha+1} t)].$$

The function $\alpha \mapsto <_\alpha$ is absolute for every adequate M (see (13.12)) and therefore for every L_δ where δ is a limit ordinal. □

Consistency of the Generalized Continuum Hypothesis

Theorem 13.20 (Gödel). *If $V = L$ then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every α.*

Proof. We shall prove that if X is a constructible subset of ω_α then there exists a $\gamma < \omega_{\alpha+1}$ such that $X \in L_\gamma$. Therefore $P^L(\omega_\alpha) \subset L_{\omega_{\alpha+1}}$, and since $|L_{\omega_{\alpha+1}}| = \aleph_{\alpha+1}$ (this is easy to show; see Exercise 13.19), we have $|P^L(\omega_\alpha)| \leq \aleph_{\alpha+1}$.

Thus let $X \subset \omega_\alpha$. There exists a limit ordinal $\delta > \omega_\alpha$ such that $X \in L_\delta$. Let M be an elementary submodel of L_δ such that $\omega_\alpha \subset M$ and $X \in M$, and

that $|M| = \aleph_\alpha$. (As we can construct M within L which satisfies AC, this can be done even if AC does not hold in the universe.)

By the Condensation Lemma 13.17, the transitive collapse N of M is L_γ for some $\gamma \leq \delta$. Clearly, γ is a limit ordinal, and $\gamma < \omega_{\alpha+1}$ because $|N| = |\gamma| = \aleph_\alpha$. As $\omega_\alpha \subset M$, the collapsing map π is the identity on ω_α and so $\pi(X) = X$. Hence $X \in L_\gamma$. □

The next theorem illustrates further the significance of Gödel's Condensation Lemma. The combinatorial principle $\diamondsuit$ was formulated by Ronald Jensen.

Theorem 13.21 (Jensen). *$V = L$ implies the Diamond Principle*:

($\diamondsuit$) *There exists a sequence of sets $\langle S_\alpha : \alpha < \omega_1\rangle$ with $S_\alpha \subset \alpha$, such that for every $X \subset \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is a stationary subset of ω_1.*

The sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ is called a $\diamondsuit$-*sequence*.

Proof. Assume $V = L$. By induction on $\alpha < \omega_1$, we define a sequence of pairs (S_α, C_α), $\alpha < \omega_1$, such that $S_\alpha \subset \alpha$ and C_α is a closed unbounded subset of α. We let $S_0 = C_0 = \emptyset$ and $S_{\alpha+1} = C_{\alpha+1} = \alpha + 1$ for all α. If α is a limit ordinal, we define:

(13.17) (S_α, C_α) is the $<_L$-least pair such that $S_\alpha \subset \alpha$, C_α is a closed unbounded subset of α, and $S_\alpha \cap \xi \neq S_\xi$ for all $\xi \in C_\alpha$; if no such pair exists, let $S_\alpha = C_\alpha = \alpha$.

We are going to show that the sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ is a $\diamondsuit$-sequence. Thus assume the contrary; then for some $X \subset \omega_1$, there exists a closed unbounded set C such that

$$X \cap \alpha \neq S_\alpha \quad \text{for all } \alpha \in C. \tag{13.18}$$

Let (X, C) be the $<_L$-least pair such that $X \subset \omega_1$, C is a closed unbounded subset of ω_1, and such that (13.18) holds.

Since $\langle (S_\alpha, C_\alpha) : \alpha < \omega_1\rangle$ is a ω_1-sequence of pairs of subsets of ω_1, it belongs to L_{ω_2}, and moreover, it satisfies the same definition (13.17) in the model $(L_{\omega_2}, \in)$. Also, $(X, C) \in L_{\omega_2}$, and (X, C) is, in $(L_{\omega_2}, \in)$, the $<_L$-least pair such that $X \subset \omega_1$, C is a closed unbounded subset of ω_1, and such that (13.18) holds.

Let N be a countable elementary submodel of $(L_{\omega_2}, \in)$. Since (X, C) and $\langle (S_\alpha, C_\alpha) : \alpha < \omega_1\rangle$ are definable in $(L_{\omega_2}, \in)$, they belong to N. The set $\omega_1 \cap N$ is an initial segment of ω_1 (see Exercise 13.18), thus let $\delta = \omega_1 \cap N$.

The transitive collapse of N is L_γ, for some $\gamma < \omega_1$, and let $\pi : N \to L_\gamma$ be the isomorphism. We have $\pi(\omega_1) = \delta$, $\pi(X) = X \cap \delta$, $\pi(C) = C \cap \delta$ and $\pi(\langle (S_\alpha, C_\alpha) : \alpha < \omega_1\rangle) = \langle (S_\alpha, C_\alpha) : \alpha < \delta\rangle$.

Therefore $(L_\delta, \in)$ satisfies

(13.19) $(X \cap \delta, C \cap \delta)$ is the $<_L$-least pair (Z, D) such that $Z \subset \delta$, $D \subset \delta$ is closed unbounded and $Z \cap \xi \neq S_\xi$ for all $\xi \in D$.

By absoluteness, (13.19) holds (in L, and $L = V$) and therefore, by (13.17), $X \cap \delta = S_\delta$. Since $C \cap \delta$ is unbounded in δ, and C is closed, it follows that $\delta \in C$. This contradicts (13.18). □

Relative Constructibility

Constructibility can be generalized by considering sets *constructible relative to* a given set A, resulting in an inner model $L[A]$. The idea is to relativize the hierarchy L_α by using the generalization

$$\mathrm{def}_A(M) = \{X \subset M : X \text{ is definable over } (M, \in, A \cap M)\} \tag{13.20}$$

where $A \cap M$ is considered a unary predicate. A generalization of Corollary 13.8 provides an alternative description of def_A: For every transitive set M,

$$\mathrm{def}_A(M) = \mathrm{cl}(M \cup \{M\} \cup \{A \cap M\}) \cap P(M). \tag{13.21}$$

The class of all sets *constructible from* A is defined as follows:

$$\begin{aligned} &L_0[A] = \emptyset, \qquad L_{\alpha+1}[A] = \mathrm{def}_A(L_\alpha[A]), \\ &L_\alpha[A] = \bigcup_{\beta<\alpha} L_\beta[A] \quad \text{if } \alpha \text{ is a limit ordinal}, \\ &L[A] = \bigcup_{\alpha \in Ord} L_\alpha[A]. \end{aligned} \tag{13.22}$$

The following theorem is the generalization of the relevant theorem on constructible sets:

Theorem 13.22. *Let A be an arbitrary set.*

(i) *$L[A]$ is a model of* ZFC.
(ii) *$L[A]$ satisfies the axiom $\exists X\,(V = L[X])$.*
(iii) *If M is an inner model of* ZF *such that $A \cap M \in M$, then $L[A] \subset M$.*
(iv) *There exists α_0 such that for all $\alpha \geq \alpha_0$,*

$$L[A] \vDash 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

Proof. The proof follows closely the corresponding proofs for L, but some additional arguments are needed.

Lemma 13.23. *Let $\bar{A} = A \cap L[A]$. Then $L[\bar{A}] = L[A]$ and moreover $\bar{A} \in L[\bar{A}]$.*

Proof. We show by induction on α that $L_\alpha[\bar{A}] = L_\alpha[A]$. The induction step is obvious if α is a limit ordinal; thus assume that $L_\alpha[\bar{A}] = L_\alpha[A]$ and let us prove $L_{\alpha+1}[\bar{A}] = L_{\alpha+1}[A]$.

If we denote $U = L_\alpha[A]$, then we have

$$A \cap U = A \cap U \cap L[A] = \bar{A} \cap U,$$

and since $\mathrm{def}_A(U) = \mathrm{def}_{A\cap U}(U)$, we have

$$L_{\alpha+1}[A] = \mathrm{def}_A(U) = \mathrm{def}_{A\cap U}(U) = \mathrm{def}_{\bar{A}}(U) = L_{\alpha+1}[\bar{A}].$$

Thus $L[\bar{A}] = L[A]$. Moreover, there is α such that $A \cap L[A] = A \cap L_\alpha[A]$ and thus $\bar{A} \in L_{\alpha+1}[A]$. □

By Lemma 13.23 we may assume that $A \in L[A]$. In this case, $L[A]$ can be well-ordered by a relation that is definable from A.

In analogy with (13.13) there exists a Π_2 sentence (in the language $\{\in, A\}$ where A is a unary predicate) such that for every transitive set M

(13.23) $(M, \in, A \cap M) \models \sigma$ if and only if $M = L_\delta$ for some limit ordinal δ.

The Condensation Lemma is generalized as follows:

Lemma 13.24. *If $M \prec (L_\delta[A], \in, A \cap L_\delta[A])$ where δ is a limit ordinal, then the transitive collapse of M is $L_\gamma[A]$ for some $\gamma \le \delta$.* □

Consequently, if $A \subset L_{\omega_\alpha}[A]$ then for every $X \subset \omega_\alpha$ in $L[A]$ there exists a $\gamma < \omega_{\alpha+1}$ such that $X \in L_\gamma[A]$, completing the proof of Theorem 13.22. □

A consequence of Theorem 13.22(iv) is that if $V = L[A]$ and $A \subset \omega$, then the Generalized Continuum Hypothesis holds. For a slightly better result, see Exercise 13.26.

A different generalization yields for every set A the smallest inner model $L(A)$ that contains A. (As an example, $L(\boldsymbol{R})$ is the smallest inner model that contains all reals.) The model $L(A)$ need not, however, satisfy the Axiom of Choice.

We define $L(A)$ as follows: Let $T = \mathrm{TC}(\{A\})$ be transitive closure of A (to ensure that the resulting class $L(A)$ is transitive), and let

(13.24)
$$\begin{aligned} &L_0(A) = T, \qquad L_{\alpha+1}(A) = \mathrm{def}(L_\alpha(A)), \\ &L_\alpha(A) = \bigcup_{\beta<\alpha} L_\beta(A) \quad \text{if } \alpha \text{ is a limit ordinal, and} \\ &L(A) = \bigcup_{\alpha\in Ord} L_\alpha(A). \end{aligned}$$

The transitive class $L(A)$ is an inner model of ZF, contains A, and is the smallest such inner model.

Ordinal-Definable Sets

A set X is *ordinal-definable* if there is a formula φ such that

$$X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n)\} \tag{13.25}$$

for some ordinal numbers $\alpha_1, \ldots, \alpha_n$.

It is not immediate clear that the property "ordinal-definable" is expressible in the language of set theory. Thus we give a different definition of ordinal definable sets and show that it is equivalent to (13.25).

We recall that $\mathrm{cl}(M)$ denotes the closure of a set M under Gödel operations. The class OD of all *ordinal-definable sets* is define as follows:

$$OD = \bigcup_{\alpha \in Ord} \mathrm{cl}\{V_\beta : \beta < \alpha\}. \tag{13.26}$$

In other words, OD is the Gödel closure of $\{V_\alpha : \alpha \in Ord\}$, that is, ordinal definable sets are obtained from the V_α by applications of Gödel operations. We shall show that the elements of the class OD are exactly the sets satisfying (13.25).

Lemma 13.25. *There exists a definable well-ordering of the class OD (and a one-to-one definable mapping F of Ord onto OD).*

Proof. Earlier we described how to construct from a given well-ordering of a set M, a well-ordering of the set $\mathrm{cl}(M)$. For every α, the set $\{V_\beta : \beta < \alpha\}$ has an obvious well-ordering, which induces a well-ordering of $\mathrm{cl}\{V_\beta : \beta < \alpha\}$. Thus we get a well-ordering of the class OD, and denote F the corresponding (definable) one-to-one mapping of Ord onto OD. □

Now it follows that every $X \in OD$ has the form (13.25). There exists α such that $X = \{u : \varphi(u, \alpha)\}$ where $\varphi(u, \alpha)$ is the formula $u \in F(\alpha)$.

We shall show that on the other hand, if φ is a formula and X is the set in (13.25), then $X \in OD$. By the Reflection Principle, let β be such that $X \subset V_\beta$, $\alpha_1, \ldots, \alpha_n < \beta$ and that V_β reflects φ. Then we have

$$X = \{u \in V_\beta : \varphi^{V_\beta}(u, \alpha_1, \ldots, \alpha_n)\}.$$

Since φ^{V_β} is a Δ_0 formula, we apply the normal form theorem and find a Gödel operation G such that $X = G(V_\beta, \alpha_1, \ldots, \alpha_n)$. Since every α is obtained (uniformly) from V_α by a Gödel operation (because $\alpha = \{x \in V_\alpha : x \text{ is an ordinal}\}$), there exists a Gödel operation H such that $X = H(V_{\alpha_1}, \ldots, V_{\alpha_n}, V_\beta)$ and therefore $X \in OD$.

Thus let HOD denote the class of *hereditarily ordinal-definable* sets

$$HOD = \{x : \mathrm{TC}(\{x\}) \subset OD\}.$$

The class HOD is transitive and contains all ordinals.

Theorem 13.26. *The class HOD is a transitive model of* ZFC.

Proof. The class HOD is transitive, and it is easy to see that it is closed under Gödel operations. Thus to show that HOD is a model of ZF, it suffices to show that HOD is almost universal. For that, it is enough to verify that $V_\alpha \cap HOD \in HOD$, for all α. For any α, the set $V_\alpha \cap HOD$ is a subset of HOD, and so it is sufficient to prove that $V_\alpha \cap HOD$ is ordinal-definable. This is indeed true because $V_\alpha \cap HOD$ is the set of all u satisfying the formula

$$u \in V_\alpha \wedge (\forall z \in \mathrm{TC}(\{u\}))\, \exists \beta\, [z \in \mathrm{cl}\{V_\gamma : \gamma < \beta\}]$$

and thus $V_\alpha \cap HOD \in OD$.

It remains to prove that HOD satisfies the Axiom of Choice. We shall show that for each α there exists a one-to-one function $g \in HOD$ of $V_\alpha \cap HOD$ into the ordinals. Since every such function is a subset of HOD, it suffices to find $g \in OD$.

By Lemma 13.25, there is a definable one-to-one mapping G of the class OD onto the ordinals. If we let g be the restriction of G to the ordinal-definable set $V_\alpha \cap HOD$, then g is ordinal-definable. □

A set X is *ordinal-definable from* A, $X \in OD[A]$, if there is a formula φ such that

$$X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, A)\} \tag{13.27}$$

for some ordinal numbers $\alpha_1, \ldots, \alpha_n$.

As above, this notion is expressible in the language of set theory:

$$OD[A] = \mathrm{cl}(\{V_\alpha : \alpha \in Ord\} \cup \{A\}). \tag{13.28}$$

The class $OD[A]$ has a well-ordering definable from A and thus every set in $OD[A]$ is of the form (13.27). Conversely (using the Reflection Principle), every set X in (13.27) belongs to $OD[A]$.

The proof of Theorem 13.26 generalizes easily to the case of $HOD[A]$. Thus $HOD[A]$, the class of all sets hereditarily ordinal-definable from A, is a transitive model of ZFC.

As a further generalization, we call X *ordinal-definable over* A, $X \in OD(A)$, if it belongs to the Gödel closure of $\{V_\alpha : \alpha \in Ord\} \cup \{A\} \cup A$. If $X \in OD(A)$, then $X \in \mathrm{cl}(\{V_\alpha : \alpha \in Ord\} \cup \{A\} \cup E)$, where $E = \{x_0, \ldots, x_k\}$ is a finite subset of A. Hence there is a finite sequence $s = \langle x_0, \ldots, x_k \rangle$ in A such that X is ordinal-definable from A and s. On the other hand, if s is a finite sequence in A, then obviously $s \in OD(A)$ and thus we have

$$OD(A) = \{X : X \in OD[A, s] \text{ for some finite sequence } s \text{ in } A\}.$$

In other words, $X \in OD(A)$ if and only if there is a formula φ such that

$$X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, A, \langle x_0, \ldots, x_k \rangle)\}$$

for some ordinal numbers $\alpha_1, \ldots, \alpha_n$ and a finite sequence $\langle x_0, \ldots, x_k \rangle$ in A.

The class $HOD(A)$ of all sets hereditarily ordinal-definable over A is a transitive model of ZF. To show that $HOD(A)$ is almost universal, it suffices to verify that $V_\alpha \cap HOD(A) \in OD(A)$. In fact, $V_\alpha \cap HOD(A)$ is ordinal-definable from A: It is the set

$$\{u \in V_\alpha : (\forall z \in \mathrm{TC}(\{u\}))\, z \in \mathrm{cl}(\{V_\beta : \beta \in Ord\} \cup \{A\} \cup A)\}.$$

More on Inner Models

We conclude this chapter with some comments on inner models of ZF.

As we remarked earlier, cardinal concepts are generally not absolute. The following theorem summarizes the relations between some of the concepts and their relativizations (see also Lemma 13.13):

Theorem 13.27. *Let M be an inner model of* ZF*. Then*

(i) $P^M(X) = P(X) \cap M$, $V_\alpha^M = V_\alpha \cap M$.
(ii) *If* $|X|^M = |Y|^M$ *then* $|X| = |Y|$.
(iii) *If α is a cardinal then α is a cardinal in M; if α is a limit cardinal, then α is a limit cardinal in M.*
(iv) $|\alpha| \le |\alpha|^M$, $\mathrm{cf}(\alpha) \le \mathrm{cf}^M(\alpha)$.
(v) *If α is a regular cardinal, then α is a regular cardinal in M; if α is weakly inaccessible, then α is weakly inaccessible in M.*
(vi) *If M is a model of* ZFC *and κ is inaccessible, then κ is inaccessible in M.* □

Concerning (vi), if $\alpha < \kappa$, then since $M \vDash$ AC, we must have either $(2^\alpha)^M < \kappa$ or $(2^\alpha)^M \ge \kappa$ and the latter is impossible since $2^\alpha < \kappa$.

If M is a transitive model of ZFC, then the Axiom of Choice in M enables us to code all sets in M by sets of ordinals and the model is determined by its sets or ordinals. The precise statement of this fact is: If M and N are two transitive models of ZFC with the same sets of ordinals, then $M = N$. In fact, a slightly stronger assertion is true. (On the other hand, one cannot prove that $M = N$ if neither model satisfies AC.)

Theorem 13.28. *Let M and N be transitive models of* ZF *and assume that the Axiom of Choice holds in M. If M and N have the same sets of ordinals, i.e., $P^M(Ord^M) = P^N(Ord^N)$, then $M = N$.*

Proof. We start with a rather trivial remark: M and N have the same sets of pairs of ordinals. To see this, use the absolute canonical one-to-one function $\Gamma : Ord \times Ord \to Ord$. If $X \subset Ord^2$ and $X \in M$, then $\Gamma(X)$ is both in M and in N, and we have $X = \Gamma_{-1}(\Gamma(X)) \in N$.

First we prove that $M \subset N$. Let $X \in M$. Since M satisfies AC, there is a one-to-one mapping $f \in M$ of some ordinal θ onto $\mathrm{TC}(\{X\})$. Let $E \in M$ be the following relation on θ:

$$\alpha \, E \, \beta \quad \text{if and only if} \quad f(\alpha) \in f(\beta).$$

E is a set of pairs of ordinals and thus we have $E \in N$. In M, E is well-founded and extensional. However, these properties are absolute and so E is well-founded and extensional in N. Applying the Collapsing Theorem (in N), we get a transitive set $T \in N$ such that $(T, \in)$ is isomorphic to (θ, E). Hence T is isomorphic to $\mathrm{TC}(\{X\})$ and since both are transitive, we have $T = \mathrm{TC}(\{X\})$. It follows that $\mathrm{TC}(\{X\}) \in N$ and so $X \in N$.

Now we prove $M = N$ by $\in$-induction. Let $X \in N$ and assume that $X \subset M$; we prove that $X \in M$. Let $Y \in M$ be such that $X \subset Y$ (for instance let $Y = V_\alpha^M$ where $\alpha = \mathrm{rank}(X)$; the rank function is absolute). Let $f \in M$ be a one-to-one function of Y into the ordinals. Since $M \subset N$, f is in N and so $f(X) \in N$. Since $M \subset N$, f is in N and so $f(X) \in N$. However, $f(X)$ is a set of ordinals and so $f(X) \in M$, and we have $X = f_{-1}(f(X)) \in M$. □

Exercises

13.1. If M is a transitive set then its closure under Gödel operations is transitive.

13.2. If M is closed under Gödel operations and extensional and if $X \in M$ is finite, then $X \subset M$. In particular, if $(x, y) \in M$, then $x \in M$ and $y \in M$.

13.3. If M is closed under Gödel operations and extensional, and π is the transitive collapse of M, then $\pi(G_i(X, Y)) = G_i(\pi X, \pi Y)$, $(i = 1, \ldots, 10)$ for all $X, Y \in M$.
[Use the Normal Form Theorem.]

13.4. The operations G_5 and G_8 are compositions of the remaining G_i.
$[G_8(X) = \mathrm{dom}(G_{10}(G_{10}(G_9(G_{10}(X \times X)))))$.]

13.5. The Axioms of Comprehension in the Bernays-Gödel set theory can be proved from a finite number of axioms of the form

$$\forall X \, \forall Y \, \exists Z \; Z = G(X, Y)$$

where the G's are operations analogous to $G_1, \ldots, G_{10}$. Thus the theory BG is finitely axiomatizable.
[Formulate and prove an analog of the Normal Form Theorem.]

13.6. Prove that for every transitive M, $V_\alpha^M = V_\alpha \cap M$ (for all $\alpha \in M$).

13.7. Show that "X is finite" is Δ_1.
[To get a Π_1 formulation, use T-finiteness from Chapter 1.]

13.8. The functions $\alpha + \beta$ and $\alpha \cdot \beta$ are Δ_1.

13.9. The canonical well-ordering of $Ord \times Ord$ is a Δ_0 relation. The function Γ is Δ_1.

13.10. The function $S \mapsto \mathrm{TC}(S)$ is Δ_1.

13.11. The function $x \mapsto \mathrm{rank}(x)$ is Δ_1.

13.12. "X is countable" is Σ_1.

13.13. $|X| \leq |Y|$, $|X| = |Y|$ are Σ_1.

13.14. The relation $\vDash_0$ is Σ_1; for each $n \geq 1$, $\vDash_n$ is Σ_n.

13.15. $M \prec_{\Sigma_0} V$ holds for every transitive set M.

13.16. Let n be a natural number. For every M_0 there exists a set $M \supset M_0$ such that $M \prec_{\Sigma_n} V$.

[Use the Reflection Principle.]

13.17. If $M \prec (L_{\omega_1}, \in)$, then $M = L_\alpha$ for some α.

[Show that M is transitive. Let $X \in M$. Let f be the $<$-least mapping of ω onto X. Since f is definable in $(L_{\omega_1}, \in)$ from X, f is in M. Hence $f(n) \in M$ for each n and we get $X \subset M$.]

13.18. If $M \prec (L_{\omega_2}, \in)$, then $\omega_1 \cap M = \alpha$ for some $\alpha \leq \omega_1$.

[Same argument as in Exercise 13.17: If $\gamma < \omega_1$ and $\gamma \in M$, then $\gamma \subset M$.]

13.19. For all $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$.

13.20. If $\alpha \geq \omega$ and X is a constructible subset of α, then $X \in L_\beta$, where β is the least cardinal in L greater than α.

13.21. The canonical well-ordering of L, restricted to the set $\boldsymbol{R}^L = \boldsymbol{R} \cap L$ of all constructible reals, has order-type ω_1^L.

[$\boldsymbol{R} \cap L \subset L_{\omega_1^L}$.]

13.22. If κ is a regular uncountable cardinal in L, then L_κ is a model of ZF^- (Zermelo-Fraenkel without the Power Set Axiom).

[Prove it in L. Replacement: (i) If $X \in L_\kappa$, then $|X| < \kappa$; (ii) if $Y \subset L_\kappa$ and $|Y| < \kappa$, then $Y \in L_\kappa$.]

13.23. If κ is inaccessible in L, then $L_\kappa = V_\kappa^L = V_\kappa \cap L$ and L_κ is a model of $\mathrm{ZFC} + (V = L)$.

13.24. If δ is a limit ordinal, then the model $(L_\delta, \in)$ has definable Skolem functions. Therefore, for every $X \subset L_\delta$, there exists a smallest $M \prec (L_\delta, \in)$ such that $X \subset M$.

[The well-ordering $<_\delta$ is definable in $(L_\delta, \in)$. Let $h_\varphi(x) =$ the $<_\delta$-least y such that $(L_\delta, \in) \vDash \varphi[x, y]$.]

13.25. If $\diamondsuit$ holds, then there exists a family $\mathcal{F}$ of stationary subsets of ω_1 such that $|\mathcal{F}| = 2^{\aleph_1}$ and $|S_1 \cap S_2| \leq \aleph_0$ whenever S_1 and S_2 are distinct elements of $\mathcal{F}$.

[Let $\mathcal{F} = \{S_X : X \subset \omega_1\}$, where $S_X = \{\alpha : X \cap \alpha = S_\alpha\}$.]

13.26. If $V = L[A]$ where $A \subset \omega_1$, then $2^{\aleph_0} = \aleph_1$. (Consequently, GCH holds.)

[Show that if $X \subset \omega$, then $X \in L_\alpha[A \cap \xi]$ for some $\alpha < \omega_1$ and $\xi < \omega_1$. It follows that $|P(\omega)| = \aleph_1$.]

13.27. For every X there is a set of ordinals A such that $L[X] = L[A]$.

[Let $\bar{X} = X \cap L[X]$, and let (θ, E) be isomorphic to $\mathrm{TC}(\{\bar{X}\})$ (in $L[X]$). Let $A = \Gamma(E)$ where Γ is the canonical mapping of Ord^2 onto Ord. Then $A \in L[X]$ and $X \in L[A]$, and hence $L[A] = L[X]$.]

13.28. Let $\alpha \geq \omega$ be a countable ordinal. There exists $A \subset \omega$ such that α is countable in $L[A]$.

[Let $W \subset \omega \times \omega$ be a well-ordering of ω of order-type α; let $A \subset \omega$ be such that $L[A] = L[W]$.]

13.29. If ω_1 (in V) is not a limit cardinal in L, then there exists $A \subset \omega$ such that $\omega_1 = \omega_1^{L[A]}$.

[There exists $\alpha < \omega_1$ such that in L, ω_1 is the successor of α. Let A be such that α is countable in $L[A]$.]

13.30 (ZFC). There exists $A \subset \omega_1$ such that $\omega_1 = \omega_1^{L[A]}$.

[For each $\alpha < \omega_1$, choose $A_\alpha \subset \omega$ such that α is countable in $L[A_\alpha]$. Let $A \subset \omega_1 \times \omega_1$ be such that $A_\alpha = \{\xi : (\alpha, \xi) \in A\}$ for all α; then $\omega_1^{L[A]} = \omega_1$.]

13.31 (ZFC). If ω_2 is not inaccessible in L, then there exists $A \subset \omega_1$ such that $\omega_1^{L[A]} = \omega_1$ and $\omega_2^{L[A]} = \omega_2$.

If A is a class, let us define $L[A]$ as in (13.22) where $\mathrm{def}_A(M)$ is defined as in (13.20).

13.32. $L[A] = L[\bar{A}]$, where $\bar{A} = A \cap L[A]$, and $L[A]$ is a model of ZFC. Moreover, $L[A]$ is the smallest inner model M such that $V_\alpha^M \cap A \in M$ for all α.

13.33. Assume that there exists a choice function F on V. Then there is a class $A \subset Ord$ such that $V = L[A]$.

13.34. Let M be a transitive model of ZF, $M \supset Ord$, and let X be a subset of M. Then there is a least model $M[X]$ of ZF such that $M \subset M[X]$ and $X \in M[X]$. If $M \vDash \mathrm{AC}$, then $M[x] \vDash \mathrm{AC}$.

[Modify the construction in (13.24).]

13.35. If $X \in OD$, then there exists γ such that X is a definable subset of $(V_\gamma, \in)$ (without parameters). Hence OD is the class of all X definable in some V_γ.

[If $X = \{u \in V_\beta : \varphi^{V_\beta}(u, \alpha)\}$, consider $\gamma = \Gamma(\alpha, \beta)$.]

13.36. If F is a definable function on Ord, then $\mathrm{ran}(F) \subset OD$. Thus: OD is the largest class for which there exists a definable one-to-one correspondence with the class of all ordinals.

13.37. HOD is the largest transitive model of ZF for which there exists a definable one-to-one correspondence with the class of all ordinals.

Historical Notes

The main results, namely consistency of the Axiom of Choice and the Generalized Continuum Hypothesis, are due to Kurt Gödel, as is the concept of constructible sets. The results were announced in [1938], and an outline of proof appeared in [1939]. Gödel's monograph [1940] contains a detailed construction of L, and the proof that L satisfies AC and GCH.

In [1939] Gödel defined constructible sets using $L_{\alpha+1}$ = the set of all subsets of L_α definable over L_α; in [1940] he used finitely many operations (and worked in the system BG).

The investigation of transitive models of set theory was of course motivated by Gödel's construction of the model L. The first systematic study of transitive models was done by Shepherdson in [1951, 1952, 1953]. Bernays in [1937], employed a finite number of operations on classes to give a finite axiomatization of BG. Theorem 13.9 is explicitly stated by Hajnal in [1956].

The Σ_n hierarchy was introduced by Lévy in [1965a]. Another result of Lévy [1965b] is that the truth predicate $\vDash_{n+1}$ is Σ_{n+1}

Karp's paper [1967] investigates Σ_1 relations and gives a detailed computation verifying that constructibility is Σ_1. The characterization of the sets L_α as transitive models of a single sentence σ is a result of Boolos [1970].

The Diamond Principle was introduced by Jensen in [1972].

Relative constructibility was investigated by Hajnal [1956], Shoenfield [1959] and most generally by Lévy [1957] and [1960a].

The concept of ordinal definability was suggested by Gödel in his talk in 1946, cf. [1965]; the theory was developed independently by Myhill and Scott in [1971] and by Vopěnka, Balcar, and Hájek in [1968].

Theorem 13.28 is due to Vopěnka and Balcar [1967].

14. Forcing

The method of forcing was introduced by Paul Cohen in his proof of independence of the Continuum Hypothesis and of the Axiom of Choice. Forcing proved to be a remarkably general technique for producing a large number of models and consistency results.

The main idea of forcing is to extend a transitive model M of set theory (the *ground model*) by adjoining a new set G (a *generic set*) in order to obtain a larger transitive model of set theory $M[G]$ called a *generic extension*. The generic set is approximated by *forcing conditions* in the ground model, and a judicious choice of forcing conditions determines what is true in the generic extension.

Cohen's original approach was to start with a countable transitive model M of ZFC (and a particular set of forcing conditions in M). A generic set can easily be proved to exist, and the main result was to show that $M[G]$ is a model of ZFC, and moreover, that the Continuum Hypothesis fails in $M[G]$.

A minor difficulty with this approach is that a countable transitive model need not exist. Its existence is unprovable, by Gödel's Second Incompleteness Theorem. The modern approach to forcing is to let the ground model be the universe V, and pretend that V has a generic extension, i.e., to postulate the existence of a generic set G, for the given set of forcing conditions. As the properties of the generic extension can be described entirely withing the ground model, statements about $V[G]$ can be understood as statements in the ground model using the language of forcing. We shall elaborate on this in due course.

Forcing Conditions and Generic Sets

Let M be a transitive model of ZFC, the *ground model*. In M, let us consider a nonempty partially ordered set $(P, <)$. We call $(P, <)$ a *notion of forcing* and the elements of P *forcing conditions*. We say that p is *stronger* than q if $p < q$. If p and q are conditions and there exists r such that both $r \leq p$ and $r \leq q$, then p and q are *compatible*; otherwise they are *incompatible*. A set $W \subset P$ is an *antichain* if its elements are pairwise incompatible. A set $D \subset P$ is *dense* in P if for every $p \in P$ there is $q \in D$ such that $q \leq p$.

Definition 14.1. A set $F \subset P$ is a *filter* on P if

(14.1) (i) F is nonempty;
(ii) if $p \leq q$ and $p \in F$, then $q \in F$;
(iii) if $p, q \in F$, then there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

A set of conditions $G \subset P$ is *generic* over M if

(14.2) (i) G is a filter on P;
(ii) if D is dense in P and $D \in M$, then $G \cap D \neq \emptyset$.

We also say that G is M-generic, or P-generic (over M), or just *generic*.

Note how genericity depends on the ground model M: What matters is which dense subsets of P are in M. Thus if $\mathcal{D}$ is any collection of sets, let us say that a set $G \subset P$ is a *$\mathcal{D}$-generic filter* on P if it is a filter and if $G \cap D \neq \emptyset$ for every dense subset of P that is in $\mathcal{D}$. Then G is generic over M just in case it is $\mathcal{D}$-generic where $\mathcal{D}$ is the collection of all $D \in M$ dense in P.

Genericity can be described in several equivalent ways. A set $D \subset P$ is *open dense* if it is dense and in addition, $p \in D$ and $q \leq p$ imply $q \in D$; D is *predense* if every $p \in P$ is compatible with some $q \in D$. If $p \in P$, then D is dense (open dense, predense, an antichain) *below* p if it is dense (open dense, predense, an antichain) in the set $\{q \in P : q \leq p\}$.

If D is either dense or a maximal antichain then D is predense. In Definition 14.1, "dense" in (14.2)(ii) can be replaced by "open dense," "predense," or "a maximal antichain"—see Exercises 14.3, 14.4, and 14.5.

Example 14.2. Let P be the following notion of forcing: The elements of P are finite 0–1 sequences $\langle p(0), \dots, p(n-1)\rangle$ and a condition p is stronger than q ($p < q$) if p extends q. Clearly, p and q are compatible if either $p \subset q$ or $q \subset p$. Let M be the ground model (note that $(P, <) \in M$), and let $G \subset P$ be generic over M. Let $f = \bigcup G$. Since G is a filter, f is a function. For every $n \in \omega$, the sets $D_n = \{p \in P : n \in \mathrm{dom}(p)\}$ is dense in P, hence it meets G, and so $\mathrm{dom}(f) = \omega$.

The 0–1 function f is the characteristic function of a set $A \subset \omega$. We claim that the function f (or the set A) is not in the ground model. For every 0–1 function g in M, let $D_g = \{p \in P : p \not\subset g\}$. The set D_g is dense, hence it meets G, and it follows that $f \neq g$. □

This example describes the simplest way of adjoining a new set of natural numbers to the ground model. A set $A \subset \omega$ obtained this way is called a *Cohen generic* real.

Except in trivial cases, a generic set does not belong to the ground model; see Exercise 14.6.

Example 14.3. In the ground model M, consider the following partially ordered set P. The elements of P are finite sequences $p = \langle \alpha_0, \dots, \alpha_{n-1}\rangle$ of

countable ordinals (in M), and a condition p is stronger than a condition q ($p < q$) if p extends q. Now if $G \subset P$ is generic over M, we let $f = \bigcup G$. As in Example 14.2, f is a function on ω, and since for every $\alpha < \omega_1^M$, the set $E_\alpha = \{p \in P : \alpha \in \mathrm{ran}(p)\}$ is dense, it follows that $\mathrm{ran}(f) = \omega_1^M$. Thus in any model $N \supset M$ that contains G, the ordinal ω_1^M is countable. □

This example describes the simplest way of *collapsing a cardinal*.

As these examples suggest, a generic set over a transitive model need not exist in general. However, if the ground model is countable, then generic sets do exist. If M is countable and $(P, <) \in M$, then the collection $\mathcal{D}$ of all $D \in M$ that are dense in P is countable and the following lemma applies:

Lemma 14.4. *If $(P, <)$ is a partially ordered set and $\mathcal{D}$ is a countable collection of dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P. In fact, for every $p \in P$ there exists a $\mathcal{D}$-generic filter G on P such that $p \in G$.*

Proof. Let $D_1, D_2, \ldots$ be the sets in $\mathcal{D}$. Let $p_0 = p$, and for each n, let p_n be such that $p_n \leq p_{n-1}$ and $p_n \in D_n$. The set

$$G = \{q \in P : q \geq p_n \text{ for some } n \in \boldsymbol{N}\}$$

is a $\mathcal{D}$-generic filter on P and $p \in G$. □

We shall now state the first of the three main theorems on generic models. We shall prove these theorems (14.5, 14.6, 14.7) later in this chapter.

Theorem 14.5 (The Generic Model Theorem). *Let M be a transitive model of* ZFC *and let $(P, <)$ be a notion of forcing in M. If $G \subset P$ is generic over P, then there exists a transitive model $M[G]$ such that*:

(i) *$M[G]$ is a model of* ZFC;
(ii) *$M \subset M[G]$ and $G \in M[G]$*;
(iii) *$Ord^{M[G]} = Ord^M$*;
(iv) *if N is a transitive model of* ZF *such that $M \subset N$ and $G \in N$, then $M[G] \subset N$.*

The model $M[G]$ is called a *generic extension* of M. The sets in $M[G]$ will be definable from G and finitely many elements of M. Each element of $M[G]$ will have a *name* in M describing how it has been constructed. An important feature of forcing is that the generic model $M[G]$ can be described within the ground model. Associated with the notion of forcing $(P, <)$ is a *forcing language*. This forcing language as well as the *forcing relation* $\Vdash$ are defined in the ground model M. The forcing language contains a *name* for every element of $M[G]$, including a constant $\dot{G}$, the name for a generic set (it is customary to denote names by dotted letters $\dot{a}$). Once we select a generic set G, then every constant of the forcing language is interpreted as an element of the model $M[G]$.

The forcing relation is a relation between the forcing conditions and sentences of the forcing language:

$$p \Vdash \sigma$$

(p *forces* σ). The forcing relation, which is defined in M, is a generalization of the notion of satisfaction. For instance, if $p \Vdash \sigma$ and if σ' is a logical consequence of σ, then $p \Vdash \sigma'$.

The second main theorem on generic models establishes the relation between forcing and truth in $M[G]$:

Theorem 14.6 (The Forcing Theorem). *Let $(P, <)$ be a notion of forcing in the ground model M. If σ is a sentence of the forcing language, then for every $G \subset P$ generic over M,*

$$M[G] \vDash \sigma \quad \textit{if and only if} \quad (\exists p \in G)\, p \Vdash \sigma. \tag{14.3}$$

[In the left-hand-side σ one interprets the constants of the forcing language according to G.]

The third main theorem lists the most important properties of the forcing relation.

Theorem 14.7 (Properties of Forcing). *Let $(P, <)$ be a notion of forcing in the ground model M, and let M^P be the class (in M) of all names.*

(i) (a) *If p forces φ and $q \leq p$, then $q \Vdash \varphi$.*
(b) *No p forces both φ and $\neg\varphi$.*
(c) *For every p there is a $q \leq p$ such that q* decides *φ, i.e., either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.*

(ii) (a) *$p \Vdash \neg\varphi$ if and only if no $q \leq p$ forces φ.*
(b) *$p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$.*
$p \Vdash \forall x\, \varphi$ if and only if $p \Vdash \varphi(\dot{a})$ for every $\dot{a} \in M^P$.
(c) *$p \Vdash \varphi \vee \psi$ if and only if $\forall q \leq p\, \exists r \leq q\, (r \Vdash \varphi$ or $r \Vdash \psi)$.*
$p \Vdash \exists x\, \varphi$ if and only if $\forall q \leq p\, \exists r \leq q\, \exists \dot{a} \in M^P\, r \Vdash \varphi(\dot{a})$.

(iii) *If $p \Vdash \exists x\, \varphi$ then for some $\dot{a} \in M^P$, $p \Vdash \varphi(\dot{a})$.*

Separative Quotients and Complete Boolean Algebras

While the forcing relation can be defined directly from the partial ordering $(P, <)$, it turns out that its properties, and the properties of the generic extension are determined by a certain complete Boolean algebra that can be associated with $(P, <)$. We shall therefore introduce the Boolean algebra $B(P)$ and then use it to define the class M^P (the P-names) and the forcing relation $\Vdash$.

Definition 14.8. A partially ordered set $(P, <)$ is *separative* if for all $p, q \in P$,

$$\text{if } p \not\leq q \text{ then there exists an } r \leq p \text{ that is incompatible with } q. \tag{14.4}$$

The forcing notions in Examples 14.2 and 14.3 are separative. On the other hand, a linear ordering is not separative (if it has more than one element). Another example of a nonseparative partial order is the set of all infinite subsets of ω, ordered by inclusion.

If B is a Boolean algebra, then $(B^+, <)$ is a separative partial order. A more general statement is true. A set $D \subset P$ is *dense* in a partially ordered set $(P, <)$ if for every $p \in P$ there is a $d \in D$ such that $d \leq p$. A set $D \subset B^+$ is *dense* in a Boolean algebra B if it is dense in $(B^+, <)$. The following lemma is easy to verify:

Lemma 14.9. *If D is a dense subset of a Boolean algebra B, then $(D, <)$ is a separative partial order.* □

Conversely, every separative partial order can be embedded densely in a complete Boolean algebra:

Theorem 14.10. *Let $(P, <)$ be a separative partially ordered set. Then there is a complete algebra B such that*:

(i) *$P \subset B^+$ and $<$ agrees with the partial ordering of B.*
(ii) *P is dense in B.*

The algebra B is unique up to isomorphism.

Proof. The proof is exactly the same as the proof of Theorem 7.13. B is the set of all regular cuts in P and separativity implies that every U_p (where $p \in P$) is regular. □

When $(P, <)$ is not separative, we can replace it by a separative partial order that will produce the same generic extension. This is the consequence of the following lemma:

Lemma 14.11. *Let $(P, <)$ be a partially ordered set. There exists a separative partially ordered set $(Q, \prec)$ and a mapping h of P onto Q such that*

(14.5) (i) *$x \leq y$ implies $h(x) \preccurlyeq h(y)$;*
(ii) *x and y are compatible in P if and only if $h(x)$ and $h(y)$ are compatible in Q.*

Proof. Let us define the following equivalence relation on P:

$x \sim y$ if and only if $\forall z$ (z is compatible with x $\leftrightarrow$ z is compatible with y).

Let $Q = P/\sim$ and let us define

$$[x] \preccurlyeq [y] \leftrightarrow (\forall z \leq x)[z \text{ and } y \text{ are compatible}].$$

The relation $\preccurlyeq$ on Q is a partial ordering, and it is easy to verify that $(Q, \prec)$ is separative. The mapping $h(x) = [x]$ satisfies (14.5). □

The partial order $(Q, \prec)$ is called the *separative quotient* of $(P, <)$ and is unique (up to isomorphism); see Exercise 14.9.

Corollary 14.12. *For every partially ordered set $(P, <)$ there is a complete Boolean algebra $B = B(P)$ and a mapping $e : P \to B^+$ such that*:

(14.6) (i) *if $p \leq q$ then $e(p) \leq e(q)$;*
(ii) *p and q are compatible if and only if $e(p) \cdot e(q) \neq 0$;*
(iii) *$\{e(p) : p \in P\}$ is dense in B.*

B is unique up to isomorphism. □

Our earlier statements about the generic extension being determined by $B(P)$ are based on the following facts:

Lemma 14.13. (i) *In the ground model M, let Q be the separative quotient of P and let h map P onto Q such that* (14.5) *holds. If $G \subset P$ is generic over M then $h(G) \subset Q$ is generic over M. Conversely, if $H \subset Q$ is generic over M then $h_{-1}(H) \subset P$ is generic over M.*

(ii) *In the ground model M, let P be a dense subset of a partially ordered set Q. If $G \subset Q$ is generic over M then $G \cap P \subset P$ is generic over M. Conversely, if $H \subset P$ is generic over M then $G = \{q \in Q : (\exists p \in G)\, p \leq q\}$ is generic over M.*

Proof. The proof is an exercise in verifying definitions (Exercise 14.1 is useful here). □

As a consequence, if $e : P \to B(P)$ is as in Corollary 14.12 then $G \subset P$ and $H = \{u \in B : \exists p \in G\, e(p) \leq u\}$ are definable from each other, and G is generic if and only if H is, and $M[G] = M[H]$. Thus P and $B(P)$ produce the same generic extension.

In the ground model M, let B be a complete Boolean algebra. Outside M, B is still a Boolean algebra, though not necessarily complete. An ultrafilter G on B is called *generic* (over M) if

(14.7) $\qquad \prod X \in G$ whenever $X \in M$ and $X \subset G$.

A routine verification (see Exercise 14.10) shows that G is a generic ultrafilter if and only if G is a generic filter on B^+.

Boolean-Valued Models

Let B be a complete Boolean algebra. A *Boolean-valued model* (of the language of set theory) $\mathfrak{A}$ consists of a *Boolean universe* A and functions of two variables with values in B,

(14.8) $\qquad \|x = y\|, \qquad \|x \in y\|$

(the *Boolean values* of $=$ and $\in$), that satisfy the following:

(14.9)
(i) $\|x = x\| = 1$,
(ii) $\|x = y\| = \|y = x\|$,
(iii) $\|x = y\| \cdot \|y = z\| \leq \|x = z\|$,
(iv) $\|x \in y\| \cdot \|v = x\| \cdot \|w = y\| \leq \|v \in w\|$.

For every formula $\varphi(x_1, \ldots, x_n)$, we define the *Boolean value* of φ

$$\|\varphi(a_1, \ldots, a_n)\| \qquad (a_1, \ldots, a_n \in A)$$

as follows:

(a) For atomic formulas, we have (14.8).

(b) If φ is a negation, conjunction, etc.,

$$\begin{aligned}
\|\neg\psi(a_1, \ldots, a_n)\| &= -\|\psi(a_1, \ldots, a_n)\|, \\
\|(\psi \wedge \chi)(a_1, \ldots, a_n)\| &= \|\psi(a_1, \ldots, a_n)\| \cdot \|\chi(a_1, \ldots, a_n)\|, \\
\|(\psi \vee \chi)(a_1, \ldots, a_n)\| &= \|\psi(a_1, \ldots, a_n)\| + \|\chi(a_1, \ldots, a_n)\|, \\
\|(\psi \to \chi)(a_1, \ldots, a_n)\| &= \|(\neg\psi \vee \chi)(a_1, \ldots, a_n)\|, \\
\|(\psi \leftrightarrow \chi)(a_1, \ldots, a_n)\| &= \|((\psi \to \chi) \wedge (\chi \to \psi))(a_1, \ldots, a_n)\|.
\end{aligned}$$

(c) If φ is $\exists x\, \psi$ or $\forall x\, \psi$,

$$\begin{aligned}
\|\exists x\, \psi(x, a_1, \ldots, a_n)\| &= \sum_{a \in A} \|\psi(a, a_1, \ldots, a_n)\|, \\
\|\forall x\, \psi(x, a_1, \ldots, a_n)\| &= \prod_{a \in A} \|\psi(a, a_1, \ldots, a_n)\|.
\end{aligned}$$

Note how the notion of a Boolean-valued model generalizes the notion of a model; the Boolean value of φ is a generalization of the satisfaction predicate $\vDash$. If B is the trivial algebra $\{0, 1\}$, then a Boolean-valued model is just a (two-valued) model; i.e., consider $A/\equiv$ where $x \equiv y$ if and only if $\|x = y\| = 1$.

We say that $\varphi(a_1, \ldots, a_n)$ is *valid in* $\mathfrak{A}$, if $\|\varphi(a_1, \ldots, a_n)\| = 1$. An implication $\varphi \to \psi$ is valid if $\|\varphi\| \leq \|\psi\|$. Hence it is postulated in (14.9) that the axioms for the equality predicate $=$ are valid in a Boolean-valued model. It can be easily verified that all the other axioms of predicate calculus are valid, and that the rules of inference applied to valid sentences result in valid sentences. Thus every sentence provable in predicate calculus has Boolean value 1, and if two formulas φ, ψ are provably equivalent, we have $\|\varphi\| = \|\psi\|$. For example, we have

$$\|x = y\| \cdot \|\varphi(x)\| \leq \|\varphi(y)\|.$$

Boolean-valued models can therefore be used in consistency proofs in much the same way as two-valued models. Let $\mathfrak{A}$ be a Boolean-valued model such

that all the axioms of ZFC are valid in $\mathfrak{A}$. (We say that $\mathfrak{A}$ is a Boolean-valued model of ZFC.) Let σ be a set-theoretical statement and assume that $\|\sigma\| \neq 0$. Then we can conclude that σ is consistent relative to ZFC; otherwise, $\neg\sigma$ would be provable in ZFC and therefore valid in $\mathfrak{A}$: $\|\neg\sigma\| = -\|\sigma\| = 1$.

There is an important special case of Boolean-valued models, and in this special case, the Boolean-valued model can be transformed into a two-valued model.

We say that a Boolean-valued model $\mathfrak{A}$ is *full* if for any formula $\varphi(x, x_1, \ldots, x_n)$ the following holds: For all $a_1, \ldots, a_n \in A$, there exists an $a \in A$ such that

$$\|\varphi(a, a_1, \ldots, a_n)\| = \|\exists x\, \varphi(x, a_1, \ldots, a_n)\|. \tag{14.10}$$

Let F be an ultrafilter on B. We define an equivalence relation on A by

$$x \equiv y \quad \text{if and only if} \quad \|x = y\| \in F, \tag{14.11}$$

and a binary relation E on $A/\!\equiv$ by

$$[x]\ E\ [y] \quad \text{if and only if} \quad \|x \in y\| \in F. \tag{14.12}$$

That $\equiv$ is an equivalence relation, and that (14.12) does not depend on the choice of representatives are easy consequences of (14.9) and the fact that F is a filter. Thus $\mathfrak{A}/F = (A/\!\equiv, E)$ is a model. Moreover, we have the following relationship between the Boolean-valued model $\mathfrak{A}$ and the model $\mathfrak{A}/F$:

Lemma 14.14. *Let $\mathfrak{A}$ be full. For any formula $\varphi(x_1, \ldots, x_n)$,*

$$\mathfrak{A}/F \vDash \varphi([a_1], \ldots, [a_n]) \quad \textit{if and only if} \quad \|\varphi(a_1, \ldots, a_n)\| \in F, \tag{14.13}$$

for all $a_1, \ldots, a_n \in A$.

Proof. (a) If φ is atomic, then (14.13) is true by definition.

(b) If φ is a negation, conjunction, etc., we use the basic properties of an ultrafilter, and the definition of $\|\ \|$; e.g., we use

$$\begin{aligned} \|\neg\psi\| \in F \quad &\text{if and only if} \quad \|\psi\| \notin F, \\ \|\psi \wedge \chi\| \in F \quad &\text{if and only if} \quad \|\psi\| \in F \text{ and } \|\chi\| \in F. \end{aligned}$$

(c) If φ is $\exists x\, \psi(x, \ldots)$, we use the fullness of $\mathfrak{A}$ to prove (14.13), assuming it holds for ψ. By (14.10), we pick some $a \in A$ such that $\|\varphi(a, \ldots)\| = \|\exists x\, \varphi(x, \ldots)\|$ and then we have

$$\|\exists x\, \varphi(x, \ldots)\| \in F \quad \text{if and only if} \quad (\exists a \in A)\, \|\varphi(a, \ldots)\| \in F,$$

which enables us to do the induction step in this case. □

The Boolean-Valued Model V^B

We now define the Boolean-valued model V^B. Let B be a complete Boolean algebra.

Our intention is to define a Boolean-valued model in which all the axioms of ZFC are valid. In particular, we want V^B to be *extensional*, i.e., the Axiom of Extensionality to be valid in V^B:

$$\|\forall u\,(u \in X \leftrightarrow u \in Y)\| \le \|X = Y\|. \tag{14.14}$$

We shall define V^B as a generalization of V: Instead of (two-valued) sets, we consider "Boolean-valued" sets, i.e., functions that assign Boolean values to its "elements." Thus we define V^B as follows:

(14.15)
(i) $V_0^B = \emptyset$,
(ii) $V_{\alpha+1}^B =$ the set of all functions x with $\mathrm{dom}(x) \subset V_\alpha^B$ and values in B,
$V_\alpha^B = \bigcup_{\beta<\alpha} V_\beta^B$ if α is a limit ordinal, and
(iii) $V^B = \bigcup_{\alpha \in Ord} V_\alpha^B$.

The definition of $\|x \in y\|$ and $\|x = y\|$ is motivated by (14.14), and the requirement that $x(t) \le \|t \in x\|$. We define Boolean values by induction. Each $x \in V^B$ is assigned the rank in V^B,

$$\rho(x) = \text{the least } \alpha \text{ such that } x \in V_{\alpha+1}^B.$$

The forthcoming definition is by induction on pairs $(\rho(x), \rho(y))$, under the canonical well-ordering.

To make the notation more suggestive, we introduce the following Boolean operation that corresponds to the implication:

$$u \Rightarrow v = -u + v$$

Let

(14.16)
(i) $\|x \in y\| = \sum_{t \in \mathrm{dom}\, y} (\|x = t\| \cdot y(t))$,
(ii) $\|x \subset y\| = \prod_{t \in \mathrm{dom}\, x} (x(t) \Rightarrow \|t \in y\|)$, and
(iii) $\|x = y\| = \|x \subset y\| \cdot \|y \subset x\|$.

We are going to show that V^B is a Boolean-valued model. To do that, we have to verify (14.9). Clause (ii) in (14.9) is trivially satisfied since the definition of $\|x = y\|$ is symmetric in x and y.

Lemma 14.15. $\|x = x\| = 1$ *for all* $x \in V^B$.

Proof. By induction on $\rho(x)$. Clearly, it suffices to show that $\|x \subset x\| = 1$, i.e., we wish to show that $x(t) \Rightarrow \|t \in x\| = 1$ for all $t \in \mathrm{dom}(x)$, or equivalently, that $x(t) \leq \|t \in x\|$. If $t \in \mathrm{dom}(x)$, then by the induction hypothesis we have $\|t = t\| = 1$ and hence, by definition of $\|t \in x\|$, $x(t) = \|t = t\| \cdot x(t) \leq \|t \in x\|$. □

Now we prove (14.9)(iii) and (iv), simultaneously by induction:

Lemma 14.16. *For all* $x, y, z \in V^B$,

(i) $\|x = y\| \cdot \|y = z\| \leq \|x = z\|$,
(ii) $\|x \in y\| \cdot \|x = z\| \leq \|z \in y\|$,
(iii) $\|y \in x\| \cdot \|x = z\| \leq \|y \in z\|$.

Proof. By induction on triples $\{\rho(x), \rho(y), \rho(z)\}$.

(i) It suffices to prove that $\|x \subset y\| \cdot \|y = z\| \leq \|x \subset z\|$. Let $t \in \mathrm{dom}(x)$ be arbitrary; we wish to show that

$$\|y = z\| \cdot (x(t) \Rightarrow \|t \in y\|) \leq x(t) \Rightarrow \|t \in z\| \tag{14.17}$$

(using the definition of $\|x \subset z\|$). By the induction hypothesis, we have $\|t \in y\| \cdot \|y = z\| \leq \|t \in z\|$. Thus $\|y = z\| \cdot (-x(t) + \|t \in y\|) = (\|y = z\| - x(t)) + (\|y = z\| \cdot \|t \in y\| \leq -x(t) + \|t \in z\|$, and (14.17) follows.

(ii) Let $t \in \mathrm{dom}(y)$ be arbitrary. By the induction hypothesis we have $\|x = z\| \cdot \|x = t\| \leq \|z = t\|$ and so

$$\|x = z\| \cdot \|x = t\| \cdot y(t) \leq \|z = t\| \cdot y(t). \tag{14.18}$$

Taking the sum of (14.18) over all $t \in \mathrm{dom}(y)$, we get

$$\|x = z\| \cdot \sum_{t \in \mathrm{dom}\, y} (\|x = t\| \cdot y(t)) \leq \sum_{t \in \mathrm{dom}\, y} (\|z = t\| \cdot y(t)),$$

that is, $\|x = z\| \cdot \|x \in y\| \leq \|z \in y\|$.

(iii) Let $t \in \mathrm{dom}(x)$. By the definition of $\|x = z\|$ we have $x(t) \cdot \|x = z\| \leq \|t \in z\|$ and so

$$\|y = t\| \cdot x(t) \cdot \|x = z\| \leq \|y = t\| \cdot \|t \in z\|.$$

By the induction hypothesis, $\|y = t\| \cdot \|t \in z\| \leq \|y \in z\|$, and therefore

$$\|y = t\| \cdot x(t) \cdot \|x = z\| \leq \|y \in z\|. \tag{14.19}$$

Taking the sum of the left-hand side of (14.19) over all $t \in \mathrm{dom}(x)$, we get

$$\sum_{t \in \mathrm{dom}\, x} (\|y = t\| \cdot x(t)) \cdot \|x = z\| \leq \|y \in z\|,$$

that is, $\|y \in x\| \cdot \|x = z\| \leq \|y \in z\|$. □

Thus V^B is a Boolean-valued model. We will show that all axioms of ZFC are valid in V^B. First we show that V^B is extensional, and full.

Lemma 14.17. *V^B is extensional.* □

Proof. Let $X, Y \in V^B$. By the definition of $a \Rightarrow b$ we observe that if $a \leq a'$, then $(a' \Rightarrow b) \leq (a \Rightarrow b)$. Thus for any $u \in V^B$ we have $(\|u \in X\| \Rightarrow \|u \in Y\|) \leq (X(u) \Rightarrow \|u \in Y\|)$ and therefore

$$\prod_{u \in V^B} (\|u \in X\| \Rightarrow \|u \in Y\|) \leq \prod_{u \in V^B} (X(u) \Rightarrow \|u \in Y\|). \tag{14.20}$$

While the left-hand side of (14.20) is equal to $\|\forall u\,(u \in X \to u \in Y)\|$, the right-hand side is easily seen to equal $\|X \subset Y\|$. Consequently,

$$\|\forall u\,(u \in X \leftrightarrow u \in Y)\| \leq \|X = Y\|. \qquad \square$$

Lemma 14.18. *If W is a set of pairwise disjoint elements of B and if a_u, $u \in W$, are elements of V^B, then there exists some $a \in V^B$ such that $u \leq \|a = a_u\|$ for all $u \in W$.*

Proof. Let $D = \bigcup_{u \in W} \operatorname{dom}(a_u)$, and for every $t \in D$, let $a(t) = \sum\{u \cdot a_u(t) : u \in W\}$. Since the u's are pairwise disjoint, we have $u \cdot a(t) = u \cdot a_u(t)$ for each $u \in W$ and each $t \in D$. In other words, $u \leq (a(t) \Rightarrow a_u(t))$ and $u \leq (a_u(t) \Rightarrow a(t))$, and so $u \leq \|a = a_u\|$. □

Lemma 14.19. *V^B is full. Given a formula $\varphi(x, \ldots)$, there exists some $a \in V^B$ such that* (14.10) *holds, i.e.,*

$$\|\varphi(a, \ldots)\| = \|\exists x\, \varphi(x, \ldots)\|.$$

Proof. In (14.10), $\leq$ holds for every a. We wish to find an $a \in V^B$ such that $\geq$ holds. Let $u_0 = \|\exists x\, \varphi(x, \ldots)\|$. Let

$$D = \{u \in B : \text{there is some } a_u \text{ such that } u \leq \|\varphi(a_u, \ldots)\|\}.$$

It is clear that D is open and dense below u_0. Let W be a maximal set of pairwise disjoint elements of D; clearly, $\sum\{u : u \in W\} \geq u_0$. By Lemma 14.18 there exists some $a \in V^B$ such that $u \leq \|a = a_u\|$ for all $u \in W$. Thus for each $u \in W$ we have $u \leq \|\varphi(a, \ldots)\|$, and hence $u_0 \leq \|\varphi(a, \ldots)\|$. □

We remark that Lemma 14.19 was the only place in this chapter where we used the Axiom of Choice.

Every set (in V) has a canonical name in the Boolean-valued model V^B:

Definition 14.20 (By $\in$-Induction).

(i) $\check{\emptyset} = \emptyset$;

(ii) for every $x \in V$, let $\check{x} \in V^B$ be the function whose domain is the set $\{\check{y} : y \in x\}$, and for all $y \in x$, $\check{x}(\check{y}) = 1$.

When calculating the Boolean value of a formula, one may find the following observation helpful (cf. Exercise 14.12):

$$\begin{aligned} \|(\exists y \in x)\, \varphi(y)\| &= \sum_{y \in \operatorname{dom} x} (x(y) \cdot \|\varphi(y)\|), \\ \|(\forall y \in x)\, \varphi(y)\| &= \prod_{y \in \operatorname{dom} x} (x(y) \Rightarrow \|\varphi(y)\|). \end{aligned} \tag{14.21}$$

The following lemma is the Boolean-valued version of absoluteness of Δ_0 formulas:

Lemma 14.21. *If $\varphi(x_1, \ldots, x_n)$ is a Δ_0 formula, then*

$$\varphi(x_1, \ldots, x_n) \quad \textit{if and only if} \quad \|\varphi(\check{x}_1, \ldots, \check{x}_n)\| = 1.$$

Proof. By induction on the complexity of φ. □

Corollary 14.22. *If φ is Σ_1, then $\varphi(x, \ldots)$ implies $\|\varphi(\check{x}, \ldots)\| = 1$.* □

The next lemma states that V and V^B "have the same ordinals:"

Lemma 14.23. *For every $x \in V^B$,*

$$\|x \text{ is an ordinal}\| = \sum_{\alpha \in Ord} \|x = \check{\alpha}\|.$$

Proof. Since "x is an ordinal" is Δ_0, we have, by Lemma 14.21,

$$\sum_{\alpha \in Ord} \|x = \check{\alpha}\| \le \|x \text{ is an ordinal}\|.$$

On the other hand, let $\|x \text{ is an ordinal}\| = u$. We first observe that if γ is an ordinal, then

$$\|x \text{ is an ordinal and } x \in \check{\gamma}\| \le \sum_{\alpha \in \gamma} \|x = \check{\alpha}\|.$$

Also, for every α, we have

$$u \le \|x \in \check{\alpha}\| + \|x = \check{\alpha}\| + \|\check{\alpha} \in x\|.$$

However, there is only a set of α's such that $\|\check{\alpha} \in x\| \neq 0$ (because $\|\check{\alpha} \in x\| = \sum_{t \in \operatorname{dom} x} (\|\check{\alpha} = t\| \cdot x(t))$). Hence there is γ such that $u \le \|x \subset \check{\gamma}\|$ and we have $u \le \sum_{\alpha \le \gamma} \|x = \check{\alpha}\|$. □

We show now that V^B is a Boolean-valued model of ZFC.

Theorem 14.24. *Every axiom of* ZFC *is valid in V^B.*

Proof. We show that $\|\sigma\| = 1$ for every axiom of ZFC.

Extensionality. See Lemma 14.17.

Pairing. Given $a, b \in V^B$, let $c = \{a, b\}^B \in V^B$ be such that $\mathrm{dom}(c) = \{a, b\}$ and $c(a) = c(b) = 1$. Then $\|a \in c \wedge b \in c\| = 1$. This, combined with Separation, suffices for the Pairing Axiom. (We could also verify directly that $\|\forall x \in c\,(x = a \vee x = b)\| = 1$.)

Separation. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|Y \subset X\| = 1 \quad \text{and} \quad \|(\forall z \in X)(\varphi(z) \leftrightarrow z \in Y)\| = 1. \tag{14.22}$$

Let $Y \in V^B$ be as follows:

$$\mathrm{dom}(Y) = \mathrm{dom}(X), \qquad Y(t) = X(t) \cdot \|\varphi(t)\|.$$

For every $x \in V^B$ we have $\|x \in Y\| = \|x \in X\| \cdot \|\varphi(x)\|$ and this gives (14.22).

Union. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|(\forall u \in X)(\forall v \in u)(v \in Y)\| = 1 \tag{14.23}$$

(this is the weak version, cf. (1.8)).

If $X \in V^B$, then letting $Y \in V^B$ as follows verifies (14.23):

$$\mathrm{dom}(Y) = \bigcup\{\mathrm{dom}(u) : u \in \mathrm{dom}(X)\}, \qquad Y(t) = 1 \quad \text{for all } t \in \mathrm{dom}(Y).$$

Power Set. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|\forall u\,(u \subset X \to u \in Y)\| = 1; \tag{14.24}$$

(cf. (1.9)). Here we let

$$\mathrm{dom}(Y) = \{u \in V^B : \mathrm{dom}(u) = \mathrm{dom}(X) \text{ and } u(t) \le X(t) \text{ for all } t\},$$
$$Y(u) = 1 \quad \text{for all } u \in \mathrm{dom}(Y).$$

To verify that Y satisfies (14.24) we use the following observation: If $u \in V^B$ is arbitrary, let $u' \in V^B$ be such that $\mathrm{dom}(u') = \mathrm{dom}(X)$ and $u'(t) = X(t) \cdot \|t \in u\|$ for all $t \in \mathrm{dom}(X)$. Then

$$\|u \subset X\| \le \|u = u'\|$$

which makes it possible to include in $\mathrm{dom}(Y)$ only the "representative" u's.

Infinity. See Lemma 14.21 for $\|\check{\omega} \text{ is an inductive set}\| = 1$.

Replacement. It suffices to verify the Collection Principle, cf. (6.5); we prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|(\forall u \in X)(\exists v\, \varphi(u, v) \to (\exists v \in Y)\, \varphi(u, v))\| = 1. \tag{14.25}$$

Here we let

$$\mathrm{dom}(Y) = \bigcup\{S_u : u \in \mathrm{dom}(X)\}, \qquad Y(t) = 1 \quad \text{for all } t \in \mathrm{dom}(Y),$$

where $S_u \subset V^B$ is some set such that

$$\sum_{v \in V^B} \|\varphi(u, v)\| = \sum_{v \in S_u} \|\varphi(u, v)\|.$$

Regularity. We prove that for every $X \in V^B$,

$$\|X \text{ is nonempty} \to (\exists y \in X)(\forall z \in y)\, z \notin X\| = 1. \tag{14.26}$$

If (14.26) is false, then

$$\|\exists u\,(u \in X) \wedge (\forall y \in X)(\exists z \in y)\, z \in X\| = b \neq 0.$$

Let $y \in V^B$ be of least $\rho(y)$ such that $\|y \in X\| \cdot b \neq 0$. Then $\|y \in X\| \cdot b \leq \|(\exists z \in y)\, z \in X\|$, so there exists a $z \in \mathrm{dom}(y)$ such that $\|z \in X\| \cdot \|y \in X\| \cdot b \neq 0$. Since $\rho(z) < \rho(y)$, this is a contradiction.

Choice. For every S, we have (by Corollary 14.22)

$$\|\check{S} \text{ can be well-ordered}\| = 1.$$

Now, we prove that for every $X \in V^B$ there exist some S and $f \in V^B$ such that

$$\|f \text{ is a function on } \check{S} \text{ and } \mathrm{ran}(f) \supset X\| = 1. \tag{14.27}$$

(This shows that $\|X$ can be well-ordered$\| = 1$.)

We let $S = \mathrm{dom}(X)$ and $f \in V^B$ as follows:

$$\mathrm{dom}(f) = \{(\check{x}, x)^B : x \in S\}, \qquad f(t) = 1 \quad \text{for all } t \in \mathrm{dom}(f)$$

(where $(a,b)^B = \{\{a\}^B, \{a,b\}^B\}^B$). These S and f satisfy (14.27). □

Among elements of V^B, one is of particular significance: the canonical name for a generic ultrafilter on B:

Definition 14.25. The *canonical name* $\dot{G}$ for a generic ultrafilter is the Boolean-valued function defined by

$$\mathrm{dom}(\dot{G}) = \{\check{u} : u \in B\}, \qquad \dot{G}(\check{u}) = u \quad \text{for every } u \in B.$$

See Exercise 14.14.

The Forcing Relation

Let M be a transitive model of ZFC (the *ground model*) and let $(P, <) \in M$ be a notion of forcing. We shall now introduce the forcing language by specifying *names*, define the forcing relation $\Vdash$ and prove the fundamental properties of $\Vdash$ (Theorem 14.7). Throughout this section we work inside the ground model.

Let $(P, <)$ be a notion of forcing. By Corollary 14.12 there exists a complete Boolean algebra $B = B(P)$ such that P embeds in B by a mapping $e : P \to B$ that satisfies (14.6) (and is not one-to-one if P is not separative). We use M^B to denote the B-valued model defined in (14.15) (inside M).

Definition 14.26. $M^P = M^{B(P)}$. The elements of M^P are called *P-names* (or just *names*). P-names are usually denoted by dotted letters. The *forcing language* is the language of set theory with names added as constants. The forcing relation $\Vdash_P$ (or just $\Vdash$) is defined by

$$p \Vdash \varphi(\dot a_1, \dots, \dot a_n) \quad \text{if and only if} \quad e(p) \le \|\varphi(\dot a_1, \dots, \dot a_n)\|$$

where φ is a formula of set theory and $\dot a_1, \dots, \dot a_n$ are names.

We remark that both names and the forcing relation can be defined directly from P without using the complete Boolean algebra. However, we find the direct definition somewhat less intuitive.

Proof of Theorem 14.7. (i) (a) If $q \le p$ then $e(q) \le e(p)$.
(b) $\|\varphi\| \cdot \|\neg\varphi\| = 0$.
(c) If $e(p) \cdot \|\varphi\| \ne 0$ then there is a $q \le p$ such that $e(q) \le \|\varphi\|$; similarly if $e(p) \cdot \|\neg\varphi\| \ne 0$.
(ii) (a) Left-to-right: Use (i)(a) and (b). Right-to-left: If p does not force $\neg\varphi$ then $e(p) \cdot \|\varphi\| \ne 0$ and proceed as in (i)(c).
(b) By (14.9)(b) and (c).
(c) For disjunction, we use $\|\varphi \vee \psi\| = \|\varphi\| + \|\psi\|$ and argue as in (ii)(a). The existential quantifier is similar, using (14.9)(c).
(iii) By Lemma 14.19, M^B is full and so $e(p) \le \|\varphi(\dot a)\|$ for some $\dot a$. □

Among P-names there are canonical names $\check x$ for sets in the ground model. In practice one often abuses the notation by dropping the háček $\check{}$ and confusing $x \in M$ with its name $\check x$.

We can also introduce a "name for M;" since $a \in M \leftrightarrow (\exists x \in M)\, a = x$, we define

$$p \Vdash \dot a \in \check M \quad \text{if and only if} \quad \forall q \le p\, \exists r \le q\, \exists x\, (r \Vdash \dot a = \check x). \tag{14.28}$$

Finally, we consider the canonical name for a generic filter on P. Using Definition 14.25 for $B(P)$ and the relation between generic filters on P and generic ultrafilters on $B(P)$ spelled out in Lemma 14.13, we arrive at the following definition:

$$p \Vdash q \in \dot G \quad \text{if and only if} \quad \forall r \le p\, \exists s \le r\ s \le q, \tag{14.29}$$

or in terms of the separative quotient mapping h (Lemma 14.11),

$$p \Vdash q \in \dot G \quad \text{if and only if} \quad h(p) \preccurlyeq h(q).$$

One final remark: By Theorem 14.24, every axiom of ZFC is forced by every condition. So is every axiom of predicate calculus, and the forcing relation is preserved by the rules of inference. Hence every condition forces every sentence provable in ZFC.

The Forcing Theorem and the Generic Model Theorem

We shall now define the generic extension $M[G]$ and prove Theorems 14.5 and 14.6. We do it first for Boolean-valued models and handle the general case afterward.

Let M be a generic transitive model of ZFC, and let B be a complete Boolean algebra in M. Let G be an *M-generic* ultrafilter on B, i.e., generic over M.

Definition 14.27 (Interpretation by G). For every $x \in M^B$ we define x^G by induction on $\rho(x)$:

(i) $\emptyset^G = \emptyset$,
(ii) $x^G = \{y^G : x(y) \in G\}$.

Using the interpretation by G, we let

$$M[G] = \{x^G : x \in M^B\}. \tag{14.30}$$

Lemma 14.28. *Let G be an M-generic ultrafilter on B. Then for all names $x, y \in M^B$*

(i) *$x^G \in y^G$ if and only if $\|x \in y\| \in G$,*
(ii) *$x^G = y^G$ if and only if $\|x = y\| \in G$.*

Proof. We prove (i) and (ii) simultaneously, by induction on pairs $(\rho(x), \rho(y))$.

$$\begin{aligned}
\text{(i)}\quad \|x \in y\| \in G &\leftrightarrow \exists t \in \operatorname{dom}(y)\,(y(t) \in G \text{ and } \|x = t\| \in G)\\
&\leftrightarrow \exists t\,(y(t) \in G \text{ and } x^G = t^G)\\
&\leftrightarrow x^G \in \{t^G : y(t) \in G\}\\
&\leftrightarrow x^G \in y^G.\\
\text{(ii)}\quad \|x \subset y\| \in G &\leftrightarrow \textstyle\prod_{t \in \operatorname{dom} x}(x(t) \Rightarrow \|t \in y\|) \in G\\
&\leftrightarrow \forall t \in \operatorname{dom}(x)\,(x(t) \in G \text{ implies } \|t \in y\| \in G)\\
&\leftrightarrow \forall t\,(x(t) \in G \text{ implies } t^G \in y^G)\\
&\leftrightarrow \{t^G : x(t) \in G\} \subset y^G\\
&\leftrightarrow x^G \subset y^G.
\end{aligned}$$

□

$M[G]$ is a transitive class. The following is the Forcing Theorem for Boolean-valued models.

Theorem 14.29. *If G is an M-generic ultrafilter on B, then for all $x_1, \dots, x_n \in M^B$,*

$$M[G] \vDash \varphi(x_1^G, \dots, x_n^G) \quad \textit{if and only if} \quad \|\varphi(x_1, \dots, x_n)\| \in G. \tag{14.31}$$

Proof. Lemma 14.28 proves (14.31) for atomic formulas. The rest of the proof is by induction on the complexity of φ.

(a) φ is $\neg\psi$, $\psi\wedge\chi$, $\psi\vee\chi$, etc. Assuming (14.31) for ψ and χ, the induction step works because G is an ultrafilter. For instance,

$$\begin{aligned} M[G]\vDash\psi\wedge\chi &\leftrightarrow M[G]\vDash\psi \text{ and } M[G]\vDash\chi \\ &\leftrightarrow \|\psi\|\in G \text{ and } \|\chi\|\in G \\ &\leftrightarrow \|\psi\|\cdot\|\chi\|\in G \\ &\leftrightarrow \|\psi\wedge\chi\|\in G. \end{aligned}$$

Similarly for $\neg$, $\vee$, etc.

(b) φ is $\exists x\,\psi(x,\dots)$ or $\forall x\,\psi(x,\dots)$. We assume (14.31) for ψ and use the genericity of G:

$$\begin{aligned} M[G]\vDash\exists x\,\psi(x,\dots) &\leftrightarrow (\exists x\in M[G])\, M[G]\vDash\psi(x,\dots) \\ &\leftrightarrow (\exists x\in M^B)\, M[G]\vDash\psi(x^G,\dots) \\ &\leftrightarrow (\exists x\in M^B)\, \|\psi(x,\dots)\|\in G \\ &\leftrightarrow \sum_{x\in M^B}\|\psi(x,\dots)\|\in G \\ &\leftrightarrow \|\exists x\,\psi(x,\dots)\|\in G. \end{aligned}$$

The penultimate equivalence holds because if we let $A=\{\|\psi(x,\dots)\| : x\in M^B\}$, then $A\subset B$ and $A\in M$, and since G is generic we have

$$(\exists a\in A)\, a\in G \quad \text{if and only if} \quad \textstyle\sum A\in G.$$

Similarly for $\forall x\,\psi(x,\dots)$. □

Corollary 14.30. *$M[G]$ is a model of* ZFC.

Proof. By Theorem 14.24, every axiom σ of ZFC is valid in M^B, therefore $\|\sigma\|=1\in G$ and hence σ is true in $M[G]$. □

The following completes the proof of both Theorems 14.5 and 14.6 when forcing with a complete Boolean algebra:

Lemma 14.31.

(i) *$M\subset M[G]$, and both models have the same ordinals.*
(ii) *$G\in M[G]$ and if $N\supset M$ is a transitive model of* ZFC *such that $G\in N$, then $N\supset M[G]$.*

Proof. (i) For every $x\in M$, the G-interpretation of the canonical name $\check{x}$ is $\check{x}^G=x$ (proved by $\in$-induction). Hence $M\subset M[G]$. To show that every ordinal in $M[G]$ is in M (that $M[G]$ is not "longer" than M), we use Lemma 14.23.

(ii) Let $\dot{G}$ be the canonical name of a generic ultrafilter (Definition 14.25). Its interpretation is $\dot{G}^G = G$ and so $G \in M[G]$. If $N \supset M$ is a transitive model containing G, then the construction of $M[G]$ can be carried out inside N, and thus $M[G] \subset N$. □

We shall now prove Theorems 14.5 and 14.6:

Let $(P, <)$ be a notion of forcing in the ground model M, and let $G \subset P$ be generic over M. Let $B = B(P)$, and let $M^P = M^B$ be the class of the P-names. First we define *G-interpretation* of P-names: For every $x \in M^P$,

$$\begin{aligned} &\text{(i)} \ \ \emptyset^G = \emptyset, \\ &\text{(ii)} \ \ x^G = \{y^G : (\exists p \in G)\, e(p) \leq x(y)\}. \end{aligned} \tag{14.32}$$

Then we let

$$M[G] = \{x^G : x \in M^P\}.$$

Now let H be the ultrafilter on B generated by $e(G)$: $H = \{u \in B : \exists p \in G\, e(p) \leq u\}$. H is M-generic, and it is easily seen that $x^G = x^H$ for all $x \in M^B$. Thus $M[G] = M[H]$.

The Forcing Theorem now follows from the definition of $\Vdash$ and Theorem 14.29. As for the Generic Model Theorem 14.5, (a), (c), (d), and the first part of (b) are immediate consequences of Lemma 14.31; it only remains to verify that $G \in M[G]$. For that, we can either observe that $G = \{p \in P : e(p) \in H\}$ is in $M[H]$, or invoke (14.29) and verify that $\dot{G}^G = G$.

Consistency Proofs

Forcing is used mainly (but not exclusively) in consistency proofs. In practice, a consistency result is usually presented as follows: Suppose that A is some sentence (in the language of set theory) and we wish to prove that A is consistent with ZFC, or more generally, that A is consistent with some extension T of ZFC. This is accomplished by assuming that T holds (in V, the universe) and by exhibiting a forcing notion P such that the generic extension $V[G]$ satisfies A.

One way to make this argument legitimate is to assume that there exists a countable transitive model M of T. Using a forcing notion $P \in M$, there exists a P-generic filter G over M, and $M[G]$ is a transitive model that satisfies A. Hence A is consistent relative to T.

The assumption of a countable transitive model is unnecessary, as statements about generic extensions can be considered merely as an informal reformulation of statements about the forcing relation. In particular, "$V[G]$ satisfies A" is to be understood to mean "every $p \in P$ forces A." Then (assuming that T is consistent), the negation ¬A is not provable: If it were then every condition would force ¬A (or, the Boolean value $\|\neg \mathrm{A}\|$ would be 1). Note

that for consistency of A, it is enough to show that *some* $p \in P$ forces A; in the language of generic extensions, one finds a $p \in P$ such that when G is generic and $p \in G$, then $V[G] \vDash \mathrm{A}$.

In some cases, forcing results are stated as *independence* results: A sentence A is *independent* of the axioms T. This usually means that both A and $\neg$A are consistent with T.

Independence of the Continuum Hypothesis

We now present Cohen's proof of independence of CH.

Theorem 14.32. *There is a generic extension $V[G]$ that satisfies $2^{\aleph_0} > \aleph_1$.*

Proof. We describe the notion of forcing that produces a generic extension with the desired property. Let P be the set of all functions p such that

(14.33) (i) $\mathrm{dom}(p)$ is a finite subset of $\omega_2 \times \omega$,
(ii) $\mathrm{ran}(p) \subset \{0, 1\}$,

and let p be stronger than q if and only if $p \supset q$.

If G is a generic set of conditions, we let $f = \bigcup G$. We claim that

(14.34) (i) f is a function;
(ii) $\mathrm{dom}(f) = \omega_2 \times \omega$.

(Of course, ω_2 means ω_2 in the ground model.)

Part (i) of (14.34) holds because G is a filter. For part (ii), the sets $D_{\alpha,n} = \{p \in P : (\alpha, n) \in \mathrm{dom}(p)\}$ are dense in P, hence G meets each of them, and so $(\alpha, n) \in \mathrm{dom}(f)$ for all $(\alpha, n) \in \omega_2 \times \omega$.

Now, for each $\alpha < \omega_2$, let $f_\alpha : \omega \to \{0, 1\}$ be the function defined as follows:

$$f_\alpha(n) = f(\alpha, n).$$

If $\alpha \neq \beta$, then $f_\alpha \neq f_\beta$; this is because the set

$$D = \{p \in P : p(\alpha, n) \neq p(\beta, n) \text{ for some } n\}$$

is dense in P and hence $G \cap D \neq \emptyset$. Thus in $V[G]$ we have a one-to-one mapping $\alpha \mapsto f_\alpha$ of ω_2 into $\{0, 1\}^\omega$.

Each f_α is the characteristic function of a set $a_\alpha \subset \omega$. As in Example 14.2, we call these sets *Cohen generic reals*. Thus P adjoins $\aleph_2$ Cohen generic reals to the ground model.

The proof of Theorem 14.32 is almost complete, except for one detail: We don't know that the ordinal ω_2^V is the cardinal $\aleph_2$ of $V[G]$. We shall complete the proof by showing that $V[G]$ has the same cardinals as the ground model (P *preserves cardinals*).

Definition 14.33. A forcing notion P satisfies the *countable chain condition* (c.c.c.) if every antichain in P is at most countable.

The following theorem is one of the basic tools of forcing:

Theorem 14.34. *If P satisfies the countable chain condition then V and $V[G]$ have the same cardinals and cofinalities.*

In other words $\mathrm{cf}^V \alpha = \mathrm{cf}^{V[G]} \alpha$ for all limit ordinals α; the statement on cardinals follows.

Proof. It suffices to show that if κ is a regular cardinal then κ remains regular in $V[G]$. Thus let $\lambda < \kappa$; we show that every function $f \in V[G]$ from λ into κ is bounded.

Let $\dot f$ be a name, let $p \in P$ and assume

$$p \Vdash \dot f \text{ is a function from } \check\lambda \text{ to } \check\kappa. \tag{14.35}$$

For every $\alpha < \lambda$ consider the set

$$A_\alpha = \{\beta < \kappa : \exists q < p \; q \Vdash \dot f(\alpha) = \beta\}.$$

We claim that every A_α is at most countable: If $W = \{q_\beta : \beta \in A_\alpha\}$ is a set of witnesses to $\beta \in A_\alpha$ then W is an antichain, and therefore countable by c.c.c. Hence $|A_\alpha| \le \aleph_0$.

Now, because κ is regular, the set $\bigcup_{\alpha<\kappa} A_\alpha$ is bounded, by some $\gamma < \kappa$. It follows that for each $\alpha < \lambda$, p forces $\dot f(\alpha) < \gamma$.

Thus for every $\dot f \in V^P$ and every $p \in P$, if (14.35) then $p \Vdash \dot f$ is bounded below κ. It follows that in $V[G]$, every function $f : \lambda \to \kappa$ is bounded. □

Now we complete the proof of Theorem 14.32 by showing that the forcing notion that we employed satisfies c.c.c. That follows from the following consequence of Theorem 9.18 on Δ-systems. □

Lemma 14.35. *Let P be a set of finite functions, with values in a given countable set C. Let $p < q$ be defined as $p \supset q$, and assume that for all $p, q \in P$, if $p \cup q$ is a function then $p \cup q \in P$ (or more generally, $\exists r \in P\,(r \supset p \cup q)$). Then P satisfies the countable chain condition.*

Proof. Let F be an uncountable subset of P, and let W be the set $\{\mathrm{dom}(p) : p \in F\}$. As C is countable, the set W must be uncountable. By Theorem 9.18 there exists an uncountable Δ-system $Z \subset W$; let $S = X \cap Y$ for any $X \ne Y$ in Z. Let G be the set of all $p \in F$ such that $\mathrm{dom}(p) \in Z$; again because C is countable there are uncountably many $p \in G$ with the same $p{\restriction}S$. Now if p and q are two such functions, i.e., $\mathrm{dom}(p) \cap \mathrm{dom}(q) = S$ and $p{\restriction}S = q{\restriction}S$, then p and q are compatible functions and therefore compatible conditions. Hence F is not an antichain. □

Independence of the Axiom of Choice

If the ground model M satisfies the Axiom of Choice, then so does the generic extension. However, we can still use the method of forcing to construct a model in which AC fails; namely, we find a suitable submodel of the generic model, a model N such that $M \subset N \subset M[G]$.

Theorem 14.36 (Cohen). *There is a model of* ZF *in which the real numbers cannot be well-ordered. Thus the Axiom of Choice is independent of the axioms of* ZF*.*

Before we construct a model without Choice, we shall prove an easy but useful lemma on automorphisms of Boolean-valued models. Let B be a complete Boolean algebra and let π be an automorphism of B. We define, by induction on $\rho(x)$ an automorphism of the Boolean-valued universe V^B, and denote it also π:

(14.36) (i) $\pi(\emptyset) = \emptyset$;
(ii) $\operatorname{dom}(\pi x) = \pi(\operatorname{dom}(x))$, and $(\pi x)(\pi y) = \pi(x(y))$ for all $\pi(y) \in \operatorname{dom}(\pi x)$.

Clearly, π is a one-to-one function of V^B onto itself, and $\pi(\check{x}) = \check{x}$ for every x.

Lemma 14.37. *Let $\varphi(x_1, \dots, x_n)$ be a formula. If π is an automorphism of B, then for all $x_1, \dots, x_n \in V^B$,*

$$\|\varphi(\pi x_1, \dots, \pi x_n)\| = \pi(\|\varphi(x_1, \dots, x_n)\|). \tag{14.37}$$

Proof. (a) If φ is an atomic formula, (14.37) is proved by induction (as in the definition of $\|x \in y\|$, $\|x = y\|$). For instance,

$$\begin{aligned}\|\pi x \in \pi y\| &= \sum_{t \in \operatorname{dom}(\pi y)} (\|\pi x = t\| \cdot (\pi y)(t)) \\ &= \sum_{z \in \operatorname{dom}(y)} (\|\pi x = \pi z\| \cdot (\pi y)(\pi z)) \\ &= \pi\Big(\sum_{z \in \operatorname{dom}(y)} (\|x = z\| \cdot y(z))\Big) = \pi(\|x \in y\|).\end{aligned}$$

(b) In general, the proof is by induction on the complexity of φ. □

In practice, (14.37) is used as follows: Let $(P, <)$ be a separative partially ordered set. If π is an automorphism of P, then π extends to an automorphism of the complete Boolean algebra $B(P)$, by $\pi(u) = \sum\{\pi(p) : p \le u\}$. Then (14.37) takes this form: For all P-names $\dot{x}_1, \dots, \dot{x}_n$,

$$p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n) \quad \text{if and only if} \quad \pi p \Vdash \varphi(\pi\dot{x}_1, \dots, \pi\dot{x}_n). \tag{14.38}$$

For the proof of Theorem 14.36, let us assume that the ground model M satisfies $V = L$. We first extend M by adding countably many Cohen generic

reals: Let P be the set of all functions p such that

(14.39) (i) $\mathrm{dom}(p)$ is a finite subset of $\omega \times \omega$,
(ii) $\mathrm{ran}(p) \subset \{0, 1\}$,

and let $p < q$ if and only if $p \supset q$.

Let G be a generic set of conditions. For each $i \in \omega$, let

$$a_i = \{n \in \omega : (\exists p \in G)\, p(i, n) = 1\}$$

and let $A = \{a_i : i \in \omega\}$. Let $\dot{a}_i$, $i \in \omega$, and $\dot{A}$ be the canonical names for a_i and A:

(14.40) $\mathrm{dom}(\dot{a}_i) = \{\check{n} : n \in \omega\}$, and $\dot{a}_i(\check{n}) = \sum\{p \in P : p(i, n) = 1\}$,

(14.41) $\mathrm{dom}(\dot{A}) = \{\dot{a}_i : i \in \omega\}$, and $\dot{A}(\dot{a}_i) = 1$.

Lemma 14.38. *If $i \neq j$, then every p forces $\dot{a}_i \neq \dot{a}_j$.*

Proof. For every p there exists a $q \supset p$ such that for some $n \in \omega$, $q(i, n) = 1$ and $q(j, n) = 0$. □

In the model $M[G]$, let N be the class of all sets hereditarily ordinal-definable over A, $N = HOD(A)$. As we have seen in Chapter 13, N is a transitive model of ZF. Since the elements of A are sets of integers, it is clear that $A \in N$. We shall show that A cannot be well-ordered in the model N. For that, it suffices to show that there is no one-to-one function $f \in N$ from A into the ordinals.

Lemma 14.39. *In $M[G]$, there is no one-to-one function $f : A \to Ord$, ordinal-definable over A.*

Proof. Assume that $f : A \to Ord$ is one-to-one and ordinal-definable over A. Then there is a finite sequence $s = \langle x_0, \ldots, x_k \rangle$ in A such that f is ordinal-definable from s and A. Since f is one-to-one, it is easy to see that every $a \in A$ is ordinal definable from s and A. In particular, pick some $a \in A$ that is not among the x_i, $i \leq k$.

Since $a \in OD[s, A]$, there is a formula φ such that

(14.42) $M[G] \vDash a$ is the unique set such that $\varphi(a, \alpha_1, \ldots, \alpha_n, s, A)$

for some ordinals $\alpha_1, \ldots, \alpha_n$. We shall show that (14.42) is impossible.

Let $\dot{a}$ be a name for a, let $\dot{x}_0, \ldots, \dot{x}_k$ be names for $x_0, \ldots, x_k$ and let $\dot{s}$ be a name for the sequence $\langle x_0, \ldots, x_k \rangle$. We shall show the following:

(14.43) For every p_0 that forces $\varphi(\dot{a}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A})$ there exist $\dot{b}$ and $q \leq p_0$ such that q forces $\dot{a} \neq \dot{b}$ and $\varphi(\dot{b}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A})$.

Let $p_0 \Vdash \varphi(\dot{a}, \alpha_1, \ldots, \alpha_n, \dot{s}, \dot{A})$. There exist i, $i_0, \ldots, i_k$ and $p_1 \leq p_0$ such that p_1 forces $\dot{a} = \dot{a}_i$, $\dot{x}_0 = \dot{a}_{i_0}, \ldots, \dot{x}_k = \dot{a}_{i_k}$. Let $j \in \omega$ be such that $j \neq i$, and that for all m, $(j, m) \notin \mathrm{dom}(p_1)$.

Now let π be the permutation of ω that interchanges i and j, and $\pi x = x$ otherwise. This permutation induces an automorphism of P: For every $p \in P$,

$$\begin{aligned} \operatorname{dom}(\pi p) &= \{(\pi x, m) : (x, m) \in \operatorname{dom}(p)\}, \\ (\pi p)(\pi x, m) &= p(x, m). \end{aligned} \tag{14.44}$$

In turn, π induces an automorphism of B, and of M^B. It is easy to see (cf. (14.40) and (14.41)) that $\pi(\dot{a}_i) = \dot{a}_j$, $\pi(\dot{a}_j) = \dot{a}_i$, $\pi(\dot{a}_x) = \dot{a}_x$ for all $x \neq i, j$, $\pi(\dot{A}) = \dot{A}$ and $\pi(\dot{s}) = \dot{s}$. Since $(j, m) \notin \operatorname{dom}(p_1)$ for all m, it follows that $(i, m) \notin \operatorname{dom}(\pi p_1)$ for all m, and thus p_1 and πp_1 are compatible. Let $q = p_1 \cup \pi p_1$.

Now, on the one hand we have

$$p_1 \Vdash \varphi(\dot{a}_i, \alpha_1, \ldots, \alpha_n, \dot{s}, \dot{A}),$$

and on the other hand, since $\pi\check{\alpha} = \check{\alpha}$, $\pi\dot{s} = \dot{s}$ and $\pi\dot{A} = \dot{A}$, we have

$$\pi p_1 \Vdash \varphi(\dot{a}_j, \alpha_1, \ldots, \alpha_n, \dot{s}, \dot{A}).$$

Hence

$$q \Vdash \varphi(\dot{a}_i, \ldots) \text{ and } \varphi(\dot{a}_j, \ldots)$$

and by Lemma 14.38, $q \Vdash \dot{a}_i \neq \dot{a}_j$. Thus we have proved (14.43), which contradicts (14.42). □

Exercises

14.1. Show that in the definition of generic set one can replace (14.1)(iii) by the following weaker property: If $p, q \in G$, then p and q are compatible.

[To prove (14.1)(iii), show that $D = \{r \in P :$ either r is incompatible with p, or r is incompatible with q, or $r \leq p$ and $r \leq q\}$ is dense.]

14.2. A filter G on P is generic over M if and only if for every $p \in G$, if $D \in M$ is dense below p then $G \cap D \neq \emptyset$.

14.3. A filter G on P is generic over M if and only if $G \cap D \neq \emptyset$ whenever $D \in M$ is open and dense in P.

14.4. A filter G on P is generic over M if and only if $G \cap D \neq \emptyset$ whenever $D \in M$ is predense in P.

14.5. A filter G on P is generic over M if and only if $G \cap D \neq \emptyset$ whenever $D \in M$ is a maximal antichain in P.

14.6. Let $(P, <)$ be a notion of forcing in M with the following property: For every $p \in P$ there exist q and r such that $q \leq p$, $r \leq p$ and such that q and r are incompatible. Show that if $G \subset P$ is generic over M, then $G \notin M$.

[If F is a filter on P, then $\{p \in P : p \notin F\}$ is dense in P.]

14.7. If $\{q : q \Vdash \varphi\}$ is dense below p then $p \Vdash \varphi$.

14.8. Assume that for every $p \in P$ there exists a $G \subset P$ generic over M such that $p \in G$ (e.g., if M is countable). Show that $p \Vdash \sigma$ if and only if $M[G] \vDash \sigma$ for all generic G such that $p \in G$.

14.9. The separative quotient is unique up to isomorphism.

[If $(Q, <)$ is separative, then $\leq$ can be defined in terms of compatibility: $x \leq y$ if and only if every z compatible with x is compatible with y.]

14.10. If B is a complete Boolean algebra in the ground model M, then $G \subset B$ is a generic ultrafilter over M if and only if G is a generic filter on B^+ over M.

14.11. An ultrafilter G on B is generic over M if and only if for every partition W of B such that $W \in M$, there exists a unique $u \in G \cap W$.

14.12. (i) $\|(\exists y \in x)\, \varphi(y)\| = \sum_{y \in \operatorname{dom} x} (x(y) \cdot \|\varphi(y)\|)$.
(ii) $\|(\forall y \in x)\, \varphi(y)\| = \prod_{y \in \operatorname{dom} x} (x(y) \Rightarrow \|\varphi(y)\|)$.

14.13. (i) If $x = y$ then $\|\check{x} = \check{y}\| = 1$ and if $x \neq y$ then $\|\check{x} = \check{y}\| = 0$.
(ii) If $x \in y$ then $\|\check{x} \in \check{y}\| = 1$ and if $x \notin y$ then $\|\check{x} \in \check{y}\| = 0$.

14.14. Let $\dot{G}$ be the canonical name for a generic ultrafilter on B. Show that
(i) $\|\dot{G}$ is an ultrafilter on $B\| = 1$.
(ii) For every $X \subset B$, $\|$if $\check{X} \subset \dot{G}$ then $\prod X \in \dot{G}\| = 1$.

14.15. If G is an M-generic ultrafilter on B, let M^B/G be defined by (14.11) and (14.12). Prove that M^B/G is isomorphic to $M[G]$.

14.16. If G is an M-generic ultrafilter on B and π an automorphism of B (in M), then $H = \pi(G)$ is M-generic and $M[H] = M[G]$.

Historical Notes

The method of forcing was invented by Paul Cohen who used it to prove the independence of the Continuum Hypothesis and the Axiom of Choice (see [1963, 1964] and the book [1966]). The Boolean-valued version of Cohen's method has been formulated by Scott, Solovay, and Vopěnka. Following an observation of Solovay that the forcing relation can be viewed as assigning Boolean-values to formulas, Scott formulated his version of Boolean-valued models in [1967]. Vopěnka developed a theory of Cohen's method of forcing, using open sets in a topological space as forcing conditions (in [1964, 1965a, 1965b, 1965c, 1966, 1967a] and Vopěnka-Hájek [1967]), eventually arriving at the Boolean-valued version of forcing more or less identical to Scott-Solovay's version (Vopěnka [1967b]).

15. Applications of Forcing

In this chapter we present some important applications of the method of forcing. These applications establish several major consistency results and illustrate the techniques involved in use of forcing. Throughout we use V to denote the ground model, and $V[G]$ for the generic extension.

Cohen Reals

In (14.23) we described a notion of forcing that adjoins $\aleph_2$ real numbers to the ground model. In general, let κ be an infinite cardinal. The following notion of forcing adjoins κ real numbers, called *Cohen reals*.

Let P be the set of all functions p such that

(15.1) (i) $\mathrm{dom}(p)$ is a finite subset of $\kappa \times \omega$,
(ii) $\mathrm{ran}(p) \subset \{0, 1\}$,

and let p be stronger than q if and only if $p \supset q$.

Let G be a generic set of conditions and let $f = \bigcup G$. By a genericity argument, f is a function from $\kappa \times \omega$ into $\{0, 1\}$. For each $\alpha < \kappa$, we let f_α be the function on ω defined by $f_\alpha(n) = f(\alpha, n)$ and let $a_\alpha = \{n \in \omega : f_\alpha(n) = 1\}$. Each a_α is a real (a subset of ω), $a_\alpha \notin V$ and if $\alpha \neq \beta$, then $a_\alpha \neq a_\beta$. This is proved as in Theorem 14.32.

Also as in Theorem 14.32 one shows that P satisfies the countable chain condition. It follows that cardinals and cofinalities are *preserved* in the generic extension.

Since P adds κ distinct Cohen reals, the size of the continuum in $V[G]$ is at least κ. In fact, it is at least $(\kappa^{\aleph_0})^V$:

$$(2^{\aleph_0})^{V[G]} = ((2^{\aleph_0})^{\aleph_0})^{V[G]} \geq (\kappa^{\aleph_0})^{V[G]} \geq (\kappa^{\aleph_0})^V.$$

It turns out that there are precisely $(\kappa^{\aleph_0})^V$ reals in $V[G]$. The following is a general estimate of the number of new sets in a generic extension:

Lemma 15.1. *Let λ be a cardinal in V. If G is a V-generic ultrafilter on B, then*

$$(2^\lambda)^{V[G]} \leq (|B|^\lambda)^V.$$

Proof. Every subset $A \subset \lambda$ in $V[G]$ has a name $\dot{A} \in V^B$; every such $\dot{A}$ determines a function $\alpha \mapsto \|\check{\alpha} \in \dot{A}\|$ from λ into B. Different subsets correspond to different functions, and thus the number of all subsets of λ in $V[G]$ is not greater than the number of all functions from λ into B in V. □

When P is the forcing (15.1) that adds κ Cohen reals, P satisfies c.c.c. and so every element of $B = B(P)$ is the Boolean sum of a countable antichain in P; hence $|B| \leq |P|^{\aleph_0} = \kappa^{\aleph_0}$. By Exercise 7.32, $|B| = |B|^{\aleph_0}$ and it follows that $|B| = \kappa^{\aleph_0}$, and consequently, $(2^{\aleph_0})^{V[G]} = (\kappa^{\aleph_0})^V$.

If we start with a ground model that satisfies GCH, and if κ is (in V) a cardinal of uncountable cofinality, then $\kappa^{\aleph_0} = \kappa$ in V, and we get a model $V[G]$ in which $2^{\aleph_0} = \kappa$.

Adding Subsets of Regular Cardinals

The forcing that adds a Cohen real generalizes easily from ω to any regular cardinal κ. Let κ be, in V, a regular cardinal and assume that $2^{<\kappa} = \kappa$.

Let P be the set of all functions p such that

(15.2) (i) $\operatorname{dom}(p) \subset \kappa$ and $|\operatorname{dom}(p)| < \kappa$;
(ii) $\operatorname{ran}(p) \subset \{0, 1\}$.

A condition p be stronger than q if and only if $p \supset q$.

Let G be a set of conditions generic over V and let $f = \bigcup G$. As before, f is a function from κ into $\{0, 1\}$, and $X = \{\alpha < \kappa : f(\alpha) = 1\}$ is a subset of κ and $X \notin V$.

In order to add more new subsets of κ, we use a generalization of (15.1): Let κ be as above, and let λ be a cardinal greater than κ such that $\lambda^\kappa = \lambda$. Let P be the set of all functions p such that:

(15.3) (i) $\operatorname{dom}(p) \subset \lambda \times \kappa$ and $|\operatorname{dom}(p)| < \kappa$,
(ii) $\operatorname{ran}(p) \subset \{0, 1\}$,

and let p be stronger than q if and only if $p \supset q$.

Let G be a generic set of conditions and let $f = \bigcup G$. For each $\alpha < \lambda$, we let

$$a_\alpha = \{\xi < \kappa : f(\alpha, \xi) = 1\}.$$

Each a_α is a subset of κ, each $a_\alpha \notin V$ and $a_\alpha \neq a_\beta$ whenever $\alpha \neq \beta$.

We claim that in the generic extension, all cardinals are preserved, and $2^\kappa = \lambda$. But to show this, we need additional results in the theory of forcing, proved in the next two sections.

The κ-Chain Condition

Definition 15.2. A forcing notion P satisfies the *κ-chain condition* (κ-c.c.) if every antichain in P has cardinality less than κ.

The $\aleph_1$-chain condition is the c.c.c. Note that P satisfies the κ-c.c. if and only if $B(P)$ satisfies the κ-c.c.

Theorem 14.34 generalizes as follows:

Theorem 15.3. *If κ is a regular cardinal and if P satisfies the κ-chain condition then κ remains a regular cardinal in the generic extension by P.*

Proof. The proof is exactly as the proof of Theorem 14.34. The only difference is that the set A_α is not necessarily countable but has cardinality less than κ. □

Consequently, all regular cardinals $\kappa \geq \mathrm{sat}(B(P))$, and in particular all regular $\kappa \geq |P|^+$ are preserved in $V[G]$.

The following lemma generalizes Lemma 14.35, and implies that the forcing notion (15.3) satisfies the κ^+-chain condition. We remark that Lemma 15.4 is related to (a generalization of) Theorem 9.18 on Δ-systems.

Lemma 15.4. *Let κ be a regular cardinal such that $2^{<\kappa} = \kappa$. Let S be an arbitrary set and let $|C| \leq \kappa$. Let P be the set of all functions p whose domains are subsets of S of size $< \kappa$, with values in C. Let $p < q$ if and only if $q \supset q$. Then P satisfies the κ^+-chain condition.*

Proof. Let $W \subset P$ be an antichain. We construct sequences $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$ $(\alpha < \kappa)$ of subsets of S, and $W_0 \subset W_1 \subset \ldots \subset W_\alpha \subset \ldots$ $(\alpha < \kappa)$ of subsets of W. If α is a limit ordinal, we let $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$ and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Given A_α and W_α, we choose for each $p \in P$ with $\mathrm{dom}(p) \subset A_\alpha$ some $q \in W$ (if there is one) such that $p = q{\restriction}A_\alpha$. Then we let $W_{\alpha+1} = W_\alpha \cup \{\text{the chosen } q\text{'s}\}$ and $A_{\alpha+1} = \bigcup\{\mathrm{dom}(q) : q \in W_{\alpha+1}\}$; finally, $A = \bigcup_{\alpha<\kappa} A_\alpha$.

Next we show that $W = \bigcup_{\alpha<\kappa} W_\alpha$: If $q \in W$, then there is an $\alpha < \kappa$ such that $\mathrm{dom}(q) \cap A = \mathrm{dom}(q) \cap A_\alpha$. Thus if $p = q{\restriction}A_\alpha$, there exists some $q' \in W_{\alpha+1}$ such that $q'{\restriction}A_\alpha = p$. Since $\mathrm{dom}(q') \subset A$, it follows that q and q' are compatible; however, both are elements of W and thus $q = q'$. Hence $q \in W_{\alpha+1}$.

The proof is completed by showing that $|A_\alpha| \leq \kappa$ and $|W_\alpha| \leq \kappa$ for each $\alpha < \kappa$. This is proved by induction on α. If $|W_\alpha| \leq \kappa$, then $|A_\alpha| \leq \kappa$ because $A_\alpha = \bigcup\{\mathrm{dom}(q) : q \in W_\alpha\}$. If α is a limit ordinal and $|W_\beta| \leq \kappa$ for all $\beta < \alpha$, then $|W_\alpha| = |\bigcup_{\beta<\alpha} W_\beta| \leq \kappa$. Thus let us assume that $|W_\alpha| \leq \kappa$ and let us show that $|W_{\alpha+1}| \leq \kappa$. The set $W_{\alpha+1}$ is obtained by adding to W_α at most one $q \in W$ for each $p \in P$ with $\mathrm{dom}(p) \subset A_\alpha$. There are at most $\kappa^{<\kappa}$ subsets X of A_α of size $< \kappa$, and since κ is regular and $2^{<\kappa} = \kappa$, we have $\kappa^{<\kappa} = \kappa$. On each X there are $|C|^{|X|}$ functions with values in C, and therefore there are at most κ elements p of P with $\mathrm{dom}(p) \subset A_\alpha$. Hence $|W_{\alpha+1}| \leq \kappa$. Then it follows that $|W| \leq \kappa$. □

Distributivity

In (7.28) we defined κ-distributivity of complete Boolean algebras. We now show that this concept plays a crucial role in the theory of forcing.

Definition 15.5. A forcing notion P is *κ-distributive* if the intersection of κ open dense sets is open dense. P is *$<\kappa$-distributive* if it is λ-distributive for all $\lambda < \kappa$.

Note that if P is dense in B then P is κ-distributive if and only if B is.

Theorem 15.6. *Let κ be an infinite cardinal and assume that $(P, <)$ is κ-distributive. Then if $f \in V[G]$ is a function from κ into V, then $f \in V$. In particular, κ has no new subsets in $V[G]$.*

Proof. Let $f : \kappa \to V$ and $f \in V[G]$, let $\dot{f}$ be a name for f. There exist some $A \in V$ and a condition $p_0 \in G$ such that p_0 forces

$$\dot{f} \text{ is a function from } \check{\kappa} \text{ into } \check{A}.$$

For each $\alpha < \kappa$, the set

$$D_\alpha = \{p \leq p_0 : (\exists x \in A)\, p \Vdash \dot{f}(\check{\alpha}) = \check{x}\}$$

is open dense below p_0. Thus $D = \bigcap_{\alpha<\kappa} D_\alpha$ is dense below p_0 and therefore there is some $p \in D \cap G$. Now we argue in V: For each $\alpha < \kappa$ there is some x_α such that $p \Vdash \dot{f}(\check{\alpha}) = \check{x}_\alpha$; let $g : \kappa \to A$ be the function defined by $g(\alpha) = x_\alpha$. However, it is easy to see that $f(\alpha) = x_\alpha = g(\alpha)$, for every $\alpha < \kappa$, and thus $f \in V$. □

See Exercise 15.5 for the converse.

The following property, stronger than distributivity, is often easy to verify:

Definition 15.7. P is *κ-closed* if for every $\lambda \leq \kappa$, every descending sequence $p_0 \geq p_1 \geq \ldots \geq p_\alpha \geq \ldots$ $(\alpha < \lambda)$ has a lower bound. P is *$<\kappa$-closed* if it is λ-closed for all $\lambda < \kappa$.

Lemma 15.8. *If P is κ-closed then it is κ-distributive.*

Proof. Let $\{D_\alpha : \alpha < \kappa\}$ be a collection of open dense sets. The intersection $D = \bigcap_{\alpha<\kappa} D_\alpha$ is clearly open; to show that D is dense, let $p \in P$ be arbitrary. By induction on $\alpha < \kappa$, we construct a descending κ-sequence of conditions $p \geq p_0 \geq p_1 \geq \ldots$. We let p_α be a condition stronger than all p_ξ, $\xi < \alpha$, and such that $p_\alpha \in D_\alpha$. Finally, we let q be a condition stronger than all p_α, $\alpha < \kappa$. Clearly, $q \in D$. □

Now we can prove the claim about the generic extension by the forcing in (15.3). The forcing P is $<\kappa$-closed, and therefore κ has no new bounded subsets; hence κ is preserved. The cardinals above κ are preserved because P satisfies the κ^+-chain condition, by Lemma 15.4. We have $|P| = \lambda$ and therefore $|B| = |P|^\kappa = \lambda$, and so, by Lemma 15.1, $(2^\kappa)^{V[G]} = \lambda^\kappa = \lambda$.

Product Forcing

Let P and Q be two notions of forcing. The *product* $P \times Q$ is the coordinate-wise partially ordered set product of P and Q:

$$(p_1, q_1) \leq (p_2, q_2) \quad \text{if and only if} \quad p_1 \leq p_2 \text{ and } q_1 \leq q_2. \tag{15.4}$$

If G is a generic filter on $P \times Q$, let

$$G_1 = \{p \in P : \exists q\, (p, q) \in G\}, \qquad G_2 = \{q \in Q : \exists p\, (p, q) \in G\}. \tag{15.5}$$

The sets G_1 and G_2 are generic on P and Q respectively, and $G = G_1 \times G_2$. The following lemma describes genericity on products:

Lemma 15.9 (The Product Lemma). *Let P and Q be two notions of forcing in M. In order that $G \subset P \times Q$ be generic over M, it is necessary and sufficient that $G = G_1 \times G_2$ where $G_1 \subset P$ is generic over M and $G_2 \subset Q$ is generic over $M[G_1]$. Moreover, $M[G] = M[G_1][G_2]$.*

As a corollary, if G_1 is generic over M and G_2 is generic over $M[G_1]$, then G_1 is generic over $M[G_2]$, and $M[G_1][G_2] = M[G_2][G_1]$.

Proof. First let G be an M-generic filter on $P \times Q$. We define G_1 and G_2 by (15.5). Clearly, G_1 and G_2 are filters, and $G \subset G_1 \times G_2$. If $(p_1, p_2) \in G_1 \times G_2$, then there are $p_1' \in G_1$ and $p_2' \in G_2$ such that $(p_1', p_2) \in G$ and $(p_1, p_2') \in G$. Since G is a filter, there exist $q_1 \leq p_1, p_1'$ and $q_2 \leq p_2, p_2'$ such $(q_1, q_2) \in G$. Hence $(p_1, p_2) \in G$ and we have $G = G_1 \times G_2$.

It is easy to see that G_1 is generic over M: If $D_1 \in M$ is dense in P, then $D_1 \times Q$ is dense in $P \times Q$; and since $(D_1 \times Q) \cap G \neq \emptyset$, we have $D_1 \cap G_1 \neq \emptyset$. To show that G_2 is generic over $M[G_1]$, let $D_2 \in M[G_1]$ be dense in Q. Let $\Vdash$ be the forcing relation corresponding to P. Let $\dot{D}_2$ be a name for D_2 and let $p_1 \in G_1$ be such that p_1 forces "$\dot{D}_2$ is dense in Q." Let $p_2 \in G_2$ be arbitrary. For every $q_1 \leq p_1$ and every $q_2 \leq p_2$ there exist $r_1 \leq q_1$ and $r_2 \leq q_2$ such that $r_1 \Vdash r_2 \in \dot{D}_2$; thus

$$D = \{(r_1, r_2) : r_1 \leq p_1 \text{ and } r_1 \Vdash r_2 \in \dot{D}_2\}$$

is dense in $P \times Q$ below (p_1, p_2) and so there exist r_1, r_2 such that $r_1 \in G_1$ and $r_1 \Vdash r_2 \in \dot{D}_2$. Hence $r_2 \in D_2 \cap G_2$.

Conversely, let $G_1 \subset P$ be M-generic and let $G_2 \subset Q$ be $M[G_1]$-generic. We let $G = G_1 \times G_2$. Clearly G is a filter on $P \times Q$. To show that G is M-generic, let $D \in M$ be dense in $P \times Q$. We let

$$D_2 = \{p_2 : (p_1, p_2) \in D \text{ for some } p_1 \in G_1\}.$$

The set D_2 is in $M[G_1]$; we shall show that D_2 is dense in Q and thus $D \cap (G_1 \times G_2) \neq \emptyset$.

Let $q_2 \in Q$ be arbitrary. Since D is dense in $P \times Q$, it follows that the set

$$D_1 = \{p_1 : (\exists p_2 \leq q_2)\,(p_1, p_2) \in D\}$$

is dense in P. Hence there is $p_1 \in G_1 \cap D_1$ and so D_2 is dense in Q. Since $G_1 \times G_2 \in M[G_1][G_2]$, it is obvious that $M[G_1 \times G_2] = M[G_1][G_2]$. □

We shall now define products of infinitely many notions of forcing. In order to simplify the notation, we will assume that every notion of forcing has a greatest element, denoted 1. In practice, the *empty condition* $\emptyset$ is often the greatest element of $(P, <)$.

Definition 15.10. Let $\{P_i : i \in I\}$ be a collection of partially ordered sets, each having a greatest element 1. The *product* $P = \prod_{i \in I} P_i$ consists of all functions p on I with values $p(i) \in P_i$, such that $p(i) = 1$ for all but finitely many $i \in I$. P is partially ordered by

$$p \leq q \quad \text{if and only if} \quad p(i) \leq q(i) \text{ for all } i \in I. \tag{15.6}$$

For each $p \in \prod_i P_i$, the finite set $s(p) = \{i \in I : p(i) \neq 1\}$ is called the *support* of p.

If G is a generic filter on $\prod_i P_i$, then for each $i \in I$, the set $G_i = \{p(i) : p \in G\}$, the *projection* of G on P_i, is a generic filter on P_i.

A natural generalization of a product is κ-product:

Definition 15.11. Let κ be a regular cardinal. The *κ-product* (the product with $<\kappa$-support) of P_i is the set of all functions p on I with $p(i) \in P_i$ such that $|s(p)| < \kappa$; the ordering is coordinatewise (15.6).

As usual, *λ-support* means $<\lambda^+$-support, *countable* support means $<\aleph_1$-support, etc.

The following lemma is immediate:

Lemma 15.12. *If P and Q are λ-closed then $P \times Q$ is λ-closed. More generally, if each P_i is λ-closed and P is the κ-product of the P_i, with $\lambda < \kappa$, then P is λ-closed.*

Proof. Let $\alpha \leq \lambda$ and let $p^\xi = \langle p_i^\xi : i \in I\rangle$, $\xi < \alpha$, be a descending α-sequence of conditions in P. If we let $s = \bigcup_{\xi<\alpha} s(p^\xi)$, then $|s| < \kappa$, and since each P_i is λ-closed, it is easy to find $p = \langle p_i : i \in I\rangle$ such that $s(p) = s$ and that $p_i \leq p_i^\xi$ for each $i \in I$ and each $\xi < \alpha$. □

Chain conditions are generally not preserved by products. While it is consistent that c.c.c. is preserved by products (we return to this in Chapter 16), it is also consistent to have a forcing P that satisfies c.c.c. but $P \times P$ does not (see Exercise 15.28).

The following property (K for Knaster) is stronger than the countable chain condition:

Definition 15.13. A notion of forcing has *property* (K) if every uncountable set of conditions has an uncountable subset of pairwise compatible elements.

Lemma 15.14. *If P and Q both have property* (K) *then so does $P \times Q$.*

Proof. Let $W \subset P \times Q$ be uncountable. If there exists a $p \in P$ such that the set $X = \{q : (p, q) \in W\}$ is uncountable, then since Q has property (K) there exists an uncountable $Y \subset X$ of pairwise compatible elements, and $\{p\} \times Y$ is such a subset of W in $P \times Q$.

The proof is similar if for some $q \in Q$, the set $\{p : (p, q) \in W\}$ is uncountable. In the remaining case, there is an uncountable set of pairs $F \subset W$ that is a one-to-one function. Applying successively property (K) to P and Q, we get an uncountable $G \subset F$ such that for any two elements (p_1, q_1) and (p_2, q_2) of G, p_1 is compatible with p_2 in P and q_1 is compatible with q_2 in Q, hence (p_1, q_1) and (p_2, q_2) are compatible. □

Theorem 15.15. *If for every $i \in I$, P_i has property* (K) *then $\prod_{i\in I} P_i$ has property* (K).

Proof. Let X be an uncountable subset of P, and let $W = \{s(p) : p \in X\}$. If W is countable, then there is a finite set $J \subset I$ such that $s(p) = J$ for uncountably many p. By Lemma 15.14, $\prod_{i\in J} P_i$ has property (K) and the theorem follows. If W is uncountable, there exist, by Theorem 9.18, an uncountable $Z \subset X$ and a finite set $J \subset I$ such that $s(p) \cap s(q) = J$ whenever $p, q \in Z$, $p \neq q$. Since $\prod_{i\in I} P_i$ has property (K), Z has an uncountable subset Y such that for any $p, q \in Y$, $p{\restriction}J$ and $q{\restriction}J$ are compatible. But such p and q are compatible in $\prod_{i\in I} P_i$. □

Corollary 15.16. *The product of any collection of countable forcing notions has property* (K) *and so it satisfies the countable chain condition.* □

The best one can say about the chain condition in products is this:

Theorem 15.17. (i) *If each P_i has size λ (infinite) then the product of the P_i satisfies the λ^+-chain condition.*

(ii) *If κ is regular, $\lambda \geq \kappa$, $\lambda^{<\kappa} = \lambda$ and $|P_i| \leq \lambda$ for all $i \in I$, then the κ-product of the P_i satisfies the λ^+-chain condition.*

(iii) *If λ is inaccessible, $\kappa < \lambda$ is regular, and $|P_i| < \lambda$ for each i, then the κ-product satisfies the λ-chain condition.*

Proof. (i) is a special case of (ii); thus consider κ-products. Let P be the κ-product, and let W be an antichain in P. If $p = \langle p_i : i \in I\rangle$ and $q = \langle q_i : i \in I\rangle$ are incompatible in P, then for some $i \in s(p) \cap s(p)$, p_i and q_i are incompatible in P_i, and in particular $p_i \neq q_i$. Thus we can regard elements of W as functions whose domain is a subset $s(p)$ of I of size $< \kappa$, with values in the P_i, and show that if W consists of pairwise incompatible functions then $|W|$ has the required bound.

We follow the proof of Lemma 15.4. As there we construct κ-sequences $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$ $(\alpha < \kappa)$ of subsets of I and $W_0 \subset W_1 \subset \ldots \subset W_\alpha \subset \ldots$ $(\alpha < \kappa)$ of subsets of W such that $A_\alpha = \bigcup\{s(p) : p \in W_\alpha\}$ for each α. And as there we show that $W = \bigcup_{\alpha<\kappa} W_\alpha$. Thus it remains to show, by induction on α, that $|W_\alpha| \leq \lambda$ (in (ii)) or that $|W_\alpha| < \lambda$ (in (iii)). Let us prove (ii); (iii) is similar.

If $|W_\alpha| \leq \lambda$, then $|A_\alpha| \leq \kappa \cdot \lambda = \lambda$. If $\alpha < \kappa$ is limit and if $|W_\beta| \leq \lambda$ for each $\beta < \alpha$, then $|W_\alpha| \leq |\alpha| \cdot \lambda = \lambda$. Thus let us assume that $|W_\alpha| \leq \lambda$ and let us show that $|W_{\alpha+1}| \leq \lambda$. The set $W_{\alpha+1}$ is obtained by adding to W_α at most one q for each $p \in P$ with $s(p) \subset A_\alpha$. However, since $|A_\alpha| \leq \lambda$, there are at most $\lambda^{<\kappa}$ functions p with $s(p) \subset A_\alpha$, $|s(p)| < \kappa$, and λ possible values for each $i \in s(p)$. Thus $|W_{\alpha+1}| \leq \lambda^{<\kappa} = \lambda$. □

Easton's Theorem

The theorem that we are about to prove shows that in ZFC alone the continuum function 2^κ can behave in any prescribed way consistent with König's Theorem, for *regular cardinals* κ. As we have seen in Chapter 8 (Silver's Theorem) and shall see again in Chapter 24, this is not the case with singular cardinals.

Theorem 15.18 (Easton). *Let M be a transitive model of* ZFC *and assume that the Generalized Continuum Hypothesis holds in M. Let F be a function (in M) whose arguments are regular cardinals and whose values are cardinals, such that for all regular κ and λ:*

(15.7)
(i) $F(\kappa) > \kappa$;
(ii) $F(\kappa) \leq F(\lambda)$ *whenever* $\kappa \leq \lambda$;
(iii) $\operatorname{cf} F(\kappa) > \kappa$.

Then there is a generic extension $M[G]$ of M such that M and $M[G]$ have the same cardinals and cofinalities, and for every regular κ,

$$M[G] \vDash 2^\kappa = F(\kappa).$$

We have to point out that the generic extension is obtained by forcing with a *class* of conditions. By Lemma 15.1, a notion of forcing can only increase the size of 2^κ for $\kappa < |B(P)|$; thus we *have* to use a class of conditions. We shall describe the appropriate generalization of the forcing method.

Since the proof of Easton's Theorem involves forcing with a class of conditions, we shall first give a proof of the special case, when the "continuum function" F is prescribed for only a *set* of regular cardinals. Thus let us work in a ground model M that satisfies the GCH and let F be a function defined on a set A of regular cardinals and having the properties (15.7)(i)–(iii).

For each $\kappa \in \operatorname{dom}(F)$, let $(P_\kappa, \supset)$ be the notion of forcing that adjoins $F(\kappa)$ subsets of κ (cf. (15.3)):

$$\operatorname{dom}(p) \subset \kappa \times F(\kappa), \qquad |\operatorname{dom}(p)| < \kappa, \quad \text{and} \quad \operatorname{ran}(p) \subset \{0,1\}. \tag{15.8}$$

We let $(P, <)$ be the *Easton product* of P_κ, $\kappa \in A$: A condition p is a function $p = \langle p_\kappa : \kappa \in A\rangle \in \prod_{\kappa\in A} P_\kappa$ such that if we denote $s(p) = \{\kappa \in A : p_\kappa \neq \emptyset\}$, the support of p, then

$$\text{for every regular cardinal } \gamma,\ |s(p) \cap \gamma| < \gamma. \tag{15.9}$$

We can regard the conditions as functions with values 0 and 1, whose domain consists of triples (κ, α, β) where $\kappa \in A$, $\alpha < \kappa$, and $\beta < F(\kappa)$, and such that for every regular cardinal γ,

$$|\{(\kappa,\alpha,\beta) \in \operatorname{dom}(p) : \kappa \leq \gamma\}| < \gamma \tag{15.10}$$

(and p is stronger than q if and only if $p \supset q$). Note that (15.10) implies that for each $\kappa \in A$, $|\operatorname{dom}(p_\kappa)| < \kappa$, where p_κ is defined by

$$p_\kappa(\alpha,\beta) = p(\kappa,\alpha,\beta).$$

Let G be a generic set of conditions, and let for each $\kappa \in A$, G_κ be the projection of G on P_κ. Each G_κ is a generic filter on P_κ and thus produces $F(\kappa)$ new subsets of κ:

$$a^\kappa_\beta = \{\alpha < \kappa : (\exists p \in G)\, p(\kappa,\alpha,\beta) = 1\} \qquad (\beta < F(\kappa)).$$

We shall show that $(P, <)$ preserves cardinals and cofinalities, and that each $\kappa \in A$ has exactly $F(\kappa)$ subsets in $M[G]$. The condition (15.10) is instrumental in the proof.

Given a regular cardinal λ, we can decompose each condition $p \in P$ into two parts:

$$p^{\leq\lambda} = p{\restriction}\{(\kappa,\alpha,\beta) : \kappa \leq \lambda\}, \qquad p^{>\lambda} = p{\restriction}\{(\kappa,\alpha,\beta) : \kappa > \lambda\}. \tag{15.11}$$

Clearly $p = p^{\leq\lambda} \cup p^{>\lambda}$. We let

$$P^{\leq\lambda} = \{p^{\leq\lambda} : p \in P\}, \qquad P^{>\lambda} = \{p^{>\lambda} : p \in P\}. \tag{15.12}$$

Obviously, $P^{\leq\lambda}$ is the Easton product of P_κ, $\kappa \in A$ and $\kappa \leq \lambda$, and $P^{>\lambda}$ is the Easton product of P_κ, $\kappa \in A$ and $\kappa > \lambda$. Moreover, P is (isomorphic to) the product $P^{\leq\lambda} \times P^{>\lambda}$.

First we notice that $P^{>\lambda}$ is λ-closed: If $C \subset P^{>\lambda}$ consists of pairwise compatible conditions and $|C| \leq \lambda$, then $p = \bigcup C$ is a condition in $P^{>\lambda}$; (15.10) holds for all regular $\gamma > \lambda$, and holds trivially for $\gamma \leq \lambda$ because if $(\kappa,\alpha,\beta) \in \operatorname{dom}(p)$, then $\kappa > \lambda$.

Furthermore, $P^{\leq\lambda}$ satisfies the λ^+-chain condition: If $W \subset P^{\leq\lambda}$ is an antichain, then $|W| \leq \lambda$. The proof given in Theorem 15.17 works in this case as well because $|\operatorname{dom}(p)| < \lambda$ for each $p \in P^{\leq\lambda}$ (and because GCH holds). Thus $P = P^{>\lambda} \times P^{\leq\lambda}$ where $P^{>\lambda}$ is λ-closed and $P^{\leq\lambda}$ satisfies the λ^+-chain condition.

Lemma 15.19. *Let $G \times H$ be an M-generic filter on $P \times Q$, where P is λ-closed and Q satisfies the λ^+-chain condition. Then every function $f : \lambda \to M$ in $M[G \times H]$ is in $M[H]$. In particular,*

$$P^{M[G\times H]}(\lambda) = P^{M[H]}(\lambda).$$

Proof. Let $\dot f$ be a name for f; let us assume, without loss of generality, that for some A, every condition forces that $\dot f$ is a function from λ into A. For each $\alpha < \lambda$, let $D_\alpha \subset P$ be defined as follows:

$$\begin{aligned} &p \in D_\alpha \text{ if and only if there exist a maximal antichain } W \subset Q \\ &\text{and a family } \{a^{(\alpha)}_{p,q} : q \in W\} \text{ such that for each } q \in W, \\ &(p,q) \Vdash \dot f(\alpha) = a^{(\alpha)}_{p,q}. \end{aligned} \tag{15.13}$$

We claim that each D_α is open dense in P. Clearly, D_α is open; thus let $p_0 \in P$ be arbitrary and let us find $p \in D_\alpha$ such that $p \leq p_0$. There exist $p_1 \leq p_0$, $q_1 \in Q$ and $a_1 \in A$ such that $(p_1, q_1) \Vdash \dot f(\alpha) = a_1$. By induction on $\gamma < \lambda^+$, we construct $p_\gamma \in P$, $q_\gamma \in Q$, and $a_\gamma \in A$ such that $p_0 \geq p_1 \geq \ldots \geq p_\gamma \geq \ldots$, that the q_γ are pairwise incompatible and that (p_γ, q_γ) forces $\dot f(\alpha) = a_\gamma$. If $\{q_\xi : \xi < \gamma\}$ is not maximal, we can find such p_γ, q_γ, and a_γ since P is λ-closed. By the λ^+-chain condition, there is some $\beta < \lambda^+$ such that $W = \{q_\gamma : \gamma < \beta\}$ is a maximal antichain; then we find $p \in P$ stronger than all p_γ, $\gamma < \beta$. Thus D_α is open dense in P.

Since P is λ-closed, it follows that $\bigcap_{\alpha<\lambda} D_\alpha$ is open dense, and so there exists some $p \in G$ such that $p \in D_\alpha$ for all $\alpha < \lambda$. We pick (in M) for each $\alpha < \lambda$ a maximal antichain $W_\alpha \subset Q$ and a family $\{a^{(\alpha)}_{p,q} : q \in W_\alpha\}$ such that (15.13) holds for each $q \in W_\alpha$. By the genericity of H, for every α there is a unique $q \in W_\alpha$ such that $q \in H$, and we have, for every $\alpha < \lambda$,

$$f(\alpha) = a^{(\alpha)}_{p,q}, \qquad \text{where } q \text{ is the unique } q \in W_\alpha \cap H. \tag{15.14}$$

However, (15.14) defines the function f in $M[H]$. □

Now we can finish the proof of Easton's Theorem, that is, at least in the case when F is defined on a set A of regular cardinals.

Let κ be a regular cardinal in M; we shall show that κ is a regular cardinal in $M[G]$. If κ fails to be a regular cardinal, then there exists a function f that maps some $\lambda < \kappa$, regular in M, cofinally into κ. We consider P as the product: $P = P^{>\lambda} \times P^{\leq\lambda}$. Then $G = G^{>\lambda} \times G^{\leq\lambda}$ and $M[G] = M[G^{>\lambda}][G^{\leq\lambda}] = M[G^{\leq\lambda}][G^{>\lambda}]$. By Lemma 15.19, f is in $M[G^{\leq\lambda}]$ and so κ is not a regular cardinal in $M[G^{\leq\lambda}]$. However, this is a contradiction since $P^{\leq\lambda}$ satisfies the κ-chain condition and hence κ is regular in $M[G^{\leq\lambda}]$.

It remains to prove that $(2^\lambda)^{M[G]} = F(\lambda)$, for each $\lambda \in A$. Again, we regard P as the product $P^{>\lambda} \times P^{\leq\lambda}$ and $G = G^{>\lambda} \times G^{\leq\lambda}$. By Lemma 15.19, every subset of λ in $M[G]$ is in $M[G^{\leq\lambda}]$ and we have $(2^\lambda)^{M[G]} = (2^\lambda)^{M[G^{\leq\lambda}]}$. However, an easy computation shows that $|P^{\leq\lambda}| = F(\lambda)$ and $|B(P^{\leq\lambda})| = F(\lambda)$, and hence $(2^\lambda)^{M[G]} \leq F(\lambda)$. On the other hand, we have exhibited $F(\lambda)$ subsets of λ for each $\lambda \in A$, and so $M[G] \models 2^\lambda = F(\lambda)$. □

Forcing with a Class of Conditions

We shall now show how to generalize the preceding construction to prove Easton's Theorem in full generality, when the function F is defined for all regular cardinals. This generalization involves forcing with a proper class of conditions. Although it is possible to give a general method of forcing with a class, we shall concentrate only on the particular example.

Thus let M be a transitive model of ZFC + GCH. Moreover, we assume that M has a well-ordering of the universe (e.g., if M satisfies $V = L$). Let F be a function (in M) defined on all regular cardinals and having the properties (15.7)(i)–(iii). We define a class P of forcing conditions as follows: P is the class of all functions p with values 0 and 1, whose domain consists of triples (κ, α, β) where κ is a regular cardinal, $\alpha < \kappa$ and $\beta < F(\kappa)$, and such that for every regular cardinal γ, (15.10) holds, i.e.,

$$|\{(\kappa, \alpha, \beta) \in \mathrm{dom}(p) : \kappa \leq \gamma\}| < \gamma$$

(and p is stronger than q if and only if $p \supset q$).

As before, we define $P^{\leq\lambda}$ and $P^{>\lambda}$ for every regular cardinal λ. Note that $P^{\leq\lambda}$ is a set. To define the Boolean-valued model M^B and the forcing relation, we use the fact that P is the Easton product of P_κ, κ a regular cardinal. For each regular λ, we let $B_\lambda = B(P^{\leq\lambda})$. If $\lambda < \mu$ then the inclusion $P^{\leq\lambda} \subset P^{\leq\mu}$ defines an obvious embedding of B_λ into B_μ; thus we arrange the definition of the B_λ so that B_λ is a complete subalgebra of B_μ whenever $\lambda < \mu$. Then we let $B = \bigcup_\lambda B_\lambda$. B is a proper class; otherwise it has all the features of a complete Boolean algebra. In particular, $\sum X$ exists for every set $X \subset B$. Also, P is dense in B.

To define M^B, we cannot quite use the inductive definition (14.15) since B is not a set. However, we simply let $M^B = \bigcup_\lambda M^{B_\lambda}$; the formal definition of M^B does not present any problem. Similarly, to define $\|x \in y\|$ and $\|x = y\|$, we first notice that if $x, y \in M^{B_\lambda}$ and $\lambda \leq \mu$, then $\|x \in y\|_{B^\lambda} = \|x \in y\|_{B_\mu}$ and so we let $\|x \in y\| = \|x \in y\|_{B_\lambda}$ where λ is such that $x, y \in M^{B_\lambda}$. The same for $\|x = y\|$.

As for the forcing relation in general, we cannot define $\|\varphi\|$ unless φ is Δ_0; this is because $\sum X$ does not generally exist if $X \subset B$ is a class. However, we can still define $p \Vdash \varphi$ using the formulas from Theorem 14.7.

Now, we call $G \subset P$ *generic* over M if (i) $p \supset q$ and $p \in G$ implies $q \in G$, (ii) $p, q \in G$ implies $p \cup q \in G$, and (iii) if D is a *class* in M and D is dense in P, then $D \cap G \neq \emptyset$.

The question of existence of a generic filter can be settled in a more or less the same way as in the case when P is a set. One possible way is to assume that M is a countable transitive model. Then there are only countably many classes in M and G exists. Another possible way is to use the canonical generic ultrafilter. It is the class $\dot{G}$ in M^B defined by $\dot{G}(\check{p}) = p$ for all $p \in P$ (here we need the assumption that M is a class in M^B).

Thus let G be an M-generic filter on P. For every regular λ, $G_\lambda = G \cap P^{\leq\lambda}$ is generic on $P^{\leq\lambda}$. If $\dot{x} \in M^{B_\lambda}$ and $\lambda \leq \mu$, then $\dot{x}^{G_\lambda} = \dot{x}^{G_\mu}$, and so we define $\dot{x}^G = \dot{x}^{G_\lambda}$ where λ is such that $\dot{x} \in M^{B_\lambda}$. Then we define $M[G] = \bigcup_\lambda M[G_\lambda]$.

Using the genericity of G and properties of the forcing relation, we get the Forcing Theorem,

$$M[G] \vDash \varphi(x_1, \ldots, x_n) \quad \text{if and only if} \quad (\exists p \in G)\, p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n) \tag{15.15}$$

where $\dot{x}_1, \ldots, \dot{x}_n \in M^B$ are names for $x_1, \ldots, x_n$.

The formula (15.15) is proved first for atomic formulas and then by induction on φ; in the induction step involving the quantifiers, we use the fact that G intersects every dense class of M.

We shall now show that $M[G]$ is a model of ZFC. The proofs of all axioms of ZFC except Power Set and Replacement go through as when we forced with a set. (Separation also needs some extra work which we leave to the reader.) It is no surprise that the Power Set and Replacement Axioms present problems. It is easy to construct either a class of forcing conditions adding a proper class of Cohen reals, or a class of conditions collapsing *Ord* onto ω (as in the following section). The present proof of the Power Set and Replacement Axioms uses the fact that for every regular λ (or at least for arbitrarily large regular λ), $P = P^{>\lambda} \times P^{\leq\lambda}$ where $P^{>\lambda}$ is λ-closed and $P^{\leq\lambda}$ is a set and satisfies the λ^+-chain condition.

Power Set. Let λ be a regular cardinal. Lemma 15.19 remains true even when applied to $P^{>\lambda} \times P^{\leq\lambda}$. It does not matter that each D_α is a class. The "sequence" of classes $\langle D_\alpha : \alpha < \lambda \rangle$ can be defined (e.g., as a class of pairs $\{(p, \alpha) : p \in D_\alpha\}$) and since $P^{>\lambda}$ is λ-closed, the intersection $\bigcap_{\alpha<\lambda} D_\alpha$ is dense, and there exists $p \in G \cap P^{>\lambda}$ such that $p \in D_\alpha$ for all $\alpha < \lambda$. The rest of the proof of Lemma 15.19 remains unchanged, and thus we have proved that every subset of λ in $M[G]$ is in $M[G]$. Since $P^{\leq\lambda}$ is a set, it follows that the Power Set Axiom holds in $M[G]$.

Replacement. To show that the Axioms of Replacement hold in $M[G]$, we combine the proof for ordinary generic extension with Lemma 15.19. It suffices to prove that if in $M[G]$, $\varphi(\alpha, v)$ defines a function $K : Ord \to M[G]$, then $\{K(\alpha) : \alpha < \lambda\}$ is a set in $M[G]$ for every regular cardinal λ. Without less of generality, let us assume that for every $p \in P$

$$p \Vdash \text{for every } \alpha \text{ there is a unique } v \text{ such that } \varphi(\alpha, v). \tag{15.16}$$

Let λ be a regular cardinal, and let us consider again $P = P^{>\lambda} \times P^{\leq\lambda}$, and $G = (G \cap P^{>\lambda}) \times G_\lambda$. As in Lemma 15.19, let us define, for each $\alpha < \lambda$, a class $D_\alpha \subset P^{>\lambda}$:

$$\begin{aligned} &p \in D_\alpha \text{ if and only if there is a maximal antichain } W \subset P^{\leq\lambda} \\ &\text{and a family } \{\dot{a}^{(\alpha)}_{p,q} : q \in W\} \text{ such that for each } q \in W, \\ &\qquad p \cup q \Vdash \varphi(\alpha, \dot{a}^{(\alpha)}_{p,q}). \end{aligned} \tag{15.17}$$

As in Lemma 15.19, each D_α, $\alpha < \lambda$, is open dense; since $P^{>\lambda}$ is λ-closed, $\bigcap_{\alpha<\lambda} D_\alpha$ is dense and there exists $p \in G \cap P^{>\lambda}$ such that $p \in D_\alpha$ for all $\alpha < \lambda$. We pick (in M) for each $\alpha < \lambda$ a maximal antichain $W_\alpha \subset P^{\leq\lambda}$ and a family $\{\dot{a}^{(\alpha)}_{p,q} : q \in W_\alpha\}$ such that (15.17) holds for each $q \in W_\alpha$. Now, if we let $S = \{\dot{a}^{(\alpha)}_{p,q} : \alpha < \lambda \text{ and } q \in W_\alpha\}$, then it follows that $\{K(\alpha) : \alpha < \lambda\} \subset \{\dot{a}^G : \dot{a} \in S\}$. However, the latter is a set in $M[G]$: There is a γ such that $S \subset M^{B_\gamma}$, and we have $\{\dot{a}^G : \dot{a} \in S\} = \{\dot{a}^{G_\gamma} : \dot{a} \in S\} \in M[G_\gamma]$.

Thus $M[G]$ is a model of ZFC and it remains to show that $M[G]$ has the same cardinals and cofinalities as M, and that in $M[G]$, $2^\kappa = F(\kappa)$ for every regular cardinal κ. However, this is proved exactly the same way as when we forced with a set of Easton conditions. □

We conclude the section with a remark on the Bernays-Gödel axiomatic set theory. If a sentence involving only set variables is provable in BGC = BG + Axiom E, then it is provable in BG + AC. This is a consequence of the following: If M is a transitive model of BG + AC, then there is a generic extension $M[G]$ that has the same sets and has a choice function F defined for all nonempty sets. The forcing conditions $p \in P$ used in the proof are choice functions whose domain is a set of nonempty sets (and $p < q$ means $p \supset q$). The proof that $M[G]$ is a model of BG is rather easy since no new sets are added (P is κ-closed for all κ). The generic filter on P defines a choice function $F = \bigcup G$, and F is defined for all nonempty sets $X \in M[G]$.

The Lévy Collapse

One of the most useful techniques provided by forcing is *collapsing cardinals*. We start with the simplest example:

Example 15.20. Let λ be an uncountable cardinal. Let P be the set of all finite sequences $\langle p(0), \ldots, p(n-1)\rangle$ of ordinals less than λ; p is stronger than q if $p \supset q$.

Let G be a generic filter on P and let $f = \bigcup G$; f is a function with domain ω and range λ. Thus P *collapses* λ: Its cardinality in $V[G]$ is $\aleph_0$.

As $|P| = \lambda$, P satisfies the λ^+-chain condition and so all cardinals greater than λ are preserved (as are all cofinalities greater than λ). □

This construction generalizes to collapsing λ to κ:

Lemma 15.21. *Let κ be a regular cardinal and let $\lambda > \kappa$ be a cardinal. There is a notion of forcing $(P, <)$ that collapses λ onto κ, i.e., λ has cardinality κ in the generic extension. Moreover,*

(i) *every cardinal $\alpha \leq \kappa$ in V remains a cardinal in $V[G]$; and*
(ii) *if $\lambda^{<\kappa} = \lambda$, then every cardinal $\alpha > \lambda$ remains a cardinal in the extension.*

[The condition in (ii) is satisfied if GCH holds and $\operatorname{cf}\lambda \geq \kappa$.]

Proof. Let P be the set of all functions p such that:

(15.18) (i) $\operatorname{dom}(p) \subset \kappa$ and $|\operatorname{dom}(p)| < \kappa$,
(ii) $\operatorname{ran}(p) \subset \lambda$,

and let $p < q$ if and only if $p \supset q$.

Let G be a generic set of conditions and let $f = \bigcup G$. Clearly, f is a function, and it maps κ onto λ.

$(P, <)$ is $<\kappa$-closed and therefore all cardinals $\leq \kappa$ are preserved. If $\lambda^{<\kappa} = \lambda$, then $|P| = \lambda$ and it follows that all cardinals $\geq \lambda^+$ are preserved. □

The following technique collapses all cardinals below an inaccessible cardinal λ while preserving λ, thus making λ a successor cardinal in the generic extension. The forcing notion P defined in (15.19) is called the *Lévy collapse*; we denote $B(P) = \operatorname{Col}(\kappa, <\lambda)$.

Theorem 15.22 (Lévy). *Let κ be a regular cardinal and let $\lambda > \kappa$ be an inaccessible cardinal. There is a notion of forcing $(P, <)$ such that*:

(i) *every α such that $\kappa \leq \alpha < \lambda$ has cardinality κ in $V[G]$; and*
(ii) *every cardinal $\leq \kappa$ and every cardinal $\geq \lambda$ remains a cardinal in $V[G]$.*

Hence $V[G] \vDash \lambda = \kappa^+$.

Proof. For each $\alpha < \lambda$, let P_α be the set of all functions p_α such that $\operatorname{dom}(p_\alpha) \subset \kappa$, $|\operatorname{dom}(p_\alpha)| < \kappa$, and $\operatorname{ran}(p_\alpha) \subset \alpha$; let $p_\alpha < q_\alpha$ if and only if $p_\alpha \supset q_\alpha$.

Let $(P, <)$ be the κ-product of the P_α, $\alpha < \lambda$. Equivalently, the conditions $p \in P$ are functions on subsets of $\lambda \times \kappa$ such that

(15.19) (i) $|\operatorname{dom}(p)| < \kappa$;
(ii) $p(\alpha, \xi) < \alpha$ for each $(\alpha, \xi) \in \operatorname{dom}(p)$.

Let G be a generic set of conditions; for each $\alpha < \lambda$, let G_α be the projection of G on P_α. Then G_α is a generic filter on P_α; and as in Lemma 15.21, the set $f_\alpha = \bigcup G_\alpha$ is a function that maps κ onto α. Thus $V[G] \vDash |\alpha| \leq |\kappa|$, for every $\alpha < \lambda$.

The notion of forcing $(P, <)$ is $<\kappa$-closed and hence it preserves all cardinals and cofinalities $\leq \kappa$. In particular, κ is a cardinal in $V[G]$.

By Theorem 15.17(iii), $(P, <)$ satisfies the λ-chain condition. Hence λ remains a cardinal in $V[G]$, and so do all cardinals greater than λ. It follows that in $V[G]$, λ is the cardinal successor of κ. □

Suslin Trees

One of the earliest applications of forcing was the solution of Suslin's Problem: The existence of a Suslin line is independent of ZFC. In this section we show how to construct a Suslin tree by forcing and in L; in Chapter 16 we will construct a generic model in which there are no Suslin trees.

Theorem 15.23. *There is a generic extension in which there exists a Suslin tree.*

Proof. Let P be the collection of all countable normal trees, i.e., all T such that for some $\alpha < \omega_1$,

(15.20)
(i) each $t \in T$ is a function $t : \beta \to \omega$ for some $\beta < \alpha$;
(ii) if $t \in T$ and s is an initial segment of t then $s \in T$;
(iii) if $\beta + 1 < \alpha$ and $t : \beta \to \omega$ is in T, then $t^\frown n \in T$ for all $n \in \omega$;
(iv) if $\beta < \alpha$ and $t : \beta \to \omega$ is in T, then for every γ such that $\beta \leq \gamma < \alpha$ there exists an $s : \gamma \to \omega$ in T such that $t \subset s$;
(v) $T \cap \omega^\beta$ is at most countable for all $\beta < \alpha$.

(See (9.9) and Exercise 9.6.) T_1 is stronger than T_2 if T_1 is an extension of T_2, i.e.,

(15.21) $\quad T_1 < T_2 \quad$ if and only if $\quad \exists \alpha < \text{height}(T_1)\ T_2 = \{t \restriction \alpha : t \in T_1\}$.

Let G be a generic set of conditions and let $\mathcal{T} = \bigcup\{T : T \in G\}$. We shall show that in $V[G]$, $\mathcal{T}$ is a normal Suslin tree.

First we note that if T_1 and T_2 are two conditions, then either one is an extension of the other, or T_1 and T_2 are incompatible. Thus G consists of pairwise comparable trees and one can easily verify that $\mathcal{T}$ is a normal tree (of height $\leq \omega_1$).

If $T_0, T_1, \ldots, T_n, \ldots$ is a sequence of conditions such that for each n, T_{n+1} is an extension of T_n, then $\bigcup_{n=0}^{\infty} T_n$ is a normal countable tree (and extends each T_n). Hence P is $\aleph_0$-closed, and consequently, the cardinal $\aleph_1$ is preserved (and $V[G]$ has the same countable sequences in V as V).

To show that the height of $\mathcal{T}$ is ω_1, we verify that for every $\alpha < \omega_1$, G contains a condition T of height at least α. We show that the set $\{T \in P : \text{height}(T) \geq \alpha\}$ is dense in P, for any $\alpha < \omega_1$. In other words, we show that for each $T_0 \in P$ and each $\alpha < \omega_1$, there is an extension $T \in P$ of T_0, of height at least α. It suffices to show that each $T_0 \in P$ has an extension $T \in P$ that has one more level; for then we can proceed by induction and take unions at limit steps.

If height(T_0) is a successor ordinal, then an extension of T_0 is easily obtained. If height(T_0) is a limit ordinal, then we first observe that for each $t \in T_0$ there exists a branch b of length α in T_0 such that $t \in b$: Using an increasing sequence $\alpha_0 < \alpha_1 < \ldots < \alpha_n \ldots$ with limit α, we use the normality

condition (15.20)(iv) to obtain such a branch. Now we construct an extension T of T_0, of height $\alpha + 1$, as follows: For each $t \in T_0$, we pick a branch b_t of length α in T_0 such that $t \in b_t$, and let $T = T_0 \cup \{s : s = \bigcup b_t \text{ for some } t \in T\}$ (we extend all the branches b_t, $t \in T_0$, of T_0). Since T_0 is countable the added level is countable, and one can verify that $T \in P$.

It remains to show that $\mathcal{T}$ has no uncountable antichain. If T is a tree and A is an antichain in T, then A is called a *maximal antichain* if there is no antichain A' in T such that $A' \supset A$: Each $t \in T$ is comparable with some $a \in A$. If A is a maximal antichain in T and if T' is an extension of T, then A is not necessarily maximal in T. Let us call a set $S \subset T$ *bounded* in T if there is some $\alpha < \text{height}(T)$ such that all elements of S are at levels $\leq \alpha$. (If the height of T is a successor ordinal, then every $S \subset T$ is bounded.)

Lemma 15.24. *If A is a maximal antichain in a normal tree T and if A is bounded in T (in particular, if the height of T is a successor ordinal), then A is maximal in every extension of T.*

Proof. Let T' be an extension of T. Let $\alpha < \text{height}(T)$ be such that each $a \in A$ is at level $\leq \alpha$. If $t' \in T' - T$, then there exists $t \in T$ at level α such that $t \subset t'$; in turn, there exists $a \in A$ such that $a \subset t$. Hence t' is comparable with some $a \in A$. □

Lemma 15.25. *Let α be a countable limit ordinal, let $T \in P$ be a normal α-tree and let A be a maximal antichain in T. Then there exists an extension $T' \in P$ of T of height $\alpha + 1$ such that A is a maximal antichain in T' (and hence A is a bounded maximal antichain in T').*

Proof. For each $t \in T$ there exists $a \in A$ such that either $t \subset a$ or $a \subset t$. In either case, there exists a branch $b = b_t$ of length α in T such that $t \in b$ and $a \in b$. Let T' be the extension of T obtained by extending the branches b_t, for all $t \in T$: $T' = T \cup \{\bigcup b_t : t \in T\}$. The tree T' is a normal $(\alpha + 1)$-tree and extends T; moreover, since every $s \in T'$ is comparable with some $a \in A$, A is maximal in T'. □

Now we finish the proof of Theorem 15.23 by showing that in $V[G]$, every antichain in $\mathcal{T}$ is countable. Since every antichain can be extended to a maximal antichain, it suffices to show that every maximal antichain is countable. Thus let A be a maximal antichain in $\mathcal{T}$. There is a name $\dot{A}$ for A and a condition $T \in G$ such that

$$T \Vdash \dot{A} \text{ is a maximal antichain in } \mathcal{T}.$$

We will show that the following set of conditions is dense below T:

$$D = \{T' \leq T : \text{there is a bounded maximal antichain } A' \text{ in } T' \text{ such that } T' \Vdash A' \subset \dot{A}\}.$$

Then some $T' \in D$ is in G and there is a bounded maximal antichain A' in T' such that $A' \subset A$. However, $\mathcal{T}$ is an extension of T', and by Lemma 15.24,

A' is maximal in $\mathcal{T}$. Consequently, $A = A'$, and since A' is countable, we are done.

To show that D is dense below T let $T_0 \leq T$ be arbitrary. We shall construct a tree $T' \leq T_0$ such that $T' \in D$. Since $T_0 \Vdash (\dot{A}$ is a maximal antichain in $\mathcal{T}$ and $\mathcal{T}$ is an extension of $T_0)$, there exist for each $s \in T_0$ an extension T_0' of T_0 and some $t_s \in T_0'$ such that

$$s \text{ and } t_s \text{ are comparable} \quad \text{and} \quad T_0' \Vdash t_s \in \dot{A}. \tag{15.22}$$

Since T_0 is countable, we repeat this countably many times and obtain an extension $T_0' < T_0$ such that (15.22) holds for every $s \in T_0$. Let $T_1 = T_0'$. Then we proceed by induction and construct a sequence of trees $T_0 \geq T_1 \geq \ldots \geq T_n \geq \ldots$ such that for each n, T_{n+1} extends T_n and

$$(\forall s \in T_n)(\exists t_s \in T_{n+1})\ s \text{ and } t_s \text{ are comparable and } T_{n+1} \Vdash t_s \in \dot{A}. \tag{15.23}$$

We let $T_\infty = \bigcup_{n=0}^{\infty} T_n$, and $A' = \{t_s : s \in T_\infty\}$. By (15.23), A' is a maximal antichain in T_∞, and $T_\infty \Vdash A' \subset \dot{A}$. Now we apply Lemma 15.25 and get an extension T' of T such that A' is a bounded maximal antichain in T'. Clearly, $T' \Vdash A' \subset \dot{A}$, and hence $T' \in D$. □

In the Exercises (15.21 and 15.22) we present another forcing notion (with finite conditions) that produces a Suslin tree. Later in the book we show that the forcing that adds a Cohen real also adds a Suslin tree.

The following theorem shows that a Suslin tree exists in L.

Theorem 15.26 (Jensen). *If $V = L$ then there exists a Suslin tree.*

Proof. We shall prove that the Diamond Principle $\diamondsuit$ implies that a Suslin tree exists. First we make the following observation. If T is a normal ω_1-tree, let $T_\alpha = \{x \in T : o(x) < \alpha\}$.

Lemma 15.27. *If A is a maximal antichain in T, then the set*

$$C = \{\alpha : A \cap T_\alpha \text{ is a maximal antichain in } T_\alpha\}$$

is closed unbounded.

Proof. It is easy to see that C is closed. To show that C is unbounded, let $\alpha_0 < \omega_1$ be arbitrary. Since T_{α_0} is countable, there exists a countable ordinal $\alpha_1 > \alpha_0$ such that every $t \in T_{\alpha_0}$ is compatible with some $a \in A \cap T_{\alpha_1}$. Then there is $\alpha_2 > \alpha_1$ such that each $t \in T_{\alpha_1}$ is comparable with some $a \in A \cap T_{\alpha_2}$, etc. If $\alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \ldots$ is constructed in this way and if $\alpha = \lim_n \alpha_n$, then $A \cap T_\alpha$ is a maximal antichain in T_α. □

We now use $\diamondsuit$ to construct a normal Suslin tree $(T, <_T)$. We proceed by induction on levels. To facilitate the use of $\diamondsuit$, we let points of T be countable

ordinals, $T = \omega_1$, and in fact each T_α (the first α levels of T) is an initial segment of ω_1.

We construct T_α, $\alpha < \omega_1$, such that each T_α is a normal α-tree and such that T_β extends T_α whenever $\beta > \alpha$. T_1 consists of one point. If α is a limit ordinal, then $(T_\alpha, <_T)$ is the union of the trees $(T_\beta, <_T)$, $\beta < \alpha$. If α is a successor ordinal, then $(T_{\alpha+1}, <_T)$ is an extension of $(T_\alpha, <_T)$ obtained by adjoining infinitely immediate successors to each x at the top level of T_α.

It remains to describe the construction of $T_{\alpha+1}$ if α is a limit ordinal. Let $\langle S_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. If S_α is a maximal antichain in $(T_\alpha, <_T)$, then we use Lemma 15.25 and find an extension $(T_{\alpha+1}, <_T)$ of T_α such that S_α is maximal in $T_{\alpha+1}$. Otherwise, we let $T_{\alpha+1}$ be any extension of T_α that is a normal $(\alpha+1)$-tree. (In either case, we let the set $T_{\alpha+1}$ be an initial segment of countable ordinals.)

We shall now show that the tree $T = \bigcup_{\alpha<\omega_1} T_\alpha$ is a normal Suslin tree. It suffices to verify that T has no uncountable antichain. If $A \subset T$ $(= \omega_1)$ is a maximal antichain in T, then by Lemma 15.27, $A \cap T_\alpha$ is a maximal antichain in T_α, for a closed unbounded set of α's. It follows that easily from the construction that for a closed unbounded set of α's, $T_\alpha = \alpha$. Thus using the Diamond Principle, we find a limit ordinal α such that $A \cap \alpha = S_\alpha$ and $A \cap \alpha$ is a maximal antichain in T_α. However, we constructed $T_{\alpha+1}$ in such a way that $A \cap \alpha$ is maximal in $T_{\alpha+1}$, and therefore in T. It follows that $A = A \cap \alpha$ and so A is countable. □

Suslin trees are a fruitful source of counterexamples in set-theoretic topology as well as in the theory of Boolean algebras. As an example, let $(T, <)$ be a Suslin tree, and consider the partial ordering $(P_T, <) = (T, >)$. Any two elements of T are incomparable in T if and only if they are incompatible in P_T. Thus P_T satisfies the countable chain condition.

Lemma 15.28. *If T is a normal Suslin tree, then P_T is $\aleph_0$-distributive.*

Proof. Let D_n, $n = 0, 1, 2 \ldots$, be open dense subsets of P_T. We shall prove that $\bigcap_{n=0}^{\infty} D_n$ is dense in P_T. First we claim that if $D \subset P_T$ is open dense, then there is an $\alpha < \omega_1$ such that D contains all levels of T above α. To prove this, let A be a maximal antichain in D. A is an antichain in T and hence countable. Thus let $\alpha < \omega_1$ be such that all $a \in A$ are below level α. Now if $x \in T$ is at level $\geq \alpha$, x is comparable with some $a \in A$ (by maximality of A), and hence $a \leq_T x$. Since D is open, we have $x \in D$.

Now if D_n, $n = 0, 1, \ldots$, are open dense, we pick countable ordinals α_n such that D_n contains all levels of T above α_n; and since T is normal, this implies that $\bigcap_{n=0}^{\infty} D_n$ is dense in P_T. □

Corollary 15.29. *If T is a normal Suslin tree, then $B = B(P_T)$ is an $\aleph_0$-distributive, c.c.c., atomless, complete Boolean algebra.* □

Random Reals

Consider the notion of forcing where forcing conditions are Borel sets of reals of positive Lebesgue measure; a condition p is stronger than q if $p \subset q$. The corresponding complete Boolean algebra is $\mathcal{B}/I_\mu$ where $\mathcal{B}$ is the σ-algebra of all Borel sets of reals and I_μ is the σ-ideal of all null sets. As I_μ is σ-saturated, $\mathcal{B}/I_\mu$ satisfies the countable chain condition, and hence the forcing preserves cardinals.

The generic extension $V[G]$ is determined by a single real, called a *random real*. Let $a \in \boldsymbol{R}^{V[G]}$ be the unique member of each rational interval $[r_1, r_2]^{V[G]}$ such that $[r_1, r_2]^V \in G$. Conversely, G can be defined from a, and so $V[G] = V[a]$. (see Exercise 13.34 for the meaning of $V[a]$.)

The following lemma illustrates one of the differences between random and generic reals. If f and g are functions from ω to ω we say that g *dominates* f if $f(n) < g(n)$ for all n.

Lemma 15.30. (i) *In the random real extension $V[G]$, every $f : \omega \to \omega$ is dominated by some $g \in V$.*

(ii) *In the Cohen real extension $V[G]$, there exists a function $f : \omega \to \omega$ that is not dominated by any $g \in V$.*

Proof. (i) Forcing conditions are Borel sets of positive measure, and we freely confuse them with their equivalence classes in $\mathcal{B}/I_\mu$.

Let $p \Vdash \dot{f} : \omega \to \omega$; we shall find a $q < p$ and some $g : \omega \to \omega$ such that q forces that g dominates $\dot{f}$. For each n, let $g(n)$ be sufficiently large, so that

$$\mu(p - \|\dot{f}(n) < g(n)\|) < \frac{1}{2^n} \cdot \frac{1}{4} \cdot \mu(p).$$

The Borel set $q = p \cap \bigcap_{n=0}^{\infty} \|\dot{f}(n) < g(n)\|$ has measure at least $\mu(p)/2$, and forces $\forall n\, \dot{f}(n) < g(n)$.

(ii) We use the following variant of Cohen forcing: Forcing conditions are finite sequences $\langle p(0), \ldots, p(n-1)\rangle$ of natural numbers, and $p < q$ if and only if $p \supset q$. (This forcing produces the same generic extension—and has the same $B(P)$—as the forcing from Example 14.2).

Let $\dot{f}$ be the name for the function $f = \bigcup G$. If p is any condition and $g : \omega \to \omega$ is in V, then there exist a stronger $q \supset p$ and some $n \in \mathrm{dom}(q)$ such that $q(n) > g(n)$. It follows that q forces $g(n) > \dot{f}(n)$ (because $q \Vdash \dot{f}(n) = q(n)$). □

To add a large number of random reals, we use *product measure*:

Example 15.31. Let κ be an infinite cardinal and let $I = \kappa \times \omega$. Let $\Omega = \{0,1\}^I$. Let T be the set of all finite 0–1 functions with $\mathrm{dom}(t) \subset I$. Let $\mathcal{S}$ be the σ-algebra generated by the sets S_t, $t \in T$, where $S_t = \{f \in \Omega : t \subset f\}$. The product measure on $\mathcal{S}$ is the unique σ-additive measure such that each S_t has measure $1/2^{|t|}$. Let $B = \mathcal{S}/I$ where I is the ideal of measure 0 sets.

If G is a generic ultrafilter on B then $f = \bigcup\{t : S_t \in G\}$ is a 0–1 function on I, and for each $\alpha < \kappa$, we define $f_\alpha(n) = f(\alpha, n)$, for all $n < \omega$. The f_α, $\alpha < \kappa$, are κ-distinct random reals, and the continuum in $V[G]$ has size at least κ. But since $|B|^{\aleph_0} = \kappa^{\aleph_0}$, we have $(2^{\aleph_0})^{V[G]} = \kappa^{\aleph_0}$. □

Forcing with Perfect Trees

This section describes forcing with perfect trees (due to Gerald Sacks) that produces a real of *minimal degree* of constructibility. If forced over L, the generic filter yields a real a such that $a \notin L$ and such that for every real $x \in L[a]$, either $x \in L$ or $a \in L[x]$.

Let $Seq(\{0,1\})$ denote the set of all finite 0–1 sequences. A *tree* is a set $T \subset Seq(\{0,1\})$ that satisfies

(15.24) if $t \in T$ and $s = t\restriction n$ for some n, then $s \in T$.

A nonempty tree T is *perfect* if for every $t \in T$ there exists an $s \supset t$ such that both $s^\frown 0$ and $s^\frown 1$ are in T. (Compare with (4.4) and Lemma 4.11.) The set of all paths in a perfect tree is a perfect set in the Cantor space $\{0,1\}^\omega$.

Definition 15.32 (Forcing with Perfect Trees). Let P be the set of all perfect trees $p \subset Seq(\{0,1\})$; p is stronger than q if and only if $p \subset q$.

If G is a generic set of perfect trees, let

(15.25) $$f = \bigcup\{s : (\forall p \in G)\, s \in p\}.$$

The function $f : \omega \to \{0,1\}$ is called a *Sacks real*. Note that $V[G] = V[f]$. Since $|P| = 2^{\aleph_0}$, if we assume CH in the ground model, P satisfies the $\aleph_2$-chain condition and all cardinals $\geq \aleph_2$ are preserved. We prove below that $\aleph_1$ is preserved as well.

Definition 15.33. A generic filter G is *minimal* over the ground model M if for every set of ordinals X in $M[G]$, either $X \in M$ or $G \in M[X]$.

Theorem 15.34 (Sacks). *When forcing with perfect trees, the generic filter is minimal over the ground model.*

The proof uses the technique of *fusion*. Let p be a perfect tree. A node $s \in p$ is a *splitting node* if both $s^\frown 0 \in p$ and $s^\frown 1 \in p$; a splitting node s is an nth splitting node if there are exactly n splitting nodes t such that $t \subset s$. (A perfect tree has 2^{n-1} nth splitting nodes.) For each $n \geq 1$, let

(15.26) $p \leq_n q$ if and only if $p \leq q$ and every nth splitting node of q is an nth splitting node of p.

A *fusion sequence* is a sequence of conditions $\{p_n\}_{n=0}^\infty$ such that $p_n \leq_n p_{n-1}$ for all $n \geq 1$. The following is the key property of fusion sequences:

Lemma 15.35. *If $\{p_n\}_{n=0}^{\infty}$ is a fusion sequence then $\bigcap_{n=0}^{\infty} p_n$ is a perfect tree.* □

If s is a node in p, let $p{\restriction}s$ denote the tree $\{t \in p : t \subset s \text{ or } t \supset s\}$. If A is a set of incompatible nodes of p and for each $s \in A$, q_s is a perfect tree such that $q_s \subset p{\restriction}s$, then the *amalgamation* of $\{q_s : s \in A\}$ into p is the perfect tree

$$\{t \in p : \text{if } t \supset s \text{ for some } s \in A \text{ then } t \in q_s\}. \tag{15.27}$$

(Replace in p each $p{\restriction}s$ by q_s.)

Proof of Theorem 15.34. Let $\dot{X}$ be a name for a set of ordinals and let $p \in P$ be a condition that forces $\dot{X} \notin V$; no stronger condition forces $\dot{X} = A$, for any $A \in V$. We shall find a condition $q \leq p$ and a set of ordinals $\{\gamma_s : s \text{ is a splitting node of } q\}$ such that $q_{s^\frown 0}$ and $q_{s^\frown 1}$ decide $\gamma_s \in \dot{X}$, but in opposite ways. Then the generic branch (15.25) can be recovered from $\dot{X}^G$, and so $V[\dot{X}^G] = V[G]$.

To construct q and $\{\gamma_s\}_s$ we build a fusion sequence $\{p_n\}_{n=0}^{\infty}$ as follows: Let $p_0 = p$. For each $n \geq 1$, let S_n be the set of all nth splitting nodes of p_{n-1}. For each $s \in S_n$, let γ_s be an ordinal such that $p_{n-1}{\restriction}s$ does not decide $\gamma_s \in \dot{X}$, and let $q_{s^\frown 0} \leq p_{n-1}{\restriction}s^\frown 0$ and $q_{s^\frown 1} \leq p_{n-1}{\restriction}s^\frown 1$ be conditions that decide $\gamma_s \in \dot{X}$ in opposite ways. Then let p_n be the amalgamation of $\{q_{s^\frown i} : s \in S_n$ and $i = 0, 1\}$ into p_{n-1}. Clearly, $p_n \leq_n p_{n-1}$, and so $\{p_n\}_{n=0}^{\infty}$ is a fusion sequence. Then we set $q = \bigcap_{n=0}^{\infty} p_n$. □

A similar argument shows that forcing with perfect trees preserves $\aleph_1$:

Lemma 15.36. *If X is a countable set of ordinals in $V[G]$ then there exists a set $A \in V$, countable in V, such that $X \subset A$.*

Proof. Let $\dot{F}$ be a name and let $p \in P$ be such that p forces "$\dot{F}$ is a function from ω into the ordinals." We build a fusion sequence $\{p_n\}_{n=0}^{\infty}$ with $p_0 = p$ as follows: For each $n \geq 1$, let S_n be the set of all nth splitting nodes of p_{n-1}. For each $s \in S_n$, let $q_{s^\frown 0}$, $q_{s^\frown 1}$, $a_{s^\frown 0}$, $a_{s^\frown 1}$ be such that (for $i = 0, 1$) $q_{s^\frown i} \leq p_{n-1}{\restriction}s^\frown i$ and $q_{s^\frown i} \Vdash \dot{F}(n-1) = a_{s^\frown i}$. Let p_n be the amalgamation of $\{q_{s^\frown i} : s \in S_n$ and $i = 0, 1\}$. Then let $q = \bigcap_{n=0}^{\infty} p_n$, and

$$A = \bigcup_{n=0}^{\infty} \{a_{s^\frown i} : s \in S_n \text{ and } i = 0, 1\}.$$

It follows that $q \Vdash \operatorname{ran}(\dot{F}) \subset A$. □

More on Generic Extensions

Properties of a generic extensions are determined by properties of the forcing notion that constructs it. For instance, if P satisfies the countable chain condition then $V[G]$ preserves cardinals. Or, if P is ω-distributive then $V[G]$ has

no new countable sets of ordinals. But since the model $V[G]$ is determined by the complete Boolean algebra $B(P)$, its properties depend on properties of the algebra. Below we illustrate the correspondence between properties of a complete Boolean algebra B and truth in the model V^B.

The first example shows the importance of distributivity.

Let κ and λ be cardinals. A complete Boolean algebra B is (κ,λ)-distributive if

$$\prod_{\alpha<\kappa}\sum_{\beta<\lambda} u_{\alpha,\beta} = \sum_{f:\kappa\to\lambda}\prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}. \tag{15.28}$$

Note that (15.28) is a special case of (7.28); B is κ-distributive if and only if it is (κ,λ)-distributive for all λ. As in Lemma 7.16 we can reformulate (κ,λ)-distributivity as follows:

Lemma 15.37. *B is (κ,λ)-distributive if and only if every collection of κ partitions of B of size at most λ has a common refinement.* □

Theorem 15.6 and Exercise 15.5 yield the following equivalence:

Theorem 15.38. *B is (κ,λ)-distributive if and only if every $f:\kappa\to\lambda$ in the generic extension by B is in the ground model.*

Proof. If $\|\dot f$ is a function from κ to $\lambda\| = 1$, then $\{\|\dot f(\alpha)=\beta\| : \beta<\lambda\}$ is a partition of B of size $\le\lambda$. □

Exercises 15.31 and 15.32 give short proofs of Boolean algebraic results using generic extensions.

A related concept is weak distributivity: B is called *weakly (κ,λ)-distributive*, if

$$\prod_{\alpha<\kappa}\sum_{\beta<\lambda} u_{\alpha,\beta} = \sum_{g:\kappa\to\lambda}\prod_{\alpha<\kappa}\sum_{\beta<g(\alpha)} u_{\alpha,\beta}. \tag{15.29}$$

A modification of Theorem 15.38 gives this:

Lemma 15.39. *B is weakly (κ,λ)-distributive if and only if every $f:\kappa\to\lambda$ in $V[G]$ is dominated by some $g:\kappa\to\lambda$ that is in V (i.e., $f(\alpha)<g(\alpha)$ for all $\alpha<\kappa$).* □

Consequently, by Lemma 15.30(i), the measure algebra $\mathcal{B}/I_\mu$ is weakly (ω,ω)-distributive.

Let B be a complete Boolean algebra and let D be a complete subalgebra of B. If G is generic on B, then it is easy to see that $G\cap D$ is generic on D, and so $V[G\cap D]$ is a model of ZFC, and $V\subset V[G\cap D]\subset V[G]$. We shall prove that every model of ZFC between V and $V[G]$ is obtained this way, and that for every subset A of V in $V[G]$ there is a complete subalgebra D of B such that $V[G\cap D]=V[A]$.

We recall (cf. Chapter 7) that a complete subalgebra B of a complete Boolean algebra D is (completely) generated by a set $X \subset D$ if B is the smallest complete subalgebra of D such that $X \subset B$. Let κ be a cardinal. We say that a complete Boolean algebra B is *κ-generated* if there exists some $X \subset B$ of size at most κ such that the complete subalgebra of B generated by X is equal to B.

Lemma 15.40. *Let X be a subset of a complete Boolean algebra B such that B is completely generated by X. Then for every generic G on B, $V[G] = V[X \cap G]$.*

Proof. We want to show that $V[G]$ is the least model such that the set $A = X \cap G$ is in $V[G]$. It suffices to show that G can be defined in terms of A.

Since B is generated by X, every element of B can be obtained from the elements of X by successive (transfinite) application of the operation $-$ and $\sum$. Thus let X_α be subsets of B defined recursively as follows:

$$X_0 = X, \qquad \overline{X_\alpha} = \{-a : a \in X_\alpha\}, \quad \text{and}$$
$$X_\alpha = \{a : a = \textstyle\sum Z \text{ where } Z \subset \bigcup_{\beta<\alpha}(X_\beta \cup \overline{X_\beta})\}.$$

Then $B = \bigcup_{\alpha<\theta} X_\alpha$ for some $\theta \leq |B|^+$. If we denote $G_\alpha = G \cap \overline{X_\alpha}$, $\overline{G_\alpha} = G \cap \overline{X_\alpha}$, we have

$$\begin{aligned} G_0 &= A, \qquad \overline{G_\alpha} = \{-a : a \in X_\alpha - G_\alpha\}, \quad \text{and} \\ G_\alpha &= \{a \in X_\alpha : a = \textstyle\sum Z \text{ where } Z \text{ contains at least one } b \\ &\qquad\qquad\qquad\qquad \text{in some } G_\beta \text{ or } \overline{G_\beta},\ \beta < \alpha\}, \end{aligned} \tag{15.30}$$

and $G = \bigcup_{\alpha<\theta} G_\alpha$. Thus given A, we define G_α and $\overline{G_\alpha}$ inductively using (15.30) and let $G = \bigcup_{\alpha<\theta} G_\alpha$. □

Corollary 15.41. *If B is κ-generated, then $V[G] = V[A]$ for some $A \subset \kappa$.* □

Corollary 15.42. *If G is generic on B and $A \in V[G]$ is a subset of κ, then there exists a κ-generated complete subalgebra D of B such that $V[D \cap G] = V[A]$ for some $A \subset \kappa$.*

Proof. Let $\dot{A}$ be a name for A. We let $X = \{u_\alpha : \alpha < \kappa\}$, where $u_\alpha = \|\check{\alpha} \in A\|$. Now let D be the complete subalgebra completely generated by X; by Lemma 15.40 we have $V[X \cap G] = V[D \cap G]$. It remains to show that $V[X \cap G] = V[A]$.

On the one hand, we have $A = \{\alpha : u_\alpha \in X \cap G\}$. On the other hand, $X \cap G = \{u_\alpha : \alpha \in A\}$. □

Lemma 15.43. *Let G be generic on B. If M is a model of* ZFC *such that $V \subset M \subset V[G]$, then there exists a complete subalgebra $D \subset B$ such that $M = V[D \cap G]$.*

Proof. We show that $M = V[A]$, where A is a set of ordinals. Then the lemma follows from Corollary 15.42. First we note that since M satisfies the Axiom of Choice, there exists for every $X \in M$ a set of ordinals $A_X \in M$ such that $X \in V[A_X]$. We let $Z = P(B) \cap M$, and let $A = A_Z$; we claim that $M = V[A]$.

If $X \in M$, consider the set of ordinals A_X; by Corollary 15.42 there exists a subalgebra $D_X \subset B$ such that $V[A_X] = V[D_X \cap G]$. Hence $D_X \cap G \in M$, and we have $D_X \cap G \in Z$. Since $Z \in V[A]$, it follows that $D_X \cap G \in V[A]$ and hence $X \in V[A]$. Thus $M = V[A]$. □

Let us now address the question under what conditions one generic extension embeds (as a submodel) into another generic extension. Of course, if $B(P) = B(Q)$, then $V^P = V^Q$ and if $B(P)$ is a complete subalgebra of $B(Q)$ then $V^P \subset V^Q$. But if B_1 is a complete subalgebra of B_2, we can have $V[G \cap B_1] = V[G]$ even if $B_1 \neq B_2$. For every $a \in B_2^+$ (not necessarily in B_1), let $B_1 \restriction a = \{x \cdot a : x \in B_1\}$. Now assume that the set $\{a \in B_2^+ : B_1 \restriction a = B_2 \restriction a\}$ is dense in B_2. Then it is easy to see that $V[G \cap B_1] = V[G]$, for every generic G on B_2. (One can show that this condition is also necessarily for B_1 to give the same generic extension as B_2.)

By $V^P \subset V^Q$ we mean the following: Whenever G is a generic filter on Q then there is some $H \in V[G]$ that is a generic filter on P. In practice there are several ways how to verify $V^P \subset V^Q$. The following two lemmas are sometimes useful:

Lemma 15.44. *Let $i : P \to Q$ be such that*

(i) *if $p_1 \leq p_2$ then $i(p_1) \leq i(p_2)$,*
(ii) *if p_1 and p_2 are incompatible then $i(p_1)$ and $i(p_2)$ are incompatible,*
(iii) *for every $q \in Q$ there is a $p \in P$ such that for all $p' \leq p$, $i(p')$ is comparable with q.*

Then $V^P \subset V^Q$.

Proof. If G is generic on Q then $i^{-1}(G)$ is generic on P. □

Lemma 15.45. *Let $h : Q \to P$ be such that*

(i) *if $q_1 \leq q_2$ then $h(q_1) \leq h(q_2)$.*
(ii) *for every $q \in Q$ and every $p \leq h(q)$ there exists a q' compatible with q such that $h(q') \leq p$.*

Then $V^P \subset V^Q$.

Proof. If $D \subset P$ is open dense then $h^{-1}(D)$ is predense in Q. It follows that if G is generic on Q then $\{p \in P : p \geq h(q) \text{ for some } q \in G\}$ is generic on P. □

We conclude this section with the following result that shows that for every set A of ordinals, the model $L[A]$ is a generic extension of *HOD*:

Theorem 15.46 (Vopěnka). *Let $V = L[A]$ where A is a set of ordinals. Then V is a generic extension of the model HOD. There is a Boolean algebra $B \in HOD$ complete in HOD, and there is an ultrafilter $G \subset B$, generic over HOD, such that $V = HOD[G]$.*

Proof. Let κ be such that $A \subset \kappa$. We let $C = OD \cap P(P(\kappa))$ be the family of all ordinal definable sets of subsets of κ. Let us consider the partial ordering $(C, \subset)$.

First we claim that there is a hereditarily ordinal definable partially ordered set $(B, \leq)$ and an ordinal definable isomorphism π between $(C, \subset)$ and $(B, \leq)$: There is a definable one-to-one mapping F of OD into the ordinals. The set C is an ordinal definable set of ordinal definable sets and so $F \restriction C$ is an OD one-to-one mapping of C onto $F(C)$. We let $B = F(C)$, and define the partial ordering of B so that $(B, \leq)$ is isomorphic to $(C, \subset)$. Since $\subset \cap C^2$ is an OD relation, we have $(B, \leq) \in HOD$.

Now $(C, \subset)$ is clearly a Boolean algebra. Moreover, if $X \subset C$ is ordinal definable, then $\bigcup X$ is ordinal definable and so $\bigcup X = \sum^C X$. Hence the algebra C is OD-complete; and using the OD isomorphism π, we can conclude that $(B, \leq)$ is a complete Boolean algebra in HOD.

Now we let $H = \{u \in C : A \in u\}$. Clearly, H is an ultrafilter on C, and if $X \subset H$ is OD, then $\bigcap X \in H$. Hence $G = \pi(H)$ is an HOD-generic ultrafilter on B.

It remains to show that $V = HOD[G]$. Let $f : \kappa \to B$ be the function defined by $f(\alpha) = \pi(\{Z \subset \kappa : \alpha \in Z\})$. Clearly, f is OD, and so $f \in HOD$. Now we note that for every $\alpha < \kappa$, $\alpha \in A$ if and only if $f(\alpha) \in G$ and therefore $A \in HOD[G]$. It follows that $V = L[A] = HOD[G]$. □

Symmetric Submodels of Generic Models

In Chapter 14 we constructed a model of set theory in which the reals cannot be well-ordered, thus showing that the Axiom of Choice is independent of the axioms of ZF. What follows is a more systematic study of models in which the Axiom of Choice fails. We shall present a general method of construction of submodels of generic extensions. The construction uses symmetry arguments similar to those used in Theorem 14.36, and the models obtained are generally models of ZF and do not satisfy the Axiom of Choice. This method has been used to obtain a number of results about the relative strength of various weaker versions and consequences of the Axiom of Choice.

The main idea of the construction of symmetric models is the use of automorphisms of the Boolean-valued model V^B and the Symmetry Lemma 14.37. In fact, the idea of using automorphisms of the universe to show that the Axiom of Choice is unprovable dates back into the preforcing era of set theory. We shall describe this older construction first.

In order to describe this method, we introduce the theory ZFA, *set theory with atoms*. In addition to sets, ZFA has additional objects called *atoms*. These atoms do not have any elements themselves but can be collected into sets. Obviously, we have to modify the Axiom of Extensionality, for any two atoms have the same elements—none.

The language of ZFA has, in addition to the predicate $\in$, a constant A. The elements of A are called *atoms*; all other objects are sets. The axioms of ZFA are the axioms 1.1–1.8 of ZF plus (15.31) and (15.32):

(15.31) *If $a \in A$, then there is no x such that $x \in a$.*

The Axiom of Extensionality takes this form:

(15.32) *If two sets X and Y have the same elements, then $X = Y$.*

Other axioms of ZF remain unchanged. In particular, the Axiom of Regularity states that every nonempty set has an $\in$-minimal element. This minimal element may be an atom.

The effect of atoms is that the universe is no longer obtained by iterated power set operation from the empty set. In ZFA, the universe is built up from atoms.

Ordinal numbers are defined as usual except that one has to add that an ordinal does not contain any atom. For any set S, let us define the following cumulative hierarchy:

(15.33)
$$\begin{aligned} P^0(S) &= S, \\ P^\alpha(S) &= \bigcup_{\beta<\alpha} P^\beta(S) \quad \text{if } \alpha \text{ is limit}, \\ P^{\alpha+1}(S) &= P^\alpha(S) \cup P(P^\alpha(S)), \\ P^\infty(S) &= \bigcup_{\alpha \in Ord} P^\alpha(S). \end{aligned}$$

It follows that $V = P^\infty(A)$, and that the *kernel*, the class $P^\infty(\emptyset)$ of "hereditary" sets, is a model of ZF. If A is empty, then we have just ZF.

Lemma 15.47. *The theory* ZFA + AC + "*A is infinite*" *is consistent relative to* ZFC.

Proof. Construct a model of ZFA. Let C be an infinite set of sets of the same rank (so that $X \notin \mathrm{TC}(Y)$ for any $X, Y \in C$). Consider one $X_0 \in C$ as the empty set, and all other $X \in C$ as atoms. Build up the model from C by iterating the operation $P^*(Z) = P(Z) - \{\emptyset\}$. □

While in ZF, the universe does not admit nontrivial automorphisms, the important feature of ZFA is that every permutation of atoms induces an

automorphism of V: If π is a one-to-one mapping of A onto A (a *permutation* of A), then we define for every x (by $\in$-induction)

$$\pi(x) = \{\pi(t) : t \in x\}.$$

Clearly, π is an $\in$-automorphism, and we have $\pi(x) = x$ for every x in the kernel $P^\infty(\emptyset)$.

We use these automorphisms to construct transitive models of ZFA. First we point out that the analog of Theorem 13.9 is true in ZFA: If M is a transitive, almost universal class closed under Gödel operations, and if $A \in M$, then M is a model of ZFA.

Let $\mathcal{G}$ be a group of permutations of a set S. A set $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a *filter on* $\mathcal{G}$, if for all subgroups H, K of $\mathcal{G}$:

(15.34)
(i) $\mathcal{G} \in \mathcal{F}$;
(ii) if $H \in \mathcal{F}$ and $H \subset K$, then $K \in \mathcal{F}$;
(iii) if $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$;
(iv) if $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$, then $\pi H \pi^{-1} \in \mathcal{F}$.

For a given group of permutations $\mathcal{G}$ of the set A of atoms and a given filter $\mathcal{F}$ on $\mathcal{G}$, we say that x is *symmetric* if the group

$$\mathrm{sym}(x) = \{\pi \in \mathcal{G} : \pi(x) = x\}$$

belongs to $\mathcal{F}$.

Let us further assume that $\mathrm{sym}(a) \in \mathcal{F}$ for all $a \in A$, that is, that all atoms are symmetric and let U be the class of all hereditarily symmetric objects:

(15.35)
$$U = \{x : \text{every } z \in \mathrm{TC}(\{x\}) \text{ is symmetric}\}.$$

The class U is called a *permutation model*. It is a transitive class and includes the kernel (because $\mathrm{sym}(x) = \mathcal{G}$ for all $x \in P^\infty(\emptyset)$), moreover, all atoms are in U, and $A \in U$.

Lemma 15.48. *U is a transitive model of* ZFA.

Proof. We show that U is closed under Gödel operations and almost universal. It is easy to see that $G_i(\pi x, \pi y) = \pi(G_i(x, y))$ for all $i = 1, \ldots, 10$, and therefore

$$\mathrm{sym}(G_i(x, y)) \supset \mathrm{sym}(x) \cap \mathrm{sym}(y) \qquad (i = 1, \ldots, 10).$$

It follows that if x and y are hereditarily symmetric, then so is $G_i(x, y)$.

To show that U is almost universal, it suffices to verify that for each α, $U \cap P^\alpha(A)$ is symmetric. For all x and all $\pi \in \mathcal{G}$ we have $\mathrm{rank}(\pi x) = \mathrm{rank}\, x$. Also, $\mathrm{sym}(\pi x) = \pi \cdot \mathrm{sym}(x) \cdot \pi^{-1}$, and hence, by property (iv) in (15.34), if x is symmetric and $\pi \in \mathcal{G}$, then $\pi(x)$ is symmetric. Thus for all $\pi \in \mathcal{G}$ we have $\pi(U \cap P^\alpha(A)) = U \cap P^\alpha(A)$ and therefore, $\mathrm{sym}(U \cap P^\alpha(A)) = \mathcal{G}$. □

In the following examples we construct permutation models as follows: For every finite $E \subset A$, we let

$$\text{fix}(E) = \{\pi \in \mathcal{G} : \pi a = a \text{ for all } a \in E\} \tag{15.36}$$

and let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by $\{\text{fix}(E) : E \subset A \text{ is finite}\}$. $\mathcal{F}$ is a filter since $\pi \cdot \text{fix}(E) \cdot \pi^{-1} = \text{fix}(\pi(E))$. Thus x is symmetric if and only if there exists a finite set of atoms E, a *support* for x, such that $\pi(x) = x$ whenever $\pi \in \mathcal{G}$ and $\pi(a) = a$ for all $a \in E$.

We shall now give two examples of permutation models.

Example 15.49. Let A be infinite, and let $\mathcal{G}$ be the group of all permutations of A. Let $\mathcal{F}$ be generated by $\{\text{fix}(E) : E \subset A \text{ is finite}\}$, and let U be the permutation model. In the model U the set A, although infinite, has no countable subset. Hence the Axiom of Choice fails in U.

Proof. Assume that there exists an $f \in U$ that is a one-to-one mapping of ω into A. Let E be a finite subset of A such that $\pi f = f$ for every $\pi \in \text{fix}(E)$. Since E is finite, there exists an $a \in A - E$ such that $a = f(n)$ for some n; also, let $b \in A - E$ be arbitrary such that $b \neq a$. Now, let π be a permutation of A such that $\pi a = b$ but $\pi x = x$ for all $x \in E$. Then $\pi f = f$, and since n is in the kernel, we have $\pi n = n$. It follows that $\pi(f(n)) = (\pi f)(\pi n) = f(n)$; however, $f(n) = a$ while $\pi(f(n)) = \pi(a) \neq a$. A contradiction. □

Example 15.50. Let A be a disjoint countable union of pairs: $A = \bigcup_{n=0}^{\infty} P_n$, $P_n = \{a_n, b_n\}$, and let $\mathcal{G}$ be the group of all permutations of A such that $\pi(\{a_n, b_n\}) = \{a_n, b_n\}$, for all n. Let $\mathcal{F}$ be generated by $\{\text{fix}(E) : E \subset A \text{ is finite}\}$, and let U be the permutation model. In the model U, $\{P_n : n \in \omega\}$ is a countable set of pairs and has no choice function.

Proof. Each P_n is a symmetric set since $\pi(P_n) = P_n$ for all $\pi \in \mathcal{G}$. For the same reason, $\pi\langle P_n : n \in \omega\rangle = \pi(\{(n, P_n) : n \in \omega\}) = \langle P_n : n \in \omega\rangle$, for all $\pi \in \mathcal{G}$, and so $\langle P_n : n \in \omega\rangle \in U$. Hence $S = \{P_n : n \in \omega\}$ is a countable set in U.

We show that there is no function $f \in U$ such that $\text{dom}(f) = S$ and $f(P_n) \in P_n$ for all n. Assume that f is such a function and let E be a support of f. There exists n such that neither a_n nor b_n is in E, and we let $\pi \in \mathcal{G}$ be such that $\pi(a_n) = b_n$ but $\pi x = x$ for all $x \in E$. Then $\pi f = f$, $\pi P_n = P_n$, and so $\pi(f(P_n)) = (\pi f)(\pi P_n) = f(P_n)$ but $\pi(f(P_n)) = b_n$ while $f(P_n) = a_n$; a contradiction. □

The method of permutation models gives numerous examples of violation of the Axiom of Choice. One usually uses the set of atoms to produce a counterexample (in the permutation model) to some consequence of the Axiom of Choice, thus showing the limitations of proofs not using the Axiom of Choice. (A typical example is a vector space that has no basis, a set that cannot be linearly ordered, etc.) However, these examples do not give any information

about the "true" sets, like real numbers, sets of real numbers, etc., since those sets are in the kernel. It is clear that a different method has to be used to investigate the role of the Axiom of Choice in ZF. We shall now describe such a method and exploit the similarities between it and permutation models.

We shall use automorphisms (symmetries) to construct submodels of generic extensions. As shown in (14.36), every automorphism π of a complete Boolean algebra B induces an automorphism of the Boolean-valued model V^B. The important property of such an automorphism is (14.36) in the Symmetry Lemma 14.37:

$$\|\varphi(\pi\dot{x}_1, \ldots, \pi\dot{x}_n)\| = \pi(\|\varphi(\dot{x}_1, \ldots, \dot{x}_n)\|).$$

for all names $\dot{x}_1, \ldots, \dot{x}_n$.

Let $\mathcal{G}$ be a group of automorphisms of B, and let $\mathcal{F}$ be a *filter on* $\mathcal{G}$, i.e., a set of subgroups that satisfies (15.34). For each $\dot{x} \in V^B$ we define its symmetry group

$$\mathrm{sym}(\dot{x}) = \{\pi \in \mathcal{G} : \pi(\dot{x}) = \dot{x}\}.$$

If π is an automorphism of B, then

$$\mathrm{sym}(\pi\dot{x}) = \pi \cdot \mathrm{sym}(\dot{x}) \cdot \pi^{-1}. \tag{15.37}$$

This is because $\sigma(\pi\dot{x}) = \pi\dot{x}$ if and only if $(\pi^{-1}\sigma\pi)(\dot{x}) = \dot{x}$. Given a filter $\mathcal{F}$ on $\mathcal{G}$, we call $\dot{x}$ *symmetric* if $\mathrm{sym}(\dot{x}) \in \mathcal{F}$. The class HS of *hereditarily symmetric names* is defined by induction on $\rho(\dot{x})$:

$$\text{if } \mathrm{dom}(\dot{x}) \subset HS \text{ and if } \dot{x} \text{ is symmetric, then } \dot{x} \in HS.$$

Note that $\pi(\check{x}) = \check{x}$ for all x and all π, and so all $\check{x}$ are in HS. If a name $\dot{x}$ is symmetric, and if $\pi \in \mathcal{G}$, then by (15.37) and (15.34)(iv), $\pi(\dot{x})$ is also symmetric. It follows that $\pi\dot{x} \in HS$ whenever $\dot{x} \in HS$ and $\pi \in \mathcal{G}$.

The class HS is a submodel of the Boolean-valued model V^B, and can be shown to satisfy all axioms of ZF. Instead, we prove that its interpretation is a transitive model of ZF.

Thus let M be the ground model, let B be a complete Boolean algebra in M, and let $\mathcal{G}$ and $\mathcal{F}$ be respectively (in M), a group of automorphisms of B and a filter on $\mathcal{G}$. Let G be an M-generic ultrafilter on B. We let

$$N = \{\dot{x}^G : \dot{x} \in HS\} \tag{15.38}$$

be the class of all elements of $M[G]$ that have a hereditarily symmetric name. N is called a *symmetric submodel* of $M[G]$. We will prove that N is a transitive model of ZF. Before we do so, we notice that HS is a Boolean-valued model (with the same $\|x \in y\|$ and $\|x = y\|$ as M^B). Thus we can define $\|\varphi\|_{HS}$ for every formula φ. Note that

$$\|\exists x\, \varphi(x)\|_{HS} = \sum_{\dot{x} \in HS} \|\varphi(\dot{x})\| \tag{15.39}$$

and that $\|\varphi\|_{HS} = \|\varphi\|$ whenever φ is a Δ_0 formula. We also have a forcing theorem for the model N:

$$N \vDash \varphi(x_1, \dots, x_n) \quad \text{if and only if} \quad \|\varphi(\dot{x}_1, \dots, \dot{x}_n)\|_{HS} \in G \tag{15.40}$$

where $\dot{x}_1, \dots, \dot{x}_n \in HS$ are names for $x_1, \dots, x_n$. Finally, since $\pi(HS) = HS$ for all $\pi \in \mathcal{G}$, we have the Symmetry Lemma for $\| \ \|_{HS}$: If $\pi \in \mathcal{G}$ and $\dot{x}_1, \dots, \dot{x}_n \in HS$, then

$$\|\varphi(\pi\dot{x}_1, \dots, \pi\dot{x}_n)\|_{HS} = \pi(\|\varphi(\dot{x}_1, \dots, \dot{x}_n)\|_{HS}). \tag{15.41}$$

Lemma 15.51. *A symmetric submodel N of $M[G]$ is a transitive model of* ZF, *and $M \subset N \subset M[G]$.*

Proof. Since $\check{x} \in HS$ for every $x \in M$, we have $M \subset N$. The heredity of HS implies that N is transitive. To verify that the axioms of ZF hold in N, we follow closely the proof of the Generic Model Theorem. As there, we have to show that certain sets exist in the model by exhibiting names for the sets; here we have to find such names in HS.

A. Extensionality, Regularity, Infinity. These axioms hold in N since N is transitive and $N \supset M$.

B. Separation. Let φ be a formula and let

$$Y = \{x \in X : N \vDash \varphi(x, p)\}$$

where $X, p \in N$. Let $\dot{X}, \dot{p} \in HS$ be names for X, p. We let $\dot{Y} \in M^B$ as follows:

$$\operatorname{dom}(\dot{Y}) = \operatorname{dom}(\dot{X}), \qquad \dot{Y}(\dot{t}) = \dot{X}(\dot{t}) \cdot \|\varphi(\dot{t}, \dot{p})\|_{HS}.$$

A routine argument shows that $\dot{Y}$ is a name for Y; it remains to show that $\dot{Y}$ is symmetric.

We shall show that $\operatorname{sym}(\dot{Y}) \supset \operatorname{sym}(\dot{X}) \cap \operatorname{sym}(\dot{p})$. Thus let $\pi \in \mathcal{G}$ be such that $\pi\dot{X} = \dot{X}$ and $\pi\dot{p} = \dot{p}$. For every $\dot{t} \in \operatorname{dom}(\dot{X})$ we have $\pi\dot{t} \in \operatorname{dom}(\pi\dot{X}) = \operatorname{dom}(\dot{X})$ and $\dot{X}(\pi\dot{t}) = (\pi\dot{X})(\pi\dot{t}) = \pi(\dot{X}(\dot{t}))$, and $\|\varphi(\pi\dot{t}, \dot{p})\|_{HS} = \pi\|\varphi(\dot{t}, \dot{p})\|_{HS}$, and so $\dot{Y}(\pi\dot{t}) = \pi(\dot{Y}(\dot{t}))$. Therefore, $\pi\dot{Y} = \dot{Y}$.

C. Pairing, Union, Power Set. Let $X \in N$ and let $\dot{X} \in HS$ be a name for X. For the union, we let $S = \bigcup\{\operatorname{dom}(\dot{y}) : \dot{y} \in \operatorname{dom}(\dot{X})\}$. If $\pi \in \operatorname{sym}(\dot{X})$ then $\pi(S) = S$ and so the set $Y = \{t^G : t \in S\}$ has a hereditarily symmetric name $\dot{Y}$: $\dot{Y}(\dot{t}) = 1$ for all $\dot{t} \in S$. Moreover, $Y \supset \bigcup X$.

Pairing and Power Set are handled similarly.

D. Replacement. We show that if $X \in N$, then there exists a $Y \in N$ such that for all $u \in X$, N satisfies

$$\exists v\, \varphi(u, v) \to (\exists v \in Y)\, \varphi(u, v).$$

We proceed as in (14.15) except that (we deal with $\| \ \|_{HS}$ instead of $\| \ \|$ and that) we look for $S \subset HS$ such that $\pi(S) = S$ for all $\pi \in \mathcal{G}$ (for then

$Y = \{t^G : t \in S\}$ has a name in HS). This is accomplished by taking for S the set $HS \cap M_\alpha^B$ for large enough α. Since every π preserves the rank and since each $\pi \in \mathcal{G}$ preserves HS, we have $\pi(S) = S$ for all $\pi \in \mathcal{G}$. □

In general, the set G is not a member of N, and N does not satisfy the Axiom of Choice.

The model in Example 15.52 is due to Cohen. It is an analog of the permutation model in Example 15.49, and in fact, it is the same model that was used in Theorem 14.36.

Example 15.52. Let $V[G]$ be the generic extension adjoining countably many Cohen reals: P is the set of all finite 0–1 functions p with domain $\mathrm{dom}(p) \subset \omega \times \omega$. We define a_n, $n \in \omega$, and $A = \{a_n : n \in \omega\}$, as well as their canonical names as in (14.40) and (14.41).

Every permutation π of ω induces an automorphism of P (and in turn an automorphism of B) by (14.44). We can view such permutations as permutations of the set $\{\dot{a}_n : n \in \omega\}$. Let $\mathcal{G}$ be the group of all automorphisms of B that are induced by such permutations. For every finite $E \subset \omega$, let

$$\mathrm{fix}(E) = \{\pi \in \mathcal{G} : \pi n = n \text{ for each } n \in E\},$$

and let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by the $\{\mathrm{fix}(E) : E \subset \omega \text{ is finite}\}$.

Now let HS be the class of all hereditarily symmetric names, and let N be the corresponding symmetric submodel of $V[G]$. It is easy to see that all $\dot{a}_n$ are in HS and so is $\dot{A}$. Moreover, the a_n are distinct subsets of ω and so A is an infinite set of reals in N.

We claim that in N, A has no countable subset. Thus assume that some $f \in N$ is a one-to-one function from ω into A. Let $\dot{f} \in HS$ and let $p_0 \in G$ be such that

$$p_0 \Vdash \dot{f} \text{ maps } \check{\omega} \text{ one-to-one into } \dot{A}.$$

The contradiction is obtained as in Lemma 14.39. We let E be a support of $\dot{f}$, i.e., a finite subset of ω such that $\mathrm{sym}(\dot{f}) \supset \mathrm{fix}(E)$. We pick $i \in \omega$ such that $i \notin E$, and find $p \leq p_0$ and $n \in \omega$ such that

$$p \Vdash \dot{f}(\check{n}) = \dot{a}_i.$$

Then we find a permutation $\pi \in \mathcal{G}$ such that:

(i) πp and p are compatible;
(ii) $\pi \in \mathrm{fix}(E)$;
(iii) $\pi i = j \neq i$.

Then $\pi\dot{f} = \dot{f}$, $\pi(\check{n}) = \check{n}$, and we have $p \cup \pi p \Vdash \dot{f}(\check{n}) = \dot{a}_i$ and $p \cup \pi p \Vdash \dot{f}(\check{n}) = \dot{a}_j$, a contradiction. □

The set A in Example 15.52 is a set of reals and is therefore linearly ordered. Lévy proved that in the model N in Example 15.52, every set can

be linearly ordered. In fact, Halpern and Lévy proved that the model even satisfies the Prime Ideal Theorem, thus establishing the independence of the Axiom of Choice from the Prime Ideal Theorem. We note that numerous consequences of the Axiom of Choice in mathematics can be proved using the Prime Ideal Theorem instead—among others the Hahn-Banach Theorem, Compactification Theorems, the Completeness Theorem, the Tikhonov Theorem for Hausdorff spaces, etc.

Another construction of Cohen yields a model that has similar properties as the permutation model in Example 15.50. The atoms are replaced not by reals, but by sets of reals.

The similarity between permutation models and symmetric submodels is made precise by the following result that shows that every permutation model can be embedded in a symmetric model of ZF, "with a prescribed degree of accuracy."

Theorem 15.53 (Jech-Sochor). *Let U be a permutation model, let A be its set of atoms, and let α be an ordinal. There exist a symmetric model N of* ZF *and an embedding $x \mapsto \tilde{x}$ of U into N such that*

$$(P_\alpha(A))^U \text{ is } \in\text{-isomorphic to } \big(P_\alpha(\tilde{A})\big)^N.$$

Proof. We work in the theory ZFA, plus the Axiom of Choice. We denote A the set of all atoms, and let M be the kernel, $M = P^\infty(\emptyset)$. We consider a group $\mathcal{G}$ of permutations of A, and a filter $\mathcal{F}$ on $\mathcal{G}$, and let U be the permutation model given by $\mathcal{G}$ and $\mathcal{F}$. Let α be an ordinal number.

We shall construct a generic extension $M[G]$ of the kernel, and then the model N as a symmetric submodel of $M[G]$. We construct $M[G]$ by adjoining to M a number of subsets of a regular cardinal κ, κ of them for each $a \in A$. We use these to embed U in $M[G]$.

Let κ be a regular cardinal such that $\kappa > |P^\alpha(A)|$. The set P of forcing conditions consists of 0–1 functions p such that $|\operatorname{dom}(p)| < \kappa$ and $\operatorname{dom}(p) \subset (A \times \kappa) \times \kappa$; as usual, $p < q$ if and only if $p \supset q$.

Let G be an M-generic filter on P. For each $a \in A$ and each $\xi < \kappa$, we let

$$x_{a,\xi} = \{\eta \in \kappa : p(a, \xi, \eta) = 1 \text{ for some } p \in G\}.$$

Each $x_{a,\xi}$ has a canonical name $\dot{x}_{a,\xi}$:

$$\dot{x}_{a,\xi}(\check{\eta}) = \sum\{p \in P : p(a, \xi, \eta) = 1\} \qquad (\eta \in \kappa).$$

Then we define, for every $a \in A$,

$$\tilde{a} = \{x_{a,\xi} : \xi < \kappa\}$$

and let $\tilde{A} = \{\tilde{a} : a \in A\}$. The sets $\tilde{a}$ and $\tilde{A}$ have obvious canonical names.

Having defined $\tilde{a}$ for each $a \in A$, we can define $\tilde{x}$ (and its canonical name $\dot{\tilde{x}}$) for each x be $\in$-induction:

$$\tilde{x} = \{\tilde{y} : y \in x\}. \tag{15.42}$$

We shall show that the function $x \mapsto \tilde{x}$ is an $\in$-isomorphism.

Lemma 15.54. *For all x and y, $x \in y$ if and only if $\tilde{x} \in \tilde{y}$, and $x = y$ if and only if $\tilde{x} = \tilde{y}$.*

Proof. First we note that $\|\dot{x}_{a,\xi} = \dot{x}_{a',\xi'}\| = 0$ whenever $(a,\xi) \neq (a',\xi')$, and that $\|\dot{x}_{a,\xi} = \check{z}\| = 0$ for all $z \in M$. Consequently, we have $\tilde{a} \neq \tilde{b}$ whenever $a \neq b$ are atoms. We claim that for all x, $\tilde{x} \neq x_{a,\xi}$ for any a, ξ. If $x \in M$, then $\tilde{x} = x$ and so $\tilde{x} \neq x_{a,\xi}$. If $x \notin M$, then $\tilde{x}$ is of higher rank than any $x_{a,\xi}$: $x_{a,\xi}$ is a subset of κ, while the transitive closure of $\tilde{x}$ contains some of the $x_{a,\xi}$.

Now we can prove the lemma, simultaneously for $\in$ and $=$, by induction on rank:

(a) If $x \in y$, then $\tilde{x} \in \tilde{y}$ follows from the definition (15.42). If $\tilde{x} \in \tilde{y}$, then y cannot be an atom because then we would have $\tilde{x} = x_{a,\xi}$ for some a, ξ, which is impossible. Hence $\tilde{x} = \tilde{z}$ for some $z \in y$ and we have $x = z$ by the induction hypothesis; thus $x \in y$.

(b) If $x = y$, then $\tilde{x} = \tilde{y}$. Conversely, if $x \neq y$, then either both x and y are atoms and then $\tilde{x} \neq \tilde{y}$; or, e.g., x contains some z that is not in y, and then, by the induction hypothesis, $\tilde{z} \in \tilde{x}$ and $\tilde{z} \notin \tilde{y}$; thus $\tilde{x} \neq \tilde{y}$. □

Note that the proof of Lemma 15.54 does not depend on the particular G and so in fact we have proved

$$x = y \quad \text{if and only if} \quad \|\dot{\tilde{x}} = \dot{\tilde{y}}\| \neq 0 \quad \text{if and only if} \quad \|\dot{\tilde{x}} = \dot{\tilde{y}}\| = 1 \tag{15.43}$$

and similarly for $\in$.

Now we shall construct a symmetric submodel N of $M[G]$. We construct N so that for every $x \in U$, $\tilde{x}$ is in N and that $(P^\alpha(A))^U$ is isomorphic to $(P^\alpha(\tilde{A}))^N$. For every permutation σ of A, let $\bar{\sigma}$ be the group of all permutations π of $A \times \kappa$ such that for all a, ξ,

$$\pi(a,\xi) = (\sigma a, \xi') \qquad \text{for some } \xi'.$$

We let $\bar{H} = \bigcup\{\bar{\sigma} : \sigma \in H\}$ for every subgroup H of $\mathcal{G}$. Since every permutation π of $A \times \kappa$ induces an automorphism of P by

$$(\pi p)(\pi(a,\xi),\eta) = p(a,\xi,\eta) \qquad (\text{all } a,\xi,\eta)$$

we consider $\bar{\mathcal{G}}$ as a group of automorphisms of $B = B(P)$. For every finite $A \subset A \times \kappa$ we let

$$\text{fix}(E) = \{\pi \in \bar{\mathcal{G}} : \pi(a,\xi) = (a,\xi) \text{ for all } (a,\xi) \in E\},$$

and we let $\bar{\mathcal{F}}$ be the filter on $\bar{\mathcal{G}}$ generated by the set

$$\{\bar{H} : H \in \mathcal{F}\} \cup \{\mathrm{fix}(E) : E \subset A \times \kappa \text{ finite}\}. \tag{15.44}$$

Let HS be the class of all hereditarily symmetric names and let N be the corresponding symmetric submodel of $M[G]$. It is an immediate consequence of (15.44) that all $\dot{x}_{a,\xi}$, all $\dot{a}$ ($a \in A$), and $\check{\tilde{A}}$ are symmetric, and so $\tilde{A}$ is in N. The following two lemmas show that for any x, $\tilde{x}$ is in N if and only if $x \in U$.

Lemma 15.55. *For all x, $x \in U$ if and only if $\dot{\tilde{x}} \in HS$.*

Proof. It suffices to show that x is symmetric if and only if $\dot{\tilde{x}}$ is symmetric. If $\sigma \in \mathcal{G}$ and $\pi \in \bar{\sigma}$, then $\pi\dot{\tilde{x}}$ is the canonical name for $(\sigma x)\tilde{}$, and so $\mathrm{sym}_{\bar{\mathcal{G}}}(\dot{\tilde{x}}) = \overline{\mathrm{sym}_{\mathcal{G}}(x)}$; thus if $\mathrm{sym}(x) \in \mathcal{F}$, then $\mathrm{sym}_{\bar{\mathcal{G}}}(\dot{\tilde{x}}) \in \bar{\mathcal{F}}$. On the other hand, if $\mathrm{sym}_{\bar{\mathcal{G}}}(\dot{\tilde{x}}) \in \bar{\mathcal{F}}$, then $\overline{\mathrm{sym}(x)} \supset \bar{H} \cap \mathrm{fix}(E)$ for some $H \in \mathcal{F}$ and a finite $E \subset A \times \kappa$. If $e = \{a \in A : (a,\xi) \in E \text{ for some } \xi\}$, then $\mathrm{sym}(x) \supset H \cap \mathrm{fix}(e)$, and since $\mathrm{fix}(e) \in \mathcal{F}$, we have $\mathrm{sym}(x) \in \mathcal{F}$. □

Lemma 15.56. *For all x, $x \in U$ if and only if $\tilde{x} \in N$.*

Proof. By Lemma 15.55, it suffices to show that if $\tilde{x} \in N$, then $x \in U$. Assume otherwise, and let x be of least rank such that $\tilde{x} \in N$ and $x \notin U$. Thus $x \subset U$, and since $\tilde{x} \in N$, there exist a name $\dot{z} \in HS$ and some $p \in G$ such that $p \Vdash \dot{z} = \dot{\tilde{x}}$. Since $\mathrm{sym}_{\bar{\mathcal{G}}}(\dot{z}) \in \bar{\mathcal{F}}$, we have $\mathrm{sym}_{\bar{\mathcal{G}}}(\dot{z}) \supset \bar{H} \cap \mathrm{fix}(E)$ for some $H \in \mathcal{F}$ and a finite $E \subset A \times \kappa$. We shall find $\sigma \in \mathcal{G}$ and $\pi \in \bar{\sigma}$ such that:

(i) πp and p are compatible;
(ii) $\pi \in \bar{H} \cap \mathrm{fix}(E)$;
(iii) $\sigma x \neq x$.

Then we have $\pi\dot{z} = \dot{z}$ by (ii), $\|\pi\dot{\tilde{x}} = \dot{\tilde{x}}\| = 0$ by (iii) and (15.43); and since $\pi p \Vdash \pi\dot{z} = \pi\dot{\tilde{x}}$, we have

$$\pi p \cup p \Vdash \dot{z} = \dot{\tilde{x}}, \qquad \pi p \cup p \Vdash \dot{z} = \pi\dot{\tilde{x}},$$

a contradiction.

To find π, note that x is not symmetric, so that there is a $\sigma \in \mathcal{G}$ such that $\sigma x \neq x$ and $\sigma \in H \cap \mathrm{fix}(e)$, where $e = \{a \in A : (a,\xi) \in E \text{ for some } \xi\}$. Since $|p| < \kappa$, there exists a $\gamma < \kappa$ such that $(a,\xi) \notin \mathrm{dom}(p)$ for all $a \in A$ and all $\xi > \gamma$. Thus we define $\pi \in \bar{\sigma}$ as follows:

$$\text{if } a \in e, \text{ then } \pi(a,\xi) = (a,\xi) \text{ for all } \xi;$$

$$\text{if } a \notin e, \text{ then } \begin{cases} \pi(a,\xi) = \pi(\sigma a, \gamma+\xi) \text{ and } \pi(a,\gamma+\xi) = \pi(\sigma a,\xi) & \text{if } \xi < \gamma; \\ \pi(a,\xi) = (\sigma a,\xi) & \text{if } \xi > \gamma \cdot 2. \end{cases}$$

It follows that $\pi \in \bar{H} \cap \mathrm{fix}(E)$ and that p and πp are compatible. □

We complete the proof of Theorem 15.53 by showing that

$$\big((P^\alpha(A))^U\big)^\sim = \big(P^\alpha(\tilde{A})\big)^N.$$

The left-hand side is clearly included in the right-hand side; we prove the converse by induction. Thus let $x \in P^\alpha(A) \cap U$ and let $y \in N$ be a subset of x; we shall show that $y = \tilde{z}$ for some $z \in U$. Let $\dot{y}$ be a name for y. The notion of forcing that we are using here is $<\kappa$-closed; and since we have chosen κ large, it follows that there is a $p \in G$ that decides $\check{t} \in \dot{y}$ for all $t \in x$. Hence $y = \tilde{z}$, where $z = \{t \in x : p \Vdash \check{t} \in \dot{y}\}$, and by Lemma 15.56 we have $z \in U$. □

As for applications of Theorem 15.53, consider a formula $\varphi(X, \gamma)$ such that the only quantifiers in φ are $\exists u \in P^\gamma(X)$ and $\forall u \in P^\gamma(X)$. Let U be a permutation model such that

$$U \vDash \exists X\, \varphi(X, \gamma).$$

Let $X \in U$ be such that $U \vDash \varphi(X, \gamma)$; let α be such that $P^\gamma(X) \subset P^\alpha(A)$. By the theorem, U can be embedded in a model N of ZF such that $(P^\alpha(A))^U$ is isomorphic to $(P^\alpha(\tilde{A}))^N$. Since the quantifiers in φ are restricted to $P^\gamma(X)$, it follows that $N \vDash \varphi(\tilde{X}, \gamma)$, and so

$$N \vDash \exists X\, \varphi(X, \gamma).$$

Therefore, if we wish to prove consistency (with ZF) of an existential statement of the kind just described, it suffices to construct a permutation model (of ZFA).

Note that "X cannot be well ordered," "X cannot be linearly ordered" are formulas of the above type and so is "X is a countable set of pairs without a choice function."

Theorem 15.53, in conjunction with the construction of permutation models, has interesting applications in algebra. One can construct various abstract counterexamples to theorems whose proofs use the Axiom of Choice. For example, one can construct a vector space that has no basis, etc.

We conclude this section by sketching two examples of models of ZF in which the Axiom of Choice fails. The first model was constructed by Feferman and Lévy, the other by Feferman.

Example 15.57. Let M be a transitive model of ZFC. There is a model $N \supset M$ such that $(\aleph_1)^N = (\aleph_\omega)^M$; hence $\aleph_1$ is singular in N.

Proof. First we construct a generic extension $M[G]$ by adjoining collapsing maps $f_n : \omega \to \omega_n$, for all $n \in \omega$: We let $(P, \supset)$ consist of finite functions with domain $\subset \omega \times \omega$, such that $p(n, i) < \omega_n$ for all $(n, i) \in \mathrm{dom}(p)$. If G is a generic filter on P, then $f = \bigcup G$ is a function on $\omega \times \omega$, and for every n,

the function f_n defined on ω by $f_n(i) = f(n,i)$ maps ω onto ω_n. We shall construct a symmetric model $N \subset M[G]$ such that each f_n is in N but $\aleph_\omega$ is a cardinal in N.

Let $\mathcal{G}$ be the group of all permutations π of $\omega \times \omega$ such that for every n, $\pi(n,i) = (n,j)$, for some j. Every π induces an automorphism of P by

$$\mathrm{dom}(\pi p) = \{\pi(n,i) : (n,i) \in \mathrm{dom}(p)\}, \qquad (\pi p)(\pi(n,i)) = p(n,i).$$

Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by $\{H_n : n \in \omega\}$, where H_n consists of all π such that $\pi(k,i) = (k,i)$ for all $k \leq n$, all $i \in \omega$. Let HS be the class of all hereditarily symmetric names and let N be the symmetric model.

It is easy to verify that for each n, the canonical name $\dot{f}_n$ of f_n is symmetric and so $f_n \in N$. To show that $\aleph_\omega$ remains a cardinal in N, we use the following lemma:

Lemma 15.58. *If* $\mathrm{sym}(\dot{x}) \supset H_n$ *and* $p \Vdash \varphi(\dot{x})$, *then* $p{\restriction}n \Vdash \varphi(\dot{x})$, *where* $p{\restriction}n$ *is the restriction of* p *to* $\{(k,i) : k \leq n\}$.

Proof. Let us assume that $p{\restriction}n$ does not force $\varphi(\dot{x})$ and let $q \supset p{\restriction}n$ be such that $q \Vdash \neg\varphi(\dot{x})$. It is easy to find some $\pi \in H_n$ such that πp and q are compatible; since $\pi p \Vdash \varphi(\pi\dot{x})$ and $\pi\dot{x} = \dot{x}$, we get a contradiction. □

Now let us assume that $g \in N$ is a function of ω onto $\aleph_\omega$, and let $\dot{g}$ be a symmetric name for g. Let $p_0 \in G$ be such that p_0 forces "$\dot{g}$ is a function from $\check{\omega}$ onto $\check{\aleph}_\omega$." Let n be such that $p_0{\restriction}n = p_0$ and that $\mathrm{sym}(\dot{g}) \supset H_n$. Since g takes $\aleph_\omega$ values, it follows that for some $k \in \omega$, there exists an incompatible set W of conditions $p \supset p_0$ such that $|W| \geq \aleph_{n+1}$, and distinct ordinals α_p, $p \in W$, such that for each $p \in W$, $p \Vdash \dot{g}(k) = \alpha_p$. By Lemma 15.58, we have $p{\restriction}n \Vdash \dot{g}(k) = \alpha_p$, for each $p \in W$, which is a contradiction: On the one hand, the conditions $p{\restriction}n$, $p \in W$, must be mutually incompatible, and on the other hand, the set $\{p{\restriction}n : p \in P\}$ has size only $\aleph_n$. □

If the ground model M in the above example satisfies GCH, then one can show that in N, the set of all reals is the countable union of countable sets.

Example 15.59. Let M be a transitive model of ZFC. There is a model $N \supset M$ such that in N, there is no nonprincipal ultrafilter on ω.

Proof. The model N is obtained by adjoining to M infinitely many generic reals a_n, $n < \omega$, without putting in N the set $\{a_n : n \in \omega\}$ (unlike in Example 15.52 where $\{a_n : n \in \omega\}$ is in N). First we construct $M[G]$ as in Example 15.52: $(P, \supset)$ is the set of all finite 0–1 functions with domain $\subset \omega \times \omega$. Let G be generic and let $a_n = \{m : p(n,m) = 1 \text{ for some } p \in G\}$, for each $n \in \omega$.

Now let N be as follows. Every $X \subset \omega \times \omega$ induces a *symmetry* σ_X, an automorphism of P defined by

$$(\sigma_X p)(n,m) = \begin{cases} p(n,m) & \text{if } (n,m) \notin X, \\ 1 - p(n,m) & \text{if } (n,m) \in X. \end{cases}$$

Let $\mathcal{G}$ be the group of all σ_X, $X \subset \omega \times \omega$, and let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by $\{\text{fix}(E) : E \subset \omega \text{ finite}\}$, where $\text{fix}(E) = \{\sigma_X : X \cap (E \times \omega) = \emptyset\}$. Let N be the symmetric model.

Let $D \in N$ be an ultrafilter on ω; we shall show that D is principal. Let $\dot{D} \in HS$ be a name for D and let $p \in G$ be such that p forces "$\dot{D}$ is an ultrafilter on $\check{\omega}$." Let $E \subset \omega$ be finite, such that $\text{sym}(\dot{D}) \supset \text{fix}(E)$, and let $n \notin E$. Then there is some $q \leq p$, $q \in G$, that decides $\dot{a}_n \in \dot{D}$ (where $\dot{a}_n$ is the canonical name for a_n). For example, assume that $q \Vdash \dot{a}_n \in \dot{D}$ (the proof is similar if $q \Vdash \dot{a}_n \notin \dot{D}$).

Let m_0 be such that for all $m \geq m_0$, $(n, m) \notin \text{dom}(q)$, and let $X = \{(n, m) : m \geq m_0\}$. Let $\dot{b}_n = \sigma_X(\dot{a}_n)$. Since for each $m \geq m_0$, $\|\check{m} \in \dot{b}_n\| = -\|\check{m} \in \dot{a}_n\|$, it follows that $a_n \cap b_n$ is a finite set. However, $\sigma_X q \Vdash \sigma_X \dot{a}_n \in \sigma_X \dot{D}$; it is fairly obvious that $\sigma_X q = q$ and since $\sigma_X \in \text{fix}(E)$, we have $\sigma_X \dot{D} = \dot{D}$. Thus $q \Vdash \dot{b}_n \in \dot{D}$ and hence $a_n \cap b_n \in D$. Consequently, D is principal. □

Exercises

15.1. If P satisfies the κ-chain condition then $|B(P)| \leq |P|^{<\kappa}$.
[Every $u \in B^+$ is $\sum W$ for some antichain in P.]

15.2. Let P be as in (15.2) and let $Q = \{p \in P : \text{dom}(p) \text{ is an initial segment of } \kappa\}$. Then Q is dense in P and hence $B(Q) = B(P)$.

15.3. Let κ be a singular cardinal and let $(P, <)$ be defined as in (15.2). Then P collapses κ to $\text{cf}(\kappa)$: In the generic extension, there is a one-to-one function g from κ into $\text{cf}(\kappa)$.
[Let $\kappa = \aleph_\omega$, and let X be the added subset of $\aleph_\omega$. For each $\alpha < \aleph_\omega$, let $g(\alpha) =$ the least n such that the order-type of $X \cap (\omega_{n+1} - \omega_n)$ is $\omega_n + \alpha$. Show that for every α and every $p \in P$ there is $q \supset p$ and some n such that $\text{dom}(q) \supset \omega_{n+1} - \omega_n$ and that the set $\{\xi \in \omega_{n+1} - \omega_n : q(\xi) = 1\}$ has the order-type $\omega_n + \alpha$. By the genericity of G, the function g is defined for every $\alpha < \aleph_\omega$; it is clearly one-to-one.]

15.4. Again let κ be singular, and let P be the set of all 0–1 functions whose domains are bounded subsets of κ; P is ordered by $\supset$. Show that P collapses κ to $\text{cf}(\kappa)$.

15.5. If every $f : \kappa \to V$ in V^B is in the ground model, then B is κ-distributive.
[Let W_α, $\alpha < \kappa$, be partitions of B. Consider $\dot{f} \in V^B$ such that $\|\dot{f}(\alpha) = u\| = u$ for $u \in W_\alpha$, and find a common refinement of the W_α.]

15.6. If $B(P_1) = B(P_2)$ and $B(Q_1) = B(Q_2)$ then $B(P_1 \times Q_1) = B(P_2 \times Q_2)$.

15.7. $B(P \times Q)$ is the completion of the direct sum of the algebras $B(P)$ and $B(Q)$.

15.8. Let P be such that for every p there exist incompatible $q \leq p$ and $r \leq p$. Show that if $G \subset P$ then $G \times G$ is not generic on $P \times P$.

15.9. If $B(P_i) = B(Q_i)$ for each $i \in I$, then $B(P) = B(Q)$ where $P = \prod_i P_i$ and $Q = \prod_i Q_i$.

15.10. Let P be the notion of forcing (15.1) that adjoins κ Cohen reals. Then P is (isomorphic to) the product of κ copies of the forcing for adding a single Cohen real (Example 14.2).

15.11. If P satisfies c.c.c. and Q has property (K) then $P \times Q$ satisfies c.c.c.

15.12. The Singular Cardinal Hypothesis holds in Easton's model.

[If κ is singular then every $f : \mathrm{cf}(\kappa) \to \kappa$ is in $N = V[G^{\leq \mathrm{cf}\,\kappa}]$, and so if $F(\mathrm{cf}\,\kappa) < \kappa$ then $(\kappa^{\mathrm{cf}\,\kappa})^{V[G]} = (\kappa^{\mathrm{cf}\,\kappa})^N \leq (2^\kappa)^N \leq |B(P^{\leq \mathrm{cf}\,\kappa})|^\kappa = (F(\mathrm{cf}\,\kappa))^\kappa = \kappa^+$.]

15.13. In (15.18), let $\kappa = \aleph_1$ and $\lambda = \aleph_\omega$. Then in $V[G]$ there is a one-to-one function $g : \aleph_\omega^{\aleph_0} \to \aleph_1$.

[If X is a countable subset of $\aleph_\omega$, let $g(X) =$ the least α such that $f(\alpha + \omega - \alpha) = X$ (where $f = \bigcup G$ is the collapsing function). Use the fact that $X \in V$.]

15.14. In (15.18), let $\kappa = \aleph_\omega$. Then in $V[G]$ there is a one-to-one function g from λ into ω.

[Let $f = \bigcup G$, and let $g(\alpha) =$ the least n such the function $f \restriction (\omega_{n+1} - \omega_n)$ is eventually equal to α.]

15.15. There is a generic extension $V[G]$ such that $V[G]$ satisfies the GCH.

[For each α, let P_α be the notion of forcing which collapses $\lambda = \beth_{\alpha+1}$ onto $\kappa = (\beth_\alpha)^+$ (see (15.18)). P_α is $\beth_\alpha$-closed and satisfies the λ^+-chain condition. Let P be an Easton product of P_α, $\alpha \in Ord$; namely, we require that $|s(p) \cap \gamma| < \gamma$ for every inaccessible $\gamma = \beth_\alpha$. Show that for each α, $\kappa = (\beth_\alpha)^+$ is a cardinal in $V[G]$, $\kappa = \aleph_{\alpha+1}^{V[G]}$, and $V[G] \models 2^{\aleph_\alpha} = \aleph_{\alpha+1}$. Apply Lemma 15.19 in two ways: (a) For each α, consider $P^{\leq\alpha} \times P^{>\alpha}$: $P^{\leq\alpha}$ satisfies the $\beth_{\alpha+1}^+$-chain condition and $P^{>\alpha}$ is $\beth_{\alpha+1}$-closed; (b) if α is inaccessible and $\alpha = \beth_\alpha$, consider $P^{<\alpha} \times P^{\geq\alpha}$: $P^{<\alpha}$ satisfies the $\beth_\alpha^+$-chain condition and $P^{\geq\alpha}$ is $\beth_\alpha$-closed.]

15.16. Let $(P, <)$ be the notion of forcing that adds a subset of ω_1 (15.2), and let $(Q, <)$ be the notion of forcing that collapses $2^{\aleph_0}$ onto $\aleph_1$ (15.18). Then $B(P) = B(Q)$.

[Let $Q' = \{q \in Q : \mathrm{dom}(q)$ is an initial segment of $\omega_1\}$; Q' is dense in Q. Show that P has a dense set P' isomorphic to Q': Use the fact that every $p \in P$ has $2^{\aleph_0}$ mutually incompatible extensions.]

[Another way to show that $(P, <)$ from (15.2) adjoins a one-to-one mapping of $2^{\aleph_0}$ into $\aleph_1$: Let $f = \bigcup G$, and for every $g \in \{0,1\}^\omega$, let $F(g) =$ least α such that $f(\alpha + n) = g(n)$ for all n.]

15.17. Let P be the forcing that adds a subset of ω_1, and let Q be the forcing that adds a Suslin tree as in (15.9). Then $B(P) = B(Q)$.

If T_1 and T_2 are trees, then an *isomorphism* $\pi : T_1 \to T_2$ between T_1 and T_2 is a one-to-one mapping of T_1 onto T_2 such that $x < y$ if and only if $\pi(x) < \pi(y)$. An isomorphism maps level α of T_1 onto level α of T_2 (for all α); and if b is a branch in T_1, then $\pi(b)$ is a branch in T_2. An *automorphism* of T is an isomorphism of T_1 onto T_2. A tree T is *rigid* if it has no nontrivial automorphism, i.e., the only automorphism of T is the identity mapping. T is *homogeneous* if for any x, y at the same level of T, there exists an automorphism π of T such that $\pi(x) = y$.

15.18. If T is a normal α-tree where $\alpha < \omega_1$ is a limit ordinal and if π is a nontrivial automorphism of T, then T has an extension $T \in P$ of height $\alpha + 1$ such that π cannot be extended to an automorphism of T'.

[Construct T' so that for some branch b in T, b is extended while $\pi(b)$ is not.]

15.19. The generic Suslin tree constructed in Theorem 15.23 is rigid.

[If $T \Vdash \dot{\rho}$ is a nontrivial automorphism of $\mathcal{T}$, then the set $\{T' \leq T :$ $\exists$automorphism π of an initial segment of T' that cannot be extended to an automorphism of T' and $T' \Vdash \pi \subset \dot{\rho}\}$ is dense below T; a contradiction.]

If $s : \alpha \to \omega$ and $t : \alpha \to \omega$, let $s \sim t$ if and only if $s(\xi) = t(\xi)$ for all but finitely many $\xi < \alpha$.

15.20. There is a generic model $V[G]$ in which there exists a homogeneous Suslin tree.

[Let the forcing conditions be normal countable trees with the additional properties: (vi) if $t \in T$ and $s \sim t$, then $s \in T$; and (vii) if $s \in T$ and $t \in T$ are at the same level, then $s \sim t$.]

Let $(P, <)$ be the notion of forcing consisting of finite trees $(T, <_T)$ such that $T \subset \omega_1$, and such that $\alpha < \beta$ if $\alpha <_T \beta$; $(T_1, <_{T_1})$ is stronger than $(T_2, <_{T_2})$ if and only if $T_1 \supset T_2$ and $<_{T_1} = <_{T_2} \cap (T_2 \times T_2)$. If G is a generic set of conditions, then $\mathcal{T} = \bigcup\{T : T \in G\}$ is a Suslin tree. The crucial properties to verify are: (a) $(P, <)$ satisfies the countable chain condition, and (b) $\mathcal{T}$ has no uncountable antichain:

15.21. $(P, <)$ satisfies c.c.c.

[Given an uncountable set W of conditions, use Δ-Lemma to find an uncountable $Z \subset W$ such that any $X, Y \in Z$ are compatible.]

15.22. $\mathcal{T}$ has no uncountable antichain.

[If $T_0 \Vdash \dot{A}$ is uncountable, we first find an uncountable set W of pairs (T, α_T) such that $T \leq T_0$ and $T \Vdash \alpha_T \in \dot{A}$. By Δ-Lemma, find an uncountable $Z \subset W$ with the property that if $T_1, T_2 \in Z$, then there is T stronger than both T_1 and T_2 such that $T \Vdash \alpha_{T_1}$ is compatible with α_{T_2}. Then some $T' \leq T_0$ forces that $\dot{A}$ is not an antichain.]

Let Q consist of all countable sequences $p = \langle S_\xi : \xi < \alpha\rangle$ $(\alpha < \omega_1)$ where $S_\xi \subset \xi$ for all $\xi < \alpha$; let $p \leq q$ if and only if p extends q. Q is $\aleph_0$-closed.

15.23. Let G be Q-generic. Then $V[G] \vDash \diamondsuit$.

[If $p \Vdash (\dot{C}$ is closed unbounded set and $\dot{X} \subset \omega_1)$, find $q \leq p$ such that $q = \langle S_\xi :$ $\xi \leq \alpha\rangle$ and $q \Vdash (\alpha \in \dot{C}$ and $\dot{X} \cap \alpha = S_\alpha)$.]

15.24. Let P be the forcing that adds a subset of ω_1 (15.2) and let Q be the forcing that adds a $\diamondsuit$-sequence (Exercise 15.23). Then $B(P) = B(Q)$.

A purely combinatorial argument can be used to show that $\diamondsuit$ is equivalent to the following statement:

($\diamondsuit'$) There exists a sequence of functions h_α, $\alpha < \omega_1$, such that for every $f : \omega_1 \to \omega_1$, the set $\{\alpha < \omega_1 : f\restriction\alpha = h_\alpha\}$ is stationary.

15.25. $V = L$ implies $\diamondsuit'$.

15.26. If $V = L$ then there exists a rigid Suslin tree.

15.27. If $V = L$ then there exists a homogeneous Suslin tree.

15.28. If T is a normal Suslin tree then $P_T \times P_T$ does not satisfy the countable chain condition.

[For each $x \in T$, pick two immediate successors p_x and q_x of x. The set $\{(p_x, q_x) :$ $x \in T\} \subset P_T \times P_T$ is an antichain in $P_T \times P_T$.]

15.29. A Cohen-generic real is not minimal over the ground model.
[Show that P is isomorphic to $P \times P$, and therefore $V[x] = V[x_1][x_2]$, where x_1 is Cohen-generic over V and x_2 is Cohen-generic over $V[x_1]$. Consequently, $x_1 \notin V$ and $x \notin V[x_1]$.]

15.30. If a is a Sacks real, then in $V[a]$, every $f : \omega \to \omega$ is dominated by some $g : \omega \to \omega$ in the ground model.

15.31. If B is $(\kappa, 2)$-distributive then it is $(\kappa, 2^\kappa)$-distributive.
[Given $f : \kappa \to P(\kappa)$, consider $\{(\alpha, \beta) : \beta \in f(\alpha)\} \in P(\kappa \times \kappa)$.]

15.32. If κ is singular and B is $<\kappa$-distributive then it is κ-distributive.
[Given a function f on κ, consider $\{f \restriction \kappa_\alpha : \alpha < \operatorname{cf} \kappa\}$.]

15.33. Let P be the forcing that adds a Cohen real. The algebra $B(P)$ is not weakly (ω, ω)-distributive.
[See Lemma 15.30(ii).]

15.34. B is weakly (ω, ω_1)-distributive if and only if ω_1 is a cardinal in $V[G]$.

15.35. If a complete Boolean algebra is κ-generated and λ-saturated, then $|B| \leq \kappa^{<\lambda}$.

15.36. Every infinite countably generated c.c.c. complete Boolean algebra has size $2^{\aleph_0}$.

15.37. Show that in either Example 15.49 or 15.50, the set A cannot be linearly ordered.

Historical Notes

The forcing that adds Cohen reals is due to Cohen. Shortly after Cohen's discoveries, Solovay (in [1963]) noticed that Cohen's construction of a model for $2^{\aleph_0} = \aleph_2$ can be generalized so that for a regular cardinal κ one obtains a model of with $2^\kappa = \lambda$ (assuming $2^{<\kappa} = \kappa$ and $\lambda^\kappa = \lambda$ in the ground model).

The relation between the chain condition and preservation of cardinals is basically due to Cohen; the observation that a λ-closed notion of forcing does not produce new subsets of λ is due to Solovay. The Product Lemma 15.9 is due to Solovay [1970]. Theorem 15.15 comes from general topology and is due to Engelking and Karłowicz [1965].

Easton's Theorem (Theorem 15.18) was published in [1970]. The generalization of Cohen's method allowing a class of forcing conditions is due to Easton. The Lévy collapse (Theorem 15.22) was constructed by Lévy; cf. [1970].

Suslin's Problem was formulated by Suslin in [1920]. Tennenbaum [1968] and Jech [1967] discovered models of set theory in which a Suslin line exists; Solovay and Tennenbaum [1971] proved that existence of a Suslin line is not provable in ZFC. Subsequently, Jensen proved that a Suslin line exists in the constructible universe (cf. [1968, 1972]).

The present proof of Theorem 15.23 is as in Jech [1967] (countable conditions); Tennenbaum's proof (finite conditions) is presented in Exercises 15.21 and 15.22.

Random reals were introduced by Solovay [1970]. Forcing with perfect trees to obtain a minimal degree (Theorem 15.34) is due to Sacks [1971].

Theorem 15.46 is due to Vopěnka and appears in the book [1972] of Vopěnka and Hájek.

The idea of using symmetry arguments to construct models in which the Axiom of Choice fails goes back to Fraenkel [1922b]; the two examples of models of ZFA (an infinite set of atoms without a countable subset, and a countable set of pairs that has no choice function) are basically due to him. Further examples of permutation models were given by Mostowski who (in [1939]) developed a theory of such models. The present definition using filters was given by Specker [1957].

Cohen incorporated the symmetry arguments into his method and constructed the model in Example 15.52. The formulation of Cohen's method in terms of symmetric submodels of Boolean-valued models is due to Scott (unpublished) and Jech [1971a]; the latter's version was a reformulation of a topological version of Vopěnka and Hájek [1965].

Theorem 15.53 is due to Jech and Sochor [1966a, 1966b]. Numerous applications of the theorem are given in the second paper [1966b]. The method has been generalized by Pincus in [1971] and in [1972], extending further the analogy between permutations models of ZFA and symmetric models of ZF.

Lévy showed that in Cohen's model in Example 15.52 every set can be linearly ordered; consequently, Halpern and Lévy [1971] proved that the Prime Ideal Theorem holds in the model. Example 15.57 (singularity of $\aleph_1$) is due to Feferman and Lévy [1963]. Example 15.59 (independence of the Prime Ideal Theorem) is due to Feferman [1964/65]. A. Blass constructed in [1977] a model, similar to Feferman's model, in which every ultrafilter is principal.

Exercise 15.15: Jensen [1965].

Exercise 15.20: Fukson [1971].

Exercises 15.25–15.28: Jensen [1969].

The results in Exercises 15.31 and 15.32 had been known before forcing; see Sikorski [1964].

16. Iterated Forcing and Martin's Axiom

In this chapter we introduce two related concepts: iterated forcing and Martin's Axiom. Iteration of forcing is one of the basic techniques used in applications of forcing. It was first used by Solovay and Tennenbaum in their proof of the independence of Suslin's Hypothesis. The idea is to repeat the generic model construction transfinitely many times. Such iterations are described in the ground model.

Martin observed that many properties of a generic extension obtained by iteration follow from a single axiom that captures the combinatorial content of the model. The general principle has become known as Martin's Axiom. Martin's Axiom has become a favorite tool in combinatorial set theory and set-theoretic topology. Its consistency is proved by iterated forcing.

Two-Step Iteration

The basic observation is that a two-step iteration can be represented by a single forcing extension. Let P be a notion of forcing, and let $\dot{Q} \in V^P$ be a name for a partial ordering in V^P.

Definition 16.1.

(i) $P * \dot{Q} = \{(p, \dot{q}) : p \in P \text{ and } \Vdash_P \dot{q} \in \dot{Q}\}$,
(ii) $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ if and only if $p_1 \leq p_2$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_2$.

In (i), $\Vdash_P \varphi$ means that every condition in P forces φ; equivalently, $\|\varphi\|_{B(P)} = 1$.

Theorem 16.2. (i) *Let G be a V-generic filter on P, let $Q = \dot{Q}^G$, and let H be a $V[G]$-generic filter on Q. Then*

$$G * H = \{(p, \dot{q}) \in P * \dot{Q} : p \in G \text{ and } \dot{q}^G \in H\}$$

is a V-generic filter on $P * \dot{Q}$ and $V[G * H] = V[G][H]$.

(ii) *Let K be a V-generic filter on $P * \dot{Q}$. Then*

$$G = \{p \in P : \exists \dot{q}\, (p, \dot{q}) \in K\} \quad \text{and} \quad H = \{\dot{q}^G : \exists p\, (p, \dot{q}) \in K\}$$

are, respectively, a V-generic filter on P and a $V[G]$-generic filter on $Q = \dot{Q}^G$, and $K = G * H$.

Proof. (i) Let us prove that if $D \in V$ is a dense subset of $P*Q$ then $D \cap (G*H)$ is nonempty. In $V[G]$, let

$$D_1 = \{\dot{q}^G : \exists p \in G \text{ such that } (p, \dot{q}) \in D\}.$$

The set D_1 is dense in Q; this is proved by showing that for every $\dot{q}_0$, the set (in V)

$$\{p \in P : \exists \dot{q}_1 \, (p \Vdash \dot{q}_1 \leq \dot{q}_0 \text{ and } (p, \dot{q}_1) \in D)\}$$

is dense in P. Hence $D_1 \cap H \neq \emptyset$ and so there exists some $q \in H$ such that for some $p \in G$ and some G-name $\dot{q}$ for q, $(p, \dot{q}) \in D$. It follows that $(p, \dot{q}) \in D \cap (G * H)$.

(ii) Let $D \in V$ be dense in P. Then $D_1 = \{(p, \dot{q}) : p \in D\}$ is dense in $P * \dot{Q}$ and so $D \cap G$ is nonempty. Hence G is a V-generic filter on P.

Let $D \in V[G]$ be dense in Q, and let $\dot{D} \in V^P$ be a G-name for D such that $\Vdash_P \dot{D}$ is dense in $\dot{Q}$. Then the set $\{(p, \dot{q}) \in P * \dot{Q} : p \Vdash \dot{q} \in \dot{D}\}$ is dense in $P * \dot{Q}$ and it follows that $D \cap H$ is nonempty. Hence H is $V[G]$-generic.

The proof of $K = G * H$ is routine. □

We shall now describe two-step iteration in terms of complete Boolean algebras. Let B be a complete Boolean algebra and let $\dot{C} \in V^B$ be such that

$$\|\dot{C} \text{ is a complete Boolean algebra}\|_B = 1.$$

Let us consider all $\dot{c} \in V^B$ such that $\|\dot{c} \in \dot{C}\| = 1$ and the equivalence relation

$$\dot{c}_1 \equiv \dot{c}_2 \quad \text{if and only if} \quad \|\dot{c}_1 = \dot{c}_2\| = 1. \tag{16.1}$$

We let D be the set of equivalence classes for (16.1). We make D a Boolean algebra as follows: If $\dot{c}_1$ and $\dot{c}_2$ are in D, there exists a unique $\dot{c} \in D$ such that $\|\dot{c} = \dot{c}_1 +_{\dot{C}} \dot{c}_2\| = 1$; we let $\dot{c} = \dot{c}_1 +_D \dot{c}_2$. The operations $\cdot_D$ and $-_D$ are defined similarly. With these operations, D is a Boolean algebra; also,

$$\dot{c}_1 \leq_D \dot{c}_2 \quad \text{if and only if} \quad \|\dot{c}_1 \leq_{\dot{C}} \dot{c}_2\| = 1.$$

Lemma 16.3. *D is a complete Boolean algebra, and B embeds in D as a complete subalgebra.*

Proof. If $X \subset D$, let $\dot{X} \in V^B$ be such that $\mathrm{dom}(\dot{X}) = X$ and $\dot{X}(\dot{c}) = 1$ for all $\dot{c} \in X$. Since $\dot{C}$ is a complete Boolean algebra in V^B and V^B is full, there exists a $\dot{c}$ such that $\|\dot{c} = \sum_{\dot{C}} \dot{X}\| = 1$. It follows that $\dot{c} = \sum_D X$.

For each $b \in B$, let $\dot{c} = \pi(b)$ be the unique $\dot{c} \in D$ such that

$$\|\dot{c} = 1_{\dot{C}}\| = b \quad \text{and} \quad \|\dot{c} = 0_{\dot{C}}\| = -b;$$

π is a complete embedding of B into D. □

We use the notation $D = B * \dot{C}$. If $B = B(P)$ and in V^B, $\dot{C} = B(\dot{Q})$, then $P * \dot{Q}$ embeds densely in $B * \dot{C}$ (Exercise 16.1).

Two-step iteration is a generalization of product: If P and Q are two notions of forcing then $P \times Q$ embeds densely in $P * \check{Q}$ (Exercise 16.2).

If B and D are complete Boolean algebras and B is a complete subalgebra of D then there exists a $\dot{C} \in V^B$ that is a complete Boolean algebra in V^B, such that $D = B * \dot{C}$: In V^B, let $\dot{F}$ be the filter on $\check{D}$ generated by the generic ultrafilter $\dot{G}$ on $\check{B}$, and let $\dot{C}$ be the quotient of $\check{D}$ by $\dot{F}$. We denote this algebra (in V^B) $\dot{C} = D : B$. $D : B$ is a complete Boolean algebra in V^B, and $B * (D : B) = D$ (Exercises 16.3 and 16.4).

It follows that if $V[G]$ and $V[H]$ are two generic extensions of V such that $V[G] \subset V[H]$, then $V[H]$ is a generic extension of $V[G]$.

Theorem 16.4. *Let κ be a regular uncountable cardinal. If P satisfies the κ-chain condition and if in V^P, $\dot{Q}$ satisfies the κ-chain condition, then $P * \dot{Q}$ satisfies the κ-chain condition.*

Proof. Assume that $(p_\alpha, \dot{q}_\alpha)$, $\alpha < \kappa$, are mutually incompatible in $P * \dot{Q}$. Let $\dot{Z} \in V^P$ be the canonical name for the set $\{\alpha : p_\alpha \in G\}$ (where G is a generic filter on P), i.e., $\|\alpha \in \dot{Z}\| = p_\alpha$. For every α and every β, either p_α and p_β are incompatible, or every stronger condition forces that $\dot{q}_\alpha$ and $\dot{q}_\beta$ are incompatible. Thus q_α and q_β are incompatible if $\alpha \in Z$ and $\beta \in Z$, and since Q satisfies the κ-chain condition in $V[G]$, we have $|Z| < \kappa$; i.e., $\Vdash_P |\dot{Z}| < \kappa$.

Since κ is regular in $V[G]$ (by Theorem 15.3), there exists a maximal antichain $W \subset P$, and for each $p \in W$ there exists some $\gamma_p < \kappa$ such that $p \Vdash \dot{Z} \subset \gamma_p$. If we let $\gamma = \sup\{\gamma_p : p \in W\}$, we have $\gamma < \kappa$, and $\Vdash_P \dot{Z} \subset \gamma$. This is a contradiction, since $p_\gamma \Vdash \gamma \in \dot{Z}$. □

The converse of Theorem 16.4 is also true:

Lemma 16.5. *If $P * \dot{Q}$ satisfies the κ-chain condition then $\Vdash_P \dot{Q}$ satisfies the κ-chain condition.*

Of course P satisfies the κ-c.c. because $B(P)$ is a complete subalgebra of $B(P * \dot{Q})$.

Proof. Let $D = B * \dot{C}$ and assume that D satisfies the κ-chain condition. Let $\dot{W} \in V^B$ and $b_0 \in B^+$ be such that

$$b_0 \Vdash \dot{W} \text{ is a subset of } \dot{C}^+ \text{ of size } \kappa.$$

We shall find a nonzero $b \leq b_0$ such that

$$b \Vdash \dot{W} \text{ is not an antichain.} \tag{16.2}$$

Let $\dot{f} \in V^B$ be such that

$$b_0 \Vdash \dot{f} \text{ is a one-to-one function of } \kappa \text{ onto } \dot{W}.$$

For every $\alpha < \kappa$, $b_0 \Vdash (\exists x \in \dot{W})\, x = \dot{f}(\check{\alpha})$; and since V^B is full, there exists a $\dot{c}_\alpha \in D$ such that $b_0 \Vdash (\dot{c}_\alpha \in W$ and $\dot{c}_\alpha = \dot{f}(\check{\alpha}))$. Let $\dot{d}_\alpha = b_0 \cdot \dot{c}_\alpha$. Since $b_0 \Vdash \dot{c}_\alpha \neq \dot{c}_\beta$, for all $\alpha \neq \beta$, the set $\{\dot{d}_\alpha : \alpha < \kappa\}$ is a subset of D of size κ. Since D satisfies the κ-chain condition, there exist α and β such that $\dot{d}_\alpha$ and $\dot{d}_\beta$ are compatible. Hence there exists a $\dot{d} \in D^+$ such that $\dot{d} \leq \dot{d}_\alpha \cdot \dot{d}_\beta$; moreover, we can find $\dot{d}$ such that $\dot{d} = b \cdot \dot{c}$, where $0 \neq b \leq b_0$ and $b \Vdash (\dot{c} \neq 0$ and $\dot{c} \leq \dot{c}_\alpha \cdot \dot{c}_\beta)$. Now (16.2) follows. □

Corollary 16.6. *If P and Q satisfy the κ-chain condition then $P \times Q$ satisfies the κ-chain condition if and only if $\Vdash_P \check{Q}$ satisfies the κ-chain condition.* □

Lemma 16.7. *If P is κ-closed and $\Vdash_P \dot{Q}$ is κ-closed, then $P * \dot{Q}$ is κ-closed.*

Proof. Let $\lambda \leq \kappa$ and let $(p_1, \dot{q}_1) \geq (p_2, \dot{q}_2) \geq \ldots \geq (p_\alpha, \dot{q}_\alpha) \geq \ldots$ $(\alpha < \lambda)$ be a descending sequence in $P * \dot{Q}$. Then $\{p_\alpha\}_{\alpha<\lambda}$ is a descending sequence in P, and has a lower bound p. The condition p forces that $\{\dot{q}_\alpha\}_{\alpha<\lambda}$ is a descending sequence in $\dot{Q}$, and has a lower bound $\dot{q}$. Then $(p, \dot{q})$ is a lower bound of $\{(p_\alpha, \dot{q}_\alpha)\}_{\alpha<\lambda}$. □

Iteration with Finite Support

The idea of transfinite iteration of forcing is to construct sequences $\{P_\alpha\}_{\alpha<\theta}$ of forcing notions so that for every α, $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$ where $\dot{Q} \in V^{P_\alpha}$, and that at limit stages, P_α is a "limit" of $\{P_\beta\}_{\beta<\alpha}$. In this section we describe iteration with finite support, where the "limit" is the direct limit.

In Definition 16.8 below, $\dot{Q}_\alpha$ is assumed to be a forcing notion in V^{P_α}, with greatest element 1. The symbol $\leq_\alpha$ denotes the partial ordering of P_α, and $\Vdash_\alpha$ denotes the corresponding forcing relation.

Definition 16.8. Let $\alpha \geq 1$. A forcing notion P_α is an *iteration* (of *length* α *with finite support*) if it is a set of α-sequences with the following properties:

(i) If $\alpha = 1$ then for some forcing notion Q_0,
  (a) P_1 is the set of all 1-sequences $\langle p(0)\rangle$ where $p(0) \in Q_0$;
  (b) $\langle p(0)\rangle \leq_1 \langle q(0)\rangle$ if and only if $p(0) \leq q(0)$ (in Q_0).

(ii) If $\alpha = \beta + 1$ then $P_\beta = P_\alpha{\restriction}\beta = \{p{\restriction}\beta : p \in P_\alpha\}$ is an iteration of length β, and there is some forcing notion $\dot{Q}_\beta \in V^{P_\beta}$ such that
  (a) $p \in P_\alpha$ if and only if $p{\restriction}\beta \in P_\beta$ and $\Vdash_\beta p(\beta) \in \dot{Q}_\beta$;
  (b) $p \leq_\alpha q$ if and only if $p{\restriction}\beta \leq_\beta q{\restriction}\beta$ and $p{\restriction}\beta \Vdash_\beta p(\beta) \leq q(\beta)$.

(iii) If α is a limit ordinal, then for every $\beta < \alpha$, $P_\beta = P_\alpha{\restriction}\beta = \{p{\restriction}\beta : p \in P_\alpha\}$ is an iteration of length β and
  (a) $p \in P_\alpha$ if and only if $\forall \beta < \alpha\ p{\restriction}\beta \in P_\beta$ and for all but finitely many $\beta < \alpha$, $\Vdash_\beta p(\beta) = 1$;
  (b) $p \leq_\alpha q$ if and only if $\forall \beta < \alpha\ p{\restriction}\beta \leq_\beta q{\restriction}\beta$.

The finite set $\{\beta < \alpha : \text{not } \Vdash_\beta p(\beta) = 1\}$ is the *support* of $p \in P_\alpha$.

An iteration with finite support is uniquely determined by the sequence $\langle \dot{Q}_\beta : \beta < \alpha \rangle$. Thus we call P_α the iteration of $\langle \dot{Q}_\beta : \beta < \alpha \rangle$. For each $\beta < \alpha$, $P_{\beta+1}$ is isomorphic to $P_\beta * \dot{Q}_\beta$. When α is a limit ordinal, $(P_\alpha, \leq_\alpha)$ is the direct limit of the P_β, $\beta < \alpha$, in the sense of Lemma 12.2. In fact $B(P_\alpha)$ is the completion of the direct limit of the $B(P_\beta)$, $\beta < \alpha$ (Exercise 16.8).

Finite support iteration preserves chain conditions:

Theorem 16.9. *Let κ be a regular uncountable cardinal. Let P_α be the iteration with finite support of $\langle \dot{Q}_\beta : \beta < \alpha \rangle$, such that for each $\beta < \alpha$, $\Vdash_\beta \dot{Q}_\beta$ satisfies the κ-chain condition. Then P_α satisfies the κ-chain condition.*

Proof. By induction on α. If $\alpha = \beta + 1$ then $P_\alpha = P_\beta * \dot{Q}_\beta$ and the assertion follows from Theorem 16.4. Thus let α be a limit ordinal. For each $p \in P_\alpha$, let $s(p)$ denote the support of p.

Let $W = \{p_\xi : \xi < \kappa\}$ be a subset of P_α of size κ. If $\operatorname{cf} \alpha \neq \kappa$ then there exist a $\beta < \alpha$ and some $Z \subset W$ of size κ such that $s(p) \subset \beta$ for each $p \in Z$. Then $\{p \restriction \beta : \beta \in Z\} \subset P_\beta$ and since P_β satisfies the κ-chain condition, there exist p and q in Z such that $p \restriction \beta$ and $q \restriction \beta$ are compatible (in P_β). Since $s(p) \subset \beta$ and $s(q) \subset \beta$, p and q are compatible.

Thus assume that $\operatorname{cf} \alpha = \kappa$, and let $\{\alpha_\xi : \xi < \kappa\}$ be a normal sequence with limit α. Let $C \subset \kappa$ be the closed unbounded set of all η such that $s(p_\xi) \subset \alpha_\eta$ for all $\xi < \eta$. For each limit $\xi \in C$ there is some $\gamma(\xi) < \xi$ such that $s(p_\xi) \cap \alpha_\xi \subset \alpha_{\gamma(\xi)}$. By Fodor's Theorem there exist a stationary set $S \subset C$ and some $\gamma < \kappa$ such that $s(p_\xi) \cap \alpha_\xi \subset \alpha_\gamma$ for all $\xi \in S$.

Now consider the set $\{p_\xi \restriction \alpha_\gamma : \xi \in S\}$. This is a subset of P_{α_γ}, of size κ, and therefore there exist ξ and η in S, $\gamma < \xi < \eta$, such that $p_\xi \restriction \alpha_\gamma$ and $p_\eta \restriction \alpha_\gamma$ are compatible. Let $q \in P_{\alpha_\gamma}$ be a condition stronger than both $p_\xi \restriction \alpha_\gamma$ and $p_\eta \restriction \alpha_\gamma$, and consider the following α-sequence r:

$$r(\beta) = \begin{cases} q(\beta) & \text{if } \beta < \alpha_\gamma, \\ p_\xi(\beta) & \text{if } \alpha_\gamma \leq \beta < \alpha_\eta, \\ p_\eta(\beta) & \text{if } \alpha_\eta \leq \beta < \alpha. \end{cases} \tag{16.3}$$

It is easily verified that r is a condition in P_α and is stronger than both p_ξ and p_η. Thus p_ξ and p_η are compatible, and W is not an antichain. □

Theorem 16.9 gives the following corollary for complete Boolean algebras:

Corollary 16.10. *Let $B_0 \subset B_1 \subset \ldots \subset B_\beta \subset \ldots$ $(\beta < \alpha)$ be a sequence of complete Boolean algebras such that for all $\beta < \gamma$, B_β is a complete subalgebra of B_γ, and that for each limit ordinal γ, $\bigcup_{\beta<\gamma} B_\beta$ is dense in B_γ. If every B_β satisfies the κ-chain condition then $\bigcup_{\beta<\alpha} B_\beta$ satisfies the κ-chain condition.* □

Martin's Axiom

Definition 16.11 (Martin's Axiom (MA)). If $(P, <)$ is partially ordered set that satisfies the countable chain condition and if $\mathcal{D}$ is a collection of fewer than $2^{\aleph_0}$ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.

By Lemma 14.4, if $(P, <)$ is any partial ordering and if $\mathcal{D}$ is a countable collection of dense subsets of P, then a $\mathcal{D}$-generic filter on P exists. Hence Martin's Axiom is a consequence of the Continuum Hypothesis. Exercises 16.10 and 16.11 show that the restriction to fewer than continuum dense sets as well as some restriction on $(P, <)$ are necessary.

If κ is an infinite cardinal, let MA_κ be the statement

(16.4) If $(P, <)$ is a partially ordered set that satisfies the countable chain condition, and if $\mathcal{D}$ is a collection of at most κ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.

$\mathrm{MA}_{\aleph_0}$ is true by Lemma 14.4, and Martin's Axiom states that MA_κ holds for all $\kappa < 2^{\aleph_0}$. Exercise 16.10 shows that MA_κ implies that $\kappa < 2^{\aleph_0}$.

Lemma 16.12. *Martin's Axiom is equivalent to its restriction to partial orders of cardinality* $< \mathfrak{c}$:

(16.5) *If $(P, <)$ is a partially ordered set that satisfies the countable chain condition and $|P| < 2^{\aleph_0}$, and if $\mathcal{D}$ is a collection of at most κ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.*

Proof. Let P be a c.c.c. partially ordered set and let us assume that (16.5) holds. Let $\mathcal{D}$ be a family of fewer than $\mathfrak{c}$ dense subsets of P. For each $D \in \mathcal{D}$, we let W_D be a maximal incompatible subset of D. Since each W_D is countable, there exists a set $Q \subset P$ of size $< \mathfrak{c}$ such that $W_D \subset Q$ for all $D \in \mathcal{D}$, and if $p, q \in Q$ are compatible, then there exists some $r \in Q$ such that $r \leq p$ and $r \leq q$. Each W_D is a maximal antichain in Q; let $E_D = \{q \in Q : q \leq w \text{ for some } w \in W_D\}$. Each E_D is dense in Q.

The partially ordered set Q has size at most κ and satisfies the countable chain condition. By (16.5) there is a filter G on Q that meets every E_D. G generates a $\mathcal{D}$-generic filter on P. □

We will now show that MA is consistent with $2^{\aleph_0} > \aleph_1$:

Theorem 16.13 (Solovay and Tennenbaum). *Assume* GCH *and let κ be a regular cardinal greater than $\aleph_1$. There exists a c.c.c. notion of forcing P such that the generic extension $V[G]$ by P satisfies Martin's Axiom and $2^{\aleph_0} = \kappa$.*

As P satisfies the countable chain condition, the model $V[G]$ preserves cardinals and cofinalities.

Proof. We construct P as a finite support iteration of length κ, of a certain (yet to be determined) sequence $\langle \dot{Q}_\alpha : \alpha < \kappa \rangle$. At each stage, we'll have $\Vdash_\alpha \dot{Q}_\alpha$ satisfies the countable chain condition, and so P will satisfy c.c.c. as well. We shall also have, for each $\alpha < \kappa$, $\Vdash_\alpha |\dot{Q}_\alpha| < \kappa$. It follows, by induction on α, that $|P_\alpha| \leq \kappa$ for every $\alpha \leq \kappa$: If α is a limit ordinal and if $|P_\beta| \leq \kappa$ for all $\beta < \alpha$, then $|P_\alpha| \leq \kappa$ since the elements of P_α are α-sequences with finite support. Thus assume that $|P_\alpha| \leq \kappa$ and let us prove $|P_{\alpha+1}| \leq \kappa$. Because P_α satisfies c.c.c. and κ is regular, there exists a $\lambda < \kappa$ such that $\Vdash_\alpha |\dot{Q}_\alpha| \leq \lambda$. Every name $\dot{q}$ for an element of $\dot{Q}_\alpha$ can be represented by a function from an antichain in P_α into λ. As every antichain in P_α is countable, the number of such functions is at most $\kappa^{\aleph_0}$ which is κ (by GCH). It follows that $|P_{\alpha+1}| \leq \kappa$; in fact $|B(P_{\alpha+1})| \leq \kappa$.

Note that because GCH holds in V, and because P_α is a c.c.c. forcing of size $\leq \kappa$, we have $\Vdash_\alpha 2^\lambda \leq \kappa$, for every $\lambda < \kappa$. In particular, $\Vdash_P 2^{\aleph_0} \leq \kappa$.

We shall now define the $\dot{Q}_\alpha$, by induction on $\alpha < \kappa$. Let us fix a function π that maps κ onto $\kappa \times \kappa$ such that if $\pi(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$. For every $\alpha < \kappa$, the model V^{P_α} has at most κ nonisomorphic partial orderings of size $< \kappa$ (because $\Vdash_\alpha \kappa^{<\kappa} = \kappa$). Since P_α satisfies c.c.c., there are at most κ distinct names in V^{P_α} for such partial orderings.

Thus let us assume that $\alpha < \kappa$ and that $\langle \dot{Q}_\beta : \beta < \alpha \rangle$ has been defined. Let $\pi(\alpha) = (\beta, \gamma)$. Let $\dot{Q}$ be the γth name in V^{P_β} for a partial order with a greatest element 1, of size $< \kappa$. Let $b = \|\dot{Q}$ satisfies the countable chain condition$\|_{P_\alpha}$ and let $\dot{Q}_\alpha \in V^{P_\alpha}$ be such that $\|\dot{Q}_\alpha = \dot{Q}\|_{P_\alpha} = b$ and $\|\dot{Q}_\alpha = \{1\}\|_{P_\alpha} = -b$.

Now let P be the finite support iteration of $\langle \dot{Q}_\alpha : \alpha < \kappa \rangle$. We shall prove that V^P satisfies Martin's Axiom as well as $2^{\aleph_0} = \kappa$. Let G be a generic filter on P, and let $G_\alpha = G \restriction P_\alpha$ for all $\alpha < \kappa$.

Lemma 16.14. *If $\lambda < \kappa$ and $X \subset \lambda$ is in $V[G]$ then $X \in V[G_\alpha]$ for some $\alpha < \kappa$.*

Proof. Let $\dot{X}$ be a name for X. Every Boolean value $\|\xi \in \dot{X}\|$ (where $\xi < \lambda$) is determined by a countable antichain in P and hence $\dot{X}$ is determined by at most λ conditions in P. Every condition has finite support which in turn is included in some $\alpha < \kappa$. Therefore there exists some $\alpha < \kappa$ such that all these λ conditions have support included in α. It follows that X has a name in V^{P_α}; hence $X \in V[G_\alpha]$. □

Lemma 16.15. *Let $(Q, <) \in V[G]$ and $\mathcal{D} \in G$ be such that $(Q, <)$ is a c.c.c. partial order, $|Q| < \kappa$ and $|\mathcal{D}| < \kappa$. There exists in $V[G]$ a $\mathcal{D}$-generic filter on Q.*

Once we prove Lemma 16.15, we finish the proof of Theorem 16.13 as follows: Let Q be the forcing for adding one Cohen generic real; Q is countable. For any set $X \subset \{0,1\}^\omega$ of size $< \kappa$, let $\mathcal{D}_X = \{D_g : g \in X\}$ where $D_g = \{q \in Q : q \not\subset g\}$ (see Exercise 16.10). Lemma 16.15 applied to $\mathcal{D}_X$ shows

that $X \neq \{0,1\}^\omega$ and therefore $V[G]$ satisfies $2^{\aleph_0} \geq \kappa$. However, we already proved that $2^{\aleph_0} \leq \kappa$, and so $V[G] \models 2^{\aleph_0} = \kappa$. Thus $V[G]$ satisfies (16.5) and therefore MA, completing the proof. □

Proof of Lemma 16.15. By Lemma 16.14, both $(Q, <)$ and $\mathcal{D}$ are in $V[G_\beta]$, for some $\beta < \kappa$. Let $\dot{Q}$ be a name for Q in V^{P_β}. We may assume that Q has a greatest element, and let γ be such that $\dot{Q}$ is the γth name for such partial order. Let α be such that $\pi(\alpha) = (\beta, \gamma)$. As Q satisfies the countable chain condition in $V[G]$, it also satisfies the countable chain condition in $V[G_\alpha]$. Thus $Q = \dot{Q}_\alpha^{G_\alpha}$.

In $V[G_{\alpha+1}]$ there is a generic filter H on Q over $V[G_\alpha]$, because $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$. The filter H meets every dense subset of Q that is in $V[G_\alpha]$, and therefore it meets every $D \in \mathcal{D}$. Hence H is $\mathcal{D}$-generic. □

Independence of Suslin's Hypothesis

Suslin's Hypothesis (SH) is the statement there are no Suslin lines. In Chapter 15 we showed that the negation of SH is consistent; by the following theorem, SH is independent.

Theorem 16.16. *If* $\mathrm{MA}_{\aleph_1}$ *holds, then there is no Suslin tree.*

Proof. Let us assume that T is a normal Suslin tree and let P_T be the partially ordered set obtained from T by reversing the order. P_T satisfies the countable chain condition. For each $\alpha < \omega_1$, let D_α be the union of all levels above α: $D_\alpha = \{x \in T : o(x) > \alpha\}$. Each D_α is dense in P_T; if we let $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ and if G is a $\mathcal{D}$-generic filter on P, then G is a branch in T of length ω_1. A contradiction. □

The proof of independence of SH was the first application of iterated forcing (and led to the formulation of Martin's Axiom). The model for SH, due to Solovay and Tennenbaum [1971], was constructed by iteration of the forcing notions P_T, for all prospective Suslin trees in the final model. The forcing P_T "kills" the Suslin tree T by forcing an ω_1-branch in T.

In the proof of the following theorem, Suslin trees are killed by a different method: by specializing the tree. Recalling the definition in Chapter 9 (and Exercise 9.9), an Aronszajn tree T is *special* if there exists a function $f : T \to \omega$ such that each $f_{-1}(\{n\})$ is an antichain.

Theorem 16.17 (Baumgartner, Malitz, and Reinhardt [1970]). *If* $\mathrm{MA}_{\aleph_1}$ *holds, then every Aronszajn tree is special.*

Lemma 16.18. *If T is an Aronszajn tree and W is an uncountable collection of finite pairwise disjoint subsets of T, then there exist $S, S' \in W$ such that any $x \in S$ is incomparable with any $y \in S'$.*

Proof. Since uncountably many elements of W have the same size, we may as well assume that there exists a natural number n such that $|S| = n$ for all $S \in W$; furthermore let us consider a fixed enumeration $\{z_1, \ldots, z_n\}$ of each set $S \in W$. Let D be an ultrafilter on W such that every $X \in D$ is uncountable.

Let us assume that the lemma is false. For each $x \in T$ and each $k = 1, \ldots, n$, let $Y_{x,k}$ be the set of all $S \in W$ such that x is comparable with the kth element of S. Since any S and S' contain comparable elements, we have

$$\bigcup_{x \in S} \bigcup_{k=1}^{n} Y_{x,k} = W$$

for every $S \in W$. Thus pick, for each $S \in W$, an element $x = x_S$ of S and $k = k_S$ such that $Y_{x,k} \in D$. Now, there is $k \leq n$ such that the set $Z = \{S \in W : k_S = k\}$ is uncountable. We shall show that the elements x_S, $S \in Z$, are pairwise comparable; and that will be a contradiction since T has no uncountable branch.

If $S_1, S_2 \in Z$ and $x = x_{S_1}$, $y = x_{S_2}$, then $Y = Y_{x,k} \cap Y_{y,k}$ is in the ultrafilter and thus uncountable. If $S \in Y$, then the kth element of S is comparable with both x and y. Since Y is uncountable, there must exist $S \in Y$ such that the kth element of S is greater than both x and y. But then it follows that x and y are comparable. □

Let T be an Aronszajn tree and let us consider the following notion of forcing $(P, <)$: Forcing conditions are functions p such that

(16.6)
(i) $\mathrm{dom}(p)$ is a finite subset of T;
(ii) $\mathrm{ran}(p) \subset \omega$;
(iii) if $x, y \in \mathrm{dom}(p)$ and x and y are comparable, then $p(x) \neq p(y)$;
(iv) p is stronger than q if and only if p extends q.

Lemma 16.19. *$(P, <)$ satisfies the countable chain condition.*

Proof. Let W be an uncountable subset of P. Note that the set $\{\mathrm{dom}(p) : p \in W\}$ is uncountable (there are only countably many functions from a finite set into ω). By Δ-Lemma, there is an uncountable $W_1 \subset W$, and a finite set $S \subset T$ such that $\mathrm{dom}(p) \cap \mathrm{dom}(q) = S$ for any distinct elements $p, q \in W_1$. Then there is an uncountable $W_2 \subset W_1$ such that $p{\restriction}S = q{\restriction}S$ for any $p, q \in W_2$. By Lemma 16.18 there exist p and $q \in W_2$ such that any $x \in \mathrm{dom}(p) - S$ is incomparable with any $y \in \mathrm{dom}(q) - S$. Then $p \cup q$ is a function that satisfies (16.6) and extends both p and q. Thus p and q are compatible elements of W and so $(P, <)$ satisfies the countable chain condition. □

Proof of Theorem 16.17. For each $x \in T$, let D_x be the set of all $p \in P$ such that $x \in \mathrm{dom}(p)$; clearly, each D_x is dense in P. Let $\mathcal{D} = \{D_x : x \in T\}$.

It follows from $\mathrm{MA}_{\aleph_1}$, that $(P, <)$ has a $\mathcal{D}$-generic filter G. The elements of G are pairwise compatible and since G is $\mathcal{D}$-generic, every $x \in T$ is in the domain of the function $f = \bigcup G$. The function f maps T into ω and witnesses that T is a special Aronszajn tree. □

More Applications of Martin's Axiom

Theorem 16.20 (Martin-Solovay). *Martin's Axiom implies that* $\mathfrak{c}$ *is regular, and* $2^\kappa = \mathfrak{c}$ *for all infinite cardinals* $\kappa < \mathfrak{c}$.

Proof. Assuming MA, we prove that $2^\kappa = 2^{\aleph_0}$ for every $\kappa < 2^{\aleph_0}$. Regularity of $\mathfrak{c}$ follows, as $\operatorname{cf} 2^{\aleph_0} = \operatorname{cf} 2^\kappa > \kappa$ for all $\kappa < 2^{\aleph_0}$. Let $\kappa < 2^{\aleph_0}$ and let $\{A_\alpha : \alpha < \kappa\}$ be an almost disjoint family of subsets of ω.

Let X be a subset of κ. We shall find a set $A \subset \omega$ such that for all $\alpha < \kappa$

$$\alpha \in X \quad \text{if and only if} \quad A \cap A_\alpha \text{ is infinite.} \tag{16.7}$$

In other words, $X = \{\alpha \in \kappa : A \cap A_\alpha \text{ is infinite}\}$ is "coded" by the set A. Therefore there exists a mapping of $P(\omega)$ onto $P(\kappa)$, and so $2^\kappa \leq 2^{\aleph_0}$.

Let $(P, <)$ be the following notion of forcing: A condition is a function p from a subset of ω into $\{0, 1\}$ such that:

(16.8) (i) $\operatorname{dom}(p) \cap A_\alpha$ is finite for every $\alpha \in X$;
(ii) $\{n : p(n) = 1\}$ is finite.

The set P is partially ordered by reverse inclusion: $p \leq q$ if and only if p extends q.

We first show that P satisfies the countable chain condition. If p and q are incompatible, then $\{n : p(n) = 1\} \neq \{n : q(n) = 1\}$ and since there are only countably many finite subsets of ω, it follows that P satisfies c.c.c.

For each $\beta \in \kappa - X$, let $D_\beta = \{p \in P : A_\beta \subset \operatorname{dom}(p)\}$. Any $q \in P$ can be extended to some $p \in D_\beta$: Simply let $p(n) = 0$ for all $n \in A_\beta - \operatorname{dom}(p)$. Since A_β is almost disjoint from all A_α, $\alpha \in X$, p has property (16.8)(i) and hence is a condition. Thus each D_β is dense.

For each $\alpha \in X$ and each $k \in \omega$, let

$$E_{\alpha,k} = \{p \in P : \{n \in A_\alpha : p(n) = 1\} \text{ has size at least } k\}.$$

It is easy to see that each $E_{\alpha,k}$ is dense in P.

Let $\mathcal{D}$ be the collection of all D_β for $\beta \in \kappa - X$ and all $E_{\alpha,k}$ for $\alpha \in X$ and $k \in \omega$. By MA, there exists a $\mathcal{D}$-generic filter G on P. Note that $f = \bigcup G$ is a function on a subset of ω. We let

$$A = \{n : f(n) = 1\} = \{n : p(n) = 1 \text{ for some } p \in G\}. \tag{16.9}$$

If $\alpha \in X$, then $A \cap A_\alpha$ is infinite because for each k there is some $p \in G \cap E_{\alpha,k}$. If $\beta \in \kappa - X$, then $A \cap A_\beta$ is finite because for some $p \in G$, $A_\beta \subset \operatorname{dom}(p)$ and $\{n : p(n) = 1\}$ is finite. □

The *almost disjoint forcing* defined in the proof of Theorem 16.20 is often used to code generically uncountable sets. A typical application is the following:

Let $V[X]$ be a generic extension where $X \subset \omega_1$; furthermore, assume that $\omega_1^{V[X]} = \omega_1$. Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ be an almost disjoint family in V, and let us consider the almost disjoint forcing P in $V[X]$. If $G \subset P$ is generic over $V[X]$, then $V[X][G] = V[X][A]$, where A is defined by (16.9). Note that $\omega_1^{V[X][A]} = \omega_1$.

Now in $V[X][A]$, the set X satisfies (16.7), and since $\mathcal{A} \in V$, it follows that $X \in V[A]$, and we have $V[X][A] = V[A]$. Thus we have found a generic extension $V[A]$ such that $A \subset \omega$ and $X \in V[A]$. See Exercise 16.15.

The next theorem shows that under $\mathrm{MA}_{\aleph_1}$, countable chain condition is preserved by products. Compare with Exercise 15.28.

Theorem 16.21. $\mathrm{MA}_{\aleph_1}$ *implies that every partially ordered set that satisfies the countable chain condition has property* (K).

Proof. Let P be a partially ordered set that satisfies the countable chain condition and let $W = \{w_\alpha : \alpha < \omega_1\}$ be an uncountable subset of P. We will use $\mathrm{MA}_{\aleph_1}$ to find a filter G such that $Z = G \cap W$ is uncountable.

First we claim that there is some $p_0 \in W$ such that every $p \leq p_0$ is compatible with uncountably many w_α. Otherwise, for each $\alpha < \omega_1$ there is $\beta > \alpha$ and some $v_\alpha \leq w_\alpha$ which is incompatible with all w_γ, $\gamma \geq \beta$; then we can construct an ω_1-sequence $\{v_{\alpha_i} : i < \omega_1\}$ of pairwise incompatible elements.

For each $\alpha < \omega_1$, let

$$D_\alpha = \{p \leq p_0 : p \leq w_\gamma \text{ for some } \gamma \geq \alpha\}.$$

By the above claim, each D_α is dense below p_0. By $\mathrm{MA}_{\aleph_1}$, there exists a filter G on P such that $p_0 \in G$ and $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. It follows that $G \cap W$ is uncountable. Hence P has property (K). □

Corollary 16.22. $\mathrm{MA}_{\aleph_1}$ *implies that if every* P_i, $i \in I$, *satisfies the countable chain condition then so does the product* $\prod_{i \in I} P_i$ *(with finite support).*

Proof. Theorem 15.15. □

The next result generalizes the Baire Category Theorem:

Theorem 16.23. *Martin's Axiom implies that the intersection of fewer than* $\mathfrak{c}$ *dense open sets of reals is dense.*

Proof. Let $\kappa < \mathfrak{c}$ and let U_α, $\alpha < \kappa$, be dense open sets of reals. Let I be a bounded open interval. We'll show that $\bigcap_{\alpha<\kappa} U_\alpha \cap I \neq \emptyset$. Let P be the following notion of forcing: Conditions are nonempty open sets p such that $\bar{p} \subset I$, with $p \leq q$ if and only if $p \subset q$. Since every collection of disjoint

open sets is at most countable, P satisfies the countable chain condition. For each $\alpha < \kappa$, let $D_\alpha = \{p \in P : \bar{p} \subset U_\alpha\}$; each D_α is dense in P. Let G be a $\mathcal{D}$-generic filter on P where $\mathcal{D} = \{D_\alpha : \alpha < \kappa\}$. Since G is a filter, the intersection $\bigcap\{\bar{p} : p \in G\}$ is nonempty, and is contained in each U_α since $G \cap D_\alpha \neq \emptyset$. □

If f and g are functions from ω to ω, we say that f *eventually dominates* g if $f(n) > g(n)$ for all but finitely many $n \in \omega$ (i.e., $f > g$ in the notation of Lemma 10.16). A set of functions $\mathcal{G}$ is eventually dominated by f if $f > g$ for all $g \in \mathcal{G}$.

Theorem 16.24. *Martin's Axiom implies that every family $\mathcal{G}$ of fewer than $\mathfrak{c}$ functions from ω to ω is eventually dominated by some $f \in \omega^\omega$.*

Corollary 16.25. MA *implies that there exists a $\mathfrak{c}$-scale.*

Proof. A scale is constructed by transfinite induction, using an enumeration of ω^ω of order-type $\mathfrak{c}$. □

Corollary 16.26. MA *implies that $\mathfrak{c}$ is not real-valued measurable.*

Proof. Lemma 10.16. □

The proof of Theorem 16.24 uses the *Hechler forcing*: Let $\mathcal{G}$ be a given family of functions $h : \omega \to \omega$. A forcing condition is a pair $p = (s, E)$, where $s = \langle s(0), \ldots, s(n-1)\rangle$ is a finite sequence of natural numbers and E is a finite subset of $\mathcal{G}$. A condition (s', E') is stronger than (s, E) if:

(16.10) (i) $s \subset s'$, and $E \subset E'$;
(ii) if $k \in \mathrm{dom}(s') - \mathrm{dom}(s)$, then $s(k) > h(k)$ for all $h \in E$.

If (s_1, E_1) and (s_2, E_2) are conditions and $s_1 = s_2$, then (s_1, E_1) and (s_2, E_2) are compatible. Hence $(P, <)$ satisfies the countable chain condition. Let $G \subset P$ be generic; we let $f = \bigcup\{s : (s, E) \in G \text{ for some } E\}$. We claim that $\mathcal{G}$ is eventually dominated by f. Let $h \in \mathcal{G}$ be arbitrary. First there is a condition $(s, E) \in G$ such that $h \in E$ (by genericity). Secondly, every condition $(s', E') < (s, E)$ satisfies (16.10)(ii), and so $f(k) > g(k)$ for all $k \notin \mathrm{dom}(s)$. Thus in $V[G]$, there is $f : \omega \to \omega$ such that $h < f$ for all $h \in \mathcal{G}$.

Proof of Theorem 16.24. If $\mathcal{G} \subset \omega^\omega$ and $|\mathcal{G}| < \mathfrak{c}$, let P be the Hechler forcing for the family $\mathcal{G}$. Let $\mathcal{D} = \{D_h : h \in \mathcal{G}\} \cup \{E_n : n \in \omega\}$ where $D_h = \{(s, E) : h \in E\}$ and $E_n = \{(s, E) : n \in \mathrm{dom}(s)\}$. Then if G is a $\mathcal{D}$-generic filter, the function $f = \bigcup\{s : (s, E) \in G \text{ for some } E\}$ eventually dominates all $h \in \mathcal{G}$. □

Theorem 16.27 (Booth). *Martin's Axiom implies that there exists a p-point.*

Proof. Let $\mathcal{A}_\alpha$, $\alpha < 2^{\aleph_0}$, be an enumeration of all decreasing sequences $\{A_n\}_{n=0}^\infty$ of subsets of ω. We construct, by induction on $\alpha < 2^{\aleph_0}$, a chain of families $\mathcal{G}_0 \subset \ldots \subset \mathcal{G}_\alpha \subset \ldots$ of nonempty subsets of ω, such that each $\mathcal{G}_\alpha$ is closed under finite intersections and $|\mathcal{G}_\alpha| < 2^{\aleph_0}$ for all α.

We let $\mathcal{G}_0 = \{X \subset \omega : \omega - X \text{ is finite}\}$. If α is a limit ordinal, we let $\mathcal{G}_\alpha = \bigcup_{\beta<\alpha} \mathcal{G}_\alpha$. Having constructed $\mathcal{G}_\alpha$, we construct $\mathcal{G}_{\alpha+1}$ as follows: Let $\mathcal{A}_\alpha = \{A_n\}_{n=0}^\infty$ be a decreasing sequence of subsets of ω. If some A_n is disjoint from some $X \in \mathcal{G}_\alpha$, then we let $\mathcal{G}_{\alpha+1} = \mathcal{G}_\alpha$. Otherwise, the family $\mathcal{G} = \mathcal{G}_\alpha \cup \{A_n : n \in \omega\}$ has the finite intersection property and we claim (see Lemma 16.28 below), that there exists a $Z \subset \omega$ such that $Z - A_n$ is finite for all n, and $\mathcal{G}' = \mathcal{G} \cup \{Z\}$ has the finite intersection property. Then we let $\mathcal{G}_{\alpha+1}$ consist of all finite intersections $X_1 \cap \ldots \cap X_k$ of elements of $\mathcal{G}'$.

Finally, we let $\mathcal{G} = \bigcup\{\mathcal{G}_\alpha : \alpha < 2^{\aleph_0}\}$, and let D be any ultrafilter such that $D \supset \mathcal{G}$. We claim that D is a p-point: If $A_0 \supset A_1 \supset \ldots A_n \supset \ldots$ is any decreasing sequence of elements of D, then $\{A_n\}_{n=0}^\infty = \mathcal{A}_\alpha$ for some $\alpha < 2^{\aleph_0}$ and we have $Z \in \mathcal{G}_{\alpha+1}$ such that $Z - A_n$ is finite for all n. By Exercise 7.7, D is a p-point. □

It remains to prove the claim:

Lemma 16.28. *Assume* MA, *and let $\mathcal{G}$ be a family of subsets of ω with the finite intersection property such that $|\mathcal{G}| < 2^{\aleph_0}$. Let $A_0 \supset A_1 \supset \ldots A_n \supset \ldots$ be a decreasing sequence of elements of $\mathcal{G}$. Then there exists a $Z \subset \omega$ such that*:

(i) *$\mathcal{G} \cup \{Z\}$ has the finite intersection property*;
(ii) *$Z - A_n$ is finite for all $n \in \omega$.*

Proof. We may assume that that if $X, Y \in \mathcal{G}$, then $X \cap Y \in \mathcal{G}$. For each $X \in \mathcal{G}$, let $h_X : \omega \to \omega$ be some function such that $h_X(n) \in X \cap A_n$. By Theorem 16.24 the family $\{h_X : X \in \mathcal{G}\}$ is eventually dominated by a function f; in particular for every $X \in \mathcal{G}$ there exists some n such that $f(n) \geq h_X(n)$. Now we let $Z = \bigcup_{n=0}^\infty \{k \in A_n : k \leq f(n)\}$. It is readily verified that $Z - A_n$ is finite for each n, and that $Z \cap X \neq \emptyset$ for every $X \in \mathcal{G}$. □

Iterated Forcing

We conclude this chapter with the general definition of iterated forcing. We shall return to the general method in later chapters. Below we follow closely Definition 16.8 of finite support iteration. As before, for each ordinal $\alpha \geq 1$, P_α denotes an iteration of length α, $\leq_\alpha$ is the partial ordering of P_α and $\Vdash_\alpha$ is the corresponding forcing relation, and $\dot{Q}_\alpha$ is a name in V^{P_α} for a forcing notion with a greatest element 1. The general definition differs from Definition 16.8 by its handling of limit stages.

Definition 16.29. Let $\alpha \geq 1$. A forcing notion P_α is an iteration (of length α) if it is a set of α-sequences with the following properties:

(i) If $\alpha = 1$ then for some forcing notion Q_0,
  (a) P_1 is the set of all 1-sequences $\langle p(0)\rangle$ where $p(0) \in Q_0$;
  (b) $\langle p(0)\rangle \leq_1 \langle q(0)\rangle$ if and only if $p(0) \leq q(0)$.

(ii) If $\alpha = \beta + 1$ then $P_\beta = P_\alpha{\restriction}\beta = \{p{\restriction}\beta : p \in P_\alpha\}$ is an iteration of length β, and there is some forcing notion $\dot{Q}_\beta \in V^{P_\beta}$ such that
  (a) $p \in P_\alpha$ if and only if $p{\restriction}\beta \in P_\beta$ and $\Vdash_\beta p(\beta) \in \dot{Q}_\beta$;
  (b) $p \leq_\alpha q$ if and only if $p{\restriction}\beta \leq_\beta q{\restriction}\beta$ and $p{\restriction}\beta \Vdash_\beta p(\beta) \leq q(\beta)$.

(iii) If α is a limit ordinal, then for every $\beta < \alpha$, $P_\beta = P_\alpha{\restriction}\beta = \{p{\restriction}\beta : p \in P_\alpha\}$ is an iteration of length β and
  (a) the α-sequence $\langle 1, 1, \ldots, 1, \ldots\rangle$ is in P_α;
  (b) if $p \in P_\alpha$, $\beta < \alpha$ and if $q \in P_\beta$ is such that $q \leq_\beta p{\restriction}\beta$, then $r \in P_\alpha$ where for all $\xi < \alpha$, $r(\xi) = q(\xi)$ if $\xi < \beta$ and $r(\xi) = p(\xi)$ if $\beta \leq \xi < \alpha$;
  (c) $p \leq_\alpha q$ if and only if $\forall \beta < \alpha\ p{\restriction}\beta \leq_\beta q{\restriction}\beta$.

Clearly, an iteration with finite support is an iteration. In general, property (iii)(b) guarantees that if $P_\beta = P_\alpha{\restriction}\beta$ then $V^{P_\beta} \subset V^{P_\alpha}$; see Exercise 16.17.

A general iteration depends not only on the $\dot{Q}_\beta$ but also on the limit stages of the iteration. Let P_α be an iteration of length α where α is a limit ordinal. P_α is a *direct limit* if for every α-sequence p,

$$p \in P_\alpha \quad \text{if and only if} \quad \exists \beta < \alpha\ p{\restriction}\beta \in P_\beta \text{ and } \forall \xi \geq \beta\ p(\xi) = 1. \tag{16.11}$$

P_α is an *inverse limit* if for every α-sequence p,

$$p \in P_\alpha \quad \text{if and only if} \quad \forall \beta < \alpha\ p{\restriction}\alpha \in P_\beta. \tag{16.12}$$

In practice, forcing iterations combine direct and inverse limits. Finite support iterations are exactly those that use direct limits at all limit stages. In general, let $s(p)$, the *support* of $p \in P_\alpha$, be the set of all $\beta < \alpha$ such that it is not the case that $\Vdash_\beta p(\beta) = 1$. If I is an ideal on α containing all finite sets then an *iteration with I-support* is an iteration that satisfies for every limit ordinal $\gamma \leq \alpha$,

$$p \in P_\gamma \quad \text{if and only if} \quad \forall \beta < \gamma\ p{\restriction}\beta \in P_\beta \text{ and } s(p) \in I. \tag{16.13}$$

One of the most useful tools in forcing are iterations with *countable support*, where in (16.13) I is the ideal of at most countable sets. A countable support iteration is an iteration such that for every limit ordinal γ if $\operatorname{cf}\gamma = \omega$ then P_γ is an inverse limit, and if $\operatorname{cf}\gamma > \omega$ then P_γ is a direct limit. We shall return to countable support iterations later in the book.

The following generalizes Theorem 16.9:

Theorem 16.30. *Let κ be a regular uncountable cardinal and let α be a limit ordinal. Let P_α be an iteration such that for each $\beta < \alpha$, $P_\beta = P_\alpha \restriction \beta$ satisfies the κ-chain condition. If P_α is a direct limit, and either $\operatorname{cf}\alpha \neq \kappa$ or (if $\operatorname{cf}\alpha = \kappa$) for a stationary set of $\beta < \alpha$, P_β is a direct limit, then P_α satisfies the κ-chain condition.*

Proof. Exactly as the proof of Theorem 16.9. The only difference is that we apply Fodor's Theorem not to C, but to the stationary subset of C consisting of all ξ such that P_{α_ξ} is a direct limit. □

Exercises

16.1. $B(P * \dot{Q}) = B(P) * B(\dot{Q})$.

16.2. $P \times Q$ embeds densely in $P * \check{Q}$.

16.3. In V^B, $D : B = D/\dot{I}$ where for each $d \in D$, $\|d \in \dot{I}\|_B = \sum\{b \in B : b \cdot d = 0\}$.

16.4. $\|D : B \text{ is a complete Boolean algebra}\|_B = 1$, and D is isomorphic to $B * (D : B)$.

[Every name for an element of $D : B$ has the form d/I where $d \in D$. To see that $D : B$ is complete in V^B, let A be a name for a subset of $D : B$, and let $e = \sum\{d : \|d/I \in A\|\} = 1$. Then $\|e/I = \sum A\| = 1$.]

16.5. Let $h : P * \dot{Q} \to P$ be defined by $h(p, \dot{q}) = p$. Then h satisfies the conditions in Lemma 15.45.

16.6. If P has property (K) and $\Vdash_P \dot{Q}$ has property (K), then $P * \dot{Q}$ has property (K).

16.7. If P is κ-distributive and $\Vdash_P \dot{Q}$ is κ-distributive then $P * \dot{Q}$ is κ-distributive.

16.8. Let P_α, α a limit ordinal, be a finite support iteration, and $B_\beta = B(P_\alpha \restriction \beta)$ for all $\beta \leq \alpha$. Then B_α is the completion of the direct limit of the algebras B_β, $\beta < \alpha$.

16.9. If P_α is a finite support iteration and $P_\beta = P_\alpha \restriction \beta$ then $V^{P_\beta} \subset V^{P_\alpha}$. The projection $h(p) = p \restriction \beta$ satisfies Lemma 15.45; $G_\beta = \{p \restriction \beta : p \in G\}$ is a generic filter on P_β.

16.10. Let $(P, <)$ be the notion of forcing producing a Cohen generic real. There is a collection $\mathcal{D}$ of size $2^{\aleph_0}$ of dense subsets of P such that there is no $\mathcal{D}$-generic filter on P.

[For each $g : \omega \to \{0, 1\}$, let $D_g = \{p \in P : p \not\subset g\}$.]

16.11. Let $(P, <)$ be the notion of forcing that collapses ω_1. There is a collection $\mathcal{D}$ of size $\aleph_1$ of dense subsets of P such that there is no $\mathcal{D}$-generic filter on P.

[For each $\alpha < \omega_1$, let $D_\alpha = \{p \in P : \alpha \in \operatorname{ran}(p)\}$.]

16.12. MA_κ is equivalent to the statement of MA_κ restricted to complete Boolean algebras.

16.13. MA_κ is equivalent to of MA_κ restricted to partial orders of cardinality $\leq \kappa$.

16.14. Let T be a Suslin tree and let P be the notion of forcing that adjoins κ Cohen generic reals. Let G be a generic filter on P. Then T is a Suslin tree in $V[G]$.

[Let P_T be the notion of forcing associated with the Suslin tree T. P satisfies the c.c.c. in any $V[H]$ where H is a generic filter on P_T. Thus $P_T \times P$ is c.c.c., and so P_T is c.c.c. in V[G].]

It follows that the existence of a Suslin tree is consistent with $2^{\aleph_0} > \aleph_1$.

16.15. There is a generic extension $V[A]$ where $A \subset \omega$, such that $\omega_1^{V[A]} = \omega_1$, and ω_2 is collapsed.

[Let f be a generic mapping of ω_1 onto ω_2 and let $X \subset \omega_1$ be such that $V[f] = V[X]$. Use almost disjoint forcing to find $A \subset \omega$ such that $V[A] = V[X][A]$.]

16.16. Assume MA_κ and let $\{X_\alpha : \alpha < \kappa\}$ be a sequence of infinite subsets of ω such that $X_\beta - X_\alpha$ is finite if $\alpha < \beta$. Show that there exists an infinite X such that $X - X_\alpha$ is finite for all $\alpha < \kappa$.

[A forcing condition is a pair (s, F) where s is a finite subset of ω and F is a finite subset of κ; $(s', F') \leq (s, F)$ just in case $s' \supset s$, $F' \supset F$, and $s' - s \subset X_\alpha$ for all $\alpha \in F$. Consider the dense sets $D_n = \{(s, F) : |s| \geq n\}$, $n < \omega$, and $E_\alpha = \{(s, F) : \alpha \in F\}$, $\alpha < \kappa$.]

16.17. If P_α is an iteration and $P_\beta = P_\alpha \restriction \beta$ then $V^{P_\beta} \subset V^{P_\alpha}$.

[Use (iii)(b) in Definition 16.29 and Lemma 15.45.]

16.18. Let P_α and P'_α be countable support iterations of $\{\dot{Q}_\beta\}_\beta$ and $\{\dot{Q}'_\beta\}_\beta$, respectively. Assume that for every $\beta < \alpha$, if $B(P_\beta) = B(P'_\beta)$ then $\Vdash_\beta B(\dot{Q}_\beta) = B(\dot{Q}'_\beta)$. Then $B(P_\alpha) = B(P'_\alpha)$.

16.19. Let I be a κ-closed ideal on α, and let P_α be an iteration of $\{\dot{Q}_\beta\}_\beta$ with I-support. If for each $\beta < \alpha$, $\Vdash_\beta \dot{Q}_\beta$ is $<\kappa$-closed, then P_α is $<\kappa$-closed.

16.20. Let $\kappa \geq \aleph_2$ be a regular cardinal. Let P be a countable support iteration of length κ such that for all $\beta < \kappa$, $P \restriction \beta$ has a dense subset of size $< \kappa$. Then P satisfies the κ-chain condition.

[Use Theorem 16.30.]

Historical Notes

Iterated forcing was introduced by Solovay and Tennenbaum [1971]. The formulation in terms of Boolean algebras is based on their paper. Our presentation of general iteration (Definitions 16.8 and 16.29) follows Baumgartner [1983].

Following Solovay and Tennenbaum's construction of a model in which there are no Suslin trees (Theorem 16.13), Martin formulated an axiom ($\mathrm{MA}_{\aleph_1}$) which implies that there are no Suslin trees, and whose consistency was obtained by Solovay-Tennenbaum's method. The consistency proof of $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$ appears in Solovay and Tennenbaum [1971].

Martin's Axiom is investigated in detail in the paper [1970] of Martin and Solovay. The paper contains various equivalent formulations of Martin's Axiom and numerous applications (including Theorem 16.20). Theorem 16.21 was discovered by Kunen, Rowbottom, Solovay and possibly others.

Baumgartner, Malitz, and Reinhardt [1970] proved that $\mathrm{MA}_{\aleph_1}$ implies that every Aronszajn tree is special (Theorem 16.17). Special Aronszajn trees have applications in model theory (this fact is due to Rowbottom and Silver) and are investigated in Mitchell's paper [1972/73].

Scales were investigated extensively by Hechler [1974]. Hechler introduced the notion of forcing used in the proof of Theorem 16.24. Hechler, among others, showed that if $\operatorname{cf} \kappa > \omega$, then there is a generic extension in which $2^{\aleph_0} > \kappa$ and a κ-scale exists.

The construction of p-points (and Ramsey ultrafilters) under the assumption of Martin's Axiom is due to Booth [1970]. Our proof of Theorem 16.27 follows Ketonen [1976].

17. Large Cardinals

The theory of large cardinals plays central role in modern set theory. In this chapter we begin a systematic study of large cardinals. In addition to combinatorial methods, the proofs use techniques from model theory.

Ultrapowers and Elementary Embeddings

We start with the following theorem that introduced the technique of ultrapowers to the study of large cardinals.

Theorem 17.1 (Scott). *If there is a measurable cardinal then $V \neq L$.*

Ultrapowers were introduced in Chapter 12. We now generalize the technique to construct ultrapowers of the universe. Let U be an ultrafilter on a set S and consider the class of all functions with domain S. Following (12.3) and (12.4) we define

$$\begin{aligned} f =^* g \quad &\text{if and only if} \quad \{x \in S : f(x) = g(x)\} \in U, \\ f \in^* g \quad &\text{if and only if} \quad \{x \in S : f(x) \in g(x)\} \in U. \end{aligned}$$

For each f, we denote $[f]$ the equivalence class of f in $=^*$ (recall (6.4)):

$$[f] = \{g : f =^* g \text{ and } \forall h\,(h =^* f \to \operatorname{rank} g \leq \operatorname{rank} h)\}.$$

We also use the notation $[f] \in^* [g]$ when $f \in^* g$.

Let $\mathrm{Ult} = \mathrm{Ult}_U(V)$ be the class of all $[f]$, where f is a function on S, and consider the model $\mathrm{Ult} = (\mathrm{Ult}, \in^*)$. Łoś's Theorem 12.3 holds in the present context as well: If $\varphi(x_1, \ldots, x_n)$ is a formula of set theory, then

$$\mathrm{Ult} \vDash \varphi([f_1], \ldots, [f_n]) \quad \text{if and only if} \quad \{x \in S : \varphi(f_1(x), \ldots, f_n(x))\} \in U.$$

If σ is a sentence, then $\mathrm{Ult} \vDash \sigma$ if and only if σ holds; the ultrapower is *elementarily equivalent* to the universe $(V, \in)$. The constant functions c_a are defined, for every set a, by (12.12), and the function $j = j_U : V \to \mathrm{Ult}$, defined by $j_U(a) = [c_a]$ is an *elementary embedding* of V in Ult:

$$\varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad \mathrm{Ult} \vDash \varphi(ja_1, \ldots, ja_n) \tag{17.1}$$

whenever $\varphi(x_1, \ldots, x_n)$ is a formula of set theory.

The most important application of ultrapowers in set theory are those in which $(\mathrm{Ult}, \in^*)$ is well-founded. As we show below, well-founded ultrapowers are closely related to measurable cardinals.

The model $\mathrm{Ult}_U(V)$ is well-founded if (i) every nonempty set $X \subset \mathrm{Ult}$ has a $\in^*$-minimal element, and (ii) $\mathrm{ext}(f)$ is a set for every f, where

$$\mathrm{ext}(f) = \{[g] : g \in^* f\}.$$

The second condition is clearly satisfied for any ultrafilter U: For every $g \in^* f$ there is some $h =^* g$ such that $\operatorname{rank} h \leq \operatorname{rank} f$. As for the condition (i), this is satisfied if and only if there exists no infinite descending $\in^*$-sequence

$$f_0 \ni^* f_1 \ni^* \ldots \ni^* f_k \ni^* \ldots \qquad (k \in \omega)$$

of elements of the ultrapower.

Lemma 17.2. *If U is a σ-complete ultrafilter, then $(\mathrm{Ult}, \in^*)$ is a well-founded model.*

Proof. We shall show that there is no infinite descending $\in^*$-sequence in Ult if U is a σ-complete ultrafilter on S. Let us assume that $f_0, f_1, \ldots, f_n, \ldots$ is such a descending sequence. Thus for each n, the set

$$X_n = \{x \in S : f_{n+1}(x) \in f_n(x)\}$$

is in the ultrafilter. Since U is σ-complete, the intersection $X = \bigcap_{n=0}^{\infty} X_n$ is also in U and hence nonempty; let x be an arbitrary element of X. Then we have

$$f_0(x) \ni f_1(x) \ni f_2(x) \ni \ldots$$

an infinite descending $\ni$-sequence, which is a contradiction. □

By the Mostowski Collapsing Theorem every well-founded model is isomorphic to a transitive model. Thus if U is σ-complete, there exists a one-to-one mapping π of Ult onto a transitive class such that $f \in^* g$ if and only $\pi([f]) \in \pi([g])$. In order to simplify notation, we shall identify each $[f]$ with its image $\pi([f])$. Thus if U is σ-complete, the symbol Ult denotes the transitive collapse of the ultrapower, and for each function f on S, $[f]$ is an element of the transitive class Ult; we say the function f *represents* $[f] \in \mathrm{Ult}$.

Thus if U is a σ-complete ultrafilter, $M = \mathrm{Ult}_U(V)$ is an inner model and $j = j_U$ is an elementary embedding $j : V \to M$.

If α is an ordinal, then since j is elementary, $j(\alpha)$ is an ordinal; moreover, $\alpha < \beta$ implies $j(\alpha) < j(\beta)$. Thus we have $\alpha \leq j(\alpha)$ for every ordinal number α. Note that $j(\alpha + 1) = j(\alpha) + 1$, and $j(n) = n$ for all natural numbers n. It is also easy to see that $j(\omega) = \omega$: If $[f] < \omega$, then $f(x) < \omega$ for almost all $x \in S$, and by σ-completeness, there exists $n < \omega$ such that $f(x) = n$ for almost all x. By the same argument, if U is λ-complete, then $j(\gamma) = \gamma$ for all $\gamma < \lambda$.

Now let κ be a measurable cardinal, and let U be a nonprincipal κ-complete ultrafilter on κ. Let d (the *diagonal function*) be the function on κ defined by

$$d(\alpha) = \alpha \qquad (\alpha < \kappa).$$

Since U is κ-complete every bounded subset of κ has measure 0 and so for every $\gamma < \kappa$, we have $d(\alpha) > \gamma$ for almost all α. Hence $[d] > \gamma$ for all $\gamma < \kappa$ and thus $[d] \geq \kappa$. However, we clearly have $[d] < j(\kappa)$ and it follows that $j(\kappa) > \kappa$.

We have thus proved that if there is a measurable cardinal, then there is an elementary embedding j of the universe in a transitive model M such that j is not the identity mapping; j is a *nontrivial elementary embedding of the universe*.

Proof of Theorem 17.1. Let us assume $V = L$ and that measurable cardinals exist; let κ be the least measurable cardinal. Let U be a nonprincipal κ-complete ultrafilter on κ and let $j : V \to M$ be the corresponding elementary embedding. As we have shown, $j(\kappa) > \kappa$.

Since $V = L$, the only transitive model containing all ordinals is the universe itself: $V = M = L$. Since j is an elementary embedding and κ is the least measurable cardinal, we have

$$M \vDash j(\kappa) \text{ is the least measurable cardinal;}$$

and hence, $j(\kappa)$ is the least measurable cardinal. This is a contradiction since $j(\kappa) > \kappa$. □

If there exists a measurable cardinal, then there exists a nontrivial elementary embedding of the universe. Let us show that conversely, if $j : V \to M$ is a nontrivial elementary embedding then there exists a measurable cardinal.

Lemma 17.3. *If j is a nontrivial elementary embedding of the universe, then there exists a measurable cardinal.*

Proof. Let $j : V \to M$ be a nontrivial embedding. Notice that there exists an ordinal α such that $j(\alpha) \neq \alpha$; otherwise, we would have $\text{rank}(jx) = \text{rank}(x)$ for all x, and then we could prove by induction on rank that $j(x) = x$ for all x.

Thus let κ be the least ordinal number such that $j(\kappa) \neq \kappa$ (and hence $j(\kappa) > \kappa$). It is clear that $j(n) = n$ for all n and $j(\omega) = \omega$ since 0, $n + 1$, and ω are absolute notions and j is elementary. Hence $\kappa > \omega$. We shall show that κ is a measurable cardinal.

Let D be the collection of subsets of κ defined as follows:

$$X \in D \quad \text{if and only if} \quad \kappa \in j(X) \qquad (X \subset \kappa). \tag{17.2}$$

Since $\kappa < j(\kappa)$, i.e., $\kappa \in j(\kappa)$, we have $\kappa \in D$; also $\emptyset \notin D$ because $j(\emptyset) = \emptyset$. Using the fact that $j(X \cap Y) = j(X) \cap j(Y)$ and that $j(X) \subset j(Y)$ whenever

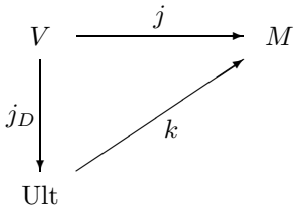
$X \subset Y$, we see that D is a filter: If $\kappa \in j(X)$ and $\kappa \in j(Y)$, then $\kappa \in j(X \cap Y)$; if $X \subset Y$ and $\kappa \in j(X)$, then $\kappa \in j(Y)$. Similarly, $j(\kappa - X) = j(\kappa) - j(X)$ and thus D is an ultrafilter.

D is a nonprincipal ultrafilter: For every $\alpha < \kappa$, we have $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$, and so $\kappa \notin j(\{\alpha\})$ and we have $\{\alpha\} \notin D$. We shall now show that D is κ-complete. Let $\gamma < \kappa$ and let $\mathcal{X} = \langle X_\alpha : \alpha < \gamma \rangle$ be a sequence of subsets of κ such that $\kappa \in j(X_\alpha)$ for each $\alpha < \gamma$. We shall show that $\bigcap_{\alpha<\gamma} X_\alpha \in D$. In M (and thus in V), $j(\mathcal{X})$ is a sequence of length $j(\gamma)$ of subsets of $j(\kappa)$; for each $\alpha < \gamma$, the $j(\alpha)$th term of $j(\mathcal{X})$ is $j(X_\alpha)$. Since $j(\alpha) = \alpha$ for all $\alpha < \gamma$ and $j(\gamma) = \gamma$, it follows that $j(\mathcal{X}) = \langle j(X_\alpha) : \alpha < \gamma \rangle$. Hence if $X = \bigcap_{\alpha<\gamma} X_\alpha$, we have $j(X) = \bigcap_{\alpha<\gamma} j(X_\alpha)$. Now it is clear that $\kappa \in j(X)$ and hence $X \in D$. □

The construction of a κ-complete ultrafilter from an elementary embedding yields the following commutative diagram (17.3):

Lemma 17.4. *Let $j : V \to M$ be a nontrivial elementary embedding, let κ be the least ordinal moved, and let D be the ultrafilter on κ defined in* (17.2)*. Let $j_D : V \to \mathrm{Ult}$ be the canonical embedding of V in the ultrapower $\mathrm{Ult}_D(V)$. Then there is an elementary embedding k of* Ult *in M such that $k(j_D(a)) = j(a)$ for all a:*

(17.3)



Proof. For each $[f] \in \mathrm{Ult}$, let

$$k([f]) = (j(f))(\kappa). \tag{17.4}$$

(Here f is a function on κ and $j(f)$ is a function on $j(\kappa)$.)

We shall first show that definition (17.4) does not depend on the choice of f representing $[f]$. If $f =_D g$, then the set $X = \{\alpha : f(\alpha) = g(\alpha)\}$ is in D and hence κ is in the set

$$j(X) = \{\alpha < j(\kappa) : (jf)(\alpha) = (jg)(\alpha)\}.$$

Therefore $(jf)(\kappa) = (jg)(\kappa)$.

Next we show that k is elementary. Let $\varphi(x)$ be a formula and let $\mathrm{Ult} \vDash \varphi([f])$; we shall show that $M \vDash \varphi(k([f]))$. The set $X = \{\alpha : \varphi(f(\alpha))\}$ is in D and hence κ belongs to the set

$$j(X) = \{\alpha < j(\kappa) : M \vDash \varphi((jf)(\alpha))\}.$$

Since $(jf)(\kappa) = k([f])$, we have $M \vDash \varphi(k([f]))$.

Finally, we show that $k(j_D(a)) = j(a)$ for all a. Since $j_D(a) = [c_a]$, where c_a is the constant function on κ with value a, we have $k(j_D(a)) = (j(c_a))(\kappa)$. Now $j(c_a)$ is the constant function on $j(\kappa)$ with value $j(a)$ and hence $(j(c_a))(\kappa) = j(a)$. □

We remark that the measure $D = \{X \subset \kappa : \kappa \in j(X)\}$ defined from an elementary embedding is normal: Let f be a regressive function on some $X \in D$. Then $(jf)(\kappa) < \kappa$, and if $\gamma = (jf)(\kappa)$, then $f(\alpha) = \gamma$ for almost all α.

Normality can be expressed in terms of ultrapowers:

Lemma 17.5. *Let D be a nonprincipal κ-complete ultrafilter on κ. Then the following are equivalent*:

(i) *D is normal.*
(ii) *In the ultrapower* $\mathrm{Ult}_D(V)$,

$$\kappa = [d]$$

where d is the diagonal function.
(iii) *For every $X \subset \kappa$, $X \in D$ if and only if $\kappa \in j_D(X)$.*

Proof. (i) implies (ii): Every function $f \in^* d$ is regressive, and hence represents an ordinal $\gamma < \kappa$.

(ii) implies (iii): If $X \subset \kappa$, then $X \in D$ if and only if $d(\alpha) \in X$ for almost all α, that is, if and only if $[d] \in j_D(X)$. If $[d] = \kappa$, we get $X \in D$ if and only if $\kappa \in j_D(X)$.

(iii) implies (i): If $D = \{X \subset \kappa : \kappa \in j_D(X)\}$ then D is normal, by the remark preceding the lemma. □

Let $j : V \to M$ be an elementary embedding. If X is a class defined by a formula φ, then $j(X)$ is the class of the model M, defined in M by the same formula φ. Note that $j(X) = \bigcup_{\alpha \in Ord} j(X \cap V_\alpha)$. In particular, $M = j(V)$.

Lemma 17.6. *Let j be an elementary embedding of the universe and let κ be the least ordinal moved (i.e., $j(\kappa) > \kappa$). If C is a closed unbounded subset of κ, then $\kappa \in j(C)$.*

Proof. Since $j(\alpha) = \alpha$ for all $\alpha < \kappa$, we have $j(C) \cap \kappa = C$. Thus $j(C) \cap \kappa$ is unbounded in κ; and because $j(C)$ is closed (in $j(V)$ and hence in the universe), we have $\kappa \in j(C)$. □

A consequence of Lemma 17.6 is that the set of all regular cardinals below a measurable cardinal κ is stationary (cf. Lemma 10.21): Let $X \subset \kappa$ be the set of all regular cardinals below κ. Since κ is regular in M, we have $\kappa \in j(X)$, and $\kappa \in j(X \cap C)$ for every closed unbounded $C \subset \kappa$. Hence X is stationary. Similarly, as κ is Mahlo, it is Mahlo in M, and if X is now the set of all Mahlo cardinals below κ, it follows that X is stationary.

More generally, if $M(X)$ denotes the *Mahlo operation*

$$M(X) = \{\alpha : X \cap \alpha \text{ is stationary in } \alpha\} \tag{17.5}$$

where X is any class of ordinals, the above argument shows that if $\kappa \in j(X)$ then $\kappa \in M(X)$ (Exercise 17.7).

The next theorem shows that there exists no nontrivial elementary embedding of V into V. As the statement "there exists an elementary embedding of V" is not expressible in the language of set theory, the theorem needs to be understood as a theorem in the following modification of ZFC: The language has, in addition to $\in$, a function symbol j, the axioms include Separation and Replacement Axioms for formulas that contain the symbol j, and axioms that state that j is an elementary embedding of V.

Theorem 17.7 (Kunen). *If $j : V \to M$ is a nontrivial elementary embedding, then $M \neq V$.*

First we prove the following lemma:

Lemma 17.8. *Let λ be an infinite cardinal such that $2^\lambda = \lambda^{\aleph_0}$. There exists a function $F : \lambda^\omega \to \lambda$ such that whenever A is a subset of λ of size λ and $\gamma < \lambda$, there exists some $s \in A^\omega$ such that $F(s) = \gamma$.*

Proof. Let $\{(A_\alpha, \gamma_\alpha) : \alpha < 2^\lambda\}$ be an enumeration of all pairs (A, γ) where $\gamma < \lambda$ and A is a subset of λ of size λ. We define, by induction on α, a sequence s_α, $\alpha < 2^\lambda$, of elements of λ^ω as follows: If $\alpha < 2^\lambda$, then since $\lambda^{\aleph_0} = 2^\lambda > |\alpha|$, there exists an $s_\alpha \in A_\alpha^\omega$ such that $s_\alpha \neq s_\beta$ for all $\beta < \alpha$.

For each $\alpha < 2^\lambda$, we define $F(s_\alpha) = \gamma_\alpha$ (and let $F(s)$ be arbitrary if s is not one of the s_α). The function F has the required property: If $A \subset \lambda$ has size λ and $\gamma < \lambda$, then $(A, \gamma) = (A_\alpha, \gamma_\alpha)$ for some α, and then $\gamma_\alpha = F(s_\alpha)$. □

Proof of Theorem 17.7. Let us assume that j is a nontrivial elementary embedding of V in V. Let $\kappa = \kappa_0$ be the least ordinal moved; κ_0 is measurable, and so are $\kappa_1 = j(\kappa_0)$, $\kappa_2 = j(\kappa_1)$, and every κ_n, where $\kappa_{n+1} = j(\kappa_n)$. Let $\lambda = \lim_{n\to\infty} \kappa_n$. Since $j(\langle \kappa_n : n < \omega\rangle) = \langle j(\kappa_n) : n < \omega\rangle = \langle \kappa_{n+1} : n < \omega\rangle$, we have $j(\lambda) = \lim_{n\to\infty} j(\kappa_n) = \lambda$. Let $G = \{j(\alpha) : \alpha < \lambda\}$; we shall use the set G and Lemma 17.8 to obtain a contradiction.

The cardinal λ is the limit of a sequence of measurable cardinals and hence is a strong limit cardinal. Since $\operatorname{cf} \lambda = \omega$, we have $2^\lambda = \lambda^{\aleph_0}$. By Lemma 17.8 there is a function $F : \lambda^\omega \to \lambda$ such that $F(A^\omega) = \lambda$ for all $A \subset \lambda$ of size λ. Since j is elementary, and $j(\omega) = \omega$ and $j(\lambda) = \lambda$, the function $j(F)$ has the same property. Thus, considering the set $A = G$, there exists $s \in G^\omega$ such that $(jF)(s) = \kappa$.

Now, s is a function, $s : \omega \to G$, and hence there is a $t : \omega \to \lambda$ such that $s(n) = j(t(n))$ for all $n < \omega$. It follows that $s = j(t)$. Thus we have $\kappa = (jF)(jt) = j(F(t))$; in other words, $\kappa = j(\alpha)$ where $\alpha = F(t)$. However, this is impossible since $j(\alpha) = \alpha$ for all $\alpha < \kappa$, and $j(\kappa) > \kappa$. □

Let us now consider ultrapowers and the corresponding elementary embeddings $j_U : V \to \mathrm{Ult}$. To introduce the following lemma, let us observe that if $j : V \to M$ and if κ is the least ordinal moved, then $j(x) = x$ for every $x \in V_\kappa$, and $j(X) \cap V_\kappa = X$ for every $X \subset V_\kappa$. Hence $V^M_{\kappa+1} = V_{\kappa+1}$ (and $P^M(\kappa) = P(\kappa)$).

Lemma 17.9. *Let U be a nonprincipal κ-complete ultrafilter on κ, let $M = \mathrm{Ult}_U(V)$ and let $j = j_U$ be the canonical elementary embedding of V in M.*

(i) *$M^\kappa \subset M$, i.e., every κ-sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of M is itself a member of M.*
(ii) *$U \notin M$.*
(iii) *$2^\kappa \le (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$.*
(iv) *If λ is a limit ordinal and if $\operatorname{cf}\lambda = \kappa$, then $j(\lambda) > \lim_{\alpha\to\lambda} j(\alpha)$; if $\operatorname{cf}\lambda \ne \kappa$, then $j(\lambda) = \lim_{\alpha\to\lambda} j(\alpha)$.*
(v) *If $\lambda > \kappa$ is a strong limit cardinal and $\operatorname{cf}\lambda \ne \kappa$, then $j(\lambda) = \lambda$.*

Proof. (i) Let $\langle a_\xi : \xi < \kappa\rangle$ be a κ-sequence of elements of M. For each $\xi < \kappa$, let g_ξ be a function that represents a_ξ, and let h be a function that represents κ:

$$[g_\xi] = a_\xi, \qquad [h] = \kappa.$$

We shall construct a function F such that $[F] = \langle a_\xi : \xi < \kappa\rangle$. We let, for each $\alpha < \kappa$,

$$F(\alpha) = \langle g_\xi : \xi < h(\alpha)\rangle.$$

Since for each α, $F(\alpha)$ is an $h(\alpha)$-sequence, $[F]$ is a κ-sequence. Let $\xi < \kappa$; we want to show that the ξth term of $[F]$ is a_ξ. Since $[h] > \xi$, we have $\xi < h(\alpha)$ for almost all α; and for each α such that $\xi < h(\alpha)$, the ξth term of $F(\alpha)$ is $g_\xi(\alpha)$. But $[c_\xi] = \xi$ and $[g_\xi] = a_\xi$, and we are done.

(ii) Assume that $U \in M$, and let us consider the mapping e of κ^κ onto $j(\kappa)$ defined by $e(f) = [f]$. Since $\kappa^\kappa \in M$ and $U \in M$, the mapping e is in M. It follows that $M \vDash |j(\kappa)| \le 2^\kappa$. This is a contradiction since $\kappa < j(\kappa)$ and $j(\kappa)$ is inaccessible in M.

(iii) $2^\kappa \le (2^\kappa)^M$ holds because $P^M(\kappa) = P(\kappa)$ and $M \subset V$; $(2^\kappa)^M$ is less than $j(\kappa)$ since $j(\kappa)$ is inaccessible in M; finally, we have $|j(\kappa)| = 2^\kappa$ and hence $j(\kappa) < (2^\kappa)^+$.

(iv) If $\operatorname{cf}\lambda = \kappa$, let $\lambda = \lim_{\alpha\to\kappa}\lambda_\alpha$ and let $f(\alpha) = \lambda_\alpha$ for all $\alpha < \kappa$. Then $[f] > j(\lambda_\alpha)$ for all $\alpha < \kappa$ and $[f] < j(\lambda)$. If $\operatorname{cf}\lambda > \kappa$, then for every $f : \kappa \to \lambda$ there exists $\alpha < \lambda$ such that $[f] < j(\alpha)$. If $\operatorname{cf}\lambda = \gamma < \kappa$, let $\lambda = \lim_{\nu\to\gamma}\lambda_\nu$; for every $f : \kappa \to \lambda$ there exists (by κ-completeness) $\nu < \gamma$ such that $[f] < j(\lambda_\nu)$.

(v) For every $\alpha < \lambda$, the ordinals below α are represented by functions $f : \kappa \to \alpha$; hence $|j(\alpha)| \le |\alpha^\kappa| < \lambda$; by (iv) we have $j(\lambda) = \lim_{\alpha\to\lambda} j(\alpha) = \lambda$. □

Note that in (v) it suffices to assume that $\operatorname{cf}\lambda \neq \kappa$ and $\alpha^\kappa < \lambda$ for all cardinals $\alpha < \lambda$.

Let us recall (Lemma 10.18) that a measurable cardinal is weakly compact. We now prove a stronger result:

Theorem 17.10. *Every measurable cardinal κ is weakly compact and if D is a normal measure on κ then the set $\{\alpha < \kappa : \alpha$ is weakly compact$\}$ is in D.*

Proof. The first statement was proved in Lemma 10.18. Let D be a normal measure on κ, and let $j_D : V \to M$ be the canonical embedding. Since $P^M(\kappa) = P(\kappa)$, it follows that κ is weakly compact in M, and since $[d]_D = \kappa$, we have $\{\alpha : \alpha$ is weakly compact$\} \in D$. □

The following two results show that the existence of measurable cardinals influences cardinal arithmetic:

Lemma 17.11. *Let κ be a measurable cardinal. If $2^\kappa > \kappa^+$, then the set $\{\alpha < \kappa : 2^\alpha > \alpha^+\}$ has measure one for every normal measure on κ.*

Consequently, if $2^\alpha = \alpha^+$ for all cardinals $\alpha < \kappa$, then $2^\kappa = \kappa^+$.

Proof. Let D be a normal measure on κ, and let $M = \operatorname{Ult}_D(V)$. If $2^\alpha = \alpha^+$ for almost all α, then, since $[d]_D = \kappa$, we have $M \vDash 2^\kappa = \kappa^+$. In other words, there is a one-to-one mapping in M between $P^M(\kappa)$ and $(\kappa^+)^M$. However, $P^M(\kappa) = P(\kappa)$ and $(\kappa^+)^M = \kappa^+$ (because $P^M(\kappa) = P(\kappa)$), and so $2^\kappa = \kappa^+$. □

Lemma 17.12. *Let κ be a measurable cardinal, let D be a normal measure on κ and let $j : V \to M$ be the corresponding elementary embedding. Let $\lambda > \kappa$ be a strong limit cardinal of cofinality κ. Then $2^\lambda < j(\lambda)$.*

Proof. Since $\operatorname{cf}\lambda = \kappa$, we have $j(\lambda) > \lambda$. We shall show that $2^\lambda = \lambda^\kappa \leq (\lambda^\kappa)^M \leq (\lambda^{j(\kappa)})^M < j(\lambda)$. The first equality holds because λ is strong limit. We have $\lambda^\kappa \leq (\lambda^\kappa)^M$ because every function $f : \kappa \to \lambda$ is in M. As for the last inequality, we have

$$M \vDash j(\lambda) \text{ is a strong limit cardinal}$$

and since $\lambda < j(\lambda)$ and $j(\kappa) < j(\lambda)$, we have $M \vDash \lambda^{j(\kappa)} < j(\lambda)$. □

See Exercises 17.12–17.16.

Weak Compactness

We shall investigate weakly compact cardinals in some detail, and give a characterization of weakly compact cardinals that explains the name "weakly compact." This aspect of weakly compact cardinals has, as many other large cardinal properties, motivation in model theory.

We shall consider infinitary languages which are generalizations of the ordinary first order language. Let κ be an infinite cardinal number. The language $\mathcal{L}_{\kappa,\omega}$ consists of

(i) κ variables;
(ii) various relation, function, and constant symbols;
(iii) logical connectives and infinitary connectives $\bigvee_{\xi<\alpha}\varphi_\xi$, $\bigwedge_{\xi<\alpha}\varphi_\xi$ for $\alpha<\kappa$ (infinite disjunction and conjunction);
(iv) quantifiers $\exists v$, $\forall v$.

The language $\mathcal{L}_{\kappa,\kappa}$ is like $\mathcal{L}_{\kappa,\omega}$ except that it also contains infinitary quantifiers:

(v) $\exists_{\xi<\alpha}v_\xi$, $\forall_{\xi<\alpha}v_\xi$ for $\alpha<\kappa$.

The interpretation of the infinitary symbols of $\mathcal{L}_{\kappa,\kappa}$ is the obvious generalization of the finitary case where $\bigvee_{\xi<n}\varphi_\xi$ is $\varphi_0\vee\ldots\vee\varphi_{n-1}$, $\exists_{\xi<n}v_\xi$ stands for $\exists v_0\ldots\exists v_{n-1}$, etc. The language $\mathcal{L}_{\omega,\omega}$ is just the language of the first order predicate calculus.

The finitary language $\mathcal{L}_{\omega,\omega}$ satisfies the Compactness Theorem: If Σ is a set of sentences such that every finite $S\subset\Sigma$ has a model, then Σ has a model. Let us say that the language $\mathcal{L}_{\kappa,\kappa}$ (or $\mathcal{L}_{\kappa,\omega}$) satisfies the Weak Compactness Theorem if whenever Σ is a set of sentences of $\mathcal{L}_{\kappa,\kappa}$ ($\mathcal{L}_{\kappa,\omega}$) such that $|\Sigma|\leq\kappa$ and that every $S\subset\Sigma$ with $|S|<\kappa$ has a model, then Σ has a model. Clearly, if $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem, then so does $\mathcal{L}_{\kappa,\omega}$ because $\mathcal{L}_{\kappa,\omega}\subset\mathcal{L}_{\kappa,\kappa}$.

Theorem 17.13.

(i) *If κ is a weakly compact cardinal, then the language $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem.*
(ii) *If κ is an inaccessible cardinal and if $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem, then κ is weakly compact.*

Proof. (i) The proof of the Weak Compactness Theorem for $\mathcal{L}_{\kappa,\kappa}$ is very much like the standard proof of the Compactness Theorem for $\mathcal{L}_{\omega,\omega}$. Let Σ be a set of sentences of $\mathcal{L}_{\kappa,\kappa}$ of size κ such that if $S\subset\Sigma$ and $|S|<\kappa$, then S has a model. Let us assume that the language $\mathcal{L}=\mathcal{L}_{\kappa,\kappa}$ has only the symbols that occur in Σ; thus $|\mathcal{L}|=\kappa$.

First we extend the language as follows: For each formula φ with free variables v_ξ, $\xi<\alpha$, we introduce new constant symbols c_ξ^φ, $\xi<\alpha$ (*Skolem constants*); let $\mathcal{L}^{(1)}$ be the extended language. Then we do the same for each formula of $\mathcal{L}^{(1)}$ and obtain $\mathcal{L}^{(2)}\supset\mathcal{L}^{(1)}$. We do the same for each $n<\omega$, and then let $\mathcal{L}^*=\bigcup_{n=1}^\infty\mathcal{L}^{(n)}$. Since κ is inaccessible, it follows that $|\mathcal{L}^*|=\kappa$. $\mathcal{L}^*$ has the property that for each formula φ with free variables v_ξ, $\xi<\alpha$, there are in $\mathcal{L}^*$ constant symbols c_ξ^φ, $\xi<\alpha$ (which do not occur in φ).

For each $\varphi(v_\xi, \ldots)_{\xi<\alpha}$ let σ_φ be the sentence (a *Skolem sentence*)

$$\exists_{\xi<\alpha} v_\xi \, \varphi(v_\xi, \ldots)_{\xi<\alpha} \to \varphi(c^\varphi_\xi, \ldots)_{\xi<\alpha} \tag{17.6}$$

and let $\Sigma^* = \Sigma \cup \{\sigma_\varphi : \varphi \text{ is a formula of } \mathcal{L}^*\}$.

Note that if $S \subset \Sigma^*$ and $|S| < \kappa$, then S has a model: Take a model for $S \cap \Sigma$ (for $\mathcal{L}$) and then expand it to a model for $\mathcal{L}^*$ by interpreting the Skolem constants so that each sentence (17.6) is true.

Let $\{\sigma_\alpha : \alpha < \kappa\}$ be an enumeration of all the sentences in $\mathcal{L}^*$. Let $(T, \subset)$ be the binary κ-tree consisting of all $t : \gamma \to \{0, 1\}$, $\gamma < \kappa$, for which there exists a model $\mathfrak{A}$ of $\Sigma \cap \{\sigma_\alpha : \alpha \in \mathrm{dom}(t)\}$ such that for all $\alpha \in \mathrm{dom}(t)$

$$t(\alpha) = 1 \quad \text{if and only if} \quad \mathfrak{A} \vDash \sigma_\alpha.$$

Since κ has the tree property, there exists a branch B in T of length κ. Let

$$\Delta = \{\sigma_\alpha : t(\alpha) = 1 \text{ for some } t \in B\}.$$

Clearly, $\Sigma^* \subset \Delta$. Let A_0 be the set of all constant terms of $\mathcal{L}^*$, and let $\approx$ be the equivalence relation on A_0 defined by

$$\tau_1 \approx \tau_2 \quad \text{if and only if} \quad (\tau_1 \approx \tau_2) \in \Delta,$$

and let $A = A_0/\!\approx$.

We make A into a model $\mathfrak{A}$ for $\mathcal{L}^*$ as follows:

$$\mathfrak{A} \vDash P[[\tau_1], \ldots, [t_n]] \quad \text{if and only if} \quad P(\tau_1, \ldots, \tau_n) \in \Delta$$

and similarly for function and constant symbols. The proof is then completed by showing that $\mathfrak{A}$ is a model for Δ (and hence for Σ). The proof of

$$\mathfrak{A} \vDash \sigma \quad \text{if and only if} \quad \sigma \in \Delta \tag{17.7}$$

is done by induction on the number of quantifier blocks in σ: If $\sigma = \exists_{\xi<\alpha} v_\xi \, \varphi(v_\xi, \ldots)$, then by induction hypothesis we have

$$\mathfrak{A} \vDash \sigma(c^\varphi_\xi, \ldots)_{\xi<\alpha} \quad \text{if and only if} \quad \sigma(c^\varphi_\xi, \ldots)_{\xi<\alpha} \in \Delta$$

and (17.7) follows.

(ii) Let κ be inaccessible and assume that the language $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem. We shall show that κ has the tree property. Let $(T, <)$ be a tree of height κ such that each level of T has size $< \kappa$. Let us consider the $\mathcal{L}_{\kappa,\omega}$ language with one unary predicate B and constant symbols c_x for all $x \in T$. Let Σ be the following set of sentences:

$$\begin{aligned} &\neg(B(c_x) \wedge B(c_y)) && \text{for all } x, y \in T \text{ that are incomparable,} \\ &\textstyle\bigvee_{x \in U_\alpha} B(c_x) && \text{for all } \alpha < \kappa, \text{ where } U_\alpha \text{ is the } \alpha\text{th level of } T \end{aligned}$$

(Σ says that B is branch in T of length κ). If $S \subset \Sigma$ and $|S| < \kappa$, then we get a model for S by taking a sufficiently large initial segment of T and some branch in this segment. By the Weak Compactness Theorem for $\mathcal{L}_{\kappa,\omega}$, Σ has a model, which obviously yields a branch of length κ. □

Indescribability

Let $n > 0$ be a natural number and let us consider the nth order predicate calculus. There are variables of orders 1, 2, ..., n, and the quantifiers are applied to variables of all orders. An nth *order formula* contains, in addition to first order symbols and higher order quantifiers, predicates $X(z)$ where X and z are variables of order $k+1$ and k respectively (for any $k < n$). Satisfaction for an nth order formula in a model $\mathfrak{A} = (A, P, \ldots, f, \ldots, c, \ldots)$ is defined as follows: Variables of first order are interpreted as elements of the set A, variables of second order as elements of $P(A)$ (as subsets of A), etc.; variables of order n are interpreted as elements of $P^{n-1}(A)$. The predicate $X(z)$ is interpreted as $z \in X$. A Π^n_m *formula* is a formula of order $n+1$ of the form

$$\underbrace{\forall X\, \exists Y\, \ldots}_{m \text{ quantifiers}}\, \psi \tag{17.8}$$

where $X, Y, \ldots$ are $(n+1)$th order variables and ψ is such that all quantified variables are of order at most n. Similarly, a Σ^n_m formula is as in (17.8), but with $\exists$ and $\forall$ interchanged.

We shall often exhibit a sentence σ and claim that it is Π^n_m (or Σ^n_m) although it is only *equivalent* to a Π^n_m (or Σ^n_m) sentence, in the following sense: We are considering a specific type of models in which σ is interpreted (e.g., the models $(V_\alpha, \in)$) and there is a Π^n_m (or Σ^n_m) sentence $\bar{\sigma}$ such that the equivalence $\sigma \leftrightarrow \bar{\sigma}$ holds in all these models.

Note that every first order formula is equivalent to some Π^0_n formula (and also to some Σ^0_k formula).

Definition 17.14. A cardinal κ is Π^n_m*-indescribable* if whenever $U \subset V_\kappa$ and σ is a Π^n_m sentence such that $(V_\kappa, \in, U) \models \sigma$, then for some $\alpha < \kappa$, $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

Lemma 17.15. *Every measurable cardinal is* Π^2_1*-indescribable.*

Proof. Let κ be a measurable cardinal, let $U \subset V_\kappa$ and let σ be a Π^2_1 sentence of the (third order) language $\{\in, U\}$. Let us assume that $(V_\kappa, \in, U) \models \sigma$.

We have $\sigma = \forall X\, \varphi(X)$ where X is a third order variable and $\varphi(X)$ contains only second and first order quantifiers. Thus

$$\forall X \subset V_{\kappa+1}\ (V_{\kappa+1}, \in, X, V_\kappa, U) \models \tilde{\varphi} \tag{17.9}$$

where $\tilde{\varphi}$ is the (first order) sentence obtained from φ by replacing the first order quantifiers by the restricted quantifiers $\forall x \in V_\kappa$ and $\exists x \in V_\kappa$.

Now let D be a normal measure on κ and let $M = \mathrm{Ult}_D(V)$. Since $V^M_{\kappa+1} = V_{\kappa+1}$, we know that (17.9) holds also in M. Using the fact that V_κ is

represented in the ultrapower by the function $\alpha \mapsto V_\alpha$, $V_{\kappa+1}$ by $\alpha \mapsto V_{\alpha+1}$, and U by $\alpha \mapsto U \cap V_\alpha$, we conclude that for almost all α,

$$\forall X \subset V_{\alpha+1}\ (V_{\alpha+1}, \in, X, V_\alpha, U \cap V_\alpha) \vDash \tilde{\varphi}. \tag{17.10}$$

Then, translating (17.10) back into the third order language, we obtain

$$(V_\alpha, \in, U \cap V_\alpha) \vDash \sigma$$

for almost all, and hence for some, $\alpha < \kappa$. □

Lemma 17.16. *If κ is not inaccessible, then it is describable by a first order sentence, i.e., Π^0_m-describable for some m.*

Proof. Let κ be a singular cardinal, and let f be a function with $\mathrm{dom}(f) = \lambda < \kappa$ and $\mathrm{ran}(f)$ cofinal in κ. Let $U_1 = f$ and $U_2 = \{\lambda\}$, and let σ be the first order sentence saying that U_2 is nonempty and that the unique element of U_2 is the domain of U_1. Clearly, κ is describable in the sense that $(V_\kappa, \in, U_1, U_2) \vDash \sigma$ and there is no $\alpha < \kappa$ such that $(V_\alpha, \in, U_1 \cap V_\alpha, U_2 \cap V_\alpha) \vDash \sigma$. It is routine to find a single $U \subset V_\kappa$ and an $(\in, U)$-sentence $\tilde{\sigma}$ attesting to the describability of κ.

If $\kappa \leq 2^\lambda$ for some $\lambda < \kappa$, there is a function f that maps $P(\lambda)$ onto κ. We let $U_1 = f$ and $U_2 = \{P(\lambda)\}$; then κ is described by the same sentence as above.

Finally, $\kappa = \omega$ is describable as follows: $(V_\kappa, \in) \vDash \forall x\, \exists y\, x \in y$. □

The converse is also true; cf. Exercise 17.23.

We shall now present a result of Hanf and Scott that shows that Π^1_1-indescribable cardinals are exactly the weakly compact cardinals. First we need a lemma.

Lemma 17.17. *If κ is a weakly compact cardinal, then for every $U \subset V_\kappa$, the model $(V_\kappa, \in, U)$ has a transitive elementary extension $(M, \in, U')$ such that $\kappa \in M$.*

Proof. Let Σ be the set of all $\mathcal{L}_{\kappa,\kappa}$ sentences true in the model $(V_\kappa, \in, U, x)_{x \in V_\kappa}$ plus the sentences

$$c \text{ is an ordinal},$$
$$c > \alpha, \qquad (\text{all } \alpha < \kappa).$$

Clearly $|\Sigma| = \kappa$, and if $S \subset \Sigma$ is such that $|S| < \kappa$, then S has a model (namely V_κ, where the constant c can be interpreted as some ordinal greater than all the α's mentioned in S).

Hence Σ has a model $\mathfrak{A} = (A, E, U^{\mathfrak{A}}, x^{\mathfrak{A}})_{x \in V_\kappa}$; we may assume that $A \supset V_\kappa$, $E \cap (V_\kappa \times V_\kappa) = \in$, $U^{\mathfrak{A}} \cap V_\kappa = U$, and $x^{\mathfrak{A}} = x$ for all $x \in V_\kappa$. Moreover, $V_\kappa \prec (A, E, U^{\mathfrak{A}})$ because $\mathfrak{A}$ satisfies all formulas true in V_κ of all $x \in V_\kappa$. If we show that the model (A, E) is well-founded, then the lemma follows.

Here we make use of the expressive power of the infinitary language $\mathcal{L}_{\kappa,\kappa}$: We consider the sentence

$$\neg\exists v_0\,\exists v_1\,\ldots\,\exists v_n\,\ldots\bigwedge_{n\in\omega}(v_{n+1}\in v_n). \tag{17.11}$$

The sentence (17.11) holds in a model $\mathfrak{A}=(A,E)$ if and only if $\mathfrak{A}$ is well-founded. Since Σ contains the sentence (17.11), every model of Σ is well-founded. □

The converse is also true; this will follow from the proof of Theorem 17.18.

Theorem 17.18 (Hanf-Scott). *A cardinal κ is Π^1_1-indescribable if and only if it is weakly compact.*

Proof. First we show that every Π^1_1-indescribable cardinal is weakly compact. If κ is Π^1_1-indescribable, then by Lemma 17.16, κ is inaccessible, and it suffices to show that κ has the tree property. In fact, by the proof of Theorem 17.13(i) it suffices to consider trees $(T,<)$ consisting of sequences $t:\gamma\to\{0,1\}$, $\gamma<\kappa$. Let T be such a tree. For every $\alpha<\kappa$, the model $(V_\alpha,\in,T\cap V_\alpha)$ satisfies the Σ^1_1 sentence

$$\exists B\,(B\subset T \text{ and } B \text{ is a branch of unbounded length}). \tag{17.12}$$

Namely, let $B=\{t\restriction\xi:\xi<\alpha\}$ where t is any $t\in T$ with domain α. Since κ is Π^1_1-indescribable, the sentence (17.12) holds in $(V_\kappa,\in,T)$ and hence T has a branch of length κ.

To show that a weakly compact cardinal is Π^1_1-indescribable, we use Lemma 17.17. Let κ be weakly compact, let $U\subset V_\kappa$ and let σ be a Π^1_1 sentence true in $(V_\kappa,\in,U)$. We have $\sigma=\forall X\,\varphi(X)$ where X is a second order variable and φ has only first order quantifiers.

Let $(M,\in,U')$ be a transitive elementary extension of $(V_\kappa,\in,U)$ such that $\kappa\in M$. Since

$$(\forall X\subset V_\kappa)\ (V_\kappa,\in,U)\vDash\varphi(X)$$

and $V_\kappa^M=V_\kappa$, we have

$$(M,\in,U')\vDash(\forall X\subset V_\kappa)\ (V_\kappa,\in,U'\cap V_\kappa)\vDash\varphi(X).$$

Therefore,

$$(M,\in,U')\vDash\exists\alpha\,(\forall X\subset V_\alpha)\ (V_\alpha,\in,U'\cap V_\alpha)\vDash\varphi(X),$$

and so

$$(V_\kappa,\in,U)\vDash\exists\alpha\,(\forall X\subset V_\alpha)\ (V_\alpha,\in,U'\cap V_\alpha)\vDash\varphi(X).$$

Hence for some $\alpha<\kappa$, $(V_\alpha,\in,U\cap V_\alpha)\vDash\sigma$. □

Corollary 17.19. *Every weakly compact cardinal κ is a Mahlo cardinal, and the set of Mahlo cardinals below κ is stationary.*

Proof. Let $C \subset \kappa$ be a closed unbounded set. Since κ is inaccessible, $(V_\kappa, \in, C)$ satisfies the following Π^1_1 sentence:

$$\neg\exists F\,(F \text{ is a function from some } \lambda < \kappa \text{ cofinally into } \kappa)$$
$$\text{and } C \text{ is unbounded in } \kappa.$$

By Π^1_1-indescribability, there exists a regular $\alpha < \kappa$ such that $C \cap \alpha$ is unbounded in α; hence $\alpha \in C$. Thus κ is Mahlo.

Being Mahlo is also expressible by a Π^1_1 sentence:

$$\forall X\,(\text{if } X \text{ is closed unbounded, then } \exists \text{ a regular } \alpha \text{ in } X)$$

and so the same argument as above shows that there is a stationary set of Mahlo cardinals below κ. □

Corollary 17.20. *If κ is weakly compact and if $S \subset \kappa$ is stationary, then there is a regular uncountable $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ.*

Proof. "κ is regular" is expressible by a Π^1_1 sentence in $(V_\kappa, \in)$ and so is "κ is uncountable." "S is stationary" is Π^1_1 in $(V_\kappa, \in, S)$: For every C, if C is closed unbounded, then $S \cap C \neq \emptyset$. □

Lemma 17.21. *If κ is weakly compact and if $A \subset \kappa$ is such that $A \cap \alpha \in L$ for every $\alpha < \kappa$, then A is constructible.*

Proof. Let $A \subset \kappa$ be such that $A \cap \alpha \in L$ for all $\alpha < \kappa$. By Lemma 17.17 there is a transitive model $(M, \in, A') \succ (V_\kappa, \in, A)$ such that $\kappa \in M$. Consider the sentence $\forall\alpha\,\exists x\,(x \text{ is constructible and } x = A \cap \alpha)$ and let $\alpha = \kappa$. □

Unlike measurability, weak compactness is consistent with $V = L$:

Theorem 17.22. *If κ is weakly compact then κ is weakly compact in L.*

Proof. In L, let $T = (\kappa, <_T)$ be a tree of height κ such that each level of T has size less than κ. If κ is weakly compact then T has a branch B (in the universe), and by Lemma 17.21, $B \in L$. Hence κ has the tree property in L, and since κ is inaccessible, it is weakly compact in L. □

Partitions and Models

Let us consider a model $\mathfrak{A} = (A, P^{\mathfrak{A}}, \ldots, F^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots)$ of a (not necessarily countable) language $\mathcal{L} = \{P, \ldots, F, \ldots, c, \ldots\}$. Let κ be an infinite cardinal and let us assume that the universe A of the model $\mathfrak{A}$ contains all ordinals $\alpha < \kappa$, i.e., $\kappa \subset A$.

Definition 17.23. A set $I \subset \kappa$ is a *set of indiscernibles* for the model $\mathfrak{A}$ if for every $n \in \omega$, and every formula $\varphi(v_1, \ldots, v_n)$,

$$\mathfrak{A} \vDash \varphi[\alpha_1, \ldots, \alpha_n] \quad \text{if and only if} \quad \mathfrak{A} \vDash \varphi[\beta_1, \ldots, \beta_n]$$

whenever $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ are two increasing sequences of elements of I.

Lemma 17.24. *Let κ be an infinite cardinal and assume that*

$$\kappa \to (\alpha)^{<\omega}_{2^\lambda}$$

where α is a limit ordinal and λ is an infinite cardinal. Let $\mathcal{L}$ be a language of size $\leq \lambda$ and let $\mathfrak{A}$ be a model of $\mathcal{L}$ such that $\kappa \subset A$. Then $\mathfrak{A}$ has a set of indiscernibles of order-type α.

Proof. Let Φ be the set of all formulas of the language $\mathcal{L}$. We consider the function $F : [\kappa]^{<\omega} \to P(\Phi)$ defined as follows: If $x \in [\kappa]^n$ and $x = \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_1 < \ldots < \alpha_n$, then

$$F(x) = \{\varphi(v_1, \ldots, v_n) \in \Phi : \mathfrak{A} \vDash \varphi[\alpha_1, \ldots, \alpha_n]\}.$$

The function F is a partition into at most 2^λ pieces and thus has a homogeneous set $I \subset \kappa$ of order-type α. It is now easy to verify that I is a set of indiscernibles for $\mathfrak{A}$. □

We shall see later that for a given limit ordinal α, the least κ that satisfies $\kappa \to (\alpha)^{<\omega}$ is inaccessible and satisfies $\kappa \to (\alpha)^{<\omega}_\lambda$ for all $\lambda < \kappa$. Now we shall prove this for Ramsey cardinals.

Lemma 17.25. *If $\kappa \to (\kappa)^{<\omega}$ and if $\lambda < \kappa$ is a cardinal, then $\kappa \to (\kappa)^{<\omega}_\lambda$.*

Proof. Let $F : [\kappa]^{<\omega} \to \lambda$ be a partition into $\lambda < \kappa$ pieces. We consider the following partition G of $[\kappa]^{<\omega}$ into two pieces: If $\alpha_1 < \ldots < \alpha_k < \alpha_{k+1} < \ldots < \alpha_{2k}$ are elements of κ and if $F(\{\alpha_1, \ldots, \alpha_k\}) = F(\{\alpha_{k+1}, \ldots, \alpha_{2k}\})$, then we let $G(\{\alpha_1, \ldots, \alpha_{2k}\}) = 1$; for all other $x \in [\kappa]^{<\omega}$, we let $G(x) = 0$.

Now, let $H \subset \kappa$ be a homogeneous set for G, $|H| = \kappa$. We claim that for each k and each $x \in [H]^{2k}$, $G(x) = 1$: This is because $|H| = \kappa > \lambda$, and therefore we can find $\alpha_1 < \ldots < \alpha_k < \alpha_{k+1} < \ldots < \alpha_{2k}$ in H such that $F(\{\alpha_1, \ldots, \alpha_k\}) = F(\{\alpha_{k+1}, \ldots, \alpha_{2k}\})$.

It follows that H is homogeneous for F: If $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ are two sequences in H, we choose a sequence $\gamma_1 < \ldots < \gamma_n$ in H such that both $\alpha_n < \gamma_1$ and $\beta_n < \gamma_1$. Then

$$G(\{\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n\}) = G(\{\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n\}) = 1,$$

and hence

$$F(\{\alpha_1, \ldots, \alpha_n\}) = F(\{\gamma_1, \ldots, \gamma_n\}) = F(\{\beta_1, \ldots, \beta_n\}).$$ □

Corollary 17.26. *If κ is a Ramsey cardinal and if $\mathfrak{A} \supset \kappa$ is a model of a language of size $< \kappa$, then $\mathfrak{A}$ has a set of indiscernibles of size κ.* □

The combinatorial methods introduced in this section will now be employed to obtain a result on measurable cardinals considerably stronger than Scott's Theorem. It will be shown that if a Ramsey cardinal exists then $V = L$ fails in a strong way. A more extensive theory will be developed in Chapter 18.

Let us make a few observations about models with definable Skolem functions. Let $\mathfrak{A}$ be a model of a language $\mathcal{L}$ such that $\mathfrak{A} \supset \kappa$ and let $I \subset \kappa$ be a set of indiscernibles for $\mathfrak{A}$. Let us assume that the model $\mathfrak{A}$ has *definable Skolem functions*; i.e., for every formula $\varphi(u, v_1, \ldots, v_n)$, where $n \geq 0$, there exists an n-ary function h_φ in $\mathfrak{A}$ such that:

(i) h_φ is definable in $\mathfrak{A}$, i.e., there is a formula ψ such that

$$y = h_\varphi(x_1, \ldots, x_n) \quad \text{if and only if} \quad \mathfrak{A} \vDash \psi[y, x_1, \ldots, x_n]$$

for all $y, x_1, \ldots, x_n \in A$; and

(ii) h_φ is a Skolem function for φ.

Let $\mathfrak{B} \subset \mathfrak{A}$ be the closure of I under all functions in $\mathcal{L}$ and the functions h_φ for all formulas φ. $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$, and in fact is the smallest elementary submodel of $\mathfrak{A}$ that includes the set I; we call $\mathfrak{B}$ the *Skolem hull* of I and say that I *generates* $\mathfrak{B}$.

We augment the language of $\mathfrak{A}$ by adding function symbols for all the Skolem functions h_φ and call *Skolem terms* the terms built from variables and constant symbols (0-ary functions) by applications of functions in $\mathcal{L}$ and the Skolem functions. Since $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$, the interpretation of each Skolem term t is the same in $\mathfrak{B}$ as in $\mathfrak{A}$. For every element $x \in \mathfrak{B}$ there is a Skolem term t and indiscernibles $\gamma_1 < \ldots < \gamma_n$, elements of I, such that $x = t^{\mathfrak{A}}[\gamma_1, \ldots, \gamma_n] = t^{\mathfrak{B}}[\gamma_1, \ldots, \gamma_n]$. Now if ψ is a formula of the augmented language, i.e., if ψ also contains the Skolem terms, it still does not distinguish between the indiscernibles: If $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ are two sequences in I, then $\psi(\alpha_1, \ldots, \alpha_n)$ holds (either in $\mathfrak{A}$ or in $\mathfrak{B}$) if and only if $\psi(\beta_1, \ldots, \beta_n)$ holds.

Theorem 17.27 (Rowbottom). *If κ is a Ramsey cardinal, then the set of all constructible reals is countable. More generally, if λ is an infinite cardinal less than κ, then $|P^L(\lambda)| = \lambda$.*

Proof. Let κ be a Ramsey cardinal and let $\lambda < \kappa$. Since κ is inaccessible, we have $P^L(\lambda) \subset L_\kappa$. Consider the model

$$\mathfrak{A} = (L_\kappa, \in, P^L(\lambda), \alpha)_{\alpha \leq \lambda}.$$

$\mathfrak{A}$ is a model of the language $\mathcal{L} = \{\in, Q, c_\alpha\}_{\alpha \leq \lambda}$ where Q is a one-place predicate (interpreted in $\mathfrak{A}$ as $P(\lambda) \cap L$) and c_α, $\alpha \leq \lambda$, are constant symbols

(interpreted as ordinals less than or equal to λ). Since κ is Ramsey, there exists a set I of size κ of indiscernibles for $\mathfrak{A}$.

The model $\mathfrak{A}$ has definable Skolem functions: Since κ is inaccessible, L_κ is a model of ZFC + $V = L$ and therefore has a definable well-ordering. Thus let $\mathfrak{B} \subset L_\kappa$ be the elementary submodel of $\mathfrak{A}$ generated by the set I. Every element $x \in \mathfrak{B}$ is expressible as $x = t(\gamma_1, \ldots, \gamma_n)$ where t is a Skolem term and $\gamma_1 < \ldots < \gamma_n$ are elements of I.

We shall now show that the set $S = P^L(\lambda) \cap \mathfrak{B}$ has at most λ elements. Since S is the interpretation in $\mathfrak{B}$ of the one-place predicate Q, it suffices to show that there are at most λ elements $x \in \mathfrak{B}$ such that $\mathfrak{B} \vDash Q(x)$.

Let t be a Skolem term. Let us consider the truth value of the formula

$$t(\alpha_1, \ldots, \alpha_n) = t(\beta_1, \ldots, \beta_n) \tag{17.13}$$

for a sequence of indiscernibles $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_n$. The formula (17.13) is either true for all increasing sequences in I or false for all increasing sequences in I. If (17.13) is true, then it is true for any two sequences $\alpha_1 < \ldots < \alpha_n$, $\beta_1 < \ldots < \beta_n$, in I: Pick $\gamma_1, \ldots, \gamma_n$ bigger than both α_n and β_n and then $t(\alpha_1, \ldots, \alpha_n) = t(\gamma_1, \ldots, \gamma_n) = t(\beta_1, \ldots, \beta_n)$. If (17.13) is false, then we choose κ increasing sequences

$$\alpha_1^0 < \ldots < \alpha_n^0 < \alpha_1^1 < \ldots < \alpha_n^1 < \ldots < \alpha_1^\xi < \ldots < \alpha_n^\xi < \ldots \qquad (\xi < \kappa)$$

in I and then $t(\alpha_1^\xi, \ldots, \alpha_n^\xi) \neq t(\alpha_1^\eta, \ldots, \alpha_n^\eta)$ whenever $\xi \neq \eta$. In conclusion, the set

$$\{t(\alpha_1, \ldots, \alpha_n) : \alpha_1 < \ldots < \alpha_n \text{ are in } I\} \tag{17.14}$$

has either one or κ elements.

Now we apply this to evaluate the size of the set S. We know that $|S| < \kappa$ because $S \subset P^L(\lambda) \subset P(\lambda)$ and κ is inaccessible. If t is a Skolem term for which the set (17.14) has size κ, then $t(\alpha_1, \ldots, \alpha_n)$ is not in S, for any $\alpha_1 < \ldots < \alpha_n$ in I; by indiscernibility, $Q(t(\alpha_1, \ldots, \alpha_n))$ is true or false simultaneously for all increasing sequences in I. Thus if $t(\alpha_1, \ldots, \alpha_n) \in S$, the set (17.14) has only one element.

However, since $|\mathcal{L}| \leq \lambda$, there are at most λ Skolem terms. And since every $x \in \mathfrak{B}$ has the form $t(\alpha_1, \ldots, \alpha_n)$ for some Skolem term and $\alpha_1 < \ldots < \alpha_n$ in I, it follows that $|S| \leq \lambda$.

Thus we have proved that $S = Q^{\mathfrak{B}} = P^L(\lambda) \cap \mathfrak{B}$ has at most λ elements. Now $\mathfrak{B} \prec L_\kappa$ and $|\mathfrak{B}| = \kappa$; hence the transitive collapse of $\mathfrak{B}$ is L_κ and we have an isomorphism

$$\pi : B \simeq L_\kappa.$$

Since each $\alpha \leq \lambda$ has a name in $\mathfrak{A}$, we have $\lambda \cup \{\lambda\} \subset \mathfrak{B}$ and so $\pi(X) = X$ for each $X \subset \lambda$ in $\mathfrak{B}$. In particular $\pi(X) = X$ for all $X \in S$; and since $Q^{\pi(\mathfrak{B})} = \pi(S) = S$, we have

$$S = P^L(\lambda) \cap \pi(\mathfrak{B}) = P^L(\lambda) \cap L_\kappa = P^L(\lambda).$$

This completes the proof: On the one hand, we proved that $|S| \leq \lambda$; and on the other hand, $|P^L(\lambda)| \geq \lambda$; thus $|P^L(\lambda)| = \lambda$. □

Every Ramsey cardinal is weakly compact. Not only is the least Ramsey cardinal greater than the least weakly compact but, as we show below, there is a hierarchy of large cardinals below each Ramsey cardinal, exceeding the least weakly compact.

Definition 17.28. For every limit ordinal α, the *Erdős cardinal* η_α is the least κ such that $\kappa \to (\alpha)^{<\omega}$.

We shall prove that each η_α, if it exists, is inaccessible, and if $\alpha < \beta$ then $\eta_\alpha < \eta_\beta$. Note that κ is a Ramsey cardinal if and only if $\kappa = \eta_\kappa$.

Lemma 17.29. *If* $\kappa \to (\alpha)^{<\omega}$, *then* $\kappa \to (\alpha)^{<\omega}_{2^{\aleph_0}}$.

Proof. Let f be a partition, $f : [\kappa]^{<\omega} \to \{0,1\}^\omega$. For each $n < \omega$, let $f_n = f \restriction [\kappa]^n$, and for each $\kappa < \omega$, let $f_{n,k} : [\kappa]^n \to \{0,1\}$ be as follows:

$$f_{n,k}(\{\alpha_1, \ldots, \alpha_n\}) = h(k), \qquad \text{where } h = f_n(\{\alpha_1, \ldots, \alpha_n\}).$$

Let π be a one-to-one correspondence between ω and $\omega \times \omega$ such that if $\pi(m) = (n,k)$, then $m \geq n$; for each m, let $g_m : [\kappa]^m \to \{0,1\}$ be the partition defined by

$$g_m(\{\alpha_1, \ldots, \alpha_m\}) = f_{n,k}(\{\alpha_1, \ldots, \alpha_n\})$$

where $(n,k) = \pi(m)$.

By the assumption, there exists $H \subset \kappa$ of order-type α which is homogeneous for all g_m. We claim that H is homogeneous for f. If not, then $f_n(\{\alpha_1, \ldots, \alpha_n\}) \neq f_n(\{\beta_1, \ldots, \beta_n\})$ for some α's and β's in H. Then for some k, $f_{n,k}(\{\alpha_1, \ldots, \alpha_n\}) \neq f_{n,k}(\{\beta_1, \ldots, \beta_n\})$, contrary to the assumption that H is homogeneous for g_m, where $\pi(m) = (n,k)$. □

Lemma 17.30. *For every* $\kappa < \eta_\alpha$, $\eta_\alpha \to (\alpha)^{<\omega}_\kappa$.

Proof. Let $\kappa < \eta_\alpha$, and let $f : [\eta_\alpha]^{<\omega} \to \kappa$. We wish to find a homogeneous set for f of order-type α. Since $\kappa < \eta_\alpha$, there exists $g : [\kappa]^{<\omega} \to \{0,1\}$ that has no homogeneous set of order-type α. For each n, let $f_n = f \restriction [\eta_\alpha]^n$ and $g_n = g \restriction [\kappa]^n$, and let $\mathfrak{A}$ be the model $(V_{\eta_\alpha}, \in, f_n, g_n)_{n=0,1,\ldots}$.

By Lemmas 17.29 and 17.24, the model $\mathfrak{A}$ has a set of indiscernibles H of order-type α. We shall show that H is homogeneous for f. It suffices to show that for each n, the formula

$$f_n(\{\alpha_1, \ldots, \alpha_n\}) = f_n(\{\beta_1, \ldots, \beta_n\}) \tag{17.15}$$

holds in $\mathfrak{A}$ for any increasing sequence $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_n$ of indiscernibles: Then if $\alpha_1 < \ldots < \alpha_n$ and $\alpha'_1 < \ldots < \alpha'_n$ are arbitrary

in H, we choose $\beta_1 < \ldots < \beta_n$ in H such that $\alpha_n < \beta_1$ and $\alpha'_n < \beta_1$, and $f_n(\{\alpha_1, \ldots, \alpha_n\}) = f_n(\{\alpha'_1, \ldots, \alpha'_n\})$ follows from (17.15).

Thus let us assume that the negation of (17.15) holds for any $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_n$ in H. Let u_ξ, $\xi < \alpha$, be increasing n-sequences in H such that the last element of u_ξ is less than the first element of u_η whenever $\xi < \eta$. Let $\gamma_\xi = f(u_\xi)$ for all $\xi < \alpha$, and let $G = \{\gamma_\xi : \xi < \alpha\}$. By indiscernibility, and since $\gamma_0 > \gamma_1 > \ldots > \gamma_\xi > \ldots$ is impossible, we have $\gamma_0 < \gamma_1 < \ldots < \gamma_\xi < \ldots$.

We shall reach a contradiction by showing that G is homogeneous for g. For each k, consider the formula

$$g(\{f(u_{\xi_1}), \ldots, f(u_{\xi_k})\}) = g(\{f(u_{\nu_1}), \ldots, f(u_{\nu_k})\}). \tag{17.16}$$

By indiscernibility, either (17.16) or its negation holds for all increasing sequences $\xi_1 < \ldots < \xi_k < \nu_1 < \ldots < \nu_k$. The inequality cannot hold because g takes only two values, 0 and 1, and three sequences $\langle \xi_1, \ldots, \xi_k \rangle$ would give three different values. Thus (17.16) holds, and the same argument as earlier in this proof shows that g is constant on $[G]^k$. □

Theorem 17.31. *Every Erdős cardinal η_α is inaccessible, and if $\alpha < \beta$ then $\eta_\alpha < \eta_\beta$.*

Proof. First we claim that η_α is a strong limit cardinal. If $\kappa < \eta_\alpha$ then because $2^\kappa \not\to (\alpha)^2_\kappa$ (by Lemma 9.3) and $\eta_\alpha \to (\alpha)^2_\kappa$, we have $2^\kappa < \eta_\alpha$. We shall show that η_α is regular.

Let us assume that η_α is singular and that $\kappa = \operatorname{cf} \eta_\alpha$; let $\eta_\alpha = \lim_{\nu \to \kappa} \lambda_\nu$. For each $\nu < \kappa$, let $f^\nu : [\lambda_\nu]^{<\omega} \to \{0, 1\}$ be such that f^ν has no homogeneous set of order-type α. For each n, let $f^\nu_n = f^\nu \restriction [\lambda_\nu]^n$; let $\mathfrak{A}$ be the model $(V_{\eta_\alpha}, \in, \lambda_\nu, f^\nu_n)_{\nu<\kappa, n=0,1,\ldots}$. Since η_α is a strong limit and $\kappa < \eta_\alpha$, the model $\mathfrak{A}$ has a set of indiscernibles H of order-type α.

Let ν be such that λ_ν is greater than the least element of H. Then by indiscernibility, all elements of H are smaller than λ_ν. Since the function f^ν takes only two values, it follows that for each n, it is the equality

$$f^\nu_n(\{\alpha_1, \ldots, \alpha_n\}) = f^\nu_n(\{\beta_1, \ldots, \beta_n\})$$

that holds for all increasing sequences $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_n$ in H, and not its negation. Hence H is homogeneous for f^ν, contrary to the assumption on f^ν.

Finally, let $\alpha < \beta$ be limit ordinals, and let us assume that $\eta_\alpha = \eta_\beta$. For each $\xi < \eta_\alpha$, there exists a function $f_\xi : [\xi]^{<\omega} \to \{0, 1\}$ that has no homogeneous subset of ξ of order-type α. Let us define $g : [\eta_\beta]^{<\omega} \to \{0, 1\}$ by

$$g(\{\xi_1, \ldots, \xi_n\}) = f_{\xi_n}(\{\xi_1, \ldots, \xi_{n-1}\}).$$

Now if H is homogeneous for g, then for each $\xi \in H$, $H \cap \xi$ is homogeneous for f_ξ. Hence the order-type of each $H \cap \xi$ is less than α, and therefore the order-type of H is at most α, which is less than β. A contradiction. □

We shall now show that the least Erdős cardinal η_ω is greater than the least weakly compact cardinal. We use the following lemma, of independent interest:

Lemma 17.32. *Let M and N be transitive models of* ZFC *and let $j : M \to N$ be a nontrivial elementary embedding; let κ be the least ordinal moved. If $P^M(\kappa) = P^N(\kappa)$, then κ is a weakly compact cardinal in M.*

Proof. We prove a somewhat stronger statement: κ is ineffable in M (see Exercise 17.26).

Let $\langle A_\alpha : \alpha < \kappa\rangle \in M$ be such that $A_\alpha \subset \alpha$ for all α. We have $j(A_\alpha) = A_\alpha$ for all $\alpha < \kappa$, and hence $j(\langle A_\alpha : \alpha < \kappa\rangle) = (\langle A_\alpha : \alpha < j(\kappa)\rangle$ (for some A_α, $\kappa \leq \alpha < j(\kappa)$). The set A_κ is in M and witnesses ineffability of κ in M. □

Theorem 17.33. *If η_ω exists then there exists a weakly compact cardinal below η_ω.*

Proof. Let h_φ, $\varphi \in Form$, be Skolem functions for the language $\{\in\}$ of set theory, and let us consider the model $\mathfrak{A} = (V_{\eta_\omega}, \in, h^{\mathfrak{A}}_\varphi)_{\varphi\in Form}$ where for each φ, $h^{\mathfrak{A}}_\varphi$ is a Skolem function for φ in $(V_{\eta_\omega}, \in)$. The model $\mathfrak{A}$ has a set of indiscernibles I of order-type ω. Let B be the closure of I under the Skolem functions $h^{\mathfrak{A}}_\varphi$.

Let us consider some nontrivial order-preserving mapping of H into H. Using the Skolem functions, we extend this mapping (in the unique way) to a nontrivial elementary embedding of B into B. Let M be the transitive set isomorphic to B and let $j : M \to M$ be the corresponding nontrivial elementary embedding.

Since η_ω is inaccessible, V_{η_ω} is a model of ZFC and thus M is a transitive model of ZFC. By Lemma 17.32 there exists a weakly compact cardinal in M, and therefore in V_{η_ω}. □

The next result shows that the Erdős cardinal η_ω is consistent with $V = L$. In Chapter 18 we show that the existence of η_{ω_1} implies $V \neq L$.

Theorem 17.34. *If $\kappa \to (\omega)^{<\omega}$ then $L \vDash \kappa \to (\omega)^{<\omega}$.*

Proof. Let f be a constructible partition $f : [\kappa]^{<\omega} \to \{0, 1\}$. We claim that if there is an infinite homogeneous set for f, then there is one in L. Let T be the set of all finite increasing sequences $t = \langle\alpha_0, \ldots, \alpha_{n-1}\rangle$ in κ such that for every $k \leq n$, f is constant on $[\{\alpha_0, \ldots, \alpha_{n-1}\}]^k$, and let us consider the tree $(T, \supset)$; clearly, T is constructible. We note that an infinite homogeneous set for f exists if and only if $(T, \supset)$ is not well-founded. However, being well-founded is an absolute property for models of ZFC; and so if the tree is not well-founded, then it is not well-founded in L, and the claim follows. □

Let us consider models of a countable language $\mathcal{L}$, with a distinguished one-place predicate Q. A model $\mathfrak{A} = (A, Q^{\mathfrak{A}}, \ldots)$ of $\mathcal{L}$ has *type* (κ, λ) if $|A| = \kappa$ and $|Q^{\mathfrak{A}}| = \lambda$.

Definition 17.35. A cardinal $\kappa > \aleph_1$ is a *Rowbottom cardinal* if for every uncountable $\lambda < \kappa$, every model of type (κ, λ) has an elementary submodel of type $(\kappa, \aleph_0)$.

An infinite cardinal is a *Jónsson cardinal* if every model of size κ has a proper elementary submodel of size κ.

Every Rowbottom cardinal is a Jónsson cardinal and the following lemma, a variation on Rowbottom's Theorem, shows that every Ramsey cardinal is a Rowbottom cardinal.

Lemma 17.36. *Let κ be a Ramsey cardinal, and let λ be an infinite cardinal less than κ. Let $\mathfrak{A} = (A, \ldots)$ be a model of a language $\mathcal{L}$ such that $|\mathcal{L}| \leq \lambda$, and let $A \supset \kappa$. If $P \subset A$ is such that $|P| < \kappa$ then $\mathfrak{A}$ has an elementary submodel $\mathfrak{B} = (B, \ldots)$ such that $|B| = \kappa$ and $|P \cap B| \leq \lambda$.*

Moreover, if $X \subset A$ is of size at most λ, then we can find $\mathfrak{B}$ such that $X \subset B$.

Moreover, if κ is a measurable and D is a normal measure on κ, then we can find $\mathfrak{B}$ such that $B \cap \kappa \in D$.

Proof. First we add to the language $\mathcal{L}$ one unary predicate whose interpretation is the set P; we also add constant symbols for all $x \in X$. Next we find some Skolem functions h_φ (in $(A, \ldots, P, x)_{x \in X}$) for every formula φ, and extend the language further by adding function symbols for the functions h_φ.

Next we find a set of indiscernibles $I \subset \kappa$, of size κ, for the expanded model $\mathfrak{A}'$; if κ is measurable and D is a normal measure, we find $I \in D$. We let B be the elementary submodel of $\mathfrak{A}'$ generated by I. As in the proof of Theorem 17.27, one proves that if $|P \cap B| < \kappa$ then $|P \cap B| \leq \lambda$. □

In Chapter 18 we show that if there exists a Jónsson cardinal then $V \neq L$.

Exercises

17.1. Let U be a nonprincipal ultrafilter on ω. Then $\mathrm{Ult}_U(V)$ is not well-founded.

[For each $k \in \omega$, let f_k be a function on ω such that $f_k(n) = n - k$ for all $n \geq k$. Then $f_0 \ni^* f_1 \ni^* f_2 \ni^* \ldots$ is a descending $\in^*$-sequence in Ult.]

17.2. If U is not σ-complete, then $\mathrm{Ult}_U(V)$ is not well-founded.

[There exists a countable partition $\{X_n : n = 0, 1, 2, \ldots\}$ of S such that $X_n \notin U$ for all n. For each k, let f_k be a function on S such that $f_k(x) = n - k$ for all $x \in X_n$.]

17.3. If Ult is well-founded, then every ordinal number α is represented by a function $f : S \to Ord$.

17.4. If U is a principal ultrafilter $\{X \in S : x_0 \in S\}$ then $[f] = f(x_0)$ for each f, and j_U is the identity mapping.

17.5. Let U be a nonprincipal σ-complete ultrafilter on S and let λ be the largest cardinal such that U is λ-complete. Then $j_U(\lambda) > \lambda$.

[Let $\{X_\alpha : \alpha < \lambda\}$ be a partition of S into sets of measure 0; let f be a function on S such that $f(x) = \alpha$ if $x \in X_\alpha$. Then $[f] \geq \lambda$.]

17.6. If j is an elementary embedding of the universe into a transitive model M, then $M = \bigcup_{\alpha \in Ord} j(V_\alpha)$.

17.7. Let j be an elementary embedding of the universe and let κ be the least ordinal moved. If X is a class of ordinals such that $\kappa \in j(X)$, then $\kappa \in M(X)$.

17.8. If $j : V \to M$ is a nontrivial elementary embedding, if κ is the least ordinal moved, and if $\lambda = \lim\{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$, then there exists $A \subset \lambda$ such that $A \notin M$.

[Assuming that M contains all bounded subsets of λ, the proof of Theorem 17.7 shows that $G \notin M$.]

17.9. If κ is measurable, then there exists a normal measure D on κ such that $\mathrm{Ult}_D(V) \vDash \kappa$ is not measurable.

[Let D be a normal measure such that $j_D(\kappa)$ is the least possible ordinal; let $M = \mathrm{Ult}_D(V)$. If κ is measurable in M, then there is a normal measure U on κ such that $U \in M$. Since $P(\kappa) \subset M$, we have $\mathrm{Ult}_U(\kappa, <) \in M$. By Lemma 17.9(iii) we have $j_U(\kappa) < ((2^\kappa)^+)^M$. Since $((2^\kappa)^+)^M < j_D(\kappa)$, we get a contradiction.]

A function f on κ is *monotone* if $f(\alpha) \leq f(\beta)$ whenever $\alpha < \beta$.

17.10. Let U be a nonprincipal κ-complete ultrafilter on κ. Then U extends the closed unbounded filter if and only if the diagonal function is the least nonconstant monotone function in Ult_U.

[If U extends the closed unbounded filter and if f is monotone and regressive on some $X \in U$, then since X is stationary, f is constant on an unbounded set and hence constant almost everywhere. If U does not extend the closed unbounded filter, then $f(\alpha) = \sup(C \cap \alpha)$ (where $C \notin U$ is closed unbounded) is a nonconstant monotone function regressive on $X \in U$.]

17.11. Let U be a κ-complete ultrafilter on κ, and let $h : \kappa \to \kappa$. If $D = h_*(U)$, then the mapping $k : \mathrm{Ult}_D(V) \to \mathrm{Ult}_U(V)$ defined by $k([f]_D) = [f \circ h]_U$ is an elementary embedding.

17.12. If D is a normal measure on κ and $\{\alpha : 2^\alpha \leq \alpha^{++}\} \in D$, then $2^\kappa \leq \kappa^{++}$. More generally, if $\beta < \kappa$ and $\{\aleph_\alpha : 2^{\aleph_\alpha} \leq \aleph_{\alpha+\beta}\} \in D$, then $2^{\aleph_\kappa} \leq \aleph_{\kappa+\beta}$.

[If f is such that $f(\aleph_\alpha) = \aleph_{\alpha+\beta}$ for all $\alpha < \kappa$, then $[f]_D = (\aleph_{\kappa+j(\beta)})^M \leq \aleph_{\kappa+\beta}$.]

17.13. If D is a normal measure on κ and $\{\alpha : 2^{\aleph_\alpha} < \aleph_{\alpha+\alpha}\} \in D$, then $2^{\aleph_\kappa} < \aleph_{\kappa+\kappa}$.

[If $f(\alpha) = \aleph_{\alpha+\alpha}$, then $[f] = (\aleph_{\kappa+\kappa})^M$.]

17.14. Let κ be measurable and let $\lambda = \aleph_{\kappa+\kappa}$ be strong limit. Then $2^\lambda < \aleph_{(2^\kappa)^+}$.

[$j(\lambda) = (\aleph_{j(\kappa+\kappa)})^M \leq \aleph_{j(\kappa)+j(\kappa)}$; $j(\kappa) + j(\kappa) < (2^\kappa)^+$.]

17.15. Let κ be measurable, let λ be strong limit, $\mathrm{cf}\,\lambda = \kappa$, such that $\lambda < \aleph_\lambda$. Then $2^\lambda < \aleph_\lambda$.

[If $\lambda = \aleph_\alpha$, then $j(\lambda) = (\aleph_{j(\alpha)})^M \leq \aleph_{j(\alpha)}$, and $j(\alpha) < (\alpha^\kappa)^+ < \lambda$.]

17.16. Let $\Phi(\alpha)$ denote the αth fixed point of $\aleph$, i.e., the αth ordinal ξ such that $\aleph_\xi = \xi$. Let κ be measurable and let $\lambda = \Phi(\kappa + \kappa)$ be strong limit. Then $2^\lambda < \Phi((2^\kappa)^+)$.

[Use the fact that $(\Phi(\alpha))^M \leq \Phi(\alpha)$ for all α.]

17.17. If $\kappa = \lambda^+$ is a successor cardinal, then the Weak Compactness Theorem for $\mathcal{L}_{\kappa,\omega}$ is false.

[Consider constants c_α, $\alpha \leq \kappa$, a binary relation $<$ and a ternary relation R. Consider the sentences saying that (a) $<$ is a linear ordering; (b) $c_\alpha < c_\beta$ for $\alpha < \beta$; (c) each f_x is a function, where $f_x(y) = z$ stands for $R(x, y, z)$. Let Σ consist of these sentences, the sentence $z < x \to \exists y\, R(x, y, z)$ (saying that $\operatorname{ran}(f_x) \supset \{z : z < x\}$), and the infinitary sentence $R(x, y, z) \to \bigvee_{\xi<\lambda}(y = c_\xi)$ (saying that $\operatorname{dom}(f_x) \subset \{c_\xi : \xi < \lambda\}$). Show that each $S \subset \Sigma$, $|S| \leq \lambda$, has a model, but Σ does not.]

17.18. If κ is a singular cardinal, then the Weak Compactness Theorem for $\mathcal{L}_{\kappa,\omega}$ is false.

[Let $A \subset \kappa$ be a cofinal subset of size $< \kappa$. Consider constants c_α, $\alpha \leq \kappa$, and a linear ordering $<$. There is Σ that says on the one hand that $\{c_\alpha : \alpha \in A\}$ is cofinal in the universe, and on the other hand that for each $\alpha < \kappa$, if $(\forall \beta < \alpha)\, c_\kappa > c_\beta$ then $c_\kappa > c_\alpha$; and each $S \subset \Sigma$, $|S| < \kappa$, has a model.]

17.19. If κ is weakly compact and if $(\mathcal{B}, \subset)$ is a κ-complete algebra of subsets of κ such that $|\mathcal{B}| = \kappa$, then every κ-complete filter F on $\mathcal{B}$ can be extended to a κ-complete ultrafilter on $\mathcal{B}$.

[Consider constants c_X for all $X \in \mathcal{B}$, and a unary predicate U. Let Σ be the following set of $\mathcal{L}_{\kappa,\omega}$-sentences: $\neg U(c_\emptyset)$, $U(c_X) \vee U(c_{\kappa - X})$ for all $X \in \mathcal{B}$, $U(c_X) \to U(c_Y)$ for all $X \subset Y$ in $\mathcal{B}$, $U(c_X)$ for all $X \in F$, and $\bigwedge_{X\in\mathcal{A}} U(c_X) \to U(c_{\bigcap \mathcal{A}})$ for all $\mathcal{A} \subset \mathcal{B}$ such that $|\mathcal{A}| < \kappa$. Show that Σ has a model.]

17.20. If κ is inaccessible and if every κ-complete filter on any κ-complete algebra $\mathcal{B}$ of subsets of κ such that $|\mathcal{B}| = \kappa$ can be extended to a κ-complete ultrafilter, then κ is weakly compact.

[As in Lemma 10.18.]

17.21. If $(P, <)$ is a linearly ordered set of size κ, and κ is weakly compact, then there is a subset $W \subset P$ of size κ that is either well-ordered or conversely well-ordered by $<$.

17.22. The least measurable cardinal is Σ^2_1-describable.

[$\exists U$ (U is κ-complete nonprincipal ultrafilter on κ).]

17.23. Every inaccessible cardinal is Π^0_m-indescribable for all m.

[Let $U \subset V_\kappa$. The model $(V_\kappa, \in, U)$ has a countable elementary submodel M_0. Let $\alpha_0 < \kappa$ be such that $M_0 \subset V_{\alpha_0}$. For each n, let M_{n+1} be an elementary submodel of $(V_\kappa, \in, U)$ such that $V_{\alpha_n} \subset M_{n+1}$, and let $M_{n+1} \subset V_{\alpha_{n+1}}$. Let $\alpha = \lim_{n\to\omega} \alpha_n$; then V_α is an elementary submodel of $(V_\kappa, \in, U)$.]

17.24. If κ is weakly compact, then there is no countably generated complete Boolean algebra B such that $|B| = \kappa$.

[Assume that B is such. Note that $\operatorname{sat}(B) = \kappa$. We may assume that $B = (\kappa, +, \cdot, -)$; let $A \subset \kappa$ be a countable set of generators. Let U_1 be the set of all pairs (u, x) such that $u \in \kappa$, $x \subset \kappa$, $|x| < \kappa$, and $u = \sum x$, let $U_2 = \{A\}$. Let σ be the conjunction of these sentences: (a) B is a Boolean algebra and $B \supset A$ (first order), (b) $\forall x\, \exists u$ (if $x \subset \kappa$, then $u = \sum x$) (first order), and (c) $\forall X$ (if $X \subset \kappa$ and X is a partition of B, then $\exists x\, (x = X)$) (here x is a first order variable; the sentence (c) is Π^1_1). Since $(V_\kappa, \in, U_1, U_2)$ satisfies σ, there is some $\alpha < \kappa$ such that $(V_\alpha, \in, U_1 \cap V_\alpha, U_2 \cap V_\alpha) \vDash \sigma$. Then $(\alpha, +, \cdot, -)$ is a complete subalgebra of B containing A.]

17.25. Let κ be a measurable cardinal. If $\langle A_\alpha : \alpha < \kappa\rangle$ is a sequence of sets such that $A_\alpha \subset \alpha$ for all $\alpha < \kappa$, then there exists an $A \subset \kappa$ such that $\{\alpha \in \kappa : A \cap \alpha = A_\alpha\}$ is stationary.

A cardinal κ with the property from Exercise 17.25 is called *ineffable*.

17.26. Let κ be ineffable and let $f : [\kappa]^2 \to \{0, 1\}$ be a partition. Then there exists a stationary homogeneous set. (Hence κ is weakly compact.)

[For each $\alpha < \kappa$ let $A_\alpha = \{\xi < \alpha : f(\{\xi, \alpha\}) = 1\}$, and let $A \subset \kappa$ be such that $S = \{\alpha : A \cap \alpha = A_\alpha\}$ is stationary. Either $S \cap A$ or $S - A$ is stationary, and is homogeneous.]

17.27. If κ is ineffable then κ is ineffable in L.

[Use Lemma 17.21.]

17.28. If κ is Ramsey then $\aleph_1$ is inaccessible in L.

[Show that $P^{L[x]}(\omega)$ is countable for every $x \subset \omega$.]

17.29. If M is a transitive model of ZFC and if $j : M \to M$ is a nontrivial elementary embedding, then the least ordinal κ moved by j is Π^n_m-indescribable in M, for all n and m.

[If $U \subset V^M_\kappa$, then $U = j(U) \cap V^M_\kappa$. If σ is a Π^n_m sentence and $M \vDash ((V_\kappa, \in, U) \vDash \sigma)$, then $M \vDash ((\exists \alpha < j(\kappa))\,(V_\alpha, \in, j(U) \cap V_\alpha) \vDash \sigma)$.]

17.30. The cardinal η_ω is not weakly compact.

[η_ω is Π^1_1-describable.]

17.31. An infinite cardinal κ is a Jónsson cardinal if and only if for every $F : [\kappa]^{<\omega} \to \kappa$ there exists a set $H \subset \kappa$ of size κ such that the image of $[H]^{<\omega}$ under F is not the whole set κ.

[To show that the condition is necessary, consider the model $(\kappa, <, F_1, F_2, \ldots)$ where $F_n = F{\restriction}[\kappa]^n$. To show that the condition is sufficient, let $\mathfrak{A} = (\kappa, \ldots)$ be a model. Let $\{h_n : n < \omega\}$ be a set of Skolem functions for $\mathfrak{A}$, closed under composition and arranged so that each h_α is n-ary. For each $x \in [\kappa]^n$, let $F(x) = h_n(x)$. If $H \subset \kappa$, then the image of $[H]^{<\omega}$ under F is an elementary submodel of $\mathfrak{A}$.]

17.32. $\aleph_0$ is not a Jónsson cardinal.

[Let $\mathfrak{A} = (\omega, f)$ where $f(n) = n - 1$ for all $n > 0$.]

17.33. If κ is a Rowbottom cardinal, then either κ is weakly inaccessible or $\operatorname{cf} \kappa = \omega$.

[To show that $\kappa = \lambda^+$ is not Rowbottom, let f_α be a one-to-one mapping of α onto λ, for each α, such that $\lambda \leq \alpha < \kappa$. Let $\mathfrak{A} = (\kappa, \lambda, <, R)$ where $R(\alpha, \beta, \gamma)$ if and only if $f_\alpha(\beta) = \gamma$. If $(B, B \cap \lambda, <, R \cap B^3) \prec \mathfrak{A}$ and $|B| = \kappa$, let α be the λth element of B; then $f_\alpha(B \cap \alpha) \subset B \cap \lambda$ and hence $|B \cap \lambda| = \lambda > \aleph_0$.

To show that κ is not Rowbottom if $\kappa > \operatorname{cf} \kappa = \lambda > \aleph_0$, let f be a nondecreasing function of κ onto λ and use f to produce a counterexample.]

Historical Notes

In [1963/64] Keisler and Tarski introduced the method of ultraproducts in the study of measurable cardinals, and it was established that the least measurable

cardinal is greater than the least inaccessible cardinal. Scott used the method of ultrapowers to prove that existence of measurable cardinals contradicts the Axiom of Constructibility. Rowbottom and Silver initiated applications of infinitary combinatorics developed by Erdős and his collaborators. Scott's Theorem appeared in [1961] and Kunen's Theorem in [1971a]. (Lemma 17.8 is due to Erdős and Hajnal [1966].)

In [1963/64a] Hanf studied compactness of infinitary languages; his work let to the systematic study of Keisler and Tarski. Hanf proved that the least inaccessible cardinal is not measurable (in fact not weakly compact); Erdős and Hajnal then pointed out (cf. [1962]) that the same result can be proved using infinitary combinatorics. Keisler and Tarski introduced the Mahlo operation and showed that the least measurable cardinal is much greater than, e.g., the least Mahlo cardinal, etc.

The equivalence of various formulations of weak compactness is a result of several papers. In [1963/64a] Hanf initiated investigations of compactness of infinitary languages. Erdős and Tarski listed in [1961] several properties that were subsequently shown mutually equivalent (for inaccessible cardinals) and proved several implications. These properties included the partition property $\kappa \to (\kappa)^2_2$, the tree property, and several other properties. Hanf and Scott [1961] introduced Π^n_m-indescribability and announced Theorem 17.18. Further contributions were made in the papers Hanf [1963/64b], Hajnal [1964], Keisler [1962], Monk and Scott [1963/64], Tarski [1962], and Keisler and Tarski [1963/64]. A complete list of equivalent formulations with the proofs appeared in Silver [1971b]. Theorem 17.22 is due to Silver [1971b]. Rowbottom's Theorem (as well as Lemma 17.36) are due to Rowbottom [1971].

The main results on Erdős cardinals are due to Rowbottom, Reinhardt, and Silver. Rowbottom proved that if η_{ω_1} exists, then there are only countably many constructible reals (see [1971]); Theorem 17.33 is due to Reinhardt and Silver [1965], and Theorem 17.34 is due to Silver [1970a].

Exercise 17.10: Ketonen [1973].

Ineffable cardinals were introduced by Jensen; Exercises 17.26 and 17.27 are due to Kunen and Jensen.

Exercise 17.29: Reinhardt and Silver [1965].

18. Large Cardinals and L

In Chapter 17 we proved that while "smaller" large cardinals (inaccessible, Mahlo, weakly compact) can exist in L, the "bigger" large cardinals (measurable, Ramsey) cannot. In this chapter we isolate and investigate the concept of $0^\sharp$ (*zero-sharp*), a great divide in the landscape of large cardinals.

Silver Indiscernibles

Theorem 18.1 (Silver). *If there exists a Ramsey cardinal, then*:

(i) *If κ and λ are uncountable cardinals and $\kappa < \lambda$, then $(L_\kappa, \in)$ is an elementary submodel of $(L_\lambda, \in)$.*

(ii) *There is a unique closed unbounded class of ordinals I containing all uncountable cardinals such that for every uncountable cardinal κ:*
   (a) $|I \cap \kappa| = \kappa$,
   (b) *$I \cap \kappa$ is a set of indiscernibles for $(L_\kappa, \in)$, and*
   (c) *every $a \in L_\kappa$ is definable in $(L_\kappa, \in)$ from $I \cap \kappa$.*

The elements of the class I are called *Silver indiscernibles*. Before giving the proof of Theorem 18.1 we state some consequences of the existence of Silver indiscernibles.

By the Reflection Principle, if φ is a formula, then there exists an uncountable cardinal κ such that

$$L \vDash \varphi(x_1, \ldots, x_n) \quad \text{if and only if} \quad L_\kappa \vDash \varphi(x_1, \ldots, x_n) \tag{18.1}$$

for all $x_1, \ldots, x_n \in L_\kappa$. By (i), the right hand side holds in and only if $L_\lambda \vDash \varphi(x_1, \ldots, x_n)$ for all cardinals $\lambda \geq \kappa$. In view of this, we can define satisfaction in L for all formulas $\varphi \in \mathit{Form}$: If $\varphi(v_1, \ldots, v_n)$ is a formula of the language $\mathcal{L} = \{\in\}$ and if $\langle a_1, \ldots, a_n \rangle$ is an n-termed sequence in L, we define

$$L \vDash \varphi[a_1, \ldots, a_n] \tag{18.2}$$

as follows: For every uncountable cardinal κ such that $a_1, \ldots, a_n \in L_\kappa$, $L_\kappa \vDash \varphi[a_1, \ldots, a_n]$.

Note that this gives us a truth definition for the constructible universe: $T = \{\#\sigma : L_{\aleph_1} \vDash \sigma\}$. If σ is a sentence, then $\sigma^L \leftrightarrow \#\sigma \in T$. (Note that the set T is constructible but not definable in L: Otherwise, T would be a truth definition in L. Hence the cardinal $\aleph_1$ is not definable in L.)

Moreover, as a consequence of (i) we have $(L_\kappa, \in) \prec (L, \in)$ for every uncountable cardinal κ. As a consequence of (ii) Silver indiscernibles are indiscernibles for L: If $\varphi(v_1, \ldots, v_n)$ is a formula, then

$$L \vDash \varphi[\alpha_1, \ldots, \alpha_n] \quad \text{if and only if} \quad L \vDash \varphi[\beta_1, \ldots, \beta_n] \tag{18.3}$$

whenever $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ are increasing sequences in I. Every constructible set is definable from I. If $a \in L$, there exists an increasing sequence $\langle \gamma_1, \ldots, \gamma_n \rangle$ of Silver indiscernibles and a formula φ such that

$$L \vDash a \text{ is the unique } x \text{ such that } \varphi(x, \gamma_1, \ldots, \gamma_n).$$

By (18.3), every formula $\varphi(v_1, \ldots, v_n)$ is either true or false in L for any increasing sequence $\langle \gamma_1, \ldots, \gamma_n \rangle$ of Silver indiscernibles; moreover, the truth value coincides with the truth value of $L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n]$ since $L_{\aleph_\omega} \prec L$ and $\aleph_1, \ldots, \aleph_n$ are Silver indiscernibles. Thus let us define

$$0^\sharp = \{\varphi : L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n]\} \tag{18.4}$$

(*zero-sharp*). Later in this section we shall give another definition of the set $0^\sharp$. We shall show that a set $0^\sharp$ satisfying the definition exists if and only if (i) and (ii) holds, and then $0^\sharp$ is as in (18.4).

Thus the conclusion of Theorem 18.1 is abbreviated as

$$0^\sharp \text{ exists.}$$

In the following corollaries we assume that $0^\sharp$ exists.

Corollary 18.2. *Every constructible set definable in L is countable.*

Proof. If $x \in L$ is definable in L by a formula φ, then the same formula defines x in $L_{\aleph_1}$ and hence $x \in L_{\aleph_1}$. □

In particular, every ordinal number definable in L is countable.

In the following corollary $\aleph_\alpha$ denotes the αth cardinal in V, not $\aleph_\alpha^L$.

Corollary 18.3. *Every uncountable cardinal is inaccessible in L.*

Proof. Since $L \vDash \aleph_1$ is regular, we have

$$L \vDash \aleph_\alpha \text{ is regular}$$

for every $\alpha \geq 1$. Similarly, $L \vDash \aleph_\omega$ is a limit cardinal, and hence

$$L \vDash \aleph_\alpha \text{ is a limit cardinal}$$

for every $\alpha \geq 1$. Thus every uncountable cardinal (and in fact every $\gamma \in I$) is an inaccessible cardinal in L. □

Corollary 18.4. *Every uncountable cardinal is a Mahlo cardinal in L.*

Proof. By Corollary 18.3, every Silver indiscernible is an inaccessible cardinal in L. Since $I \cap \omega_1$ is closed unbounded in ω_1, $\aleph_1$ is a Mahlo cardinal in L. □

Corollary 18.5. *For every $\alpha \geq \omega$, $|V_\alpha \cap L| \leq |\alpha|$. In particular, the set of all constructible reals is countable.*

Proof. The set $V_\alpha \cap L$ is definable in L from α. Thus $V_\alpha \cap L$ is also definable from α in L_κ where κ is the least cardinal $> \alpha$. Hence $V_\alpha \cap L \subset L_\beta$ for some β such that $|\alpha| = |\beta|$. However, $|L_\beta| = |\beta|$. □

Models with Indiscernibles

The proof of Silver's Theorem is based on a theorem of Ehrenfeucht and Mostowski in model theory, stating that every infinite model is elementarily equivalent to a model that has a set of indiscernibles of prescribed order-type. We shall deal only with models $(L_\lambda, \in)$ (and models elementarily equivalent to these); we shall prove below a special case of the Ehrenfeucht-Mostowski Theorem.

We shall use the canonical well-ordering of L to endow the models $(L_\lambda, \in)$ with definable Skolem functions. For each formula $\varphi(u, v_1, \ldots, v_n)$, let h_φ be the n-ary function defined as follows:

$$h_\varphi(v_1, \ldots, v_n) = \begin{cases} \text{the } <_L\text{-least } u \text{ such that } \varphi(u, v_1, \ldots, v_n), \\ \emptyset \qquad \text{otherwise.} \end{cases} \tag{18.5}$$

We call h_φ, $\varphi \in \mathit{Form}$, the *canonical Skolem functions.*

For each limit ordinal λ, $h_\varphi^{L_\lambda}$ is an n-ary function on L_λ, the L_λ-interpretation of h_φ, and is definable in $(L_\lambda, \in)$.

When dealing with models $(L_\lambda, \in)$ we shall freely use terms and formulas involving the h_φ since they as definable functions can be eliminated and the formulas can be replaced by $\in$-formulas. For each limit ordinal λ, the functions $h_\varphi^{L_\lambda}$, $\varphi \in \mathit{Form}$, are Skolem functions for $(L_\lambda, \in)$ and so a set $M \subset L_\lambda$ is an elementary submodel of $(L_\lambda, \in)$ if and only if M is closed under the $h_\varphi^{L_\lambda}$. If $X \subset L_\lambda$, then the closure of X under the $h_\varphi^{L_\lambda}$ is the smallest elementary submodel $M \prec L_\lambda$ such that $X \subset M$, and is the collection of all elements of L_λ definable in L_λ from X.

The fact that the well-ordering $<_\lambda$ of L_λ is definable in L_λ uniformly for all limit ordinals λ (by the same formula) implies the following:

Lemma 18.6. *If α and β are limit ordinals and if $j : L_\alpha \to L_\beta$ is an elementary embedding of $(L_\alpha, \in)$ in $(L_\beta, \in)$, then for each formula φ and all $x_1, \ldots, x_n \in L_\alpha$,*

$$h_\varphi^{L_\beta}(j(x_1), \ldots, j(x_n)) = j(h_\varphi^{L_\alpha}(x_1, \ldots, x_n)). \tag{18.6}$$

Hence j remains elementary with respect to the augmented language $\mathcal{L}^* = \{\in\} \cup \{h_\varphi : \varphi \in Form\}$. □

Let λ be a limit ordinal, and let $\mathfrak{A} = (A, E)$ be a model elementarily equivalent to $(L_\lambda, \in)$. The set $Ord^{\mathfrak{A}}$ of all ordinal numbers of the model $\mathfrak{A}$ is linearly ordered by E; let us use $x < y$ rather than $x\ E\ y$ for $x, y \in Ord^{\mathfrak{A}}$. A set $I \subset Ord^{\mathfrak{A}}$ is a *set of indiscernibles* for $\mathfrak{A}$ if for every formula φ,

$$\mathfrak{A} \vDash \varphi[x_1, \ldots, x_n] \quad \text{if and only if} \quad \mathfrak{A} \vDash \varphi[y_1, \ldots, y_n] \tag{18.7}$$

whenever $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ are elements of I. Let $h_\varphi^{\mathfrak{A}}$ denote the $\mathfrak{A}$-interpretation of the canonical Skolem functions (18.5). Given a set $X \subset A$, let us denote $H^{\mathfrak{A}}(X)$ the closure of X under all $h_\varphi^{\mathfrak{A}}$, $\varphi \in Form$. The set $H^{\mathfrak{A}}(X)$ is the *Skolem hull* of X and is an elementary submodel of $\mathfrak{A}$.

If I is a set of indiscernibles for $\mathfrak{A}$, let $\Sigma(\mathfrak{A}, I)$ be the set of all formulas $\varphi(v_1, \ldots, v_n)$ true in $\mathfrak{A}$ for increasing sequences of elements of I:

$$\varphi(v_1, \ldots, v_n) \in \Sigma(\mathfrak{A}, I) \leftrightarrow \mathfrak{A} \vDash \varphi[x_1, \ldots, x_n] \text{ for some } x_1, \ldots, x_n \in I \text{ such that } x_1 < \ldots < x_n. \tag{18.8}$$

A set of formulas Σ is called an *E.M.* set (Ehrenfeucht-Mostowski) if there exists a model $\mathfrak{A}$ elementarily equivalent to some L_λ, λ a limit ordinal, and an infinite set I of indiscernibles for $\mathfrak{A}$ such that $\Sigma = \Sigma(\mathfrak{A}, I)$.

Lemma 18.7. *If Σ is an E.M. set and α an infinite ordinal number, then there exists a model $\mathfrak{A}$ and a set of indiscernibles I for $\mathfrak{A}$ such that*:

(i) $\Sigma = \Sigma(\mathfrak{A}, I)$;
(ii) *the order-type of I is α*;
(iii) $\mathfrak{A} = H^{\mathfrak{A}}(I)$.

Moreover, the pair $(\mathfrak{A}, I)$ is unique up to isomorphism.

Proof. We prove uniqueness first. Let $(\mathfrak{A}, I)$ and $(\mathfrak{B}, J)$ be two pairs, each satisfying (i), (ii), (iii). Since both I and J have order-type α, let π be the isomorphism between I and J. We shall extend π to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Since $\mathfrak{A}$ is the Skolem hull of I, there is for each $a \in \mathfrak{A}$ a Skolem term $t(v_1, \ldots, v_n)$ (a combination of the Skolem functions h_φ) such that $a = t^{\mathfrak{A}}[x_1, \ldots, x_n]$ for some $x_1 < \ldots < x_n$ in I; similarly for $\mathfrak{B}$, J. Thus we define

$$\pi(t^{\mathfrak{A}}[x_1, \ldots, x_n]) = t^{\mathfrak{B}}[\pi(x_1), \ldots, \pi(x_n)] \tag{18.9}$$

for each Skolem term t and all $x_1, \ldots, x_n \in I$ such that $x_1 < \ldots < x_n$. Since $\Sigma(\mathfrak{A}, I) = \Sigma(\mathfrak{B}, J)$, we have

(18.10)

$$t_1^{\mathfrak{A}}[x_1, \ldots, x_n] = t_2^{\mathfrak{A}}[y_1, \ldots, y_n] \leftrightarrow t_1^{\mathfrak{B}}[\pi x_1, \ldots, \pi x_n] = t_2^{\mathfrak{B}}[\pi y_1, \ldots, \pi y_n],$$
$$t_1^{\mathfrak{A}}[x_1, \ldots, x_n]\ E^{\mathfrak{A}}\ t_2^{\mathfrak{A}}[y_1, \ldots, y_n] \leftrightarrow t_1^{\mathfrak{B}}[\pi x_1, \ldots, \pi x_n]\ E^{\mathfrak{B}}\ t_2^{\mathfrak{B}}[\pi y_1, \ldots, \pi y_n]$$

for any terms t_1, t_2 and indiscernibles x, y: Let $z_1, \ldots, z_{n+m}$ be the enumeration of the set $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ in increasing order. Then the equality in (18.10) holds (simultaneously in $\mathfrak{A}$ and $\mathfrak{B}$) just in case $\varphi(v_1, \ldots, v_{n+m}) \in \Sigma$ where $\varphi(z_1, \ldots, z_{n+m})$ is the formula that says that $t_1[x_1, \ldots, x_n] = t_2[y_1, \ldots, y_n]$. Hence π is well-defined by (18.9) and is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ extending the order-isomorphism of I and J.

To prove the existence of a model with indiscernibles with properties (i), (ii), and (iii), we use the Compactness Theorem. Since Σ is an E.M. set, there exists $(\mathfrak{A}_0, I_0)$ such that $\Sigma = \Sigma(\mathfrak{A}_0, I_0)$. Let us extend the language $\{\in\}$ by adding α constant symbols c_ξ, $\xi < \alpha$. Let Δ be the following set of sentences:

$$
\begin{aligned}
(18.11) \qquad & c_\xi \text{ is an ordinal} && (\text{all } \xi < \alpha),\\
& c_\xi < c_\eta && (\text{all } \xi, \eta \text{ such that } \xi < \eta < \alpha),\\
& \varphi(c_{\xi_1}, \ldots, c_{\xi_n}) && (\text{all } \varphi \in \Sigma \text{ and all } \xi_1 < \ldots < \xi_n < \alpha).
\end{aligned}
$$

We shall show that every finite subset of Δ has a model. Let $D \subset \Delta$ be finite. There exist $\xi_1 < \ldots < \xi_k$ such that $c_{\xi_1}, \ldots, c_{\xi_k}$ are the only constants mentioned in D. Let $\sigma(c_{\xi_1}, \ldots, c_{\xi_k})$ be the sentence that is the conjunction of all sentences in D.

Since I_0 is infinite, there are $i_1, \ldots, i_k \in I_0$ such that $i_1 < \ldots < i_k$. Let us take the model $\mathfrak{A}_0$ and expand it by interpreting the constant symbols $c_{\xi_1}, \ldots, c_{\xi_k}$ as $i_1, \ldots, i_k$. Since $\Sigma = \Sigma(\mathfrak{A}_0, I_0)$ and $D \subset \Delta$, it is clear that $\mathfrak{A}_0 \vDash \sigma[i_1, \ldots, i_k]$ and hence the expansion $(\mathfrak{A}_0, i_1, \ldots, i_k)$ is a model of σ, hence of D.

By the Compactness Theorem, the set Δ has a model $\mathfrak{M} = (M, E, c_\xi^{\mathfrak{M}})_{\xi<\alpha}$. Let $I = \{c_\xi^{\mathfrak{M}} : \xi < \alpha\}$. I is a set of ordinals of $\mathfrak{M}$ and has order-type α. It is clear that if $\varphi(v_1, \ldots, v_n)$ is an $\in$-formula and $\xi_1 < \ldots < \xi_n$, then $(M, E) \vDash \varphi[c_{\xi_1}^{\mathfrak{M}}, \ldots, c_{\xi_n}^{\mathfrak{M}}]$ if and only if $\varphi \in \Sigma$. Thus I is a set of indiscernibles for (M, E). Now we let A be the Skolem hull of I in (M, E). Since $\mathfrak{A} = (A, E)$ is an elementary submodel of (M, E), it follows that I is a set of indiscernibles for $\mathfrak{A}$, $\Sigma(\mathfrak{A}, I) = \Sigma$, and that $H^{\mathfrak{A}}(I) = H^{(M,E)}(I) = A$. Hence $(\mathfrak{A}, I)$ satisfies (i), (ii), and (iii). □

For each E.M. set Σ and each ordinal α, let us call the (Σ, α)-*model* the unique pair $(\mathfrak{A}, I)$ given by Lemma 18.7. The uniqueness proof of Lemma 18.7 easily extends to give the following:

Lemma 18.8. *Let Σ be an E.M. set, let $\alpha \leq \beta$, and let $j : \alpha \to \beta$ be order-preserving. Then j can be extended to an elementary embedding of the (Σ, α)-model into the (Σ, β)-model.*

Proof. Extend j as in (18.9). □

We shall eventually show that the existence of Ramsey cardinal implies the existence of an E.M. set Σ having a certain syntactical property (*remarkability*) and such that every (Σ, α)-model is well-founded. Let us investigate well-foundedness first.

Lemma 18.9. *The following are equivalent, for an E.M. set* Σ*:*

(i) *For every ordinal* α*, the* (Σ, α)*-model is well-founded.*
(ii) *For some ordinal* $\alpha \geq \omega_1$*, the* (Σ, α)*-model is well-founded.*
(iii) *For every ordinal* $\alpha < \omega_1$*, the* (Σ, α)*-model is well-founded.*

Proof. (i) $\rightarrow$ (ii) is trivial.

(ii) $\rightarrow$ (iii): If $(\mathfrak{A}, I)$ is the (Σ, α)-model and if $\beta \leq \alpha$, let J be the initial segment of the first β elements of I; let $\mathfrak{B} = H^{\mathfrak{A}}(J)$. Clearly, $(\mathfrak{B}, J)$ is the (Σ, β)-model. Since a submodel of a well-founded model is well-founded, it follows that if $\beta \leq \alpha$ and the (Σ, α)-model is well-founded, then the (Σ, β)-model is also well-founded, and thus (ii) implies (iii).

(iii) $\rightarrow$ (i): Let us assume that there is a limit ordinal α such that the (Σ, α)-model is not well-founded; let $(\mathfrak{A}, I)$ be the model. There is an infinite sequence $a_0, a_1, a_2, \ldots$ in $\mathfrak{A}$ such that $a_1 \; E \; a_2$, $a_2 \; E \; a_1$, etc. Each a_n is definable from I; that is, for each n there is a Skolem term t_n such that $a_n = t_n^{\mathfrak{A}}[x_1, \ldots, x_{k_n}]$ for some $x_1, \ldots, x_{k_n} \in I$. Therefore there is a countable subset I_0 of I such that $a_n \in H^{\mathfrak{A}}(I_0)$ for all $n \in \omega$. The order-type of I_0 is a countable ordinal β and $(H^{\mathfrak{A}}(I_0), I_0)$ is the (Σ, β)-model. This model is clearly non-well-founded since it contains all the a_n. Hence for some countable β, the (Σ, β)-model is not well-founded. □

We shall now define remarkability. We consider only (Σ, α)-models where α is an infinite limit ordinal.

Let us say that a (Σ, α)-model $(\mathfrak{A}, I)$ is *unbounded* if the set I is unbounded in the ordinals of $\mathfrak{A}$, that is, if for every $x \in Ord^{\mathfrak{A}}$ there is $y \in I$ such that $x < y$.

Lemma 18.10. *The following are equivalent, for any E.M. set* Σ*:*

(i) *For all* α*,* (Σ, α) *is unbounded.*
(ii) *For some* α*,* (Σ, α) *is unbounded.*
(iii) *For every Skolem term* $t(v_1, \ldots, v_n)$ *the set* Σ *contains the formula*

(18.12) *if* $t(v_1, \ldots, v_n)$ *is an ordinal, then* $t(v_1, \ldots, v_n) < v_{n+1}$.

Proof. (i) $\rightarrow$ (ii) is trivial.

(ii) $\rightarrow$ (iii): Let $(\mathfrak{A}, I)$ be a (Σ, α)-model, where α is a limit ordinal, and assume that I is unbounded in $Ord^{\mathfrak{A}}$. To prove (iii), it suffices to show that for any term t, (18.12) is true in $\mathfrak{A}$ for some increasing sequence $x_1 < \ldots < x_{n+1}$ in I. Let t be a Skolem term. Let us choose $x_1 < \ldots < x_n \in I$ and let $y = t^{\mathfrak{A}}[x_1, \ldots, x_n]$. If $y \notin Ord^{\mathfrak{A}}$, then (18.12) is vacuously true; if $y \in Ord^{\mathfrak{A}}$, then there exists $x_{n+1} \in I$ such that $y < x_{n+1}$, and we have $\mathfrak{A} \vDash t[x_1, \ldots, x_n] < x_{n+1}$.

(iii) $\rightarrow$ (i): Let $(\mathfrak{A}, I)$ be a (Σ, α)-model, where α is a limit ordinal, and assume (iii). To prove that I is unbounded in $Ord^{\mathfrak{A}}$, let $y \in Ord^{\mathfrak{A}}$. There exist a Skolem term t and $x_1 < \ldots < x_n \in I$ such that $y = t^{\mathfrak{A}}[x_1, \ldots, x_n]$. Now if x_{n+1} is any element of I greater than x_n, (iii) implies that $y < x_{n+1}$. □

Thus we say that an E.M. set Σ is unbounded if it contains the formulas (18.12) for all Skolem terms t.

Let α be a limit ordinal, $\alpha > \omega$, and let $(\mathfrak{A}, I)$ be the (Σ, α)-model. For each $\xi < \alpha$, let i_ξ denote the ξth element of I. We say that $(\mathfrak{A}, I)$ is *remarkable* if it is unbounded and if every ordinal x of $\mathfrak{A}$ less than i_ω is in $H^{\mathfrak{A}}(\{i_n : n \in \omega\})$.

Lemma 18.11. *The following are equivalent for any unbounded E.M. set* Σ:

(i) *For all* $\alpha > \omega$, *the* (Σ, α)*-model is remarkable.*
(ii) *For some* $\alpha > \omega$, *the* (Σ, α)*-model is remarkable.*
(iii) *For every Skolem term* $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$, *the set* Σ *contains the formula*

(18.13) *if* $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ *is an ordinal smaller than* y_1, *then* $t(x_1, \ldots, x_m, y_1, \ldots, y_n) = t(x_1, \ldots, x_m, z_1, \ldots, z_n)$.

Moreover, if $(\mathfrak{A}, I)$ *is a remarkable* (Σ, α)*-model and* $\gamma < \alpha$ *is a limit ordinal, then every ordinal* x *of* $\mathfrak{A}$ *less than* i_γ *is in* $H^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$.

Proof. (i) $\rightarrow$ (ii) is trivial.

(ii) $\rightarrow$ (iii): Let $\alpha > \omega$ be a limit ordinal and let $(\mathfrak{A}, I)$ be a remarkable (Σ, α)-model. To prove (iii), it suffices to show that for any t, (18.13) is true in $\mathfrak{A}$ for some increasing sequence $x_1 < \ldots < x_m < y_1 < \ldots < y_n < z_1 < \ldots < z_n$ in I. Let t be a Skolem term. We let $x_1 < \ldots < x_m < y_1 < \ldots < y_n < z_1 < \ldots < z_n \in I$ be such that $x_1, \ldots, x_m$ are the first m members of I and that y_1 is the ωth member of I, $y_1 = i_\omega$. Now if $a = t^{\mathfrak{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is an ordinal of $\mathfrak{A}$ and less than y_1, we have, by remarkability of $(\mathfrak{A}, I)$, $a \in H^{\mathfrak{A}}(\{i_n : n < \omega\})$. Hence there is $k < \omega$, $k \geq m$, and a term s such that

$$\mathfrak{A} \vDash t[x_1, \ldots, x_m, y_1, \ldots, y_n] = s[i_0, \ldots, i_k]. \tag{18.14}$$

In other words (18.14) says that $\mathfrak{A}$ satisfies a certain formula $\varphi[i_0, \ldots, i_k, y_1, \ldots, y_n]$. By indiscernibility, $\mathfrak{A}$ also satisfies $\varphi[i_0, \ldots, i_k, z_1, \ldots, z_n]$, i.e.,

$$\mathfrak{A} \vDash t[x_1, \ldots, x_m, z_1, \ldots, z_n] = s[i_0, \ldots, i_k].$$

Therefore $t^{\mathfrak{A}}[x_1, \ldots, x_m, y_1, \ldots, y_n] = t^{\mathfrak{A}}[x_1, \ldots, x_m, z_1, \ldots, z_n]$.

(iii) $\rightarrow$ (i) and "moreover:" Let $(\mathfrak{A}, I)$ be a (Σ, α)-model, where $\alpha > \omega$ is a limit ordinal, and assume (iii). Let $\gamma \geq \omega$ be a limit ordinal and let $x \in Ord^{\mathfrak{A}}$ be less than i_γ, the γth element of I. We shall show that $x \in H^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$. Since $\mathfrak{A} = H^{\mathfrak{A}}(I)$, there is a Skolem term t and $x_1 < \ldots < x_m < y_1 \ldots < y_n \in I$ such that $y_1 = i_\gamma$ and $x = t^{\mathfrak{A}}[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Let us choose $w_1, \ldots, w_n$ and $z_1, \ldots, z_n$ in I such that

$$x_1 < \ldots < x_m < w_1 < \ldots < w_n < y_1 < \ldots < y_n < z_1 \ldots < z_n.$$

Now since $x < y_1$, it follows from (18.13) that

$$\mathfrak{A} \vDash t[x_1, \ldots, x_m, y_1, \ldots, y_n] = t[x_1, \ldots, x_m, z_1, \ldots, z_n].$$

However, by indiscernibility, this implies that

$$\mathfrak{A} \vDash t[x_1, \ldots, x_m, w_1, \ldots, w_n] = t[x_1, \ldots, x_m, z_1, \ldots, z_n],$$

and hence $x = t^{\mathfrak{A}}[x_1, \ldots, x_m, w_1, \ldots, w_n]$. Therefore $x \in H^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$. □

Thus we say that an E.M. set Σ is *remarkable* if it is unbounded and contains the formulas (18.13) for all Skolem terms t.

An important consequence of remarkability is the following: Let $(\mathfrak{A}, I)$ be a remarkable (Σ, α)-model and let $\gamma < \alpha$ be a limit ordinal. Let $J = \{i_\xi : \xi < \gamma\}$ and let $\mathfrak{B} = H^{\mathfrak{A}}(J)$. Then $(\mathfrak{B}, J)$ is the (Σ, γ)-model, and the ordinals of $\mathfrak{B}$ form an initial segment of the ordinals of $\mathfrak{A}$.

Another consequence of remarkability is that the indiscernibles form a closed unbounded subset of ordinals. Let $(\mathfrak{A}, I)$ be the (Σ, α)-model. We say that the set I is *closed* in $Ord^{\mathfrak{A}}$ if for every limit $\gamma < \alpha$, i_γ is the least upper bound (in the linearly ordered set $Ord^{\mathfrak{A}}$) of the set $\{i_\xi : \xi < \gamma\}$.

Lemma 18.12. *If $(\mathfrak{A}, I)$ is remarkable, then I is closed in $Ord^{\mathfrak{A}}$.*

Proof. Let $\gamma < \alpha$ be a limit ordinal. If x is an ordinal of $\mathfrak{A}$ less than i_γ, then by remarkability, x is in the (Σ, γ)-model $\mathfrak{B} = H^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$. However, since Σ is unbounded, $\mathfrak{B}$ is an unbounded (Σ, γ)-model and hence $x < i_\xi$ for some $\xi < \gamma$. Hence i_γ is the least upper bound of $\{i_\xi : \xi < \gamma\}$. □

Proof of Silver's Theorem and $0^\sharp$

Let us call an E.M. set Σ *well-founded* if every (Σ, α)-model is well-founded, and let us consider the statement:

(18.15) There exists a well-founded remarkable E.M. set.

We shall prove Theorem 18.1 in two steps: First we shall show that both (i) and (ii) are consequences of the assumption that there exists a well-founded remarkable E.M. set, and then we shall show that if there exists a Ramsey cardinal, then (18.15) holds. (Note that by Lemma 18.9 it suffices to find a well-founded remarkable model with uncountably many indiscernibles.)

Thus let us assume that there exists a well-founded remarkable E.M. set and let Σ be such a set.

For every limit ordinal α, the (Σ, α)-model is a well-founded model elementarily equivalent to some L_γ, and so by (13.13) is (isomorphic to) some L_β.

Lemma 18.13. *If κ is an uncountable cardinal, then the universe of the (Σ, κ)-model is L_κ.*

Proof. The (Σ, κ)-model is (L_β, I) for some β; since $|I| = \kappa$, we clearly have $\beta \geq \kappa$. To prove that $\beta = \kappa$, assume that $\beta > \kappa$. Since I is unbounded in β and has order-type κ, there is a limit ordinal $\gamma < \kappa$ such that $\kappa < i_\gamma$. By remarkability, all ordinals less than i_γ are in the (Σ, γ)-model $\mathfrak{A} = H(\{i_\xi : \xi < \gamma\})$. This is a contradiction since on the one hand we have $\kappa \subset \mathfrak{A}$, and on the other hand $|\mathfrak{A}| = |\gamma| < \kappa$. □

For each uncountable cardinal κ, let I_κ be the unique subset of κ such that (L_κ, I_κ) is the (Σ, κ)-model. By Lemma 18.12, I_κ is closed and unbounded in κ.

Lemma 18.14. *If $\kappa < \lambda$ are uncountable cardinals, then $I_\lambda \cap \kappa = I_\kappa$, and $H^{L_\lambda}(I_\kappa) = L_\kappa$.*

Proof. Let J be the set consisting of the first κ members of I_λ and let $\mathfrak{A} = H^{L_\lambda}(J)$. Then $(\mathfrak{A}, J)$ is a (Σ, κ)-model and the ordinals of $\mathfrak{A}$ are an initial segment of λ, say $Ord^{\mathfrak{A}} = \beta$. Since $(\mathfrak{A}, J)$ is isomorphic to (L_κ, I_κ), it is clear that $\beta = \kappa$ and $J = I_\kappa$. Hence $I_\lambda \cap \kappa = I_\kappa$.

Now since $\mathfrak{A} \prec L_\lambda$, $\mathfrak{A}$ is closed under the definable function $F(\alpha) =$ the αth set in the well-ordering $<_L$, and since $Ord^{\mathfrak{A}} = \kappa$, we have $A = \{F(\alpha) : \alpha < \kappa\} = L_\kappa$. □

Using this lemma, we can now prove both (i) and (ii) of Theorem 18.1 except for the uniqueness of Silver indiscernibles. We let

$$I = \bigcup\{I_\kappa : \kappa \text{ is an uncountable cardinal}\}. \tag{18.16}$$

For each uncountable cardinal κ, $I \cap \kappa = I_\kappa$ is a closed unbounded set of order-type κ, and is a set of indiscernibles for $(L_\kappa, \in)$; moreover, by Lemma 18.7(iii), every $a \in L_\kappa$ is definable in L_κ from I_κ. Let $\kappa < \lambda$ be uncountable cardinals. Since I_λ is closed in L_λ and $I_\lambda \cap \kappa = I_\kappa$, it follows that $\kappa \in I_\lambda$; hence I contains all uncountable cardinals. Also, since $L_\kappa = H^{L_\lambda}(I_\kappa)$, we have $L_\kappa \prec L_\lambda$.

The next two lemmas prove the uniqueness of Silver indiscernibles and of the corresponding E.M. set.

Lemma 18.15. *There is at most one well-founded remarkable E.M. set.*

Proof. Assuming that there is one such Σ, we define the class I in (18.16). Now since $L_{\aleph_\omega}$ is the $(\Sigma, \aleph_\omega)$-model and $\aleph_n \in I$ for each $n \geq 1$, we have

$$\varphi(v_1, \ldots, v_n) \in \Sigma \quad \text{if and only if} \quad L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n] \tag{18.17}$$

which proves that Σ is unique. □

We therefore define $0^\sharp$ (*zero-sharp*):

(18.18) $0^\sharp$ is the unique well-founded remarkable E.M. set if it exists.

The uniqueness of Silver indiscernibles now follows from:

Lemma 18.16. *For every regular uncountable cardinal κ there is at most one closed unbounded set of indiscernibles X for L_κ such that $L_\kappa = H^{L_\kappa}(X)$.*

Proof. Let $\Sigma = \Sigma(L_\kappa, X)$. Since X is closed unbounded, it follows that $X \cap I$ is infinite, and $\Sigma(L_\kappa, X) = \Sigma(L_\kappa, X \cap I) = \Sigma(L_\kappa, I \cap \kappa)$. Hence $\Sigma = 0^\sharp$ and since (L_κ, X) is the (Σ, κ)-model, we have $X = I \cap \kappa$. □

Thus we have proved (i) and (ii) of Theorem 18.1 under the assumption that $0^\sharp$ exists. On the other hand, if (ii) holds, then $0^\sharp$ exists because, e.g., $(L_{\omega_1}, I \cap \omega_1)$ is a remarkable well-founded model with $\aleph_1$ indiscernibles. To complete the proof of Theorem 18.1, it remains to show that if there is a Ramsey cardinal, then $0^\sharp$ exists. That will follow from:

Lemma 18.17. *Let κ be an uncountable cardinal. If there exists a limit ordinal λ such that $(L_\lambda, \in)$ has a set of indiscernibles of order-type κ, then there exist a limit ordinal γ and a set $I \subset \gamma$ of order-type κ such that (L_γ, I) is remarkable.*

It follows that if κ is Ramsey, then by Corollary 17.26 $(L_\kappa, \in)$ has a set of indiscernibles of order-type κ. By Lemma 18.17, there exists a remarkable model (L_γ, I) where I has order-type κ. By Lemma 18.9, $\Sigma(L_\gamma, I)$ is well-founded and remarkable and hence $0^\sharp$ exists.

Proof. Let λ be the least limit ordinal such that $(L_\lambda, \in)$ has a set of indiscernibles $I \subset \lambda$ of order-type κ. We shall show first that there is a set of indiscernibles $I \subset \lambda$ for L_λ, of order-type κ, such that $H^{L_\lambda}(I) = L_\lambda$. Let J be any set of indiscernibles for L_λ, of order-type κ, and let $\mathfrak{A} = H^{L_\lambda}(J)$. Then $\mathfrak{A} \prec L_\lambda$ and hence $\mathfrak{A}$ is isomorphic to some L_β, $\beta \leq \lambda$, by the collapsing map π. Now $I = \pi(J)$ is a set of indiscernibles for L_β, and $H^{L_\beta}(I) = L_\beta$. By the minimality of λ, we have $\beta = \lambda$ and hence I is as claimed.

Next we show that any such set I is unbounded in λ. If not, there is a limit ordinal $\alpha < \lambda$ such that $I \subset \alpha$. There is a Skolem term t and $\gamma_1 < \ldots < \gamma_n \in I$ such that $\alpha = t^{L_\lambda}[\gamma_1, \ldots, \gamma_n]$. We claim that the set $J = \{i \in I : i > \gamma_n\}$ is a set of indiscernibles for $(L_\alpha, \in)$. If $\varphi(v_1, \ldots, v_n)$ is a formula, then for any $i_1 < \ldots < i_k \in J$, L_α satisfies $\varphi[i_1, \ldots, i_k]$ if and only if L_λ satisfies the formula

$$L_\alpha \vDash \varphi[i_1, \ldots, i_k]. \tag{18.19}$$

The formula (18.19) is a formula about α, i_1, …, i_k, and since $\alpha = t^{L_\lambda}[\gamma_1, \ldots, \gamma_n]$ there is a formula $\psi(u_1, \ldots, u_n, v_1, \ldots, v_k)$ such that L_λ satisfies (18.19) if and only if

$$L_\lambda \vDash \psi[\gamma_1, \ldots, \gamma_n, i_1, \ldots, i_k]. \tag{18.20}$$

By the indiscernibility of I, the truth of (18.20) is independent of the choice of $i_1 < \ldots < i_k$ in I provided $\gamma_n < i_1$. Hence the truth of (18.19) is independent of the choice of $i_1 < \ldots < i_k$ in J. Hence J is a set of indiscernibles for L_α, and this contradicts the minimality of λ since $\alpha < \lambda$ and the order-type of J is κ.

Finally, let I be a set of indiscernibles for L_λ of order-type κ such that $H^{L_\lambda}(I) = L_\lambda$, and that i_ω, the ωth element of I, is least possible. We will show that (L_λ, I) is remarkable.

Let us assume that (L_λ, I) is not remarkable. Then there is a Skolem term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ such that the following holds in L_λ for any $x_1 < \ldots < x_m < y_1 < \ldots < y_n < z_1 < \ldots < z_n$:

$$t(x_1, \ldots, x_m, y_1, \ldots, y_n) < y_1 \tag{18.21}$$

and

$$t(x_1, \ldots, x_m, y_1, \ldots, y_n) \neq t(x_1, \ldots, x_m, z_1, \ldots, z_n). \tag{18.22}$$

Let $x_1, \ldots, x_m$ be the first m elements of I. We now consider the following increasing n-termed sequences in I: Let u_0 be the sequence of first n indiscernibles after x_m, let u_1 be the first n indiscernibles after u_0, etc.; for each $\alpha < \kappa$, let

$$\gamma_\alpha = t(x_1, \ldots, x_m, u_\alpha).$$

By indiscernibility, applied to the formula (18.22), we have $\gamma_\alpha \neq \gamma_\beta$ whenever $\alpha \neq \beta$. In fact, in (18.22) we have either $<$ or $>$ (in place of $\neq$); but $>$ is impossible since that would mean that $\gamma_\alpha > \gamma_\beta$ whenever $\alpha < \beta$. Thus $\langle \gamma_\alpha : \alpha < \kappa \rangle$ is an increasing sequence of ordinals.

We claim that $J = \{\gamma_\alpha : \alpha < \kappa\}$ is a set of indiscernibles for L_λ. This is so because for any formula φ, the truth value of $\varphi(\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_k})$ in L_λ does not depend on the choice of $\gamma_{\alpha_1} < \ldots < \gamma_{\alpha_k}$ in J because by the definition of the u_α, the truth value of $\varphi(t(x_1 \ldots, x_m, u_{\alpha_1}), \ldots, t(x_1 \ldots, x_m, u_{\alpha_k}))$ does not depend on the choice of $\alpha_1 < \ldots < \alpha_k$.

Hence $\{\gamma_\alpha : \alpha < \kappa\}$ is a set of indiscernibles for L_λ. Since i_ω is the first member of u_ω, it follows by (18.21) that $\gamma_\omega < i_\omega$. Now if $A = H(J)$ and π is the transitive collapse of A, then, as we proved in the first paragraph, $\pi(A) = L_\lambda$, and $K = \pi(J)$ is a set of indiscernibles for L_λ of order-type κ such that $H^{L_\lambda}(K) = L_\lambda$. However, $\pi(\gamma_\omega) \leq \gamma_\omega < i_\omega$, and so the ωth member of K is smaller than i_ω, contrary to our assumption. Hence (L_λ, I) is remarkable. □

This completes the proof of Theorem 18.1. Lemma 18.17 also gives the following equivalence:

Corollary 18.18. $0^\sharp$ *exists if and only if for some limit ordinal* λ, *the model* $(L_\lambda, \in)$ *has an uncountable set of indiscernibles.* □

The set $0^\sharp$ is, strictly speaking, a set of formulas. But as formulas can be coded by natural numbers, we can regard $0^\sharp$ as a subset of ω. This convention has become standard. Moreover, in Chapter 25 we show that $0^\sharp$ is a Π^1_2 singleton, and so $\{0^\sharp\}$ is a Δ^1_3 set. At this point we outline the proof of absoluteness of $0^\sharp$:

Lemma 18.19. *The property "Σ is a well-founded remarkable E.M. set" is absolute for every inner model of* ZF. *Hence $M \vDash 0^\sharp$ exists if and only if $0^\sharp \in M$ in which case $(0^\sharp)^M = 0^\sharp$.*

Proof. We first replace the property "Σ is an E.M. set" by a syntactical condition.

Let $\hat{\mathcal{L}}$ be the language $\{\in, c_1, c_2, \ldots, c_n, \ldots\}$ where c_n, $n < \omega$, are constant symbols. For every $\in$-formula $\varphi(v_1, \ldots, v_n)$ let $\hat{\varphi}$ be the sentence $\varphi(c_1, \ldots, c_n)$ of $\hat{\mathcal{L}}$. For each set of formulas Σ, let $\hat{\Sigma}$ be the set containing (i) all $\hat{\varphi}$ for $\varphi \in \Sigma$, (ii) the sentence "c_1 is an ordinal and $c_1 < c_2$," and (iii) the sentence "$\varphi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_n})$" for every $\varphi \in \Sigma$ and any $i_1 < \ldots < i_n$, $j_1 < \ldots < j_n$, (iv) all axioms of $\mathrm{ZFC} + V = L$. Let us consider the condition

(18.23) $\hat{\Sigma}$ is consistent.

Clearly, if Σ is an E.M. set, then $\hat{\Sigma}$ is consistent, for we simply interpret the constants c_n, $n < \omega$, as some Silver indiscernibles. Conversely, if $\hat{\Sigma}$ is consistent, then $\hat{\Sigma}$ has a model and that model provides us with a (Σ, ω)-model (with indiscernibles c_n, $n < \omega$) and the proof of Lemma 18.7 goes through. Therefore (18.23) holds if and only if Σ is an E.M. set.

As remarkability can also be expressed as a syntactical property, it follows that "Σ is a remarkable E.M. set" can be written as a Δ_0 property (with parameters V_ω and *Form*). As such it is absolute for transitive models.

If Σ is a remarkable E.M. set, then for every limit ordinal α there is a unique (up to isomorphism) (Σ, α)-model and we can find one $((A, E), I)$ such that $I = \alpha$ and that $<^{\mathfrak{A}}$ (i.e., E) agrees with $<$ on α. If $((A, E), \alpha)$ is such, we say that "$((A, E), \alpha)$ is a (Σ, α)-model." This last property is a Δ_1 property of Σ, (A, E), α, V_ω and *Form*. Then $\Sigma = 0^\sharp$ if and only if

(18.24) $\forall \alpha\ \forall (A, E)$ (if $((A, E), \alpha)$ is a (Σ, α)-model, then (A, E) is well-founded).

As well-foundedness is absolute for transitive models of ZF, it follows that (18.24) is absolute for inner models of ZF (which contain all ordinals), and therefore "$\Sigma = 0^\sharp$" is absolute. □

Elementary Embeddings of L

In Chapter 17 we proved that a well-founded ultrapower of the universe induces an elementary embedding $j_U : V \to \mathrm{Ult}$, and conversely, if $j : V \to M$

is a nontrivial elementary embedding, then (17.2) defines a normal measure on the least ordinal moved by j.

Let j be a nontrivial elementary embedding of the universe, and let M be a transitive model of ZFC, containing all ordinals. Let $N = j(M) = \bigcup_{\alpha \in Ord} j(M \cap V_\alpha)$. Then N is a transitive model of ZF and $j : M \to N$ is elementary:

$$M \vDash \varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad N \vDash \varphi(j(a_1), \ldots, j(a_n)). \tag{18.25}$$

((18.25) is proved by induction on the complexity of φ). In particular, if $M = L$, then $j(V) \vDash$ (N is the constructible universe), and so $N = L$, and $j \restriction L$ is an elementary embedding of L in L. Note that by Scott's Theorem, the function $j \restriction L$ is not a class in L; thus if there exists an elementary embedding of L (into L), then $V \neq L$.

If $0^\sharp$ exists, then there are nontrivial elementary embeddings of L. In fact, let j be any order-preserving function from the class I of all Silver indiscernibles into itself. Then j can be extended to an elementary embedding of L; we simply let

$$j(t^L[\gamma_1, \ldots, \gamma_n]) = t^L[j(\gamma_1), \ldots, j(\gamma_n)] \tag{18.26}$$

for every Skolem term t and any Silver indiscernibles $\gamma_1 < \ldots < \gamma_n$. We shall prove that the converse is true, that if there is a nontrivial elementary embedding of L, then $0^\sharp$ exists:

Theorem 18.20 (Kunen). *The following are equivalent*:

(i) $0^\sharp$ *exists.*
(ii) *There is a nontrivial elementary embedding* $j : L \to L$.

Toward the proof of Kunen's Theorem, let us investigate elementary embeddings $j : M \to N$ where M is a transitive model of ZFC.

Definition 18.21. Let M be a transitive model of ZFC, and let κ be a cardinal in M. An *M-ultrafilter* on κ is a collection $D \subset P^M(\kappa)$ that is an ultrafilter on the algebra of sets $P^M(\kappa)$. Explicitly,

(18.27)
(i) $\kappa \in D$ and $\emptyset \notin D$;
(ii) if $X \in D$ and $Y \in D$, then $X \cap Y \in D$;
(iii) if $X \in D$ and $Y \in M$ is such that $X \subset Y$, then $Y \in D$;
(iv) for every $X \subset \kappa$ such that $X \in M$, either X or $\kappa - X$ is in D.

D is *κ-complete* if whenever $\alpha < \kappa$ and $\{X_\xi : \xi < \alpha\} \in M$ is such that $X_\xi \in D$ for all $\xi < \alpha$, then $\bigcap_{\xi < \alpha} X_\alpha \in D$; D is *normal* if whenever $f \in M$ is a regressive function on $X \in D$, then f is constant on some $Y \in D$.

If $j : M \to N$ is an elementary embedding, then the least ordinal moved by j is called the *critical point* of j.

Lemma 18.22. *If $j : M \to N$ is an elementary embedding and κ is the critical point of j then κ is a regular uncountable cardinal in M, and $D = \{X \in P^M(\kappa) : \kappa \in j(X)\}$ is a nonprincipal normal κ-complete M-ultrafilter on κ.*

Proof. Exactly as the proof of Lemma 17.2. Note that κ-completeness of D implies that κ is regular in M. □

If D is an M-ultrafilter on κ, one can construct the ultrapower of M by D as follows: Consider, in M, the class of all functions f with domain κ. Using D, define an equivalence relation $=^*$ and the relation $\in^*$ as usual;

$$f =^* g \leftrightarrow \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D,$$
$$f \in^* g \leftrightarrow \{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in D.$$

Then define equivalence classes mod $=^*$, and the model $\mathrm{Ult} = \mathrm{Ult}_D(M)$. An analog of Theorem 12.3 is easily verified:

$$\mathrm{Ult} \vDash \varphi([f_1], \ldots, [f_n]) \text{ if and only if } \{\alpha < \gamma : M \vDash \varphi(f_1(\alpha), \ldots, f_n(\alpha))\} \in D.$$

If for each $a \in M$, c_a denotes the constant function with value a, then

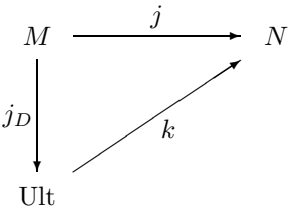
$$j_D(a) = [c_a]$$

defines an elementary embedding of M in Ult.

The ultrapower of M by an M-ultrafilter D is not necessarily well-founded, even if D is countably complete.

If $j : M \to N$ is an elementary embedding with M and N being transitive models, and if D is the M-ultrafilter $\{X : \kappa \in j(X)\}$, then, as in Lemma 17.4, we have the commutative diagram

(18.28)



and it follows that $\mathrm{Ult}_D(M)$ is well-founded. (If $[f_0] \ni^* [f_1] \ni^* \ldots$ were a descending sequence in Ult, then $k([f_0]) \ni k([f_1]) \ni \ldots$ would be a descending sequence in N.)

We proceed with the proof of Kunen's Theorem.

Let $j : L \to L$ be an elementary embedding. We shall first replace j by a more manageable embedding. We let D be the L-ultrafilter $\{X \in P^L(\gamma) : \gamma \in j(X)\}$ where γ is the critical point of j. The ultrapower $\mathrm{Ult}_D(L)$ is well-founded and so we identify Ult with its transitive collapse L; let j_D be the canonical embedding, $j_D : L \to L$. The critical point of j_D is γ because D is γ-complete.

Lemma 18.23. *If κ is a limit cardinal such that $\operatorname{cf}\kappa > \gamma$, then $j_D(\kappa) = \kappa$.*

Proof. Every constructible function $f : \gamma \to \kappa$ is bounded by some $\alpha < \kappa$ and hence $[f] < [c_\alpha]$ (where c_α is the constant function with value α). Thus $j_D(\kappa) = \lim_{\alpha\to\kappa} j_D(\alpha)$. Now if $\alpha < \kappa$, then $|j_D(\alpha)| \le |(\alpha^\gamma)^L|$, hence $j_D(\alpha) < \kappa$. It follows that $j_D(\kappa) = \kappa$. □

Let us drop the subscript D and simply assume that $j : L \to L$ is an elementary embedding, that γ is its critical point and that $j(\kappa) = \kappa$ for every limit cardinal κ such that $\operatorname{cf}\kappa > \gamma$.

Let U_0 be the class of all limit cardinals κ with $\operatorname{cf}\kappa > \gamma$; by transfinite induction we define a sequence of classes $U_0 \supset U_1 \supset \ldots \supset U_\alpha \supset \ldots$ as follows:

$$U_{\alpha+1} = \{\kappa \in U_\alpha : |U_\alpha \cap \kappa| = \kappa\}, \tag{18.29}$$
$$U_\lambda = \bigcap_{\alpha<\lambda} U_\alpha \qquad (\lambda \text{ limit}).$$

(That is, $U_{\alpha+1}$ consists of fixed points of the increasing enumeration of U_α.) Each U_α is nonempty, and in fact a proper class. To see this, verify, by induction on α, that each U_α is a proper class and is δ-*closed*, for each δ with $\operatorname{cf}\delta > \gamma$; that is, whenever $\langle\kappa_\xi : \xi < \delta\rangle$ is an increasing sequence in U_α, then $\lim_{\xi\to\delta}\kappa_\xi \in U_\alpha$. Hence each U_α is nonempty, and we choose a cardinal $\kappa \in U_{\omega_1}$.

Thus κ is such that $\operatorname{cf}\kappa > \gamma$ and κ is the κth element of each U_α, $\alpha < \omega_1$. We shall find a set of $\aleph_1$ indiscernibles for $(L_\kappa, \in)$.

Since $j : L \to L$ is an elementary and $j(\kappa) = \kappa$, it is clear that the mapping $i = j{\restriction}L_\kappa$ is an elementary embedding of $(L_\kappa, \in)$ into $(L_\kappa, \in)$. We shall use i and the sets $U_\alpha \cap \kappa$, $\alpha < \omega_1$, to produce indiscernibles γ_α, $\alpha < \omega_1$, for L_κ. Let $X_\alpha = U_\alpha \cap \kappa$ for each $\alpha < \omega_1$, and recall that γ is the critical point of i.

For each $\alpha < \omega_1$, we let

$$M_\alpha = H^{L_\alpha}(\gamma \cup X_\alpha). \tag{18.30}$$

M_α is an elementary submodel of L_κ.

If π_α is the transitive collapse of M_α, then because $|X_\alpha| = \kappa$, we have $\pi_\alpha(M_\alpha) = L_\kappa$. Thus if we denote $i_\alpha = \pi_\alpha^{-1}$, then i_α is an elementary embedding of L_κ in L_κ. Let $\gamma_\alpha = i_\alpha(\gamma)$.

Lemma 18.24.

(i) *The ordinal γ_α is the least ordinal greater than γ in M_α.*
(ii) *If $\alpha < \beta$ and $x \in M_\beta$, then $i_\alpha(x) = x$. In particular, $i_\alpha(\gamma_\beta) = \gamma_\beta$.*
(iii) *If $\alpha < \beta$, then $\gamma_\alpha < \gamma_\beta$.*

Proof. (i) Since $\gamma \subset M_\alpha$, $i_\alpha(\gamma)$ is the least ordinal in M_α greater than or equal to γ; thus it suffices to show that $\gamma \notin M_\alpha$. If $x \in M_\alpha$, then $x = t[\eta_1, \ldots, \eta_n]$ where t is a Skolem term and the η's are either smaller than γ or elements of X_α. For all such η, $i(\eta) = \eta$ and hence $i(x) = i(t(\eta_1, \ldots, \eta_n)) = t(i(\eta_1), \ldots, i(\eta_n)) = x$. However, $i(\gamma) \neq \gamma$ and so $\gamma \notin M_\alpha$.

(ii) Each $x \in M_\beta$ is of the form $t[\eta_1, \ldots, \eta_n]$ where the η's are either $< \gamma$ or in X_β. If $\eta < \gamma$, then clearly $i_\alpha(\eta) = \eta$. If $\eta \in X_\beta$, then because $\alpha < \beta$, we have $|X_\alpha \cap \eta| = \eta$ and hence $\pi_\alpha(\eta) = \eta$; in other words, $i_\alpha(\eta) = \eta$. Therefore $i_\alpha(x) = x$.

(iii) If $\alpha < \beta$, then $M_\alpha \supset M_\beta$ and hence $\gamma_\alpha \leq \gamma_\beta$. To see that $\gamma_\alpha \neq \gamma_\beta$, note that because $\gamma_\alpha > \gamma$, we have $i_\alpha(\gamma_\alpha) > i_\alpha(\gamma) = \gamma_\alpha$, while $i_\alpha(\gamma_\beta) = \gamma_\beta$. □

Lemma 18.25. *If $\alpha < \beta$, then there is an elementary embedding $i_{\alpha,\beta} : L_\kappa \to L_\kappa$ such that for every ξ that is either smaller than α or greater than β we have $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$, and $i_{\alpha,\beta}(\gamma_\alpha) = \gamma_\beta$.*

Proof. Let $M_{\alpha,\beta} = H^{L_\kappa}(\gamma_\alpha \cup X_\beta)$, and let $i_{\alpha,\beta} = \pi^{-1}_{\alpha,\beta}$ where $\pi_{\alpha,\beta}$ is the transitive collapse of $M_{\alpha,\beta}$. The mapping $i_{\alpha,\beta}$ is an elementary embedding of L_κ in L_κ.

If $\eta < \gamma_\alpha$, then clearly $i_{\alpha,\beta}(\eta) = \eta$; in particular $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$ if $\xi < \alpha$. If $x \in M_{\beta+1}$, then $x = t(\eta_1, \ldots, \eta_n)$ where the η's are either smaller than γ or elements of $X_{\beta+1}$. If $\eta \in X_{\beta+1}$, then $|X_\beta \cap \eta| = \eta$ and therefore $i_{\alpha,\beta}(\eta) = \eta$. Hence $i_{\alpha,\beta}(x) = x$ for every $x \in M_{\beta+1}$, and in particular $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$ if $\xi > \beta$.

Now we shall show that $i_{\alpha,\beta}(\gamma_\alpha) = \gamma_\beta$. Since $M_{\alpha,\beta} \supset M_\beta$, we have $\gamma_\beta \in M_{\alpha,\beta}$; and since $\gamma_\alpha \subset M_{\alpha,\beta}$, $i_{\alpha,\beta}(\gamma_\alpha)$ is the least ordinal in $M_{\alpha,\beta}$ greater than or equal to γ_α; hence we have $\gamma_\alpha \leq i_{\alpha,\beta}(\gamma_\alpha) \leq \gamma_\beta$.

Thus it suffices to show that there is no ordinal $\delta \in M_{\alpha,\beta}$ such that $\gamma_\alpha \leq \delta < \gamma_\beta$. Otherwise there is some $\delta = t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k)$ such that the ξ's are $< \gamma_\alpha$ and the η's are in X_β (and t is a Skolem term) and that $\gamma_\alpha \leq \delta < \gamma_\beta$. Thus we have:

$$(L_\kappa, \in) \vDash \exists \xi < \gamma_\alpha \text{ such that } \gamma_\alpha \leq t(\xi, \eta) < \gamma_\beta. \tag{18.31}$$

The formula in (18.31) is a formula φ about γ_α, η, and γ_β. At this point, we apply the elementary embedding $i_\alpha : L_\kappa \to L_\kappa$ backward. That is, γ_α, the η's and γ_β are all in the range of i_α: $\gamma_\alpha = i_\alpha(\gamma)$, $\eta = i_\alpha(\eta)$, and $\gamma_\beta = i_\alpha(\gamma_\beta)$; and since $L_\kappa \vDash \varphi[i_\alpha(\gamma), i_\alpha(\eta), i_\alpha(\gamma_\beta)]$, we conclude that $L_\kappa \vDash \varphi[\gamma, \eta, \gamma_\beta]$, namely

$$(L_\kappa, \in) \vDash \exists \xi < \gamma \text{ such that } \gamma \leq t(\xi, \eta) < \gamma_\beta.$$

Thus pick some ξ's less than γ such that $\gamma \leq t(\xi, \eta) < \gamma_\beta$. Since $\xi \in \gamma$ and $\eta \in X_\beta$, we have $t(\xi, \eta) \in M_\beta$, which means that $t(\xi, \eta)$ is an ordinal in M_β between γ and γ_β, and that contradicts Lemma 18.24(i). □

The proof of Kunen's Theorem will be complete when we show:

Lemma 18.26. *The set $\{\gamma_\alpha : \alpha < \omega_1\}$ is a set of indiscernibles for $(L_\kappa, \in)$.*

Proof. Let φ be a formula and let $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ be two sequences of countable ordinals. We wish to show that

$$L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}] \quad \text{if and only if} \quad L_\kappa \vDash \varphi[\gamma_{\beta_1}, \ldots, \gamma_{\beta_n}]. \tag{18.32}$$

Let us pick $\delta_1 < \ldots < \delta_n$ such that $\alpha_n < \delta_1$ and $\beta_n < \delta_1$. First we apply the elementary embedding i_{α_n, δ_n} and get

$$L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_{n-1}}, \gamma_{\alpha_n}] \quad \text{if and only if} \quad L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_{n-1}}, \gamma_{\delta_n}]$$

because $i_{\alpha_n, \delta_n}(\gamma_{\alpha_n}) = \gamma_{\delta_n}$, and preserves the other γ's. The we apply $i_{\alpha_{n-1}, \delta_{n-1}}$ with a similar effect, and by a successive application of $i_{\alpha_{n-2}, \delta_{n-2}}$, $\ldots$, i_{α_1, δ_1} we get

$$L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}] \quad \text{if and only if} \quad L_\kappa \vDash \varphi[\gamma_{\delta_1}, \ldots, \gamma_{\delta_n}].$$

Then we do the same for the β's and δ's as we did for the α's and δ's, and (18.32) follows. □

This completes the proof of Theorem 18.20.

The following result is related to Kunen's Theorem:

Theorem 18.27. *Let $j : L_\alpha \to L_\beta$ be an elementary embedding and let γ be the critical point of j. If $\gamma < |\alpha|$, then $0^\sharp$ exists.*

Proof. Let γ be the critical point of j. Since $\gamma < |\alpha|$, every $X \subset \gamma$ is in L_α, and so $D = \{X \subset \gamma : \gamma \in j(X)\}$ is an L-ultrafilter.

Let us consider the ultrapower $\mathrm{Ult}_D(L)$. If the ultrapower is well-founded, then we are done because then the canonical embedding $j_D : L \to \mathrm{Ult}_D(L)$ is a nontrivial elementary embedding of L in L. Thus we shall prove that $\mathrm{Ult}_D(L)$ is well-founded.

Let us assume that f_0, f_1, $\ldots$, f_n, $\ldots$ is a counterexample to well-foundedness of the ultrapower. Each f_n is a constructible function on γ and $\{\xi < \gamma : f_{n+1}(\xi) \in f_n(\xi)\} \in D$ for all $n < \omega$. Let θ be a limit ordinal such that $f_n \in L_\theta$ for all n and let M be an elementary submodel of $(L_\theta, \in)$ such that $|M| = |\gamma|$, $\gamma \subset M$, and $f_n \in M$ for all n. Let π be the transitive collapse of M, $\pi(M) = L_\eta$, and let $g_n = \pi(f_n)$, for all n.

Since $\pi(\xi) = \xi$ for all $\xi < \gamma$, we see that for each $\xi < \gamma$ and all n, $g_{n+1}(\xi) \in g_n(\xi)$ if and only if $f_{n+1}(\xi) \in f_n(\xi)$, and hence g_0, g_1, $\ldots$, g_n, $\ldots$ is also a counterexample to well-foundedness of the ultrapower. However, since each g_n is in L and $|\eta| = |\gamma| < |\alpha|$, we have $g_n \in L_\alpha$ for all n. Thus $j(g_n)$ is defined for all n, and we have,

$$\{\xi < \gamma : g_{n+1}(\xi) \in g_n(\xi)\} \in D \quad \text{if and only if} \quad (j(g_{n+1}))(\gamma) \in (j(g_n))(\gamma).$$

Now we reached a contradiction because $(j(g_0))(\gamma) \ni (j(g_1))(\gamma) \ni \ldots$ would be a descending sequence. □

Corollary 18.28. *If there is a Jónsson cardinal, then* $0^\sharp$ *exists.*

Proof. Let κ be a Jónsson cardinal and let us consider the model $(L_\kappa, \in)$. Let A be an elementary submodel, of size κ, such that $A \neq L_\kappa$. Let π be the transitive collapse of A; clearly, $\pi(A) = L_\kappa$. Thus $j = \pi^{-1}$ is a nontrivial elementary embedding of L_κ in L_κ. Since κ is a cardinal, $0^\sharp$ exists by Theorem 18.27. □

Chang's Conjecture is the statement that every model of type $(\aleph_2, \aleph_1)$ has an elementary submodel of type $(\aleph_1, \aleph_0)$.

Corollary 18.29. *Chang's Conjecture implies that* $0^\sharp$ *exists.*

Proof. Consider the model $(L_{\omega_2}, \omega_1, \in)$, and let $\mathfrak{A} = (A, \omega_1 \cap A, \in)$ be its elementary submodel such that $|A| = \aleph_1$ and $|\omega_1 \cap A| = \aleph_0$. Let π be the transitive collapse of A; we have $\pi(A) = L_\alpha$ for some α such that $\omega_1 \leq \alpha < \omega_2$. Also, $\pi(\omega_1 \cap A)$ is a countable ordinal, and hence $\pi(\omega_1) < \omega_1$. Then $j = \pi^{-1}$ is an elementary embedding of L_α in L_{ω_2}, and its critical point is a countable ordinal. Hence $0^\sharp$ exists. □

All results about $0^\sharp$ and Silver indiscernibles for L proved in the present section can be relativized to obtain similar results for the models $L[x]$, where $x \subset \omega$.

In particular, if there exists a Ramsey cardinal there is for every $x \subset \omega$ a unique class I_x containing all uncountable cardinals such that for each uncountable cardinal κ, $I_x \cap \kappa$ is a set of indiscernibles for the model $(L_\kappa[x], \in, x)$ and all elements of $L_\kappa[x]$ are definable in the model from $I_x \cap \kappa$. Here x is considered a one-place predicate. Also, for every regular uncountable cardinal κ, $I_x \cap \kappa$ is closed unbounded in κ.

The proof of the relativization of Silver's Theorem uses models with indiscernibles $(\mathfrak{A}, I)$ where $\mathfrak{A}$ is elementarily equivalent to some $(L_\lambda[x], \in, x)$ where $\lambda > \omega$ is a limit ordinal. If κ is a Ramsey cardinal, then $(L_\kappa[x], \in, x)$ has a set of indiscernibles of size κ, and the theorem follows.

We define $x^\sharp$ as the unique set $\Sigma = \Sigma((L_\lambda[x], \in, x), I)$ that is well-founded and remarkable. If $x^\sharp$ exists, then we have

$$x^\sharp = \{\varphi : (L_{\aleph_\omega}[x], \in, x) \vDash \varphi[\aleph_1, \ldots, \aleph_n]\}.$$

Here φ is a formula of the language $\{\in, P\}$ where P is a one-place predicate (interpreted as x). Note that x is definable in the model $(L_\lambda[x], \in, x)$ (by the formula $P(v)$).

The real $x^\sharp$ is absolute for all transitive models M of ZF containing all ordinals such that $x^\sharp \in M$.

Also, "$x^\sharp$ exists" is equivalent to the existence of a nontrivial elementary embedding $j : L[x] \to L[x]$.

Jensen's Covering Theorem

The theorem presented in this section shows that in the absence of $0^\sharp$ the universe does not differ drastically from the constructible model. In particular, the cofinality function is closely related to the cofinality function in L, and every singular cardinal is a singular cardinal in L. Moreover, the Singular Cardinal Hypothesis holds and cardinal exponentiation is determined by the continuum function on regular cardinals.

Theorem 18.30 (Jensen's Covering Theorem). *If $0^\sharp$ does not exist, then for every uncountable set X of ordinals there exists a constructible set $Y \supset X$ such that $|Y| = |X|$.*

The Covering Theorem expresses the closeness between V and L: Every uncountable set of ordinals can be *covered* by a constructible set of the same cardinality. In other words, every set X of ordinals can be covered by some $Y \in L$ such that $|Y| \leq |X| \cdot \aleph_1$. (This is best possible: In Chapter 28 we give an example of a forcing extension of L in which there is a countable set of ordinals that cannot be covered by a countable (in V) constructible set.)

The converse of the Covering Theorem is also true: If $0^\sharp$ exists then every uncountable cardinal is regular in L, and in particular, since $\aleph_\omega$ is regular in L, the countable set $\{\aleph_n : n < \omega\}$ cannot be covered by a constructible set of cardinality less than $\aleph_\omega$. This shows:

$$0^\sharp \text{ exists if and only if } \aleph_\omega \text{ is regular in } L.$$

Corollary 18.31. *If $0^\sharp$ does not exist then for every $\lambda \geq \aleph_2$, if λ is a regular cardinal in L then $\operatorname{cf} \lambda = |\lambda|$. Consequently, every singular cardinal is a singular cardinal in L.*

The assumption $\lambda \geq \aleph_2$ is necessary: The forcing mentioned above yields a model where $\lambda = \aleph_2^L$ is such that $|\lambda| = \aleph_1$ and $\operatorname{cf} \lambda = \omega$.

Proof. Let λ be a limit ordinal such that $\lambda \geq \omega_2$ and that λ is a regular cardinal in L. Let X be an unbounded subset of λ such that $|X| = \operatorname{cf} \lambda$. By the Covering Theorem, there exists a constructible set Y such that $X \subset Y \subset \lambda$ and that $|Y| = |X| \cdot \aleph_1$. Since Y is unbounded in λ and λ is a regular cardinal in L, we have $|Y| = |\lambda|$. This gives $|\lambda| = \aleph_1 \cdot \operatorname{cf} \lambda$ and since $\lambda \geq \aleph_2$, we have $|\lambda| = \operatorname{cf} \lambda$. □

Corollary 18.32. *If $0^\sharp$ does not exist then for every singular cardinal κ, $(\kappa^+)^L = \kappa^+$.*

Proof. Let κ be a singular cardinal and let λ be the successor cardinal of κ in L; we want to show that $\lambda = \kappa^+$. If not, then $|\lambda| = \kappa$, and since κ is singular, we have $\operatorname{cf} \lambda < \kappa$. However, this means that $\operatorname{cf} \lambda < |\lambda|$ which contradicts Corollary 18.31. □

Corollary 18.33. *If $0^\sharp$ does not exist then the Singular Cardinal Hypothesis holds.*

Proof. Let κ be such that $2^{\operatorname{cf}\kappa} < \kappa$, and let $A = [\kappa]^{\operatorname{cf}\kappa}$ be the set of all subsets of κ of size $\operatorname{cf}\kappa$. We shall show that $|A| \le \kappa^+$. By the Covering Theorem, for every $X \in A$ there exists a constructible $Y \subset \kappa$ such that $X \subset Y$ and $|Y| = \lambda$ where $\lambda = \aleph_1 \cdot \operatorname{cf}\kappa$. Thus

$$A \subset \bigcup\{[Y]^{\operatorname{cf}\kappa} : Y \in C\} \tag{18.33}$$

where $C = \{Y \subset \kappa : |Y| = \lambda \text{ and } Y \in L\}$. If $Y \in C$, then $|[Y]^{\operatorname{cf}\kappa}| = \lambda^{\operatorname{cf}\kappa} = (\aleph_1 \cdot \operatorname{cf}\kappa)^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa} < \kappa$. Since $|C| \le |P^L(\kappa)| = |(\kappa^+)^L| \le \kappa^+$ it follows from (18.33) that $|A| \le \kappa^+$. □

Corollary 18.34. *If $0^\sharp$ does not exist then if κ is a singular cardinal and if there exists a nonconstructible subset of κ, then some $\alpha < \kappa$ has a nonconstructible subset.*

Proof. Let κ be a singular cardinal and assume that each $\alpha < \kappa$ has only constructible subsets; we shall show that every subset of κ is constructible. It suffices to show that each subset of κ of size $\operatorname{cf}\kappa$ is constructible: If $A \subset \kappa$, let $\{\alpha_\nu : \nu < \operatorname{cf}\kappa\}$ be such that $\lim_\nu \alpha_\nu = \kappa$; then $\mathcal{A} = \{A \cap \alpha_\nu : \nu < \operatorname{cf}\kappa\}$ is a subset of L_κ of size $\le \operatorname{cf}\kappa$ and hence constructible. It follows that A is constructible.

Let $X \subset \kappa$ be such that $|X| \le \operatorname{cf}(\kappa)$. By the Covering Theorem, there exists a constructible set of ordinals $Y \supset X$ such that $|Y| < \kappa$. Let π be the isomorphism between Y and its order-type α; the function π is constructible and one-to-one. Since $|\alpha| = |Y| < \kappa$, we have $\alpha < \kappa$.

Let $Z = \pi(X)$. Then $Z \subset \alpha$ is constructible by the assumption, and hence $X = \pi^{-1}(Z)$ is also constructible. □

The rest of this chapter is devoted to the proof of the Covering Theorem. Jensen's proof of the Covering Theorem used a detailed analysis of construction of sets in L, the *fine structure theory*, see [1972]. The proof appeared in Devlin and Jensen [1975]. Subsequently, Silver and Magidor gave proofs that did not use the fine structure. The outline below is based on Magidor [1990] (and on Kanamori's presentation in [∞]).

Let us assume that there exists an uncountable set X of ordinals that cannot be covered by a constructible set of the same size. The goal is to produce a nontrivial elementary embedding from L into L. In fact, by Theorem 18.27 it suffices to find some $j : L_\alpha \to L_\beta$ with critical point below $|\alpha|$.

Let τ be the least ordinal such that there exists a set $X \subset \tau$ that cannot be covered, and let $X \subset \tau$ be such a set with $|X|$ least possible. Let $\nu = |X|$.

Lemma 18.35.

(i) *τ is a cardinal in L.*
(ii) *If $Y \in L$ covers X then $|Y|^L \geq \tau$.*
(iii) *ν is a regular cardinal, $\nu < \tau$, and $\nu = \aleph_1 \cdot \operatorname{cf} \tau$.*

Proof. (i) and (ii) follow from the minimality of τ.

(iii) $|X| < \tau$, because otherwise, $Y = \tau$ would cover X. Clearly, $|X| \geq \aleph_1 \cdot \operatorname{cf} \tau$; thus assume that $\nu > \aleph_1 \cdot \operatorname{cf} \tau$. Let $\tau = \lim_{\xi \to \operatorname{cf} \tau} \tau_\xi$. For each ξ, let $Y_\xi \in L$ cover $X \cap \tau_\xi$. Let $\{E_\alpha : \alpha < \tau\}$ be a constructible enumeration of all bounded constructible subsets of τ, and let $Z = \{\alpha < \tau : E_\alpha = Y_\xi$ for some $\xi\}$. By the minimality of ν, Z can be covered by some $W \in L$ of size $\aleph_1 \cdot \operatorname{cf} \tau$. Then the set $Y = \bigcup_{\alpha \in Z} E_\alpha$ covers X, a contradiction. □

Now let M be an elementary submodel of $(L_\tau, \in)$ such that $X \subset M$ and $|M| = \nu$. Let L_η be the transitive collapse of M, and let $j = \pi^{-1}$ where π is the collapsing isomorphism. Hence $j : L_\eta \to L_\tau$ is an elementary embedding. As X is cofinal in τ, and $|\eta| = \nu < \tau$, j is nontrivial.

The goal is to extend $j : L_\eta \to L_\tau$ to an elementary embedding $J : L_\delta \to L_\varepsilon$ where $|\delta|$ is greater than the critical point of j. This can be achieved by finding $M \prec L_\tau$ that satisfies certain closure conditions. These closure conditions guarantee that if L_η is the transitive collapse of M then η is a cardinal in L, and furthermore, that for any $\delta > \tau$, j extends to an elementary embedding J with domain L_δ.

The precise nature of the closure conditions will be spelled out in (18.41). For the remainder of this chapter, we use the phrase "M is sufficiently closed" to indicate that M satisfies (18.41).

We defer the issue of η being a cardinal in L, as its proof requires a finer analysis of the constructible hierarchy. We start with the proof of extendibility of j.

Lemma 18.36. *Let M be sufficiently closed, $X \subset M \prec L_\tau$ such that $|X| = \nu = |M|$, let $\pi : M \simeq L_\eta$ be the transitive collapse, let $j = \pi^{-1}$, and assume that η is a cardinal in L. Then for every limit ordinal $\delta \geq \eta$ there exists an elementary embedding $J : L_\delta \to L_\varepsilon$ such that $J \restriction L_\eta = j$.*

Proof. Let $\delta \geq \eta$ be a limit ordinal. We consider the following directed system of models: Let D be the set of all pairs $i = (\alpha, p)$ where $\alpha < \eta$ and p is a finite subset of L_δ, ordered by $(\alpha, p) \leq (\beta, q)$ if and only if $\alpha \leq \beta$ and $p \subset q$. $(D, <)$ is a directed set. Let $i = (\alpha, p)$, and let $M_i = H^\delta(\alpha \cup p)$ be the Skolem hull of $\alpha \cup p$ in $(L_\delta, \in)$. Let L_{η_i} be the transitive collapse of M_i and let $e_i : L_{\eta_i} \to L_\delta$ be the inverse of the collapsing map $\pi_i : M \simeq L_{\eta_i}$. For $i \leq k$, let $e_{i,k} = \pi_k \circ e_i$.

Let us consider the directed system of models

$$\{L_{\eta_i}, e_{i,k} : i, k \in D\}. \tag{18.34}$$

Clearly, every $x \in L_\delta$ is in some M_i, and so L_δ is the direct limit of $\{L_{\eta_i}, e_{i,k}\}_{i,k\in D}$. For every $i \in D$, $|M_i|^L < \eta$, and since η is a cardinal in L, we have $\eta_i < \eta$. We claim that for all $i, k \in D$, $e_{i,k} \in L_\eta$. This is because $L_{\eta_i} = H^{\eta_i}(\alpha \cup \pi_i(p))$, $L_{\eta_k} = H^{\eta_k}(\beta \cup \pi_k(q))$, and for every Skolem term t, $e_{i,k}(t^{L_{\eta_i}}(\xi, x)) = t^{L_{\eta_k}}(\xi, e_{i,k}(x))$, so $e_{i,k}$ is definable in L_η from η_i, η_k, $\pi_i(p)$, and $\pi_k(q)$.

Now we consider the directed system

$$\{j(L_{\eta_i}), j(e_{i,k}) : i, k \in D\}. \tag{18.35}$$

The closure properties (18.41) of M guarantee that the direct limit of the system (18.35) is well-founded. Let N be the direct limit, and for each $i \in D$, $\tilde{e}_i : L_{j(\eta_i)} \to N$ be an elementary embedding such that $\tilde{e}_i = \tilde{e}_k \circ j(\tilde{e}_{i,k})$ whenever $i < k$. As N is well-founded, we may assume that N is transitive, and then (by (13.13)), $N = L_\varepsilon$ for some limit ordinal ε.

We can extend $j : L_\eta \to L_\tau$ to $J : L_\delta \to L_\varepsilon$ as follows:

$$J(x) = \tilde{e}_i(j(e_i^{-1}(x))) \tag{18.36}$$

where i is some (any) $i \in D$ such that $x \in M_i$.

It remains to show that $J(x) = j(x)$ for all $x \in L_\eta$. So let $x \in L_\eta$, and let $\alpha < \eta$ be such that $x \in L_\alpha$. Let $i = (\alpha, \{x\})$. Since $L_\alpha \subset M_i = H^\delta(\alpha \cup \{x\})$, it follows that $e_i \restriction L_\alpha$ is the identity, as is $e_{k,l} \restriction L_\alpha$ whenever $i \le k \le l$. Thus $j(e_{k,l}) \restriction j(L_\alpha)$ is the identity, for all $l \ge k \ge i$, and therefore $\tilde{e}_i \restriction j(L_\alpha)$ is the identity. Hence $e_i(x) = x$ and $\tilde{e}_i(jx) = jx$, and therefore $J(x) = j(x)$. □

The crucial step in the proof of the Covering Theorem is the following.

Lemma 18.37. *Let M be sufficiently closed, $X \subset M \prec L_\tau$, such that $|X| = \nu = |M|$, and let L_η be the transitive collapse of M. Then η is a cardinal in L.*

The proof is by contradiction. Assuming that η is not a cardinal in L, we shall produce a constructible set of size ν that covers X. It is in this proof that we need a finer analysis of constructibility. We start by refining Gödel's Condensation Lemma:

Lemma 18.38. *For every infinite ordinal ρ, if $M \prec_{\Sigma_1} (L_\rho, \in)$ then the transitive collapse of M is L_γ for some γ. Moreover, there is a Π_2 sentence σ such that for every transitive set M, $(M, \in) \vDash \sigma$ if and only if $M = L_\rho$ for some infinite ordinal ρ.* □

We omit the proof of Lemma 18.38. It can be found in Magidor [1990] or in Kanamori [∞]. A related fact is the following lemma that is not difficult to deduce from Lemma 18.38:

Lemma 18.39. *Let $\{(L_{\eta_i}, \in), e_{i,k} : i, k \in D\}$ be a directed system of models, $e_{i,k}$ being Σ_0-elementary embeddings. If the direct limit of this system is well-founded, then it is isomorphic to some L_γ.* □

We also need the concept of Σ_n *Skolem terms* and Σ_n *Skolem hull*:

Definition 18.40. Let $n \geq 1$.

(i) A Σ_n *Skolem term* is a composition of canonical Skolem functions (18.5) for Σ_n formulas.
(ii) If $Z \subset L_\rho$, the Σ_n *Skolem hull* of Z is the set $H_n^\rho(Z) = \{t^{L_\rho}[z_1, \ldots, z_k] : t \text{ is a } \Sigma_n \text{ Skolem term and } z_1, \ldots, z_n \in Z\}$.

While a Σ_n Skolem function is not necessarily a Σ_n function, we have the following:

Lemma 18.41.

(i) *$H_n^\rho(Z)$ is a Σ_n-elementary submodel of L_ρ.*
(ii) *If $j : L_\alpha \to L_\beta$ is Σ_n-elementary, then for every Σ_n Skolem term t and all $x_1, \ldots, x_k \in L_\alpha$, $j(t^{L_\alpha}[x_1, \ldots, x_k]) = t^{L_\beta}[j(x_1), \ldots, j(x_k)]$.* □

Proof of Lemma 18.37. Let us assume that η is not a cardinal in L. Then there exists a constructible function that maps some $\alpha < \eta$ onto η. Consequently, there exists an ordinal $\rho \geq \eta$, such that for some $\alpha < \eta$ and some finite set $p \subset L_\rho$,

$$H^\rho(\alpha \cup p) \supset \eta. \tag{18.37}$$

We say that η is not a cardinal at ρ. Let ρ be the least ordinal such that η is not a cardinal at ρ.

There are three possible cases.

Case I. There exists some $n > 1$ such that $H_n^\rho(\alpha \cup p) \supset \eta$ for some $\alpha < \eta$ and some finite $p \subset L_\rho$, but $H_{n-1}^\rho(\beta \cup q) \not\supset \eta$, for all $\beta < \eta$ and all finite $q \subset L_\rho$.

Case II. $H_1^\rho(\alpha \cup p) \supset \eta$ for some $\alpha < \eta$ and some finite $p \subset L_\rho$.

Case III. $H_n^\rho(\alpha \cup p) \not\supset \eta$, for all $\alpha < \eta$ and all finite $p \subset L_\rho$.

We start with Case I.

Case I. We consider the following directed system of models. Let D be the set of all pairs $i = (\alpha, p)$ where $i < \eta$ and $p \subset L_\rho$ is finite, ordered by $(\alpha, p) \leq (\beta, q)$ if and only if $\alpha \leq \beta$ and $p \subset q$. For each $i \in D$, let $M_i = H_{n-1}^\rho(\alpha \cup p)$. Let L_{η_i} be the transitive collapse of M_i and let $e_i : L_{\eta_i} \to L_\rho$ be the inverse of the collapsing map. For $i \leq k$, let $e_{i,k} = e_k^{-1} \circ e_i$. Clearly, L_ρ is the direct limit of the directed system

$$\{L_{\eta_i}, e_{i,k} : i, k \in D\}, \tag{18.38}$$

with $e_{i,k}$ being Σ_{n-1}-elementary embeddings.

For each $i \in D$, $\eta_i < \eta$ because otherwise $\eta \subset H_{n-1}^\rho(\alpha \cup e^{-1}(p))$, contradicting the assumption about n. Also, $e_{i,k} \in L_\eta$ for all $i, k \in D$, because

$e_{i,k}$ is defined in L_η by its action on Σ_{n-1} Skolem terms: $e_{i,k}(t^{L_{\eta_i}}(\xi, x)) = t^{L_{\eta_k}}(\xi, e_{i,k}(x))$.

Now we consider the directed system

$$\{j(L_{\eta_i}), j(e_{i,k}) : i, k \in D\}. \tag{18.39}$$

The closure properties (18.41) of M guarantee that the direct limit of (18.39) is well-founded, and by Lemma 18.39, it is equal to L_γ for some γ. Let $\tilde{e}_i$ be the embedding of $j(L_{\eta_i})$ into L_γ; $\tilde{e}_i$ is Σ_{n-1}-elementary. We extend $j : L_\eta \to L_\tau$ to $J : L_\rho \to L_\gamma$ as follows:

$$J(x) = \tilde{e}_i(j(e_i^{-1}(x))) \tag{18.40}$$

where $i \in D$ is such that $x \in M_i$. As in the proof of Lemma 18.36, J extends j, and it is easily verified that J is Σ_{n-1}-elementary.

The key observation is that J is even Σ_n-elementary. To prove that, it is enough to show that for every Σ_{n-1} formula φ, if $L_\gamma \vDash \exists x\, \varphi(x, J(y))$ then $L_\rho \vDash \exists x\, \varphi(x, y)$. Thus let $y \in L_\rho$ and $x \in L_\gamma$ be such that $L_\gamma \vDash \varphi(x, J(y))$. Let $i \in D$ be such that $x \in \operatorname{ran}(\tilde{e}_i)$ and $y \in \operatorname{ran}(e_i)$. If $u \in L_{j(\eta_i)}$ and $v \in L_{\eta_i}$ are such that $x = \tilde{e}_i(u)$ $y = e_i(v)$ then $J(y) = \tilde{e}_i(j(v))$, and $L_\gamma \vDash \varphi(\tilde{e}_i(u), \tilde{e}_i(j(v)))$. Since $\tilde{e}_i$ is Σ_{n-1}-elementary, we have $L_{j(\eta_i)} \vDash \varphi(u, j(v))$. The statement $L_{j(\eta_i)} \vDash \exists z\, \varphi(z, j(v))$ is Σ_0 (with parameters $j(L_{\eta_i})$ and $j(v)$) and true in L_τ; hence in L_η, $L_{\eta_i} \vDash \exists z\, \varphi(z, v)$. Let $z \in L_{\eta_i}$ be such that $L_{\eta_i} \vDash \varphi(z, v)$; since e_i is Σ_{n-1}-elementary, we get $L_\rho \vDash \varphi(e_i(z), e_i(v))$, and so $L_\rho \vDash \exists x\, \varphi(x, y)$.

Now we reach a contradiction. Let $\alpha < \eta$ and a finite $p \subset L_\rho$ be such that $\eta \subset H_n^\rho(\alpha \cup p)$. First we have

$$X \subset M \cap \tau = j\text{“}\eta = J\text{“}\eta,$$

and since J is Σ_n-elementary, Lemma 18.41 gives

$$J\text{“}\eta \subset J\text{“}H_n^\rho(\alpha \cup p) = H^\gamma(J\text{“}\alpha \cup J\text{“}p).$$

By the minimality of τ, the set $J\text{“}\alpha \subset j(\alpha) < \tau$ can be covered by a constructible set Y of size $|Y| \le \nu$. Hence X can be covered by the constructible set $H_n^\gamma(Y \cup J\text{“}p)$, which has cardinality $\le \nu$, contrary to Lemma 18.35.

This completes the proof of Case I.

Case II. We use the fact that in this case, ρ must be a limit ordinal. This is an immediate consequence of this:

Lemma 18.42. *If γ is infinite, $\alpha < \gamma$ and $p \subset L_{\gamma+1}$ is finite, then there exists a finite set $q \subset L_\gamma$ such that*

$$H_1^{\gamma+1}(\alpha \cup q) \cap L_\gamma \subset H^\gamma(\alpha \cup p).$$

Proof. This is quite routine when $p = \emptyset$. When p is nonempty, the idea is to replace members of p by the parameters used in their definitions over L_γ. We omit the proof. □

Continuing Case II, we consider the directed system of models (18.38) with $\eta_i < \eta$ where all embeddings are Σ_0-elementary embeddings. The index set D is the set of all $i = (\alpha, p, \xi)$ where $\alpha < \eta$, $p \subset L_\rho$ is finite and $\xi < \rho$ such that $p \in L_\xi$. Each model L_{η_i} is the transitive collapse of $H^\xi(\alpha \cup p)$.

The closure properties (18.41) of M guarantee that the direct limit of the system (18.39) is well-founded, say L_γ. We extend j to $J : L_\rho \to L_\gamma$ as before, and as in Case I prove that J is not just Σ_0-elementary, but Σ_1-elementary. As in Case I, we reach a contradiction by covering X by a constructible set of size $\leq \nu$.

Case III. In this case, we consider the directed system (18.38) indexed by triples $i = (\alpha, p, n)$ where α and p are as before and $n \geq 1$; $(\alpha, p, n) \leq (\beta, q, m)$ means $\alpha \leq \beta$, $p \subset q$ and $n \leq m$. For each $i = (\alpha, p, n)$, $M_i = H_n^\rho(\alpha \cup p)$; by the assumption on ρ, the transitive collapse of M_i is some L_{η_i} with $\eta_i < \eta$, and if for each $k \geq i$, $e_{i,k}$ is Σ_n-elementary (and $e_{i,k} \in L_\eta$).

Again, by (18.41) the direct limit of (18.39) is some L_γ, and for each $i = (\alpha, p, n)$, $\tilde{e}_i$ is Σ_n-elementary. Extending j to $J : L_\rho \to L_\gamma$ as before, we get J elementary, and reach a contradiction in much the same way as before. □

It remains to find a model $M \supset X$ with the right closure conditions. This is provided by the following technical lemma:

Lemma 18.43. *There exists a model $M \prec L_\tau$ such that $X \subset M$, $|M| = \nu = |X|$, and if j^{-1} is the transitive collapse of M onto L_η, then*

(18.41) *for every directed system $\{L_{\eta_i}, e_{i,k} : i, k \in D\}$ with $L_{\eta_i}, e_{i,k} \in L_\eta$, with limit L_ρ for some $\rho \geq \eta$, and D as in the proof of Lemmas 18.36 and 18.37, the direct limit of $\{j(L_{\eta_i}), j(e_{i,k}) : i, k \in D\}$ is well-founded.* □

The construction of M proceeds in ν steps. At each step $\xi < \nu$ let $(\eta(\xi), \rho(\xi))$ be the least (η, ρ) such that for some increasing $\{i_n\}_{n=0}^\infty \subset D$, there are ordinals $\beta_n \in L_{\eta_{i_n}}$ such that $\beta_{n+1} < e_{i_n, i_{n+1}}(\beta_n)$ for $n = 0, 1, 2, \ldots$. We add the ordinals β_n to M at this stage ξ. Using the fact that ν is a regular uncountable cardinal, one can verify that the resulting model M satisfies (18.41). As the proof is rather long and tedious, we omit it and refer the reader to either Magidor [1990] or Chapter 32 in Kanamori's book.

Exercises

18.1. If there exists a cardinal κ such that $\kappa \to (\omega_1)^{<\omega}$ then $0^\sharp$ exists.

18.2. Let M be a transitive model of ZFC, let B be a complete Boolean algebra in M and let G be an M-generic ultrafilter on B. If $M \vDash 0^\sharp$ does not exist, then $M[G] \vDash 0^\sharp$ does not exist.

[All cardinals $\geq |B^+|$ remain cardinals in $M[G]$. Let $\gamma_1 < \gamma_2 < \ldots < \gamma_n < \ldots < \gamma_\omega$ be an increasing sequence of cardinals in M such that $\gamma_1 \geq |B^+|$. If $0^\sharp$ exists in $M[G]$, then $0^\sharp = \{\varphi : L_{\gamma_\omega} \vDash \varphi[\gamma_1, \ldots, \gamma_n]\}$ and hence $0^\sharp \in M$.]

18.3. Assume that $0^\sharp$ exists. If $A \subset \omega_1$ is such that $A \cap \alpha \in L$ for every $\alpha < \omega_1$, then $A \in L$.

[For every $\alpha \in I \cap \omega_1$ there is t_α such that $A \cap \alpha = t_\alpha(\gamma_1^\alpha, \ldots, \gamma_{n(\alpha)}^\alpha, \alpha, \delta_1^\alpha, \ldots, \delta_{k(\alpha)}^\alpha)$. Clearly $A \cap \alpha = t_\alpha(\gamma_1^\alpha, \ldots, \gamma_{n(\alpha)}^\alpha, \alpha, \aleph_2, \ldots, \aleph_{k(\alpha)+1})$. Since there are only countably many Skolem terms, and by Fodor's Theorem, there is a stationary subset X of $I \cap \omega_1$ and t, $\gamma_1, \ldots, \gamma_n$ such that for all $\alpha \in X$, $A \cap \alpha = t(\gamma_1, \ldots, \gamma_n, \alpha, \aleph_2, \ldots, \aleph_{k+1})$. Show that $A = t(\gamma_1, \ldots, \gamma_n, \aleph_1, \aleph_2, \ldots, \aleph_{k+1})$.]

18.4. Let κ be an uncountable regular cardinal. If $0^\sharp$ exists, then for every constructible set $X \subset \kappa$, either X or $\kappa - X$ contains a closed unbounded subset.

[Let $X = t(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ where $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_m$ are Silver indiscernibles such that $\alpha_n < \kappa \leq \beta_1$. Show that either X or $\kappa - X$ contains all Silver indiscernibles γ such that $\alpha_n < \gamma < \kappa$: The truth value of $\gamma \in t(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ is the same for all such γ.]

18.5. Let us assume that for some uncountable regular cardinal κ, every constructible $X \subset \kappa$ either contains or is disjoint from a closed unbounded set. Then $0^\sharp$ exists.

[Let D be the collection of all constructible subsets of κ containing a closed unbounded subset. D is an L-ultrafilter and every intersection of less than κ elements of D is nonempty; hence the ultrapower $\mathrm{Ult}_D(L)$ is well-founded and gives an elementary embedding of L in L.]

18.6. If κ is weakly compact and if $|(\kappa^+)^L| = \kappa$, then $0^\sharp$ exists.

[Let B be the least nontrivial κ-complete algebra of subsets of κ closed under inverses of constructible functions $f : \kappa \to \kappa$; we have $|B| = \kappa$. Let U be a κ-complete ultrafilter on B containing all final segments $\{\alpha : \aleph_0 \leq \alpha < \kappa\}$. $U \cap L$ is a nonprincipal L-ultrafilter, and $\mathrm{Ult}_{U \cap L}(L)$ is well-founded. Thus there is a nontrivial elementary embedding of L in L.]

18.7. Let i_n $(n \leq \omega)$ be the nth Silver indiscernible, and let $j : I \to I$ be order-preserving such that $j(i_n) = i_n$ for $n < \omega$ and $j(i_\omega) > i_\omega$. Then j extends to an elementary embedding $j : L \to L$ with i_ω its critical point.

18.8. Every Silver indiscernible is ineffable (hence weakly compact) in L.

[Show that i_ω is ineffable in L, by Lemma 17.32.]

18.9. If $0^\sharp$ exists then $L \vDash \exists \kappa\ \kappa \to (\omega)^{<\omega}$.

[Let $\kappa = i_\omega$. If $f : [\kappa]^{<\omega} \to \{0, 1\}$ is in L, there is some $n < \omega$ such that the set $\{i_k : k \leq n < \omega\}$ is homogeneous for f.]

18.10. If $0^\sharp$ exists then the Erdős cardinal η_ω in L is smaller than the least Silver indiscernible.

[$(\eta_\omega)^L$ is definable in L.]

18.11. If $j : L \to L$ is elementary, then the critical point of j is a Silver indiscernible.

[Let κ be the critical point, let $D = \{X : \kappa \in j(X)\}$, and let $j_D : L \to \mathrm{Ult}_D(L) = L$ be the canonical embedding. κ is the critical point of j_D, and $j_D(\lambda) = \lambda$ for all regular $\lambda \geq \kappa^+$. If $\kappa \notin I$ then $\kappa = t(\alpha_1, \ldots, \alpha_k, \lambda_1, \ldots, \lambda_n)$ where $\alpha_i < \kappa < \lambda_j$ and $j_D(\alpha_i) = \alpha_i$, $j_D(\lambda_j) = \lambda_j$. Hence $j_D(\kappa) = \kappa$, a contradiction.]

18.12. If both ω_1 and ω_2 are singular, then $0^\sharp$ exists.

[Let $\kappa = \omega_1$ and let λ be the successor cardinal of κ in L. Since $\mathrm{cf}\,\kappa = \mathrm{cf}\,\lambda = \omega$, there are sets $X \subset \kappa$ and $Y \subset \lambda$, both of order-type ω such that $\sup X = \kappa$ and $\sup Y = \lambda$. Let $M = L[X, Y]$; M is a model of ZFC and in M, κ is a singular cardinal, and λ is not a cardinal. Hence $0^\sharp$ exists in M.]

18.13. For every $x \subset \omega$, either $0^\sharp \in L[x]$ or $x^\sharp \in L[0^\sharp, x]$.

[If $0^\sharp \notin L[x]$, then the Covering Theorem for L holds in $L[x]$ but fails in $L[0^\sharp, x]$, and hence the Covering Theorem for $L[x]$ fails in $L[0^\sharp, x]$. Therefore $x^\sharp \in L[0^\sharp, x]$.]

Historical Notes

Theorem 18.1 was discovered by Gaifman (assuming the existence of a measurable cardinal). Gaifman's results were announced in [1964] and the proof was published in [1974], Gaifman's proof used iterated ultrapowers (see also Gaifman [1967]). Silver in his thesis (1966, published in [1971b]) developed the present method of proof, using infinitary combinatorics, and proved the theorem under the weaker assumption of existence of κ with the property $\kappa \to (\aleph_1)^{<\omega}$. Gaifman proved that if there is a measurable cardinal, then there exists $A \subset \omega$ such that the conclusion of Theorem 18.1 holds in $L[A]$. Solovay formulated $0^\sharp$ and proved that it is a Δ^1_3 set of integers; Silver deduced the existence of $0^\sharp$ under weaker assumptions.

Construction of models with indiscernibles was introduced by Ehrenfeucht and Mostowski in [1956].

The equivalence of the existence of $0^\sharp$ with the existence of a nontrivial elementary embedding of L (Theorem 18.20) is due to Kunen; the present proof is due to Silver. Kunen also derived $0^\sharp$ from the existence of Jónsson cardinals and from Chang's Conjecture.

Theorem 18.30 (and its corollaries) is due to Jensen. A proof of the theorem appeared in Devlin and Jensen [1975]. Jensen's proof makes use of his fine structure theory, see Jensen [1972]. The present proof is due to Magidor [1990]. Lemma 18.38 appears in Magidor [1990] and in Kanamori's book [∞]; Magidor attributes the proof to Boolos [1970].

Exercise 18.3: Solovay.

Exercise 18.6: Kunen.

Exercise 18.12: Magidor.

19. Iterated Ultrapowers and $L[U]$

In this chapter we investigate inner models for measurable cardinals, using Kunen's technique of iterated ultrapowers.

The Model $L[U]$

Let κ be a measurable cardinal and let U be a κ-complete nonprincipal ultrafilter on κ. Let us consider the model $L[U]$. By Lemma 13.23, $L[U] = L[\bar{U}]$, where $\bar{U} = U \cap L[U]$.

Lemma 19.1. *In $L[U]$, $\bar{U}$ is a κ-complete nonprincipal ultrafilter on κ. Moreover, if U is normal, then $L[U] \vDash \bar{U}$ is normal.*

Proof. A straightforward verification. For instance if U is normal and $f \in L[U]$ is a regressive function on κ, then for some $\gamma < \kappa$, the set $X = \{\alpha : f(\alpha) = \gamma\}$ is in U; since $X \in L[U]$, $L[U] \vDash f$ is constant on some $X \in \bar{U}$. □

We shall eventually prove, among others, that the model $L[U]$ satisfies GCH. For now, we recall Theorem 13.22(iv) by which $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds in $L[U]$ for all sufficiently large α. Specifically, using the Condensation Lemma 13.24, we get:

Lemma 19.2. *If $V = L[A]$, and if $A \subset P(\omega_\alpha)$, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.*

Proof. Let X be a subset of ω_α. Let λ be a cardinal such that $A \in L_\lambda[A]$ and $X \in L_\lambda[A]$. Let M be an elementary submodel of $(L_\lambda[A], \in)$ such that $\omega_\alpha \subset M$, $A \in M$, $X \in M$, and $|M| = \aleph_\alpha$. Let π be the transitive collapse of M, and let $N = \pi(M)$. Since $\omega_\alpha \subset M$, we have $\pi(Z) = Z$ for every $Z \subset \omega_\alpha$ that is in M and in particular $\pi(X) = X$; also, $\pi(A) = \pi(A \cap M) = A \cap N$. Now $N = L_\gamma[A \cap N]$ for some γ, and hence $N = L_\gamma[A]$. Since $|N| = \aleph_\alpha$, we have $\gamma < \omega_{\alpha+1}$ and hence $X \in L_{\omega_{\alpha+1}}[A]$. It follows that every subset of ω_α is in $L_{\omega_{\alpha+1}}[A]$ and therefore $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. □

Theorem 19.3 (Silver). *If $V = L[D]$ where D is a normal measure on a measurable cardinal κ, then the Generalized Continuum Hypothesis holds.*

Proof. If $\lambda \geq \kappa$, then $D \subset P(\lambda)$ and hence $2^\lambda = \lambda^+$ by Lemma 19.2. Thus it suffices to show that $2^\lambda = \lambda^+$ for every infinite cardinal $\lambda < \kappa$. Let $\lambda < \kappa$ and let us assume that there are more than λ^+ subsets of λ; we shall reach a contradiction. If $2^\lambda > \lambda^+$, then there exists a set $X \subset \lambda$ that is the λ^+th subset of λ in the canonical well-ordering $<_{L[D]}$ of $L[D]$. Let α be the least ordinal such that $X \in L_\alpha[D]$. Since the well-ordering $<_{L[D]}$ has the property that each $L_\xi[D]$ is an initial segment of $<_{L[D]}$ every subset of λ preceding X is also in $L_\alpha[D]$ and hence the set $P(\lambda) \cap L_\alpha[D]$ has size at least λ^+.

We shall now apply Lemma 17.36. Let η be a cardinal such that $\eta > \alpha$ and that $D \in L_\eta[D]$, and consider the model $\mathfrak{A} = (A, \in)$ where $A = L_\eta[D]$. We have $\kappa \subset A$, and we consider the set $P = P(\lambda) \cap A$. Since $2^\lambda < \kappa$, we have $|P| < \kappa$. By Lemma 17.36, there is an elementary submodel $\mathfrak{B} = (B, \in) \prec \mathfrak{A}$ such that $\lambda \cup \{D, X, \alpha\} \subset B$, $\kappa \cap B \in D$ and $|P \cap B| \leq \lambda$. Let π be the transitive collapse of $\mathfrak{B}$ onto a transitive set M; we have $M = L_\gamma[\pi(D)]$ for some γ.

Using the normality of D, we show that $\pi(D) = D \cap M$. Clearly, $\pi(\kappa) = \kappa$ because $|\kappa \cap B| = \kappa$. The function π is one-to-one, and for every $\xi < \kappa$, $\pi(\xi) \leq \xi$. Since D is normal, there is a set $Z \in D$ such that $\pi(\xi) = \xi$ for all $\xi \in Z$. Hence if $Y \in B$ is a set in D, then $\pi(Y) \supset \pi(Y \cap Z) = Y \cap Z$, and so $\pi(Y)$ is also in D; similarly, if $Y \in B$ and $\pi(Y) \in D$, then $Y \in D$. It follows that $\pi(D) = D \cap M$.

Hence $M = L_\gamma[D]$. Since $\lambda \subset B$, π maps every subset of λ onto itself, and so $P(\lambda) \cap M = P(\lambda) \cap B$. In particular, we have $\pi(X) = X$ and so $X \in L_\gamma[D]$. By the minimality assumption on α, we have $\alpha \leq \gamma$, and this is a contradiction since on the one hand $|P(\lambda) \cap L_\alpha[D]| \geq \lambda^+$, and on the other hand $|P(\lambda) \cap L_\gamma[D]| \leq \lambda$. □

One proves rather easily that the model $L[D]$ has only one measurable cardinal:

Lemma 19.4. *If $V = L[D]$ and D is a normal measure on κ, then κ is the only measurable cardinal.*

Proof. Let us assume that there is a measurable cardinal $\lambda \neq \kappa$ and let us consider the elementary embedding $j_U : V \to M$ where U is some nonprincipal λ-complete ultrafilter on λ. We shall prove that $M = L[D] = V$ thus getting a contradiction since $U \notin M$ by Lemma 17.9(ii).

Since j is elementary, it is clear that $M = L[j(D)]$. If $\lambda > \kappa$, then $j(D) = D$ and so $M = L[D]$. Thus assume that $\lambda < \kappa$.

Since κ is measurable, the set $Z = \{\alpha < \kappa : \alpha \text{ is inaccessible and } \alpha > \lambda\}$ belongs to D. By Lemma 17.9(v), $j(\kappa) = \kappa$ and $j(\alpha) = \alpha$ for all $\alpha \in Z$. We shall show that $j(D) = D \cap M$. It suffices to show that $j(D) \subset D \cap M$ since $j(D)$ is (in M) an ultrafilter. Let $X \in j(D)$ be represented by $f : \lambda \to D$. Let $Y = \bigcap_{\xi<\lambda} f(\xi)$; we have $Y \in D$, and clearly $j(Y) \subset X$. Now if $\alpha \in Y \cap Z$, then $j(\alpha) = \alpha$ and so $X \supset j(Y) \supset j\text{“}(Y \cap Z) = Y \cap Z \in D$ and hence $X \in D$.

Thus $j(D) = D \cap M$, and we have $M = L[j(D)] = L[D \cap M] = L[D]$. □

Iterated Ultrapowers

Let κ be a measurable cardinal and let U be a κ-complete nonprincipal ultrafilter on κ. Using U, we construct an ultrapower of V mod U; and since the ultrapower is well-founded, we identify the ultrapower with its transitive collapse, a transitive model $M = \mathrm{Ult}_U(V)$. Let us denote this transitive model $\mathrm{Ult}_U^{(1)}(V)$ or just $\mathrm{Ult}^{(1)}$. Let $j^{(0)} = j_U$ be the canonical embedding of V in $\mathrm{Ult}^{(1)}$, and let $\kappa^{(1)} = j^{(0)}(\kappa)$ and $U^{(1)} = j^{(0)}(U)$.

In the model $\mathrm{Ult}^{(1)}$, the ordinal $\kappa^{(1)}$ is a measurable cardinal and $U^{(1)}$ is a $\kappa^{(1)}$-complete nonprincipal ultrafilter on $\kappa^{(1)}$. Thus working inside $\mathrm{Ult}^{(1)}$, we can construct an ultrapower mod $U^{(1)}$: $\mathrm{Ult}_{U^{(1)}}(\mathrm{Ult}^{(1)})$. Let us denote this ultrapower $\mathrm{Ult}^{(2)}$, and let $j^{(1)}$ be the canonical embedding of $\mathrm{Ult}^{(1)}$ in $\mathrm{Ult}^{(2)}$ given by this ultrapower. Let $\kappa^{(2)} = j^{(1)}(\kappa^1)$ and $U^{(2)} = j^{(1)}(U^{(1)})$.

We can continue this procedure and obtain transitive models

$$\mathrm{Ult}^{(1)}, \quad \mathrm{Ult}^{(2)}, \quad \ldots, \quad \mathrm{Ult}^{(n)}, \quad \ldots \qquad (n < \omega).$$

[That we can indeed construct such a sequence of classes follows from the observation that for each α, the initial segment $V_\alpha \cap \mathrm{Ult}^{(n)}$ of each ultrapower in the sequence is defined from an initial segment V_β of the universe (where β is something like $\kappa + \alpha + 1$).]

Thus we get a sequence of models $\mathrm{Ult}^{(n)}$, $n < \omega$ (where $\mathrm{Ult}^{(0)} = V$). For any $n < m$, we have an elementary embedding $i_{n,m} : \mathrm{Ult}^{(n)} \to \mathrm{Ult}^{(m)}$ which is the composition of the embeddings $j^{(n)}$, $j^{(n+1)}$, $\ldots$, $j^{(m-1)}$:

$$i_{n,m}(x) = j^{(m-1)} j^{(m-2)} \ldots j^{(n)}(x) \qquad (x \in \mathrm{Ult}^{(n)}).$$

These embeddings form a commutative system; that is,

$$i_{m,k} \circ i_{n,m} = i_{n,k} \qquad (m < n < k).$$

We also let $\kappa^{(n)} = i_{0,n}(\kappa)$, and $U^{(n)} = i_{0,n}(U)$. Note that $\kappa^{(0)} < \kappa^{(1)} < \ldots < \kappa^{(n)} < \ldots$, and $\mathrm{Ult}^{(0)} \supset \mathrm{Ult}^{(1)} \supset \ldots \supset \mathrm{Ult}^{(n)} \supset \ldots$.

Thus we have a directed system of models and elementary embeddings

$$\{\mathrm{Ult}^{(n)}, i_{m,n} : m, n \in \omega\}. \tag{19.1}$$

Even though the models are proper classes, the technique of Lemma 12.2 is still applicable and we can consider the *direct limit*

$$(M, E) = \lim \mathrm{dir}_{n \to \omega} \{\mathrm{Ult}^{(n)}, i_{n,m}\}, \tag{19.2}$$

along with elementary embeddings $i_{n,\omega} : \mathrm{Ult}^{(n)} \to (M, E)$. The direct limit is a model of ZFC and we shall prove below that it is well-founded. Thus we identify it with a transitive model $\mathrm{Ult}^{(\omega)}$. (We shall also prove that $\mathrm{Ult}^{(\omega)} \subset \mathrm{Ult}^{(n)}$ for every n.)

Let $\kappa^{(\omega)} = i_{0,\omega}(\kappa)$ and $U^{(\omega)} = i_{0,\omega}(U)$. Since $\mathrm{Ult}^{(\omega)}$ satisfies that $U^{(\omega)}$ is a $\kappa^{(\omega)}$-complete nonprincipal ultrafilter on $\kappa^{(\omega)}$, we can construct, working inside the model $\mathrm{Ult}^{(\omega)}$, the ultrapower of $\mathrm{Ult}^{(\omega)}$ mod $U^{(\omega)}$ and the corresponding canonical embedding $j^{(\omega)}$.

Let us denote $\mathrm{Ult}^{(\omega+1)}$ the ultrapower of $\mathrm{Ult}^{(\omega)}$ mod $U^{(\omega)}$ and let $i_{\omega,\omega+1}$ be the corresponding canonical elementary embedding. For $n < \omega$, let $i_{n,\omega+1} = i_{\omega,\omega+1} \circ i_{n,\omega}$.

This procedure can be continued, and so we define the *iterated ultrapower* as follows:

$$\begin{aligned}(\mathrm{Ult}^{(0)}, E^{(0)}) &= (V, \in),\\ (\mathrm{Ult}^{(\alpha+1)}, E^{(\alpha+1)}) &= \mathrm{Ult}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)}, E^{(\alpha)}),\\ (\mathrm{Ult}^{(\lambda)}, E^{(\lambda)}) &= \lim \mathrm{dir}_{\alpha\to\lambda}\{(\mathrm{Ult}^{(\alpha)}, E^{(\alpha)}), i_{\alpha,\beta}\} \qquad (\lambda \text{ limit})\end{aligned}$$

where $U^{(\alpha)} = i_{0,\alpha}(U)$, for each α. We do not know yet that all the models $\mathrm{Ult}^{(\alpha)}$ are well-founded; but we make a convention that if $\mathrm{Ult}^{(\alpha)}$ is well-founded, then we identify it with its transitive collapse.

If M is a transitive model of set theory and U is (in M) a κ-complete nonprincipal ultrafilter on κ, we can construct, within M, the iterated ultrapowers. Let us denote by $\mathrm{Ult}_U^{(\alpha)}(M)$ the αth iterated ultrapower, constructed in M.

Lemma 19.5 (The Factor Lemma). *Let us assume that* $\mathrm{Ult}^{(\alpha)}$ *is well-founded. Then for each* β*, the iterated ultrapower* $\mathrm{Ult}^{(\beta)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)})$ *taken in* $\mathrm{Ult}^{(\alpha)}$ *is isomorphic to the iterated ultrapower* $\mathrm{Ult}^{(\alpha+\beta)}$.

Moreover, there is for each β *an isomorphism* $e^{(\alpha)}_\beta$ *such that if for all* ξ *and* η, $i^{(\alpha)}_{\xi,\eta}$ *denotes the elementary embedding of* $\mathrm{Ult}^{(\xi)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)})$ *into* $\mathrm{Ult}^{(\eta)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)})$*, then the following diagram commutes*:

$$\begin{array}{ccc}
\mathrm{Ult}^{(\xi)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)}) & \xrightarrow{i^{(\alpha)}_{\xi,\eta}} & \mathrm{Ult}^{(\eta)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)}) \\
{\scriptstyle e^{(\alpha)}_\xi}\big\downarrow & & \big\downarrow{\scriptstyle e^{(\alpha)}_\eta} \\
\mathrm{Ult}^{(\alpha+\xi)}_U & \xrightarrow[i^{(\alpha)}_{\alpha+\xi,\alpha+\eta}]{} & \mathrm{Ult}^{(\alpha+\eta)}_U
\end{array}$$

Proof. The proof is by induction on β. If $\beta = 0$, then the 0th iterated ultrapower in $\mathrm{Ult}^{(\alpha)}$ is $\mathrm{Ult}^{(\alpha)}$; and we let $e^{(\alpha)}_0$ be the identity mapping. If $\mathrm{Ult}^{(\beta)}_{U^{(\alpha)}}$ and $\mathrm{Ult}^{(\alpha+\beta)}_U$ are isomorphic and $e^{(\alpha)}_\beta$ is the isomorphism, then $\mathrm{Ult}^{(\beta+1)}_{U^{(\alpha)}}$ and $\mathrm{Ult}^{(\alpha+\beta+1)}_U$ are ultrapowers of $\mathrm{Ult}^{(\beta)}_{U^{(\alpha)}}$ and $\mathrm{Ult}^{(\alpha+\beta)}_U$, respectively, mod $i^{(\alpha)}_{0,\beta}(U^{(\alpha)})$ and mod $i_{0,\alpha+\beta}(U)$, respectively; and since $i_{0,\alpha+\beta}(U) = e^{(\alpha)}_\beta(i^{(\alpha)}_{0,\beta}(U^{(\alpha)}))$, the isomorphism $e^{(\alpha)}_\beta$ induces an isomorphism $e^{(\alpha)}_{\beta+1}$ between $\mathrm{Ult}^{(\beta+1)}_{U^{(\alpha)}}$ and $\mathrm{Ult}^{(\alpha+\beta+1)}_U$.

If λ is a limit ordinal, then $\mathrm{Ult}^{(\lambda)}_{U^{(\alpha)}}$ and $\mathrm{Ult}^{(\alpha+\lambda)}_{U}$ are (in $\mathrm{Ult}^{(\alpha)}$) the direct limits of $\{\mathrm{Ult}^{(\beta)}_{U^{(\alpha)}}, i^{(\alpha)}_{\beta,\gamma} : \beta,\gamma < \lambda\}$ and $\{\mathrm{Ult}^{(\alpha+\beta)}_{U}, i_{\alpha+\beta,\alpha+\gamma} : \beta,\gamma < \lambda\}$, respectively. It is clear that the isomorphisms $e^{(\alpha)}_{\beta}$, $\beta < \lambda$, induce an isomorphism $e^{(\alpha)}_{\lambda}$ between $\mathrm{Ult}^{(\lambda)}_{U^{(\alpha)}}$ and $\mathrm{Ult}^{(\alpha+\lambda)}_{U}$. □

Corollary 19.6. *For every limit ordinal λ, if $\mathrm{Ult}^{(\lambda)}$ is well-founded then $\mathrm{Ult}^{(\lambda)} \subset \mathrm{Ult}^{(\alpha)}$ for all $\alpha < \lambda$.*

Proof. $\mathrm{Ult}^{(\lambda)}$ is a class in $\mathrm{Ult}^{(\alpha)}$; it is the iterated ultrapower $\mathrm{Ult}^{(\beta)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)})$ where $\alpha + \beta = \lambda$. □

Next we show that the iterated ultrapowers $\mathrm{Ult}^{(\alpha)}_{U}$ are all well-founded.

Theorem 19.7 (Gaifman). *Let U be a κ-complete nonprincipal ultrafilter on κ. Then for every α, the αth iterated ultrapower $\mathrm{Ult}^{(\alpha)}$ is well-founded.*

Proof. Clearly, if $\mathrm{Ult}^{(\alpha)}$ is well-founded, then $\mathrm{Ult}^{(\alpha+1)}$ is well-founded. Thus if γ is the least γ such that $\mathrm{Ult}^{(\gamma)}$ is not well-founded, then γ is a limit ordinal. The ordinals of the model $\mathrm{Ult}^{(\gamma)}$ are not well-ordered; let ξ be the least ordinal such that the ordinals of Ult^{γ} below $i_{0,\gamma}(\xi)$ are not well-ordered.

Let $x_0, x_1, x_2, \ldots$ be a descending sequence of ordinals in the model $\mathrm{Ult}^{(\gamma)}$ such that x_0 is less than $i_{0,\gamma}(\xi)$. Since $\mathrm{Ult}^{(\gamma)}$ is the direct limit of $\mathrm{Ult}^{(\alpha)}$, $\alpha < \gamma$, there is an $\alpha < \gamma$ and an ordinal ν (less than $i_{0,\alpha}(\xi)$) such that $x_0 = i_{\alpha,\gamma}(\nu)$. Let β be such that $\alpha + \beta = \gamma$.

By our assumptions, the following is true (in V):

(19.3) $(\forall\gamma' \leq \gamma)(\forall\xi' < \xi)$ the ordinals below $i_{0,\gamma'}(\xi')$ in $\mathrm{Ult}^{(\gamma')}$ are well-ordered.

When we apply the elementary embedding $i_{0,\alpha}$ to (19.3), we get:

(19.4) $\mathrm{Ult}^{(\alpha)} \vDash (\forall\gamma' \leq i_{0,\alpha}(\gamma))(\forall\xi' < i_{0,\alpha}(\xi))$ the ordinals below $i^{(\alpha)}_{0,\gamma'}(\xi')$ in $\mathrm{Ult}^{(\gamma')}_{U^{(\alpha)}}$ are well-ordered.

Now $\beta \leq \gamma \leq i_{0,\alpha}(\gamma)$, and $\nu < i_{0,\alpha}(\xi)$. Hence if we let $\gamma' = \beta$ and $\xi' = \nu$ in (19.4), we get

$$\mathrm{Ult}^{(\alpha)} \vDash \text{the ordinals below } i^{(\alpha)}_{0,\beta}(\nu) \text{ in } \mathrm{Ult}^{(\beta)}_{U^{(\alpha)}} \text{ are well-ordered.}$$

By the Factor Lemma, $\mathrm{Ult}^{(\beta)}_{U^{(\alpha)}}$ is (isomorphic to) $\mathrm{Ult}^{(\alpha+\beta)}$, and $i^{(\alpha)}_{0,\beta}(\nu)$ is $i_{\alpha,\alpha+\beta}(\nu)$. Since $\alpha + \beta = \gamma$ and $i_{\alpha,\gamma}(\nu) = x_0$, and since being well-ordered is absolute (for the transitive model $\mathrm{Ult}^{(\alpha)}$), we have:

$$\text{The ordinals below } x_0 \text{ in } \mathrm{Ult}^{(\gamma)} \text{ are well-ordered.}$$

But this is a contradiction since x_1, x_2, x_3, ... is a descending sequence of ordinals below x_0 in $\mathrm{Ult}^{(\gamma)}$. □

Thus for any given κ-complete nonprincipal ultrafilter U on κ we have a transfinite sequence of transitive models, the iterated ultrapowers $\mathrm{Ult}_U^{(\alpha)}(V)$, and the elementary embeddings $i_{\alpha,\beta} : \mathrm{Ult}^{(\alpha)} \to \mathrm{Ult}^{(\beta)}$. Let $\kappa^{(\alpha)} = i_{0,\alpha}(\kappa)$ for each α; we shall show that the sequence $\kappa^{(\alpha)}$, $\alpha \in Ord$, is a normal sequence.

Lemma 19.8.

(i) *If* $\gamma < \kappa^{(\alpha)}$, *then* $i_{\alpha,\beta}(\gamma) = \gamma$ *for all* $\beta \geq \alpha$.
(ii) *If* $X \subset \kappa^{(\alpha)}$ *and* $X \in \mathrm{Ult}^{(\alpha)}$, *then* $X \subset i_{\alpha,\beta}(X)$ *for all* $\beta \geq \alpha$; *in fact* $X = \kappa^{(\alpha)} \cap i_{\alpha,\beta}(X)$.

Proof. By the Factor Lemma, it suffices to give the proof for $\alpha = 0$.

(i) As we know, $i_{0,1}(\gamma) = \gamma$ for all $\gamma < \kappa$. By induction on β, if $i_{0,\beta}(\gamma) = \gamma$, then $i_{0,\beta+1}(\gamma) = i_{\beta,\beta+1}(\gamma) = \gamma$ because $\gamma < \kappa^{(\beta)}$; if λ is limit and $i_{0,\beta}(\xi) = \xi$ for all $\xi \leq \gamma$ and $\beta < \lambda$, then $i_{0,\lambda}(\gamma) = \gamma$.

(ii) Follows from (i). □

Lemma 19.9. *The sequence* $\langle \kappa^{(\alpha)} : \alpha \in Ord \rangle$ *is normal*; *i.e.*, *increasing and continuous.*

Proof. For each α, $\kappa^{(\alpha+1)} = i_{\alpha,\alpha+1}(\kappa^{(\alpha)}) > \kappa^{(\alpha)}$. To show that the sequence is continuous, let λ be a limit ordinal; we want to show that $\kappa^{(\lambda)} = \lim_{\alpha\to\lambda} \kappa^{(\alpha)}$. If $\gamma < \kappa^{(\lambda)}$, then $\gamma = i_{\alpha,\lambda}(\delta)$ for some $\alpha < \lambda$ and $\delta < \kappa^{(\alpha)}$. Hence $\gamma = \delta$ and so $\gamma < \kappa^{(\alpha)}$. □

Lemma 19.10. *Let* D *be a normal measure on* κ, *and let for each* α, $\mathrm{Ult}^{(\alpha)}$ *be the* α*th iterated ultrapower* mod D, $\kappa^{(\alpha)} = i_{0,\alpha}(\kappa)$, *and* $D^{(\alpha)} = i_{0,\alpha}(D)$. *Let* λ *be an infinite limit ordinal. Then for each* $X \in \mathrm{Ult}^{(\lambda)}$, $X \subset \kappa^{(\lambda)}$,

$$X \in D^{(\lambda)} \quad \textit{if and only if} \quad (\exists \alpha < \lambda)\, X \supset \{\kappa^{(\gamma)} : \alpha \leq \gamma < \lambda\}. \tag{19.5}$$

Proof. Since for no X can both X and its complement contain a final segment of the sequence $\langle \kappa^{(\gamma)} : \gamma < \lambda \rangle$, it suffices to show that if $X \in D^{(\lambda)}$, then there is an α such that $\kappa^{(\gamma)} \in X$ for all $\gamma \geq \alpha$.

There exists an $\alpha < \lambda$ such that $X = i_{\alpha,\lambda}(Y)$ for some $Y \in D^{(\alpha)}$. Let us show that $\kappa^{(\gamma)} \in X$ for all γ, $\alpha \leq \gamma < \lambda$. Let $\gamma \geq \alpha$ and let $Z = i_{\alpha,\gamma}(Y)$. Then $Z \in D^{(\gamma)}$ and since $D^{(\gamma)}$ is a normal measure on $\kappa^{(\gamma)}$ in $\mathrm{Ult}^{(\gamma)}$, we have $\kappa^{(\gamma)} \in i_{\gamma,\gamma+1}(Z)$. However, $i_{\gamma,\gamma+1}(Z) \subset i_{\gamma+1,\lambda}(i_{\gamma,\gamma+1}(Z)) = X$ and hence $\kappa^{(\gamma)} \in X$. □

Representation of Iterated Ultrapowers

We shall now give an alternative description of each of the models $\mathrm{Ult}^{(\alpha)}$ by means of a single ultrapower of the universe, using an ultrafilter on a certain Boolean algebra of subsets of κ^α. This will enable us to obtain more precise information about the embeddings $i_{0,\alpha} : V \to \mathrm{Ult}^{(\alpha)}$.

We shall deal first with the finite iterations. Let U be a κ-complete nonprincipal ultrafilter on κ. Let us use the symbol $\forall^* \alpha$ for "almost all $\alpha < \kappa$:"

$$\forall^* \alpha \, \varphi(\alpha) \quad \text{if and only if} \quad \{\alpha < \kappa : \varphi(\alpha)\} \in U.$$

If $X \subset \kappa^n$ and $\alpha < \kappa$, let

$$X_{(\alpha)} = \{\langle \alpha_1, \ldots, \alpha_{n-1} \rangle : \langle \alpha, \alpha_1, \ldots, \alpha_{n-1} \rangle \in X\}.$$

We define ultrafilters U_n on κ^n, by induction on n:

$$\begin{aligned} U_1 &= U, \\ U_{n+1} &= \{X \subset \kappa^{n+1} : \forall^* \alpha \, X_{(\alpha)} \in U_n\}. \end{aligned}$$

Each U_n is a nonprincipal κ-complete ultrafilter on κ^n, and if $Z \in U$, then $Z^n \in U_n$. It is easy to see that for all $X \subset \kappa^n$,

$$X \in U_n \quad \text{if and only if} \quad \forall^* \alpha_0 \, \forall^* \alpha_1 \, \ldots \, \forall^* \alpha_{n-1} \, \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in X.$$

Note that U_n concentrates on *increasing* n-sequences:

$$\{\langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in \kappa^n : \alpha_0 < \ldots < \alpha_{n-1}\} \in U_n$$

(because $\forall \alpha_0 \, (\forall \alpha_1 > \alpha_0) \ldots (\forall \alpha_{n-1} > \alpha_{n-2}) \, \alpha_0 < \ldots < \alpha_{n-1}$).

Lemma 19.11. *For every n,*

$$\mathrm{Ult}_{U_n}(V) = \mathrm{Ult}^{(n)}(V)$$

and $j_{U_n} = i_{0,n}$.

Here j_{U_n} is the canonical embedding $j : V \to \mathrm{Ult}_{U_n}(V)$.

Proof. By induction on n. The case $n = 1$ is trivial. Let us assume that the lemma is true for n and let us consider $\mathrm{Ult}_{U_{n+1}}$. Let f be a function on κ^{n+1}. For each $t = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in \kappa^n$, let $f_{(t)}$ be the function on κ defined by $f_{(t)}(\xi) = f(\alpha_0, \ldots, \alpha_{n-1}, \xi)$ and let F be a function on κ^n such that $F(t) = f_{(t)}$ for all $t \in \kappa^n$. In $\mathrm{Ult}_{U_n} = \mathrm{Ult}^{(n)}$, the function F represents a function on $j_{U_n}(\kappa) = \kappa^{(n)}$: Let $\tilde{f} = [F]_{U_n}$. This way we assign to each function f on κ^{n+1} a function $\tilde{f} \in \mathrm{Ult}^{(n)}$ on $\kappa^{(n)}$.

Conversely, if $h \in \mathrm{Ult}^{(n)}$ is a function on $\kappa^{(n)}$, there is an f on κ^{n+1} such that $h = \tilde{f}$: There exists some F on κ^n such that $h = [F]_{U_n}$ and that for each $t \in \kappa^n$, $F(t)$ is a function on κ; thus we let $f(\alpha_0, \ldots, \alpha_n)$ be the value of $F(\alpha_0, \ldots, \alpha_{n-1})$ at α_n.

We shall show that the correspondence $[f]_{U_{n+1}} \mapsto [\tilde{f}]_{U^{(n)}}$ is an isomorphism between $\mathrm{Ult}_{U_{n+1}}$ and $\mathrm{Ult}^{(n+1)} = \mathrm{Ult}_{U^{(n)}}(\mathrm{Ult}^{(n)})$. We have

$$\begin{aligned}[f]_{U_{n+1}} = [g]_{U_{n+1}} &\leftrightarrow \forall^* \alpha_0 \ \ldots \ \forall^* \alpha_{n-1} \ \forall^* \xi \ f(\alpha_0, \ldots, \alpha_{n-1}, \xi) = g(\alpha_0, \ldots, \\ &\quad \alpha_{n-1}, \xi) \\ &\leftrightarrow \forall^* t \, \{\xi < \kappa : f_{(t)}(\xi) = g_{(t)}(\xi)\} \in U \\ &\leftrightarrow \mathrm{Ult}_{U_n} \vDash \{\xi < j_{U_n}(\kappa) : \tilde{f}(\xi) = \tilde{g}(\xi)\} \in j_{U_n}(U) \\ &\leftrightarrow \mathrm{Ult}^{(n)} \vDash \{\xi < \kappa^{(n)} : \tilde{f}(\xi) = \tilde{g}(\xi)\} \in U^{(n)} \\ &\leftrightarrow [\tilde{f}]_{U^{(n)}} = [\tilde{g}]_{U^{(n)}}\end{aligned}$$

and similarly for $\in$ in place of $=$.

Thus $\mathrm{Ult}_{U_{n+1}} = \mathrm{Ult}^{(n+1)}$. To show that $j_{U_{n+1}} = i_{0,n+1}$, let $f = c_x$ be the constant function on κ^{n+1} with value x. It follows that $\tilde{f}$ is the constant function on $\kappa^{(n)}$ with value $i_{0,n}(x)$, and therefore

$$j_{U_{n+1}}(x) = [c_x]_{U_{n+1}} = i_{n,n+1}(i_{0,n}(x)) = i_{0,n+1}(x). \qquad \square$$

The infinite iterations are described with the help of ultrafilters U_E on κ^E, where E ranges over finite sets of ordinal numbers. If E is a finite set of ordinals, then the order isomorphism π between $n = |E|$ and E induces, in a natural way, an ultrafilter U_E corresponding to U_n:

$$U_E = \{\pi(X) : X \subset \kappa^n\}$$

where $\pi(\langle \alpha_0, \ldots, \alpha_{n-1} \rangle) = t \in \kappa^E$ with $t(\pi(k)) = \alpha_k$ for all $k = 0, \ldots, n-1$.

If S is any set of ordinals and $E \subset S$ is a finite set, we define a mapping $\mathrm{in}_{E,S}$ (an *inclusion map*) of $P(\kappa^E)$ into $P(\kappa^S)$ as follows:

$$\mathrm{in}_{E,S}(X) = \{t \in \kappa^S : t{\restriction}E \in X\} \qquad (\text{all } X \subset \kappa^E).$$

Lemma 19.12. *If $E \subset F$ are finite sets of ordinals, then for each $X \subset \kappa^E$,*

$$X \in U_E \quad \textit{if and only if} \quad \mathrm{in}_{E,F}(X) \in U_F.$$

Proof. By induction on (m, n) where $m = |E|$ and $n = |F|$. Let $E \subset F$ be finite sets of ordinals. Let a be the least element of F, and let us assume that $a \in E$ (if $a \notin E$, then the proof is similar). Let $E' = E - \{a\}$ and $F' = F - \{a\}$.

If $X \subset \kappa^E$, let us define for each $\alpha < \kappa$, the set $X_{(\alpha)} \subset \kappa^{E'}$ as follows: $X_{(\alpha)} = \{t{\restriction}E' : t \in X \text{ and } t(a) = \alpha\}$; for $Z \subset \kappa^F$, let us define $Z_{(\alpha)} \subset \kappa^{F'}$ similarly (for all $\alpha < \kappa$). It should be clear that

$$X \in U_E \leftrightarrow \forall^* \alpha \, X_{(\alpha)} \in U_{E'} \quad \text{and} \quad Z \in U_F \leftrightarrow \forall^* \alpha \, Z_{(\alpha)} \in U_{F'}. \tag{19.6}$$

Now we observe that if $Z = \mathrm{in}_{E,F}(X)$, then $Z_{(\alpha)} = \mathrm{in}_{E',F'}(X_{(\alpha)})$, and the lemma for E, F follows from (19.6) and the induction hypothesis. $\square$

Let us now consider an ordinal number α. If $E \subset \alpha$ is a finite set, let us say that a set $Z \subset \kappa^\alpha$ has *support* E if $Z = \mathrm{in}_{E,\alpha}(X)$ for some $X \subset \kappa^E$. Note that if Z has support E and $E \subset F$, then Z also has support F. Let B_α denote the collection of all subsets of κ^α that have finite support. $(B_\alpha, \subset)$ is a Boolean algebra.

Let U_α be the following ultrafilter on B_α: For each $Z \in B_\alpha$, if $Z = \mathrm{in}_{E,\alpha}(X)$ where $X \subset \kappa^E$, let

$$Z \in U_\alpha \quad \text{if and only if} \quad X \in U_E.$$

By Lemma 19.12, the definition of U_α does not depend on the choice of support E of Z.

We shall now construct an ultrapower mod U_α. If f is a function on κ^α, let us say that f has a *finite support* $E \subset \alpha$ if $f(t) = f(s)$ whenever $t, s \in \kappa^\alpha$ are such that $t{\restriction}E = s{\restriction}E$. In other words, there is g on κ^E such that $f(t) = g(t{\restriction}E)$ for all $t \in \kappa^\alpha$. Let us consider only functions f on κ^α with finite support and define

$$\begin{aligned} f =_\alpha g \quad &\text{if and only if} \quad \{t : f(t) = g(t)\} \in U_\alpha, \\ f\ E_\alpha\ g \quad &\text{if and only if} \quad \{t : f(t) \in g(t)\} \in U_\alpha. \end{aligned} \tag{19.7}$$

The sets on the right-hand side of (19.7) have finite support, namely $E \cup F$ where E and F are, respectively, supports of f and g.

Let $(\mathrm{Ult}_{U_\alpha}(V), E_\alpha)$ be the model whose elements are equivalence classes mod $=_\alpha$ of functions on κ^α with finite support.

We are now in a position to state the main lemma.

Lemma 19.13 (The Representation Lemma). *For every α, the model $(\mathrm{Ult}_{U_\alpha}(V), E_\alpha)$ is (isomorphic to) the αth iterated ultrapower $\mathrm{Ult}_U^{(\alpha)}(V)$, and the canonical embedding $j_{U_\alpha} : V \to \mathrm{Ult}_{U_\alpha}$ is equal to $i_{0,\alpha}$. Moreover, if $\alpha \leq \beta$ and $[f]_{U_\alpha} \in \mathrm{Ult}^{(\alpha)}$, then $i_{\alpha,\beta}([f]_{U_\alpha}) = [g]_{U_\beta}$ where g is defined by $g(t) = f(t{\restriction}\alpha)$ for all $t \in \kappa^\beta$.*

Proof. By induction on α. The induction step from α to $\alpha + 1$ follows closely the proof of Lemma 19.11; thus let us describe only how to assign to $[f]_{U_{\alpha+1}}$ the corresponding $[\tilde{f}]_{U^{(\alpha)}}$ in $\mathrm{Ult}^{(\alpha+1)}$. Let f be a function on $\kappa^{\alpha+1}$ with support $E \cup \{\alpha\}$ where $E \subset \alpha$. For each $t \in \kappa^\alpha$ let $f_{(t)}(\xi) = f(t^\frown\xi)$ for all $\xi < \kappa$, and let F be a function on κ^α (with support E) such that $F(t) = f_{(t)}$ for all $t \in \kappa^\alpha$. Let $\tilde{f} = [F]_{U_\alpha}$; $\tilde{f}$ is in $\mathrm{Ult}^{(\alpha)}$ and is a function on $\kappa^{(\alpha)}$.

When λ is a limit ordinal, a routine verification shows that Ult_{U_λ} is the direct limit of $\{\mathrm{Ult}_{U_\alpha}, i_{\alpha,\beta} : \alpha, \beta < \lambda\}$ and that the embeddings $i_{\alpha,\lambda}$ commute with the $i_{\alpha,\beta}$. □

Uniqueness of the Model $L[D]$

Theorem 19.14 (Kunen).

(i) *If $V = L[D]$ and D is a normal measure on κ, then κ is the only measurable cardinal and D is the only normal measure on κ.*

(ii) *For every ordinal κ, there is at most one $D \subset P(\kappa)$ such that $D \in L[D]$ and*

$$L[D] \vDash D \text{ is a normal measure on } \kappa.$$

(iii) *If $\kappa_1 < \kappa_2$ are ordinals and if D_1, D_2 are such that*

$$L[D_i] \vDash D_i \text{ is a normal measure on } \kappa_i \qquad (i = 1,\ 2)$$

then $L[D_2]$ is an iterated ultrapower of $L[D_1]$; i.e., there is α such that $L[D_2] = \mathrm{Ult}_{D_1}^{(\alpha)}(L[D_1])$, and $D_2 = i_{0,\alpha}(D_1)$.

The proof of Theorem 19.14 uses iterated ultrapowers. The following lemma uses the representation of iterated ultrapowers.

Lemma 19.15. *Let U be a κ-complete nonprincipal ultrafilter on κ and let, for each α, $i_{0,\alpha} : V \to \mathrm{Ult}^{(\alpha)}$ be the embedding of V in its αth iterated ultrapower.*

(i) *If α is a cardinal and $\alpha > 2^\kappa$, then $i_{0,\alpha}(\kappa) = \alpha$.*

(ii) *If λ is a strong limit cardinal, $\lambda > \alpha$, and if $\mathrm{cf}\,\lambda > \kappa$, then $i_{0,\alpha}(\lambda) = \lambda$.*

Proof. It follows from the Representation Lemma that for all ξ, η, the ordinals below $i_{0,\xi}(\eta)$ are represented by functions with finite support from κ^ξ into η and hence $|i_{0,\xi}(\eta)| \le |\xi| \cdot |\eta|^\kappa$.

(i) We have $i_{0,\alpha}(\kappa) = \lim_{\xi\to\alpha} i_{0,\xi}(\kappa)$, and for each $\xi < \alpha$, $|i_{0,\xi}(\kappa)| \le |\xi| \cdot 2^\kappa < \alpha$. Hence $i_{0,\kappa}(\kappa) = \alpha$.

(ii) Since $\mathrm{cf}\,\lambda > \kappa$, every function $f : \kappa^\alpha \to \lambda$ with finite support is bounded below λ: There exists $\gamma < \lambda$ such that $f(t) < \gamma$ for all $t \in \kappa^\alpha$. Hence $i_{0,\alpha}(\lambda) = \lim_{\gamma\to\lambda} i_{0,\alpha}(\gamma)$. Since λ is strong limit, we have $|i_{0,\alpha}(\gamma)| < \lambda$ for all $\gamma < \lambda$ and hence $i_{0,\lambda} = \lambda$. □

It is clear from the proof that in (ii) it is enough to assume that $\gamma^\kappa < \lambda$ for all cardinals $\gamma < \lambda$, instead of that λ is a strong limit cardinal.

Let $U \subset P(\kappa)$. If θ is a cardinal and $U \in L_\theta[U]$, then by absoluteness of relative constructibility, every elementary submodel of $(L_\theta[U], \in)$ that contains U and all ordinals $< \kappa$, is isomorphic to $M = L_\alpha[U]$ for some α. (If π is the transitive collapse of the submodel, then $\pi(U) = U \cap M \in M$, and $M = L_\alpha[U]$.) Let θ be a cardinal such that $U \in L_\theta[U]$ and let us consider the model $(L_\theta[U], \in, U)$ where U is regarded as a constant. This model has a definable well-ordering, hence definable Skolem functions, and so we can talk about Skolem hulls of subsets of $L_\theta[U]$.

Lemma 19.16. *Assume that in $L[D]$, D is a normal measure on κ. Let A be a set of ordinals of size at least κ^+ and let θ be a cardinal such that $D \in L_\theta[D]$ and $A \subset L_\theta[D]$. Let $M \prec (L_\theta[D], \in, D)$ be the Skolem hull of $\kappa \cup A$. Then M contains all subsets of κ in $L[D]$.*

For every $X \subset \kappa$ in $L[D]$ there is a Skolem term t such that for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\gamma_1, \ldots, \gamma_m \in A$,

$$L_\theta[D] \vDash X = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, D].$$

Proof. Let π be the transitive collapse of M. We have $\pi(M) = L_\alpha[D]$ for some α, and since $A \subset M$, we have necessarily $\alpha \geq \kappa^+$. By Lemma 19.2, every $X \subset \kappa$ in $L[D]$ is in $L_{\kappa^+}[D]$, and since π is the identity on κ, we have $X \in M$ for all $X \subset \kappa$ in $L[D]$. □

The following is the key lemma in the proof of uniqueness of $L[D]$:

Lemma 19.17. *Let $D \subset P(\kappa)$ be such that $D \in L[D]$ and*

$$L[D] \vDash D \text{ is a normal measure on } \kappa.$$

For each α, let $\mathrm{Ult}_D^{(\alpha)}(L[D])$ denote the αth iterated ultrapower, constructed inside $L[D]$. Let $i_{0,\alpha}$ be the corresponding elementary embedding. Let λ be a regular cardinal greater than κ^+, and let F be the closed unbounded filter on λ. Then:

(i) $i_{0,\lambda}(D) = F \cap \mathrm{Ult}_D^{(\lambda)}(L[D])$;
(ii) $\mathrm{Ult}_D^{(\lambda)}(L[D]) = L[F]$.

Proof. First, we have $i_{0,\lambda}(\kappa) = \lambda$ by Lemma 19.15(i) because $\lambda > \kappa^+ \geq (\kappa^+)^{L[D]} = (2^\kappa)^{L[D]}$. Let $D^{(\lambda)} = i_{0,\lambda}(D)$ and let $M = \mathrm{Ult}_D^{(\lambda)}(L[D])$. If $X \in D^{(\lambda)}$, then by (19.5), X contains a closed unbounded subset and hence $X \in F$. Since $D^{(\lambda)}$ is an ultrafilter in M and F is a filter, it follows that $D^{(\lambda)} = F \cap M$.

As for (ii) we have

$$M = \mathrm{Ult}^{(\lambda)}(L[D]) = L[D^{(\lambda)}] = L[F \cap M] = L[F].$$ □

We shall now prove parts (i) and (ii) of Kunen's Theorem. We already know by Lemma 19.4 that in $L[D]$, κ is the only measurable cardinal. Thus (i) and (ii) follow from this lemma:

Lemma 19.18. *Let $D_1, D_2 \subset P(\kappa)$ be such that $D_1 \in L[D_1]$, $D_2 \in L[D_2]$ and*

$$L[D_i] \vDash D_i \text{ is a normal measure on } \kappa \qquad (i = 1,\ 2).$$

Then $D_1 = D_2$.

Proof. Let $D_1, D_2 \subset P(\kappa)$ be such that $L[D_i] \vDash D_i$ is a normal measure on κ, for $i = 1, 2$; we want to show that $D_1 = D_2$. By symmetry, it suffices to show that if $X \subset \kappa$ is in D_1, then $X \in D_2$.

Let λ be a regular cardinal greater than κ^+ and let F be the closed unbounded filter on λ. Let us consider the λth iterated ultrapowers $M_i = \mathrm{Ult}^{(\lambda)}_{D_i}(L[D_i])$ $(i = 1, 2)$, and the corresponding embeddings $i^1_{0,\lambda}$, $i^2_{0,\lambda}$.

By Lemma 19.17, $M_1 = M_2 = L[F]$, and $i^1_{0,\lambda}(D_1) = i^2_{0,\lambda}(D_2) = F \cap L[F]$. Let $G = F \cap L[F]$.

Let A be a set of ordinals, $|A| = \kappa^+$, such that all $\gamma \in A$ are greater than λ and that $i^1_{0,\lambda}(\gamma) = i^2_{0,\lambda}(\gamma)$ for all $\gamma \in A$; such a set exists by Lemma 19.15(ii). Let θ be a cardinal greater than all $\gamma \in A$ such that $i^1_{0,\lambda}(\theta) = i^2_{0,\lambda}(\theta) = \theta$.

Now let X be a subset of κ such that $X \in D_1$. By Lemma 19.16, X belongs to the Skolem hull of $\kappa \cup A$ in $(L_\theta[D_1], \in, D_1)$. Thus there is a Skolem term t such that for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\gamma_1, \ldots, \gamma_m \in A$,

$$L_\theta[D_1] \vDash X = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, D_1]. \tag{19.8}$$

Let $Y \in L_\theta[D_2]$ be such that

$$L_\theta[D_2] \vDash Y = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, D_2]. \tag{19.9}$$

We shall show that $Y \in D_2$ and $Y = X$, hence $X \in D_2$.

First we observe that $i^1_{0,\lambda}(X) = i^2_{0,\lambda}(Y)$: Let $Z_1 = i^1_{0,\lambda}(X)$ and $Z_2 = i^2_{0,\lambda}(Y)$. We have $i^1_{0,\lambda}(\alpha) = \alpha$, $i^1_{0,\lambda}(\gamma) = \gamma$, $i^1_{0,\lambda}(\theta) = \theta$, and $i^1_{0,\lambda}(D_1) = G$; and thus when we apply $i^1_{0,\lambda}$ to (19.8), we get

$$L_\theta[G] \vDash Z_1 = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, G]. \tag{19.10}$$

Similarly, when we apply $i^2_{0,\lambda}$ to (19.8), we get (19.10) with Z_2 instead of Z_1. Thus $Z_1 = Z_2$.

Now, by Lemma 19.8(ii), we have $X = Z_1 \cap \kappa$ and $Y = Z_2 \cap \kappa$. Hence $X = Y$.

Finally, since $i^2_{0,\lambda}(Y) \in F$, it follows that $i^2_{0,\lambda}(Y) \in i^2_{0,\lambda}(D_2)$ and hence $Y \in D_2$. Thus $X \in D_2$ and this completes the proof of $D_1 = D_2$. □

The key lemma in the proof of Theorem 19.14(iii) is the following:

Lemma 19.19. *Let κ, D be such that $L[D] \vDash D$ is a normal measure on κ, and let γ be an ordinal such that $\kappa < \gamma < i_{0,1}(\kappa)$, where $i_{0,1}$ is the embedding of $L[D]$ in $\mathrm{Ult}_D(L[D])$. Then there is no $U \subset P(\gamma)$ such that $L[U] \vDash U$ is a normal measure on γ.*

Proof. Let us assume that on the contrary there is such a U. Let j be the canonical embedding of $L[U]$ in $\mathrm{Ult}_U(L[U])$. Let $\lambda = |\gamma|^{++}$, and let F be the closed unbounded filter on λ. Let $G = F \cap L[F]$.

Since $L[U] \vDash \mathrm{GCH}$, we have $j(\lambda) = \lambda$ (see the remark following Lemma 19.15). In $L[U]$, G is the λth iterate of U, and in $L[j(U)]$, G is the $j(\lambda)$th iterate of $j(U)$; hence $j(G) = G$.

Let $f : \kappa \to \kappa$ be a function in $L[D]$ such that f represents γ in $\mathrm{Ult}_D(L[D])$. Since D is normal, the diagonal $d(\alpha) = \alpha$ represents κ, and thus we have $(i_{0,1}(f))(\kappa) = \gamma$. Let $i_{0,\lambda}$ be the embedding of $L[D]$ in $\mathrm{Ult}_D^{(\lambda)}(L[D]) = L[G]$. It is clear that $(i_{0,\lambda}(f))(\kappa) = \gamma$.

Now let A be a set of ordinals such that $|A| = \kappa^+$, that all $\xi \in A$ are greater than λ, and that $i_{0,\lambda}(\xi) = \xi$ and $j(\xi) = \xi$ for all $\xi \in A$. Let θ be a cardinal greater than all $\xi \in A$, such that $i_{0,\lambda}(\theta) = \theta$ and $j(\theta) = \theta$.

By Lemma 19.16, the function f is definable in $L_\theta[D]$ from $A \cup \kappa \cup \{D\}$; thus $i_{0,\lambda}(f)$ is definable in $L_\theta[G]$ from $A \cup \kappa \cup \{G\}$. Hence γ is definable in $L_\theta[G]$ from $A \cup \kappa \cup \{G\} \cup \{\kappa\}$, and so there is a Skolem term t such that

$$L_\theta[G] \vDash \gamma = t[\alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_m, G, \kappa]. \tag{19.11}$$

for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\xi_1, \ldots, \xi_m \in A$.

Now we apply the elementary embedding j to (19.11); and since $j(\theta) = \theta$, $j(G) = G$, $j(\xi) = \xi$ for $\xi \in A$, and $j(\alpha) = \alpha$ for all $\alpha < \gamma$ (hence $j(\kappa) = \kappa$), we have

$$L_\theta[G] \vDash j(\gamma) = t[\alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_m, G, \kappa].$$

which is a contradiction because $j(\gamma) > \gamma$. □

Proof of Theorem 19.14(iii). Let $\kappa_1 < \kappa_2$ and let D_1, D_2 be such that $L[D_i] \vDash D$ is a normal measure on κ_i $(i = 1, 2)$. Let $i_{0,\alpha}$ denote the embedding of $L[D_1]$ in $\mathrm{Ult}_{D_1}^{(\alpha)}(L[D_1])$ and let α be the unique α such that $i_{0,\alpha}(\kappa_1) \le \kappa_2 < i_{0,\alpha+1}(\kappa_1)$. By Lemma 19.19 (if we let $\kappa = i_{0,\alpha}(\kappa_1)$, $D = i_{0,\alpha}(D_1)$, and $\gamma = \kappa_2$), it is necessary that $\kappa_2 = i_{0,\alpha}(\kappa_1)$. Now the statement follows from the uniqueness of $i_{0,\alpha}(D_1)$. □

Thus we have proved that the model $V = L[D]$ (where D is a normal measure on κ) is unique, has only one measurable cardinal and only one normal measure on κ, and it satisfies the Generalized Continuum Hypothesis. The next lemma completes the characterization of $L[D]$ by showing that for every κ-complete nonprincipal ultrafilter U on κ, $L[U]$ is equal to $L[D]$.

Lemma 19.20. *Let U be a nonprincipal κ-complete ultrafilter on κ. Then $L[U] = L[D]$ where D is the normal measure on κ in $L[D]$.*

Proof. By the absoluteness of $L[D]$, we have $L[D] \subset L[U]$ because $L[U]$ satisfies that κ is measurable. Thus it suffices to prove that $U \cap L[D] \in L[D]$. Let $j = j_U$ be the canonical embedding $j : V \to \mathrm{Ult}_U(V)$, and let $\gamma = j(\kappa)$. Let $d(\alpha) = \alpha$ be the diagonal function and let δ be the ordinal represented in $\mathrm{Ult}_U(V)$ by d; thus

$$X \in U \quad \text{if and only if} \quad \delta \in j(X) \tag{19.12}$$

for all $X \subset \kappa$.

Since $L[j(D)] \vDash j(D)$ is a normal measure on γ, there exists an α such that $\gamma = i_{0,\alpha}(\kappa)$, $j(D) = i_{0,\alpha}(D)$, and $L[j(D)] = \mathrm{Ult}_D^{(\alpha)}(L[D])$. We shall show that for every $X \subset \kappa$ in $L[D]$,

$$j(X) = i_{0,\alpha}(X). \tag{19.13}$$

This, together with (19.12), gives

$$U \cap L[D] = \{X \in L[D] : X \subset \kappa \text{ and } \delta \in i_{0,\alpha}(X)\} \tag{19.14}$$

and therefore $U \cap L[D] \in L[D]$.

The proof of (19.13) uses Lemma 19.16 again. We let A be a set of size κ^+ of ordinals greater than α such that $i_{0,\alpha}(\xi) = j(\xi) = \xi$ for all $\xi \in A$, and let θ be a cardinal greater than all $\xi \in A$, such that $i_{0,\lambda}(\theta) = j(\theta) = \theta$.

If $X \subset \kappa$ is in $L[D]$, then there is a Skolem term t such that

$$L_\theta[D] \vDash X = t[\alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_m, D].$$

for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\xi_1, \ldots, \xi_m \in A$. Since $i_{0,\alpha}$ and j agree on $\kappa \cup A \cup \{\theta\}$, and $i_{0,\alpha}(D) = j(D)$, it follows that $i_{0,\alpha}(X) = j(X)$. □

The proof of Lemma 19.20 gives additional information about κ-complete ultrafilters in $L[D]$. Let us assume that $V = L[D]$ and let U be a nonprincipal κ-complete ultrafilter on κ. By (19.14), we have

$$U = \{X \subset \kappa : \delta \in i_{0,\alpha}(X)\} \tag{19.15}$$

where α is such that $j(\kappa) = i_{0,\alpha}(\kappa)$, and $\delta < j(\kappa)$. Note that for any $\beta \geq \alpha$, we also have $U = \{X \subset \kappa : \delta \in i_{0,\beta}(X)\}$. Now a simple observation gives the following characterization of κ-complete ultrafilters on κ in $L[D]$:

Lemma 19.21. *Assume $V = L[D]$. If U is a nonprincipal κ-ultrafilter on κ, then there exists some $\delta < i_{0,\omega}(\kappa)$ such that*

$$U = \{X \subset \kappa : \delta \in i_{0,\omega}(X)\}.$$

Proof. Let $j = j_U$ be the canonical embedding of $V = L[D]$ in Ult_U. We have $j(\kappa) = i_{0,\alpha}(\kappa)$ for some α. We shall show that α is a finite number; then the lemma follows by (19.15).

First we note that because $V = L[D] = L[U]$, we have $\mathrm{Ult}_D^{(\alpha)} = \mathrm{Ult}_U = L[i_{0,\alpha}(D)] = L[j(U)]$. Now if $\alpha \geq \omega$, then in $\mathrm{Ult}_D^{(\alpha)}$, $i_{0,\omega}(\kappa)$ is an inaccessible cardinal (because it is measurable in $\mathrm{Ult}_D^{(\omega)}$), while in Ult_U, $i_{0,\omega}(\kappa)$ has cofinality ω (because it has cofinality ω in V and Ult_U contains all ω-sequences of ordinals). Hence $\alpha < \omega$. □

Corollary 19.22. *If $V = L[D]$, there are exactly κ^+ nonprincipal κ-complete ultrafilters on κ.*

Proof. If κ is measurable, then it is easy to obtain 2^κ nonprincipal κ-complete ultrafilters on κ (because there are 2^κ subsets of κ of size κ such that $|X \cap Y| < \kappa$ for any two of them). By Lemma 19.21, if $V = L[D]$, there are at most $|i_{0,\omega}(\kappa)| = \kappa^+$ of them. □

Indiscernibles for $L[D]$

If there exist two measurable cardinals, $\kappa < \lambda$, then it is possible to prove analogous theorems for the model $L[D]$ as we did in Chapter 18 for L under the assumption of one measurable cardinal. More specifically, one can prove the existence of a closed unbounded set $I \subset \kappa$ and a closed unbounded class J of ordinals bigger than κ, such that $I \cup J$ contains all uncountable cardinals except κ, that every $X \in L[D]$ is definable in D from $I \cup J$, and that the elements of $I \cup J$ are indiscernibles for $L[D]$ in the following sense: The truth value of

$$L[D] \vDash \varphi[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m]$$

is independent of the choice of $\alpha_1 < \ldots < \alpha_n \in I$ and $\beta_1 < \ldots < \beta_m \in J$. In analogy with Silver indiscernibles, the above situations can be described by means of a certain set of formulas $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, which is called $0^\dagger$ (*zero-dagger*).

If $0^\dagger$ exists, then one can prove the consistency of the theory ZFC + "there exists a measurable cardinal;" and hence one cannot prove the relative consistency of "$0^\dagger$ exists" with ZFC + "there exists a measurable cardinal."

We shall not give details of the theory of indiscernibles for $L[D]$. Instead, let us present an argument showing that if there exist two measurable cardinals, $\kappa < \lambda$, then there is a proper class of cardinals that are inaccessible in $L[D]$.

Let U be a normal measure on λ and let for each α, $i_{0,\alpha}$ be the elementary embedding of V in $\mathrm{Ult}_U^{(\alpha)}$; let $i_{\alpha,\beta} : \mathrm{Ult}^{(\alpha)} \to \mathrm{Ult}^{(\beta)}$. Let C be the class of all cardinals α such that $\mathrm{cf}\,\alpha > \lambda$ and $\gamma^\lambda < \alpha$ for all $\gamma < \alpha$. By Lemma 19.15, if $\alpha \in C$, then $i_{0,\alpha}(\kappa) = \alpha$ and $i_{0,\alpha}(\beta) = \beta$ for all $\beta \in C$ greater than α. Hence if $\alpha, \beta \in C$, then $i_{\alpha,\beta}(\alpha) = \beta$ and $i_{\alpha,\beta}(\gamma) = \gamma$ for all $\gamma \in C$ that are greater than β or less than α.

Now if D is a normal measure on κ, then because $\kappa < \lambda$, we have $i_{\alpha,\beta}(D) = D$ for all $\alpha, \beta \in C$. Thus each $i_{\alpha,\beta}$ $(\alpha, \beta \in C)$, restricted to $L[D]$, is an elementary embedding of $L[D]$ in $L[D]$ such that $i_{\alpha,\beta}(\alpha) = \beta$ and $i_{\alpha,\beta}(\gamma) = \gamma$ for all $\gamma \in C$ below α or above β. Using these embeddings $i_{\alpha,\beta}$ (as in the proof of Lemma 18.26), one shows that the elements of C are indiscernibles for the model $L[D]$.

Since some elements of C are regular cardinals, and some are limit cardinals, it follows that all elements of C are inaccessible cardinals in $L[D]$.

In the above argument, it was not necessary that κ be a measurable cardinal, only that κ be measurable in $L[D]$. Thus we have proved:

Lemma 19.23. *Let κ be a measurable cardinal, and assume that*:

(19.16) *For some $\gamma < \kappa$, there exists a $D \subset P(\gamma)$ such that $L[D] \vDash D$ is a normal measure on γ.*

Then there are arbitrarily large successor cardinals that are inaccessible in $L[D]$.

We have proved in Lemma 19.21 that if U is a nonprincipal κ-complete ultrafilter on κ, then $j_U(\kappa) < i_{0,\omega}(\kappa)$, where $i_{0,\omega}$ is the embedding of $L[D]$ in $\mathrm{Ult}_D^{(\omega)}(L[D])$. We can prove a stronger statement:

Lemma 19.24. *If there is a κ-complete nonprincipal ultrafilter U on κ such that $j_U(\kappa) \geq i_{0,\omega}(\kappa)$, then* (19.16) *holds.*

Proof. Let us work in the model $M = \mathrm{Ult}_U(V)$. The cardinal $j(\kappa)$ is measurable while $i_{0,\omega}(\kappa)$ has cofinality ω, and so $i_{0,\omega}(\kappa) < j(\kappa)$. Let F be the collection of all subsets X of $i_{0,\omega}(\kappa)$ such that $X \supset \{i_{0,n}(\kappa) : n \geq n_0\}$ for some n_0. Using Lemma 19.10, we proceed as in the proof of Lemma 19.17 to show that

$$L[F] \vDash F \cap L[F] \text{ is a normal measure on } i_{0,\omega}(\kappa).$$

Thus (19.16) holds in M for $j(\kappa)$. Since j is an elementary embedding, (19.16) holds in V for κ. □

Corollary 19.25. *If κ is a measurable cardinal and $2^\kappa > \kappa^+$, then* (19.16) *holds. Consequently, it is impossible to prove the consistency of "κ is measurable and $2^\kappa > \kappa^+$" relative to* ZFC + "*κ is a measurable cardinal.*"

Proof. On the one hand, $|i_{0,\omega}(\kappa)| = (\kappa^+)^{L[D]} \leq \kappa^+$; on the other hand, if U is any κ-complete ultrafilter on κ, we have $j_U(\kappa) > 2^\kappa > \kappa^+$. □

General Iterations

We shall now describe two generalizations of iterated ultrapowers. The first deals with iteration of ultrapowers of transitive models by ultrafilters that are not necessarily members of the model.

Let M be a transitive model of set theory. In fact, it is not necessary for the theory of iterated ultrapowers to assume that M satisfies all axioms of ZFC. It is enough to assume that M is a model of ZFC^-, set theory without the Power Set Axiom. Thus M can be a set (e.g., $(L_\alpha, \in)$ is a model of ZFC^- when α is a regular uncountable cardinal in L).

Let κ be a cardinal in M, and let U be an M-ultrafilter on κ (Definition 18.21).

Definition 19.26. An M-ultrafilter U on κ is *iterable* if

$$\{\alpha < \kappa : X_\alpha \in U\} \in M \quad \text{whenever} \quad \langle X_\alpha : \alpha < \kappa\rangle \in M. \tag{19.17}$$

We shall consider *normal iterable* M-ultrafilters, i.e., M-ultrafilters that are nonprincipal, κ-complete, normal (as in Definition 18.21) and iterable.

Let U be a normal iterable M-ultrafilter on κ. Using functions in M, we form an ultrapower $\mathrm{Ult}_U(M)$, which may or may not be well-founded. Let $j = j_U$ be the canonical elementary embedding $j : M \to \mathrm{Ult}_U(M)$.

Lemma 19.27. *If* $\mathrm{Ult}_U(M)$ *is well-founded, and* N *is the transitive collapse of the ultrapower, then*

(i) $P^M(\kappa) = P^N(\kappa)$.
(ii) $j\text{“}U$ *is a normal iterable* N*-ultrafilter on* $j(\kappa)$.

Proof. (i) It is a routine verification by induction that $j(\alpha) = \alpha$ for all $\alpha < \kappa$. For every $X \in P^M(\kappa)$, we have $X = j(X) \cap \kappa$, and therefore $X \in P^N(\kappa)$, verifying $P^M(\kappa) \subset P^N(\kappa)$.

If $Y \in P^N(\kappa)$, let $f \in M$ be such that $Y \in [f]_U$. Then $Y \in P^M(\kappa)$ follows (by (19.17)) because for all $\alpha < \kappa$,

$$\alpha \in Y \quad \text{if and only if} \quad \{\xi < \kappa : \alpha \in f(\xi)\} \in U.$$

(ii) Let $W = j\text{“}U$. To verify that N and W satisfy (19.17), let $\langle X_\alpha : \alpha < j(\kappa)\rangle \in N$ be represented in the ultrapower by $f \in M$. We may assume that for each α, $X_\alpha \subset j(\kappa)$, and that $f(\xi) = \langle X^\xi_\eta : \eta < \kappa\rangle$ for each $\xi < \kappa$. By (19.17), we have $\{(\xi, \eta) : X^\xi_\eta \in U\} \in M$. Thus if we define $g(\xi) = \{\eta < \kappa : X^\xi_\eta \in U\}$, we have $g \in M$. Now it is routine to show that $[g]_U = \{\alpha < j(\kappa) : X_\alpha \in W\}$. □

If j is an elementary embedding $j : M \to N$ with critical point κ, and if $P^M(\kappa) = P^N(\kappa)$, then the M-ultrafilter $\{X : \kappa \in j(X)\}$ is iterable; see Exercise 19.8.

Let U be a normal iterable M-ultrafilter on κ. If the ultrapower $\mathrm{Ult}_U(M)$ is well-founded, let M_1 be its transitive collapse, let $j : M \to M_1$ be the canonical elementary embedding, and let $U^{(1)} = j\text{“}U$; $U^{(1)}$ is a normal iterable M_1-ultrafilter on $\kappa^{(1)} = j(\kappa)$. We can now proceed with the iteration as when $M = V$ and $U \in M$, as long as the iterated ultrapowers are well-founded. At limit stages we take direct limits, and use the following lemma that is quite routine to verify:

Lemma 19.28. *Let* α *be a limit ordinal, and let for each* $\beta < \alpha$, $U^{(\beta)}$ *be a normal iterable* M_β*-ultrafilter on* $\kappa^{(\beta)}$, *and assume that the direct limit of* $\{(M_\beta, \in, U^{(\beta)}), i_{\beta,\gamma} : \beta, \gamma < \alpha\}$ *is well-founded. If* $(M_\alpha, \in, U^{(\alpha)})$ *is the transitive direct limit then* $U^{(\alpha)}$ *is a normal iterable* M_α*-ultrafilter on* $\kappa^{(\alpha)} = \lim_{\beta\to\alpha} \kappa^{(\beta)}$. □

The Representation Lemma 19.13 holds true in the present context as well. The M-ultrafilters U_α are defined as before, starting with M-ultrafilters U_n on $P^M(\kappa^n)$:

$$X \in U_{n+1} \quad \text{if and only if} \quad \{\xi < \kappa : X_{(\xi)} \in U_n\} \in U \tag{19.18}$$

where $X_{(\xi)} = \{\langle \xi_1, \ldots, \xi_n\rangle : \langle \xi, \xi_1, \ldots, \xi_n\rangle \in X\}$. By induction on n one proves that each U_n is an iterable M-ultrafilter on κ^n.

To define the ultrafilters U_α and the ultrapowers $\mathrm{Ult}_{U_\alpha}(M)$, we restrict ourselves, as before, to sets $Z \subset \kappa^\alpha$ and functions f on κ^α with finite support,

with the additional restriction imposed by M: If $E = \{\alpha_1, \ldots, \alpha_n\}$ with $\alpha_1 < \ldots < \alpha_n$ is the support of Z or f then the restriction of Z or f to κ^E is such that its isomorph $\bar{Z} \subset \kappa^n$ or $\bar{f} : \kappa^n \to M$ is an element of M.

In general, iterated ultrapowers of M by an M-ultrafilter need not be well-founded. If, however, all countable iterations are well-founded then all iterations are well-founded (Exercise 19.9). An important sufficient condition for well-foundedness of iterated ultrapowers is the following (external σ-completeness):

(19.19) For any $\{X_n\}_{n\in\omega} \subset U$, $\bigcap_{n=0}^{\infty} X_n$ is nonempty

(see Exercise 19.10).

The other generalization deals with iterated ultrapowers of an inner model where each successor step $\alpha + 1$ of the iteration is obtained as an ultrapower of M_α by an arbitrary measure in M_α.

Definition 19.29. An *iterated ultrapower* of an inner model M is a sequence $\langle M_\gamma : \gamma \leq \lambda \rangle$ constructed as follows:

(i) $M_0 = M$.
(ii) $M_{\gamma+1} = \mathrm{Ult}_{U^{(\gamma)}}(M_\gamma)$ where $U^{(\gamma)} \in M_\gamma$ is a κ_γ-complete ultrafilter on κ_γ, and the ultrapower is constructed in M_γ; $i_{\gamma,\gamma+1} : M_\gamma \to M_{\gamma+1}$ is the canonical embedding, and for all $\alpha < \gamma$, $i_{\alpha,\gamma+1} = i_{\gamma,\gamma+1} \circ i_{\alpha,\gamma}$.
(iii) If γ is a limit ordinal, then M_γ is the direct limit of $\{M_\alpha, i_{\alpha,\beta} : \alpha \leq \beta < \gamma\}$.

Theorem 19.30 (Mitchell). *Let M be an inner model of* ZFC. *Every iterated ultrapower of M is well-founded.*

Proof. First we outline the proof of the theorem for $M = V$. The idea is to represent each iterated ultrapower M_γ as an ultrapower by an ultrafilter U_γ. The ultrafilters U_γ are defined by induction on γ. For each γ we define an ordinal function k_γ (that represents κ_γ in the ultraproduct by U_γ), the set D_γ (the domain of k_γ), the algebra P_γ of subsets of D_γ, the class F_γ of functions on D_γ and the ultrafilter U_γ on P_γ.

The domain D_γ of k_γ is the set

$$\{p \in Ord^\gamma : \forall \alpha < \gamma \; p(\alpha) < k_\alpha(p{\restriction}\alpha)\}.$$

The algebra P_γ and the class F_γ are

$$P_\gamma = \{X \subset D_\gamma : X \text{ has finite support}\},$$
$$F_\gamma = \{f \in V^{D_\gamma} : f \text{ has finite support}\}.$$

If γ is a limit ordinal, we let $U_\gamma = \bigcup_{\alpha<\gamma} U_\alpha$. If $\gamma = \alpha + 1$, then assume that M_α is transitive and isomorphic to $\mathrm{Ult}_{U_\alpha}(V)$. Let $k_\alpha \in F_\alpha$ be a function that represents κ_α, and let $g \in F_\alpha$ be a function that represents $U^{(\alpha)}$, in

the ultrapower M_α, i.e., $[k_\alpha]_{U_\alpha} = \kappa_\alpha$, $[g]_{U_\alpha} = U^{(\alpha)}$. Thus for U_α-almost all $p \in D_\alpha$, $g_\alpha(p)$ is an ultrafilter on $k_\alpha(p)$. For $X \in P_{\alpha+1}$ we let

$$X \in U_{\alpha+1} \quad \text{if and only if} \quad \{p \in D_\alpha : X_{(p)} \in g(p)\} \in U_\alpha$$

where $X_{(p)} = \{\xi < k_\alpha(p) : p \cup \{(\alpha, \xi)\} \in X\}$. It is now routine to verify that $\mathrm{Ult}_{U_{\alpha+1}}(V)$ is isomorphic to $\mathrm{Ult}_{U^{(\alpha)}}(M_\alpha)$.

The proof that each $\mathrm{Ult}_{U_\alpha}(V)$ is well-founded uses the argument presented in Exercise 19.10.

Now if M is an arbitrary inner model, and $\langle M_\gamma : \gamma \leq \lambda \rangle$ is an iterated ultrapower that is not necessarily defined inside M, we use an absoluteness argument. We can still use the representation of M_γ by $\mathrm{Ult}_{U_\alpha}(M)$; in this case the functions $p \in Ord^\gamma$, the sets $X \subset D_\gamma$ and the functions $f \in V^{D_\gamma}$ are all assumed to be members of M.

If $E \subset \gamma$ is a finite set, then P_E and F_E denote the subsets of P_γ and F_γ, respectively, of those sets or functions whose support is E. Let U_E be the restriction of U_γ to P_E, and let M_E be the ultrapower of M mod U_E (using functions in F_E). For $E \subset E'$, let $i_{E,E'}$ be the canonical elementary embedding of M_E in $M_{E'}$ and let $i_{E,\gamma}$ be the embedding of M_E in M_γ.

If some iterated ultrapower of M is not well-founded, then, as in Exercise 19.9, one can show that there is a countable λ such that an iterated ultrapower $\langle M_\gamma : \gamma \leq \lambda \rangle$ is not well-founded. Let κ be the supremum of all the κ_γ, $\gamma \leq \lambda$, in this iteration. Let $\{a_n\}_{n<\omega}$ be a decreasing sequence of ordinals in M_λ, and let $E_0 \subset E_1 \subset \ldots \subset E_n \subset \ldots$ be a sequence of finite subsets of λ such that $\bigcup_{n=0}^\infty E_n = \lambda$, and that each E_n is a support for (a function representing) a_n. For each n, let $b_n \in M_{E_n}$ be such that $a_n = i_{E_n,\lambda}(b_n)$. Let η be sufficiently large so that $b_n \in V_\eta^M$ for all n. Thus there exists a sequence $\{(E_n, M_n, b_n)\}_{n=0}^\infty$ such that $E_0 \subset E_1 \subset \ldots \subset E_n \subset \ldots$ are finite subsets of λ, that each M_n is an iterated ultrapower of M indexed by E_n, b_n is an ordinal in $M_n = \mathrm{Ult}_{E_n}(M)$, and for each n, $M_{n+1} \vDash i_{E_n,E_{n+1}}(b_n) > b_{n+1}$.

As each M_n is a finite iteration, it is clear that it is a class in M. Consider, in M, the set of all triples (E, N, b) such that E is a finite subset of λ, N is a finite ultrapower iteration indexed by E and using measures on ordinals $\leq \kappa$, and b is an ordinal in N represented by a function in V_η. Let $(E', N', b') < (E, N, b)$ if $E' \supset E$ and if $N' \vDash i_{E,E'}(b) > b'$. We have established that this relation $<$ is not well-founded (in the universe). Thus by absoluteness of well-foundedness, this relation is not well-founded in M. However, that means that there is an iterated ultrapower constructed in M that is not well-founded, contrary to the result of the first part of this proof. □

The Mitchell Order

Definition 19.31. Let κ be a measurable cardinal. If U_1 and U_2 are normal measures on κ, let

$$U_1 < U_2 \quad \text{if and only if} \quad U_1 \in \mathrm{Ult}_{U_2}(V).$$

The relation $U_1 < U_2$ is called the *Mitchell order*.

The Mitchell order is transitive, and by Lemma 17.9(ii) is irreflexive. Moreover, it is well-founded:

Lemma 19.32. *The Mitchell order is well-founded.*

Proof. Toward a contradiction, let κ be the least measurable cardinal on which the Mitchell order is not well-founded, and let $U_0 > U_1 > \ldots > U_n > \ldots$ be a descending sequence of normal measures on κ. Let $M = \mathrm{Ult}_{U_0}(V)$ and let $j : V \to M$ be the canonical elementary embedding. As $\kappa < j(\kappa)$, and $j(\kappa)$ is the least measurable cardinal in M on which the Mitchell order is not well-founded, we reach a contradiction once we show that $U_1 > U_2 > \ldots > U_n > \ldots$ is a descending sequence in M.

The measures U_n, $n \geq 1$, are in M, and so is the sequence $\{U_n\}_{n=1}^{\infty}$, so we need to verify that $U_{n+1} < U_n$ still holds in M. Since $U_{n+1} \in \mathrm{Ult}_{U_n}(V)$, U_{n+1} is represented in the ultrapower by a function $f = \langle u_\alpha : \alpha < \kappa \rangle$. As $P^M(\kappa) = P(\kappa)$ and $M^\kappa \subset M$, the function f is in M, and represents U_{n+1} in the ultrapower $\mathrm{Ult}_{U_n}^M(M)$. Hence $M \vDash U_{n+1} < U_n$. □

Definition 19.33. If U is a normal measure on κ, let $o(U)$, the *order* of U, denote the rank of U in $<$. Let $o(\kappa)$, the *order* of κ, denote the height of $<$.

Lemma 19.34. *Let o be the function $\langle o(\alpha) : \alpha < \kappa \rangle$. If U is a normal measure on κ then $o(U) = [o]_U$.*

Proof. Clearly, $[o]_U = o^M(\kappa)$ where $M = \mathrm{Ult}_U(V)$. The set $\{U' : U' < U\}$ is the set of all normal measures in M, and since $<$ is absolute for M (see Lemma 19.32), the order of U in V is the order of κ in M. □

Thus $o(U) > 0$ if and only if U-almost all $\alpha < \kappa$ are measurable. If κ is a measurable cardinal of order ≥ 2 then κ has a normal measure that concentrates on measurable cardinals $\alpha < \kappa$. Thus the consistency strength of $o(\kappa) \geq 2$ is more than measurability. Measurable cardinals of higher order provide a hierarchy of large cardinal axioms. A consequence of Lemma 19.34 is that $|o(U)| \leq 2^\kappa$ and therefore $o(\kappa) \leq (2^\kappa)^+$. In particular, if GCH holds, then $o(\kappa) \leq \kappa^{++}$ for every measurable cardinal κ.

There exist canonical inner models for measurable cardinals of higher order, analogous to the model $L[U]$. We shall now outline the theory of these inner models.

The key technical device is the technique of *coiteration*. It is the method used in the proof of Lemma 19.35 below. Let $\mathcal{U}$ be a set of normal measures (on possibly different cardinals). $\mathcal{U}$ is *closed* if for every measure $U \in \mathcal{U}$ on κ, every normal measure on κ in $j_U(\mathcal{U})$ is in $\mathcal{U}$. If $\mathcal{U}$ is a closed set of normal measures and $U, W \in \mathcal{U}$, let $U <_{\mathcal{U}} W$ mean that $U \in j_W(\mathcal{U})$. As $<_{\mathcal{U}}$ is a suborder of the Mitchell order it is well-founded and we define $o^{\mathcal{U}}(U)$

and $o^{\mathcal{U}}(\kappa)$ accordingly. The *length* of $\mathcal{U}$, $l(\mathcal{U})$, is the least ϑ such that $\kappa < \vartheta$ for all κ with $o^{\mathcal{U}}(\kappa) > 0$.

Let M and N be inner models of ZFC, and let $\mathcal{U} \in M$ and $\mathcal{W} \in N$ be closed sets of normal measures in M and N, respectively. We say that $\mathcal{U}$ is an *initial segment* of $\mathcal{W}$ if

(19.20)
(i) $l(\mathcal{U}) \leq l(\mathcal{W})$,
(ii) for every $\alpha < l(\mathcal{U})$, $o^{\mathcal{U}}(\alpha) = o^{\mathcal{W}}(\alpha)$,
(iii) for every $\kappa < l(\mathcal{U})$, if $U \in M$ and $W \in N$ are on κ and $o^{\mathcal{U}}(U) = o^{\mathcal{W}}(W)$, then $U \cap M \cap N = W \cap M \cap N$.

Lemma 19.35. *Let M and N be inner models of* ZFC *and let $\mathcal{U}$ and $\mathcal{W}$ be closed sets of measures in M and N, respectively. Then there exist iterated ultrapowers $i_{0,\lambda} : M \to M_\lambda$ and $j_{0,\lambda} : N \to N_\lambda$, using measures in $\mathcal{U}$ and $\mathcal{W}$, respectively, such that either $i_{0,\lambda}(\mathcal{U})$ is an initial segment of $j_{0,\lambda}(\mathcal{W})$, or vice versa.*

Proof. By induction on γ, we define iterated ultrapowers M_γ and N_γ, and the embeddings $i_{\beta,\gamma} : M_\beta \to M_\gamma$ and $j_{\beta,\gamma} : N_\beta \to N_\gamma$. We let $M_0 = M$ and $N_0 = N$, and if λ is a limit ordinal, M_λ and N_λ are direct limits of $\{M_\gamma, i_{\beta,\gamma} : \beta, \gamma < \lambda\}$ and $\{N_\gamma, j_{\beta,\gamma} : \beta, \gamma < \lambda\}$, respectively.

If at stage γ, $\mathcal{U}_\gamma = i_{0,\gamma}(\mathcal{U})$ and $\mathcal{W}_\gamma = j_{0,\gamma}(\mathcal{W})$ are not initial segments of one another, then there exist ordinals α_γ and δ_γ such that $\alpha_\gamma < l(\mathcal{U}_\gamma)$, $\alpha_\gamma < \mathcal{W}_\gamma$, $\mathcal{U}_\gamma \restriction \alpha_\gamma$ and $\mathcal{W}_\gamma \restriction \alpha_\gamma$ agree on $M_\gamma \cap N_\gamma$, the measures on α_γ of order $< \delta_\gamma$ in $\mathcal{U}_\gamma$ and in $\mathcal{W}_\gamma$ agree on $M_\gamma \cap N_\gamma$, and

(19.21) either (i) $\delta_\gamma = o^{\mathcal{U}_\gamma}(\alpha) < o^{\mathcal{W}_\gamma}(\alpha)$, or
(ii) $\delta_\gamma = o^{\mathcal{W}_\gamma}(\alpha_\gamma) < o^{\mathcal{U}_\gamma}(\alpha_\gamma)$, or
(iii) $\delta_\gamma < o^{\mathcal{W}_\gamma}(\alpha_\gamma)$, $\delta_\gamma < o^{\mathcal{U}_\gamma}(\alpha_\gamma)$ and for some $U_\gamma \in \mathcal{U}_\gamma$ and $W_\gamma \in \mathcal{W}_\gamma$ of order δ_γ there exists an $X_\gamma \in M_\gamma \cap N_\gamma$ such that $X_\gamma \in U_\gamma$ but $X_\gamma \notin W_\gamma$.

If (i) occurs, let $i_{\gamma,\gamma+1}$ be the identity and $j_{\gamma,\gamma+1} : N_\gamma \to N_{\gamma+1} = \mathrm{Ult}_W(N_\gamma)$ where W is any $W \in \mathcal{W}_\gamma$ such that $o^{\mathcal{W}_\gamma}(W) = \delta_\gamma$. Similarly, if (ii) occurs, then $j_{\gamma,\gamma+1}$ is the identity and $M_{\gamma+1}$ is an ultrapower. If (iii) occurs, let $i_{\gamma,\gamma+1} : M_\gamma \to M_{\gamma+1} = \mathrm{Ult}_{U_\gamma}(M_\gamma)$ and $j_{\gamma,\gamma+1} : N_\gamma \to N_{\gamma+1} = \mathrm{Ult}_{W_\gamma}(N_\gamma)$.

Note that if $\beta < \gamma$ then $\alpha_\beta \leq \alpha_\gamma$. Moreover, in cases (i) and (ii) we have $\alpha_{\gamma+1} > \alpha_\gamma$ as $o^{\mathcal{U}_{\gamma+1}}(\alpha_\gamma) = o^{\mathcal{W}_{\gamma+1}}(\alpha_\gamma) = \delta_\gamma$, and the measures of order $< \delta_\gamma$ agree.

We will show that the process eventually stops. Thus assume the contrary.

For every limit ordinal γ, M_γ is a direct limit, and so there exists some $\beta = \beta(\gamma) < \gamma$ such that α_γ is in the range of $i_{\beta,\gamma}$, $\alpha_\gamma = i_{\beta,\gamma}(\alpha)$ for some $\alpha = \alpha(\gamma) < l(\mathcal{U}_\beta)$. There is a stationary class Γ_1 of ordinals such that $\beta(\gamma)$ is the same β for all $\gamma \in \Gamma_1$. Also, there is a stationary class $\Gamma_2 \subset \Gamma_1$ such that $\alpha(\gamma)$ is the same $\alpha < l(\mathcal{U}_\beta)$ for all $\gamma \in \Gamma_2$. It follows that if $\beta < \gamma$ are in Γ_2 then $i_{\beta,\gamma}(\alpha_\beta) = \alpha_\gamma$. Similarly there is a stationary class $\Gamma_3 \subset \Gamma_2$ such

that $j_{\beta,\gamma}(\alpha_\beta) = \alpha_\gamma$ whenever $\beta < \gamma$ are in Γ_3. Continuing in this manner, we find a stationary class $\Gamma \subset \Gamma_3$ such that for all $\beta < \gamma$ in Γ, $i_{\beta,\gamma}(\alpha_\beta, \delta_\beta) = j_{\beta,\gamma}(\alpha_\beta, \delta_\beta) = (\alpha_\gamma, \delta_\gamma)$, and (in Case (iii)) $i_{\beta,\gamma}(X_\beta) = j_{\beta,\gamma}(X_\beta)$.

Let $\beta \in \Gamma$ and assume that (19.21)(i) occurs. Let $\gamma \in \Gamma$ be greater than β. Since $i_{\beta,\beta+1}$ is the identity, and $\text{crit}(U_\xi) = \alpha_\xi \geq \alpha_{\beta+1} > \alpha_\beta$ for all $\xi > \beta$, we have $i_{\beta,\gamma}(\alpha_\beta) = \alpha_\beta$, while $j_{\beta,\gamma}(\alpha_\beta) \geq j_{\beta,\beta+1}(\alpha_\beta) > \alpha_\beta$, contrary to $i_{\beta,\gamma}(\alpha_\beta) = j_{\beta,\gamma}(\alpha_\beta)$. Thus (i) does not occur, and similarly, (ii) leads to a contradiction.

Case (iii) gives a contradiction as follows: Let $\gamma > \beta$ be in Γ. Since $X_\beta \in U_\beta$, we have $\alpha_\beta \in i_{\beta,\gamma}(X_\beta)$, and since $X_\beta \notin W_\beta$, we have $\alpha_\beta \notin j_{\beta,\gamma}(X_\beta)$. This contradicts $i_{\beta,\gamma}(X_\beta) = j_{\beta,\gamma}(X_\beta)$, and therefore the process must eventually stop. □

The Models $L[\mathcal{U}]$

If A_α, $\alpha < \theta$, is a sequence of sets, let us define the model

$$L\langle A_\alpha : \alpha < \theta \rangle \tag{19.22}$$

as the model $L[A]$ where $A = \{(\alpha, X) : X \in A_\alpha\}$. Under this definition, $L\langle A_\alpha : \alpha < \theta\rangle = L[\langle B_\alpha : \alpha < \theta\rangle]$, where $B_\alpha = A_\alpha \cap L\langle A_\alpha : \alpha < \theta\rangle$ for all $\alpha < \theta$.

If κ_α, $\alpha < \theta$, is a sequence of measurable cardinals, and for each α, U_α is a κ_α-complete nonprincipal ultrafilter on κ_α, then in $L\langle U_\alpha : \alpha < \theta\rangle$, each $U_\alpha \cap L\langle U_\alpha : \alpha < \theta\rangle$ is again a κ_α-complete nonprincipal ultrafilter on κ_α.

More generally, let $\mathcal{U}$ be a set of normal measures indexed by pairs of ordinals (α, β) such that $U_{\alpha,\beta}$ is a measure on α. Then $L[\mathcal{U}]$ denotes the model $L\langle U_{\alpha,\beta} : \alpha, \beta\rangle$.

The technique described in the preceding section can be used to generalize many results about the model $L[U]$ to obtain canonical inner models for measurable cardinals of higher order. We shall illustrate the method by constructing a model with exactly two normal measures on a measurable cardinal of order 2.

Definition 19.36. A *canonical inner model* for a measurable cardinal κ of order 2 is a model

$$L[\mathcal{U}] = L\langle U_\alpha, U^0, U^1\rangle_{\alpha \in A} \tag{19.23}$$

such that in $L[\mathcal{U}]$

(19.24)
- (i) U^1 is a normal measure on κ of order 1.
- (ii) U^0 is a normal measure on κ of order 0 and $U^0 < U^1$.
- (iii) $A \in U^1$, each U_α is a normal measure on α of order 0, and $\langle U_\alpha : \alpha \in A\rangle$ represents U^0 in the ultrapower by U^1.

If $o(\kappa) \geq 2$ then a canonical model $L[\mathcal{U}]$ is obtained as follows: Let $A \subset \kappa$ be the set of all measurable cardinals below κ, let U^1 be a normal measure on κ of order 1, let U^0 be a normal measure on κ such that $U^0 < U^1$, and let U_α, $\alpha \in A$, be normal measures such that $[\langle U_\alpha : \alpha \in A\rangle]_{U^1} = U^0$. Then $L\langle U_\alpha, U^0, U^1\rangle_{\alpha\in A}$ is a canonical inner model (with $\mathcal{U} = \langle U_\alpha \cap L[\mathcal{U}], U^0 \cap L[\mathcal{U}], U^1 \cap L[\mathcal{U}]\rangle_{\alpha\in A}$).

The canonical model is unique (for the particular choice of the set A), in the sense that if $\mathcal{W} = \langle W_\alpha, W^0, W^1\rangle_{\alpha\in A}$ is any other sequence that satisfies (19.24), then $L[\mathcal{U}] = L[\mathcal{W}]$. We prove below a more general result.

Theorem 19.37 (Mitchell). *Let $A \subset \kappa$, and let $\mathcal{U} = \langle U_\alpha, U^0, U^1\rangle_{\alpha\in A}$ and $\mathcal{W} = \langle W_\alpha, W^0, W^1\rangle_{\alpha\in A}$ be such that for each $\alpha \in A$, U_α and W_α are normal measures on α of order 0, U^0 and W^0 are normal measures on κ of order 0, and U^1 and W^1 are normal measures on κ of order 1. Then $L[\mathcal{U}] = L[\mathcal{W}]$ and*

$$\begin{aligned} &\text{(i)}\ \ U_\alpha \cap L[\mathcal{U}] = W_\alpha \cap L[\mathcal{W}] \qquad (\text{all } \alpha \in A), \\ &\text{(ii)}\ \ U^\varepsilon \cap L[\mathcal{U}] = W^\varepsilon \cap L[\mathcal{W}] \qquad (\varepsilon = 0,\ 1). \end{aligned} \tag{19.25}$$

Proof. We use Lemma 19.35. Let $\mathcal{D}$ be the following set of measures: The U_α's, the W_α's, U^1, U^0, W^1, W^0, and all the normal measures on κ in $j_{U^1}(\mathcal{U} \cup \mathcal{W})$ and $j_{W^1}(\mathcal{U} \cup \mathcal{W})$ (so that $\mathcal{D}$ is a closed set of measures).

By Lemma 19.35 (applied to $\mathcal{D}$) there exist iterated ultrapowers $i = i_{0,\lambda} : V \to M$ and $j = j_{0,\lambda} : V \to N$ such that $i(\mathcal{D})$ is an initial segment of $j(\mathcal{D})$.

We have $l(i(\mathcal{D})) = i(\kappa) + 1$, and by (19.20), $o^{i(\mathcal{D})}(i(\kappa)) = o^{j(\mathcal{D})}(j(\kappa)) = 2$ and for all $\alpha < i(\kappa)$, $o^{i(\mathcal{D})}(\alpha) = o^{j(\mathcal{D})}(\alpha) = 1$ if $\alpha \in i(A)$ and $o^{i(\mathcal{D})}(\alpha) = o^{j(\mathcal{D})}(\alpha) = 0$ if $\alpha \notin i(A)$. It follows that $i(\kappa) = j(\kappa)$ and $i(A) = j(A)$.

By (19.20)(iii), if $D \in i(\mathcal{D})$ and $E \in j(\mathcal{D})$ are normal measures on some $\alpha \in i(A)$ then $D \cap M \cap N = E \cap M \cap N$; the same is true if $D \in i(\mathcal{D})$ and $E \in j(\mathcal{D})$ are measures on $i(\kappa)$ and $o^{i(\mathcal{D})}(D) = o^{i(\mathcal{D})}(E)$. It follows that $L[i(\mathcal{U})] = L[j(\mathcal{U})] = L[j(\mathcal{W})] = L[i(\mathcal{W})] \subset M \cap N$, $i(U^\varepsilon) \cap L[i(\mathcal{U})] = j(U^\varepsilon) \cap L[i(\mathcal{U})] = j(W^\varepsilon) \cap L[i(\mathcal{U})] = i(W^\varepsilon) \cap L[i(\mathcal{U})]$ ($\varepsilon = 0,\ 1$), and for every $\alpha \in i(A)$, $(i\mathcal{U})_\alpha = (i\mathcal{W})_\alpha$, where $\langle (i\mathcal{U})_\alpha, iU^0, iU^1\rangle_{\alpha\in iA} = i\mathcal{U} = i(\langle U_\alpha, U^0, U^1\rangle_{\alpha\in A})$. (By induction on γ, one shows that $L_\gamma[i(\mathcal{U})] = L_\gamma[i(\mathcal{W})]$).

Now (19.25) follows since i is an elementary embedding, and $i : L[\mathcal{U}] \to L[i(\mathcal{U})]$, $i : L[\mathcal{W}] \to L[i(\mathcal{W})]$. □

The analog of Theorem 19.14(i) for $L[\mathcal{U}]$ is the following:

Theorem 19.38 (Mitchell). *In $L[\mathcal{U}]$, κ and $\alpha \in A$ are the only measurable cardinals, and U_α, U^0 and U^1 are the only normal measures.*

Proof. For every ordinal $\gamma \leq \kappa$, let $\mathcal{U}\restriction\gamma = \langle U_\alpha : \alpha \in A \cap \gamma\rangle$; if $\gamma > \kappa$, $\mathcal{U}\restriction\gamma = \mathcal{U}$. Toward a contradiction, let γ be the least ordinal such that in $L[\mathcal{U}\restriction\gamma]$ there are normal measures other than those in $\mathcal{U}\restriction\gamma$, and let $\mathcal{D} = \mathcal{U}\restriction\gamma$. Let α be the least cardinal in $L[\mathcal{D}]$ that carries a normal measure not in $\mathcal{D}$, and let D be such a measure of least Mitchell order.

If $\alpha \notin A \cup \{\kappa\}$, let $M = L[\mathcal{D}]$, $N = \mathrm{Ult}_D(L[\mathcal{D}]) = L[j_D(\mathcal{D})]$, and apply Lemma 19.35 to M, N and (closed) sets of measures $\mathcal{D}$ and $j_D(\mathcal{D})$. There are iterated ultrapowers $i = i_{0,\lambda} : M \to M_\lambda$ and $j = j_{0,\lambda} : N \to N_\lambda$ such that $i(\mathcal{D})$ is an initial segment of $j(j_D(\mathcal{D}))$ or vice versa. Now because of the choice of $\mathcal{D}$ as a minimal counterexample in $L[\mathcal{D}]$ to the theorem, no proper initial segment of either $i(\mathcal{D})$ or $j(j_D(\mathcal{D}))$ can be a counterexample, and consequently, $M_\lambda = N_\lambda = L[i(\mathcal{D})] = L[j(j_D(\mathcal{D}))]$. As $j_D(\mathcal{D})\restriction(\alpha+1) = \mathcal{D}\restriction(\alpha+1)$, we have $i(\alpha) = j(\alpha) = \alpha$, which contradicts the fact that $D \in M$ but $D \notin N$.

The same argument works if $\alpha \in A$ and $U_\alpha < D$, or if $\alpha = \kappa$ and $U^1 < D$.

If $\alpha \in A$ and $o(D) = 0$, let $U = U_\alpha$; if $\alpha = \kappa$ and $o(D) = \varepsilon$ ($\varepsilon = 0, 1$), let $U = U^\varepsilon$. Let $M = \mathrm{Ult}_U(L[\mathcal{D}]) = L[j_U(\mathcal{D})]$ and $N = \mathrm{Ult}_D(L[\mathcal{D}]) = L[j_D(\mathcal{D})]$. By Lemma 19.35 there are iterated ultrapowers $i : M \to M_\lambda$ and $j : N \to N_\lambda$ such that $i(j_U(\mathcal{D}))$ is an initial segment of $j(j_D(\mathcal{D}))$ or vice versa. Using the minimality argument again, we get $M_\lambda = N_\lambda = L[i(j_U(\mathcal{D}))] = L(\mathcal{E})$ where $\mathcal{E} = i(j_U(\mathcal{D})) = j(j_D(\mathcal{D}))$. Again, $j_U(\mathcal{D})\restriction(\alpha+1) = j_D(\mathcal{D})\restriction(\alpha+1)$, so $i(\alpha) = j(\alpha) = \alpha$.

To reach a contradiction we show that $X \in U$ if and only if $X \in D$, for every $X \subset \alpha$ in $L[\mathcal{D}]$. We proceed as in the proof of Lemma 19.18. If $X \in P^{L[\mathcal{D}]}(\alpha)$ then X is definable in $L[\mathcal{D}]$ from $\mathcal{D}$ and ordinals that are not moved by j_U, j_D, i or j. As in (19.8)–(19.10) it follows that $X = Z \cap \alpha$ where $Z = i(X)$, that $Z = j(Z \cap \alpha) = j(X)$ and that $X \in U$ if and only if $\alpha \in i(X)$ if and only if $\alpha \in j(X)$ if and only if $X \in D$. □

Theorems 19.37 and 19.38 admit a generalization to yield canonical inner models for measurable cardinals of higher order. We shall state the following result without proof:

Theorem 19.39 (Mitchell). *There exists an inner model $L[\mathcal{U}]$ such that*

(i) *for every α, $o^{L[\mathcal{U}]}(\alpha) = o^{\mathcal{U}}(\alpha) = \min\{o(\alpha), (\alpha^{++})^{L[\mathcal{U}]}\}$;*
(ii) *$\mathcal{U} = \langle U_{\alpha,\beta} : \beta < o^{\mathcal{U}}(\alpha)\rangle$;*
(iii) *each $U_{\alpha,\beta}$ is in $L[\mathcal{U}]$ a normal measure of order β;*
(iv) *every normal measure in $L[\mathcal{U}]$ is $U_{\alpha,\beta}$ for some α and β;*
(v) *$L[\mathcal{U}] \models \mathrm{GCH}$.* □

Exercises

19.1. Let κ be a measurable cardinal and $j : V \to M$ be the corresponding elementary embedding. Let $M_0 = V$, $M_1 = M$, and for each $n < \omega$, $M_{n+1} = j(M_n)$ and $i_{n,n+1} = j\restriction M_n$. The direct limit of $\{M_n, i_{n,m} : n, m < \omega\}$ is not well-founded.

[$i_{0,\omega}(\kappa)$, $i_{1,\omega}(\kappa)$, ..., $i_{n,\omega}(\kappa)$, ... is a descending sequence of ordinals in the model.]

19.2. Show that if $m \leq n$, then for each f on κ^m, $i_{m,n}([f]_{U_m}) = [g]_{U_n}$ where g is the function on κ^n defined by $g(\alpha_0, \ldots, \alpha_{n-1}) = f(\alpha_0, \ldots, \alpha_{m-1})$.

19.3. Prove this version of Łoś's Theorem (for functions with finite support): $(\mathrm{Ult}_{U_\alpha}, E_\alpha) \vDash \varphi([f_1], \ldots, [f_n])$ if and only if $\{t : \varphi(f_1(t), \ldots, f_n(t))\} \in U_\alpha$.

Let D_n be the measure on κ^n defined from a normal measure D, and let $\kappa^{(n)} = i_{0,n}(\kappa)$ where $i_{0,n} : V \to \mathrm{Ult}_D^{(n)}$.

19.4. The ordinal $\kappa^{(n-1)}$ is represented in Ult_{D_n} by the function $d_n(\alpha_1, \ldots, \alpha_n) = \alpha_n$.

19.5. $A \in D_n$ if and only if $(\kappa, \kappa^{(1)}, \ldots, \kappa^{(n-1)}) \in j_{D_n}(A)$.

19.6. If $A \in D_n$ then there exists a $B \in D$ such that $[B]^n \subset A$.

[Let $n = 3$. Let $B_1 = \{\alpha_1 : (\alpha_1, \kappa, \kappa^{(1)}) \in i_{0,2}(A)\}$, $B_2 = \{\alpha_2 \in B_1 : (\forall \alpha_1 \in B_1 \cap \alpha_2)\,(\alpha_1, \alpha_2, \kappa) \in i_{0,1}(A)\}$ and $B = B_3 = \{\alpha_3 \in B_2 : (\forall \alpha_2 \in B_2 \cap \alpha_3)(\forall \alpha_1 \in B_2 \cap \alpha_2)\,(\alpha_1, \alpha_2, \alpha_3) \in A\}$.]

Compare with Theorem 10.22.

19.7. Assume $V = L[D]$. If U is a κ-complete nonprincipal ultrafilter on κ and if $U \neq D$, then there is a monotone function $f : \kappa \to \kappa$ such that $\kappa \leq [f]_U < [d]_U$. (Hence U does not extend the closed unbounded filter.)

[U satisfies (19.15) for some δ; if $\delta = \kappa^{(n)}$ for some n, then $U = D$. Let n be such that $\kappa^{(n-1)} < \delta < \kappa^{(n)}$; let $g : \kappa^n \to \kappa$ represents δ in Ult_{D_n}. Let $f(\xi) =$ least α such that $g(\alpha_1, \ldots, \alpha_{n-1}, \alpha) \geq \xi$ for some $\alpha_1 < \ldots < \alpha_{n-1} < \alpha$. The function f is monotone. To show that $[f]_U < [d]_U$, we argue as follows: For almost all (mod D_n) $\alpha_1, \ldots, \alpha_n$, $g(\alpha_1, \ldots, \alpha_n) > \alpha_n$; hence for almost all $\alpha_1, \ldots, \alpha_n$, $f(g(\alpha_1, \ldots, \alpha_n)) < g(\alpha_1, \ldots, \alpha_n)$. Hence $(j_{D_n}(f))(\delta) < \delta$, and hence for almost all ξ (mod U), $f(\xi) < \xi$. Thus $[f]_U < [d]_U$.]

19.8. If M and N are transitive models of ZFC^-, if $j : M \to N$ is an elementary embedding with critical point κ, and if $P^M(\kappa) = P^N(\kappa)$, then $\{X \in P^M(\kappa) : \kappa \in j(X)\}$ is a normal iterable M-ultrafilter.

19.9. If $\mathrm{Ult}_{U_\alpha}(M)$ is well-founded for all $\alpha < \omega_1$, then $\mathrm{Ult}_{U_\alpha}(M)$ is well-founded for all α.

[Assume that $\mathrm{Ult}_{U_\alpha}(M)$ is not well-founded and let $f_0, f_1, \ldots, f_n, \ldots$ constitute a counterexample. Each f_n has a finite support E_n. Let β be the order-type of $\bigcup_{n=0}^\infty E_n$; we have $\beta < \omega_1$. Produce a counterexample in $\mathrm{Ult}_{U_\beta}(M)$.]

19.10. If arbitrary countable intersections of elements of U are nonempty, then $\mathrm{Ult}_{U_\alpha}(M)$ is well-founded for all α.

[Let $f_0, f_1, \ldots, f_n, \ldots$ be a counterexample, let $X_n = \{t \in \kappa^\alpha : f_n(t) \ni f_{n+1}(t)\}$. To reach a contradiction, find $t \in \bigcap_{n=0}^\infty X_n$. Construct t by induction such that for each $\nu < \alpha$ if $\alpha = \nu + \eta$, then $t{\restriction}\nu$ has the property that for all n, $\{s \in \kappa^n : (t{\restriction}\nu)^\frown s \in X_n\} \in U_\eta$: Given $t{\restriction}\nu$, there is $t(\nu)$ such that the condition is satisfied for $t{\restriction}(\nu+1)$. Then $t \in \bigcap_{n=0}^\infty X_n$.]

19.11. Assume that every constructible subset of ω_1 either contains or is disjoint from a closed unbounded set. Let F be the closed unbounded filter on ω_1. Then $D = F \cap L$ is an iterable L-ultrafilter and $\mathrm{Ult}_{D_\alpha}(L)$ is well-founded (and hence equal to L) for all α.

19.12. If $L[\mathcal{U}]$, $\mathcal{U} = \langle U_\alpha, U^0, U^1\rangle_{\alpha \in A}$, is a canonical inner model for a measurable cardinal of order 2, if $B \in U^1$ is a subset of A, and if $\mathcal{W} = \langle W_\alpha, W^0, W^1\rangle_{\alpha \in B}$, $W_\alpha = U_\alpha \cap L[\mathcal{W}]$, $W^\varepsilon = U^\varepsilon \cap L[\mathcal{W}]$, then $L[\mathcal{W}]$ is also a canonical inner model.

19.13. If there exist two different normal measures of order 1 on κ, then there exist canonical inner models $L[\mathcal{U}]$ and $L[\mathcal{W}]$ such that $\mathcal{U} = \langle U_\alpha, U^0, U^1\rangle_{\alpha \in A}$, $\mathcal{W} = \langle W_\beta, W^0, W^1\rangle_{\beta \in B}$ and such that $A = U^1$ and $B = W^1$ are disjoint subsets of κ.

Historical Notes

Most of the results in the first part of Chapter 19 are due to Kunen, who in [1970] developed the method of iterated ultraproducts invented by Gaifman (cf. [1964] and [1974]). Kunen found the representation of iterated ultraproducts (Lemma 19.13) and generalized the construction for M-ultrafilters. Kunen applied the method to obtain the main results of the model $L[D]$ (Theorem 19.14).

Theorem 19.3 (the proof of the GCH in $L[D]$) is due to Silver [1971d].

The description of κ-complete ultrafilters on κ in $L[D]$ (Lemma 19.21) is due to Kunen [1970] and Paris [1969]. Lemma 19.4 was first proved by Solovay. Theorem 19.7 is due to Gaifman; cf. [1974]. The proof of well-foundedness in Exercise 19.10 is due to Kunen. Lemmas 19.20 and 19.24 are results of Kunen [1970]. $0^\dagger$ was formulated by Solovay.

Kunen generalized the basic results on $L[D]$ to the model $L\langle D_\alpha : \alpha < \theta\rangle$ constructed from a sequence of measures (with $\theta <$ the least measurable cardinal in the sequence). Mitchell [1974] and [1983] generalized the theory of $L[D]$ to inner models for sequences of measures. The definition of $o(\kappa)$, Theorem 19.30 (well-foundedness of iterated ultrapowers) as well as the results on $L[\mathcal{U}]$ are all due to Mitchell.

The results in Exercises 19.9, 19.10 and 19.11 are due to Kunen [1970].

Exercise 19.7: Jech [1972/73].

20. Very Large Cardinals

This chapter studies properties of large cardinals that generalize measurability. We are particularly interested in the method of elementary embeddings, and introduce two concepts that have become crucial in the theory of large cardinals: supercompact and Woodin cardinals.

Strongly Compact Cardinals

In Chapter 9 we proved that weakly compact cardinals are inaccessible cardinals satisfying the Weak Compactness Theorem for the infinitary language $\mathcal{L}_{\kappa,\omega}$. If we remove the restriction on the size of sets of sentences in the model theoretic characterization of weakly compact cardinals, we obtain a considerably stronger notion. This notion, *strong compactness*, turns out to be much stronger than measurability.

Strongly compact cardinals can be characterized in several different ways. Let us use, as a definition, the property that is a natural generalization of the Ultrafilter Theorem:

Definition 20.1. An uncountable regular cardinal κ is *strongly compact* if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

Obviously, every strongly compact cardinal κ is measurable, for any ultrafilter on κ that extends the filter $\{X : |\kappa - X| < \kappa\}$ is nonprincipal.

Let us say that the language $\mathcal{L}_{\kappa,\omega}$ (or $\mathcal{L}_{\kappa,\kappa}$) satisfies the *Compactness Theorem* if whenever Σ is a set of sentences of $\mathcal{L}_{\kappa,\omega}$ ($\mathcal{L}_{\kappa,\kappa}$) such that every $S \subset \Sigma$ with $|S| < \kappa$ has a model, then Σ has a model.

Let A be a set of cardinality greater than or equal to κ. For each $x \in P_\kappa(A)$, let $\hat{x} = \{y \in P_\kappa(A) : x \subset y\}$, and let us consider the filter on $P_\kappa(A)$ generated by the sets $\hat{x}$ for all $x \in P_\kappa(A)$; that is, the filter

$$\{X \subset P_\kappa(A) : X \supset \hat{x} \text{ for some } x \in P_\kappa(A)\}. \tag{20.1}$$

If κ is a regular cardinal, then the filter (20.1) is κ-complete. We call U a *fine measure* on $P_\kappa(A)$ if U is a κ-complete ultrafilter on $P_\kappa(A)$ that extends the filter (20.1); i.e., $\hat{x} \in U$ for all $x \in P_\kappa(A)$.

Lemma 20.2. *The following are equivalent, for any regular cardinal* κ:

(i) *For any set* S, *every* κ*-complete filter on* S *can be extended to a* κ*-complete ultrafilter on* S.
(ii) *For any* A *such that* $|A| \geq \kappa$, *there exists a fine measure on* $P_\kappa(A)$.
(iii) *The language* $\mathcal{L}_{\kappa,\omega}$ *satisfies the compactness theorem.*

Proof. (i) $\rightarrow$ (ii) is clear.

(ii) $\rightarrow$ (iii): Let Σ be a set of sentences of $\mathcal{L}_{\kappa,\omega}$ and assume that every $S \subset \Sigma$ of size less than κ has a model, say $\mathfrak{A}_S$. Let U be a fine measure on $P_\kappa(\Sigma)$, and let us consider the ultraproduct $\mathfrak{A} = \mathrm{Ult}_U\{\mathfrak{A}_S : S \in P_\kappa(\Sigma)\}$. It is routine to verify that Łoś's Theorem holds for the language $\mathcal{L}_{\kappa,\omega}$ provided the ultrafilter is κ-complete; in order to prove the induction step for infinitary connective $\bigwedge_{\xi<\alpha}\varphi_\xi$, one uses the κ-completeness of U. Thus we have, for any sentence σ of $\mathcal{L}_{\kappa,\omega}$,

$$\mathfrak{A} \vDash \sigma \quad \text{if and only if} \quad \{S : \mathfrak{A}_S \vDash \sigma\} \in U. \tag{20.2}$$

Now if $\sigma \in \Sigma$, then $\{\sigma\}^\wedge \in U$ and since $\mathfrak{A}_S \vDash \sigma$ whenever $S \ni \sigma$, (20.2) implies that σ holds in $\mathfrak{A}$. Hence $\mathfrak{A}$ is a model of Σ.

(iii) $\rightarrow$ (i): Let S be a set and let F be a κ-complete filter on S. Let us consider the $\mathcal{L}_{\kappa,\omega}$-language which has a unary predicate symbol $\dot{X}$ for each $X \subset S$, and a constant symbol c. Let Σ be the set of $\mathcal{L}_{\kappa,\omega}$ sentences consisting of:

(a) all sentences true in $(S, X)_{X\subset S}$,
(b) $\dot{X}(c)$ for all $X \in F$.

Every set of less than κ sentences in Σ has a model: Take S as the universe, interpret each $\dot{X}$ as X and let c be some element of S that lies in every X whose name is mentioned in the given set of sentences; since F is κ-complete, such c exists.

Hence Σ has a model $\mathfrak{A} = (A, X^{\mathfrak{A}}, c)_{X\subset S}$. Let us define $U \subset P(S)$ as follows:

$$X \in U \quad \text{if and only if} \quad \mathfrak{A} \vDash \dot{X}(c).$$

It is easy to verify that U is a κ-complete ultrafilter and that $U \supset F$: For instance, U is κ-complete because if $\alpha < \kappa$ and $X = \bigcap_{\xi<\alpha} X_\xi$, then $\mathfrak{A}$ satisfies the sentence $\bigwedge_{\xi<\alpha}\dot{X}_\xi(c) \rightarrow \dot{X}(c)$. □

Every strongly compact cardinal is measurable, but not every measurable cardinal is strongly compact (although it is consistent that there is exactly one measurable cardinal which is also strongly compact). We shall show that the existence of strongly compact cardinals is a much stronger assumption than the existence of measurable cardinals. We start with the following theorem:

Theorem 20.3 (Vopěnka-Hrbáček). *If there exists a strongly compact cardinal, then there is no set* A *such that* $V = L[A]$.

Proof. Let us assume that $V = L[A]$ for some set A. Since there is a set of ordinals A' such that $L[A] = L[A']$, we may assume that A is a set of ordinals. Let κ be a strongly compact cardinal, and let $\lambda \geq \kappa$ be a cardinal such that $A \subset \lambda$. There exists a κ-complete ultrafilter U on λ^+ such that $|X| = \lambda^+$ for every $X \in U$ (let U extend the filter $\{X : |\lambda^+ - X| \leq \lambda\}$).

Since U is κ-complete, the ultrapower $\mathrm{Ult}_U(V)$ is well-founded, and thus can be identified with a transitive model M. As usual, if f is a function on λ^+, then $[f]$ denotes the element of M represented by f. Let $j = j_U$ be the elementary embedding of V into M given by U.

Let us now consider another version of ultrapower. Let us consider only those functions on λ^+ that assume at most λ values. For these functions, we still define $f =^* g \pmod U$ and $f \in^* g \pmod U$ in the usual way, and therefore obtain a model of the language of set theory, which we denote $\mathrm{Ult}_U^-(V)$. Łoś's Theorem holds for this version of ultrapower too: If $f, \ldots$ are functions on λ^+ with $|\operatorname{ran}(f)| \leq \lambda$, then

$$\mathrm{Ult}^- \vDash \varphi(f, \ldots) \quad \text{if and only if} \quad \{\alpha : \varphi(f(\alpha), \ldots)\} \in U. \tag{20.3}$$

(Check the induction step for $\exists$.) Hence Ult^- is a model of ZFC, elementarily equivalent to V. Also, since U is κ-complete, Ult^- is well-founded and thus is isomorphic to a transitive model N. Every element of N is represented by a function f on λ^+ such that $|\operatorname{ran}(f)| \leq \lambda$. We denote $[f]^-$ the element of N represented by f. We also define an elementary embedding $i : V \to N$ by $i(x) = [c_x]^-$ where c_x is the constant function on λ^+ with value x.

For every function f on λ^+ with $|\operatorname{ran} f| \leq \lambda$, we let

$$k([f]^-) = [f]. \tag{20.4}$$

It is easy to see that the definition of $k([f]^-)$ does not depend on the choice of f representing $[f]^-$ in N, and that k is an elementary embedding of N into M. In fact, $j = k \circ i$.

If $\gamma < \lambda^+$, then every function from λ^+ into γ has at most λ values, and hence $[f]^- = [f]$ for all $f : \lambda^+ \to \gamma$. If $f : \lambda^+ \to \lambda^+$ has at most λ values, then $f : \lambda^+ \to \gamma$ for some $\gamma < \lambda^+$; it follows that $i(\lambda^+) = \lim_{\gamma \to \lambda^+} i(\gamma)$, and we have $k(\xi) = \xi$ for all $\xi < i(\lambda^+)$.

Similarly, $i(A) = j(A)$, and we have $M = L[j(A)] = L[i(A)] = N$.

Now we reach a contradiction by observing that $j(\lambda^+) > i(\lambda^+)$: Since the diagonal function $d(\alpha) = \alpha$ represents in M an ordinal greater than each $j(\gamma)$, $\gamma < \lambda^+$, we have $j(\lambda^+) > \lim_{\gamma \to \lambda^+} j(\gamma)$. While N thinks that $i(\lambda^+)$ is the successor of $i(\lambda)$, M thinks that $j(\lambda^+)$ is the successor of $j(\lambda)$ (and $j(\lambda) = i(\lambda)$). Thus $M \neq N$, a contradiction. □

The following theorem shows that the consistency strength of strong compactness exceeds the strength of measurability:

Theorem 20.4 (Kunen). *If there exists a strongly compact cardinal then there exists an inner model with two measurable cardinals.*

Kunen proved a stronger version (and the proof can be so modified): For every ordinal ϑ there exists an inner model with ϑ measurable cardinals. This was improved by Mitchell who showed that the existence of a strongly compact cardinal leads to an inner model that has a measurable cardinal κ of Mitchell order κ^{++}.

We begin with a combinatorial lemma:

Lemma 20.5. *Let κ be an inaccessible cardinal. There exists a family $\mathcal{G}$ of functions $g : \kappa \to \kappa$ such that $|\mathcal{G}| = 2^\kappa$, and whenever $\mathcal{H} \subset \mathcal{G}$ is a subfamily of size $< \kappa$ and $\{\beta_g : g \in \mathcal{H}\}$ is any collection of ordinals $< \kappa$, then there exists an α such that $g(\alpha) = \beta_g$ for all $g \in \mathcal{H}$.*

Proof. Let $\mathcal{A}$ be a family of almost disjoint subsets of κ (i.e., $|A| = \kappa$ for each $A \in \mathcal{A}$ and $|A \cap B| < \kappa$ for any distinct $A, B \in \mathcal{A}$), such that $|\mathcal{A}| = 2^\kappa$. For each $A \in \mathcal{A}$, let f_A be a mapping of A onto κ such that for each $\beta < \kappa$, the set $\{a \in A : f_A(a) = \beta\}$ has size κ. Let s_α, $\alpha < \kappa$, enumerate all subsets $s \subset \kappa$ of size $< \kappa$.

For each $A \in \mathcal{A}$, let $g_A : \kappa \to \kappa$ be defined as follows: If $s_\alpha \cap A = \{x\}$, then $g_A(\alpha) = f_A(x)$; $g_A(\alpha) = 0$ otherwise. Let $\mathcal{G} = \{g_A : A \in \mathcal{A}\}$.

If $A \neq B \in \mathcal{A}$, then it is easy to find s_α such that $g_A(\alpha) \neq 0$ and $g_B(\alpha) = 0$; hence $|\mathcal{G}| = 2^\kappa$. If $\mathcal{H} \subset \mathcal{A}$ has size $< \kappa$ and if $\{\beta_A : A \in \mathcal{H}\}$ are given, then for each $A \in \mathcal{H}$ we choose $x_A \in A$ such that $x_A \notin B$ for any other $B \in \mathcal{H}$ and that $f_A(x_A) = \beta_A$. Then if α is such that $s_\alpha = \{x_A : A \in \mathcal{A}\}$, we have $g_A(\alpha) = \beta_A$ for every $A \in \mathcal{H}$. □

Lemma 20.6. *Let κ be a strongly compact cardinal. For every $\delta < (2^\kappa)^+$ there exists a κ-complete ultrafilter U on κ such that $j_U(\kappa) > \delta$.*

Proof. Let $\delta < (2^\kappa)^+$. Let $\mathcal{G}$ be a family of functions $g : \kappa \to \kappa$ of size $|\delta|$ with the property stated in Lemma 20.5; let us enumerate $\mathcal{G} = \{g_\alpha : \alpha \leq \delta\}$.

For any $\alpha < \beta \leq \delta$, let $X_{\alpha,\beta} = \{\xi : g_\alpha(\xi) < g_\beta(\xi)\}$. Using the property of $\mathcal{G}$ from Lemma 20.5, we can see that any collection of less than κ of the $X_{\alpha,\beta}$ has a nonempty intersection and hence $F = \{X : X \supset X_{\alpha,\beta}$ for some $\alpha < \beta \leq \delta\}$ is a κ-complete filter on κ. There exists a κ-complete ultrafilter U extending F. It is clear that if $\alpha < \beta \leq \delta$, then $g_\alpha < g_\beta \bmod U$, and hence $j_U(\kappa) > \delta$. □

Combining Lemma 20.6 with Lemmas 19.23 and 19.24, we already have a strong consequence of strong compactness.

We shall apply the technique of iterated ultrapowers to construct an inner model with two measurable cardinals.

Let D be a normal measure on κ, and let $i_{0,\alpha}$ denote, for each α, the elementary embedding $i_{0,\alpha} : V \to \mathrm{Ult}^{(\alpha)}$; let $\kappa^{(\alpha)} = i_{0,\alpha}(\kappa)$ and $D^{(\alpha)} = i_{0,\alpha}(D)$.

First recall (19.5): If λ is a limit ordinal, then $X \in \mathrm{Ult}^{(\lambda)}$ belongs to $D^{(\lambda)}$ if and only if $X \supset \{\kappa^{(\gamma)} : \alpha \leq \gamma < \lambda\}$ for some $\alpha < \lambda$. Let

$$C = \{\nu : \nu \text{ is a strong limit cardinal, } \nu > 2^\kappa, \text{ and } \mathrm{cf}\,\nu > \kappa\}. \tag{20.5}$$

By Lemma 19.15, if $\nu \in C$ then $\kappa^{(\nu)} = \nu$, and $i_{0,\alpha}(\nu) = \nu$ for all $\alpha < \nu$. Thus if $\gamma_0 < \gamma_1 < \ldots < \gamma_n < \ldots$ are elements of the class C, and if $\lambda = \lim_{n\to\infty} \gamma_n$, then $\kappa^{(\lambda)} = \lambda$, and $X \in \mathrm{Ult}^{(\lambda)}$ belongs to $D^{(\lambda)}$ just in case $X \supset \{\gamma_n : n_0 \leq n\}$ for some n_0.

If A is a set of ordinals of order-type ω, $A = \{\gamma_n : n \in \omega\}$, we define a filter $F(A)$ on $\lambda = \sup A$ as follows:

$$X \in F(A) \quad \text{if and only if} \quad \exists n_0\, (\forall n \geq n_0)\, \gamma_n \in X. \tag{20.6}$$

The above discussion leads us to this: If $A \subset C$ has order-type ω, and if $\lambda = \sup A$, then for every $X \in \mathrm{Ult}^{(\lambda)}$, $X \in D^{(\lambda)}$ if and only if $X \in F(A)$. In other words,

$$D^{(\lambda)} = F(A) \cap \mathrm{Ult}^{(\lambda)}. \tag{20.7}$$

Hence $F(A) \cap \mathrm{Ult}^{(\lambda)} \in \mathrm{Ult}^{(\lambda)}$; and so, $L[F(A)] = L[D^{(\lambda)}]$. Thus $F(A) \cap L[F(A)] = D^{(\lambda)} \cap L[D^{(\lambda)}]$, and we have

$$L[F(A)] \vDash F(A) \cap L[F(A)] \text{ is a normal measure on } \lambda. \tag{20.8}$$

The only assumption needed to derive (20.8) is that κ is measurable and A is a subset of the class C. We shall now use Lemma 20.6 and a similar construction to obtain a model with two measurable cardinals.

Suppose that $A = \{\gamma_n : n \in \omega\}$ is as above, and that $A' = \{\gamma'_n : n \in \omega\}$ is another subset of C of order-type ω, such that $\gamma'_0 > \lambda = \sup A$; let $\lambda' = \sup A'$. Let $F = F(A)$ and $F' = F(A')$. Our intention is to choose A and A' such that the model $L[F, F']$ has two measurable cardinals, namely λ and λ', and that $F \cap L[F, F']$ and $F' \cap L[F, F']$ are normal measures on λ and λ', respectively.

The argument leading to (20.8) can again be used to show that $F \cap L[F, F']$ is a normal measure on λ in $L[F, F']$. This is because we have again

$$D^{(\lambda)} = F \cap \mathrm{Ult}^{(\lambda)};$$

moreover, $i_{0,\lambda}(\gamma'_n) = \gamma'_n$ for each n, and hence $i_{0,\lambda}(A') = A'$ and we have

$$i_{0,\lambda}(F') = F' \cap \mathrm{Ult}^{(\lambda)}. \tag{20.9}$$

Therefore

$$L[F, F'] = L[D^{(\lambda)}, F'] = L[D^{(\lambda)}, i_{0,\lambda}(F')]$$

and

$$F \cap L[F, F'] = D^{(\lambda)} \cap L[D^{(\lambda)}, i_{0,\lambda}(F')], \tag{20.10}$$

which gives

$$L[F, F'] \vDash F \cap L[F, F'] \text{ is a normal measure on } \lambda. \tag{20.11}$$

In order to find A, A' so that F' also gives a normal measure in $L[F, F']$, let us make the following observation: Let us think for a moment that $A \subset \kappa$ and

$A' \subset C$. Then $i_{0,\lambda'}(A) = A$ and $D^{(\lambda')} = F' \cap \mathrm{Ult}^{(\lambda')}$, and the same argument as above shows that

$$L[F, F'] \vDash F' \cap L[F, F'] \text{ is a normal measure on } \lambda'. \tag{20.12}$$

We shall use this observation below.

Let us define the following classes of cardinals (compare with (18.29)):

$$\begin{aligned} &C_0 = C, \qquad C_{\alpha+1} = \{\nu \in C_\alpha : |C_\alpha \cap \nu| = \nu\}, \\ &C_\gamma = \bigcap_{\alpha<\gamma} C_\alpha \quad (\gamma \text{ limit}). \end{aligned} \tag{20.13}$$

Each C_α is nonempty; in fact each C_α is unbounded and δ-closed for all δ of cofinality $> \kappa$.

Now we let

$$\begin{aligned} &\gamma_n = \text{the least element of } C_n, \\ &A = \{\gamma_n : n \in \omega\}, \qquad \lambda = \lim_{n\to\infty} \gamma_n \end{aligned} \tag{20.14}$$

and let $A' = \{\gamma'_n : n \in \omega\}$ be a subset of $C_{\omega+1}$.

Let us consider the model $L[A, A']$, and let for each $n \leq \omega$

$$\begin{aligned} M_n &= \text{the Skolem hull of } C_n \text{ in } L[A, A'] \\ &= \text{the class of all } x \in L[A, A'] \text{ such that} \\ &\quad L[A, A'] \vDash x = t[\nu_1, \dots, \nu_k, \gamma_0, \dots, \gamma_k, \gamma'_0, \dots, \gamma'_k, A, A'] \\ &\quad \text{where } t \text{ is a Skolem term and } \nu_1, \dots, \nu_k \in C_n. \end{aligned} \tag{20.15}$$

(Let us not worry about the problem whether (20.15) is expressible in the language of set theory; it can be shown that it is, similarly as in the case of ordinal definable sets. Alternatively, we can consider the model $L_\theta[A, A']$ where θ is some large enough cardinal in $C_{\omega+1}$.)

Each M_n is an elementary submodel of $L[A, A']$; let π_n be the transitive collapse of M_n; then $\pi_n(M_n) = L[\pi_n(A), \pi_n(A')]$ and $j_n = \pi_n^{-1}$ is an elementary embedding

$$j_n : L[\pi_n(A), \pi_n(A')] \to L[A, A'].$$

Lemma 20.7. *For each $n < \omega$, $\pi_n(\gamma_n) < (2^\kappa)^+$.*

Proof. By induction on n. First let $n = 0$. Let $\alpha < \gamma_0$ be in M_0. Then $\alpha = t(\nu_1, \dots, \nu_k, A, A')$ for some Skolem term t and some $\nu_1, \dots, \nu_k \in C_0$. Let $i_{0,\alpha}$ be the elementary embedding into $\mathrm{Ult}_U^{(\alpha)}$ for some U on κ. Since γ_0 is the least element of C_0, we have $\alpha < \nu$ for all $\nu \in C_0$ and hence $i_{0,\alpha}(\nu) = \nu$ for all $\nu \in C_0$. Hence also $i_{0,\alpha}(A) = A$ and $i_{0,\alpha}(A') = A'$ and it follows that $i_{0,\alpha}(\alpha) = \alpha$. Now $i_{0,\alpha}(\alpha) = \alpha$ is possible only if $\alpha < \kappa$. Hence each $\alpha < \gamma_0$ in M_0 is less than κ and therefore $\pi_0(\gamma_0) \leq \kappa$.

Now let us assume that $\pi_n(\gamma_n) < (2^\kappa)^+$ and let us show that $\pi_{n+1}(\gamma_{n+1}) < (2^\kappa)^+$. By Lemma 20.6 there exists a U such that $j_U(\kappa) > \pi_n(\gamma_n)$. We shall show that $\pi_n(\alpha) < j_U(\kappa)$ for all $\alpha < \gamma_{n+1}$ in M_{n+1}; since $\pi_{n+1}(\alpha) \leq \pi_n(\alpha)$ it follows that $\pi_{n+1}(\gamma_{n+1}) = \sup\{\pi_{n+1}(\alpha) : \alpha < \gamma_{n+1} \text{ and } \alpha \in M_{n+1}\} \leq j_U(\kappa) < (2^\kappa)^+$.

First notice that it follows from the definition of C_{n+1} in (20.13) that $\pi_n(\nu) = \nu$ for all $\nu \in C_{n+1}$. Note also that $\gamma_m \in C_{n+1}$ for all $m \geq n+1$, and $A' \subset C_{n+1}$.

Let $\alpha < \gamma_{n+1}$ be in M_{n+1}. Then (in $L[A, A']$),

$$\alpha = t(\gamma_0, \ldots, \gamma_n, \nu_1, \ldots, \nu_k, A, A')$$

where t is some Skolem term and $\nu_1, \ldots, \nu_k \in C_{n+1}$. Hence (in $L[\pi_n(A), \pi_n(A')]$)

$$\pi_n(\alpha) = t(\pi_n(\gamma_0), \ldots, \pi_n(\gamma_n), \nu_1, \ldots, \nu_k, \pi_n(A), A').$$

Now we argue inside the model $\mathrm{Ult}_U(V)$ (which contains both $\pi_n(A)$ and A'): Consider the αth iterated ultrapower (modulo some measure on $j_U(\kappa)$). Since $\pi_n(\gamma_0), \ldots, \pi_n(\gamma_n)$ are all less than $j_U(\kappa)$, we have $i_{0,\alpha}(\pi_n(\gamma_i)) = \pi_n(\gamma_i)$ for all $i = 0, \ldots, n$. We also have $i_{0,\alpha}(\nu) = \nu$ for each $\nu \in C_{n+1}$ (because $\alpha < \nu$ for each $\nu \in C_{n+1}$ and $C_{n+1} \subset C$). It follows that $i_{0,\alpha}(\pi_n(\alpha)) = \pi_n(\alpha)$. Now (because $\pi_n(\alpha) \leq \alpha$) this is only possible if $\pi_n(\alpha) < j_U(\kappa)$. □

We can now complete the proof of Theorem 20.4. Let us consider the model M_ω, the Skolem hull in $L[A, A']$ of C_ω. Let π_ω be the transitive collapse of M_ω and $B = \pi_\omega(A)$. Since $A' \subset C_{\omega+1}$, we have $\pi_\omega(A') = A'$, and $j_\omega = \pi_\omega^{-1}$ is an elementary embedding

$$j_\omega : L[B, A'] \to L[A, A'].$$

By Lemma 20.7, $\pi_\omega(\gamma_n) \leq \pi_n(\gamma_n) < (2^\kappa)^+$ for all n, and hence $\pi_\omega(\lambda) < (2^\kappa)^+$. Let U be a κ-complete ultrafilter on κ such that $j_U(\kappa) > \pi_\omega(\lambda)$.

In Ult_U, B is a subset of $j_U(\kappa)$ and A' is a subset of the class C. Thus we can apply (20.12) and get

$$\mathrm{Ult}_U \vDash (L[F(B), F(A')] \vDash F(A') \cap L[F(B), F(A')] \text{ is a normal measure on } \lambda').$$

Hence

$$L[B, A'] \vDash (L[F(B), F(A')] \vDash F(A') \cap L[F(B), F(A')] \text{ is a normal measure on } \lambda'),$$

and applying j_ω, we get

$$L[A, A'] \vDash (L[F(A), F(A')] \vDash F(A') \cap L[F(A), F(A')] \text{ is a normal measure on } \lambda').$$

Therefore $F' \cap L[F, F']$ is (in $L[F, F']$) a normal measure on λ'. This completes the proof of Theorem 20.4. □

The following theorem provides further evidence of the effect of large cardinals on cardinal arithmetic.

Theorem 20.8 (Solovay). *If κ is a strongly compact cardinal, then the Singular Cardinal Hypothesis holds above κ. That is, if $\lambda > \kappa$ is a singular cardinal, then $2^{\mathrm{cf}\,\lambda} < \lambda$ implies $\lambda^{\mathrm{cf}\,\lambda} = \lambda^+$. (Consequently, if $\lambda > \kappa$ is a singular strong limit cardinal, then $2^\lambda = \lambda^+$.)*

We shall prove the theorem in a sequence of lemmas. An ultrafilter on λ is *uniform* if every set in the ultrafilter has size λ.

Lemma 20.9. *If κ is a strongly compact cardinal and $\lambda > \kappa$ is a regular cardinal, then there exists a κ-complete uniform ultrafilter D on λ with the property that almost all* (mod D) *ordinals $\alpha < \lambda$ have cofinality less than κ.*

Proof. Let U be a fine measure on $P_\kappa(\lambda)$. Since U is fine, every $\alpha < \lambda$ belongs to almost all (mod U) $x \in P_\kappa(\lambda)$. Let us consider the ultrapower $\mathrm{Ult}_U(V)$ and let f be the least ordinal function in Ult_U greater than all the constant functions c_γ, $\gamma < \lambda$:

$$[f] = \lim_{\gamma\to\lambda} j_U(\gamma). \tag{20.16}$$

We note first that $f(x) < \lambda$ for almost all x: Let $g : P_\kappa(\lambda) \to \lambda$ be the function $g(x) = \sup x$. If $\gamma < \lambda$, then $\gamma \le g(x)$ for almost all x and hence $j(\gamma) \le [g]$; thus $[f] \le [g] \le j(\lambda)$.

Let D be the ultrafilter on λ defined as follows:

$$X \in D \quad \text{if and only if} \quad f_{-1}(X) \in U \qquad (X \subset \lambda). \tag{20.17}$$

It is clear that D is κ-complete, and since f is greater than the constant function, D is nonprincipal. For the same reason, the diagonal function $d(\alpha) = \alpha$ is greater (in Ult_D) than all the constant functions c_γ, $\gamma < \lambda$, and since λ is regular, D is uniform. In order to show that almost all (mod D) $\alpha < \lambda$ have cofinality $< \kappa$, it suffices by (20.17), to show that $\mathrm{cf}(f(x)) < \kappa$ for almost all x (mod U).

That will follow immediately once we show that for almost all x (mod U),

$$f(x) = \sup\{\alpha \in x : \alpha < f(x)\}. \tag{20.18}$$

We clearly have $\ge$ in (20.18). To prove $\le$, consider the function $h(x) = \sup\{\alpha \in x : \alpha < f(x)\}$. For each $\gamma < \lambda$, γ is in almost every x and hence $\gamma \le h(x)$ almost everywhere. Thus $[h] \ge j_U(\gamma)$ for all $\gamma < \lambda$ and so $f(x) \le h(x)$ almost everywhere. □

Lemma 20.10. *If κ is strongly compact and $\lambda > \kappa$ is a regular cardinal, then there exist a κ-complete nonprincipal ultrafilter D on λ and a collection $\{M_\alpha : \alpha < \lambda\}$ such that*

(20.19) (i) *$|M_\alpha| < \kappa$ for all $\alpha < \lambda$,*
(ii) *for every $\gamma < \lambda$, γ belongs to M_α for almost all α* (mod D).

(An ultrafilter D that has a family $\{M_\alpha : \alpha < \lambda\}$ with property (20.19) is called (κ, λ)-*regular*.)

Proof. Let D be the ultrafilter on λ constructed in Lemma 20.9. It follows from the construction of D that $[d]_D = \lim_{\gamma\to\lambda} j_D(\gamma)$. For almost all α (mod D), there exists an $A_\alpha \subset \alpha$ of size less than κ and cofinal in α. If $\operatorname{cf}\alpha \geq \kappa$, let $A_\alpha = \emptyset$. Let A be the set of ordinals represented in $\operatorname{Ult}_D(V)$ by the function $\langle A_\alpha : \alpha < \lambda\rangle$. The set A is cofinal in the ordinal represented by the diagonal function d; and since $[d] = \lim_{\gamma\to\lambda} j_D(\gamma)$, it follows that for each $\eta < \lambda$ there is $\eta' > \eta$ such that $A \cap \{\xi : j_D(\eta) \leq \xi < j_D(\eta')\}$ is nonempty.

We construct a sequence $\langle \eta_\gamma : \gamma < \lambda\rangle$ of ordinals $< \lambda$ as follows: Let $\eta_0 = 0$ and $\eta_\gamma = \lim_{\delta\to\gamma} \eta_\delta$ if γ is limit; let $\eta_{\gamma+1}$ be some ordinal such that there exists $\xi \in A$ such that $j_D(\eta_\gamma) \leq \xi < j_D(\eta_{\gamma+1})$.

In other words, if we denote I_γ the interval $\{\xi : \eta_\gamma \leq \xi < \eta_{\gamma+1}\}$, then for every γ, the interval I_γ has nonempty intersection with almost every A_α. Thus if we let

$$M_\alpha = \{\gamma < \lambda : I_\gamma \cap A_\alpha \neq \emptyset\}$$

for each $\alpha < \lambda$, then $\{M_\alpha : \alpha < \lambda\}$ has property (20.19)(ii). To see that M_α has property (i) as well, notice that $|A_\alpha| < \kappa$ for all α and that since the I_γ are mutually disjoint, each A_α intersects less than κ of them. □

Lemma 20.11. *If κ is strongly compact and $\lambda > \kappa$ is a regular cardinal, then there exists a collection $\{M_\alpha : \alpha < \lambda\} \subset P_\kappa(\lambda)$ such that*

$$P_\kappa(\lambda) = \bigcup_{\alpha<\lambda} P(M_\alpha). \tag{20.20}$$

Consequently, $\lambda^{<\kappa} = \lambda$.

Proof. Let $\{M_\alpha : \alpha < \lambda\}$ be as in Lemma 20.10. If x is a subset of λ of size less than κ, then by (20.19)(ii) and by κ-completeness of D, $x \subset M_\alpha$ for almost all α. Hence $x \in P(M_\alpha)$ for some $\alpha < \lambda$. This proves (20.20); since κ is inaccessible, it follows that $|P_\kappa(\lambda)| = \lambda$. □

Proof of Theorem 20.8. Let κ be a strongly compact cardinal. If $\lambda > \kappa$ is an arbitrary cardinal, then we have, by Lemma 20.11

$$\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+.$$

In particular, we have $\lambda^{\aleph_0} \leq \lambda^+$ for every $\lambda > \kappa$. This implies that the Singular Cardinal Hypothesis holds for every $\lambda > \kappa$. □

Supercompact Cardinals

We proved in Lemma 20.2 that a strongly compact cardinal κ is characterized by the property that every $P_\kappa(A)$ has a fine measure. If we require the fine

measure to satisfy a normality condition, then we obtain a stronger notion—a *supercompact* cardinal. Ultrapowers by normal measures on $P_\kappa(A)$ induce elementary embeddings that can be used to derive strong consequences of supercompact cardinals. For instance, Theorems 20.3 and 20.4 become almost trivial if the existence of a strongly compact cardinal is replaced by the existence of a supercompact cardinal. It is consistent to assume that a strongly compact cardinal is not supercompact, or that every strongly compact cardinal is supercompact, but it is not known whether supercompact cardinals are consistent relative to strongly compact cardinals.

Definition 20.12. A fine measure U on $P_\kappa(A)$ is *normal* if whenever $f : P_\kappa(A) \to A$ is such that $f(x) \in x$ for almost all x, then f is constant on a set in U. A cardinal κ is *supercompact* if for every A such that $|A| \geq \kappa$, there exists a normal measure on $P_\kappa(A)$.

Let $\lambda \geq \kappa$ be a cardinal and let us consider the ultrapower $\mathrm{Ult}_U(V)$ by a normal measure U on $P_\kappa(\lambda)$; let $j = j_U$ be the corresponding elementary embedding. Clearly, a set $X \subset P_\kappa(\lambda)$ belongs to U if and only if $[d] \in j(X)$, where d, the *diagonal function*, is the function $d(x) = x$.

Lemma 20.13. *If U is a normal measure on $P_\kappa(\lambda)$, then $[d] = \{j(\gamma) : \gamma < \lambda\} = j\text{“}\lambda$, and hence for every $X \subset P_\kappa(\lambda)$,*

$$X \in U \quad \text{if and only if} \quad j\text{“}\lambda \in j(X). \tag{20.21}$$

Proof. On the one hand, if $\gamma < \lambda$, then $\gamma \in x$ for almost all x and hence $j(\gamma) \in [d]$. On the other hand, if $[f] \in [d]$, then $f(x) \in x$ for almost all x and by normality, there is $\gamma < \lambda$ such that $[f] = j(\gamma)$. □

It follows from (20.21) that if f and g are functions on $P_\kappa(\lambda)$, then

$$[f] = [g] \quad \text{if and only if} \quad (jf)(j\text{“}\lambda) = (jg)(j\text{“}\lambda).$$

and

$$[f] \in [g] \quad \text{if and only if} \quad (jf)(j\text{“}\lambda) \in (jg)(j\text{“}\lambda).$$

Consequently,

$$[f] = (jf)(j\text{“}\lambda) \tag{20.22}$$

for every function f on $P_\kappa(\lambda)$.

For each $x \in P_\kappa(\lambda)$, let us denote

$$\begin{aligned} &\kappa_x = x \cap \kappa, \quad \text{and} \\ &\lambda_x = \text{the order-type of } x. \end{aligned} \tag{20.23}$$

Note that the order-type of $j\text{“}\lambda$ is λ and hence by (20.22), λ is represented in the ultrapower by the function $x \mapsto \lambda_x$. Also, since $\lambda_x < \kappa$ for all x, we

have $j(\kappa) > \lambda$. By the κ-completeness of U, we have $j(\gamma) = \gamma$ for all $\gamma < \kappa$; and since κ is moved by j, it follows that $j``\lambda \cap j(\kappa) = \kappa$ and therefore κ is represented by the function $x \mapsto \kappa_x$.

This gives the following characterization of supercompact cardinals:

Lemma 20.14. *Let $\lambda \geq \kappa$. A normal measure on $P_\kappa(\lambda)$ exists if and only if there exists an elementary embedding $j : V \to M$ such that*

(20.24)
(i) $j(\gamma) = \gamma$ *for all* $\gamma < \kappa$;
(ii) $j(\kappa) > \lambda$;
(iii) $M^\lambda \subset M$; *i.e., every sequence* $\langle a_\alpha : \alpha < \lambda \rangle$ *of elements of* M *is a member of* M.

A cardinal κ is called λ-*supercompact* if it satisfies (20.24).

Proof. (a) Let U be a normal measure on $P_\kappa(\lambda)$. We let $M = \mathrm{Ult}_U(V)$ and let j be the canonical elementary embedding $j : V \to \mathrm{Ult}$. We have already proved (i) and (ii). To prove (iii), it suffices to show that whenever $\langle a_\alpha : \alpha < \lambda \rangle$ is such that $a_\alpha \in M$ for all $\alpha < \lambda$, then the set $\{a_\alpha : \alpha < \lambda\}$ belongs to M. Let f_α, $\alpha < \lambda$, be functions representing elements of M: $[f_\alpha] \in M$. We consider the function f on $P_\kappa(\lambda)$ defined as follows: $f(x) = \{f_\alpha(x) : \alpha \in x\}$; we claim that $[f] = \{a_\alpha : \alpha < \lambda\}$.

On the one hand, if $\alpha < \lambda$, then $\alpha \in x$ for almost all x and hence $[f_\alpha] \in [f]$. On the other hand, if $[g] \in [f]$, then for almost all x, $g(x) = f_\alpha(x)$ for some $\alpha \in x$. By normality, there exists some $\gamma < \lambda$ such that $g(x) = f_\gamma(x)$ for almost all x, and hence $[g] = a_\gamma$.

(b) Let $j : V \to M$ be an elementary embedding that satisfies (i), (ii), and (iii). By (iii), the set $\{j(\gamma) : \gamma < \lambda\}$ belongs to M and so the following defines an ultrafilter on $P_\kappa(\lambda)$:

$$X \in U \quad \text{if and only if} \quad j``\lambda \in j(X). \tag{20.25}$$

A standard argument shows that U is a κ-complete ultrafilter. U is a fine measure because for every $\alpha \in \lambda$, $\{x : \alpha \in x\}$ is in U. Finally, U is normal: If $f(x) \in x$ for almost all x, then $(jf)(j``\lambda) \in j``\lambda$. Hence $(jf)(j``\lambda) = j(\gamma)$ for some $\gamma < \lambda$, and so $f(x) = \gamma$ for almost all x. □

We have seen several examples how large cardinals restrict the behavior of the continuum function (e.g., if κ is measurable and $2^\kappa > \kappa^+$, then $2^\alpha > \alpha^+$ for cofinally many $\alpha < \kappa$). This is more so for supercompact cardinals:

Lemma 20.15. *If κ is λ-supercompact and $2^\alpha = \alpha^+$ for every $\alpha < \kappa$, then $2^\alpha = \alpha^+$ for every $\alpha \leq \lambda$.*

Proof. Let $j : V \to M$ witness that κ is λ-supercompact. If $\alpha \leq \lambda$, then because $\lambda < j(\kappa)$ and by elementarity, $(2^\alpha)^M = (\alpha^+)^M$. Now $M^\lambda \subset M$ implies that $P^M(\alpha) = P(\alpha)$ and so $2^\alpha \leq (2^\alpha)^M = (\alpha^+)^M = \alpha^+$. □

See Exercises 20.5–20.7 for a more general statement.

Lemma 20.16. *If κ is supercompact, then there exists a normal measure D on κ such that almost every $\alpha < \kappa$ (mod D) is measurable. In particular, κ is the κth measurable cardinal.*

Proof. Let $\lambda = 2^\kappa$ and let $j : V \to M$ witness the λ-supercompactness of κ. Let D be defined by $D = \{X : \kappa \in j(X)\}$, and let $j_D : V \to \mathrm{Ult}_D$ be the corresponding elementary embedding. Let $k : \mathrm{Ult}_D \to M$ be the elementary embedding defined in Lemma 17.4:

$$k([f]_D) = (jf)(\kappa).$$

Note that $k(\kappa) = \kappa$.

Now, $P(\kappa) \subset M$ and every subset of M of size λ is in M; hence every $U \subset P(\kappa)$ is in M and it follows that in M, κ is a measurable cardinal. Since k is elementary and $k(\kappa) = \kappa$, we have $\mathrm{Ult}_D \vDash \kappa$ is a measurable cardinal, and the lemma follows. □

In contrast to Lemma 20.16, it is consistent that the least strongly compact cardinal is the least measurable. The following lemma and corollary also show that strongly compactness and supercompactness are not equivalent.

Lemma 20.17. *Let κ be a measurable cardinal such that there are κ strongly compact cardinals below κ. Then κ is strongly compact.*

Proof. Let F be a nonprincipal κ-complete ultrafilter on κ such that $C \in F$ where $C = \{\alpha < \kappa : \alpha \text{ is strongly compact}\}$. Let A be such that $|A| \geq \kappa$; we shall show that there is a fine measure on $P_\kappa(A)$.

For each $\alpha \in C$, let U_α be a fine measure on $P_\alpha(A)$, and let us define $U \subset P_\kappa(A)$ as follows:

$$X \in U \quad \text{if and only if} \quad \{\alpha \in C : X \cap P_\alpha(A) \in U_\alpha\} \in F.$$

It is easy to verify that U is a fine measure on $P_\kappa(A)$. □

Corollary 20.18. *If there exists a measurable cardinal that is a limit of strongly compact cardinals, then the least such cardinal is strongly compact but not supercompact.*

Proof. Let κ be the least measurable limit of compact cardinals. By Lemma 20.17, κ is strongly compact. Let us assume that κ is supercompact. Let $\lambda = 2^\kappa$ and let $j : V \to M$ be an elementary embedding such that κ is the least ordinal moved, and that $M^\lambda \subset M$. If $\alpha < \kappa$ is strongly compact, then $M \vDash j(\alpha)$ is strongly compact, but $j(\alpha) = \alpha$ and therefore $M \vDash \kappa$ is a limit of strongly compact cardinals. Since every $U \subset P(\kappa)$ is in M, κ is measurable in M and hence in M, κ is a measurable limit of strongly compact cardinals. This is a contradiction because M thinks that $j(\kappa)$ is the least measurable limit of strongly compact cardinals. □

The assumption of Corollary 20.18 holds if there are extendible cardinals (defined in the next section). There is also a consistency proof showing that not every strongly compact cardinal is supercompact. (And another consistency proof gives a model that has exactly one strongly compact cardinal and the cardinal is supercompact.)

The construction of a normal measure from an elementary embedding in (20.25) yields a commutative diagram analogous to (17.3). Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subset M$, cf. (20.24). Let

$$U = \{X \in P_\kappa(\lambda) : j\text{“}\lambda \in j(X)\}$$

be the normal measure defined from j. Let $\mathrm{Ult} = \mathrm{Ult}_U(V)$, and $j_U : V \to \mathrm{Ult}$.

For each $[f] \in \mathrm{Ult}$, let

$$k([f]) = (jf)(j\text{“}\lambda). \tag{20.26}$$

As in Lemma 17.4, one verifies that $k : \mathrm{Ult} \to M$ is an elementary embedding, and $j = k \circ j_U$.

We claim that

$$k(\alpha) = \alpha \quad \text{for all } \alpha \leq \lambda. \tag{20.27}$$

To prove (20.27), let $\alpha \leq \lambda$, and let us denote, for each $x \in P_\kappa(\lambda)$,

$$\alpha_x = \text{the order-type of } x \cap \alpha$$

(compare with (20.23)). Since the order-type of $j_U\text{“}\lambda \cap j_U(\alpha)$ is α, it follows from (20.22) that the function $f(x) = \alpha_x$ represents α in the ultrapower: $[f] = (j_U f)(j\text{“}\lambda) =$ the order-type of $j_U\text{“}\lambda \cap j_U(\alpha) = \alpha$. Now (20.27) follows:

$$k(\alpha) = k([f]) = (jf)(j\text{“}\lambda) = \text{the order-type of } j\text{“}\lambda \cap j(\alpha) = \alpha.$$

Lemma 20.19.

(i) *If $\lambda \geq \kappa$ and if κ is μ-supercompact, where $\mu = 2^{\lambda^{<\kappa}}$, then for every $\mathcal{X} \subset P(P_\kappa(\lambda))$ there exists a normal measure on $P_\kappa(\lambda)$ such that $\mathcal{X} \in \mathrm{Ult}_U(V)$.*

(ii) *If κ is 2^κ-supercompact, then for every $\mathcal{X} \subset P(\kappa)$ there exists a normal measure D on κ such that $\mathcal{X} \in \mathrm{Ult}_D(V)$.*

Proof. (i) Assume on the contrary that there exists some $\mathcal{X} \subset P(P_\kappa(\lambda))$ such that $\varphi(\mathcal{X}, \kappa, \lambda)$ where φ is the statement

$$\mathcal{X} \notin \mathrm{Ult}_U \text{ for every normal measure } \mathcal{U} \text{ on } P_\kappa(\lambda). \tag{20.28}$$

Let $j : V \to M$ be a witness to the μ-supercompactness of κ. As $M^\mu = M$, the ultrapowers by normal measures on $P_\kappa(\lambda)$ are correctly computed in M, and so $M \vDash \exists \mathcal{X}\, \varphi(\mathcal{X}, \kappa, \lambda)$.

Let $U = \{X \in P_\kappa(\lambda) : j``\lambda \in j(X)\}$ and let $k : \mathrm{Ult}_U \to M$ be such that $j = k \circ j_U$. By (20.27), $k(\kappa) = \kappa$ and $k(\lambda) = \lambda$, and since $k : \mathrm{Ult} \to M$ is elementary, we have $\mathrm{Ult} \vDash \exists \mathcal{X}\, \varphi(\mathcal{X}, \kappa, \lambda)$. Let $\mathcal{X} \in \mathrm{Ult}$ be such that $\mathrm{Ult} \vDash \varphi(\mathcal{X}, \kappa, \lambda)$. By (20.27) again, $k(\alpha) = \alpha$ for all $\alpha \leq \lambda$, and it follows that $k(\mathcal{X}) = \mathcal{X}$. By elementarity again, $M \vDash \varphi(k(\mathcal{X}), k(\kappa), k(\lambda))$ and so $M \vDash \varphi(\mathcal{X}, \kappa, \lambda)$. This contradicts (20.28) because $\mathcal{X} \in \mathrm{Ult}_U$.

(ii) Similar, using (17.3). □

Corollary 20.20.

(i) *If κ is supercompact then there are 2^{2^κ} normal measures on κ.*
(ii) *If κ is supercompact then for every $\lambda \geq \kappa$ there are $2^{2^{\lambda^{<\kappa}}}$ normal measures on $P_\kappa(\lambda)$.*
(iii) *If κ is supercompact then the Mitchell order of κ is $(2^\kappa)^+ \geq \kappa^{++}$.*

Proof. (i) If D is a normal measure on κ and $\mathcal{X} \subset P(\kappa)$ is in Ult_D, then $\mathcal{X}$ is represented by a function f on κ such that $f(\alpha) \subset P(\alpha)$ for all $\alpha < \kappa$. Since the number of such functions is 2^κ, it follows that Ult_D contains only 2^κ subsets of $P(\kappa)$. However, by Lemma 20.19(ii), each $\mathcal{X} \subset P(\kappa)$ is contained in some ultrapower Ult_D where D is a normal measure on κ, and therefore there must exist 2^{2^κ} normal measures on κ.

(ii) Similar, using Lemma 20.19(i).

(iii) There is an increasing chain of length $(2^\kappa)^+$ of normal measures on κ in the Mitchell order: Given at most 2^κ such measures, one can code them as some $\mathcal{X} \subset P(\kappa)$. By Lemma 20.19(ii) there exists a normal measure U on κ such that $\mathcal{X} \in \mathrm{Ult}_U$. □

We conclude this section with the following theorem reminiscent of the Diamond Principle.

Theorem 20.21 (Laver). *Let κ be a supercompact cardinal. There exists a function $f : \kappa \to V_\kappa$ such that for every set x and every $\lambda \geq \kappa$ such that $\lambda \geq |\mathrm{TC}(x)|$ there exists a normal measure U on $P_\kappa(\lambda)$ such that $j_U(f)(\kappa) = x$.*

(Such an f is called a *Laver function.*)

Proof. Assume that the theorem is false. For each $f : \kappa \to V_\kappa$, let λ_f be the least cardinal $\lambda_f \geq \kappa$ for which there exists an x with $|\mathrm{TC}(x)| \leq \lambda_x$ such that $j_U(f)(\kappa) \neq x$ for every normal measure U on $P_\kappa(\lambda_f)$. Let ν be greater than all the λ_f and let $j : V \to M$ be a witness to the ν-supercompactness of κ.

Let $\varphi(g, \delta)$ be the statement that for some cardinal α, $g : \alpha \to V_\alpha$ and δ is the least cardinal $\delta \geq \alpha$ for which there exists an x with $|\mathrm{TC}(x)| \leq \delta$ such that there is no normal measure U on $P_\alpha(\delta)$ with $(j_U g)(\alpha) = x$. (Let λ_g denote this δ.) Since $M^\nu \subset M$, we have $M \vDash \varphi(f, \lambda_f)$, for all $f : \kappa \to V_\kappa$.

Let A be the set of all $\alpha < \kappa$ such that $\varphi(g, \lambda_g)$ holds for all $g : \alpha \to V_\alpha$. Clearly, $\kappa \in j(A)$.

Now we define $f : \kappa \to V_\kappa$ inductively as follows. If $\alpha \in A$, we let $f(\alpha) = x_\alpha$ where x_α witnesses $\varphi(f{\restriction}\alpha, \lambda_{f{\restriction}\alpha})$; otherwise, $f(\alpha) = \emptyset$.

Let $x = (jf)(\kappa)$. It follows from the construction of f that x witnesses $\varphi(f, \lambda_f)$ in M, and hence in V. Let $U = \{X \in P_\kappa(\lambda) : j\text{“}\lambda \in j(X)\}$; we shall reach a contradiction by showing that $(j_U f)(\kappa) = x$. Let $k : \mathrm{Ult}_U \to M$ be the elementary embedding from (20.26) such that $j = k \circ j_U$. By (20.27), $k(x) = x$, and therefore

$$(j_U f)(\kappa) = k^{-1}((jf)(\kappa)) = k^{-1}(x) = x. \qquad \square$$

Beyond Supercompactness

Elementary embeddings can be used to define large cardinals that are stronger than supercompact.

Definition 20.22. A cardinal κ is *extendible* if for every $\alpha > \kappa$ there exist an ordinal β and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ.

Lemma 20.23. *Let $\lambda \geq \kappa$ be a regular cardinal and let κ be λ-supercompact. Let $\alpha < \kappa$. If α is γ-supercompact for all $\gamma < \kappa$, then α is λ-supercompact.*

Proof. Let U be a normal measure on $P_\kappa(\lambda)$, and let us consider $j_U : V \to \mathrm{Ult}_U$. Since $j(\alpha) = \alpha$, we have $\mathrm{Ult} \vDash (\alpha$ is γ-supercompact for all $\gamma < j(\kappa))$; in particular, $\mathrm{Ult} \vDash \alpha$ is λ-supercompact. Hence there is D such that $\mathrm{Ult} \vDash D$ is a normal measure on $P_\alpha(\lambda)$. Now, $|P_\alpha(\lambda)| = \lambda$ and $\mathrm{Ult}^\lambda \subset \mathrm{Ult}$, and hence every subset of $P_\alpha(\lambda)$ is in Ult. It follows that D is a normal measure on $P_\alpha(\lambda)$. $\square$

Theorem 20.24.

(i) *If κ is extendible, then κ is supercompact.*
(ii) *If κ is extendible, then there is a normal measure D on κ such that $\{\alpha < \kappa : \alpha$ is supercompact$\} \in D$.*

Proof. (i) Let $\alpha > \kappa$ be a limit cardinal with the property that if $V_\alpha \vDash (\kappa$ is λ-supercompact for all $\lambda)$, then κ is supercompact. (Such an α exists by the Reflection Principle.) Thus it suffices to show that κ is λ-supercompact for all regular $\lambda < \alpha$.

Let $j : V_\alpha \to V_\beta$ be such that κ is the critical point. Consider the sequence $\kappa_0 = \kappa$, $\kappa_1 = j(\kappa)$, $\ldots$, $\kappa_{n+1} = j(\kappa_n)$, $\ldots$, as long as $j(\kappa_n)$ is defined. First we note that by Exercise 17.8 either there is some n such that $\kappa_n < \alpha \leq j(\kappa_n)$ or $\alpha = \lim_{n\to\infty} \kappa_n$. Therefore, it is sufficient to prove, by induction on n, that κ is λ-supercompact for each regular $\lambda < \kappa_n$ (if $\lambda < \alpha$).

Clearly, κ is λ-supercompact for each $\lambda < \kappa_1$. Thus let $n \geq 1$ and let us assume that κ is λ-supercompact for all $\lambda < \kappa_n$. Applying j, we get: $V_\beta \vDash (j(\kappa)$ is λ-supercompact for all regular $\lambda < \kappa_{n+1})$. Now we also have

$V_\beta \vDash$ (κ is γ-supercompact for all $\gamma < j(\kappa)$) and we can apply Lemma 20.23 (in V_β) to conclude that $V_\beta \vDash$ (κ is λ-supercompact for all regular $\lambda < \kappa_{n+1}$). This completes the induction step.

(ii) Let α be some limit ordinal greater than κ and let $j : V_\alpha \to V_\beta$ be such that κ is the critical point. Let $D = \{X \subset \kappa : \kappa \in j(X)\}$. By (i), κ is supercompact, and so $V_\beta \vDash$ (κ is γ-supercompact for all $\gamma < j(\kappa)$). Hence $A \in D$, where $A = \{\alpha < \kappa : \alpha$ is γ-supercompact for all $\gamma < \kappa\}$. By Lemma 20.23, every $\alpha \in A$ is supercompact. □

Let us consider now the following axiom schema called *Vopěnka's Principle* (VP):

(20.29) Let C be a proper class of models of the same language. Then there exist two members $\mathfrak{A}$, $\mathfrak{B}$ of the class C such that $\mathfrak{A}$ can be elementarily embedded in $\mathfrak{B}$.

Lemma 20.25. *If Vopěnka's Principle holds, then there exists an extendible cardinal.*

Proof. Let A be the class of all limit ordinals α such that $\operatorname{cf}\alpha = \omega$ and that for every $\kappa < \alpha$, if $V_\alpha \vDash$ (κ is extendible), then κ is extendible; and for $\kappa < \gamma < \alpha$, if there is an elementary embedding $j : V_\gamma \to V_\delta$ with critical point κ, then $V_\alpha \vDash$ (there is an elementary embedding). Using the Reflection Principle, we see that A is a proper class. Let C consist of the models $(V_{\alpha+1}, \in)$, for $\alpha \in A$.

By Vopěnka's Principle, there exist $\alpha, \beta \in A$ and an elementary embedding $j : V_{\alpha+1} \to V_{\beta+1}$. Since $j(\alpha) = \beta$, j moves some ordinal; its critical point is measurable and so it is not α (which has cofinality ω). Let κ be the critical point.

Now $V_\alpha \vDash$ (κ is extendible) because for every $\gamma < \alpha$, $j{\restriction}V_\gamma$ reflects to a witness to extendibility. By definition of A, κ is extendible. □

A similar argument shows that Vopěnka's Principle implies existence of arbitrarily large extendible cardinals.

Definition 20.26. A cardinal κ is *huge* if there exists an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(\kappa)} \subset M$.

"Huge" is expressible in ZF: see Exercise 20.11.

Lemma 20.27. *If κ is a huge cardinal, then Vopěnka's Principle is consistent*: $(V_\kappa, \in)$ *is a model of* VP.

Proof. We shall show that if C is a set of models and $\operatorname{rank}(C) = \kappa$, then there exist two members $\mathfrak{A}, \mathfrak{B} \in C$ and an elementary embedding $h : \mathfrak{A} \to \mathfrak{B}$.

Let $j : V \to M$ be such that κ is the least cardinal moved and that $M^{j(\kappa)} \subset M$. Since $\operatorname{rank}(C) = \kappa$, there exists an $\mathfrak{A}_0 \in j(C)$ such that $\mathfrak{A}_0 \notin C$. It follows that $j(\mathfrak{A}_0) \neq \mathfrak{A}_0$.

Let $e_0 = j\restriction\mathfrak{A}_0$; it is easy to see that e_0 is an elementary embedding of $\mathfrak{A}_0$ into $j(\mathfrak{A}_0)$, and since $|\mathfrak{A}_0| < j(\kappa)$, we have $e_0 \in M$. Hence

$$M \vDash \text{there exists an } \mathfrak{A} \in j(C),\ \mathfrak{A} \neq j(\mathfrak{A}_0), \text{ and there exists an elementary } e : \mathfrak{A} \to j(\mathfrak{A}_0);$$

and so there exists some $\mathfrak{A} \in C$, $\mathfrak{A} \neq \mathfrak{A}_0$, and there exists an elementary $e : \mathfrak{A} \to \mathfrak{A}_0$. Let $\mathfrak{A}$, e be such; clearly,

$$M \vDash e \text{ is an elementary embedding of } \mathfrak{A} \text{ into } \mathfrak{A}_0,$$

and because $\operatorname{rank}(\mathfrak{A}) < \kappa$, we have $\mathfrak{A} = j(\mathfrak{A})$, and hence $\mathfrak{A} \in j(C)$, and so

$$M \vDash \text{there exist distinct } \mathfrak{A}, \mathfrak{B} \in j(C), \text{ and there exists an elementary } h : \mathfrak{A} \to \mathfrak{B}.$$

It follows that there exist distinct $\mathfrak{A}, \mathfrak{B} \in C$, and an elementary embedding $h : \mathfrak{A} \to \mathfrak{B}$. □

While the least huge cardinal is greater than the least measurable cardinal (see Exercise 20.13), it is smaller than the least supercompact cardinal (if both exist) even though the consistency of "there exists a huge cardinal" is stronger than the consistency of "there exists a supercompact cardinal." See Exercise 20.12.

Finally, consider the following axiom:

(20.30) There exists a nontrivial elementary embedding $j : V_\lambda \to V_\lambda$ where λ is a limit ordinal.

Let κ be the critical point of an elementary embedding $j : V_\lambda \to V_\lambda$. The necessarily $\lambda \geq \kappa_n$ for each n (where κ_n are as in Theorem 17.7), and it follows from Exercise 17.8 that $\lambda = \lim_{n\to\infty} \kappa_n$. It is easily seen that κ is huge (by Exercise 20.11), in fact n-huge for all n; see Exercise 20.15.

In view of Kunen's Theorem 17.7, axiom (20.30) (and its variants) is the strongest possible large cardinal axiom.

Extenders and Strong Cardinals

In this section we show how elementary embeddings can be analyzed using direct limits of ultrapowers. An elementary embedding can be approximated by a system of measures called *extenders*. The theory of extenders plays a crucial role in the inner model theory. While this theory is too weak to describe supercompactness, it is strong enough to describe a weak version of it that is considerably stronger than measurability.

Definition 20.28. A cardinal κ is a *strong cardinal* if for every set x there exists an elementary embedding $j : V \to M$ with critical point κ such that $x \in M$.

Clearly, every supercompact cardinal is strong. Strong cardinals have more consistency strength than measurable cardinals, and allow some of the techniques associated with supercompactness; see Exercises 20.16–20.19.

It follows from the theory of extenders below that "strongness" is expressible in ZF. As with supercompactness, one can also define local versions of strongness: κ is λ-*strong*, where $\lambda \geq \kappa$, if there exists some $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $V_\lambda \subset M$. A cardinal κ is strong if and only if it is λ-strong for all $\lambda \geq \kappa$.

Let $j : V \to M$ be an elementary embedding with critical point κ and let $\kappa \leq \lambda \leq j(\kappa)$. We shall define the (κ, λ)-*extender* derived from j.

For every finite subset $a \subset \lambda$, let E_a be the measure on $[\kappa]^{<\omega}$ defined as follows:

$$X \in E_a \quad \text{if and only if} \quad a \in j(X); \tag{20.31}$$

note that E_a concentrates on $[\kappa]^{|a|}$. The (κ, λ)-*extender* derived from j is the collection

$$E = \{E_a : a \in [\lambda]^{<\omega}\}; \tag{20.32}$$

κ is the *critical point* of E and λ is the *length* of the extender.

Let $a \in [\lambda]^{<\omega}$. The measure E_a on $[\kappa]^{<\omega}$ is κ-complete; let Ult_{E_a} denote the ultrapower of V by E_a and let $j_a : V \to \mathrm{Ult}_{E_a}$ be the corresponding elementary embedding. If for each equivalence class $[f]$ of a function f on $[\kappa]^{<\omega}$ we let

$$k_a([f]) = j(f)(a), \tag{20.33}$$

then k_a is an elementary embedding $k_a : \mathrm{Ult}_{E_a} \to M$ and $k_a \circ j_a = j$.

The measures E_a, $a \in [\lambda]^{<\omega}$, are *coherent*, in the following sense: Let $a \subset b$, where $b = \{\alpha_1, \ldots, \alpha_n\}$ with $\alpha_1 < \ldots < \alpha_n$. Then $\pi_{b,a} : [\lambda]^{|b|} \to [\lambda]^{|a|}$ is defined by

$$\pi_{b,a}(\{\xi_1, \ldots, \xi_n\}) = \{\xi_{i_1}, \ldots, \xi_{i_m}\}, \qquad (\xi_1 < \ldots < \xi_n) \tag{20.34}$$

where $a = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$, and

$$X \in E_a \quad \text{if and only if} \quad \{s : \pi_{b,a}(s) \in X\} \in E_b. \tag{20.35}$$

(Compare with Lemma 19.12.)

It follows that $i_{a,b} : \mathrm{Ult}_{E_a} \to \mathrm{Ult}_{E_b}$ defined by

$$i_{a,b}([f]_{E_a}) = [f \circ \pi_{b,a}]_{E_b}$$

is an elementary embedding, and

$$\{\mathrm{Ult}_{E_a}, i_{a,b} : a \subset b \in [\lambda]^{<\omega}\} \tag{20.36}$$

is a directed system. The direct limit Ult_E of (20.36) is well-founded: Note that the embeddings k_a have a direct limit $k : \mathrm{Ult}_E \to M$ such that $k \circ j_E = j$ where j_E is the elementary embedding $j_E : V \to \mathrm{Ult}_E$.

There is another description of the direct limit Ult_E: The elements of Ult_E are equivalence classes $[a,f]_E$ where $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to V$. Here (a,f) and (b,g) are equivalent if $\{s \in [\kappa]^{|a\cup b|} : \tilde{f}(s) = \tilde{g}(s)\} \in E_{a\cup b}$, where $\tilde{f} = f \circ \pi_{a\cup b,a}$ and $\tilde{g} = g \circ \pi_{a\cup b,b}$. The embedding $j_E : V \to \mathrm{Ult}_E$ is defined by $j_E(x) = [\emptyset, c_x]$ where c_x is the constant function with value x. The embedding $k : \mathrm{Ult}_E \to M$ is defined by

$$k([a,f]) = j(f)(a). \tag{20.37}$$

Now $k \circ j_E = j$ follows.

Lemma 20.29.

(i) $k(\alpha) = \alpha$ *for all* $\alpha < \lambda$.
(ii) j_E *has critical point* κ *and* $j_E(\kappa) \geq \lambda$.
(iii) $\mathrm{Ult}_E = \{j_E(f)(a) : a \in [\lambda]^{<\omega},\ f : [\kappa]^{<\omega} \to V\}$.

Proof. For each $a \in [\lambda]^{<\omega}$, let $j_{a,\infty} : \mathrm{Ult}_{E_a} \to \mathrm{Ult}_E$ be the direct limit embedding such that $j_{a,\infty} \circ j_a = j_E$; then $k \circ j_{a,\infty} = k_a$. If $x \in \mathrm{Ult}_E$ then $x = j_{a,\infty}([f])$ for some $[f] \in \mathrm{Ult}_{E_a}$, and

$$k(x) = k(j_{a,\infty}([f])) = k_a([f]) = j(f)(a)$$

(see also (20.37)). Hence

$$k\text{“}\,\mathrm{Ult}_E = \{j(f)(a) : a \in [\lambda]^{<\omega},\ f \in [\kappa]^{<\omega} \to V\}. \tag{20.38}$$

(i) By letting f be the identity function, we get from (20.38) that $a \in k\text{“}\,\mathrm{Ult}_E$, for each $a \in [\lambda]^{<\omega}$. Hence $\lambda \subset k\text{“}\,\mathrm{Ult}_E$, and therefore $k(\alpha) = \alpha$ for all $\alpha < \lambda$.

(ii) This follows from (i), because $j = k \circ j_E$.

(iii) Since $k(a) = a$ for every $a \in [\lambda]^{<\omega}$, it follows from (20.38) that for every $x \in \mathrm{Ult}_E$, $k(x) = j(f)(a) = k(j_E(f))(k(a)) = k(j_E(f)(a))$ for some a and f, and hence $x = j_E(f)(a)$. □

Hence j_E is an elementary embedding, $j_E : V \to \mathrm{Ult}_E$, with critical point κ. Since $j = k \circ j_E$ and since $k(a) = a$ for all $a \in [\lambda]^{<\omega}$, it follows that for all $X \in [\kappa]^{|a|}$, $a \in j_E(X)$ if and only if $a \in j(X)$. Hence E is the extender derived from j_E.

Extenders can be defined directly, without reference to an embedding j. The following, somewhat technical, properties guarantee that the (κ,λ)-extender is derived from the direct limit embedding j_E: Let $\kappa \leq \lambda$, and

let $E = \{E_a : a \in [\lambda]^{<\omega}\}$. E is a (κ, λ)-extender if

(20.39)
(i) Each E_a is a κ-complete measure on $[\kappa]^{|a|}$, and
  (a) at least one E_a is not κ^+-complete,
  (b) for each $\alpha \in \kappa$, at least one E_a contains the set $\{s \in [\kappa]^{|a|} : \alpha \in s\}$.
(ii) (Coherence) The E_a's are coherent, i.e., satisfy (20.35).
(iii) (Normality) If $\{s \in [\kappa]^{|a|} : f(s) \in \max s\} \in E_a$, then for some $b \supset a$, $\{t \in [\kappa]^{|b|} : (f \circ \pi_{b,a})(t) \in t\} \in E_b$.
(iv) The limit ultrapower Ult_E is well-founded.

We leave out the verification that the extender derived from some j satisfies (20.39), and that the properties (20.39) suffice to prove (ii) and (iii) of Lemma 20.29, and that E is derived from j_E.

An immediate consequence of the above technique is the following characterization of strong cardinals:

Lemma 20.30. *A cardinal κ is strong if and only if for every $\lambda \geq \kappa$ there is a $(\kappa, |V_\lambda|^+)$-extender E such that $V_\lambda \subset \mathrm{Ult}_E$ and $\lambda < j_E(\kappa)$.* □

Hence "strongness" is expressible in ZFC.

We conclude by introducing a large cardinal property that was isolated by Woodin and that has played a central role in the study of determinacy and inner models:

Definition 20.31. A cardinal δ is a *Woodin cardinal* if for all $A \subset V_\delta$ there are arbitrarily large $\kappa < \delta$ such that for all $\lambda < \delta$ there exists an elementary embedding $j : V \to M$ with critical point κ, such that $j(\kappa) > \lambda$, $V_\lambda \subset M$, and $A \cap V_\lambda = j(A) \cap V_\lambda$.

Being a Woodin cardinal is expressible in ZFC, in terms of extenders. Every supercompact cardinal is Woodin, and below a Woodin cardinal δ, there are δ cardinals that are λ-strong for every $\lambda < \delta$. While Woodin cardinals are inaccessible (and Mahlo), the least Woodin cardinal is not weakly compact, as δ being Woodin is a Π^1_1 property of $(V_\delta, \in)$.

Exercises

20.1. If κ is strongly compact then $\mathcal{L}_{\kappa,\kappa}$ satisfies the Compactness Theorem.
[Verify Łoś's Theorem]

20.2. If κ is strongly compact, $\lambda \geq \kappa$, and $A \subset \lambda$, then λ^+ is an ineffable cardinal in $L[A]$.

[Let U be as in Theorem 20.3, let $M = \mathrm{Ult}_U(L[A])$, $N = \mathrm{Ult}^-_U(L[A])$, let $j : L[A] \to M$, $i : L[A] \to N$, and let $k : N \to M$ be as there. Again, $M = N$, and $i(\lambda^+)$ is the least ordinal moved. By Lemma 17.32, N thinks that $i(\lambda^+)$ is ineffable; hence λ^+ is ineffable in $L[A]$.]

20.3. The following are equivalent, for $\kappa \leq \lambda$:

(i) There is a fine measure on $P_\kappa(\lambda)$.
(ii) For any set S, every κ-complete filter on S generated by at most λ sets can be extended to a κ-complete ultrafilter on S.

20.4. If U is a normal measure on $P_\kappa(\lambda)$, then every closed unbounded subset of $P_\kappa(\lambda)$ is in U.

[If $C \subset P_\kappa(\lambda)$ is closed unbounded, then $D = \{j(x) : x \in C\}$ is a directed subset of $j(C)$ and $|D| = \lambda^{<\kappa} < j(\kappa)$. Hence $\bigcup D \in j(C)$, and since $\bigcup D = \{j(\gamma) : \gamma < \lambda\}$, we have $C \in U$.]

20.5. Let $\lambda \geq \kappa$ and let U be a normal measure on $P_\kappa(\lambda)$. The ultraproduct $\mathrm{Ult}_U\{(V_{\lambda_x}, \in) : x \in P_\kappa(\lambda)\}$ is isomorphic to $(V_\lambda, \in)$.

20.6. If κ is inaccessible then $V_\kappa \prec_{\Sigma_1} V$.

20.7. If κ is supercompact then $V_\kappa \prec_{\Sigma_2} V$.

[Let $x \in V_\kappa$ such that $\exists y\, \varphi(x, y)$ where φ is Π_1. Let $j : V \to M$ be such that $y \in M \cap V_{j(\kappa)}$. In M, $V_{j(\kappa)} \vDash \exists y\, \varphi(x, y)$, hence $V_\kappa \vDash \exists y\, \varphi(x, y)$.]

Let κ be supercompact and let $\lambda \geq \kappa$ be a cardinal. A normal measure D on $P_\kappa(\lambda)$ is *strongly normal* if there exists $X \in D$ such that for every function f on X, if for each nonempty $x \in X$, $f(x)$ is in X, $f(x) \subset x$ and $f(x) \neq x$, then f is constant on some $Y \in D$.

20.8. The following are equivalent:

(i) D is strongly normal.
(ii) There is $X \in D$ such that if $\{Z_x : x \in X\} \subset D$, then $\triangle_{x \in X} Z_x \in D$ where $\triangle_{x \in X} Z_x = \{y : y \in Z_x$ for each $x \subset y$ such that $x \neq y$ and $x \in X\}$.
(iii) D has this partition property: If $F : [P_\kappa(\lambda)]^2 \to \{0, 1\}$ is a partition, then there is $X \in D$ such that F is constant on $\{\{x, y\} \in [X]^2 : x \subsetneqq y$ or $y \subsetneqq x\}$.
(iv) There is $X \in D$ such that if $x, y \in X$, $x \neq y$ and $x \subset y$, then $\lambda_x < \kappa_y$.

[(i) $\to$ (ii): Let $X \in D$ be a witness to strong normality. Prove by contradiction that D is closed under $\triangle_{x \in X} Z_x$.

(ii) $\to$ (iii): Let $F : [P_\kappa(\lambda)]^2 \to \{0, 1\}$; for each x, let $F_x : \hat{x} \to \{0, 1\}$ be $F_x(y) = F(x, y)$. For each x there is $Z_x \subset \hat{x}$, $Z_x \in D$, such that F_x is constant on Z_x. Let $X \in D$ be as in (ii) and such that the constant value of F_x is the same for all $x \in X$. Then $X \cap \triangle_{x \in X} Z_x$ is homogeneous for F in the sense of (iii).

(iii) $\to$ (iv): Note that if $X \in D$, then there exist $x, y \in X$ such that $x \subsetneqq y$ and $\lambda_x < \kappa_y$.

(iv) $\to$ (i): Let $X \in D$ be as in (iv) and let $f : X \to X$ be such that $f(x) \subset x$ and $f(x) \neq x$ for all x. In Ult_D, if $x \in j_D(X)$ and $x \subset j\text{“}\lambda$, then $|x| < j\text{“}\lambda \cap \kappa = \kappa$ and hence $x = j(y)$ for some $y \in P_\kappa(\lambda)$. Hence $(jf)(j\text{“}\lambda) = j(y)$ for some y and so $f(x) = y$ for almost all x.]

It has been proved that if κ is supercompact, then every $P_\kappa(\lambda)$ has a strongly normal measure; however, not every normal measure is necessarily strongly normal:

20.9. If $\lambda > \kappa$ is measurable, then there is a normal measure U on $P_\kappa(\lambda)$ that is not strongly normal.

[Let $j : V \to M$ be elementary, κ least moved, $j(\kappa) > \lambda$, and $M^\lambda \subset M$. Let D be a normal measure on λ. Let us define a normal measure U on $P_\kappa(\lambda)$ as follows: $X \in U$ if and only if $\{\alpha < \lambda : j\text{“}\alpha \in j(X)\} \in D$. If $X \in U$, then there exist $\alpha < \beta$ such that $j\text{“}\alpha$ and $j\text{“}\beta$ are in $j(X)$; hence $M \vDash \exists x, y \in j(X)$ such that x is an initial segment of y. Thus $\exists x, y \in X$ such that x is an initial segment of y, and so $\lambda_x \geq \kappa_y$.]

20.10. If κ is extendible then $V_\kappa \prec_{\Sigma_3} V$.
[Use Exercise 20.7, and show that there are arbitrarily large inaccessible $\lambda > \kappa$ such that $V_\kappa \prec V_\lambda$.]

20.11. A cardinal κ is huge, with $j : V \to M$ and $j(\kappa) = \lambda$ if and only if there is a normal κ-complete ultrafilter U on $\{X \subset \lambda : \text{order-type}(X) = \kappa\}$.
[$X \in U$ if and only if $j\text{“}\lambda \in j(X)$.]

20.12. Let κ be the least huge cardinal and let μ be the least supercompact cardinal. Then $\kappa < \mu$.
[If $\kappa = \mu$ then by 20.27, 20.25, 20.24, and 20.23 we get: $V_\mu \vDash \text{VP}$, $V_\mu \vDash (\exists$ supercompact $\alpha)$, there is a supercompact $\alpha < \mu$, a contradiction. If $\kappa < \mu$, let $j : V \to M$ with $\lambda = j(\kappa)$ and $M^\lambda \subset M$. Since μ is supercompact, let $i : V \to N$ be such that $i(\mu) > \lambda$ and $V_{\lambda+2} \subset N$. If U is a normal measure witnessing the hugeness of κ, then $U \in N$, and hence $N \vDash (\exists$ huge cardinal below $i(\mu))$. Thus there exists a huge cardinal below μ, a contradiction.]

20.13. The least huge cardinal is greater than the least measurable cardinal.
[Show that $M \vDash \kappa$ is measurable; hence there exists a measurable cardinal less than κ.]

A cardinal κ is *n-huge* if there exists an elementary $j : V \to M$ with critical point κ such that $M^{j^n(\kappa)} \subset M$.

20.14. If κ is $(n+1)$-huge then there is a normal measure D on κ such that $\{\alpha < \kappa : \alpha \text{ is } n\text{-huge}\} \in D$.

20.15. If there exists an elementary $j : V_\lambda \to V_\lambda$ with critical point κ then κ is n-huge for every n.

20.16. If there is a strong cardinal, then $V \neq L[A]$ for any set A.

20.17. If κ is strong then $o(\kappa) = (2^\kappa)^+$.
[As in Corollary 20.20(iii).]

20.18. If κ is strong then $V_\kappa \prec_{\Sigma_2} V$.
[As in Exercise 20.7.]

20.19. If κ is strong, then there exists a function $g : \kappa \to V_\kappa$ such that for every x and every $\lambda \geq \kappa$ such that $\lambda \geq |\,\text{TC}(x)|$ there exists a (κ, λ)-extender E such that $j_E(g)(\kappa) = x$.

20.20. A (κ, λ)-extender $\{E_a : a \in [\lambda]^{<\omega}\}$ has well-founded limit ultrapower if and only if for every $\langle a_m : m \in \omega\rangle$ and every sequence $\langle X_m : m \in \omega\rangle$ such that $X_m \in E_{a_m}$, there exists a function $h : \bigcup_{m\in\omega} a_m \to \kappa$ such that $h\text{“}a_m \in X_m$ for all m.

Historical Notes

Strongly compact cardinals were introduced by Keisler and Tarski in [1963/64]; supercompact cardinals were defined by Reinhardt and Solovay. Theorem 20.3 is due to Vopěnka and Hrbáček [1966]; Theorem 20.4 is due to Kunen [1971b].

Solovay discovered that the Singular Cardinal Hypothesis holds above a compact cardinal (Theorem 20.8); see [1974].

Menas and Magidor obtained several results on the relative strength of compact and supercompact cardinals. Menas in [1974/75] showed that it is consistent (relative to existence of compact cardinals) that there is a compact cardinal that is not supercompact. Magidor in [1976] improved Menas' result by showing that it is possible that the least measurable cardinal is strongly compact (while by Lemma 20.16 it is not supercompact) and also showed that it is consistent (relative to supercompact cardinals) that there exists just one compact cardinal and is supercompact.

Kunen's proof of Theorem 20.4 uses a lemma of Ketonen (Lemma 20.5); Lemmas 20.9 and 20.10 which Solovay used in his proof of Theorem 20.8, are also due to Ketonen; see [1972/73].

Most results on supercompact cardinals (e.g., Lemmas 20.16 and 20.19) are due to Solovay; see Solovay, Reinhardt, and Kanamori [1978]; Magidor's paper [1971a] gives a number of normal measures on $P_\kappa(\lambda)$ (Corollary 20.20(ii)). The example of a strongly compact nonsupercompact cardinal (Lemma 20.17 and Corollary 20.18) is due to Menas [1974/75]. Theorem 20.21 is due to Laver [1978].

Extendible cardinals were introduced by Reinhardt; he proved that extendible cardinals are supercompact; see [1974]. The present proof of Theorem 20.24, as well as Lemmas 20.23 and 20.25 are due to Magidor [1971b].

Lemma 20.27: Powell [1972].

A good source for further results on very large cardinals is the paper [1978] of Kanamori and Magidor, as well as Kanamori's book [1994].

Strong cardinals were used by Mitchell [1979a] to develop a theory of inner models for weak versions of supercompactness, and further studied by Baldwin [1986]. Extenders were introduced by Jensen and Dodd; see Dodd [1982].

Woodin cardinals were introduced by Woodin in 1984. They were used, among others, in the proof of projective determinacy by Martin and Steel [1989].

Exercises 20.7, 20.10, 20.11, 20.14, and 20.15: Solovay, Reinhardt and Kanamori [1978].

Exercise 20.9: Solovay.

Exercise 20.12: Morgenstern [1977].

Exercise 20.19: Gitik and Shelah [1989].

21. Large Cardinals and Forcing

Many forcing techniques have been developed specifically for use with large cardinals. Firstly, when investigating the effect of large cardinals on cardinal arithmetic, it is desirable to establish the relative consistency of statements about the continuum function with the existence of various large cardinals. The main technical question here is whether large cardinal properties are preserved under various forcing extensions. Secondly, it follows from the Covering Theorem that an attempt to violate the Singular Cardinal Hypothesis, or even to change cofinalities, necessarily involves large cardinals. Indeed, forcing techniques have been developed for changing cofinalities and for violating SCH that use large cardinals. And thirdly, as the large cardinal hierarchy serves as a gauge of consistency strength, forcing that uses large cardinals provides an upper bound for the consistency strength of the problems that the forcing proves consistent.

Mild Extensions

We begin with an early discovery that "mild" forcing extensions do not effect large cardinal properties, i.e., whether κ is a large cardinal is not changed by forcing of size less than κ.

Theorem 21.1 (Lévy-Solovay). *Let κ be a measurable cardinal in the ground model. Let $(P, <)$ be a notion of forcing such that $|P| < \kappa$. Then κ is measurable in the generic extension.*

Proof. We give a proof using elementary embeddings since similar arguments will be used in subsequent constructions; for a direct proof, see Exercise 21.1. Let $B = B(P)$; since $|B| < \kappa$, we may assume $B \in V_\kappa$. We can also assume that P is a dense subset of B. Let G be a generic ultrafilter on B; let us work in $V[G]$.

Since κ is measurable in V, there is an elementary embedding $j : V \to M$ with critical point κ, and M transitive. We shall extend j to an elementary embedding (denoted also j) of $V[G]$ into $M[G]$, thus showing that κ is measurable in $V[G]$.

Since $B \in V_\kappa$, we have $j(B) = B$, and B is a complete Boolean algebra in M. Since G is generic over V, G is also generic over M. Note that the

interpretation of Boolean names in M^B by G is the same whether computed in V or in M. We define $j(x)$ for $x \in V[G]$ as follows: Let $\dot{x} \in V^B$ be a name for x, $x = \dot{x}^G$. Let

$$j(x) = (j(\dot{x}))^G. \tag{21.1}$$

Since $\dot{x} \in V^B$, we have $j(\dot{x}) \in M^B$ and so $(j(\dot{x}))^G \in M[G]$. However, we have to show that the definition (21.1) does not depend on which name for x we choose.

Let $\dot{y}$ be another B-valued name and let $p \in G$ be such that

$$p \Vdash \dot{x} = \dot{y}. \tag{21.2}$$

When we apply j to (21.2), we have (in M)

$$j(p) \Vdash j(\dot{x}) = j(\dot{y}).$$

But $j(p) = p \in G$ and therefore

$$(j(\dot{x}))^G = (j(\dot{y}))^G.$$

Finally we show that $j : V[G] \to M[G]$ is elementary. Let φ be a formula such that

$$V[G] \vDash \varphi(x, \ldots).$$

Let $\dot{x}, \ldots$ be such that $(\dot{x})^G = x, \ldots$. There is some $p \in G$ such that

$$p \Vdash \varphi(\dot{x}, \ldots). \tag{21.3}$$

Applying j to (21.3), we get (in M)

$$p \vDash \varphi(j(\dot{x}), \ldots)$$

(because $j(p) = p$). Hence

$$M[G] \vDash \varphi(j(\dot{x}), \ldots)$$

and since φ was arbitrary, j is elementary. □

It turns out that practically every large cardinal property is unchanged by mild extension:

Theorem 21.2. *Let κ be an infinite cardinal, and let $(P, <)$ be a notion of forcing such that $|P| < \kappa$. Let G be a V-generic filter on P. Then κ is inaccessible (Mahlo, weakly compact, Ramsey, measurable, strongly compact, supercompact, huge, strong, Woodin) in V if and only if it is inaccessible (Mahlo, weakly compact, Ramsey, measurable, strongly compact, supercompact, huge, strong, Woodin) in $V[G]$.*

Proof. If κ is inaccessible in V, then firstly κ is regular in $V[G]$ because all cardinals and cofinalities above $|P|$ are preserved. Secondly, if $\alpha < \kappa$, then $(2^\alpha)^{V[G]} \leq |B(P)|^\alpha < \kappa$ and hence κ is inaccessible in $V[G]$. Conversely, if κ is inaccessible in $V[G]$, then κ is inaccessible in V.

If κ is Mahlo in V, then firstly κ is inaccessible in $V[G]$. Secondly, every $\alpha > |P|$ is a regular cardinal in $V[G]$ if and only if it is a regular cardinal in V and so the set $S = \{\alpha < \kappa : |P| < \alpha$ and α is regular in $V[G]\}$ is stationary in V. It is easy to see that every closed unbounded set $C \subset \kappa$ in $V[G]$ has a closed unbounded subset D in V (Exercise 21.2). Thus, S is also stationary in $V[G]$ and hence κ is Mahlo in $V[G]$. Conversely, if κ is Mahlo in $V[G]$, then κ is Mahlo in V because $V \subset V[G]$.

If κ is weakly compact in V, let $F : [\kappa]^2 \to \{0,1\}$ be a partition of $[\kappa]^2$ in $V[G]$. Let $\dot{F} \in V^B$ be its name such that $\|\dot{F} : [\kappa]^2 \to \{0,1\}\| = 1$. For all $\alpha \neq \beta \in \kappa$, let $G(\alpha,\beta) = \|\dot{F}(\alpha,\beta) = 0\|$; G is (in V) a partition of $[\kappa]^2$ into $|B(P)|$ pieces. Since $|B(P)| < \kappa$, there is an $H \subset \kappa$ of size κ homogeneous for G, and it is easy to see that H is homogeneous for F.

Conversely, if κ is weakly compact in $V[G]$, let $F : [\kappa]^2 \to \{0,1\}$ be a partition of $[\kappa]^2$ in V. There is, in $V[G]$, a set $K \subset \kappa$ of size κ, homogeneous for F. As in Exercise 21.2, K has an unbounded subset $H \in V$; hence F has in V a homogeneous set of size κ.

The argument for Ramsey cardinals is exactly the same as for weakly compact cardinals.

If κ is measurable in V, then it is measurable in $V[G]$ by Theorem 21.1. Conversely, if κ is measurable in $V[G]$, let $U \in V[G]$ be a κ-complete nonprincipal ultrafilter on κ, let J be the dual prime ideal, and let $\dot{J} \in V^B$ be its name. (Without loss of generality we assume that $\|\dot{J}$ is a κ-complete nonprincipal prime ideal$\| = 1$.) Let $I = \{X \subset \kappa : \|X \in \dot{J}\| = 1\}$. It is easy to verify that I is a κ-complete ideal containing all singletons. We claim that I is $|P|^+$-saturated: If $p \Vdash \check{X} \notin \dot{J}$ and $p \Vdash \check{Y} \notin \dot{J}$, then $p \Vdash \check{X} \cap \check{Y} \notin \dot{J}$ (because $\dot{J}$ is prime). Hence if X and Y are such that $X \notin I$, $Y \notin I$, and $X \cap Y \in I$, then $\|\check{X} \notin \dot{J}\| \cdot \|\check{Y} \notin \dot{J}\| = 0$, and it follows that I is $|P|^+$-saturated. However, since I is ν-saturated for some $\nu < \kappa$ and κ is inaccessible, κ is measurable, by Exercise 21.3.

If κ is strongly compact, let $\lambda \geq \kappa$ and let us show that in $V[G]$, there is a fine measure on $P_\kappa(\lambda)$. Let U be a fine measure on $P_\kappa(\lambda)$ in V, and let $j = j_U$ be the canonical elementary embedding $j_U : V \to \mathrm{Ult}_U(V)$. The standard argument shows that $X \in U$ if and only if $H \in j(X)$, where H is the set in $\mathrm{Ult}_U(V)$ represented by the function $d(Z) = Z$ on $P_\kappa(\alpha)$; also, $H \supset j\text{“}\lambda$ (and is equal to it if U is normal). Similarly, as in the proof of Theorem 21.1 we extend j to $V[G]$ as follows:

$$j(x) = (j(\dot{x}))^G$$

where $\dot{x}$ is a name for x; the definition is legitimate because we assume, without loss of generality, that $P \in V_\kappa$ and hence $j(p) = p$ for all $p \in P$, and

$j(P) = P$. Now we define, in $V[G]$, an ultrafilter W on $P_\kappa(\lambda)$ as follows:

$$X \in W \quad \text{if and only if} \quad H \in j(X).$$

It is routine to verify that W is a fine measure on $P_\kappa(\lambda)$; for instance, if $Z_0 \in P_\kappa(\lambda)$, then $\{Z \in P_\kappa(\lambda) : Z \supset Z_0\} \in W$ because $j(Z_0) = \{j(\alpha) : \alpha \in Z_0\} \subset H$.

Conversely, if κ is strongly compact in $V[G]$, let S be a set in V and let F be a κ-complete filter on S (in V); let us show that there is a κ-complete ultrafilter in V extending F. Every set $X \subset F$ of size $< \kappa$ in $V[G]$ is included in some $Y \subset F$ of size $< \kappa$ such that $Y \in V$ (this is because $|P| < \kappa$). Hence F generates a κ-complete filter in $V[G]$ and that in turn is included in a κ-complete ultrafilter on U. Let J be the dual prime ideal. As in the proof for measurable cardinals above, the ideal $I = \{X \subset S : \|X \in \dot{J}\| = 1\}$ is κ-complete and $|P|^+$-saturated, and clearly $X \in F$ implies $S - X \in I$. Since κ is inaccessible and I is ν-saturated for some $\nu < \kappa$, I has an atom A. If $X \in F$, then $X \cap A \notin I$ and so $\{X \subset S : X \cap A \notin I\}$ is a κ-complete ultrafilter extending F.

The proofs for the remaining large cardinal properties are similar. □

There are numerous examples when κ ceases to be large when we use a notion of forcing of size $\geq \kappa$ (a good example is the Lévy collapse). The example in Exercise 21.4 is quite interesting since inaccessibility of κ is preserved by any notion of forcing that is α-distributive for all $\alpha < \kappa$.

Kunen-Paris Forcing

It is an immediate consequence of the Lévy-Solovay Theorem that if κ is a measurable cardinal, $\lambda < \kappa$, and F is a function on regular cardinals below λ such that (i) $F(\alpha) \leq F(\beta)$ if $\alpha \leq \beta$, (ii) $\mathrm{cf}(F(\alpha)) > \alpha$, and (iii) $F(\alpha) < \kappa$ for all α in its domain, then there is a model in which κ is measurable and $2^\alpha = F(\alpha)$ for all $\alpha \in \mathrm{dom}(F)$.

One can also prescribe the values of the continuum function on regular cardinals greater than the measurable cardinal; this can be done by a κ-closed forcing; see Exercise 21.5.

The proof of the next theorem uses a method that preserves measurability of κ while adding subsets to an unbounded set of cardinals below κ. It is vital however that the set $A \subset \kappa$ has a normal measure 0.

Theorem 21.3 (Kunen-Paris). *Assume* GCH *and let κ be a measurable cardinal. Let D be a normal measure on κ and let A be a set of regular cardinals below κ such that $A \notin D$; let F be a function on A such that $F(\alpha) < \kappa$ for all $\alpha \in A$, and*:

(i) $\mathrm{cf}\, F(\alpha) > \alpha$;

(ii) $F(\alpha_1) \leq F(\alpha_2)$ *whenever* $\alpha_1 \leq \alpha_2$.

Then there is a generic extension $V[G]$ *of* V *with the same cardinals and cofinalities, such that* κ *is measurable in* $V[G]$, *and for every* $\alpha \in A$,

$$V[G] \vDash 2^\alpha = F(\alpha). \tag{21.4}$$

Moreover, given any regular cardinal $\lambda > \kappa^+$, *we can find* $V[G]$ *so that there are* λ *normal measures on* κ *in* $V[G]$.

Proof. Let $j : V \to M$ be the elementary embedding given by the ultrapower Ult_D. As we assume that $A \notin D$, we have $\kappa \notin j(A)$.

Let $(P, <)$ be the Easton product of P_α, $\alpha \in A$, where each P_α is the notion of forcing that adjoins $F(\alpha)$ subsets of α. Thus conditions are 0–1 functions whose domain consists of triples (α, ξ, η) where $\alpha \in A$, $\xi < \alpha$, and $\eta < F(\alpha)$, and such that for every regular cardinal γ,

$$|\{(\alpha, \xi, \eta) \in \mathrm{dom}(p) : \alpha \leq \gamma\}| < \gamma.$$

In particular, $|p| < \kappa$ for all $p \in P$, hence $P \subset V_\kappa$ and so $j(p) = p$ for each $p \in P$.

We shall however use not P but $j(P)$ as our notion of forcing. Thus $j(P)$ is, in M, the Easton product of P_α for $\alpha \in j(A)$. Note that $P \subset j(P)$ and that $j(P)$ is isomorphic to $P \times Q$ where $P = (jP)^{<\kappa}$ and $Q = (jP)^{\geq \kappa}$.

Let G be a V-generic filter on $j(P)$. We claim that $V[G]$ has the same cardinals and cofinalities as V and satisfies (21.4) and that κ is a measurable cardinal in $V[G]$. Let $G_1 = G \cap P$; since $j(P)$ is isomorphic to $P \times Q$, there is a $V[G_1]$-generic filter G_2 on Q such that $V[G] = V[G_1 \times G_2]$.

As we have noted before, $\kappa \notin j(A)$, and so $Q = (jP)^{\geq \kappa}$ is in fact $= (jP)^{>\kappa}$, and hence is, in M, κ-closed. But since $M^\kappa \subset M$, Q is κ-closed in V. Moreover, we have $|P| = \kappa$ and $|j(P)| = |j(\kappa)| = \kappa^+$. Thus for each regular $\lambda \leq \kappa$, we can break $j(P)$ into a product of two notions of forcing, one that satisfies the λ^+-chain condition and one that is λ-closed, and hence all cardinals $\leq \kappa^+$ are preserved. Since $|j(P)| = \kappa^+$, all cardinals $> \kappa^+$ are also preserved.

We prove (21.4) similarly to Easton's Theorem. For $\alpha \in A$, we regard $j(P)$ as a product $(jP)^{>\alpha} \times (jP)^{\leq \alpha}$; and since $(jP)^{\leq \alpha} = P^{\leq \alpha}$ and $(jP)^{>\alpha}$ is α-closed, we conclude that $(2^\alpha)^{V[G]} = F(\alpha)$.

The crucial step is to show that κ is a measurable cardinal in $V[G]$. We shall first extend $j : V \to M$ to an elementary embedding

$$j : V[G_1] \to M[G]. \tag{21.5}$$

We define $j(x)$ for $x \in V[G_1]$ as follows: If $\dot{x} \in V^P$ is a name for x, we let

$$j(x) = (j(\dot{x}))^G \tag{21.6}$$

where $j(\dot{x})$ is, in M, a $j(P)$-name. As in the proof of Theorem 21.1, we show that (21.6) does not depend on the choice of $\dot{x}$. Since $j(p) = p$ for all $p \in P$

and because $G_1 \subset G$, it follows that if some $p \in G_1$ forces $\dot{x} = \dot{y}$, then (in M) $p \Vdash j(\dot{x}) = j(\dot{y})$ and therefore $(j(\dot{x}))^G = (j(\dot{y}))^G$. The same reasoning shows that $j : V[G_1] \to M[G]$ is elementary.

Using (21.5), we define an $V[G_1]$-ultrafilter U on κ as follows:

$$X \in U \quad \text{if and only if} \quad \kappa \in j(X), \tag{21.7}$$

for all $X \subset \kappa$ in $V[G_1]$. A standard argument shows that U is κ-complete; and since j extends the original $j = j_D$, U is nonprincipal.

Now we use again the fact that $j(P)$ is isomorphic to $P \times Q$, where $|P| = \kappa$ and Q is κ-closed. Thus every subset of κ is in $V[G_1]$ and therefore U is in $V[G]$ a κ-complete nonprincipal ultrafilter on κ.

To get a model with λ normal measures on κ we modify the construction above as follows: Let R be the κ^+-product of λ copies of Q, i.e., the set of all functions $f \in Q^\lambda$ such that $|\{i < \lambda : f(i) \neq \emptyset\}| \leq \kappa$. We consider the notion of forcing $P \times R$.

Let $G \times H$ be a generic filter on $P \times R$. We claim that the model $V[G \times H]$ has the required properties.

Since R is κ-closed, all subsets of κ are contributed by G; hence $2^\kappa = \kappa^+$ holds in $V[G \times H]$. Standard arguments show that cardinals are preserved and $2^{\kappa^+} = \lambda$ in $V[G \times H]$.

To find λ distinct normal measures, let us look more closely at the definition (21.7) of U: U has a name $\dot{U} \in V^{P \times Q}$ such that for all $p \in P$ and $q \in Q$, and any name $\dot{X} \in V^P$ for a subset of κ,

$$p \cup q \Vdash (\dot{X} \in \dot{U} \leftrightarrow \kappa \in j(\dot{X})). \tag{21.8}$$

If q is represented in Ult_D by $\langle q_\alpha : \alpha < \kappa \rangle$ (with $q_\alpha \in P^\alpha$ for each $\alpha < \kappa$), we have

$$p \cup q \Vdash \dot{X} \in \dot{U} \quad \text{if and only if} \quad \{\alpha : p \cup q_\alpha \Vdash \alpha \in \dot{X}\} \in D. \tag{21.9}$$

For each $i < \lambda$, let Q_i denote the ith copy of Q and let $\dot{U}_i$ be the canonical name for a normal measure using Q_i instead of Q in (21.8). It suffices to show that for any $i \neq k < \lambda$, $\dot{U}_i$ and $\dot{U}_k$ denote different measures in $V[G \times H]$.

The last assertion follows by a standard argument using genericity, and we leave its proof to the reader: Given $i \neq k$ and a condition (p, r) in $P \times R$, use (21.9) to find a stronger condition (p', r') and some P-valued name $\dot{X}$ such that (p', r') forces $\dot{X} \in \dot{U}_i$ but $\dot{X} \notin \dot{U}_k$. □

Silver's Forcing

We shall now construct a model that has a measurable cardinal κ for which $2^\kappa > \kappa^+$. By Corollary 19.25, the consistency strength of this is more than measurability. It has been established that the failure of GCH at a measurable

cardinal is equiconsistent with the existence of a measurable cardinal κ of Mitchell's order κ^{++}. The lower bound is obtained by Mitchell's method of iterated ultrapowers from Chapter 19, while the upper bound follows from improvements (due to Woodin and Gitik) on Silver's forcing construction presented below.

Theorem 21.4 (Silver). *If there exists a supercompact cardinal κ, then there is a generic extension in which κ is a measurable cardinal and $2^\kappa > \kappa^+$.*

Silver's construction uses iterated forcing. As $2^\kappa > \kappa^+$ for a measurable cardinal implies that $2^\alpha > \alpha^+$ for many α below κ, the iteration adjoins not only subsets of κ, but, iteratively, subsets of regular cardinals below κ. The iteration combines direct and inverse limits, in a manner similar to Easton's forcing.

Definition 21.5. Let $\alpha \geq 1$, and let P_α be an iterated forcing of length α (see Definition 16.29). P_α is an iteration with *Easton support* if for every $p \in P_\alpha$ and every regular cardinal $\gamma \leq \alpha$, $|s(p) \cap \gamma| < \gamma$. Equivalently, for every limit ordinal $\gamma \leq \alpha$, P_γ is a direct limit if γ is regular, and an inverse limit otherwise.

When using iterated forcing that combines direct and inverse limits, we can apply Theorem 16.30 to calculate the chain condition, and Exercise 16.19 to calculate the degree of closedness. We shall need the following variant:

Definition 21.6. A notion of forcing $(P, <)$ is λ-*directed closed* if whenever $D \subset P$ is such that $|D| \leq \lambda$ and for any $d_1, d_2 \in D$ there is some $e \in D$ with $e \leq d_1$ and $e \leq d_2$, then there exists a $p \in P$ such that $p \leq d$ for all $d \in D$.

Lemma 21.7.

(i) *If P is λ-directed closed, and if $\Vdash_P \dot{Q}$ is λ-directed closed, then $P * \dot{Q}$ is λ-directed closed.*

(ii) *If $\operatorname{cf}\alpha > \lambda$, if P_α is a direct limit and if for each $\beta < \alpha$, P_β is λ-directed closed, then P_α is λ-directed closed.*

(iii) *Let P_α be a forcing iteration of $\langle \dot{Q}_\beta : \beta < \alpha \rangle$ such that for each limit ordinal $\beta \leq \alpha$, P_β is either a direct limit or an inverse limit. Assume that for each $\beta < \alpha$, $\dot{Q}_\beta$ is a λ-directed closed forcing in V^{P_β}. If for every limit ordinal $\beta \leq \alpha$ such that $\operatorname{cf}\beta \leq \lambda$, P_β is an inverse limit, then P_α is λ-directed closed.*

Proof. (i) Let $D = \{(p_\alpha, \dot{q}_\alpha) : \alpha < \lambda\}$ be a directed subset of $P * \dot{Q}$. Clearly, $D_1 = \{p_\alpha : \alpha < \lambda\}$ is a directed subset of P and hence there is $p \in P$ stronger than all p_α, $\alpha < \lambda$. Since for any $\alpha, \beta < \lambda$ there is $\gamma < \lambda$ such that $(p_\gamma, \dot{q}_\gamma) \leq (p_\alpha, \dot{q}_\alpha)$ and $(p_\gamma, \dot{q}_\gamma) \leq (p_\beta, \dot{q}_\beta)$, it is clear that $p \Vdash (\dot{q}_\gamma \leq \dot{q}_\alpha$ and $\dot{q}_\gamma \leq \dot{q}_\beta)$ and thus p forces that $\{\dot{q}_\alpha : \alpha < \lambda\}$ is a directed subset of $\dot{Q}$. Hence

$$p \Vdash \exists q \in \dot{Q} \text{ stronger than all the } \dot{q}_\alpha$$

and therefore there is a $\dot{q}$ such that $\|\dot{q} \in \dot{Q}\| = 1$ and

$$p \Vdash \dot{q} \leq \dot{q}_\alpha \text{ for all } \alpha < \lambda.$$

It follows that $(p, \dot{q}) \leq (p_\alpha, \dot{q}_\alpha)$ for all $\alpha < \lambda$.

(ii) Let D be a directed subset of P, $|D| \leq \lambda$. For each $d \in D$ there is $\gamma_d < \alpha$ such that if $d = \langle p_\beta : \beta < \alpha \rangle$, then $p_i = 1$ for all $i \geq \gamma_d$. Since $\lambda < \operatorname{cf} \alpha$, there is $\gamma < \alpha$ such that each $d \in D$ is as follows:

$$d = (d{\restriction}\gamma)^\frown 1^\frown 1^\frown 1^\frown \dots.$$

Now $D_\gamma = \{(d{\restriction}\gamma) : d \in D\}$ is a directed subset of P_γ; and since P_γ is λ-directed closed, D_γ has a lower bound $p \in P_\gamma$. Then $p^\frown 1^\frown 1^\frown 1^\frown \dots$ is a lower bound for D in P.

(iii) By induction on α. It follows from (i) and (ii) that the assertion is true if α is a successor or if P_α is the direct limit. Thus assume that P_α is an inverse limit.

Let $D = \langle p^\nu : \nu < \lambda \rangle$ be a directed subset of P_α; for each ν let $p^\nu = \langle p^\nu_\beta : \beta < \alpha \rangle$. We shall construct, by induction on $\beta < \alpha$, a function $p = \langle p_\beta : \beta < \alpha \rangle \in P_\alpha$ stronger than all p^ν, $\nu < \lambda$.

We construct p such that for each $\beta < \alpha$, $p{\restriction}\beta$ is in P_β and is stronger than all $p^\nu{\restriction}\beta$, $\nu < \lambda$. Having constructed $p{\restriction}\beta$, we let p_β be such that

$$p{\restriction}\beta \Vdash p_\beta \leq p^\nu_\beta \text{ for all } \nu < \lambda.$$

Moreover, if $p^\nu_\beta = 1$ for all $\nu < \lambda$, we let $p_\beta = 1$ too.

If $\gamma \leq \alpha$ is a limit ordinal, we have to show that $\langle p_\beta : \beta < \gamma \rangle \in P_\gamma$. If P_γ is the inverse limit, then there is nothing to prove, so let us assume that P_γ is the direct limit. By the assumption, we have $\operatorname{cf} \gamma > \lambda$ and therefore there is a $\delta < \gamma$ such that for all $\nu < \lambda$, $p^\nu_\beta = 1$ for all β such that $\delta \leq \beta < \gamma$. Hence we have $p_\beta = 1$ for all β such that $\delta \leq \beta < \gamma$, and so $\langle p_\beta : \beta < \gamma \rangle \in P_\gamma$. Thus we have $p = \langle p_\beta : \beta < \alpha \rangle \in P_\alpha$, and it is clear from the construction that $p \leq p^\nu$ for all $\nu < \lambda$. □

An important feature of iterated forcing is that often, under reasonable assumptions, an iteration $P_{\alpha+\beta}$ is equivalent to $P_\alpha * \dot{P}^{(\alpha)}_\beta$ where $P^{(\alpha)}_\beta$ is an iteration of length β inside V^{P_α}. The following lemma is used in applications of iteration with Easton support:

Lemma 21.8 (The Factor Lemma). *Let $P_{\alpha+\beta}$ be a forcing iteration of $\langle \dot{Q}_\xi : \xi < \alpha + \beta \rangle$, where each P_ξ, $\xi \leq \alpha + \beta$ is either a direct limit or inverse limit. In V^{P_α}, let $\dot{P}^{(\alpha)}_\beta$ be the forcing iteration of $\langle \dot{Q}_{\alpha+\xi} : \xi < \beta \rangle$ such that for every limit ordinal $\xi < \beta$, $\dot{P}^{(\alpha)}_\xi$ is either a direct or inverse limit, according to whether $P_{\alpha+\xi}$ is a direct limit or inverse limit. If $P_{\alpha+\xi}$ is an inverse limit for every limit ordinal $\xi \leq \beta$ such that $\operatorname{cf} \xi \leq |P_\alpha|$, then $P_{\alpha+\beta}$ is isomorphic to $P_\alpha * \dot{P}^{(\alpha)}_\beta$.*

This formulation is not quite accurate. The name $\dot{Q}_{\alpha+\xi}$ is in $V^{P_{\alpha+\xi}}$ while $P_\xi^{(\alpha)}$ is an iteration in $V^{P_{\alpha+\beta}}$ that at stage ξ should use a V^{P_α}-name for a name $\dot{Q}_\xi^{(\alpha)} \in V^{\dot{P}_\xi^{(\alpha)}}$. However, the Factor Lemma yields, for each ξ, an isomorphism between $V^{P_{\alpha+\xi}}$ and the Boolean-valued model $V^{\dot{P}_\xi^{(\alpha)}}$ inside V^{P_α}, and so $\dot{Q}_{\alpha+\xi}$ is identified with a V^{P_α}-name for some $\dot{Q}_\xi^{(\alpha)} \in V^{\dot{P}_\xi^{(\alpha)}}$.

Proof. By induction on β. Let β be an ordinal number; we shall construct an isomorphism π between $P_\alpha * \dot{P}_\beta^{(\alpha)}$ and $P_{\alpha+\beta}$.

If $\beta = 0$, then $P_\alpha * \dot{P}_\beta^{(\alpha)} = \{(p,1) : p \in P_\alpha\}$ and we let $\pi(p,1) = p$. Thus let $\beta > 0$. A typical element of $P_\alpha * \dot{P}_\beta^{(\alpha)}$ is a pair $(p, \dot{q})$ where $p \in P_\alpha$ and $\dot{q}$ is an element of V^{P_α} such that in V^{P_α}, $\dot{q}$ is a β-sequence and satisfies the conditions on iterated forcing; in particular, for each $\xi < \beta$, $\dot{q}{\restriction}\xi$ is in $\dot{P}_\xi^{(\alpha)}$ and the ξth term of $\dot{q}$ is in $\dot{Q}_{\alpha+\xi}$.

We shall define a β-sequence $\langle p_{\alpha+\xi} : \xi < \beta\rangle$ and let $\pi(p,\dot{q}) = p^\frown\langle p_\alpha, p_{\alpha+1}, \ldots, p_{\alpha+\xi}, \ldots\rangle$. This mapping π will be an isomorphism between $P_\alpha * \dot{P}_\beta^{(\alpha)}$ and $P_{\alpha+\beta}$. For $\xi < \beta$, let $\dot{q}_\xi \in V^{P_\alpha}$ be such that $\dot{q}_\xi$ is the ξth term of $\dot{q}$. Hence

$$\Vdash_{P_\alpha} (\Vdash_{\dot{P}_\xi^{(\alpha)}} \dot{q}_\xi \in \dot{Q}_{\alpha+\xi}).$$

By the induction hypothesis, $P_{\alpha+\xi}$ is isomorphic to $P_\alpha * \dot{P}_\xi^{(\alpha)}$. Let $p_{\alpha+\xi} \in V^{P_{\alpha+\xi}}$ be the element corresponding to $\dot{q}_\xi$ under the isomorphism between $(V^{P\alpha})^{\dot{P}_\xi^{(\alpha)}}$ and $V^{P_{\alpha+\xi}}$.

Let $\pi(p,\dot{q}) = p^\frown\langle p_\alpha, p_{\alpha+1}, \ldots, p_{\alpha+i}, \ldots\rangle$. All we have to do now is to show that π is an isomorphism between $P_\alpha * \dot{P}_\beta^{(\alpha)}$ and $P_{\alpha+\beta}$. We shall show that for each $(p,\dot{q}) \in P_\alpha * \dot{P}_\beta^{(\alpha)}$, $\pi(p,\dot{q})$ is in $P_{\alpha+\beta}$ and leave the rest to the reader, namely to show that

$$(p,\dot{q}) \le (p',\dot{q}') \quad \text{if and only if} \quad \pi(p,\dot{q}) \le \pi(p',\dot{q}').$$

We want to show that for each $\gamma \le \beta$, $p^\frown\langle p_{\alpha+\xi} : \xi < \gamma\rangle$ is an element of $P_{\alpha+\gamma}$. We need to show that if γ is a limit ordinal and $P_{\alpha+\gamma}$ is the direct limit, then there exists $i_0 < \gamma$ such that $p_{\alpha+i} = 1$ for all i, $i_0 \le i < \gamma$.

Thus let $\gamma \le \beta$ be a limit ordinal such that $P_{\alpha+\gamma}$ is the direct limit of $P_{\alpha+\xi}$, $\xi < \gamma$. Hence in V^{P_α}, $\dot{P}_\gamma^{(\alpha)}$ is the direct limit, and therefore

$$\Vdash_{P_\alpha} (\exists \xi_0 < \gamma)(\forall \xi \ge \xi_0) \text{ the } \xi\text{th term of } \dot{q} \text{ is } 1. \tag{21.10}$$

Now we have made an assumption that if $P_{\alpha+\gamma}$ is the direct limit, then $\operatorname{cf}\gamma > |P_\alpha|$. It is easy to see that because $|P_\alpha| < \operatorname{cf}\gamma$, (21.10) implies that there exists $\xi_0 < \gamma$ such that for all $\xi \ge \xi_0$, $\Vdash_{P_\alpha} \dot{q}_i = 1$. Thus for all $\xi \ge \xi_0$, $\Vdash_{P_{\alpha+\xi}} p_{\alpha+\xi} = 1$ and hence $p_{\alpha+\xi} = 1$ for all $\xi \ge \xi_0$. □

Proof of Theorem 21.4. Let κ be a supercompact cardinal and assume $2^\kappa = \kappa^+$. We shall construct a generic extension in which κ is measurable and $2^\kappa = \kappa^{++}$.

We use iterated forcing with Easton support, successively adjoining to each inaccessible cardinal $\alpha \leq \kappa$, α^{++} subsets of α. At limit stages of the iteration we use direct limits when the ordinal is a regular cardinal and inverse limits otherwise.

Let us define, by induction on α, the αth iterate P_α (and the corresponding forcing relation $\Vdash_\alpha$ and the algebra $B_\alpha = B(P_\alpha)$) and the B_α-valued notion of forcing $\dot{Q}_\alpha$:

(21.11) (i) If α is an inaccessible cardinal, let $\dot{Q}_\alpha$ be the notion of forcing in V^{P_α} that adjoins α^{++} subsets of α; that is, we let $\dot{Q}_\alpha$ be in V^{P_α}, the set of all 0–1 functions p whose domain is a subset of size $< \alpha$ of $\alpha \times \alpha^{++}$ (and $\dot{Q}_\alpha$ is ordered by $\supset$). If α is not an inaccessible cardinal, let $\dot{Q}_\alpha = \{1\}$ (as usual, 1 denotes the greatest element of each notion of forcing).

(ii) P_α is the set of all α-sequences $\langle p_\xi : \xi < \alpha\rangle$ satisfying the following:
  (a) For every $\gamma < \alpha$, $p{\restriction}\gamma \in P_\gamma$ and $\Vdash_\gamma p_\gamma \in \dot{Q}_\gamma$.
  (b) If α is a regular cardinal, then $\exists \xi_0\, \forall \xi \geq \xi_0\, p_\xi = 1$.

(iii) If $p, q \in P_\alpha$, then $p \leq_\alpha q$ if and only if

$$(\forall \gamma < \alpha)(p{\restriction}\gamma \leq_\gamma q{\restriction}\gamma \text{ and } p{\restriction}\gamma \Vdash_\gamma p_\gamma \text{ is stronger than } q_\gamma).$$

Finally, let $P = P_{\kappa+1}$, and let $B = B(P)$.

Let G be a generic filter on P and let $V[G]$ be the generic extension of V by G. We shall prove that κ is a measurable cardinal in $V[G]$ and that $V[G] \Vdash 2^\kappa = \kappa^{++}$. Since P is isomorphic to the two-step iteration $P_\kappa * \dot{Q}_\kappa$, we have $V[G] = V[G_\kappa][H_\kappa]$, where G_κ is V-generic on P_κ and H_κ is $V[G_\kappa]$-generic on $Q_\kappa = (\dot{Q}_\kappa)^{G_\kappa}$. Now P_κ is the direct limit of P_α, $\alpha < \kappa$; and since κ is a Mahlo cardinal, there is a stationary set of $\alpha < \kappa$ such that P_α is also a direct limit. Since $|P_\alpha| < \kappa$ for all $\alpha < \kappa$, it follows by Theorem 16.30 that P_κ satisfies the κ-chain condition and hence κ is a regular cardinal in $V[G_\kappa]$. Also, $|P_\kappa| = \kappa$, and hence $V[G_\kappa]$ satisfies $(\forall \alpha < \kappa)\ 2^\alpha \leq \kappa$. In $V[G_\kappa]$, Q_κ is a notion of forcing that adjoins κ^{++} subsets of κ and preserves all cardinals. Thus $V[G] \Vdash (\kappa$ is a regular cardinal and $2^\kappa = \kappa^{++})$.

It remains to prove that κ is a measurable cardinal in $V[G]$. This will be done by first constructing an elementary embedding of $V[G]$ and then showing that the induced measure is in $V[G]$.

Let $\lambda = \kappa^{++}$. Since κ is supercompact, there is an elementary embedding $j : V \to M$ with critical point κ such that $M^\lambda \subset M$ and $j(\kappa) > \lambda$. It follows that $|P| = \lambda$, $P \in M$, and moreover, P is defined in M by the same definition (21.11).

Since $P \in M$, G is also an M-generic filter on P, and we can consider the model $M[G]$. We need the following lemma:

Lemma 21.9. $(M[G])^\lambda \cap V[G] \subset M[G]$.

Proof. It suffices to show that if $f \in V[G]$ is a function from λ into ordinals, then $f \in M[G]$. Let $\dot{f}$ be a name for f and let $p_0 \in G$ be a condition that

forces that $\dot{f}$ is a function from λ into the ordinals. For each $\alpha < \lambda$, let

$$A_\alpha = \{p \leq p_0 : \exists \beta\, p \Vdash \dot{f}(\alpha) = \beta\}.$$

Each A_α is dense below p_0 (and hence $A_\alpha \cap G \neq \emptyset$). For each $\alpha < \lambda$ and each $p \in A_\alpha$, let $g(\alpha, p)$ be the unique β such that $p \Vdash \dot{f}(\alpha) = \beta$. Since $|P| = \lambda$, we have $|g| = \lambda$ and hence $g \in M$. Now it is easy to see that $f \in M[G]$ because it is defined in $M[G]$ as follows: $f(\alpha) =$ the unique β such that for some $p \in G$, $g(\alpha, p) = \beta$. □

Let us now consider $j(P)$. In M, $j(P)$ is a notion of forcing obtained by iteration up to $j(\kappa) + 1$. We claim that in M we can apply the Factor Lemma to $j(P)$ at $\alpha = \kappa + 1$. First we note that $(jP)_\alpha = P_\alpha$ for all $\alpha < \kappa$, and since $(jP)_\kappa$ is the direct limit, we have $(jP)_\kappa = P_\kappa$. Since $\dot{Q}_\kappa$ is the same in V and M, it follows that $(jP)_{\kappa+1} = P_{\kappa+1}$. The first nontrivial step above α in the iteration occurs at the least inaccessible cardinal (in M) above κ, thus the first nontrivial direct limit is taken far above λ and then only at regular cardinals. Since $|P_{\kappa+1}| = \lambda$, the assumption of the Factor Lemma is satisfied.

Hence $j(P)$ is isomorphic to a two-step iteration (in M)

$$(jP)_{\kappa+1} * (jP)^{(\kappa+1)}_{j(\kappa)+1}. \tag{21.12}$$

Now the first factor of (21.12) is equal to $P_{\kappa+1} = P$. Let us denote $\dot{Q}$ the second factor. By the Factor Lemma, $\dot{Q}$ is, in M^P, a notion of forcing obtained by iteration, with Easton support, from $\kappa + 1$ to $j(\kappa) + 1$. At each $\xi > \kappa$, the iteration uses a notion of forcing in M^{P_ξ} that is either trivial or adjoins ξ^{++} subsets of ξ (if ξ is inaccessible in M); in either case, the notion of forcing is λ-directed closed in M^{P_ξ}. By Lemma 21.7, $\dot{Q}$ is λ-directed closed in M^P. Thus we can write

$$j(P) = P * \dot{Q} \tag{21.13}$$

where $\dot{Q} \in M^P$ is a λ-directed closed notion of forcing. Thus $Q = \dot{Q}^G$ is a λ-directed closed notion of forcing in $M[G]$.

Let $p \in P$. Then by (21.13), $j(p)$ is (represented by) a pair $(s, \dot{q})$ where $s \in P$ and $\dot{q} \in M^P$ is in $\dot{Q}$. By the definition of P, $p = \langle p_\xi : \xi < \kappa + 1\rangle$ and there is $\xi_0 < \kappa$ such that $p_\xi = 1$ for all ξ, $\xi_0 \leq \xi < \kappa$. Thus $j(p) = \langle p'_\xi : \xi < j(\kappa) + 1\rangle$ and $p'_\xi = 1$ for all ξ, $\xi_0 \leq \xi < j(\kappa)$. In particular, $p'_\kappa = 1$; and since $p'_\xi = p_\xi$ for all $\xi < \kappa$, and $s = j(p)\restriction(\kappa + 1)$, we have $s = (p\restriction\kappa)^\frown 1$. This implies that if $p \in G$ and $j(p) = (s, \dot{q})$, then $s \in G$.

Now let

$$D = \{q \in Q : \text{for some } p \in G,\ q = (\dot{q})^G \text{ where } j(p) = (s, \dot{q})\}.$$

Since P has size λ, we have $j\restriction P \in M$ and therefore $D \in M[G]$. It is easy to see that D is directed, i.e., if $q_1, q_2 \in D$, then there is $q \in D$ such that

$q \leq q_1$ and $q \leq q_2$ (this is because G is directed). We have (in $M[G]$), $|D| \leq |G| \leq |P| = \lambda$; and because Q is λ-directed closed, there exists some $a \in Q$ (a *master condition*) such that $a \leq q$ for all $q \in D$.

We shall now consider a generic extension of $V[G]$. Let H be a $V[G]$-generic filter on Q such that H contains the master condition a. Since H is also $M[G]$-generic, and $j(P) = P * \dot{Q}$, there is an M-generic filter K on $j(P)$ such that $M[K] = M[G][H]$; in fact

$$K = \{(s, \dot{q}) : s \in G \text{ and } (\dot{q})^G \in H\}.$$

Now we extend the elementary embedding $j : V \to M$ to an embedding of $V[G]$ into $M[K]$. We work in $V[G][H]$ and define, for all $x \in V[G]$,

$$j(x) = (j(\dot{x}))^K \tag{21.14}$$

where $\dot{x}$ is some P-name for x.

We have to show that the definition (21.14) does not depend on the choice of the name $\dot{x}$; the verification of elementarity of j is then straightforward. Here we use the master condition a. If $p \in G$, then $j(p) = (s, \dot{q})$ where $s \in P$ and $\Vdash \dot{q} \in \dot{Q}$. We have shown that $s \in G$, and if $q = (\dot{q})^G$, then, because $p \in G$, we have $q \geq a$ and therefore $q \in H$. Thus $(s, \dot{q}) \in K$, and it follows that

$$j\text{“}G \subset K.$$

Now if $p \in G$ forces $\dot{x} = \dot{y}$, then $j(p) \in K$ forces $j(\dot{x}) = j(\dot{y})$ and hence $(j(\dot{x}))^K = (j(\dot{y}))^K$.

Thus we have (in $V[G][H]$) an elementary embedding

$$j : V[G] \to M[K]$$

and we can define, in the usual way, a $V[G]$-ultrafilter on κ:

$$U = \{X \subset \kappa : \kappa \in j(X)\}.$$

U is nonprincipal and κ-complete. It suffices to show that $U \in V[G]$; then $V[G]$ satisfies that κ is a measurable cardinal.

By Lemma 21.9, Q is λ-closed not only in $M[G]$, but also in $V[G]$. Thus the generic extension $V[G][H]$ of $V[G]$ does not add any new λ-sequences in $V[G]$, and because $|U| = \lambda$ we have $U \in V[G]$. □

Prikry Forcing

Let us address the following problem: Can one construct a generic extension in which all cardinals are preserved but the cofinality of some cardinals is different from their cofinality in the ground model? Obviously, in order to do this we have to change some (weakly) inaccessible cardinal into a singular

cardinal. Corollary 18.31 of Jensen's Covering Theorem tells us that for this, it is necessary to assume at least $0^\sharp$ in the ground model. Thus we formulate the problem as follows: Let κ be some large cardinal and let $\lambda < \kappa$ be a regular cardinal. Find a cardinal preserving generic extension, in which $\operatorname{cf} \kappa = \lambda$.

The forcing presented below was devised by Karel Prikry and has become a standard tool of the large cardinal theory.

Theorem 21.10 (Prikry). *Let κ be a measurable cardinal. There is a generic extension in which $\operatorname{cf} \kappa = \omega$ and no cardinals are collapsed. Moreover, every bounded subset of κ in $V[G]$ is in the ground model.*

Proof. Let κ be a measurable cardinal and let D be a normal measure on κ. Let $(P, <)$ be the following notion of forcing. A forcing condition is a pair $p = (s, A)$ where $s \in [\kappa]^{<\omega}$, i.e., s is a finite subset of κ, and $A \in D$. A condition (s, A) is stronger than a condition (t, B) if:

(21.15) (i) t is an initial segment of s, i.e., $t = s \cap \alpha$ for some α;
(ii) $A \subset B$;
(iii) $s - t \subset B$.

We immediately note that any two conditions (s, A), (s, B) with the same first coordinate are compatible, and hence any antichain $W \subset P$ has size at most κ. We also note that if (s, A) and (t, B) are compatible, then either s is an initial segment of t, or t is an initial segment of s.

Let G be a generic filter on P. We shall show that in $V[G]$, κ has cofinality ω, that every bounded subset of κ is in V, and that all cardinals and cofinalities above κ are preserved.

The last statement is immediate since P satisfies the κ^+-chain condition. It is also easy to show that $\operatorname{cf} \kappa = \omega$ in $V[G]$: If (s, A) and (t, B) are in G, then either s is an initial segment of t or vice versa; hence $S = \bigcup\{s : (s, A) \in G \text{ for some } A\}$ is a subset of κ of order type ω. By the genericity of G, S is clearly an unbounded subset of κ, and hence $\operatorname{cf} \kappa = \omega$.

It remains to show that if $X \in V[G]$ is such that $X \subset \lambda$ for some $\lambda < \kappa$, then $X \in V$. For this, we need the property of P stated in Lemma 21.12 below. The proof uses Theorem 10.22 which states that every partition of $[\kappa]^{<\omega}$ into less than κ pieces has a homogeneous set $H \in D$.

Lemma 21.11. *Let σ be a sentence of the forcing language. There exists a set $A \in D$ such that the condition $(\emptyset, A)$ decides σ, i.e., either $(\emptyset, A) \Vdash \sigma$, or $(\emptyset, A) \Vdash \neg\sigma$.*

Proof. Let S^+ be the set of all $s \in [\kappa]^{<\omega}$ such that $(s, X) \Vdash \sigma$ for some X and let $S^- = \{s : \exists X\, (s, X) \Vdash \neg\sigma\}$. Let $T = [\kappa]^{<\omega} - (S^+ \cup S^-)$. By Theorem 10.22, there is a set $A \in D$ such that for every n, either $[A]^n \subset S^+$ or $[A]^n \subset S^-$ or $[A]^n \subset T$. We shall prove that $(\emptyset, A)$ decides σ.

If not, then there are conditions (s, X) and (t, Y), both stronger than $(\emptyset, A)$ such that one forces σ and the other forces $\neg\sigma$. We may assume that $|s| = |t|$,

say $|s| = |t| = n$. Since $(s, X) \leq (\emptyset, A)$, we have $s \in [A]^n$; and similarly, $t \in [A]^n$. But $s \in S^+$ and $t \in S^-$, which is a contradiction since S^+ and S^- are disjoint and therefore cannot both have a nonempty intersection with $[A]^n$. □

Lemma 21.12. *Let σ be a sentence of the forcing language and let (s_0, A_0) be a condition. Then there exists a set $A \subset A_0$ in D such that the condition (s_0, A) decides σ.*

Proof. A slight modification of the preceding proof; we may assume that $\min(A_0) > \max(s_0)$. Let S^+ be the set of all $s \in [A_0]^{<\omega}$ such that $(s_0 \cup s, X) \Vdash \sigma$ for some $X \subset A_0$ and let S^- be defined similarly. As before, there exists some $A \subset A_0$ in D such that for no n, $[A]^n$ intersects both S^+ and S^-. It follows that (s_0, A) decides σ. □

Now let $\lambda < \kappa$ and let $X \subset \lambda$; we will show that $X \in V$. Let $\dot{X}$ be a name for X, and let $p_0 \in G$ be a condition such that $p \Vdash \dot{X} \subset \lambda$. It suffices to show that for each $p \leq p_0$ there is a $q \leq p$ and a set $Z \subset \lambda$ such that $q \Vdash \dot{X} = Z$.

Let $p \leq p_0$, $p = (s, A)$. For each $\alpha < \lambda$, there is, by Lemma 21.12, a set $A_\alpha \subset A$ in D such that (s, A_α) decides the sentence $\alpha \in \dot{X}$. Let $B = \bigcap_{\alpha<\lambda} A_\alpha$; we have $B \in D$ and $q = (s, B)$ decides $\alpha \in \dot{X}$ for each $\alpha < \lambda$. Thus if $Z = \{\alpha < \lambda : q \Vdash \alpha \in \dot{X}\}$, we have $q \Vdash \dot{X} = Z$.

This completeness the proof of Theorem 21.10. □

An immediate consequence of Theorems 21.4 and 21.10 is the independence of SCH:

Corollary 21.13. *It is consistent (relative to the existence of a supercompact cardinal) that there is a strong limit singular cardinal κ such that $2^\kappa > \kappa^+$.*

Proof. Let κ be a supercompact cardinal. First we construct a generic extension in which κ is measurable and $2^\kappa > \kappa^+$. Then we extend the model further to make κ a singular cardinal. The new model still satisfies $2^\kappa > \kappa^+$, and κ is a strong limit cardinal. □

We now prove a characterization of Prikry generic sequences due to Mathias. Let us first generalize the *diagonal intersection* as follows: If $\{A_s : s \in [\kappa]^{<\omega}\}$ is a collection of subsets of κ, let

$$\triangle_s A_s = \{\alpha < \kappa : \alpha \in \bigcap\{A_s : \max(s) < \alpha\}\}. \tag{21.16}$$

It is routine to show that every normal ultrafilter on κ is closed under diagonal intersections (21.16).

Theorem 21.14 (Mathias). *Let M be a transitive model of* ZFC, *let U be, in M, a normal measure on κ, and let P be the Prikry forcing defined from U. For every set $S \subset \kappa$ of order-type ω, S is P-generic over M if and only if for every $X \in U$, $S - X$ is finite.*

Proof. In the easy direction, let G be a generic filter on P and let $S = \bigcup\{s : (s, A) \in G\}$. For every $X \in U$, $S - X$ is finite because for every condition (s, A), the stronger condition $(s, A \cap X)$ forces that every $\alpha \in S$ above s is in X.

For the other direction, let $S \subset \kappa$, of order-type ω, be such that $S - X$ is finite for all $X \in U$. We want to show that the filter

$$G = \{(s, A) \in P : s \text{ is an initial segment of } S \text{ and } S - s \subset A\}$$

is M-generic; let $D \in M$ be an open dense subset of P and let us show that $G \cap D \neq \emptyset$.

For each $s \in [\kappa]^{<\omega}$, let $F : [\kappa]^{<\omega} \to \{0, 1\}$ be a partition such that $F(t) = 1$ if and only if $\max(s) < \min(t)$ and $\exists X\ (s \cup t, X) \in D$. Let $A_s \in U$ be a homogeneous set for F; if there is an X such that $(s, X) \in D$, let $B_s = A_s \cap X$, and otherwise, let $B_s = A_s$. Let $A = \triangle_s B_s$ be the diagonal intersection. Since D is open dense, we have for all $s \in [\kappa]^{<\omega}$:

$$\text{If } \exists X\ (s, X) \in D \text{ then } (s, A \smallsetminus s) \in D \tag{21.17}$$

where $A \smallsetminus s = A - (\max(s) + 1)$.

By the assumption on S, S has an initial segment s such that $S - s \subset A$. By density of D there exist a $t \in [B \smallsetminus s]^{<\omega}$ and X such that $(s \cup t, X) \in D$. Let $u \subset S - s$ be such that $|u| = |t|$; the homogeneity of $A \smallsetminus s \subset A_s$ for F_s implies that for some Y, $(s \cup u, Y) \in D$. By (21.17) we have $(s \cup u, A \smallsetminus u) \in D$ and since $(s \cup u, A \smallsetminus u) \in G$, $D \cap G \neq \emptyset$. □

The following theorem shows the relationship between Prikry forcing and iterated ultrapowers. Let U be a normal measure on κ, and consider the iterated ultrapowers $M_\alpha = \mathrm{Ult}^{(\alpha)}$, and the embeddings $i_{\alpha,\beta} : M_\alpha \to M_\beta$. Let $\kappa_\alpha = \kappa^{(\alpha)} = i_{0,\alpha}(\kappa)$, and let $U^{(\alpha)} = i_{0,\alpha}(U)$ be the measure on κ_α in M_α.

Theorem 21.15. *Let $M = M_\omega$, $N = \bigcap_{n<\omega} M_n$ and let $P \in M$ be the Prikry forcing for the measure $U^{(\omega)}$ on κ_ω in M. The set $S = \{\kappa_n : n < \omega\}$ is P-generic over M, and $M[S] = N$.*

Proof. The genericity follows from Lemma 19.10 and Theorem 21.14. N is easily seen to be a model of ZF, and since $M[S] \subset M_n$ for all n, we have $M[S] \subset N$. In order to prove $N \subset M[S]$, it suffices, by Theorem 13.28, to prove that every set of ordinals in N is in $M[S]$.

First we claim that for every ordinal ξ

$$\text{for eventually all } n < \omega,\ i_{n,\omega}(\xi) = i_{\omega,\omega+\omega}(\xi). \tag{21.18}$$

As $\xi \in M_\omega$, let n be such that $\xi = i_{n,\omega}(\xi_n)$ for some ξ_n. Thus

$$M_n \vDash \xi \text{ is the image of } \xi_n \text{ under the embedding from } M_n \text{ into } M_\omega.$$

Applying $i_{n,\omega}$, we have

$$i_{n,\omega}(M_n) \vDash i_{n,\omega}(\xi) \text{ is the image of } i_{n,\omega}(\xi_n) \text{ under the embedding from } i_{n,\omega}(M_n) \text{ into } i_{n,\omega}(M_\omega).$$

Since $i_{n,\omega}(M_n) = M_\omega$ and $i_{n,\omega}(M_\omega) = M_{\omega+\omega}$, we get

$$i_{n,\omega}(\xi) \text{ is the image of } \xi \text{ under } i_{\omega,\omega+\omega},$$

establishing (21.18).

Now let x be a set of ordinals in N. Hence $x \in M_n$ for each n. By the representation of iterated ultrapowers, there is for each $n < \omega$ a function f_n on $[\kappa]^n$ such that $x = i_{0,n}(f_n)(\kappa_0, \ldots, \kappa_{n-1})$. Since $i_{n,\omega}(\kappa_i) = \kappa_i$ for $i < n$, we have $i_{n,\omega}(x) = i_{0,\omega}(f)(\kappa_0, \ldots, \kappa_{n-1})$. Now the sequence $\langle i_{0,\omega}(f_n) : n < \omega\rangle = i_{0,\omega}(\langle f_n : n < \omega\rangle)$ is in M_ω and therefore the sequence $\langle i_{n,\omega}(x) : n < \omega\rangle$ is in $M_\omega[\langle \kappa_n : n < \omega\rangle] = M[S]$.

If ξ is an ordinal then $\xi \in x$ if and only if for any n, $i_{n,\omega}(\xi) \in i_{n,\omega}(x)$, and by (21.18), if for eventually all $n < \omega$, $i_{\omega,\omega+\omega}(\xi) \in i_{n,\omega}(x)$. Since $i_{\omega,\omega+\omega}$ is definable in M, x is definable from the sequence $\langle i_{n,\omega}(x) : n < \omega\rangle$ in $M[S]$. Hence $x \in M[S]$, and $N \subset M[S]$. □

Measurability of $\aleph_1$ in ZF

In ZF (without the Axiom of Choice), one can still define measurability in the usual way: An uncountable cardinal κ is measurable if there exists a κ-complete nonprincipal ultrafilter on κ. In the absence of the Axiom of Choice, a measurable cardinal is still regular, but not necessarily a limit cardinal. The absence of AC has no effect on the consistency strength: If U is a nonprincipal κ-complete ultrafilter on $\kappa > \omega$, then in $L[U]$ (a model of ZFC), κ is a measurable cardinal. The following theorem shows that in ZF, $\aleph_1$ can be measurable:

Theorem 21.16. *Let M be a transitive model of* ZFC + "*there is a measurable cardinal.*" *There is a symmetric model $N \supset M$ of* ZF *such that $N \vDash \aleph_1$ is measurable.*

We shall construct a symmetric extension of M. Recall the theory of symmetric models from Chapter 15. We consider a complete Boolean algebra B, a group $\mathcal{G}$ of automorphisms of B, and a normal filter $\mathcal{F}$ on $\mathcal{G}$ (see (15.34)). For every $\dot{x} \in M^B$ we let $\mathrm{sym}(\dot{x})$ be the symmetry group of $\dot{x}$, $\mathrm{sym}(\dot{x}) = \{\pi \in \mathcal{G} : \pi(\dot{x}) = \dot{x}\}$, and call $\dot{x}$ symmetric if $\mathrm{sym}(\dot{x}) \in \mathcal{F}$. We denote HS the class of all hereditarily symmetric names. If G is an M-generic ultrafilter on B, we let N be the G-interpretation of the class HS; N is a model of ZF and $N \supset M$.

Let us call a subset $A \subset B$ *symmetric* if

$$\{\pi \in \mathcal{G} : \pi(a) = a \text{ for all } a \in A\} \in \mathcal{F}.$$

Lemma 21.17. *Let κ be measurable in M, and let N be a symmetric extension of M (via B, G, $\mathcal{F}$, $\mathcal{G}$). If every symmetric subset of B has size $< \kappa$, then κ is measurable in N.*

Proof. Let U be, in M, a κ-complete nonprincipal ultrafilter on κ. We show that U generates a κ-complete nonprincipal ultrafilter on κ in N. It suffices to show that if $\gamma < \kappa$ and $\{X_\alpha : \alpha < \gamma\}$ is a partition of κ in N, then for some $\alpha < \gamma$, X_α includes some $Y \in U$.

We give the proof for $\gamma = 2$ since the general case is analogous. Let $X \in N$ be a subset of κ, and let $\dot{X} \in HS$ be a symmetric name for X. Let $A = \{\|\alpha \in \dot{X}\| : \alpha < \kappa\}$. If $\pi \in \mathcal{G}$ is such that $\pi(\dot{X}) = \dot{X}$, then (because $\pi(\check{\alpha}) = \check{\alpha}$ for all α) $\pi(a) = a$ for all $a \in A$; thus A is a symmetric subset of B.

Hence $|A| < \kappa$. For each $a \in A$, let $Y_a = \{\alpha : \|\alpha \in \dot{X}\| = a\}$. Clearly, $\{Y_a : a \in A\}$ is a partition of κ into fewer than κ pieces, and hence one $Y = Y_\alpha$ is in U. Now if $a \in G$, then we have $Y \subset X$, and if $a \notin G$, then $Y \subset \kappa - X$. Hence either X or $\kappa - X$ has a subset that is in U. □

Proof of Theorem 21.16. Let κ be a measurable cardinal in M. Let P be the set of all one-to-one finite sequences $p = \langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle$ of ordinals less than κ; p is stronger than q if p extends q (P collapses κ; cf. Example 15.20). Let $\mathcal{G}$ be the set of all permutations of κ; every $\pi \in \mathcal{G}$ induces an automorphism of $(P, <)$ as follows:

$$\pi(\langle \alpha_0, \ldots, \alpha_{n-1} \rangle) = \langle \pi(\alpha_0), \ldots, \pi(\alpha_{n-1}) \rangle$$

and, in turn, an automorphism of $B = B(P)$. Thus we identify $\mathcal{G}$ with the group of automorphisms so induced.

For each $\gamma < \kappa$, let

$$H_\gamma = \{\pi \in \mathcal{G} : \pi(\alpha) = \alpha \text{ for all } \alpha < \gamma\}$$

and let $\mathcal{F}$ be the normal filter on $\mathcal{G}$ generated by $\{H_\gamma : \gamma < \kappa\}$. Thus $\dot{x} \in M^B$ is symmetric if and only if there is some $\gamma < \kappa$ such that $\pi(\dot{x}) = \dot{x}$ whenever $\pi(\alpha) = \alpha$ for all $\alpha < \gamma$.

Let G be an M-generic filter on B and let N be the symmetric model given by B, G, $\mathcal{F}$, $\mathcal{G}$. We shall show that $\kappa = (\aleph_1)^N$ and that κ is measurable in N.

If $\gamma < \kappa$, then γ is countable in N: Let $\dot{f}$ be the name such that

$$\|\dot{f}(n) = \alpha\| = \sum\{p \in P : p(n) = \alpha\}$$

for all $n < \omega$ and $\alpha < \gamma$. Clearly, $\dot{f}$ is symmetric because $\pi(\dot{f}) = \dot{f}$ for every $\pi \in H_\gamma$, and hence $\dot{f} \in HS$. The interpretation of $\dot{f}$ is a one-to-one function of a subset of ω onto γ.

It remains to show that κ is measurable in N. By Lemma 21.17 it suffices to show that every symmetric $A \subset B$ has size $< \kappa$. Let $A \subset B$ be symmetric. There exists a $\gamma < \kappa$ such that $\pi(a) = a$ for all $a \in A$ and all $\pi \in H_\gamma$.

For every $a \in A$, let $S_a = \{p \in P : p \leq a\}$. If $\pi \in H_\gamma$ and $p \in S_a$, then $\pi(p) \in S_a$ because $\pi(p) \leq \pi(a) = a$. Let $T_a = \{p \in S_a : p(n) < \gamma + \omega$ for all $n \in \mathrm{dom}(p)\}$. If $\pi \in H_\gamma$ and $p \in T_a$, then $\pi(p) \in S_a$. Conversely, if $p \in S_a$, there is a $\pi \in H_\gamma$ that maps all $\alpha \in \mathrm{ran}(p)$ greater than γ into $\gamma + \omega$; since $\pi \in H_\gamma$ and $p \in S_a$, we have $\pi(p) \in S_a$ and hence $\pi(p) \in T_a$. Thus $S_a = \{\pi(p) : p \in T_a$ and $\pi \in H_\gamma\}$, and consequently

$$\text{if } a \neq b \in A, \text{ then } T_a \neq T_b. \tag{21.19}$$

However, each T_a is a set of finite sequences in $\gamma + \omega$; and since κ is inaccessible, (21.19) implies that $|A| < \kappa$. □

Exercises

21.1. Let κ be measurable and $|P| < \kappa$; let U be a κ-complete ultrafilter on κ. Then in $V[G]$, the filter $W = \{X \subset \kappa : X \supset Y$ for some $Y \in U\}$ is a κ-complete ultrafilter on κ.

[For instance, to show that W is an ultrafilter, consider $\dot{X} \in V^B$ such that $\|\dot{X} \subset \kappa\| \in G$. The function $\alpha \mapsto \|\alpha \in \dot{X}\|$ is a partition of κ into $|B| < \kappa$ pieces, and by the κ-completeness of U there is a $Y \in U$ such that $\|\alpha \in \dot{X}\|$ is the same for all $\alpha \in Y$. Now either $X \in W$ or $\kappa - X \in W$ according to whether this B-value is in G or not.]

21.2. If κ is an inaccessible cardinal and $|P| < \kappa$, then every closed unbounded set $C \subset \kappa$ in $V[G]$ has a closed unbounded subset in V. (See Lemma 22.25 for a stronger result.)

[For each $b \in B(P)$, let $C_b = \{\alpha : \|\alpha \in \dot{C}\| = b\}$. Using $|B| < \kappa$, show that for some $b \in G$, C_b is unbounded. Let D be the closure of C_b.]

21.3. Let κ be a regular uncountable cardinal and let $\nu < \kappa$. If I is a κ-complete ν-saturated ideal on κ then either κ is measurable or $\kappa \leq 2^\nu$.

[Use the proof of Lemma 10.9.]

21.4. Let κ be an inaccessible cardinal. There is a notion of forcing $(P, <)$ such that $|P| = \kappa$ and P is α-distributive for all $\alpha < \kappa$, and such that κ is not a Mahlo cardinal in the generic extension.

[Forcing conditions are sets $p \subset \kappa$ such that $|p \cap \gamma| < \gamma$ for every regular $\gamma \leq \kappa$; $p \leq q$ if and only if p is an end-extension of q, i.e., if $q = p \cap \alpha$ for some α. To show that for any $\alpha < \kappa$, P does not add any new α-sequence, observe that for every p there is a $q \leq p$ such that $P_q = \{r \in P : r \leq q\}$ is α-closed.]

21.5. If κ is a measurable cardinal and P is a κ-closed notion of forcing (or just κ-distributive), then κ is measurable in the generic extension.

21.6. It is consistent that $2^{\mathrm{cf}\,\kappa} < \kappa$ and $\kappa^+ < \kappa^{\mathrm{cf}\,\kappa} < 2^\kappa$.

[Extend the model in Corollary 21.13 by adding a large number of subsets of ω_1.]

The Prikry model $V[G]$ of Theorem 21.10 provides an example of a singular Rowbottom cardinal. The exercise below shows that κ has in $V[G]$ the combinatorial property equivalent to being a Rowbottom cardinal.

21.7. In the Prikry model, for every partition $F : [\kappa]^{<\omega} \to \lambda$ into $\lambda < \kappa$ pieces there exists a set $H \subset \kappa$ of size κ such that F takes at most $\aleph_0$ values on $[H]^{<\omega}$.

[Let $\dot{F}$ be a name for F and let (s_0, A_0) be a condition (such that $\max(s_0) < \min(A_0)$). Let g be a partition of $[\kappa]^{<\omega} \times [\kappa]^{<\omega}$ into λ pieces, defined as follows: If $s \in [A_0]^{<\omega}$ and for some $X \subset A_0$, $(s_0 \cup s, X) \Vdash \dot{F}(t) = \alpha$, then let $g(s,t) = \alpha$; otherwise, let $g(s,t) = 0$. Show that there is $A \subset A_0$ in D and a countable $S \subset \lambda$ such that $g([A]^{<\omega} \times [A^{<\omega}]) \subset S$. Then $(s_0, A) \Vdash \dot{F}([A]^{<\omega}) \subset S$.]

Historical Notes

Theorem 21.1 is due to Lévy and Solovay [1967]. Theorem 21.3 was proved by Kunen and Paris in [1970/71] Theorem 21.4 is an unpublished result of Silver; an account of Silver's forcing appeared in Menas [1976]. Theorem 21.10 is due to Prikry [1970]. The characterization of Prikry sequences (Theorem 21.14) appeared in Mathias [1973]. Theorem 21.15 was proved by Bukovský [1973, 1977] and by Dehornoy [1975, 1978]. Theorem 21.16 is due to Jech [1968] and Takeuti [1970].

Exercise 21.4: Jensen.

Exercise 21.7: Prikry [1970].

22. Saturated Ideals

One of the key concepts in the theory of large cardinals is *saturation* of ideals. In this chapter we investigate σ-saturated, κ-saturated and κ^+-saturated κ-complete ideals on κ.

Let κ be a regular uncountable cardinal. Let I be a κ-complete ideal on κ containing all singletons; thus $X \in I$ whenever $X \subset \kappa$ is such that $|X| < \kappa$. We shall be using the following terminology: X has *measure zero* if $X \in I$, *measure one* if $\kappa - X \in I$, and *positive measure* if $X \notin I$; the phrase *almost all* α means that the set of all contrary α's has measure 0.

Let us consider the Boolean algebra $B = P(\kappa)/I$. Recall that if λ is a cardinal, then B is called λ-*saturated* if every pairwise disjoint family of elements of B has size less than λ; $\mathrm{sat}(B)$ is the least λ such that B is λ-saturated. Let us say that I is λ-*saturated* if B is λ-saturated and let

$$\mathrm{sat}(I) = \mathrm{sat}(B).$$

In other words, I is λ-saturated just in case there exists no collection W of size λ of subsets of κ such that $X \notin I$ for all $X \in W$ and $X \cap Y \in I$ whenever X and Y are distinct members of W. If $\mathrm{sat}(I)$ is finite, then κ is the union of finitely many atoms of I; if $\mathrm{sat}(I)$ is infinite, then it is uncountable and regular, by Theorem 7.15. If $\lambda \leq \kappa$, then I is λ-saturated if and only if there is no disjoint collection W of size λ of subsets X of κ such that $X \notin I$ (see Exercise 22.1). Clearly, every I on κ is $(2^\kappa)^+$-saturated. Thus if I is atomless, then $\mathrm{sat}(I)$ is a regular cardinal and

$$\aleph_1 \leq \mathrm{sat}(I) \leq (2^\kappa)^+.$$

Since I is κ-complete, it follows that $B = P(\kappa)/I$ is a κ-complete Boolean algebra.

Real-Valued Measurable Cardinals

By Ulam's Theorem 10.1, if there exists a nontrivial σ-additive measure then either there exists a measurable cardinal or there exists a real-valued measurable cardinal.

In this section we prove the following theorems:

Theorem 22.1 (Solovay).

(i) *If κ is a real-valued measurable cardinal, then there is a transitive model of set theory in which κ is measurable.*
(ii) *If κ is a measurable cardinal, then there exists a generic extension in which $\kappa = 2^{\aleph_0}$ and κ is real-valued measurable.*

Theorem 22.2 (Prikry). *If $2^{\aleph_0}$ is real-valued measurable, then $2^\lambda = 2^{\aleph_0}$ for all infinite $\lambda < 2^{\aleph_0}$.*

If μ is a κ-additive real-valued measure on κ, then the ideal I_μ of all sets of measure 0 is a σ-saturated κ-complete ideal on κ. We have proved that if an uncountable cardinal κ carries a σ-saturated κ-complete ideal, then κ is weakly inaccessible.

We shall prove Theorem 22.1(i) and Theorem 22.2 for this generalization of real-valued measurability, namely under the assumption that κ is uncountable and carries a σ-saturated κ-complete ideal. Thus let κ be an uncountable cardinal and let I be a σ-saturated κ-complete ideal. Thus let κ be an uncountable cardinal and let I be a σ-saturated κ-complete ideal on κ.

Let us call $A \subset \kappa$ an *atom* if A has positive measure and is not the union of two disjoint sets of positive measure. I is *atomless* if it has no atoms. What we proved in Lemma 10.9(ii) can be formulated as follows: If I is atomless, then $\kappa \leq 2^{\aleph_0}$. It follows that if $2^{\aleph_0} < \kappa$, then every set X of positive measure contains an atom $A \subset X$, and hence there exists an at most countable disjoint collection W of atoms such that $\kappa = \bigcup\{A : A \in W\}$.

We start with the following analog of Theorem 10.20. We recall that a κ-complete ideal on κ is normal if every function $f : S \to \kappa$ regressive on a set $S \subset \kappa$ of positive measure is constant on some $T \subset S$ of positive measure. A real-valued measure μ is *normal* if I_μ is normal.

Lemma 22.3.

(i) *If I is a σ-saturated κ-complete ideal on an uncountable cardinal κ, then there exists a function $f : \kappa \to \kappa$ such that*

$$J = f_*(I) = \{X \subset \kappa : f_{-1}(X) \in I\}$$

is a normal σ-saturated κ-complete ideal on κ.
(ii) *If μ is a κ-additive real-valued measure on κ, then there exists a function $f : \kappa \to \kappa$ such that $\nu = f_*(\mu)$ defined by*

$$\nu(X) = \mu(f_{-1}(X)) \qquad (X \subset \kappa)$$

is a normal κ-additive real-valued measure on κ.

Proof. We shall prove (i) and leave the completely analogous proof of (ii) to the reader. Let us say that a function $g : S \to \kappa$ is *unbounded* on a set S of positive measure if there is no $\gamma < \kappa$ and no $T \subset S$ of positive measure such

that $g(\alpha) < \gamma$ for all $\gamma \in T$. Let us consider the family $\mathcal{F}$ of all functions g into κ defined on a set of positive measure and unbounded on its domain. Let us define $g < h$ if $\mathrm{dom}(g) \subset \mathrm{dom}(h)$ and if $g(\alpha) \leq h(\alpha)$ everywhere on $\mathrm{dom}(g)$. Let us also define $g \leq h$ if $\mathrm{dom}(g) \subset \mathrm{dom}(h)$ and if $g(\alpha) \leq h(\alpha)$ everywhere on $\mathrm{dom}(g)$. Let us call $g \in \mathcal{F}$ *minimal* if there is no $h \in \mathcal{F}$ such that $h < g$.

We shall first show that there exists a minimal $g \in \mathcal{F}$. Otherwise, for every $g \in \mathcal{F}$ there is $h \in \mathcal{F}$ such that $h < g$. Thus let $g \in \mathcal{F}$ be arbitrary. Let W be a maximal collection of elements of $\mathcal{F}$ such that $h < g$ for each $h \in W$, and that $\mathrm{dom}(h_1) \cap \mathrm{dom}(h_2) = \emptyset$ whenever h_1 and h_2 are distinct elements of W. Since I is σ-saturated, W is at most countable and by our assumption, the set $\mathrm{dom}(g) - \bigcup\{\mathrm{dom}(h) : h \in W\}$ has measure zero. Thus if we let $f = \bigcup\{h : h \in W\}$, we have $\mathrm{dom}(g) - \mathrm{dom}(f) \in I$, and $f < g$. Since g was arbitrary, we can construct a countable sequence $g_0 > g_1 > \ldots > g_n > \ldots$ such that $\mathrm{dom}(g_n) - \mathrm{dom}(g_{n+1}) \in I$ for each n. It follows that $\bigcap_{n=0}^{\infty} \mathrm{dom}(g_n)$ has positive measure and we get a contradiction since for any $\alpha \in \bigcap_{n=0}^{\infty} \mathrm{dom}(g_n)$ we would have $g_0(\alpha) > g_1(\alpha) > \ldots$.

The same argument shows that for every $h \in \mathcal{F}$ there exists a minimal $g \in \mathcal{F}$ such that $g \leq h$. Thus if W is a maximal family of minimal functions $g \in \mathcal{F}$ such that $\mathrm{dom}(g_1) \cap \mathrm{dom}(g_2) = \emptyset$ whenever g_1 and g_2 are distinct elements of W, W is at most countable and $\bigcup\{\mathrm{dom}(g) : g \in W\}$ has measure one. Thus if we let $f = \bigcup\{g : g \in W\}$, then $\mathrm{dom}(f)$ has measure one and f is a *least* unbounded function: On the one hand, if $\gamma < \kappa$, then there is no $S \subset \kappa$ of positive measure such that $f(\alpha) < \gamma$ everywhere on S; on the other hand, if S is a set of positive measure and g is a function on S such that $g(\alpha) < f(\alpha)$ everywhere on S, then g is constant on some $T \subset S$ of positive measure. We can clearly assume that $\mathrm{dom}(f) = \kappa$.

Let $f : \kappa \to \kappa$ be a least unbounded function; we shall show that $J = f_*(I)$ is a normal σ-saturated κ-complete ideal on κ. It is obvious that J is a κ-complete ideal. For every $\gamma \in \kappa$, $f_{-1}(\{\gamma\})$ has measure zero and hence $\{\gamma\} \in J$. If $X \notin J$, then $f_{-1}(X) \notin I$, and if $X \cap Y = \emptyset$, then $f_{-1}(X) \cap f_{-1}(Y) = \emptyset$, and hence J is σ-saturated because I is σ-saturated.

To show that J is normal, let $S \notin I$, and let $g(\alpha) < \alpha$ for all $\alpha \in S$. Then $g(f(\xi)) < f(\xi)$ for all $\xi \in f_{-1}(S)$ and since f is a least unbounded function, $g(f(\xi))$ is constant on some $X \subset f_{-1}(S)$ of positive I-measure. Hence g is constant on $f(X)$ and $f(X) \notin J$. □

Lemma 22.4. *Let I be a normal σ-saturated κ-complete ideal on κ. If S is a set of positive measure and $f : S \to \kappa$ is regressive on S, then f is bounded almost everywhere on S; that is, there exists $\gamma < \kappa$ such that $\{\alpha \in S : f(\alpha) \geq \gamma\} \in I$.*

Proof. For every $X \subset S$ of positive measure there exists $Y \subset X$ of positive measure such that f is constant on Y. Thus let W be a maximal disjoint family of sets $X \subset S$ of positive measure such that f is constant on X. Let

$T = \bigcup\{X : X \in W\}$. The family W is at most countable and hence there is γ such that $f(\alpha) < \gamma$ for all $\alpha \in T$. Clearly, $S - T$ has measure 0. □

Corollary 22.5. *If κ is real-valued measurable (or if κ carries a σ-saturated κ-complete ideal), then κ is a weakly Mahlo cardinal.*

Proof. Let I be a normal σ-saturated κ-complete ideal on κ. Since I is normal, every closed unbounded set has I-measure one (see Lemma 8.11). Because κ is weakly inaccessible, it suffices to show that the set of all regular cardinals $\alpha < \kappa$ has measure one.

Let us assume that the set S of all limit ordinals $\alpha < \kappa$ such that $\operatorname{cf}\alpha < \alpha$ has positive measure. Considering the regressive function $\alpha \mapsto \operatorname{cf}\alpha$, we find a set T of positive measure and some $\lambda < \kappa$ such that $\operatorname{cf}\alpha = \lambda$ for all $\alpha \in T$. For each $\alpha \in T$, let $\langle \alpha_\nu : \nu < \lambda\rangle$ be an increasing λ-sequence with limit α.

For each $\nu < \lambda$, the function $\alpha \mapsto \alpha_\nu$ is regressive on T and so, by Lemma 22.4 there is γ_ν such that $\alpha_\nu < \gamma_\nu$ for almost all $\alpha \in T$. Let $\gamma = \sup\{\gamma_\nu : \nu < \lambda\}$. Since $\lambda < \kappa$, we conclude, by κ-completeness of I, that for almost all $\alpha \in T$, $\alpha_\nu < \gamma$ of all $\nu < \lambda$. But this means that for almost all $\alpha \in T$, $\alpha = \lim_\nu \alpha_\nu \leq \gamma$. This is a contradiction since T is unbounded. □

Since every closed unbounded set has measure one (if I is a normal σ-saturated κ-complete ideal on κ), every set of positive measure is stationary. It can even be proved that if S has positive measure, then $S \cap \alpha$ is stationary in α for almost all α. Then it follows that κ is the κth weakly Mahlo cardinal, κth cardinal which is a limit of weakly Mahlo cardinals, etc. We shall return to this subject later in this chapter.

We shall now show that every real-valued measurable cardinal is a Rowbottom cardinal; we shall show that the statement of Lemma 17.36 for a measurable cardinal holds under the weaker assumption that κ carries a σ-saturated κ-complete ideal.

Lemma 22.6. *Let I be a normal σ-saturated κ-complete ideal on κ, and let λ be an infinite cardinal less than κ. Let $\mathfrak{A} = (A, \ldots)$ be a model of a language $\mathcal{L}$ such that $|\mathcal{L}| \leq \lambda$, and let $A \supset \kappa$. If $P \subset A$ is such that $|P| < \kappa$, then $\mathfrak{A}$ has an elementary submodel $\mathfrak{B} = (B, \ldots)$ such that $B \cap \kappa$ has measure one and $|P \cap B| \leq \lambda$. Moreover, if $X \subset A$ has size at most λ, then we can find $\mathfrak{B}$ such that $X \subset B$.*

The proof of Lemma 22.6 uses Skolem functions and arguments similar to those in Theorem 17.27 and Lemma 17.36. The key ingredient is the following lemma:

Lemma 22.7. *Let I be a normal σ-saturated κ-complete ideal on κ, let $\gamma < \kappa$ and let $f : [\kappa]^{<\omega} \to \gamma$ be a partition. Then there exists $H \subset \kappa$ of measure one such that the image of $[H]^{<\omega}$ under f is at most countable.*

Proof. We proceed as in the proof of Theorem 10.22. It suffices to show that for each $n = 1, 2, \ldots$ there is H_n of measure one such that $f([H]^n)$ is at most countable; then we take $H = \bigcap_{n=1}^{\infty} H_n$.

We prove, by induction on n, that for every partition of $[\kappa]^n$ into less than κ pieces there is $H \subset \kappa$ of measure one such that $f([H]^n)$ is at most countable. For $n = 1$, let $f : \kappa \to \gamma$ and $\gamma < \kappa$; let W be a maximal pairwise disjoint family of subsets $X \subset \kappa$ such that X has positive measure and f is constant on X. Let $H = \bigcup\{X : X \in W\}$. Since $|W| \leq \aleph_0$, we have $|f(H)| \leq \aleph_0$, and since $\gamma < \kappa$ and I is κ-complete, we clearly have $\kappa - H \in I$.

Let us assume that the assertion is true for n and let us prove that it holds also for $n+1$. Let $f : [\kappa]^{n+1} \to \gamma$ where $\gamma < \kappa$. For each $\alpha < \kappa$, we define f_α on $[\kappa - \{\alpha\}]^n$ by $f_\alpha(x) = f(\{\alpha\} \cup x)$. By the induction hypothesis, there exists for each $\alpha < \kappa$ a set X_α of measure one such that $f_\alpha([X_\alpha]^n)$ is at most countable; let A_α be the image of $[X_\alpha]^n$ under f_α. Let X be the diagonal intersection

$$X = \{\alpha < \kappa : \alpha \in \bigcap_{\xi<\alpha} X_\xi\}$$

The set X has measure one since I is normal; also if $\alpha < \alpha_1 < \ldots < \alpha_n$ are in X, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\alpha]^n$ and so $f(\{\alpha, \alpha_1, \ldots, \alpha_n\}) = f_\alpha(\{\alpha_1, \ldots, \alpha_n\}) \in A_\alpha$.

For each $\alpha \in X$, let $A_\alpha = \{a_{\alpha,n} : n < \omega\}$. For each n, consider the function $g_n : X \to \gamma$ defined by $g_n(\alpha) = a_{\alpha,n}$. There exists a set $H_n \subset X$ of measure one such that $g_n(H_n)$ is at most countable. Thus let $H = \bigcap_{n=0}^{\infty} H_n$; the set H has measure one, and moreover $\bigcup\{A_\alpha : \alpha \in H\} = \bigcup_{n=0}^{\infty} g_n(H)$ is at most countable. It follows that $f([H]^{n+1})$ is at most countable. □

We can now proceed as in Theorem 19.3 and prove that if I is a normal σ-saturated κ-complete ideal on κ and $V = L[I]$, then GCH holds. In fact, if D denotes the filter dual to I, that is, the filter of all sets of I-measure one, then the proof of Theorem 19.3 goes through in the present context (use $P = \{Y \subset \lambda : Y \leq_{L[D]} X\}$).

Now we recall the results of Chapter 18: If κ carries a σ-saturated κ-complete ideal then either $\kappa \leq 2^{\aleph_0}$ or κ is measurable. Thus we conclude: If I is a normal σ-saturated κ-complete ideal on κ and $V = L[I]$, then κ is measurable.

Proof of Theorem 22.1(i). Let κ be real-valued measurable. Then there is a normal κ-additive measure μ on κ by Lemma 22.3. Let I be the ideal of sets of measure zero. I is a normal σ-saturated κ-complete ideal on κ. Let $J = I \cap L[I]$. We have $L[J] = L[I]$, and in $L[I]$, J is a normal $\aleph_1$-saturated κ-complete ideal on κ. (If we could assume that $\aleph_1^{L[I]} = \aleph_1$, it would now follow that κ is a measurable cardinal in $L[I]$.)

Since we are not able to show directly that if I is σ-saturated, then $I \cap L[I]$ is σ-saturated in $L[I]$, let us consider a somewhat more general situation.

Let ν be a regular uncountable cardinal less than κ, and let us consider ν-saturated κ-complete ideals on κ.

Lemmas 22.3 and 22.4 hold again; in Lemma 22.7 we have to replace "at most countable" by "of size less than ν." Lemma 22.6 holds for all $\lambda \geq \nu$ and the analog of Theorem 19.3 is: If $V = L[I]$ and I is normal, then $2^{<\nu} = \nu$ and $2^\lambda = \lambda^+$ for all $\lambda \geq \nu$.

Lemma 10.9 can also be generalized, and we get: If I is atomless, then $\kappa \leq 2^{<\nu}$. Hence if $V = L[I]$, every set of positive measure contains a subset that is an atom, and therefore κ is the union of a disjoint family W of atoms such that $|W| < \nu$.

Lemma 22.8. *Let $\nu < \kappa$ be a regular uncountable cardinal, and let I be a normal ν-saturated κ-complete ideal on κ; let F be the dual filter. Then in $L[F]$, $F \cap L[F]$ is a normal measure on κ (and $L[F]$ is the model $L[D]$ of Chapter 19).*

Proof. It is easy to verify that $L[F] = L[I]$, and that in $L[F]$, $I \cap L[I]$ is a normal ν-saturated κ-complete ideal on κ. Thus we may assume that $V = L[F]$; we want to show that F is an ultrafilter.

We know that κ is the union of a disjoint family W of atoms. (What we want to show is that W has only one element.) For $A \in W$, let

$$F_A = \{X \subset \kappa : X \cap A \text{ has positive measure}\}.$$

Since A is an atom, F_A is a filter, and F_A is in fact a normal measure on κ. Hence $F_A \cap L[F_A]$ is the unique normal measure D in $L[F_A]$, and $L[F_A]$ is *the* model $L[D]$.

We shall now show that $F \cap L[D] = D$. Let $X \in L[D]$ be a subset of κ. If $X \in F$, then $X \in F_A$ for all $A \in W$ and hence $X \in F_A \cap L[F_A] = D$. If $X \notin F$, then there is $A \in W$ such that $X \notin F_A$ and hence $X \notin F_A \cap L[F_A] = D$. It follows that $F \cap L[D] = D$ and so $F \cap L[D] \in L[D]$.

Consequently, $L[F] = L[D]$; since we assumed that $V = L[F]$ and because $F \cap L[D] = D$, we have $F = D$. □

Proof of Theorem 22.1(ii). Let κ be a measurable cardinal, and let $\lambda \geq \kappa$ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. We shall construct a generic extension in which $2^{\aleph_0} = \lambda$ and κ is real-valued measurable.

Let F be a σ-algebra of sets and let μ be a measure on F. Let $I \subset F$ be the ideal of sets of measure 0 and let us consider the Boolean algebra $B = F/I$. That is, the members of B are equivalence classes $[X]$ where $X \in F$ and where $X \equiv Y$ if and only if $\mu(X \triangle Y) = 0$.

Since both F and I are countably complete, it follows that B is countably saturated and $\sum_{n=0}^{\infty}[X_n] = [\bigcup_{n=0}^{\infty} X_n]$. Since μ is a measure, I is countably saturated and so B satisfies the countable chain condition. Now a Boolean algebra that is both σ-complete and σ-saturated is complete, and so B is a complete Boolean algebra.

For $[X] \in B$, let us define $m([X]) = \mu(X)$. Clearly, the definition of m does not depend on the particular choice of X, and furthermore, m has the following properties:

(22.1) (i) m is a real-valued function on B;
(ii) $m(0) = 0$, $m(a) > 0$ if $a \neq 0$, and $m(1) = 1$;
(iii) if $a \leq b$, then $m(a) \leq m(b)$;
(iv) if a_n, $n = 0, 1, \ldots$, are pairwise disjoint, then

$$m\Big(\sum_{n=0}^{\infty} a_n\Big) = \sum_{n=0}^{\infty} m(a_n).$$

A Boolean algebra B with a measure m (satisfying (22.1)) is called a *measure algebra*; a set S with a field of sets $\mathcal{F}$ and a measure μ on $\mathcal{F}$ is called a *measure space*.

We need from measure theory the following basic fact about *products* of measure spaces. Let I be a set (of indices), and for each $i \in I$ let $(S_i, \mathcal{F}_i, \mu_i)$ be a measure space. Let us consider the product $S = \prod_{i\in I} S_i$, and let us consider the following σ-algebra of subsets of S: Let E be a finite subset of I, and for each $i \in E$, let $Z_i \in \mathcal{F}_i$. Let $Z \subset S$ be as follows: If $t \in \prod_{i\in I} S_i$, then

(22.2) $$t \in Z \quad \text{if and only if} \quad t(i) \in Z_i \text{ for all } i \in E.$$

Let $\mathcal{F}$ be the least σ-algebra of subsets of S such that $\mathcal{F}$ contains every $Z \subset S$ of the form (22.2), for any finite $E \subset I$ and any $Z_i \in \mathcal{F}_i$, $i \in E$.

There exists a unique measure μ on $\mathcal{F}$ (the *product measure*) such that for every Z of the form (22.2), $\mu(Z)$ is the product of $\mu_i(Z_i)$, $i \in E$. (In case of the product $S = S_1 \times S_2$, the measure of a "rectangle" $Z_1 \times Z_2$ is equal to $\mu(Z_1) \cdot \mu(Z_2)$.)

We shall use the following simple example of a product measure space (cf. Example 15.31). Let I be an infinite set, and for each $i \in I$ let us consider the space $\{0, 1\}$ of two elements. We give measure $1/2$ to both $\{0\}$ and $\{1\}$:

(22.3) $$S_i = \{0, 1\}, \qquad \mathcal{F}_i = P(S_i),$$
$$\mu_i(\{0\}) = \mu_i(\{1\}) = 1/2, \qquad \mu_i(\emptyset) = 0, \qquad \mu_i(\{0, 1\}) = 1.$$

Let $S = \prod_{i\in I} S_i$, and let μ be the product measure on $\mathcal{F}$, the least σ-algebra of subsets of S containing the sets $\{t \in \{0, 1\}^I : t(i) = 0\}$ for all $i \in I$.

Let M be a transitive model of ZFC (the ground model). In M let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$. Let $(S, \mathcal{F}, \mu)$ be the product measure space $\{0, 1\}^I$ defined above, where $I = \lambda \times \omega$. Let B be the corresponding measure algebra $\mathcal{F}$/the ideal of sets of measure 0.

Let G be an M-generic ultrafilter on B. Since B satisfies the countable chain condition, the generic extension $M[G]$ preserves cardinals. We shall show that in $M[G]$, $2^{\aleph_0} = \lambda$.

On the one hand, an easy computation gives $|\mathcal{F}| = \lambda$ (because $\lambda^{\aleph_0} = \lambda$) and since B satisfies the c.c.c., we get $|B| = \lambda$. Therefore

$$(2^{\aleph_0})^{M[G]} \leq (|B|^{\aleph_0})^M = \lambda$$

and we have $(2^{\aleph_0})^{M[G]} \leq \lambda$.

On the other hand, we shall exhibit λ distinct subsets of ω in $M[G]$. For each $\alpha < \lambda$ and each $n < \omega$, let $u_{\alpha,n} = [U_{\alpha,n}]$, where $U_{\alpha,n}$ is as follows:

$$U_{\alpha,n} = \{t \in \{0,1\}^{\lambda\times\omega} : t(\alpha, n) = 1\}. \tag{22.4}$$

For $\alpha < \lambda$, let $\dot{x}_\alpha$ be the B-valued subset of ω such that

$$\|n \in \dot{x}_\alpha\| = u_{\alpha,n} \qquad (n < \omega). \tag{22.5}$$

Let x_α be the G-interpretation of $\dot{x}_\alpha$.

We shall show that $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$, and in fact that $\|\dot{x}_\alpha = \dot{x}_\beta\| = 0$. Let k be any natural number. Then

$$\|\dot{x}_\alpha \cap k = \dot{x}_\beta \cap k\| = [N_{\alpha,\beta,k}],$$

where

$$N_{\alpha,\beta,k} = \{t : t(\alpha, n) = t(\beta, n) \text{ for all } n < k\}.$$

It is easy to verify that for each k, $\mu(N_{\alpha,\beta,k}) = 1/2^k$. But $\|\dot{x}_\alpha = \dot{x}_\beta\| = \prod_{k=0}^{\infty}[N_{\alpha,\beta,k}] = [\bigcap_{k=0}^{\infty} N_{\alpha,\beta,k}] = 0$ since $\mu(\bigcap_{k=0}^{\infty} N_{\alpha,\beta,k}) = 0$. This completes the proof that $2^{\aleph_0} = \lambda$ in $M[G]$.

Now let us assume that κ is a measurable cardinal in the ground model, and let $\lambda \geq \kappa$ be such that $\lambda^{\aleph_0} = \lambda$. We construct a generic extension $M[G]$ of M, using the measure algebra described above. In $M[G]$, we have $2^{\aleph_0} = \lambda$, and we show that κ is real-valued measurable in $M[G]$. This follows from this general lemma:

Lemma 22.9. *Let κ be a measurable cardinal in the ground model M, let B be (in M) a measure algebra, and let G be an M-generic ultrafilter on B. Then in $M[G]$, there exists a nontrivial κ-additive measure on κ.*

Proof. Let U be a κ-complete nonprincipal ultrafilter on κ. Let B be a complete Boolean algebra and let m be a measure on B. We shall define a B-valued name $\dot{\mu}$ and show that if G is a generic ultrafilter, then the G-interpretation of $\dot{\mu}$ is a nontrivial κ-additive measure on κ.

Let a be a nonzero element of B, and let $\dot{A} \in M^B$ be a B-valued name such that $a \Vdash \dot{A} \subset \kappa$. For each $\alpha < \kappa$, we let

$$f_a(\dot{A}, \alpha) = \frac{m(a \cdot \|\alpha \in \dot{A}\|)}{m(a)}. \tag{22.6}$$

Since U is a κ-complete, there is a unique real number r such that $f_a(\dot{A}, \alpha) = r$ for almost all α (mod U). Thus let

(22.7) $\mu_a(A) =$ the unique r such that $f_a(A, \alpha) = r$ almost everywhere (mod U).

Note that if $a \Vdash \dot{A} = \dot{A}'$, then $\mu_a(\dot{A}) = \mu_a(\dot{A}')$. Also, if $a \Vdash \dot{A}_1 \subset \dot{A}_2$, then $\mu_a(\dot{A}_1) \leq \mu_a(\dot{A}_2)$. If $X \subset \kappa$ is in M, then $f_a(\check{X}, \alpha) = 1$ for all $\alpha \in X$ and $f_a(\check{X}, \alpha) = 0$ for all $\alpha \notin X$. Hence $\mu_a(\check{X}) = 1$ if $X \in U$ and $\mu_a(\check{X}) = 0$ if $X \notin U$.

Let $\gamma < \kappa$ and let $\dot{A}_\xi$, $\xi < \gamma$, be such that $a \Vdash \dot{A}_\xi \subset \kappa$ for all $\xi < \gamma$, and that $a \Vdash \dot{A}_\xi \cap \dot{A}_\eta = \emptyset$ whenever $\xi \neq \eta$. Let $\dot{A}$ be such that $a \Vdash \dot{A} = \bigcup_{\xi<\gamma} \dot{A}_\xi$. Then $f_a(\dot{A}, \alpha) = \sum_{\xi<\gamma} f_a(\dot{A}_\xi, \alpha)$ for all $\alpha < \kappa$, and hence (because U is κ-complete),

$$\mu_a(\dot{A}) = \sum_{\xi<\gamma} \mu_a(\dot{A}_\xi). \tag{22.8}$$

Let r be a real number, $0 \leq r \leq 1$, and let $\{a_n\}_{n=0}^{\infty}$ be a partition of $a \in B$. If $\mu_{a_n}(\dot{A}) < r$ for all n, then for almost all α, $m(a_n \cdot \|\alpha \in \dot{A}\|) < r \cdot m(a_n)$, and it follows that for almost all α, $m(a \cdot \|\alpha \in \dot{A}\|) < r \cdot m(a)$; hence $\mu_a(\dot{A}) < r$.

As a consequence, we obtain:

(22.9) If for every nonzero $b \leq a$ there is a nonzero $c \leq b$ such that $\mu_c(\dot{A}) < r$, then $\mu_a(\dot{A}) < r$.

(And a similar statement holds when $<$ is replaced by $\leq$, $>$ or $\geq$.)

Now if $b \Vdash \dot{A} \subset \kappa$, we define

$$\mu_b^*(\dot{A}) = \inf_{a \leq b} \mu_a(\dot{A}). \tag{22.10}$$

Again, if $b \Vdash \dot{A}_1 \subset \dot{A}_2$, then $\mu_b^*(\dot{A}_1) \leq \mu_b^*(\dot{A}_2)$, and if $X \in M$, then $\mu_b^*(\check{X}) = 1$ if $X \in U$ and $\mu_b^*(\check{X}) = 0$ if $X \notin U$. However, μ_b^* is not additive and we only have (using (22.8)), for $\gamma < \kappa$:

$$\mu_b^*(\dot{A}) \geq \sum_{\xi<\gamma} \mu_b^*(\dot{A}_\xi) \tag{22.11}$$

under the assumption that $b \Vdash \dot{A}_\xi \cap \dot{A}_\eta = \emptyset$ whenever $\xi \neq \eta$, and that $b \Vdash \dot{A} = \bigcup_{\xi<\gamma} \dot{A}_\xi$.

Note that if $b_1 \leq b_2$, then $\mu_{b_1}(\dot{A}) \geq \mu_{b_2}(\dot{A})$.

Now we are ready to define $\dot{\mu}$. Let G be a generic ultrafilter; in $M[G]$, we define $\mu : P(\kappa) \to [0, 1]$ as follows:

$$\mu(A) = \sup_{b \in G} \mu_b^*(\dot{A}) \tag{22.12}$$

where $\dot{A}$ is a name for A. Let $\dot{\mu}$ be the canonical name for μ (defined in M^B by (22.12) using the canonical $\dot{G}$).

It is clear that μ does not depend on the name $\dot{A}$ for A, that $A_1 \subset A_2$ implies $\mu(A_1) \leq \mu(A_2)$ and that if $X \in M$, then $\mu(X) = 1$ if $X \in U$ and $\mu(X) = 0$ if $X \notin U$. It remains to show that μ is κ-additive.

Let r be a real number (in M) such that $0 \leq r \leq 1$. We claim that

$$\mu_b^*(\dot{A}) \geq r \quad \text{if and only if} \quad b \Vdash \dot{\mu}(\dot{A}) \geq \check{r}. \tag{22.13}$$

If $\mu_b^*(\dot{A}) \geq r$, then for every generic G such that $b \in G$, $\mu(A) \geq r$, and hence $b \Vdash \dot{\mu}(\dot{A}) \geq \check{r}$. Thus assume that $b \Vdash \dot{\mu}(\dot{A}) \geq \check{r}$. Then

$$b \Vdash \forall q < \check{r}\ \exists d \in \dot{G}\ \mu_d^*(\dot{A}) \geq q,$$

that is,

$$\forall q < r\ \forall c \leq b\ \exists d \leq c\ \mu_d^*(\dot{A}) \geq q. \tag{22.14}$$

Let $q < r$; we claim that $\mu_b^*(\dot{A}) \geq q$. If $a \leq b$, then $\forall c \leq a\ \exists d \leq c$ such that $\mu_d^*(\dot{A}) \geq q$ and hence (by a variant of (22.9)), $\mu_a(\dot{A}) \geq q$. Thus $\mu_b^*(\dot{A}) \geq q$. Since this holds for any $q < r$, we have $\mu_b^*(\dot{A}) \geq r$.

Next we show that μ is finitely additive. Let $\dot{A}$, $\dot{A}_1$, and $\dot{A}_2$ be such that every condition forces that $\dot{A}$ is the disjoint union of $\dot{A}_1$ and $\dot{A}_2$. If r_1 and r_2 are real numbers and if $b \Vdash (\dot{\mu}(\dot{A}_1) \geq \check{r}_1$ and $\mu(\dot{A}_2) \geq \check{r}_2)$, then by (22.13) and (22.11), $b \Vdash \dot{\mu}(\dot{A}) \geq \check{r}_1 + \check{r}_2$; hence $\mu(A) \geq \mu(A_1) + \mu(A_2)$. Conversely, let us assume that $\mu(A) > \mu(A_1) + \mu(A_2)$. There are reals $r_1, r_2 \in M$ and $b \in G$ such that

$$b \Vdash \dot{\mu}(\dot{A}_1) < \check{r}_1,\ \dot{\mu}(\dot{A}_2) < \check{r}_2, \text{ and } \dot{\mu}(\dot{A}) \geq \check{r}_1 + \check{r}_2.$$

Since $b \Vdash \dot{\mu}(\dot{A}_1) < \check{r}_1$, there is for each $c \leq b$ some $d \leq c$ such that $\mu_d(\dot{A}_1) < r_1$; hence by (22.9), $\mu_b(\dot{A}_1) < r_1$. Similarly, $\mu_b(\dot{A}_2) < r_2$, and so $\mu_b^*(\dot{A}) \leq \mu_b(\dot{A}) < r_1 + r_2$. This is a contradiction.

Now when we know that μ is finitely additive, it suffices to show that $\mu(\bigcup_{\xi<\gamma} A_\xi) \leq \sum_{\xi<\gamma} \mu(A_\xi)$ for any family $\{A_\xi : \xi < \gamma\}$ of fewer than κ subsets of κ. Thus let $\gamma < \kappa$ and let $\dot{A}_\xi$, $\xi < \gamma$, and $\dot{A}$ be such that $\|\dot{A} = \bigcup_{\xi<\gamma} \dot{A}_\xi\| = 1$, and let us assume that $\mu(A) > \sum_{\xi<\gamma} \mu(A_\xi)$. Then there exist $r \in M$ and $b \in G$ such that

$$b \Vdash \sum_{\xi<\gamma} \dot{\mu}(\dot{A}_\xi) < \check{r} \text{ and } \dot{\mu}(\dot{A}) > \check{r}.$$

Let $E \subset \gamma$ be an arbitrary finite set, let $A_E = \bigcup_{\xi\in E} A_\xi$. Since $\|\dot{\mu}(\dot{A}_E) \leq \sum_{\xi\in E} \dot{\mu}(\dot{A}_\xi)\| = 1$, we have $b \Vdash \dot{\mu}(\dot{A}_E) < \check{r}$. By (22.9), we get $\mu_b(\dot{A}_E) < r$.

Since $\mu_b(\dot{A}_E) < r$ for all finite $E \subset \gamma$, it follows from (22.8) that $\mu_b(\dot{A}) \leq r$. Hence $\mu_b^*(\dot{A}) \leq r$, a contradiction.

This completes the proof that in $M[G]$ μ is a nontrivial κ-additive measure on κ. □

Example 22.10 (A model in which $2^{\aleph_0}$ carries a σ-saturated ideal). Let κ be a measurable cardinal, and let $\lambda \geq \kappa$ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. We shall construct a generic extension that satisfies $2^{\aleph_0} = \lambda$ and such that there is a σ-saturated κ-complete ideal on κ.

Let P be the notion of forcing that adjoins λ Cohen reals; i.e., a condition is a finite 0–1 function p with $\mathrm{dom}(p) \subset \lambda$. If G is a generic filter on P, then $V[G] \vDash 2^{\aleph_0} = \lambda$, and all cardinals are preserved in $V[G]$ because P satisfies the countable chain condition. That κ carries in $V[G]$ a σ-saturated ideal follows from this lemma:

Lemma 22.11. *Let κ be a measurable cardinal and let I be a nonprincipal κ-complete prime ideal on κ. Let P be a notion of forcing that satisfies the countable chain condition. Then in $V[G]$, the ideal J generated by I is a σ-saturated κ-complete ideal on κ.*

Proof. Let J be the ideal in $V[G]$ defined as follows:

$$X \in J \quad \text{if and only if} \quad X \subset Y \text{ for some } Y \in I.$$

First we show that J is κ-complete. Let $\mathcal{X} = \{X_\xi : \xi < \gamma\}$ be a family of fewer than κ elements of J; let $\dot{\mathcal{X}}$ be a name for $\mathcal{X}$ and let $p_0 \in G$ be such that $p_0 \Vdash \forall \xi < \gamma\ \dot{X}_\xi \in \dot{J}$.

For each $\xi < \gamma$ and each $p \leq p_0$, there exist $q \leq p$ and some $Y \in I$ such that $q \Vdash \dot{X}_\xi \subset \check{Y}$. Let W_ξ be a maximal antichain of $q \leq p_0$ for which there is $Y_{\xi,q}$ such that $q \Vdash \dot{X}_\xi \subset \check{Y}_{\xi,q}$. Since P satisfies the countable chain condition, each W_ξ is countable, and hence $Y = \bigcup\{Y_{\xi,q} : \xi < \gamma \text{ and } q \in W_\xi\}$ belongs to I. Now it is easy to verify that $p_0 \Vdash \bigcup_{\xi<\gamma} \dot{X}_\xi \subset \check{Y}$ and hence $\bigcup \mathcal{X} \in J$.

To prove that J is countably saturated, let us assume that $\mathcal{X} = \{X_\xi : \xi < \omega_1\}$ is a family of pairwise disjoint sets of positive J-measure. Let $\dot{\mathcal{X}}$ be a name for $\mathcal{X}$ and let $p \in G$ be such that $p \Vdash \dot{X}_\xi \notin \dot{J}$, for each $\xi < \omega_1$, and $p \Vdash \dot{X}_\xi \cap \dot{X}_\eta = \emptyset$ for all $\xi \neq \eta$.

For each $\xi < \omega_1$, let $Y_\xi = \{\alpha < \kappa : \text{some } q \leq p \text{ forces } \alpha \in \dot{X}_\xi\}$. Clearly $p \Vdash \dot{X}_\xi \subset \check{Y}_\xi$, and so $Y_\xi \notin I$. By the κ-completeness of I, we have $Y = \bigcap_{\xi<\omega_1} Y_\xi \notin I$. Thus $Y \neq \emptyset$, and let α be some element of Y. For each $\xi < \omega_1$, let $q_\xi \leq p$ be such that $q_\xi \Vdash \alpha \in \dot{X}_\xi$. Since P satisfies the countable chain condition, there are ξ, η such that q_ξ and q_η are compatible. Let q be stronger than both q_ξ and q_η; then $q \Vdash \alpha \in \dot{X}_\xi \cap \dot{X}_\eta$, a contradiction. □

Proof of Theorem 22.2. We shall prove that if $2^{\aleph_0}$ carries a σ-saturated $2^{\aleph_0}$-complete ideal then $2^\lambda = 2^{\aleph_0}$ for all $\lambda < 2^{\aleph_0}$. Let λ be a regular cardinal; two functions f, g on λ are *almost disjoint* if there is $\gamma < \lambda$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \gamma$.

Lemma 22.12. *Let κ carry a σ-saturated κ-complete ideal, and let $\lambda < \kappa$ be a regular uncountable cardinal. If $\mathcal{F}$ is a family of almost disjoint functions $f : \lambda \to \kappa$, then $|\mathcal{F}| \leq \kappa$.*

Proof. If $|\mathcal{F}| > \kappa$, then because every $f : \lambda \to \kappa$ is bounded by some $\beta < \kappa$, there exist some $\mathcal{G} \subset \mathcal{F}$ and some $\beta < \kappa$ such that $|\mathcal{G}| = \kappa$ and every $f \in \mathcal{G}$ is bounded by β.

Let $F : [\mathcal{G}]^2 \to \lambda$ be the following partition: $F(\{f, g\}) =$ some γ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \gamma$. By Lemma 22.7, there exists $\mathcal{H} \subset \mathcal{G}$ of size κ such that the image A of $[\mathcal{H}]^2$ under F is at most countable. Let $\alpha > \sup(A)$. Then $f(\alpha) \neq g(\alpha)$ whenever $f, g \in \mathcal{H}$, which is a contradiction since $|\mathcal{H}| = \kappa$ and $f(\alpha) < \beta$ for all $f \in \mathcal{H}$. □

Now we are ready to prove the theorem. Let $\kappa = 2^{\aleph_0}$. We prove $2^\lambda = \kappa$ by induction on $\lambda < \kappa$. If λ is a singular cardinal and $2^\nu = \kappa$ for all $\nu < \lambda$, then $2^\lambda = \kappa$ by Corollary 5.17. Thus let $\lambda < \kappa$ be regular and let us assume that $2^\nu = \kappa$ for all $\nu < \lambda$.

For each $X \subset \lambda$, let $f_X = \langle X \cap \alpha : \alpha < \lambda \rangle$. If $X \neq Y$, then f_X and f_Y are almost disjoint. For each $\alpha < \lambda$, the set $\{f_X(\alpha) : X \subset \lambda\}$ has size κ, and hence $\{f_X : X \subset \lambda\}$ yields a family $\mathcal{F}$ of 2^λ almost disjoint functions from λ into κ. By Lemma 22.12 we get $|\mathcal{F}| \leq \kappa$ and therefore $2^\lambda = \kappa$. □

Generic Ultrapowers

We shall now introduce a powerful method for dealing with ideals on regular uncountable cardinals. The method is a generalization of the method of ultrapowers from Chapter 10.

Let κ be a regular uncountable cardinal and let I be an ideal on κ. Let us view the universe as a ground model, let us denote this ground model M and let us consider the generic extension of M given by the completion of the Boolean algebra $P(\kappa)/I$. In other words, consider the notion of forcing $(P, \subset)$, where P is the collection of all subsets of κ of positive measure:

(22.15) (i) $X \in P$ if and only if $X \notin I$;
(ii) X is stronger than Y if and only if $X \subset Y$

Let G be a generic filter on P.

Lemma 22.13.

(i) *G is an M-ultrafilter on κ extending the filter dual to I.*
(ii) *If I is κ-complete in M, then G is a κ-complete M-ultrafilter.*
(iii) *If I is normal, then G is normal.*

Proof. (i) If $X \subset \kappa$ has measure one, then $\{Y \in P : Y \subset X\}$ is dense in P and hence $X \in G$. That G is an M-ultrafilter is obvious.

(ii) If $\{X_\alpha : \alpha < \gamma\}$, $\gamma < \kappa$, is (in M) a partition of κ, then by the κ-completeness of I, the set $\{Y \in P : Y \subset \text{some } X_\alpha\}$ is dense in P and hence $X_\alpha \in G$ for some α.

(iii) If $X \in G$ and if $f \in M$ is a regressive function on X, then $\{Y \subset X : f \text{ is constant on } Y\}$ is dense below X, and hence f is constant on some $Y \in G$. □

From now on assume that I is a κ-complete ideal on κ containing all singletons. Then G is a nonprincipal κ-complete M-ultrafilter on κ. Note that if I is atomless, then $G \notin M$ (if I is prime, then G is the dual of I and so $G \in M$).

Let us consider (in $M[G]$) the ultrapower $\mathrm{Ult}_G(M)$; let us call this ultrapower a *generic ultrapower*. The generic ultrapower is a model of ZFC, but is not necessarily well-founded. We have the analog of Łoś's Theorem, in this form

$$\mathrm{Ult}_G(M) \vDash \varphi([f_1], \ldots, [f_n]) \quad \text{if and only if} \quad \{\alpha : M \vDash \varphi(f_1(\alpha), \ldots, f_n(\alpha))\} \in G \tag{22.16}$$

whenever $f_1, \ldots, f_n \in M$ are functions defined on a set $X \in G$. In particular, we have an elementary embedding, the *canonical embedding* $j_G : M \to \mathrm{Ult}_G(M)$, defined by

$$j_G(x) = [c_x]$$

where c_x is the constant function on κ with value x, and $[c_x]$ is its equivalence class in the ultrapower.

Let us denote the generic ultrapower by N and $j_G = j$. The ordinal numbers of the model N form a linearly ordered class, not necessarily well-ordered, but we shall show that (because I is κ-complete), Ord^N has an initial segment of order-type κ. If $x \in Ord^N$, let us call the *order-type* of x the order-type of the set $\{y \in Ord^N : y <^N x\}$. If the order-type of x is an ordinal number, we take the liberty of identifying x with this ordinal.

Lemma 22.14.

(i) *For every $\gamma < \kappa$, $j(\gamma) = \gamma$; hence Ord^N has an initial segment of order-type κ.*
(ii) *$j(\kappa) \neq \kappa$.*
(iii) *If I is normal, then there exists $x \in Ord^N$ such that $x = \kappa$; in fact $[d] = \kappa$ where d is the diagonal function $d(\alpha) = \alpha$.*

Proof. G is κ-complete, nonprincipal, and if I is normal then G is normal. □

Let us mention again the fact that we already mentioned and that is fairly easy to verify: If P is the notion of forcing (22.15) then $B(P) = B(P(\kappa)/I)$; the mapping $X \mapsto [X]$ gives the natural correspondence.

To illustrate the method of generic ultrapowers we present two examples. The first is (a modification of) Silver's proof of Theorem 8.12; the other is a theorem of Jech and Prikry.

Example 22.15 (Proof of Silver's Theorem 8.12). Let us consider this typical special case: Let κ be a singular cardinal of cofinality $\aleph_1$, and assume that $2^\lambda = \lambda^+$ for all $\lambda < \kappa$. We shall show that $2^\kappa = \kappa^+$, using a generic ultrapower.

Let I be the ideal of nonstationary subsets of ω_1, let P be the corresponding notion of forcing (i.e., forcing conditions are stationary sets) and let G be a generic filter on P. Note that since $|P| = 2^{\aleph_1} < \kappa$ (in M), all cardinals $\geq \kappa$ remain cardinals in $M[G]$.

Let us work in $M[G]$. G is a normal σ-complete M-ultrafilter on ω_1^M. Let $N = \mathrm{Ult}_G(M)$ be the generic ultrapower and let $j : M \to N$ be the canonical elementary embedding. N is not necessarily well-founded.

Let $\langle \kappa_\alpha : \alpha < \omega_1 \rangle$ be (in M) an increasing continuous sequence of cardinals converging to κ. Let e be the cardinal number in N represented by the function $e(\alpha) = \kappa_\alpha$. Let e^+ denote the successor cardinal of e in N.

For each $X \subset \kappa$ in M let f_X be the function on ω_1^M defined by $f_X(\alpha) = X \cap \kappa_\alpha$. Clearly, each f_X represents in N a subset of e. Moreover, if $X \neq Y$, then f_X and f_Y are almost disjoint and hence represent distinct subsets of e. It follows that $|P^M(\kappa)| \leq |P^N(e)|$, where $P^N(e)$ denotes the collection of all subsets of e in N.

Now $N \vDash 2^e = e^+$ (because $M \vDash 2^{\kappa_\alpha} = \kappa_\alpha^+$ for all α), which means that in the model N there is a one-to-one correspondence between the power set of e and e^+. It follows that there is a one-to-one correspondence between $P^N(e)$ and the set $\mathrm{ext}(e^+) = \{x \in Ord^N : x <^N e^+\}$. Thus we have so far $|P^M(\kappa)| \leq |\mathrm{ext}(e^+)|$.

Next we observe that $e = \sup\{j(\kappa_\gamma) : \gamma < \omega_1^M\}$. This is because if f represents an ordinal less than e, then there is a set of limit ordinals $X \in G$ such that $f(\alpha) < \kappa_\alpha$ for all $\alpha \in X$; thus $f(\alpha) < \kappa_{\gamma(\alpha)}$ for some $\gamma(\alpha) < \alpha$, and by normality of G, there is γ such that $[f] <^N \kappa_\gamma$. Now for each $\gamma < \omega_1^M$, $|\mathrm{ext}(j(\kappa_\gamma))| \leq |(\kappa_\gamma^{\aleph_1})^M| < \kappa$, and therefore $|\mathrm{ext}(e)| \leq \kappa$.

If $x <^N e^+$, then there is in N a one-to-one mapping of x into e, and therefore, $|\mathrm{ext}(x)| \leq |\mathrm{ext}(e)| \leq \kappa$. Thus $\mathrm{ext}(e^+)$ is a linearly ordered set whose each initial segment has size at most κ. Therefore $|\mathrm{ext}(e^+)| \leq \kappa^+$, and we have

$$|P^M(\kappa)| \leq \kappa^+.$$

We have argued so far in $M[G]$; in other words, we have proved that $|P^M(\kappa)|^{M[G]} \leq (\kappa^+)^{M[G]}$. But since all cardinals $\geq \kappa$ in M remain cardinals in $M[G]$, it is necessary that $|P^M(\kappa)|^M \leq (\kappa^+)^M$; in other words we have proved that $2^\kappa = \kappa^+$ (in M). □

Theorem 22.16. *Let I be a σ-complete ideal on ω_1. If $2^{\aleph_0} < \aleph_{\omega_1}$ then $2^{\aleph_1} \leq 2^{\aleph_0} \cdot \mathrm{sat}(I)$.*

Corollary 22.17. *If there exists an $\aleph_2$-saturated ideal on ω_1, then*

(i) $2^{\aleph_0} = \aleph_1$ *implies* $2^{\aleph_1} = \aleph_2$;

(ii) $\aleph_1 < 2^{\aleph_0} < \aleph_{\omega_1}$ *implies* $2^{\aleph_0} = 2^{\aleph_1}$. □

Proof of Theorem 22.16. Let $2^{\aleph_0} = \aleph_\gamma < \aleph_{\omega_1}$ and let I be a σ-complete λ-saturated ideal on ω_1. We shall show that $2^{\aleph_1} \leq \aleph_\gamma \cdot \lambda$.

Let P be the notion of forcing corresponding to I, and let G be generic on P. Since $\mathrm{sat}(P) = \mathrm{sat}(I) \leq \lambda$, all cardinals $\geq \lambda$ in M are cardinals in $M[G]$.

Let us work in $M[G]$, and let $N = \mathrm{Ult}_G(M)$ and $j = j_G : M \to N$. For each $X \subset \omega_1$ in M let f_X be the function on ω_1^M defined by $f_X(\alpha) = X \cap \alpha$. Each f_X represents in N a subset of the countable ordinal d represented by the function $d(\alpha) = \alpha$; moreover, if $X = Y$, then f_X and f_Y are almost disjoint and hence $[f_X] \neq [f_Y]$. It follows that $|P^M(\omega_1^M)| \leq |P^N(d)|$. Let e be the cardinal number in N such that $N \vDash 2^{\aleph_0} = e$ (we recall that $\omega^N = \omega$). Since $N \vDash |P(d)| = e$, we have $|P^N(d)| = |\operatorname{ext}(e)|$ and so

$$|P^M(\omega_1^M)| \leq |\operatorname{ext}(e)|. \tag{22.17}$$

Next we shall compute the size of $\operatorname{ext}(e)$. Since $M \vDash 2^{\aleph_0} = \aleph_\gamma$ and $j : M \to N$ is elementary, we have $e = j(\omega_\gamma)$. We shall now prove by induction on $\gamma < \omega_1^M$ that

$$|\operatorname{ext}(j(\omega_\gamma^M))| \leq \lambda \cdot |\omega_\gamma^M|. \tag{22.18}$$

Let us denote $j(\omega_\gamma^M) = e_\gamma$ for all $\gamma < \omega_1^M$. By Lemma 22.14, the ordinals of N have an initial segment of order-type ω_1^M; thus the *infinite cardinals* of N also have an initial segment of order-type ω_1^M, namely $\{e_\gamma : \gamma < \omega_1^M\}$.

If $\gamma = 0$, then $e_\gamma = \omega$ and (22.18) is true. If γ is a limit ordinal, then $e_\gamma = \sup\{e_\delta : \delta < \gamma\}$ and (22.18) is again true provided it is true for all $\delta < \gamma$. If $\gamma = 1$, then $\operatorname{ext}(e_1)$ is a linearly ordered set whose each initial segment is countable, and hence $|\operatorname{ext}(e_1)| \leq \aleph_1$. Since λ is a cardinal (now we are in $M[G]$), we have $\aleph_1 \leq \lambda$, and (22.18) holds.

Let us assume that (22.18) holds for γ and let us show that it also holds for $\gamma + 1$. Every function $f : \omega_1 \to \omega_{\gamma+1}$ in M is bounded by some constant function, and therefore $j(\omega_{\gamma+1}) = \sup\{j(\xi) : \xi < \omega_{\gamma+1}\}$. Hence the linearly ordered set $\operatorname{ext}(e_{\gamma+1})$ has a cofinal set of order-type $\omega_{\gamma+1}^M$ and each its initial segment has size $\leq \lambda \cdot |\omega_\gamma^M|$ (because if $\xi < \omega_{\gamma+1}$, then $|\operatorname{ext}(j(\xi))| \leq |\operatorname{ext}(e_\gamma)| \leq \lambda \cdot |\omega_\gamma^M|$). It follows that $|\operatorname{ext}(e_{\gamma+1})| \leq \lambda \cdot |\omega_{\gamma+1}^M|$.

Now we put (22.17) and (22.18) together and get

$$|P^M(\omega_1^M)| \leq \lambda \cdot \aleph_\gamma^M.$$

This we proved in $M[G]$; but since all cardinals $\geq \lambda$ in M remains cardinals in $M[G]$, the same must be true in M. Hence (in M)

$$2^{\aleph_1} \leq \lambda \cdot \aleph_\gamma. \qquad \square$$

Precipitous Ideals

In an early application of generic ultrapowers, Solovay proved that if I is κ^+-saturated then the generic ultrapower is well-founded (see Lemma 22.22 in the next section). It has been recognized that this property of ideals is important enough to single out and study such ideals.

Definition 22.18. A κ-complete ideal on κ is *precipitous* if the generic ultrapower $\mathrm{Ult}_G(M)$ is well-founded.

We give below several necessary and sufficient (combinatorial) conditions on I to be precipitous.

Let I be a κ-complete ideal on κ containing all singletons. Let S be a set of positive measure. An *I-partition* of S is a maximal family W of subsets of S of positive measure such that $X \cap Y \in I$ for any distinct $X, Y \in W$. An I-partition W_1 of S is a *refinement* of an I-partition W_2 of S, $W_1 \leq W_2$, if every $X \in W_1$ is a subset of some $Y \in W_2$. A *functional* on S is a collection F of functions such that $W_F = \{\mathrm{dom}(f) : f \in F\}$ is an I-partition of S and $\mathrm{dom}(f) \neq \mathrm{dom}(g)$ whenever $f \neq g \in F$.

We define $F < G$ for two functionals on S to mean that:

(i) each $f \in F \cup G$ is a function into the ordinals;
(ii) $W_F \leq W_G$; and
(iii) if $f \in F$ and $g \in G$ are such that $\mathrm{dom}(f) \subset \mathrm{dom}(g)$, then $f(\alpha) < g(\alpha)$ for all $\alpha \in \mathrm{dom}(f)$.

The reason we define functionals is that they represent functions in the Boolean-valued model M^B (and so are canonical representatives for elements of $\mathrm{Ult}_G(M)$): Let $\dot{f} \in M^B$ be such that

$$S \Vdash \dot{f} \text{ is a function with } \mathrm{dom}(\dot{f}) \in \dot{G} \text{ and } \dot{f} \in M. \tag{22.19}$$

Then there is an I-partition W of S, and for each $X \in W$ a function f_X on X such that for all $X \in W$, $X \Vdash \dot{f} \restriction \check{X} = \check{f}_X$. Thus the functional $\{f_X : X \in W\}$ represents the Boolean-valued $\dot{f}$ on S.

Conversely, if F is functional on S, then there is some $\dot{f} \in M^B$ such that (22.19) holds; and for each $f \in F$, if $X = \mathrm{dom}(f)$, then $X \Vdash \dot{f} \restriction \check{X} = \check{f}$.

Note also that if $F < G$ are functionals on S and $\dot{f}$, $\dot{g}$ are corresponding Boolean-valued names, then

$$S \Vdash \dot{f}, \dot{g} \in M \text{ and } \mathrm{dom}(\dot{f}) \subset \mathrm{dom}\,\dot{g} \text{ and } \dot{f}(\alpha) < \dot{g}(\alpha) \text{ for all } \alpha \in \mathrm{dom}(\dot{f}). \tag{22.20}$$

Conversely, if $\dot{f}$ and $\dot{g}$ satisfy (22.20), then there are functionals F and G that represent $\dot{f}$ and $\dot{g}$, and $F < G$.

Lemma 22.19. *The following are equivalent*:

(i) *I is precipitous.*
(ii) *Whenever S is a set of positive measure and $\{W_n : n < \omega\}$ are I-partitions of S such that $W_0 \geq W_1 \geq \ldots \geq W_n \geq \ldots$, then there exists a sequence of sets $X_0 \supset X_1 \supset \ldots \supset X_n \supset \ldots$ such that $X_n \in W_n$ for each n, and $\bigcap_{n=0}^{\infty} X_n$ is nonempty.*
(iii) *For no set S of positive measure is there a sequence of functionals on S such that $F_0 > F_1 > \ldots > F_n > \ldots$.*

Proof. In view of the preceding discussion on functionals, (ii) is equivalent to (i): If $F_0 > F_1 > \ldots$ are functionals on S, and $\dot{f}_0, \dot{f}_1, \ldots$, the corresponding elements of M^B, then S forces that $[\dot{f}_0], [\dot{f}_1], \ldots$ is a descending sequence of ordinals in the generic ultrapower. Conversely, if S forces that $\mathrm{Ult}_G(M)$ has a descending sequence of ordinals, we construct F_0, F_1, ... on S such that $F_0 > F_1 > \ldots$.

The implication (ii) $\rightarrow$ (iii) is easy. If $F_0 > F_1 > \ldots$ are functionals on S, then the partitions W_{F_0}, W_{F_1}, ... constitute a counterexample: If $X_0 \supset X_1 \supset \ldots$ are elements of $W_{F_0}, W_{F_1}, \ldots$, let $f_0 \in F_0$ be the function with domain X_0, $f_1 \in F_1$ with domain X_1, etc.; now if $\bigcap_{n=0}^{\infty} X_n$ were nonempty, we would get $f_0(\alpha) > f_1(\alpha) > \ldots$ for $\alpha \in \bigcap_{n=0}^{\infty} X_n$.

To show (iii) $\rightarrow$ (ii), let $W_0 \geq W_1 \geq \ldots$ be partitions of some $S \notin I$ that fail (ii). We shall construct functionals on S such that $F_0 > F_1 > \ldots$.

Without loss of generality, let us assume that if $X \in W_{n+1}$, $Y \in W_n$, and $X \subset Y$, then $X \neq Y$. Let $T = \bigcup_{n=0}^{\infty} W_n$; note that the partially ordered set $(T, \subset)$ is an upside-down tree (of height ω).

For each $z \in S$, let us consider the set $T_z = \{X \in T : z \in X\}$. Since every descending sequence $X_0 \supset X_1 \supset \ldots$ in T has empty intersection, it follows that for every z, T_z has no infinite descending sequence $X_0 \supset X_1 \supset \ldots$; hence the relation $\subset$ on T_z is well-founded. Thus there is, for each z, an ordinal function ρ_z on T (the rank function) such that $\rho_z(X) < \rho_z(Y)$ when $X \subset Y$. It is clear that if $X \in W_{n+1}$, $Y \in W_n$, and $z \in X \subset Y$, then $\rho_z(X) < \rho_z(Y)$.

Thus we define, for each $X \in T$, a function f_X on X as follows:

$$f_X(z) = \rho_z(X) \qquad (\text{all } z \in X).$$

Now it is clear that if we let $F_n = \{f_X : X \in W_n\}$ for each n, then F_0, F_1, ... are functionals on S and $F_0 > F_1 > \ldots$. □

Lemma 22.20. *Let κ be a regular uncountable cardinal. The ideal $I = \{X \subset \kappa : |X| < \kappa\}$ is not precipitous.*

Proof. Let $I = \{X \subset \kappa : |X| < \kappa\}$. A set $X \subset \kappa$ has positive measure just in case $|X| = \kappa$. For each such X, let f_X be the unique order-preserving function from X onto κ.

For each set X of positive measure there exists a set $Y \subset X$ of positive measure such that $f_Y(\alpha) < f_X(\alpha)$ for all $\alpha \in Y$; namely if we let $Y =$

$\{\alpha \in X : f_X(\alpha)$ is a successor ordinal$\}$, then $f_X(\alpha) = f_Y(\alpha) + 1$ for all $\alpha \in Y$. Thus for each $X \notin I$ there is an I-partition W_X of X such that for all $Y \in W_X$, $f_Y(\alpha) < f_X(\alpha)$ on Y.

Now we construct I-partitions $W_0 \geq W_1 \geq \ldots$ as follows: We let $W_0 = \{\kappa\}$, and for each n, we let $W_{n+1} = \bigcup\{W_X : X \in W_n\}$. For each n, we let F_n be the functional $F_n = \{f_X : X \in W_n\}$. It is clear that $F_0 > F_1 > \ldots > F_n > \ldots$, and therefore I is not precipitous. □

An alternate characterization of precipitousness is in terms of infinite games. $\mathcal{G}_I$ is the infinite game played by two players, Empty and Nonempty, who alternately choose sets S_n of positive I-measure such that $S_{n+1} \subset S_n$. Empty plays first and wins if $\bigcap_{n=0}^{\infty} S_n = \emptyset$.

Lemma 22.21. *I is precipitous if and only if Empty has no winning strategy in the game $\mathcal{G}_I$.*

Proof. If I is not precipitous then there is a set S of positive measure and a sequence of functionals on S such that $F_0 > F_1 > \ldots > F_n > \ldots$. Empty chooses $S_0 = S$ for his first move. When Nonempty plays S_{2n-1}, Empty finds some $f \in F_n$ such that the set $X = \mathrm{dom}(f) \cap S_{2n-1}$ has positive measure and chooses $S_{2n} = X$ for his move. It follows that $\bigcap_{n=0}^{\infty} S_n$ is empty, and hence Empty wins.

Now suppose that I is precipitous and σ is a strategy for Empty; we will show that σ is not a winning strategy. Let S_0 be Empty's first move by σ. Then S_0 forces that in $M[G]$ there is an infinite sequence $\langle S_n : n \in \omega\rangle$ of moves in which Empty follows σ and each $S_n \in G$. If $j : M \to \mathrm{Ult}_G(M)$ is the canonical embedding then $\langle j(S_n) : n \in \omega\rangle$ is an infinite sequence of moves (of $j(\mathcal{G}_I)$) in which Empty follows $j(\sigma)$ and $\kappa \in \bigcap_{n=0}^{\infty} j(S_n)$. Since $\mathrm{Ult}_G(M)$ is well-founded, there exists (by absoluteness) such a sequence in $\mathrm{Ult}_G(M)$, and since j is elementary, there exists a sequence $\langle S_n : n \in \omega\rangle$ in M in which Empty follows σ but for some $\alpha < \kappa$, $\alpha \in \bigcap_{n=0}^{\infty} S_n$. Thus σ is not a winning strategy. □

Saturated Ideals

Results from Chapter 10 and those proved earlier in this chapter establish the following facts about the existence of a σ-saturated κ-complete ideal on κ: If κ carries a σ-saturated κ-complete ideal then either κ is measurable, or $\kappa \leq 2^{\aleph_0}$ and κ is weakly inaccessible (Lemma 10.9 and 10.14). If a σ-saturated ideal exists then there exists a normal one (Lemma 22.3), and its consistency strength is that of a measurable cardinal (Lemma 22.8). These results generalize easily to ν-saturated ideals for $\nu < \kappa$; the analog of Lemma 10.9 (with the same proof) is that either κ is measurable, or $\kappa \leq 2^{<\nu}$.

In this section we investigate κ-saturated and κ^+-saturated ideals. We shall employ the technique of generic ultrapowers; this is particularly useful because the generic ultrapower is well-founded:

Lemma 22.22. *Let κ be a regular uncountable cardinal. Every κ^+-saturated κ-complete ideal on κ is precipitous.*

Proof. Let I be a κ^+-saturated κ-complete ideal on κ. Let S be a set of positive measure and let $W_0 \geq W_1 \geq \ldots$ be I-partitions of S. We shall find $X_0 \supset X_1 \supset \ldots$ in $W_0, W_1, \ldots$ such that $\bigcap_{n=0}^{\infty} X_n$ is nonempty.

We shall first modify each W_n to obtain a new I-partition W_n' that is almost like W_n but is disjoint. We proceed by induction on n. Since $|W_0| \leq \kappa$, let $W_0 = \{X_\alpha : \alpha < \theta\}$ where $\theta \leq \kappa$, and for each $\alpha < \theta$, let $X_\alpha' = X_\alpha - \bigcup_{\beta<\alpha} X_\beta$; then we let $W_0' = \{X' : X \in W\}$. Since I is κ-complete, we have $X - X' \in I$ for all $X \in W_0$ and thus W_0' is an I-partition of S; moreover, W_0' is disjoint, and is a partition of $S_0 = \bigcup W_0'$ and $S - S_0 \in I$. Having constructed W_n', we enumerate $W_{n+1} = \{X_\alpha : \alpha < \theta\}$ where $\theta \leq \kappa$, and for each $\alpha < \theta$, let $X_\alpha' = (X_\alpha - \bigcup_{\beta<\alpha} X_\beta) \cap Z$ where Z is the unique $Z \in W_n'$ that is almost all of the unique $Y \in W_n$ such that $X_\alpha \subset Y$. We let $W_{n+1}' = \{X' : X \in W_{n+1}\}$; W_n' is a partition of $S_{n+1} = \bigcup W_{n+1}'$, $S - S_{n+1} \in I$, and $X - X' \in I$ for all $X \in W_{n+1}$.

Since each S_n is almost all of S, the set $\bigcap_{n=0}^{\infty} S_n$ is nonempty; let z be an element of this intersection. For each n there is a unique $Y_n \in W_n'$ such that $z \in Y_n$; let X_n be the unique $X_n \in W_n$ such that $Y_n \subset X_n$. It is clear that $X_0 \supset X_1 \supset \ldots \supset X_n \supset \ldots$, and $\bigcap_{n=0}^{\infty} X_n \neq \emptyset$. □

In the next section we shall generalize Lemma 22.8: We shall prove that if κ carries a precipitous ideal then it is measurable in an inner model.

Let us first consider κ-saturated ideals. First we observe that the proof of Lemmas 22.3(i) and 22.4 works as well when I is only κ-saturated, and so we have:

Lemma 22.23.

(i) *If there exists a κ-saturated κ-complete ideal on an uncountable cardinal κ, then there exists a normal κ-saturated κ-complete ideal on κ.*

(ii) *Let I be a normal κ-saturated κ-complete ideal on κ. If $S \notin I$ and if $f : S \to \kappa$ is regressive on S, then there is $\gamma < \kappa$ such that $f(\alpha) < \gamma$ for almost all $\alpha \in S$.* □

Also, the proof of Lemma 10.14 works also for κ-saturated ideals, and so we have:

Lemma 22.24. *If κ carries a κ-saturated κ-complete ideal then κ is weakly inaccessible.* □

It is consistent (relative to the existence of a measurable cardinal) that an inaccessible cardinal κ carries an ideal I such that $\mathrm{sat}(I) = \kappa$. (Such κ cannot be weakly compact, see Exercise 22.13.)

We shall now prove the main result on κ-saturated ideals, using generic ultrapowers. First we need a lemma on preservation of stationary sets by forcing:

Lemma 22.25. *Let κ be a regular uncountable cardinal. Let $V[G]$ be a generic extension of V by a κ-c.c. notion of forcing. Then every closed unbounded $C \subset \kappa$ in $V[G]$ has a closed unbounded subset $D \in V$. Consequently, if $S \in V$ is stationary in V, then S remains stationary in $V[G]$.*

Proof. Let $\dot{C}$ be a name such that every condition forces that $\dot{C}$ is a closed unbounded subset of κ. Let $D = \{\alpha : \|\alpha \in \dot{C}\| = 1\}$. Clearly, D is a subset of C and is closed; we have to prove only that D is unbounded.

Let $\alpha_0 < \kappa$; we wish to find $\alpha > \alpha_0$ such that every condition forces $\alpha \in \dot{C}$. For every p, there is $q \leq p$ and some $\beta > \alpha_0$ such that $q \Vdash \beta \in \dot{C}$. Thus there is a maximal incompatible set W of conditions, and for each $q \in W$ an ordinal $\beta = \beta_q$ such that $q \Vdash \beta \in \dot{C}$. Since $|W| < \kappa$, we let $\alpha_1 = \sup\{\beta_q : q \in W\}$; we have $\alpha_1 < \kappa$ and

$$p \Vdash (\exists \beta \in \dot{C})\ \alpha_0 < \beta \leq \alpha_1$$

for all conditions p. Similarly, we find $\alpha_1 < \alpha_2 < \alpha_3 < \dots$ such that for every n and every condition p,

$$p \Vdash (\exists \beta \in \dot{C})\ \alpha_n < \beta \leq \alpha_{n+1}.$$

If we let $\alpha = \lim_n \alpha_n$, it is clear that $\|\alpha \in \dot{C}\| = 1$. □

Theorem 22.26 (Solovay). *Let κ be a regular uncountable cardinal and assume that κ carries a κ-saturated ideal.*

(i) *κ is weakly Mahlo;*
(ii) *$\{\alpha < \kappa : \alpha$ is weakly Mahlo$\}$ is stationary;*
(iii) *if $X \subset \kappa$ has measure one in a normal κ-saturated ideal, then $X \cap M(X)$ has measure one, where*

$$M(X) = \{\alpha < \kappa : \operatorname{cf} \alpha > \omega \text{ and } X \cap \alpha \text{ is stationary in } \alpha\}.$$

Proof. If there exists a κ-saturated ideal on κ, then κ is weakly inaccessible by Lemma 22.24, and there exists a normal κ-saturated ideal on κ (by Lemma 22.23). Let I be a normal κ-saturated ideal on κ. We first prove:

Lemma 22.27. *If $S \subset \kappa$ is stationary, then for I-almost all $\alpha < \kappa$, $S \cap \alpha$ is stationary in α.*

Proof. If not, then there is a set X of positive measure such that $S \cap \alpha$ is not stationary in α (or $\operatorname{cf} \alpha = \omega$) for all $\alpha \in X$. Let G be a generic ultrafilter on κ (corresponding to I) such that $X \in G$. Let $N = \operatorname{Ult}_G(M)$. N is a transitive model. Since I is normal, κ is represented in N by the function $d(\alpha) = \alpha$. Since $S = j(S) \cap \kappa$, we have $N \models S$ is not stationary.

However, the notion of forcing is κ-saturated and hence κ is a regular cardinal in $M[G]$, and by Lemma 22.25, $M[G] \models S$ is stationary. Now $N \subset M[G]$ and so $N \models S$ is stationary. A contradiction. □

Since I is normal, every set of positive measure is stationary. Thus (iii) follows since if X has measure one then $M(X)$ has measure one by Lemma 22.27, and so does $X \cap M(X)$.

To prove (i), it suffices to show that almost all $\alpha < \kappa$ are regular cardinals. Otherwise, let X be a set of positive measure such that all $\alpha \in X$ are singular. Let $G \ni X$ be generic and let $N = \mathrm{Ult}_G(M)$. Then $N \models \kappa$ is singular, contrary to the fact that κ is regular in $M[G]$ and $N \subset M[G]$.

Now (ii) follows by an application of (iii): Let $X = \{\alpha < \kappa : \alpha \text{ is regular}\}$, then $X \cap M(X) = \{\alpha < \kappa : \alpha \text{ is weakly Mahlo}\}$. □

As a corollary of Theorem 22.26 we have Solovay's original proof of Theorem 8.10:

Let κ be a regular uncountable cardinal and let S be a stationary subset of κ. We claim that S is the disjoint union of κ stationary subsets.

Otherwise, the ideal $I = \{X \subset \kappa : X \cap S \text{ is nonstationary}\}$ is a normal κ-saturated ideal. By Lemma 22.27, $S - M(S)$ has measure zero and hence is nonstationary, which contradicts Lemma 8.9. □

Now let us consider κ^+-saturated ideals.

Lemma 22.28. *Let I be a κ^+-saturated κ-complete ideal on κ.*

(i) *There exists a least unbounded function, i.e., a function $f : \kappa \to \kappa$ such that for any $\gamma < \kappa$ there is no S of positive measure such that $f(\alpha) < \gamma$ on S (unbounded) and that for any $g : \kappa \to \kappa$, if $g(\alpha) < f(\alpha)$ on a set of positive measure then g is constant on a set of positive measure.*

(ii) *There exists a normal κ^+-saturated κ-complete ideal on κ.*

Proof. By Lemma 22.22 I is precipitous. Since I is κ^+-saturated, the Boolean-valued names for functions on κ in the ground model can be represented not by functionals but by ordinary functions: Let F be a functional (on κ). Let $W = \{\mathrm{dom}(f) : f \in F\}$; since I is κ^+-saturated, W can be replaced by a *disjoint* W' such that for each $X \in W$ there is $X' \in W'$ such that $X' \subset X$ and $X - X' \in I$. If we replace each $f \in F$ by its restriction to the corresponding $X' \in W'$, we get a functional F' whose elements have disjoint domains. Then $f = \bigcup F'$ is a function, and if $\dot{f} \in V^B$ is the name corresponding to F, then $\|\dot{f} = \check{f}\| = 1$.

Let $\dot{f} \in V^B$ be such that $\|\dot{f}$ represents $\check{\kappa}$ in the generic ultrapower$\| = 1$ and let $f : \kappa \to \kappa$ be such that $\|\dot{f} = \check{f}\| = 1$. Then f is the least unbounded function.

If f is the least unbounded function then $f_*(I) = \{X \subset \kappa : f_{-1}(X) \in I\}$ is a normal κ-complete ideal and is κ^+-saturated. □

Unlike κ-saturation, the existence of a κ^+-saturated ideal on κ does not imply that κ be a limit cardinal. However, the consistency strength of a κ^+-saturated ideal on a successor cardinal κ is considerably stronger than measurability (while the existence of an ideal I on an inaccessible κ such that

$\mathrm{sat}(I) = \kappa^+$ is equiconsistent with measurability). It is consistent, relative to a Woodin cardinal, that the nonstationary ideal on $\aleph_1$ is $\aleph_2$-saturated; we shall study this problem in Part III. We shall also return to the subject of saturation of the nonstationary ideal in general in Chapter 23.

Saturated ideals have influence on cardinal arithmetic, similar to measurable cardinals:

Lemma 22.29. *Let κ be a regular uncountable cardinal and let I be a κ^+-saturated ideal on κ. If $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$.*

Proof. Let M be the ground model. Let P be the notion of forcing corresponding to I, let G be generic on P, and let $N = \mathrm{Ult}_G(M)$. Since I is κ^+-saturated, N is well-founded and hence we identify it with a transitive model $N \subset M[G]$. Let $j : M \to N$ be the canonical embedding. We have $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and $j(\kappa) > \kappa$. If $X \subset \kappa$ and $X \in M$, then $X \in N$ because $X = j(X) \cap \kappa$. Thus $P^M(\kappa) \subset P^N(\kappa)$.

We assume that $M \vDash (2^\lambda = \lambda^+$ for all $\lambda < \kappa)$ and hence $N \vDash (2^\lambda = \lambda^+$ for all $\lambda < j(\kappa))$, and in particular, $N \vDash |P(\kappa)| = \kappa^+$, where α^+ denotes the least cardinal greater than α. Now $(\kappa^+)^N \leq (\kappa^+)^{M[G]}$; and because $\mathrm{sat}(P) = \kappa^+$ (in M), $(\kappa^+)^M$ is a cardinal in $M[G]$ and we have also $(\kappa^+)^{M[G]} = (\kappa^+)^M$. Thus we have, in $M[G]$,

$$|P^M(\kappa)| \leq (\kappa^+)^M$$

and since all cardinals above κ^+ in M are preserved, the last formula is also true in M, and we have $2^\kappa = \kappa^+$. □

Lemma 22.30. *Let I be an $\aleph_2$-saturated ideal on ω_1. Then*

(i) *If $2^{\aleph_0} = \aleph_1$, then $2^{\aleph_1} = \aleph_2$.*
(ii) *If $\aleph_1 < 2^{\aleph_0} < \aleph^{\omega_1}$, then $2^{\aleph_1} = 2^{\aleph_0}$.*
(iii) *If $2^{\aleph_0} = \aleph_{\omega_1}$, then $2^{\aleph_1} \leq \aleph_{\omega_2}$.*
(iv) *If $\aleph_{\omega_1}$ is strong limit, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$.*
(v) *Let $\Phi(\alpha)$ denote the αth member of the class $\{\kappa : \aleph_\kappa = \kappa\}$. If $\Phi(\omega_1)$ is strong limit, then $2^{\Phi(\omega_1)} < \Phi(\omega_2)$.*

Proof. (i) and (ii) are as in Corollary 22.17.

Let G be a generic ultrafilter on ω_1, let $N = \mathrm{Ult}_G(M)$ and let $j : M \to N$. N is a transitive model, $N \subset M[G]$.

Let us denote $\kappa = \omega_1^M$. We have $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and $j(\kappa) > \kappa$. Thus κ is a countable ordinal in N. Moreover, every $f : \kappa \to Ord$ in M belongs to N, and so every $\gamma < \omega_2^M$ is countable in N. Since $\mathrm{sat}(I) = \aleph_2$, ω_2^M is a cardinal in $M[G]$, hence in N, and so

$$j(\omega_1^M) = \omega_2^M.$$

We shall now prove (iii), (iv), and (v). To prove (iii), let us assume that $M \vDash 2^{\aleph_0} = \aleph_{\omega_1}$. Since $N \vDash |\omega_1^M| = \aleph_0$, and j is elementary, we have $M[G] \vDash$

$|P^M(\kappa)| \leq (2^{\aleph_0})^N$ and $N \vDash 2^{\aleph_0} = \aleph_{j(\kappa)}$. Now $\aleph^N_{j(\kappa)} \leq \aleph^{M[G]}_{j(\kappa)}$; and since $j(\kappa) = \omega_2^M$, $\aleph^{M[G]}_{j(\kappa)}$ is the ω_2^Mth cardinal in $M[G]$. However, all cardinals $\geq \aleph_2^M$ are preserved and hence $\aleph^{M[G]}_{j(\kappa)} = \aleph^M_{\omega_2}$. Thus

$$|P^M(\omega_1^M)| \leq \aleph^M_{\omega_2}$$

holds in $M[G]$; and because cardinals above $\aleph_2$ are preserved, this also holds in M.

To prove (iv) or (v), note that if $\aleph_{\omega_1}$ (or $\Phi(\omega_1)$) is strong limit, then $2^{\aleph_{\omega_1}} = \aleph^{\aleph_1}_{\omega_1}$ ($2^{\Phi(\omega_1)} = (\Phi(\omega_1))^{\aleph_1}$). Let λ denote $\aleph_{\omega_1}$ in (iv) and $\Phi(\omega_1)$ in (v). It is easy to see that $j(\lambda) > \lambda$. Now $N \vDash \forall \alpha < j(\lambda)\ |\alpha^\kappa| < j(\lambda)$, and because $(\lambda^\kappa)^M \subset (\lambda^\kappa)^N$, we have $M[G] \vDash |(\lambda^\kappa)^M| \leq j(\lambda)$.

In case (iv) we have $j(\lambda) \leq \aleph^M_{\omega_2}$ as in (iii); and in case (v) we obtain similarly $j(\lambda) \leq \Phi(j(\kappa)) \leq \Phi(\omega_2)$. The rest of the proof of either (iv) or (v) is as before. □

To conclude this section we prove a generalization of Lemma 22.11; we show in Lemma 22.32 that for all $\lambda \leq \kappa^+$, if $V[G]$ is a generic extension by λ-saturated forcing, then a λ-saturated ideal in V generates a λ-saturated ideal in $V[G]$.

Lemma 22.31. *Let I be a κ^+-saturated κ-complete ideal on κ. Let G be a corresponding generic ultrafilter and let $N = \mathrm{Ult}_G(M)$ be the generic ultrapower. Then every $s : \kappa \to M$ in $M[G]$ is in N.*

Proof. Let $\dot{s}$ be a name for s; for each $\alpha < \kappa$, let $\dot{s}_\alpha$ be a name such that $\|\dot{s}(\alpha) = \dot{s}_\alpha\| = 1$. Each $\dot{s}_\alpha$ is represented by a function $f_\alpha \in M$ on κ. Let $h : \kappa \to \kappa$ be the least unbounded function. Let f be the function on κ defined by $f(\alpha) = \langle f_\beta(\alpha) : \beta < h(\alpha)\rangle$. Then f represents $\dot{s}$ in the generic ultrapower. □

Lemma 22.32. *Let B be a complete Boolean algebra, let G be a V-generic ultrafilter on B and let κ be an uncountable regular cardinal. Let $\lambda \leq \kappa^+$ be regular and assume that $\mathrm{sat}(B) \leq \lambda$ and $\mathrm{sat}(B) < \kappa$. If I is a λ-saturated κ-complete ideal on κ, then in $V[G]$, I generates a λ-saturated κ-complete ideal.*

Proof. Let $J \in V[G]$ be the ideal generated by I. Since $\mathrm{sat}\, B \leq \kappa$, J is κ-complete. Let $\dot{J} \in V^B$ be the canonical name for J, and let $\dot{C} \in V^B$ be the Boolean algebra $\dot{C} = P(\check{\kappa})/\dot{J}$.

We want to show that $V^B \vDash \dot{C}$ is $\check{\lambda}$-saturated; by Lemma 16.5 it suffices to show that $B * \dot{C}$ is λ-saturated because B is λ-saturated. Let $D = P(\kappa)/I$. We shall find in V^D a Boolean algebra $\dot{E}$ such that $V^D \vDash \dot{E}$ is $\check{\lambda}$-saturated, and such that $D * \dot{E}$ is isomorphic to $B * \dot{C}$. Since D is λ-saturated, it will follow that $D * \dot{E}$ is λ-saturated and we shall be done.

In V^D, consider the generic ultrapower $N = \mathrm{Ult}_{\dot{G}}(V)$, where $\dot{G}$ is the canonical ultrafilter on $\check{D}$. Let $j : V \to N$ be the corresponding elementary embedding. Let $\dot{E} = j(B)$.

Let $\mathrm{sat}(B) = \nu < \kappa$. Since j is elementary, we have $N \vDash \mathrm{sat}(j(\check{B})) = j(\nu)$; and since $j(\nu) = \nu$ and by Lemma 22.31 all ν-sequences in V^D are in N, we have $V^D \vDash \mathrm{sat}(\dot{E}) = \nu$. Thus $V^D \vDash \dot{E}$ is λ-saturated.

It remains to show that $B * \dot{C}$ and $D * \dot{E}$ are isomorphic. Let $\dot{c} \in B * \dot{C}$. Then $\dot{c} \in V^B$ and $\|\dot{c} \in \dot{C}\|_B = 1$. Thus there is some $\dot{X} \in V^B$ such that $\|\dot{X} \subset \kappa\|_B = 1$ and that

$$\|\dot{c} \text{ is the equivalence class of } \dot{X} \text{ mod } \dot{J}\|_B = 1.$$

Let $f : \kappa \to B$ be the function $f(\alpha) = \|\alpha \in \dot{X}\|$. Since $f(\alpha) \in B$ for all $\alpha < \kappa$, f represents in $N = \mathrm{Ult}_G(B) \subset V^D$ an element $\dot{e} \in j(\check{B}) = \dot{E}$; and $\dot{e} \in D * \dot{E}$. We let $h(\dot{c}) = \dot{e}$.

The proof is completed by verifying that the definition of $h(\dot{c})$ does not depend on the choice of $\dot{X}$ and that h is an isomorphism. □

Consistency Strength of Precipitousness

Theorem 22.33.

(i) *If κ is a regular uncountable cardinal that carries a precipitous ideal, then κ is measurable in an inner model of* ZFC.

(ii) *If κ is a measurable cardinal, then there exists a generic extension in which $\kappa = \aleph_1$, and κ carries a precipitous ideal.*

The proof of (i) uses the technique of iterated ultrapowers (compare with (20.5)–(20.8)).

Let κ be a regular uncountable cardinal, and let I be a precipitous ideal on κ. Let C be the class of all strong limit cardinals $\nu > 2^\kappa$ such that $\mathrm{cf}\,\nu \geq \mathrm{sat}(I)$. Let $\gamma_0 < \gamma_1 < \ldots < \gamma_n < \ldots$ $(n < \omega)$, be elements of C such that $|\gamma_n \cap C| = \gamma_n$, let $A = \{\gamma_n : n = 0, 1, \ldots\}$ and let $\lambda = \sup(A)$.

Lemma 22.34. *There exists an $L[A]$-ultrafilter W on κ such that W is nonprincipal, κ-complete normal and iterable, and every iterated ultrapower $\mathrm{Ult}_W^{(\alpha)}(L[A])$ is well-founded.*

Proof. Since I is precipitous, the generic ultrapower is well-founded, and so the diagonal function $d(\alpha) = \alpha$ represents some ordinal number in $\mathrm{Ult}_G(V)$. Thus there is a set S of positive measure, and an ordinal γ such that

$$S \Vdash \check{d} \text{ represents } \check{\gamma} \text{ in } \mathrm{Ult}_G(V). \tag{22.21}$$

We shall first show that for every $X \in L[A]$, $X \subset \kappa$, either $S \cap X$ or $S - X$ has measure 0, and so

$$U = \{X \in P(\kappa) \cap L[A] : X \cap S \text{ has positive measure}\} \tag{22.22}$$

is an $L[A]$-ultrafilter.

Let $H = H^{L[A]}(\kappa \cup C \cup \{A\})$ be the class of all sets definable in $L[A]$ from elements of $\kappa \cup C \cup \{A\}$ (this is expressible in ZF similarly to the way in which ordinal definability is; or we can use $L_\theta[A]$ for some large θ). Since $|C \cap \gamma_n| = \gamma_n$ for each n, it follows that if π is the transitive collapse of H, then $\pi(A) = A$ and $\pi(H) = L[A]$. Now if $X \subset \kappa$ and $X \in L[A]$, then because π is the identity on κ, we have $X = \pi(Y) = Y \cap \kappa$ for some $Y \in H$, and $Y = \{\xi : L[A] \vDash \varphi(\xi, E, A)\}$. Thus for every $X \in P(\kappa) \cap L[A]$ there is a formula φ and a finite set $E \subset \kappa \cup K$ such that

$$X = \{\xi < \kappa : L[A] \vDash \varphi(\xi, E, A)\}. \tag{22.23}$$

We shall now show that (22.22) defines an $L[A]$-ultrafilter. Recall that for any generic ultrafilter G on κ, j_G is the identity on κ, and moreover, $j_G(\nu) = \nu$ for all $\nu \in C$ (this follows from the definition of C).

If $X \in L[A]$, and $X \cap S$ has positive measure, then because $X \Vdash \check{X} \in \dot{G}$ and because (22.21) holds, we have

$$X \cap S \Vdash \check{\gamma} \in j_G(\dot{X}).$$

Now using (22.23), and the fact that $j_G(A) = A$ and $j_G(E) = E$, we have

$$X \cap S \Vdash (L[\check{A}] \Vdash \varphi(\check{\gamma}, \check{E}, \check{A})). \tag{22.24}$$

But the formula forced by $X \cap S$ in (22.24) is about V, and thus true. Hence

$$\|\check{\gamma} \in j_G(\check{X})\| = 1,$$

and by (22.21),

$$S \Vdash \check{X} \in \dot{G}.$$

This, however, means that $S - X$ has measure 0.

Since I is κ-complete, it is clear that U is $L[A]$-κ-complete, and moreover the intersection of any countable family of elements of U is nonempty. It is less clear that U is iterable: Let $\langle X_\alpha : \alpha < \kappa \rangle \in L[A]$; it suffices to show that $S \Vdash (\{\alpha : X_\alpha \in U\} \in L[A])$. If G is generic such that $S \in G$, then $\{\alpha < \kappa : X_\alpha \in U\} = \{\alpha < \kappa : \gamma \in j_G(X_\alpha)\}$, but this is in $L[A]$ because $j_G(\langle X_\alpha : \alpha < \kappa \rangle) \in L[j_G(A)]$ and $j_G(A) = A$.

By Exercise 19.10, $\mathrm{Ult}_U(L[A])$ is well-founded; let $f : \kappa \to \kappa$ be the function that represents κ in $\mathrm{Ult}_U(L[A])$. Let $W = f_*(U)$.

It is easy to verify that W is a normal, $L[A]$-κ-complete, iterable $L[A]$-ultrafilter on κ, and that the intersection of any countable family of elements of W is nonempty. By Exercise 19.10, every iterated ultrapower $\mathrm{Ult}_W^{(\alpha)}(L[A])$ is well-founded. □

Proof of Theorem 22.33(i). Let $A = \{\gamma_n : n = 0, 1, \dots\}$ be as above, let $\lambda = \sup(A)$, and let W be an $L[A]$-ultrafilter as in Lemma 22.34. Let us define in $L[A]$

$$F = \{X \subset \lambda : \exists n_0 \; \forall n \geq n_0 \; \gamma_n \in X\} \tag{22.25}$$

(compare with (20.6)). We claim that $D = F \cap L[F]$ is a normal measure on λ in $L[D]$.

For each α, let $i_{0,\alpha} : L[A] \to \mathrm{Ult}_W^{(\alpha)}(L[A])$ be the canonical elementary embedding. It follows from the definition of the class C that:

(22.26) if $\alpha < \gamma_n$, then $i_{0,\alpha}(\gamma_n) = \gamma_n$;
$i_{0,\gamma_n}(\kappa) = \gamma_n$;
if $\alpha < \lambda$, then $i_{0,\alpha}(\lambda) = \lambda$.

Hence for all $\alpha < \lambda$, $i_{0,\alpha}(L[A]) = L[A]$, $i_{0,\alpha}(F) = F$, and $i_{0,\alpha}(D) = D$.

We shall now prove that D is an ultrafilter in $L[D]$. Otherwise, let $X \subset \lambda$ be the least X (in the canonical well-ordering of $L[D]$) such that $X \notin D$ and $\lambda - X \notin D$. Since $i_{0,\alpha}(D) = D$ for all $\alpha < \lambda$, we have $i_{0,\alpha}(X) = X$ for all $\alpha < \lambda$; in particular, $i_{0,\gamma_n}(X) = X$ for all n. Now for any n, if $\gamma_n \in X$, then $i_{0,\gamma_n}(\kappa) \in i_{0,\gamma_n}(X)$ and hence $\kappa \in X$, and vice versa. Hence either all γ_n are in X or none, and so either $X \in F$ or $\lambda - X \in F$, a contradiction.

The proof that D is λ-complete (in $L[D]$) and normal is similar and is left to the reader.

Thus we have proved that there exists a D in $L[A]$ such that

(22.27) $L[D] \vDash D$ is a normal measure on λ.

The proof will be complete if we find a transitive model M and an elementary embedding $i : M \to L[D]$ such that $i(\kappa) = \lambda$. Then κ is measurable in some transitive model.

Let us recall that for each α, $i_{0,\alpha}$ is the elementary embedding $i_{0,\alpha} : L[A] \to \mathrm{Ult}_W^{(\alpha)}(L[A])$. As we have seen, if $\alpha < \lambda$, then $i_{0,\alpha}(\lambda) = \lambda$ and $i_{0,\alpha}(L[A]) = L[A]$. Let C_1 be a proper class of ordinals, greater than λ such that $i_{0,\lambda}(\nu) = \nu$ for all $\nu \in C_1$.

Let $H = H^{L[D]}(\kappa \cup \{\lambda\} \cup C_1)$ be the class of all sets definable in $L[D]$ from elements of $\kappa \cup \{\lambda\} \cup C_1$. (As before, the problem of expressibility of H in ZF can be overcome by replacing $L[D]$ by a suitable large segment $L_\theta[D]$.) H is an elementary submodel of $L[D]$.

If $\alpha < \lambda$, then $i_{0,\alpha}(\nu) = \nu$ for all $\nu \in \kappa \cup \{\lambda\} \cup C_1$; it follows that $i_{0,\alpha}(x) = x$ for all $x \in H$. Observing that for every ν such that $\kappa \leq \nu < \lambda$ there exists $\alpha < \lambda$ such that $i_{0,\alpha}(\nu) > \nu$, we conclude that H contains no ordinal ν such that $\kappa \leq \nu < \lambda$. Hence if π is the transitive collapse of H, and $M = \pi(H)$, then $\pi(\lambda) = \kappa$; thus $i = \pi^{-1}$ is an elementary embedding of some transitive model M into $L[D]$, and $i(\kappa) = \lambda$. □

The proof of (ii) uses the notion of forcing which collapses all $\alpha < \kappa$ onto ω and makes $\kappa = \aleph_1$ (the Lévy collapse).

Proof of (ii). Let κ be a measurable cardinal. We shall show that if $V[G]$ is the generic extension by the Lévy collapse such that κ becomes $\aleph_1$, then $V[G]$ has a precipitous ideal on $\aleph_1$.

Let P be the set of all functions p such that $\mathrm{dom}(p)$ is a finite subset of $\kappa \times \omega$ and such that $p(\alpha, n) < \alpha$ for all $(\alpha, n) \in \mathrm{dom}(p)$; p is stronger than q if $p \supset q$. Let G be a V-generic filter on P. In $V[G]$, κ is $\aleph_1$.

Let D be a normal measure on κ, let $M = \mathrm{Ult}_D(V)$ and let $j : V \to M$ be the elementary embedding $j = j_D$. In $V[G]$, let I be the ideal on κ generated by the dual of D; i.e.,

$$X \in I \quad \text{if and only if} \quad X \cap Y = \emptyset \text{ for some } Y \in D. \tag{22.28}$$

A routine argument (using $\mathrm{sat}\, P = \kappa$) shows that I is in $V[G]$ a countably complete ideal containing all singletons. It can be proved that I is precipitous; instead, we shall prove a weaker (but sufficient) property, namely that there exists an $S \subset \aleph_1$, $S \notin I$, such that $I{\restriction}S = \{X \subset \aleph_1 : X \cap S \in I\}$ is a precipitous ideal.

For that, it suffices to show that there exists an $S \notin I$ such that (when forcing with sets $X \notin I$) S forces that the generic ultrapower is well-founded. In turn, it suffices to construct an extension of $V[G]$ in which there exists a $V[G]$-ultrafilter W on κ, generic over $V[G]$ (with respect to forcing with sets $X \notin I$) such that the generic ultrapower $\mathrm{Ult}_W(V[G])$ is well-founded.

For every ν, let P_ν be the set of all $p \in P$ such that $\alpha < \nu$ whenever $(\alpha, n) \in \mathrm{dom}(p)$, and let $P^\nu = \{p \in P : \alpha \geq \nu \text{ for all } (\alpha, n) \in \mathrm{dom}(p)\}$; P is isomorphic to the product $P_\nu \times P^\nu$.

Let us consider the notion of forcing $j(P)$. Clearly, $(j(P))_\kappa = P$, and thus $j(P)$ is isomorphic to $P \times Q$ where $Q = (j(P))^\kappa$. Every $q \in Q$ is represented in the ultrapower M by a function $\langle q_\alpha : \alpha < \kappa \rangle$ such that $q_\alpha \in P^\alpha$ for all $\alpha < \kappa$.

Let H be a $V[G]$-generic filter on Q; thus $G \times H$ is V-generic on $P \times Q$. As in Theorem 21.3 we define in $V[G \times H]$ a $V[G]$-ultrafilter W on κ as follows:

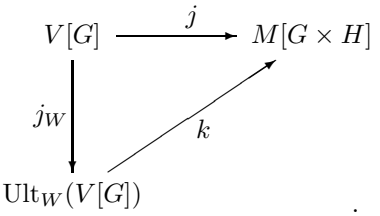
$$X \in W \quad \text{if and only if} \quad \kappa \in (j(\dot{X}))^{G \times H}. \tag{22.29}$$

The definition (22.29) does not depend on the choice of the name $\dot{X}$ because $p \in G$ implies $j(p) \in G \times H$. Let $\dot{W}$ be the canonical name for W. As in (21.8) we have for any $p \in P$, $q \in Q$,

$$(p, q) \Vdash \dot{X} \in \dot{W} \quad \text{if and only if} \quad \text{for almost all } \alpha,\ p \cup q_\alpha \Vdash \check{\alpha} \in \dot{X} \tag{22.30}$$

(here $\dot{X}$ is a P-valued name and $\langle q_\alpha : \alpha < \kappa \rangle$ represents q in M; "almost all" refers to the normal measure D).

First we observe that the ultrapower $\mathrm{Ult}_W(V[G])$ is well-founded. This is because the following commutative diagram holds:



In the diagram, j is the extension of $j : V \to M$ defined by

$$j(x) = (j(\dot{x}))^{G\times H}$$

and k is defined as follows: If $f \in V[G]$ is a function on κ representing $[f]$ in $\mathrm{Ult}_W(V[G])$, then

$$k([f]) = (j(f))(\kappa).$$

Both j and k are elementary and the diagram commutes.

It remains to show that W is $V[G]$-generic with respect to forcing with sets $X \notin I$. It suffices to show that if $\mathcal{X} = \{X_i : i < \theta\}$ is an I-partition of κ, then $X_i \in W$ for some i. Let $\dot{\mathcal{X}} \in V^P$ be a name for $\mathcal{X}$ and let $\dot{X}_i$, $i < \theta$, be names for the X_i. Let us assume that there are conditions $p \in G$ and $q \in H$ such that

$$p \Vdash \dot{\mathcal{X}} \text{ is an } \dot{I}\text{-partition of } \check{\kappa}$$

and for each $i < \theta$,

$$(p, q) \Vdash \dot{X}_i \notin \dot{W}.$$

We shall derive a contradiction.

Let q be represented in M by $\langle q_\alpha : \alpha < \kappa\rangle$. By (22.30) there is for each i a set $A_i \in D$ such that for all $\alpha \in A_i$,

$$p \cup q_\alpha \Vdash \alpha \notin \dot{X}_i. \tag{22.31}$$

Let us define (in $V[G]$),

$$T = \{\alpha < \kappa : q_\alpha \in G\}. \tag{22.32}$$

We shall prove that $T \notin I$ and that $T \cap X_i \in I$ for all $i < \theta$, thus reaching a contradiction since $\mathcal{X}$ is an I-partition.

For each $i < \theta$, if $\alpha \in T$ and $\alpha \in A_i$, then $p \cup q_\alpha \in G$ and hence, by (22.31), $\alpha \notin X_i$. It follows that $T \cap X_i \cap A_i = \emptyset$, and so by (22.28), $T \cap X_i \in I$.

Let us finally show that $T \notin I$. It suffices to show that $T \cap Z \neq \emptyset$ whenever $Z \in D$. Thus let $Z \in D$, and let us prove that $q_\alpha \in G$ for some $\alpha \in Z$. Let

$$E = \{r \in P : r \leq q_\alpha \text{ for some } \alpha \in Z\}.$$

It is easy to see that E is dense in P because Z is unbounded and $q_\alpha \in P^\alpha$ for each $\alpha < \kappa$. Thus $E \cap G \neq \emptyset$ and hence $T \cap Z \neq \emptyset$. □

Exercises

22.1. Let I be a κ-complete ideal and let $\lambda \leq \kappa$. If I is not λ-saturated then there exists a family $\{Z_\alpha : \alpha < \lambda\}$ of pairwise disjoint sets of positive I-measure.

[If $\{X_\alpha : \alpha < \lambda\}$ is such that $X_\alpha \cap X_\beta \in I$ whenever $\alpha \neq \beta$, let $Z_\alpha = X_\alpha - \bigcup_{\beta<\alpha} X_\beta$.]

22.2. Let I be a κ-complete σ-saturated ideal on κ. If g is a minimal unbounded function then $g_*(I)$ is a normal κ-complete σ-saturated ideal.

22.3. Let $\dot{x}_\alpha$ be as in (22.5). Show that $x_\alpha \notin M$.

[Let $a \in M$. Show that for each k, $\|\dot{x}_\alpha \cap k = \check{a} \cap k\| = [D_k]$ where $\mu(D_k) = 1/2^k$.]

22.4. Let I be a σ-saturated κ-complete ideal on κ, and let $\{Y_\xi : \xi < \omega_1\}$ be a family of sets of positive measure. Then there is an uncountable $W \subset \omega_1$ such that $\bigcap_{\xi\in W} Y_\xi$ is nonempty.

[Assume that $\{Y_\xi : \xi < \omega_1\}$ is a counterexample. For each $\nu < \omega_1$, let $Z_\nu = \bigcap_{\xi\geq\nu}(\kappa - Y_\xi)$. Show that $Z_0 \subset Z_1 \subset \ldots \subset Z_\nu \subset \ldots$ and that $\bigcup_{\nu<\omega_1} Z_\nu = \kappa$. Hence there is $\gamma < \omega_1$ such that $Z = \bigcup_{\nu<\gamma} Z_\nu$, then $\kappa - Z \in I$. This is a contradiction since $Y_\gamma \subset \kappa - Z$.]

22.5. If I is a σ-saturated κ-complete ideal on κ and P is a σ-saturated notion of forcing then in $V[G]$, the ideal generated by I is a σ-saturated κ-complete ideal.

[Proceed as in Lemma 22.11 and use Exercise 22.4 to show that J is σ-saturated.]

If κ is measurable and if we adjoin $\lambda \geq \kappa$ Cohen reals, then κ carries a σ-saturated κ-complete ideal but is not real-valued measurable:

22.6. Show that in $V[G]$ there are functions $f_\alpha : \omega \to \omega$, $\alpha < \lambda$, such that whenever $g : \omega \to \omega$, then for at most countably many α's we have $f_\alpha(n) \leq g(n)$ for all n.

[$V[G]$ is also obtained by forcing with the product of λ copies of the notion of forcing that adjoins a generic function $f : \omega \to \omega$, thus $V[G] = V[\langle f_\alpha : \alpha < \lambda\rangle]$. Show that if $g : \omega \to \omega$, then there is a countable $A \subset \lambda$ such that $g \in V[\langle f_\alpha : \alpha < A\rangle]$; if $\beta \notin A$, use the genericity of f_β over $V[\langle f_\alpha : \alpha < A\rangle]$ to show that $f_\beta(n) > g(n)$ for some n.]

22.7. In $V[G]$, κ is not real-valued measurable.

[Use Exercise 22.6 and the proof of Lemma 10.16.]

22.8. If I is κ^+-saturated, then $P(\kappa)/I$ is a complete Boolean algebra.

[By Exercise 7.33 it suffices to show that B is κ^+-complete. Show that $\sum W$ exists in B for every incompatible $W \subset B$. Extend W to a partition Z of B; $Z = \{[X_\alpha] : \alpha < \kappa\}$. Let $Y_\alpha = X_\alpha - \bigcup_{\beta<\alpha} X_\beta$, and $Y = \bigcup\{Y_\alpha : [X_\alpha] \in W\}$. Show that $[Y] = \sum W$ in B.]

22.9. If the GCH holds and $B = P(\kappa)/I$ is complete, then I is κ^+-saturated.

[If B is not κ^+-saturated, let W be an incompatible subset of B of size κ^+. For each $X \subset W$ let $u_x = \sum X$. It follows that $|B| \geq 2^{\kappa^+}$, but clearly $|B| \leq 2^\kappa$; a contradiction.]

22.10. If I is normal, then $P(\kappa)/I$ is κ^+-complete.

[Let X_α, $\alpha < \kappa$, be disjoint subsets of κ such that $X_\alpha \notin I$ for all α. For each $\alpha < \kappa$ let Y_α be X_α without the least element of X_α; let $Y = \bigcup_{\alpha<\kappa} Y_\alpha$. On the one hand, $[Y] \geq [X_\alpha]$ for all α; on the other hand, if $Z \subset Y$ and $Z \notin I$, let f be the function on Z defined such that for all $x \in Y_\alpha$, $f(x) =$ the least element of X_α. Since f is regressive, and I is normal, f is constant on some $S \notin I$, and hence $Z \cap Y_\alpha \notin I$ for some α. Thus $[Y] = \sum_{\alpha<\kappa}[X_\alpha]$.]

22.11. Let I be a normal κ-complete ideal on κ. If I is not κ^+-saturated, then there exists an almost disjoint family of κ^+ sets of positive measure.

[Let X_i, $i < \kappa^+$, be sets of positive measure such that $X_i \cap X_j$ has measure zero. For each $i < \kappa^+$, enumerate $\{X_j : j < i\}$ by $\{Z_\alpha : \alpha < \kappa\}$, and let Y_i be the diagonal intersection of $\{X_i - Z_\alpha : \alpha < \kappa\}$. Now Y_i contains almost all elements of X_i, and $Y_i \cap Z_\alpha \subset \alpha + 1$ for every $\alpha < \kappa$. Thus any Y_i, Y_j are almost disjoint.]

22.12. If I is a κ-complete ideal on κ with the property that every regressive function is bounded almost everywhere (i.e., if $f(\alpha) < \alpha$ for almost all α, then there is $\gamma < \kappa$ such that $f(\alpha) < \gamma$ for almost all α), then I is κ-saturated (and normal).

[Otherwise, let X_α, $\alpha < \kappa$, be a partition of κ into disjoint sets of positive measure. For $\alpha > 0$, let $Y_\alpha = X_\alpha - \{a_\alpha\}$ where $a_\alpha = \min X_\alpha$, and let $Y_0 = X_0 \cup \{a_\alpha : \alpha > 0\}$. The function f that has value a_α on each Y_α is regressive almost everywhere but is not bounded almost everywhere.]

22.13. If I is an atomless κ-complete κ-saturated ideal on an inaccessible cardinal κ, then κ is not weakly compact.

[Show that κ does not have the tree property. Use I to construct a tree $(T, \supset)$ whose elements are sets of positive measure. At successor steps, split each X on the top level into two disjoint sets of positive measure. At limit steps, take all those intersections along branches that have positive measure. Since I is κ-saturated, each level has size $< \kappa$; each level $\alpha < \kappa$ is nonempty because κ is inaccessible and I is κ-complete. Then use $\mathrm{sat}(I) \leq \kappa$ to show that T has no branch of length κ.]

22.14. If I is a precipitous ideal on κ, then there exists a minimal unbounded function.

[There is a set X of positive measure and a function f on X such that X forces that f represents κ in the generic ultrapower.]

Historical Notes

Saturated ideals, a concept introduced by Tarski in [1945], were brought to prominence in Solovay's work [1971]. Solovay introduced the technique of generic ultrapowers and proved Theorems 22.1 and 22.26 (as well as Theorem 8.10).

Theorem 22.2 is due to Prikry [1975], and so is the model in Example 22.10 in which κ carries a σ-saturated ideal [1970]. Theorem 22.16 is due to Jech and Prikry ([1976] and [1979]).

Precipitous ideals were introduced by Jech and Prikry in [1976]; their paper [1979] investigates generic ultrapowers. Lemma 22.21: Jech [1976]. Lemma 22.28: Solovay [1971]. Lemmas 22.29 and 22.30: Jech and Prikry [1976, 1979].

Kunen's paper [1978] contains a number of results on saturated ideals. Kunen constructs several generic extensions with saturated ideals, including a model (using a huge cardinal) in which $\aleph_1$ carries an $\aleph_2$-saturated ideal. In [1970], Kunen proves that if κ carries a κ^+-saturated ideal then there is an inner model with a measurable cardinal; in [1971a] Kunen shows that if moreover κ is a successor cardinal then there is an inner model with many measurable cardinals. Mitchell [1983] improved this to measurable cardinals of order κ^+. Part (i) of Theorem 22.33 is due to Jech and Prikry and part (ii) was proved by Mitchell; see Jech et al. [1980].

Exercise 22.4: Silver.

Exercises 22.5, 22.6 and 22.7: Prikry [1970].

Exercise 22.8: Smith and Tarski.

Exercise 22.9: Solovay [1971].
Exercise 22.11: Baumgartner, Hajnal, and Máté [1975].
Exercise 22.12: Kanamori [1976]
Exercise 22.13: Lévy, Silver.

23. The Nonstationary Ideal

Stationary sets play a fundamental role in modern set theory. In particular, the analysis of the nonstationary ideal $I_{\rm NS}$ on ω_1 has been used in the study of forcing axioms, large cardinals and determinacy. These will be dealt with in later chapters; this chapter continues the investigations began in Chapters 8 and 22. Throughout this chapter "almost all" means all except nonstationary many.

Some Combinatorial Principles

We begin with combinatorial principles that involve stationary sets. Let us recall Jensen's Principle ($\diamondsuit$): There exist sets $S_\alpha \subset \alpha$ such that for every $X \subset \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is stationary. There are several variants of $\diamondsuit$ (see e.g. Exercise 15.25); most notably the following weak version:

Lemma 23.1. *The following principle is equivalent to $\diamondsuit$: There exists a sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of countable sets such that for each $X \subset \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha \in S_\alpha\}$ is stationary.*

Proof. Let $\langle S_\alpha : \alpha < \omega_1\rangle$ be a sequence as in the lemma; we shall produce a diamond sequence. First, let f be a one-to-one mapping of ω_1 onto $\omega_1 \times \omega$ such that $f\text{``}\alpha = \alpha \times \omega$ for all limit ordinals $\alpha$. For every limit ordinal $\alpha$, let $A_\alpha = \{f\text{``}x : x \in S_\alpha\}$ (and $A_\alpha = \emptyset$ otherwise). Note that for each $Y \subset \omega_1 \times \omega$ the set $\{\alpha < \omega_1 : Y \cap (\alpha \times \omega) \in A_\alpha\}$ is stationary.

For each α, let $A_\alpha = \{a^n_\alpha : n \in \omega\}$. It follows that for each $X \subset \omega_1 \times \omega$ there exists some n such that the set $\{\alpha : X \cap (\alpha \times \omega) = a^n_\alpha\}$ is stationary.

For each $\alpha < \omega_1$ and each n, let $S^n_\alpha = \{\xi < \alpha : (\xi, n) \in a^n_\alpha\}$. We complete the proof by showing that for some n, $\langle S^n_\alpha : \alpha < \omega_1\rangle$ is a diamond sequence. If not, there exist sets $X_n \subset \omega_1$ such that $\{\alpha < \omega_1 : X_n \cap \alpha = S^n_\alpha\}$ are nonstationary. Letting $X = \bigcup_{n\in\omega}(X_n \times \{n\})$, it follows that for each n, $\{\alpha < \omega_1 : X \cap (\alpha \times \omega) \neq a^n_\alpha\}$ is nonstationary; a contradiction. □

The Diamond Principle admits a generalization from ω_1 to any regular cardinal κ. Even more generally, let E be a stationary subset of a regular

cardinal κ. $\diamondsuit(E)$ is the following principle (and $\diamondsuit_\kappa$ is $\diamondsuit(\kappa)$):

(23.1) There exists a sequence of sets $\langle S_\alpha : \alpha \in E\rangle$ with $S_\alpha \subset \alpha$ such that for every $X \subset \kappa$, the set $\{\alpha \in E : X \cap \alpha = S_\alpha\}$ is a stationary subset of κ.

The proof of Theorem 13.21 generalizes to show that if $V = L$, then $\diamondsuit(E)$ holds for any regular cardinal κ and any stationary set $E \subset \kappa$.

For a successor cardinal κ^+ and a stationary subset E, consider the following:

(23.2) There exists a sequence of sets $\langle S_\alpha : \alpha \in E\rangle$ such that $|S_\alpha| \leq \kappa$ for each α, and for every $X \subset \kappa^+$, the set $\{\alpha \in E : X \cap \alpha \in S_\alpha\}$ is stationary.

The proof of Lemma 23.1 generalizes and shows that (23.2) is equivalent to $\diamondsuit(E)$.

While the Diamond Principle holds in L, as well as in $L[U]$ and other inner models for large cardinals, restrictions of $\diamondsuit$ to various stationary sets can be proved just from assumptions on cardinal arithmetic. Let $\lambda < \kappa^+$ be a regular cardinal, and recall (8.4) that $E^{\kappa^+}_\lambda$ is the set of all ordinals $\alpha < \kappa^+$ of cofinality λ.

Theorem 23.2 (Gregory). *If λ is regular such that $\kappa^\lambda = \kappa$ and if $2^\kappa = \kappa^+$, then $\diamondsuit(E^{\kappa^+}_\lambda)$ holds.*

In particular, if $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ then $\diamondsuit(E^{\aleph_2}_{\aleph_0})$ holds.

Proof. We prove the version of $\diamondsuit(E)$ from (23.2) where $E = E^{\kappa^+}_\lambda$; by Lemma 23.1, $\diamondsuit(E)$ follows. Let $\langle x_\alpha : \alpha < \kappa^+\rangle$ enumerate all bounded subsets of κ^+ (this is possible by $2^\kappa = \kappa^+$). For each $\alpha \in E$, we let S_α be the set of all $Y \subset \alpha$ such that Y is the union of at most λ elements of the set $\{x_\beta : \beta < \alpha\}$. Since $\kappa^\lambda = \kappa$, we have $|S_\alpha| \leq \kappa$.

We claim that $\langle S_\alpha : \alpha \in E\rangle$ satisfies (23.2). Let $X \subset \kappa^+$; we will show that $X \cap \alpha \in S_\alpha$ for almost all $\alpha \in E$. Let C be the set of all $\alpha < \kappa^+$ such that for every $\beta < \alpha$, $X \cap \beta = x_\gamma$ for some $\gamma < \alpha$. The set C is closed unbounded.

We claim that if $\alpha \in C \cap E$ then $X \cap \alpha \in S_\alpha$. Let $Z \subset \alpha$ be a set cofinal in α such that $|Z| = \lambda$. If for each $\beta \in Z$, $\gamma(\beta) < \alpha$ is such that $X \cap \beta = x_{\gamma(\beta)}$, then $X \cap \alpha = \bigcup\{x_{\gamma(\beta)} : \beta \in Z\}$, and hence $X \cap \alpha \in S_\alpha$. □

A property related to $\diamondsuit$ is *club-guessing*. This has been introduced and investigated in detail by Shelah. Let κ be a regular uncountable cardinal and let E be a stationary subset of κ. If $C \subset \kappa$ is closed unbounded and if each c_α, $\alpha \in E$, is cofinal in α, we say that $\langle c_\alpha : \alpha \in E\rangle$ *guesses* C if for all $\alpha \in E$, C contains an end segment of c_α, i.e., $C \supset c_\alpha - \beta$ for some $\beta < \alpha$.

Theorem 23.3 (Shelah). *Let $\kappa \geq \aleph_3$ be a regular uncountable cardinal, and let λ be a regular uncountable cardinal such that $\lambda^+ < \kappa$. Then there exists a sequence $\langle c_\alpha : \alpha \in E^\kappa_\lambda\rangle$ with each $c_\alpha \subset \alpha$ closed unbounded, such that for every closed unbounded set $C \subset \kappa$, the set $\{\alpha \in E^\kappa_\lambda : c_\alpha \subset C\}$ is stationary.*

Proof. It suffices to find a family $\{c_\alpha : \alpha \in E^\kappa_\lambda\}$ such that each c_α is a closed subset of α, and for every closed unbounded $C \subset \kappa$, the set $\{\alpha \in E^\kappa_\lambda : c_\alpha \text{ is unbounded in } \alpha \text{ and } c_\alpha \subset C\}$ is stationary.

Assume that no such $\{c_\alpha : \alpha \in E^\kappa_\lambda\}$ exists. Let $\{c^0_\alpha : \alpha \in E^\kappa_\lambda\}$ be any collection of closed unbounded subsets of the α's of order-type λ. By induction on $\nu < \lambda^+$, we construct closed unbounded sets $C_\nu \subset \kappa$ and collections $\{c^\nu_\alpha : \alpha \in E^\kappa_\lambda\}$ as follows: $c^\nu_\alpha = c^0_\alpha \cap \bigcap_{\xi<\nu} C_\xi$, and C_ν is such that the set $\{\alpha \in E^\kappa_\lambda : c^\nu_\alpha \text{ is unbounded in } \alpha \text{ and } c^\nu_\alpha \subset C_\nu\}$ is nonstationary.

Let C be the closed unbounded set $C = \bigcap_{\nu<\lambda^+} C_\nu$, and for each α let $c_\alpha = c^0_\alpha \cap C$. The set $S = \{\alpha \in E^\kappa_\lambda : C \cap \alpha \text{ is unbounded in } \alpha\}$ is stationary, and for each $\alpha \in S$ there exists a $\nu(\alpha) < \lambda^+$ such that $c_\alpha = c^{\nu(\alpha)}_\alpha$ (because $c^0_\alpha \supset c^1_\alpha \supset \ldots$ of length λ^+).

There exist a $\nu < \lambda^+$ and a stationary set $T \subset S$ such that $c_\alpha = c^\nu_\alpha$ for all $\alpha \in T$. If $\alpha \in T$ then $c^\nu_\alpha = c^{\nu+1}_\alpha = c^\nu_\alpha \cap C_\nu$, and so $c^\nu_\alpha \subset C_\nu$, contrary to the choice of C_ν. □

The sequence $\langle c_\alpha : \alpha \in E^\kappa_\lambda\rangle$ guesses every closed unbounded set at stationary many α's. The same proof shows that for every stationary $E \subset E^\kappa_\lambda$ there exists a sequence $\langle c_\alpha : \alpha \in E\rangle$ that guesses every closed unbounded set at stationary many $\alpha \in E$ (Exercise 23.1). We state, without proof, a further refinement that will be used later in this chapter in the proof of the Gitik-Shelah Theorem 23.17.

Lemma 23.4. *Let κ and λ be regular uncountable cardinals such that $\lambda^+ < \kappa$. For every stationary set $E \subset E^\kappa_\lambda$ there exists a sequence $\langle c_\alpha : \alpha \in E\rangle$ with each $c_\alpha \subset \alpha$ closed unbounded, such that for every closed unbounded $C \subset \kappa$, the set $\{\alpha \in E : c_\alpha \in C\}$ is stationary, and moreover,*

(23.3) *if $\alpha \in E$ is a limit of ordinals of cofinality greater than λ, then all nonlimit elements of c_α have cofinality greater than λ.*

Proof. For proof, see Gitik and Shelah [1997]. □

This cannot be improved much further; see Exercise 23.2.

One of the most fundamental combinatorial principles is Jensen's *Square Principle*. Let κ be an uncountable cardinal; $\square_\kappa$ (*square-kappa*) is as follows:

(23.4) ($\square_\kappa$) There exists a sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ such that
- (i) C_α is a closed unbounded subset of α;
- (ii) if $\beta \in \mathrm{Lim}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$;
- (iii) if $\mathrm{cf}\,\alpha < \kappa$ then $|C_\alpha| < \kappa$.

The sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ is called a *square-sequence*. Note that by (ii) and (iii), the order-type of every C_α is at most κ.

Using the fine structure theory of L, Jensen proved that in L, $\square_\kappa$ holds for every uncountable cardinal κ. (We elaborate on this in Part III). This has

been extended to most inner models for large cardinals: the Square Principles hold in $L[U]$, $L[\mathcal{U}]$, an in more general inner models.

Squares are relatively easy to obtain by forcing; as an example, see Exercise 23.3.

Definition 23.5. Let κ be a regular uncountable cardinal and let $\alpha < \kappa$ be a limit ordinal of uncountable cofinality. We say that a stationary set $S \subset \kappa$ *reflects* at α if $S \cap \alpha$ is a stationary subset of α.

Corollary 17.20 states that if κ is a weakly compact cardinal then every stationary subset of κ reflects. We address the subject of reflection of stationary sets later in this chapter. See also Exercises 23.4 and 23.5. In general, squares provide examples of nonreflecting stationary sets:

Lemma 23.6. $\square_{\omega_1}$ *implies that there exists a stationary set* $S \subset E^{\aleph_2}_{\aleph_0}$ *that does not reflect.*

Proof. Let $\langle C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ be a square-sequence. For each $\alpha < \omega_2$ of cofinality ω_1, the order-type of C_α is ω_1. It follows that there exists a countable limit ordinal η such that the set $S = \{\gamma \in E^{\aleph_2}_{\aleph_0} : \gamma \text{ is the } \eta\text{th element of some } C_\alpha\}$ is stationary. But for every α of cofinality ω_1, S has at most one element in common with C_α. Hence S does not reflect. □

Stationary Sets in Generic Extensions

If κ is a regular uncountable cardinal then a closed unbounded subset of κ in the ground model remains a closed unbounded subset of κ in a generic extensions (but κ may fail to remain a cardinal or its cofinality may change). It follows that if $S \in V$ is stationary in $V[G]$ then S is stationary in V. A stationary set in V may, however, be no longer stationary in $V[G]$, as there may exist a new closed unbounded set in $V[G]$ that is disjoint from it.

We recall Lemma 22.25 that states that if the forcing satisfies the κ-chain condition then every stationary subset of κ in V is preserved; i.e., remains stationary in $V[G]$. Another condition on preservation of stationary sets is the following:

Lemma 23.7. *Let* κ *be a regular uncountable cardinal and let* P *be a notion of forcing. If* P *is* $<\kappa$*-closed then every stationary* $S \subset \kappa$ *remains stationary in* $V[G]$.

Proof. Let $p \Vdash \dot{C}$ is closed unbounded; we find a $\gamma \in S$ and a $q \leq p$ such that $q \Vdash \gamma \in \dot{C}$ as follows: We construct an increasing continuous ordinal sequence $\langle \gamma_\alpha : \alpha < \kappa\rangle$ and a decreasing sequence $\langle p_\alpha : \alpha < \kappa\rangle$ of conditions such that $p_{\alpha+1} \Vdash \gamma_{\alpha+1} \in \dot{C}$. If α is a limit ordinal then $\gamma_\alpha = \lim_{\xi<\alpha} \gamma_\xi$ and p_α is a lower bound of $\{p_\xi : \xi < \alpha\}$. There exists a limit ordinal α such that $\gamma_\alpha \in S$. It follows that $p_\alpha \Vdash \gamma_\alpha \in \dot{C}$. □

The basic method for destroying stationary sets by forcing is the following forcing known as "shooting a closed unbounded set."

Theorem 23.8. *Let S be a stationary subset of ω_1. There is a notion of forcing P_S that adds generically a closed unbounded set $C \subset \omega_1$ such that $C \subset S$, and such that P_S adds no new countable sets.*

Since P_S adds no countable sets, $\aleph_1$ is preserved. The set $\omega_1 - S$ is nonstationary in $V[G]$; thus if S is chosen so that its complement is stationary, the forcing destroys some stationary set.

Proof. P_S consists of all bounded closed sets of ordinals p such that $p \subset S$; p is stronger than q if p is an end-extension of q (if $q = p \cap \alpha$ for some α).

If G is a generic filter, let $C = \bigcup G$. Clearly, C is a subset of S, and because for every $\alpha < \omega_1$ the set $\{p \in P : \max(p) \geq \alpha\}$ is dense in P_S, C is an unbounded subset of ω_1. Also, $\sup(C \cap \alpha) \in C$ holds for every $\alpha < \omega_1$, and so C is a closed unbounded set. It remains to prove that $\aleph_1$ is preserved and that there are no new countable sets of ordinals.

Lemma 23.9. *P_S is ω-distributive.*

Proof. Let $p \Vdash \dot{f} : \omega \to Ord$; we shall find a $q \leq p$ and some f so that $q \Vdash \dot{f} = f$.

By induction on α we construct a chain $\{A_\alpha : \alpha < \omega_1\}$ of countable subsets of P_S. Let $A_0 = \{p\}$, and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ if α is a limit ordinal. Given A_α, let $\gamma_\alpha = \sup\{\max(q) : q \in A_\alpha\}$. For each $q \in A_\alpha$ and each n, we choose some $r = r(q,n) \in P_S$ so that $r \leq q$, r decides $\dot{f}(n)$, and $\max(r) > \gamma_\alpha$. Then we let $A_{\alpha+1} = A_\alpha \cup \{r(q,n) : q \in A_\alpha,\ n < \omega\}$.

The sequence $\langle \gamma_\alpha : \alpha < \omega_1 \rangle$ is increasing and continuous. Let $C = \{\lambda :$ if $\alpha < \lambda$ then $\gamma_\alpha < \lambda\}$. As C is closed unbounded, there exists a limit ordinal λ such that $\lambda \in C \cap S$. Let $\langle \alpha_n : n < \omega \rangle$ be an increasing sequence with limit λ; then $\lim_n \gamma_{\alpha_n} = \lambda$ as well.

There is a sequence of conditions $\langle p_n : n < \omega \rangle$ such that $p_0 = p$ and that for every n, $p_{n+1} \in A_{\alpha_{n+1}}$, $p_{n+1} \leq p_n$, and p_{n+1} decides $\dot{f}(n)$. Since $\gamma_{\alpha_n} < \max(p_{n+1}) \leq \gamma_{\alpha_{n+1}}$, we have $\lim_n \max(p_n) = \lambda$, and because $\lambda \in S$, the closed set $q \in \bigcup_{n=0}^{\infty} p_n \cup \{\lambda\}$ is a condition. Since $q \leq p_n$ for all n, q decides each $\dot{f}(n)$, and so there exists some f such that $q \Vdash \dot{f} = f$. □

The forcing P_S can be generalized for cardinals κ greater than $\aleph_1$ but additional assumptions on S must be made in order to preserve κ. See, e.g., Exercises 23.7 and 23.8.

Precipitousness of the Nonstationary Ideal

In Theorem 22.33(ii) we showed that if κ is a measurable cardinal and P is the Lévy collapse (with finite conditions, making $\kappa = \aleph_1$) then in $V[G]$,

there exists a precipitous ideal on $\aleph_1$. We now improve this to making the nonstationary ideal precipitous:

Theorem 23.10 (Magidor). *It is consistent, relative to the existence of a measurable cardinal, that the nonstationary ideal on $\aleph_1$ is precipitous.*

Proof. Let κ be a measurable cardinal, and let us assume that $2^\kappa = \kappa^+$. Let U be a normal measure on κ. Let M be the ultrapower $M = \mathrm{Ult}_U(V)$ with $j : V \to M$ the canonical embedding.

Let P be the Lévy collapse: a condition $p \in P$ is a finite function with $\mathrm{dom}(p) \subset \kappa \times \omega$ such that $p(\alpha, n) < \alpha$ for every $(\alpha, n) \in \mathrm{dom}(p)$.

Let G be a P-generic filter. In $V[G]$ (where $\kappa = \aleph_1$) let I_0 be the ideal generated by the dual of U: $I_0 = \{X \subset \kappa : X \cap Y = \emptyset \text{ for some } Y \in U\}$. The proof of Theorem 22.33(ii) shows that in $V[G]$, for every $X \subset \kappa$, $X \notin I_0$, there exists an I_0-generic ultrafilter D_0 with $X \in D_0$ such that $\mathrm{Ult}_{D_0}(V[G])$ is well-founded. Let $G \times H$ be $j(P)$-generic that contains some condition (p, q) with $(p, q) \Vdash_{j(P)} \kappa \in j(\dot{X})$; then $j : V \to M$ extends in $V[G \times H]$ to an elementary $j : V[G] \to M[G \times H]$ by setting $j(\dot{x}^G) = (j(\dot{x}))^{G \times H}$ for every P-name $\dot{x}$, and $D_0 = \{\dot{X}^G : \kappa \in (j(\dot{X}))^{G \times H}\}$.

Our model will be of the form $V[G, \mathcal{C}]$ where G is P-generic and $\mathcal{C} = \langle C_\alpha : \alpha < \kappa^+ \rangle$, with each C_α a closed unbounded subset of κ, is $V[G]$-generic on a set Q_{κ^+} of conditions. The sets Q_α, $\alpha \le \kappa^+$, will be defined by induction on α, together with ideals I_α on κ in $V[G, \mathcal{C}{\restriction}\alpha]$.

Since $2^\kappa = \kappa^+$, we can define a sequence $\langle \dot{A}_\alpha : \alpha < \kappa^+ \rangle$ such that for each $\alpha < \kappa^+$, $\dot{A}_\alpha$ is a name for a subset of κ in $V[G, \mathcal{C}{\restriction}\alpha]$ and for all $\alpha < \kappa^+$ every subset of κ in $V[G, \mathcal{C}{\restriction}\alpha]$ has a name $\dot{A}_\gamma$ for some $\gamma \ge \alpha$. We will show that Q_{κ^+} satisfies the κ-chain condition; it will follow that every subset of κ in $V[G, \mathcal{C}]$ is in $V[G, \mathcal{C}{\restriction}\alpha]$ for some $\alpha < \kappa^+$.

The forcing Q_{κ^+} is, in $V[G]$, a countable support iteration of shooting a closed unbounded subset C_α of $\kappa - \dot{A}_\alpha$, if $\dot{A}_\alpha \in I_\alpha$. More precisely:

(23.5) A condition $q \in Q_\alpha$ is a sequence $\langle q_\beta : \beta < \alpha \rangle$ in $V[G]$ such that
- (i) $q_\beta = \emptyset$ for all but countably many $\beta < \alpha$,
- (ii) q_β is a closed countable subset of κ, for all $\beta < \alpha$,
- (iii) $q_\beta{\restriction}\beta \in Q_\beta$ for all $\beta < \alpha$,
- (iv) if $\alpha = \beta + 1$ then either $q{\restriction}\beta \Vdash \dot{A}_\beta \notin I_\beta$ or $q{\restriction}\beta \Vdash q_\beta \cap \dot{A}_\beta = \emptyset$.

(The ideals I_α, $\alpha < \kappa^+$, will be defined in Lemma 23.12 below.) If $q, q' \in Q_\alpha$ then $q \le q'$ if for each $\beta < \alpha$, q_β is an end-extension of q'_β.

In a generic extension of $V[G]$ by Q_{κ^+}, for every $\alpha < \kappa^+$ the union of all q_α, with $q = \langle q_\alpha : \alpha < \kappa^+ \rangle$ in the generic filter, is a closed unbounded subset of κ.

Lemma 23.11. *Q_{κ^+} satisfies the κ^+-chain condition.*

Proof. Let W be a maximal antichain. Since $|Q_\alpha| \le \kappa$ for each $\alpha < \kappa^+$ there exists an $\alpha < \kappa^+$ such that for every $q \in W$ there is some $q' \in W \cap Q_\alpha$ with

$q' \leq q{\restriction}\alpha$. But $q \cup q'$ is a condition, so q' and q are compatible, and so $q = q'$. Hence $W \subset Q_\alpha$ and $|W| \leq \kappa$. □

Working in $V[G \times H]$, let us consider again the elementary embedding $j : V[G] \to M[G \times H]$. If $\alpha < \kappa^+$, then $|P(Q_\alpha)^{V[G]}| = \kappa^+ < j(\kappa)$ and therefore $P(Q_\alpha)^{V[G]}$ is countable in $V[G \times H]$, and hence there exists a Q_α-generic set $\mathcal{C} = \langle C_\beta : \beta < \alpha \rangle$. Because each $C_\beta \subset \kappa$ and $|\alpha| \leq \kappa$, it follows that $\mathcal{C} \in M[G \times H]$ (for the proof see Lemma 21.9).

In the following arguments, we consider sequences $\mathcal{C} = \langle C_\beta : \beta < \alpha \rangle$ of closed unbounded subsets of κ, and $q = \langle q_\beta : \beta < \alpha \rangle$ of conditions in Q_α, and use the notation $q \in \mathcal{C}$ to mean that each q_β is an initial segment of C_β.

For any $\alpha < \kappa^+$, we define (in $M[G \times H]$), for any Q_α-generic sequence $\mathcal{C}$, the sequence $q^{\mathcal{C}} = \langle q^{\mathcal{C}}_\gamma : \gamma < j(\alpha) \rangle$ by

$$q^{\mathcal{C}}_\gamma = \begin{cases} C_\beta \cup \{\kappa\} & \text{if } \gamma = j(\beta), \\ \emptyset & \text{otherwise.} \end{cases} \tag{23.6}$$

Lemma 23.12. *$q^{\mathcal{C}} \in j(Q_\alpha)$ and $q^{\mathcal{C}} \leq j(q)$ for any $q \in \mathcal{C}$.*

Proof. By induction on α. Simultaneously, we define the ideals I_α for $\alpha > 0$. Assuming that the lemma holds, we define I_α as follows: If $X \in V[G, \mathcal{C}{\restriction}\alpha]$, then $X \in I_\alpha$ if and only if for some $p \in G$ and some $q \in \mathcal{C}{\restriction}\alpha$

$$p \Vdash_{j(P)} (\text{for every } \mathcal{C} \ni q,\ Q_\alpha\text{-generic over } M[G],\ q^{\mathcal{C}} \Vdash_{j(Q_\alpha)} \kappa \notin j(\dot{X})). \tag{23.7}$$

Now assume that the lemma has been proved for all $\beta < \alpha$. When we define $q^{\mathcal{C}}$ by (23.7), we have $q^{\mathcal{C}} \in M[G \times H]$, and once we verify that $q^{\mathcal{C}}$ is a condition in $j(Q_\alpha)$ then the rest of the lemma follows. The only nontrivial verification of $q^{\mathcal{C}} \in j(Q_\alpha)$ is clause (iv) of (23.5).

Thus let $\alpha = \beta + 1$ and assume that $q^{\mathcal{C}}{\restriction}j(\beta)$ does not force $j(\dot{A}_\beta) \notin j(I_\beta)$; we want $q^{\mathcal{C}}{\restriction}j(\beta) \Vdash q^{\mathcal{C}}_{j(\beta)} \cap j(\dot{A}_\beta) = \emptyset$.

Let $\xi \leq \kappa = \max(q^{\mathcal{C}}_{j(\beta)})$, and first consider the case $\xi < \kappa$. Assume that (in some extension of $M[G \cap H]$ by a generic filter containing $q^{\mathcal{C}}{\restriction}j(\beta)$), $\xi \in q^{\mathcal{C}}_{j(\beta)} \cap j(\dot{A}_\beta)$. Since $\xi = j(\xi)$, we have $\xi \in C_\beta$, and there is some $q \in Q_\alpha$ such that $q \in \mathcal{C}$, $\xi \in q_\beta$, and $q{\restriction}\beta \Vdash \xi \in \dot{A}_\beta$. Hence $q{\restriction}\beta \Vdash \dot{A}_\beta \notin I_\beta$, therefore $j(q{\restriction}\beta) \Vdash j(\dot{A}_\beta) \notin I_{j(\beta)}$ and because $q^{\mathcal{C}}{\restriction}j(\beta) \leq j(q{\restriction}\beta)$, we have $q^{\mathcal{C}}{\restriction}j(\beta) \Vdash j(\dot{A}_\beta) \notin I_{j(\beta)}$, a contradiction.

Now consider the case $\xi = \kappa$. Let $q \in \mathcal{C}{\restriction}\beta$ be such that q decides $\dot{A}_\beta \in I_\beta$. The assumption $q \Vdash \dot{A}_\beta \notin I_\beta$ leads to a contradiction as in the preceding case; thus assume that $q \Vdash \dot{A}_\beta \in I_\beta$. Then, by definition of I_β, we have $q^{\mathcal{C}}{\restriction}j(\beta) \Vdash \kappa \notin j(\dot{A}_\beta)$, and so $q^{\mathcal{C}}{\restriction}j(\beta) \Vdash q^{\mathcal{C}}_{j(\beta)} \cap j(\dot{A}_\beta) = \emptyset$.

Hence $q^{\mathcal{C}}$ satisfies (23.5)(iv). □

Lemma 23.13. *If $\beta < \alpha$ then $I_\beta = I_\alpha \cap V[G, \mathcal{C}{\restriction}\beta]$.*

Proof. It is clear that $I_\beta \subset I_\alpha$. Thus let $\dot X$ be a $P * Q_\beta$-name for a subset of κ, and let $p \in P$ and $q \in Q_\alpha$ be such that (p, q) forces $\dot X \in I_\alpha$ and $\dot X \notin I_\beta$. By the latter,

$$p \Vdash_{j(P)} \exists Q_\beta\text{-generic } \mathcal{C}' \text{ over } M[G] \text{ with } q{\restriction}\beta \in \mathcal{C} \text{ and } \exists q' \le q^{\mathcal{C}'} \text{ such that } q' \Vdash_{j(Q_\beta)} \kappa \in j(\dot X).$$

In $M[G \times H]$, Q_α is countable, so $\mathcal{C}' \in M[G, H{\restriction}\delta]$ for some $\delta < j(\kappa)$. In $M[G \times H]$, find a Q_α-generic $\mathcal{C}''$ over $M[G]$ such that $\mathcal{C}{\restriction}\beta = \mathcal{C}'$ and $q \in \mathcal{C}''$. Then $q' \cup q^{\mathcal{C}''}$ is stronger than $q^{\mathcal{C}''}$ and forces $\kappa \in j(\dot X)$. It follows that (p, q) does not force $\dot X \in I_\alpha$, a contradiction. □

We let $I = \bigcup_{\alpha<\kappa^+} I_\alpha$.

Lemma 23.14. *I is a normal ideal.*

Proof. Let $f \in V[G, \mathcal{C}]$ be a function $f : \kappa \to \kappa$, and assume that

$$(p, q) \Vdash \{\alpha : f(\alpha) < \alpha\} \notin I \quad \text{and} \quad (\forall \gamma < \kappa)\ \{\alpha : f(\alpha) = \gamma\} \in I. \tag{23.8}$$

By Lemma 23.11, $f \in V[G, \mathcal{C}{\restriction}\alpha]$ for some $\alpha < \kappa^+$, and (23.8) holds for I_α in place of I. Let $G \times H$ be $j(P)$-generic with $p \in G$, and work in $M[G \times H]$. There exists a Q_α-generic $\mathcal{C}'$ with $q \in \mathcal{C}'$ such that some $q' < q^{\mathcal{C}'}$ forces $j(f)(\kappa) < \kappa$. Then some $q'' < q'$ forces $j(f)(\kappa) = \gamma$, for some γ, and hence (p, q) does not force $\{\alpha : f(\alpha) = \gamma\} \in I_\alpha$, a contradiction. □

Lemma 23.15. *I is, in $V[G, \mathcal{C}]$, the nonstationary ideal on $\aleph_1 = \kappa$.*

Proof. That $\kappa = \aleph_1$ in $V[G, \mathcal{C}]$ follows from the normality of I. Each C_α is a closed unbounded subset of ω_1, and since $C_\alpha \cap A_\alpha = \emptyset$ if $A_\alpha \in I$, every set in I is nonstationary. On the other hand let $\dot C$ be a name for a subset of κ and let $q \in \mathcal{C}$ be such that $q \Vdash \dot C$ is a closed unbounded set. Then for every $\mathcal{C}' \ni q$,

$$q^{\mathcal{C}'} \Vdash j(\dot C) \text{ is closed and } j(\dot C) \cap \kappa \text{ is unbounded in } \kappa$$

and so $q^{\mathcal{C}'} \Vdash \kappa \in j(\dot C)$. It follows that $q \Vdash \kappa - \dot C \in I_\alpha$ and so every nonstationary set is in I. □

It remains to show that I is precipitous.

Let $R(I)$ denote the forcing with I-positive sets; a generic filter on $R(I)$ is an ultrafilter that extends the dual of I. Let (p_1, q_1) be a condition in $P * Q$ and let $\dot X$ be a name for a subset of κ, such that $(p_1, q_1) \Vdash \dot X \notin I$. We want to find generic G and $\mathcal{C}$ with $p_1 \in G$ and $q_1 \in \mathcal{C}$, and an $R(I)$-generic D with $\dot X \in D$ such that $\mathrm{Ult}_D V[G, \mathcal{C}]$ is well-founded.

Since (p_1, q_1) forces $\dot X \notin I$, there exist $p_1' \in j(P)$ and $\alpha < \kappa^+$ such that $p_1' < p_1$, $p_1' \Vdash \dot X \in V[G, \mathcal{C}{\restriction}\alpha]$ and

$$p_1' \Vdash_{j(P)} \exists Q_\alpha\text{-generic } \mathcal{C}' \text{ over } M[G] \text{ with } q_1 \in \mathcal{C}' \text{ such that } q^{\mathcal{C}'} \Vdash_{j(Q_\alpha)} \kappa \notin j(\dot X). \tag{23.9}$$

Let $G \times H$ be $j(P)$-generic over V with $p_1' \in G \times H$. Let $\mathcal{C}' \in M[G \times H]$ be as in (23.9), and pick $q_1' \in j(Q_\alpha)$ such that $q_1' \le q^{\mathcal{C}'}$ and $q_1' \Vdash \kappa \in j(\dot X)$.

We shall find $\mathcal{C}$ and $\mathcal{C}^*$ so that j extends to $j : V[G, \mathcal{C}] \to M[G \times H, \mathcal{C}^*]$. We require that $j(q) \in \mathcal{C}^*$ whenever $q \in \mathcal{C}$, or $\mathcal{C} = \{q \in Q : j(q) \in \mathcal{C}^*\}$.

Let Q^* be the following subordering of $j(Q)$ in $V[G \times H]$. For each $q \in j(Q)$ let $\mathcal{C}_q = \{q' \in Q : j(q') \geq q\}$, and let

$$Q^* = \{q \in j(Q) : (\exists \alpha < \kappa^+)\ \mathcal{C}_q \subset Q_\alpha \text{ and } \mathcal{C}_q \text{ is } Q_\alpha\text{-generic over } V[G]\}. \tag{23.10}$$

Then $q_1' \in Q^*$, so we can find a set $\mathcal{C}^*$ that is Q^*-generic over $V[G \times H]$ with $q_1^* \in \mathcal{C}^*$, and let $\mathcal{C} = j_{-1}(\mathcal{C}^*)$.

Lemma 23.16. *$\mathcal{C}$ is Q-generic over $V[G]$ and $\mathcal{C}^*$ is $j(Q)$-generic over $M[G \times H]$.*

Proof. We first show that for all $\alpha < \kappa^+$, the set

$$B_\alpha = \{q \in Q^* : \mathcal{C}_q \restriction \alpha \text{ is } Q_\alpha\text{-generic over } V[G]\}$$

is dense in Q^*. Let $q \in Q^*$. If $q \notin B_\alpha$ then by (23.10) there exists some $\beta < \alpha$ such that $C_q \subset Q_\beta$ and C_q is Q_β-generic over $V[G]$. But $P(Q_\alpha) \cap V[G]$ is countable in $M[G \times H]$ so there exists a $\mathcal{C}'$ Q_α-generic over $V[G]$ such that $\mathcal{C}' \restriction \beta = C_q$. Since $|Q_\alpha| = \kappa$, this $\mathcal{C}'$ is in $M[G \times H]$ and we can take $q' = q \cup q^{\mathcal{C}'}$. Then $q' \leq q$ and $q' \in Q^* \cap B_\alpha$.

Now let A be an open dense subset of Q in $V[G]$. Since Q satisfies the κ^+-chain condition in $V[G]$, A contains a maximal antichain of cardinality κ. Thus for some $\alpha < \kappa^+$, $A \cap Q_\alpha$ is dense in Q_α. Since B_α is dense in Q^*, there is a $q \in \mathcal{C}^*$ such that $\mathcal{C}_q \restriction \alpha$ is Q_α-generic over $V[G]$ and hence $\mathcal{C} \cap A \supset \mathcal{C}_q \cap A \neq \emptyset$, and so $\mathcal{C}$ is Q-generic over $V[G]$.

Similarly, if $A \in M[G \times H]$ is open dense in $j(Q)$ then (because $j(Q)$ satisfies the $j(\kappa^+)$-chain condition in $M[G \times H]$ and $j(\kappa^+) = \bigcup_{\alpha < \kappa^+} j(\alpha)$) $A \cap j(Q_\alpha)$ is dense in $j(Q_\alpha)$ for some $\alpha < \kappa^+$. Since B_α is dense in Q^*, $\mathcal{C}^* \restriction j(\alpha)$ is $j(Q_\alpha)$-generic over $V[G \times H]$ and hence $A \cap \mathcal{C}^* \neq \emptyset$, and so $\mathcal{C}^*$ is $j(Q)$-generic over $M[G \times H]$. □

Hence j extends to an elementary embedding $j : V[G, \mathcal{C}] \to M[G \times H, \mathcal{C}^*]$. Let

$$D = \{z \in P(\kappa) \cap V[G, \mathcal{C}] : \kappa \in j(z)\};$$

D is an ultrafilter extending the dual of I, and $\mathrm{Ult}_D\, V[G, \mathcal{C}]$ is well-founded. Also, $\dot{X} \in D$ because $(p_1', q_1') \in G \times H \times \mathcal{C}^*$; it remains to show that D is $R(I)$-generic over $V[G, \mathcal{C}]$.

Toward a contradiction, let W be a subset of $R(I)$ in $V[G, \mathcal{C}]$ such that $W \cap D = \emptyset$; we will show that W is not dense in $R(I)$. Since W is disjoint from D, there exist $p_2 \in G \times H$ and $q_2 \in \mathcal{C}^*$, and some $\dot{A}$ such that $(p_2, q_2) \leq (p_1, q_1)$ and

$$p_2 \Vdash_{j(P)} q_2 \Vdash_{Q^*} \dot{A} \in W \text{ and } \kappa \notin j(\dot{A}). \tag{23.11}$$

Since $(p_2, q_2) \in M = \mathrm{Ult}_U(V)$, there is a function $f : \kappa \to P * Q$ such that $(p_2, q_2) = j(f)(\kappa)$. Let $T = \{\alpha : f(\alpha) \in G \times \mathcal{C}\}$. Then since $\kappa \in j(T)$ if and only if $(p_2, q_2) \in G \times H \times \mathcal{C}^*$, we can rewrite (23.11) as

$$\Vdash_{j(P)} \Vdash_{Q^*} (\forall A \in W)\ \kappa \notin j(A \cap T). \tag{23.12}$$

For any $A \in W$, let α_A be such that $q_2 \in j(Q_{\alpha_A})$ and $A \in V[G, \mathcal{C} \restriction \alpha_A]$. By (23.12) and (23.10) we have

$$\Vdash_{j(P)} \text{ for every } Q_{\alpha_A}\text{-generic } \mathcal{C}' \text{ over } V[G],\ q^{\mathcal{C}'} \Vdash_{j(Q)} \kappa \notin j(A \cap T).$$

This says that $A \cap T \in I$ for all $A \in W$. But $T \in D$, and hence $T \notin I$. This contradicts W being dense. □

Thus the consistency strength of "I_{NS} on ω_1 is precipitous" is exactly the existence of a measurable cardinal. For cardinals greater than ω_1 the consistency is considerably stronger. For instance, "I_{NS} on ω_2 is precipitous" is equiconsistent with a measurable cardinal of order 2 (Gitik); for larger cardinals it is much stronger. Most of the best results to date are due to Gitik.

Saturation of the Nonstationary Ideal

By Solovay's Theorem 8.10, the nonstationary ideal I_{NS} on κ is nowhere κ-saturated. For $\kappa = \aleph_1$ it is consistent that I_{NS} is κ^+-saturated; its consistency strength is roughly that of a Woodin cardinal. We shall return to this subject in Part III.

For κ greater than $\aleph_1$, the nonstationary ideal is not κ^+-saturated:

Theorem 23.17 (Gitik-Shelah). *For every regular cardinal $\kappa \geq \aleph_2$, the ideal I_{NS} on κ is not κ^+-saturated.*

The proof of Theorem 23.17 appears in Gitik and Shelah [1997]. Most special cases were proved earlier by Shelah, and we present this proof first, as it is somewhat easier. The complete proof will follow.

The results presented here are somewhat more general as they apply to other normal ideals. If I is a normal ideal, I^+ denotes the collection $\{S \subset \kappa : S \notin I\}$ of sets of positive I-measure. For $S \in I^+$, $I \restriction S$ denotes the ideal $\{X \subset \kappa : X \cap S \in I\}$; we say that $I \restriction S$ *concentrates* on S.

We shall use the method of generic ultrapowers, and start with several observations. Let I be a normal κ^+-saturated κ-complete ideal on a regular uncountable cardinal κ. The generic ultrapower $M = \mathrm{Ult}_G(V)$ is well-founded, and since the forcing with sets of positive I-measure satisfies the κ^+-chain condition, κ^+ is a cardinal in $V[G]$, and hence in M.

Lemma 23.18. *Let I be a normal κ^+-saturated κ-complete ideal on κ, let $R(I)$ be the forcing with I-positive sets, let G be the $R(I)$-generic ultrafilter and let $M = \mathrm{Ult}_G(V)$. Then $P^M(\kappa) = P^{V[G]}(\kappa)$, and all cardinals (and cofinalities) $< \kappa$ are preserved in $V[G]$.*

Proof. The Boolean algebra $B = P(\kappa)/I$ is complete (see Exercise 22.9). If $\dot{A}$ is a name for a subset $A = \dot{A}^G$ of κ in $V[G]$, let $S_\alpha \in I^+$ be, for each $\alpha < \kappa$, such that $\|\alpha \in \dot{A}\| = [S_\alpha]$. If $j : V \to M$ is the canonical embedding, we have, for each α, $\alpha \in A$ if and only if $S_\alpha \in G$ if and only if $\kappa \in j(S_\alpha)$, and so the set $A = \{\alpha \in \kappa : \kappa \in j(S_\alpha)\}$ is in M.

If $\lambda < \kappa$ is a cardinal then since κ is the critical point of j, λ is a cardinal in M. Since $P^{V[G]}(\lambda) = P^M(\lambda)$, λ is a cardinal in $V[G]$. □

We shall use a combinatorial lemma due to Shelah. Let λ be a cardinal and let $\alpha < \lambda^+$ be a limit ordinal. A family $\{X_\xi : \xi < \lambda^+\}$ of subsets of α is *strongly almost disjoint* if every $X_\xi \subset \alpha$ is unbounded, and if for every $\vartheta < \lambda^+$ there exist ordinals $\delta_\xi < \alpha$, for $\xi < \vartheta$, such that the sets $X_\xi - \delta_\xi$, $\xi < \vartheta$, are pairwise disjoint. If κ is a regular cardinal then there exists a strongly almost disjoint family of κ^+ subsets of κ (see Exercise 23.10).

Lemma 23.19. *If $\alpha < \lambda^+$ and $\mathrm{cf}\,\alpha \neq \mathrm{cf}\,\lambda$ then there exists no strongly almost disjoint family of subsets of α.*

Proof. Assume to the contrary that $\{X_\xi : \xi < \lambda^+\}$ is a strongly almost disjoint family of subsets of α. We may assume that each X_ξ has order-type $\mathrm{cf}\,\alpha$. Let f be a function that maps λ onto α. Since $\mathrm{cf}\,\lambda \neq \mathrm{cf}\,\alpha$ there exists for each ξ some $\gamma_\xi < \lambda$ such that $X_\xi \cap f\text{“}\gamma_\xi$ is cofinal in α. There exist some γ and a set $W \subset \lambda^+$ of size λ such that $\gamma_\xi = \gamma$ for all $\xi \in W$. Let $\vartheta > \sup W$. By the assumption on the X_ξ there exist ordinals $\delta_\xi < \alpha$, $\xi < \vartheta$, such that the $X_\xi - \delta_\xi$ are pairwise disjoint. Thus $f^{-1}(X_\xi - \delta_\xi)$, $\xi \in W$, are λ pairwise disjoint nonempty subsets of γ. A contradiction. □

Corollary 23.20. *If κ is a regular cardinal and if a notion of forcing P makes $\mathrm{cf}\,\kappa \neq \mathrm{cf}\,|\kappa|$, then P collapses κ^+.*

Proof. Assume that κ^+ is preserved; thus in $V[G]$, $(\kappa^+)^V = \lambda^+$ where $\lambda = |\kappa|$. In V there is a strongly almost disjoint family $\{X_\xi : \xi < (\kappa^+)^V\}$, and it remains strongly almost disjoint in $V[G]$, and has size λ^+. Since $\mathrm{cf}\,\kappa \neq \mathrm{cf}\,\lambda$ (in $V[G]$), this contradicts Lemma 23.19. □

Corollary 23.21. *If $\kappa = \lambda^+$, if $\nu \neq \mathrm{cf}\,\lambda$ is a regular cardinal, and if I is a normal κ-complete κ^+-saturated ideal on κ, then $E^\kappa_\nu = \{\alpha < \kappa : \mathrm{cf}\,\alpha = \nu\} \in I$.*

Proof. Assume that $E^\kappa_\nu \in I^+$, and let G be a generic ultrafilter on $P(\kappa)/I$. By Lemma 23.18, all cardinals $\leq \lambda$, as well as κ^+, are preserved in $V[G]$. If $E^\kappa_\nu \in G$, then in M, $\mathrm{cf}\,\kappa = \nu$, and so (by Lemma 23.18) $\mathrm{cf}\,\kappa = \nu$ in $V[G]$. Thus we have, in $V[G]$, $\mathrm{cf}\,\kappa = \nu$ and $|\kappa| = \lambda$ while κ^+ is preserved, contradictory Corollary 23.20. □

It follows that if κ is a successor cardinal greater than $\aleph_1$ then the nonstationary ideal on κ is not κ^+-saturated: In fact $I_{\rm NS}\restriction E^\kappa_\nu$ is not κ^+-saturated, for all regular $\nu \neq \operatorname{cf}\lambda$ where λ is the predecessor of κ.

We complete the proof of Theorem 23.17 using Lemma 23.4 on club-guessing. We shall show that for every regular $\kappa \geq \aleph_3$ and every uncountable regular λ such that $\lambda^+ < \kappa$, the ideal $I_{\rm NS}\restriction E^\kappa_\lambda$ is not κ^+-saturated.

Thus let κ and λ be regular uncountable such that $\lambda^+ < \kappa$. Let E be a stationary subset of E^κ_λ. By Lemma 23.4 there exists a sequence $\langle c_\alpha : \alpha \in E\rangle$ with each c_α cofinal in α, that satisfies (23.3) and such that for every closed unbounded C, the set

$$\mathcal{G}(C) = \{\alpha \in E : (\exists \beta < \alpha)\ C \supset c_\alpha - \beta\}$$

is stationary.

Lemma 23.22. *If $I_{\rm NS}\restriction E^\kappa_\lambda$ is κ^+-saturated then there exists a stationary set $\tilde{E} \subset E$ such that for every closed unbounded C, $\tilde{E} - \mathcal{G}(C)$ is nonstationary (C is guessed at almost every $\alpha \in \tilde{E}$).*

Proof. If not, then for every stationary $S \subset E$ there exists a closed unbounded set C such that $S - \mathcal{G}(C)$ is stationary. By the κ^+-saturation, there exists a collection $\{(S_i, C_i) : i < \kappa\}$ such that $W = \{S_i - \mathcal{G}(C_i) : i < \kappa\}$ is a maximal antichain in $P(\kappa)/I_{\rm NS}$ below E. Let $C = \triangle_{i<\kappa} C_i$. For every $i < \kappa$, C_i contains an end-segment of C, and hence $\mathcal{G}(C_i)$ contains an end-segment of $\mathcal{G}(C)$. As $\mathcal{G}(C)$ is stationary, this contradicts the maximality of W. □

Now we use the κ^+-saturation again, and using Lemma 23.22 obtain a maximal antichain $\{S_i : i < \kappa\}$ of pairwise disjoint stationary subsets of E^κ_λ, and for each i a sequence $\langle c_\alpha : \alpha \in S_i\rangle$ of cofinal c_α satisfying (23.3) such that every closed unbounded C is guessed at almost every $\alpha \in S_i$. Then $\langle c_\alpha : \alpha \in \bigcup_{i<\kappa} S_i\rangle$ guesses every C almost everywhere, contrary to Exercise 23.2.

This completes the proof of Theorem 23.17. □

The question whether various restrictions of the nonstationary ideal can be κ^+-saturated has been studied extensively. For instance, it is proved in Jech and Woodin [1985] that it is consistent, relative to a measurable cardinal, that κ is a Mahlo cardinal and $I_{\rm NS}\restriction \mathrm{Reg}$ is κ^+-saturated, where $\mathrm{Reg} = \{\alpha < \kappa : \alpha \text{ is a regular cardinal}\}$. It is open whether (for instance) $I_{\rm NS}\restriction E^{\aleph_2}_{\aleph_1}$ can be $\aleph_3$-saturated.

Reflection

There has been a large number of results on reflecting stationary sets. Let us recall that a stationary set S *reflects* at α if $S \cap \alpha$ is a stationary subset of α. In this section we investigate the simplest case, namely $\kappa = \aleph_2$.

There are two kinds of limit ordinals below ω_2: those of cofinality $\aleph_0$ and those of cofinality $\aleph_1$; the sets $E^{\aleph_2}_{\aleph_0}$ and $E^{\aleph_2}_{\aleph_1}$. By Exercise 23.4, the set $E^{\aleph_2}_{\aleph_1}$ does not reflect (at any ordinal $\alpha < \omega_2$). By Exercise 23.5, the set $E^{\aleph_2}_{\aleph_0}$ reflects at every $\alpha \in E^{\aleph_2}_{\aleph_1}$; the question is whether every stationary $S \subset E^{\aleph_2}_{\aleph_0}$ can reflect. By Lemma 23.6, if every $S \subset E^{\aleph_2}_{\aleph_0}$ reflects then $\square_{\omega_1}$ fails, and this is known (due to Jensen) to imply that $\aleph_2$ is a Mahlo cardinal in L. On the other hand, it is consistent relative to the existence of a Mahlo cardinal, that every stationary $S \subset E^{\aleph_2}_{\aleph_1}$ reflects (Harrington and Shelah [1985]).

The following theorem shows that a stronger version of reflection is consistent, if fact equiconsistent with weak compactness:

Theorem 23.23 (Magidor). *The following are equiconsistent*:

(i) *the existence of a weakly compact cardinal*,
(ii) *every stationary set $S \subset E^{\aleph_2}_{\aleph_0}$ reflects at almost all $\alpha \in E^{\aleph_2}_{\aleph_1}$.*

This result does not generalize to cardinals greater than $\aleph_2$; see Exercise 23.12. Reflection for stationary subsets of $\kappa > \aleph_2$ is considerably more complicated.

We shall prove that (ii) implies that $\aleph_2$ is weakly compact in L, and then give a brief account of the consistency proof of (ii). If every stationary set $S \subset E^{\aleph_2}_{\aleph_0}$ reflects then $\aleph_2$ is a Mahlo cardinal in L. Using Jensen's Theorem 27.1 we prove a somewhat weaker statement.

Lemma 23.24. *If every stationary $S \subset E^{\aleph_2}_{\aleph_0}$ reflects then $\aleph_2$ is inaccessible in L.*

Proof. Let $\kappa = \aleph_2$. Assume that κ is in L the successor of some λ, $\kappa = (\lambda^+)^L$. In L, there exists a square-sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa)\rangle$, and the order-type of each C_α is at most λ. By Fodor's Theorem, there exists a stationary set $A \subset E^{\aleph_2}_{\aleph_1}$ such that all C_α, $\alpha \in A$, have the same order-type.

The set $\bigcup\{C_\alpha : \alpha \in A\}$ is stationary, and it follows that there exists a limit ordinal η such that the set $S = \{\gamma \in E^{\aleph_2}_{\aleph_0} : \gamma$ is the ηth element of some $C_\alpha\}$ is stationary. As in Lemma 23.6, S does not reflect. □

Note that if every stationary $S \subset E^{\aleph_2}_{\aleph_0}$ reflects at almost every $\alpha \in E^{\aleph_2}_{\aleph_1}$ then every two stationary sets $S_1, S_2 \subset E^{\aleph_2}_{\aleph_1}$ reflect at the same α. The following lemma completes the proof:

Lemma 23.25. *If for any stationary sets $S_1, S_2 \subset E^{\aleph_2}_{\aleph_0}$ there exists an $\delta \in E^{\aleph_2}_{\aleph_1}$ such that both $S_1 \cap \delta$ and $S_2 \cap \delta$ are stationary, then $\aleph_2$ is Π^1_1-indescribable in L.*

Proof. Let $\varphi(X)$ be a second order formula with only first order quantifiers and assume that for each $\alpha < \omega_2$ there exists some $X_\alpha \in L$, $X_\alpha \subset \alpha$, such that $L_\alpha \vDash \varphi(X_\alpha)$. We shall find an $X \in L$, $X \subset \omega_2$, such that $L_{\omega_2} \vDash \varphi(X)$.

Let X_α be the least such X_α in L. There exists a $\beta < (\alpha^+)^L$ such that $X_\alpha \in L_\beta$, and let β be the least such β. Let $Z_\alpha \in L$ be such that $Z_\alpha \in \{0,1\}^\alpha$ and that Z_α codes the model $(L_\beta, \in, X_\alpha)$.

For every $\delta < \omega_2$ of cofinality of ω_1, let

$$C_\delta = \{\alpha < \delta : Z_\alpha = Z_\delta{\restriction}\alpha \text{ and } X_\alpha = X_\delta{\restriction}\alpha\}.$$

The set C_δ is a closed unbounded subset of δ.

For each $\gamma < \omega_2$ and each $t \in L$ such that $t \in \{0,1\}^\delta$, let

$$S_t = \{\alpha \in E^{\aleph_2}_{\aleph_0} : t \subset Z_\alpha\}.$$

Since $\aleph_2$ is inaccessible in L, there exists for each $\gamma < \omega_2$ some $t \in \{0,1\}^\gamma$ such that S_t is stationary. Now let $\gamma_1 \leq \gamma_2$ and $t_i \in \{0,1\}^{\gamma_i}$ $(i = 1, 2)$, and assume that both S_{t_1} and S_{t_2} are stationary. By the assumption of the lemma, there exists a $\delta < \omega_2$ of cofinality ω_1 such that both $S_{t_1} \cap \delta$ and $S_{t_2} \cap \delta$ are stationary. Let $\alpha_1, \alpha_2 \in C_\delta$ be such that $\alpha_i \in S_{t_i}$ $(i = 1, 2)$. Since $t_i \subset Z_{\alpha_i} \subset Z_\delta$, it follows that $t_1 \subset t_2$.

Hence for each $\gamma < \kappa$ there is a unique t_γ such that S_{t_γ} contains almost all ordinals in $E^{\aleph_2}_{\aleph_0}$; $S_{t_\gamma} \supset E^{\aleph_2}_{\aleph_0} \cap D_\gamma$ with D_γ closed unbounded. Let $D = \triangle_\gamma D_\gamma$; then for every $\alpha \in E^{\aleph_2}_{\aleph_0} \cap D$ we have $t_\alpha = Z_\alpha$. Now let $Z = \bigcup\{t_\gamma : \gamma < \omega_2\}$. The set Z codes some model $(L_\eta, \in, X)$ with $X \subset \omega_2$ and $X \in L$. It follows that $X \cap \alpha = X_\alpha$ for almost all $\alpha \in E^{\aleph_2}_{\aleph_0}$.

We finish the proof by verifying $L_{\omega_2} \vDash \varphi(X_\alpha)$. This holds because $L_\alpha \vDash \varphi(X_\alpha)$ for all α and therefore $L_{\omega_2} \vDash \varphi(X \cap \alpha)$ for almost all $\alpha \in E^{\aleph_2}_{\aleph_0}$. □

This completes the proof that the existence of a weakly compact cardinal is necessary for the consistency of (ii). We shall not present the consistency proof of (ii) and instead give a brief description of the methods involved.

One starts with a ground model where κ is a weakly compact cardinal, and GCH holds. First one uses the Lévy collapse Q with countable conditions that makes $\kappa = \aleph_2$ (all cardinals between $\aleph_1$ and κ are collapsed). In V^Q, one constructs a forcing iteration P of length κ^+, with $\aleph_1$-support. At every stage α of the iteration, one considers (in V^Q) a P_α-name for a stationary set $S \subset E^\kappa_\omega$ and shoots a closed unbounded set through the set $T = \mathrm{Tr}(S) \cup E^\kappa_\omega$. Forcing conditions are closed bounded subsets of T. It is not difficult to verify that such forcing is ω-closed, and that the iteration satisfies the κ-chain condition. Thus one can arrange the iteration so that every potential stationary set $S \subset E^\kappa_\omega$ is considered.

The main point of the proof is to show that $\aleph_1$ is preserved by the iteration, and that at each stage, if $S \subset E^\kappa_\omega$ is stationary then $\mathrm{Tr}(S) \cap E^\kappa_{\omega_1}$ is unbounded. This is proved using arguments similar to those used in Theorem 23.10.

Weak compactness of κ is used as follows: At a given stage α of the iteration, there is a transitive model $M \supset \alpha$ of size κ of a sufficiently large fragment of ZFC, and (by weak compactness) there is an elementary $j : M \to N$, cf. Lemma 17.17. This j extends to $j : M^Q \to N^{j(Q)}$.

For details, consult Magidor [1982].

Exercises

23.1. Let κ and λ be regular, $\lambda \geq \aleph_1$ and $\lambda^+ < \kappa$. For every stationary $E \subset E^\kappa_\lambda$ there exists a sequence $\langle C_\alpha : \alpha \in E\rangle$ of closed unbounded subsets of the α's such that for every closed unbounded $C \subset \kappa$, the set $\{\alpha \in E : c_\alpha \subset C\}$ is stationary.

23.2. Let κ and λ be regular, $\lambda \geq \aleph_1$ and $\lambda^+ < \kappa$. There exists no sequence $\langle C_\alpha : \alpha \in E^\kappa_\lambda\rangle$ with each $c_\alpha \subset \alpha$ closed unbounded, that guesses every closed unbounded $C \subset \kappa$ almost everywhere (i.e., C contains an end-segment of c_α for almost all $\alpha \in E^\kappa_\lambda$) and satisfies (23.3).

[Assume $\langle c_\alpha : \alpha \in E^\kappa_\lambda\rangle$ is such. Let $E = \{\xi < \kappa : \operatorname{cf}\xi > \lambda\}$ and let $C_0 = E'$. For each n, let $C_{n+1} \subset C'_n$ be closed unbounded such that C'_n contains an end-segment of c_α, for all $\alpha \in E^\kappa_\lambda \cap C_{n+1}$. Let $C = \bigcap_{n<\omega} C_n$ and let α be the least element of $C \cap E^\kappa_\lambda$; C contains an end-segment of C_α. There is a $\beta \in C \cap c_\alpha$ such that $\operatorname{cf}\beta > \lambda$. It follows that there exists some $\gamma \in C \cap \beta \cap E^\kappa_\lambda$, contradicting the minimality of α.]

23.3. There exists an $\aleph_0$-closed, $\aleph_1$-distributive notion of forcing such that $V[G]$ satisfies $\square_{\omega_1}$.

[A forcing condition is a sequence $p = \langle C_\alpha : \alpha \leq \gamma\rangle$, where $\gamma < \omega_2$ is a limit ordinal, and the C_α satisfy (23.4). A condition $\langle C_\alpha : \alpha \leq \gamma\rangle$ is stronger than $\langle C'_\alpha : \alpha \leq \gamma'\rangle$ if $\gamma \geq \gamma'$ and $C_\alpha = C'_\alpha$ for all $\alpha < \gamma'$. To verify $\aleph_1$-distributivity, let $\dot f$ be a name for a function on ω_1 and let p_0 be a condition. Construct an ω_1-chain of conditions $p_0 \subset p_1 \subset \ldots \subset p_\alpha \subset \ldots$, $\alpha < \omega_1$, such that each $p_{\alpha+1}$ decides $\dot f(\alpha)$ and that each limit ordinal $\alpha < \omega_1$, if $\gamma_\alpha = \lim_{\xi\to\alpha}(\operatorname{dom} p_\xi)$, then $\gamma_\alpha \in \operatorname{dom} p_\alpha$, and for each limit ordinal $\beta < \alpha$, C_{γ_β} is an initial segment of C_{γ_α}. Then if $\gamma = \lim_{\alpha\to\omega_1}\gamma_\alpha$, let $C_\gamma = \bigcup_{\alpha<\omega_1} C_{\gamma_\alpha}$ and $p = \langle C_\xi : \xi \leq \gamma\rangle$; p is a condition and decides $\dot f(\alpha)$ for all $\alpha < \omega_1$.]

23.4. Let κ be regular uncountable, $\alpha < \kappa$ and $\operatorname{cf}\alpha > \omega$. If $S \subset \kappa$ is stationary and if $\operatorname{cf}\beta \geq \operatorname{cf}\alpha$ for all $\beta \in S$, then S does not reflect at α.

[There is a closed unbounded $C \subset \alpha$ such that $\operatorname{cf}\beta < \operatorname{cf}\alpha$ for all $\beta \in C$.]

23.5. Let κ and α be as above, let $\lambda < \kappa$ be regular and $\lambda < \operatorname{cf}\alpha$. Then E^κ_λ reflects at α.

23.6. Let P_S be the forcing (in Theorem 23.8) for shooting a closed unbounded subset of S. Show that every stationary subset of S (in V) remains stationary.

[Let $T \subset S$ be stationary and let $p \Vdash \dot C$ is closed unbounded; find a $q \leq p$ and some $\lambda \in T$ such that $q \Vdash \lambda \in \dot C$: As in Lemma 23.9, construct a chain $\{A_\alpha\}_\alpha$ of countable subsets of P_S and an increasing continuous sequence $\langle\gamma_\alpha : \alpha < \omega_1\rangle$, such that for each $q \in A_\alpha$ there exist some stronger $r(q) \in A_{\alpha+1}$ and $\beta(q) > \gamma_\alpha$ with $r(q) \Vdash \beta \in \dot C$. Then find $\lambda \in T$, and a sequence $\langle p_n : n \in \omega\rangle$ of conditions such that $\lim_n \max(p_n) = \lim_n \beta(p_n) = \lambda$.]

23.7. Let S be a stationary subset of ω_2 such that $S \supset E^{\aleph_2}_{\aleph_0}$ and that $S \cap E^{\aleph_2}_{\aleph_1}$ is stationary. Let P_S be the set of all bounded closed subsets of S (ordered by end-extension). Then P_S preserves $\aleph_2$.

23.8. Let κ be inaccessible and let $S \subset \kappa$ be such that S contains every singular limit ordinal $\alpha < \kappa$. Then P_S is *essentially* $<\kappa$*-closed*, i.e., for every regular $\lambda < \kappa$, P_S has a dense subset that is λ-closed. Hence P_S preserves κ (and adds no λ-sequences for $\lambda < \kappa$).

[For each $\lambda < \kappa$, consider $\{p \in P_S : \max(p) > \lambda\}$.]

23.9. If $I_{\rm NS}$ on ω_1 is $\aleph_2$-saturated then every nontrivial normal κ-complete ideal on ω_1 is $\aleph_2$-saturated.

[Use Exercise 22.11, and that every $S \in I^+$ is stationary.]

23.10. If κ is a regular cardinal then there exists a strongly almost disjoint family $\{X_\xi : \xi < \kappa^+\}$ of subsets of κ.

23.11. It is consistent that $\mathrm{sat}(I_{\rm NS}) < 2^{\aleph_1}$.

[Assume GCH and add more than $\aleph_4$ Cohen reals. Let $\{S_i : i < \omega_4\} \in V[G]$ be a family of stationary sets such that each $S_i \cap S_j$ is nonstationary. Let $i \neq j$. There exists a nonstationary set $A_{i,j} \supset S_i \cap S_j$ in V. Since the forcing notion is c.c.c., there exists an $A_{i,j} \in I_{\rm NS}$ such that $\Vdash S_i \cap S_j \subset A_{i,j}$. Apply the Erdős-Rado Theorem (namely $\aleph_4 \to (\aleph_3)^2_{\aleph_2}$) to find some set $H \subset \omega_4$ of size $\aleph_3$ and some $A \in I_{\rm NS}$ such that $\Vdash S_i \cap S_j \subset A$ for all $i, j \in H$. Get $\aleph_3$ disjoint subsets $S_i - A$ of ω_1 in $V[G]$, a contradiction.]

23.12. There exist stationary sets $S \subset E^{\aleph_3}_{\aleph_0}$ and $A \subset E^{\aleph_3}_{\aleph_1}$ such that S does not reflect at any $\alpha \in A$.

[Let S_i, $i < \omega_2$, be pairwise disjoint stationary subsets of $E^{\aleph_3}_{\aleph_0}$. For each $\alpha \in E^{\aleph_3}_{\aleph_1}$, let $C_\alpha \subset \alpha$ be closed unbounded of size $\aleph_1$. For every α there exists an i_α such that $S_i \cap C_\alpha = \emptyset$ for all $i \geq i_\alpha$. There exists a stationary set $A \subset E^{\aleph_3}_{\aleph_1}$ such that i_α is constant on A, $i_\alpha = i$. The set S_i does not reflect at any $\alpha \in A$.]

Historical Notes

The equivalence in Lemma 23.1 is due to Kunen. Theorem 23.2 is due to Gregory [1976]. Club-guessing principles were introduced by Shelah; see Gitik and Shelah [1997] for details. Lemma 23.6 is due to Jensen.

The construction in Theorem 23.8 (shooting a closed unbounded set) appears in Baumgartner et al. [1976]. Theorem 23.10 uses a construction of Magidor, see Jech et al. [1980]. There is a sequence of results on the strength of precipitousness of $I_{\rm NS}$ on cardinals $\kappa > \aleph_1$: Jech [1984], Gitik [1984, 1995, 1997] See the detailed discussion in Jech [∞].

Theorem 23.17 uses the work of Shelah [1982] (Lemma 23.19 and Corollaries 23.20 and 23.21) and Gitik and Shelah [1997]. The paper Jech and Woodin [1985] investigates saturation of $I_{\rm NS}\restriction \mathrm{Reg}$ for inaccessible cardinals.

Theorem 23.23 appears in Magidor [1982].

Exercise 23.2: Gitik and Shelah [1997].

Exercise 23.9: Baumgartner et al. [1977],

Exercise 23.11: Baumgartner.

Exercise 23.12: Shelah.

24. The Singular Cardinal Problem

In this chapter we use combinatorial methods to prove theorems (in ZFC) on cardinal arithmetic of singular cardinals. We introduce a powerful theory of Shelah, the *pcf* theory, and apply the theory to present a most remarkable result of Shelah on powers of singular cardinals.

The Galvin-Hajnal Theorem

Following Silver's Theorem 8.12 on singular cardinals of uncountable cofinality, Galvin and Hajnal proved a related result:

Theorem 24.1 (Galvin-Hajnal [1975]). *Let $\aleph_\alpha$ be a strong limit singular cardinal of uncountable cofinality. Then $2^{\aleph_\alpha} < \aleph_\gamma$ where $\gamma = (2^{|\alpha|})^+$.*

Note that the theorem gives a nontrivial information only if $\aleph_\alpha$ is not a fixed point of the aleph function.

In order to simplify the notation, we consider the special case $\alpha = \omega_1$. The following lemma implies the theorem (as in the proof of Silver's Theorem). Two functions f and g on ω_1 are almost disjoint if $\{\alpha : f(\alpha) = g(\alpha)\}$ is at most countable.

Lemma 24.2. *Assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Let F be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha$$

such that $|A_\alpha| < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Then $|F| < \aleph_\gamma$ where $\gamma = (2^{\aleph_1})^+$.

Proof. We first introduce the following relation among functions $\varphi : \omega_1 \to \omega_1$

(24.1) $\quad \varphi < \psi \quad$ if and only if $\quad \{\alpha < \omega_1 : \varphi(\alpha) \geq \psi(\alpha)\}$ is nonstationary.

Since the closed unbounded filter is σ-complete, it follows that there is no infinite descending sequence

$$\varphi_0 > \varphi_1 > \varphi_2 > \dots .$$

Otherwise, the set $\{\alpha < \omega_1 : \varphi_n(\alpha) \leq \varphi_{n+1}(\alpha) \text{ for some } n\}$ is nonstationary and so there is an α such that

$$\varphi_0(\alpha) > \varphi_1(\alpha) > \varphi_2(\alpha) > \dots,$$

a contradiction.

Hence the relation $\varphi < \psi$ is well-founded and we can define the rank $\|\varphi\|$ of φ in this relation (called the *norm* of φ) such that

$$\|\varphi\| = \sup\{\|\psi\| + 1 : \psi < \varphi\}.$$

Note that $\|\varphi\| = 0$ if and only if $\varphi(\alpha) = 0$ for a stationary set of α's.

Lemma 24.2 follows from

Lemma 24.3. *Assume that* $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ *for all* $\alpha < \omega_1$. *Let* $\varphi : \omega_1 \to \omega_1$ *and let* F *be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha$$

such that

$$|A_\alpha| \leq \aleph_{\alpha+\varphi(\alpha)}$$

for every $\alpha < \omega_1$. *Then* $|F| \leq \aleph_{\omega_1+\|\varphi\|}$.

To prove Lemma 24.2 from Lemma 24.3, we let φ be such that $|A_\alpha| \leq \aleph_{\alpha+\varphi(\alpha)}$. If ϑ is the length of the well-founded relation $\varphi < \psi$, then certainly $|\vartheta| \leq 2^{\aleph_1}$ and so $\vartheta < (2^{\aleph_1})^+$. Hence $\omega_1 + \|\varphi\| < (2^{\aleph_1})^+$ for every φ and Lemma 24.2 follows. □

Proof of Lemma 24.3. By induction on $\|\varphi\|$. If $\|\varphi\| = 0$, then $\varphi(\alpha) = 0$ on a stationary set and the statement is precisely Lemma 8.16.

To handle the case $\|\varphi\| > 0$, we first generalize the definition of $\varphi < \psi$. Let $S \subset \omega_1$ be a stationary set. We define

$$\varphi <_S \psi \quad \text{if and only if} \quad \{\alpha \in S : \varphi(\alpha) \geq \psi(\alpha)\} \text{ is nonstationary.} \tag{24.2}$$

The same argument as before shows that $\varphi <_S \psi$ is a well-founded relation and so we define the norm $\|\varphi\|_S$ accordingly. Note that if $S \subset T$, then $\|\varphi\|_T \leq \|\varphi\|_S$. In particular, $\|\varphi\| \leq \|\varphi\|_S$, for any stationary S. Moreover,

$$\|\varphi\|_{S\cup T} = \min\{\|\varphi\|_S, \|\varphi\|_T\} \tag{24.3}$$

as can easily be verified.

For every $\varphi : \omega_1 \to \omega_1$, we let I_φ be the collection of all nonstationary sets along with those stationary S such that $\|\varphi\| < \|\varphi\|_S$. If S is stationary and X is nonstationary, then $\|\varphi\|_{S\cup X} = \|\varphi\|_S$. This and (24.3) imply that I_φ is a proper ideal on ω_1.

If $\|\varphi\|$ is a limit ordinal, then

$$S = \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor ordinal}\} \in I_\varphi$$

because if $S \notin I_\varphi$, then $\|\varphi\| = \|\varphi\|_S = \|\psi\|_S + 1$, where $\psi(\alpha) = \varphi(\alpha) - 1$ for all $\alpha \in S$. Hence

$$\{\alpha < \omega_1 : \varphi(\alpha) \text{ is a limit ordinal}\} \notin I_\varphi.$$

Similarly, if $\|\varphi\|$ is a successor ordinal, then

$$\{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor ordinal}\} \notin I_\varphi.$$

Now we are ready to proceed with the induction.

(a) Let $\|\varphi\|$ be a limit ordinal, $\|\varphi\| > 0$. Let

$$S = \{\alpha < \omega_1 : \varphi(\alpha) > 0 \text{ and is a limit ordinal}\}.$$

It follows that $S \notin I_\varphi$.

We may assume that $A_\alpha \subset \aleph_{\alpha+\varphi(\alpha)}$ for every α, and so we have $f(\alpha) < \aleph_{\alpha+\varphi(\alpha)}$ for every $f \in F$. Given $f \in F$, we can find for each $\alpha \in S$ some $\beta < \varphi(\alpha)$ such that $f(\alpha) < \omega_{\alpha+\beta}$; call this $\beta = \psi(\alpha)$. For $\alpha \notin S$, let $\psi(\alpha) = \varphi(\alpha)$. Since $S \notin I_\varphi$, we have $\|\psi\| \leq \|\psi\|_S < \|\varphi\|_S = \|\varphi\|$. We also have $f \in F_\psi$, where

$$F_\psi = \{f \in F : f(\alpha) < \omega_{\alpha+\psi(\alpha)} \text{ for all } \alpha\},$$

and so

$$F = \bigcup\{F_\psi : \|\psi\| < \|\varphi\|\}.$$

By the induction hypothesis, $|F_\psi| \leq \aleph_{\omega_1+\|\psi\|} < \aleph_{\omega_1+\|\varphi\|}$ for every ψ such that $\|\psi\| < \|\varphi\|$. Since the number of functions $\psi : \omega_1 \to \omega_1$ is $2^{\aleph_1}$, and $2^{\aleph_1} < \aleph_{\omega_1}$, we have $|F| \leq \aleph_{\omega_1+\|\varphi\|}$.

(b) Let $\|\varphi\|$ be a successor ordinal, $\|\varphi\| = \gamma + 1$. Let

$$S_0 = \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor}\}.$$

It follows that $S_0 \notin I_\varphi$.

Again, we may assume that $A_\alpha \subset \omega_{\alpha+\varphi(\alpha)}$ for each $\alpha < \omega_1$. First we prove that for every $f \in F$, the set

$$F_f = \{g \in F : \exists S \subset S_0,\ S \notin I_\varphi,\ (\forall \alpha \in S)\ g(\alpha) \leq f(\alpha)\}$$

has cardinality $\aleph_{\omega_1+\gamma}$. If $S \subset S_0$ and $S \notin I_\varphi$, let

$$F_{f,S} = \{g \in F : (\forall \alpha \in S)\ g(\alpha) \leq f(\alpha)\}.$$

Let $\psi : \omega_1 \to \omega_1$ be such that $\psi(\alpha) = \varphi(\alpha) - 1$ for $\alpha \in S$, and $\psi(\alpha) = \varphi(\alpha)$ otherwise. Since $S \notin I_\varphi$, we have $\|\psi\| \leq \|\psi\|_S < \|\varphi\|_S = \|\varphi\| = \gamma + 1$ and so $\|\psi\| = \gamma$. Since $F_{f,S} \subset \prod_{\alpha<\omega_1} B_\alpha$, where $|B_\alpha| \leq \aleph_{\alpha+\psi(\alpha)}$ for all α, we use the

induction hypothesis to conclude that $|F_{f,S}| \leq \aleph_{\omega_1+\gamma}$. Then it follows that $|F_f| \leq \aleph_{\omega_1+\gamma}$.

To complete the proof, we construct a sequence

$$\langle f_\xi : \xi < \vartheta \rangle \tag{24.4}$$

such that $\vartheta \leq \aleph_{\omega_1+\gamma+1}$ and

$$F = \bigcup \{F_{f_\xi} : \xi < \vartheta\}. \tag{24.5}$$

Given f_ν, $\nu < \xi$, we let $f_\xi \in F$ (if it exists) be such that $f_\xi \notin F_{f_\nu}$, for all $\nu < \xi$. Then the set

$$\{\alpha \in S_0 : f_\xi(\alpha) \leq f_\nu(\alpha)\}$$

belongs to I_φ, and so $f_\nu \in F_{f_\xi}$, for each $\nu < \xi$.

Since $|F_{f_\xi}| \leq \aleph_{\omega_1+\gamma}$ and $F_{f_\xi} \supset \{f_\nu : \nu < \xi\}$, it follows that $\xi < \aleph_{\omega_1+\gamma+1}$ if f_ξ exists. Thus the sequence (24.4) has length $\vartheta \leq \aleph_{\omega_1+\gamma+1}$. Then we have

$$F = \bigcup \{F_{f_\xi} : \xi < \vartheta\}$$

and so $|F| \leq \aleph_{\omega_1+\gamma+1}$. □

Ordinal Functions and Scales

The proof of the Galvin-Hajnal Theorem suggests that ordinal functions play an important role in arithmetic of singular cardinals. We shall now embark on a systematic study of ordinal functions and introduce Shelah's pcf theory.

Let A be an infinite set and let I be an ideal on A.

Definition 24.4. For ordinal functions f, g on A, let

$$\begin{aligned} f =_I g \quad &\text{if and only if} \quad \{a \in A : f(a) \neq g(a)\} \in I, \\ f \leq_I g \quad &\text{if and only if} \quad \{a \in A : f(a) > g(a)\} \in I, \\ f <_I g \quad &\text{if and only if} \quad \{a \in A : f(a) \geq g(a)\} \in I. \end{aligned}$$

If F is a filter on A, then $f <_F g$ means $f <_I g$ where I is the dual ideal, and similarly for $f \leq_F g$ and $f =_F g$.

The relation $\leq_I$ is a partial ordering (of equivalence classes). If S is a set of ordinal functions on A then g is an *upper bound* of S if $f \leq_I g$ for all $f \in S$, and g is a *least upper bound* of S if it is an upper bound and if $g \leq_I h$ for every upper bound h.

The relation $<_I$ is also a partial ordering (different from $\leq_I$ unless I is a prime ideal), and if I is σ-complete then $<_I$ is well-founded. If I is the nonstationary ideal on a regular uncountable cardinal κ, then the rank of an ordinal function f on κ is the (Galvin-Hajnal) *norm* $\|f\|$.

The following lemma shows that for every $\eta < \kappa^+$ there is a *canonical* function f_η on κ of norm η:

Lemma 24.5. *Let κ be a regular uncountable cardinal. There exist ordinal functions f_η, $\eta < \kappa^+$, on κ such that*

(i) $f_0(\alpha) = 0$ *and* $f_{\eta+1}(\alpha) = f_\eta(\alpha) + 1$, *for all* $\alpha < \kappa$,
(ii) *if η is a limit ordinal then f_η is a least upper bound of $\{f_\xi : \xi < \eta\}$ in $\leq_{I_{\rm NS}}$.*

The functions are unique up to $=_{I_{\rm NS}}$, and for every stationary set $S \subset \kappa$, $\|f_\eta\|_S = \eta$.

Proof. Let $\langle \xi_\nu : \nu < \operatorname{cf}\eta\rangle$ be some sequence with limit η. If $\operatorname{cf}\eta < \kappa$, let $f_\eta(\alpha) = \sup\{f_{\xi_\nu}(\alpha) : \nu < \operatorname{cf}\eta\}$, and if $\operatorname{cf}\eta = \kappa$, let $f_\eta(\alpha) = \sup\{f_{\xi_\nu}(\alpha) : \nu < \alpha\}$ (for every limit ordinal α), the *diagonal limit* of f_ξ, $\xi < \eta$. □

For $\eta \geq \kappa^+$, canonical functions may or may not exist. The existence of f_η for all ordinals η is equiconsistent with a measurable cardinal. For the relation between canonical functions and canonical stationary sets, see Exercise 24.10.

A subset A of a partially ordered set $(P, <)$ is *cofinal* if for every $p \in P$ there exists some $a \in A$ such that $p \leq a$. The *cofinality* of $(P, <)$ is the smallest size of a cofinal set (it need not be a regular cardinal—see Exercise 24.11). The *true cofinality* of $(P, <)$ is the least cardinality of a cofinal chain (if it exists—see Exercise 24.12). The true cofinality is a regular cardinal (or 1 if P has a greatest element).

Consider again an infinite set A, an ideal I on A, and an indexed set $\{\gamma_a : a \in A\}$ of limit ordinals.

Definition 24.6. A *scale* in $\prod_{a\in A}\gamma_a$ is a $<_I$-increasing transfinite sequence $\langle f_\alpha : \alpha < \lambda\rangle$ of functions in $\prod_{a\in A}\gamma_a$ that is cofinal in $\prod_{a\in A}\gamma_a$ in the partial ordering $<_I$.

If $\prod_{a\in A}\gamma_a$ has a λ-*scale* (i.e., a scale of length λ) and λ is a regular cardinal then it has true cofinality λ, and is λ-*directed*, i.e., every set $B \subset \prod_{a\in A}\gamma_a$ of size $< \gamma$ has an upper bound. The ordinal function $\langle \gamma_a : a \in A\rangle$ is the least upper bound of $\prod_{a\in A}\gamma_a$; moreover, it is an *exact upper bound*:

Definition 24.7. In a partially ordered set $(P, <)$, g is an *exact upper bound* of a set S if S is cofinal in the set $\{f \in P : f < g\}$.

The following theorem is a precursor of the pcf theory. We note that the pcf theory shows, among others, that different sequences $\langle \lambda_n : n < \omega\rangle$ with the same limit will generally result in different cofinalities of $\prod_{n<\omega}\lambda_n$.

Theorem 24.8 (Shelah). *Let κ be a strong limit cardinal of cofinality ω. There exists an increasing sequence $\langle \lambda_n : n < \omega\rangle$ of regular cardinals with limit κ such that the true cofinality of $\prod_{n<\omega}\lambda_n$ modulo the ideal of finite sets is equal to κ^+.*

Proof. Let I be the ideal of finite subsets of ω. We shall find the λ_n's and a κ^+-scale in $\prod_n \lambda_n$ in the partial ordering $<_I$.

First we choose any increasing sequence κ_n, $n < \omega$, of regular cardinals with limit κ. As every subset of $\prod_{n<\omega} \kappa_n$ of size κ has an upper bound in $(\prod_{n<\omega} \kappa_n, <_I)$, we can construct inductively a $<_I$-increasing κ^+-sequence $F = \langle f_\xi : \xi < \kappa^+ \rangle$ of functions in $\prod_n \kappa_n$.

Lemma 24.9. *There exists a function $g : \omega \to \kappa$ that is an upper bound of F in $<_I$, and is $\leq_I$-minimal among such upper bounds.*

Proof. Let $g_0 = \langle \kappa_n : n < \omega \rangle$; we shall construct a maximal transfinite $\leq_I$-decreasing sequence $\langle g_\nu \rangle_\nu$ of upper bounds of F. It suffices to show that the length of the sequence $\langle g_\nu \rangle_\nu$ is not a limit ordinal: Then the last function is $\leq_I$-minimal.

Thus let ϑ be a limit ordinal, and let $\langle g_\nu : \nu < \vartheta \rangle$ be a $\leq_I$-decreasing sequence of upper bounds for F. We shall find a function g such that $g >_I f_\xi$ for all $\xi < \kappa^+$, and $g \leq_I g_\nu$ for all $\nu < \vartheta$.

First we claim that $|\vartheta| \leq 2^{\aleph_0}$. Thus assume that $|\vartheta| \geq (2^{\aleph_0})^+$ and consider the partition $G : [\vartheta]^2 \to \omega$ defined as follows (for $\alpha < \beta$):

$$G(\alpha, \beta) = \text{the least } n \text{ such that } g_\alpha(n) > g_\beta(n).$$

By the Erdős-Rado Partition Theorem 9.6 there exists an infinite set of ordinals $\alpha_0 < \alpha_1 < \alpha_2 < \ldots$ such that for some n, $g_{\alpha_0}(n) > g_{\alpha_1}(n) > g_{\alpha_2}(n) > \ldots$, a contradiction.

Let $A = \bigcup_{\nu<\vartheta} \operatorname{ran}(g_\nu)$ and let $S = A^\omega$. Since $|\vartheta| \leq 2^{\aleph_0}$, we have $|S| \leq 2^{\aleph_0}$. For every $g \in S$, if g is not an upper bound for F, let ξ_g be such that $f_{\xi_g} \not<_I g$. Since $|S| \leq 2^{\aleph_0}$, there is some $\eta < \kappa^+$ greater than all the ξ_g's. Now let

$$g(n) = \text{the least } \gamma \in A \text{ such that } \gamma > f_\eta(n).$$

The function g is an upper bound for F: If not then $f_{\xi_g} \not<_I g$ but $f_{\xi_g} <_I f_\eta <_I g$. We complete the proof of the lemma by showing that $g \leq_I g_\nu$ for all $\nu < \vartheta$. If $\nu < \vartheta$ then $g_\nu(n) > f_\eta(n)$ for all but finitely many n and, since $g_\nu(n) \in A$, we have $g_\nu \geq g$. □

Let g be the function given by Lemma 24.9. We claim that g is an exact upper bound of F. If not, let $f <_I g$ be such that $f \not<_I f_\xi$ for all ξ. For each $\xi < \kappa^+$, let A_ξ be the infinite set of all n such that $f(n) > f_\xi(n)$. Since $2^{\aleph_0} < \kappa$, there exists an infinite set A, such that for κ^+ many ξ's, $f(n) > f_\xi(n)$ for all $a \in A$. It follows that $f{\restriction}A >_I f_\xi{\restriction}A$ for every $\xi < \kappa^+$, and therefore the function $g' = f{\restriction}A \cup g{\restriction}(\omega - A) \leq_I g$ is an upper bound of F but $g' \neq_I g$, a contradiction.

Now, if g is increasing with limit κ and if every $g(n)$ is a regular cardinal, then we let $\lambda_n = g(n)$ and are done. In general, all but finitely many $g(n)$ are limit ordinals; without loss of generality, all are. For each n, let Y_n be a closed unbounded subset of $g(n)$ whose order-type is a regular cardinal γ_a. Note that

$\sup_n \gamma_n = \kappa$; otherwise, $|\prod_n Y_n| < \kappa$ and hence bounded by some f_ξ. So let $\langle \lambda_n : n < \omega \rangle = \langle \gamma_{k_n} : n < \omega \rangle$ be an increasing subsequence of $\langle \gamma_n \rangle_n$.

For each $f \in F$, let h_f be the function

$$h_f(n) = \text{the least } \alpha \in Y_{k_n} \text{ such that } \alpha \geq f(k_n).$$

and let $H = \{h_f : f \in F\}$. For every $f \in \prod_n Y_n$ there exists some $h \in H$ such that $f <_I h$. Also, $|H| = \kappa^+$ since every smaller set of functions is bounded by some f_ξ. Thus we can find in H a $<_I$-increasing transfinite sequence $\langle h_\xi : \xi < \kappa^+ \rangle$ such that for every $f \in \prod_n Y_n$, there is a ξ with $f <_I h_\xi$. By copying $\prod_n Y_n$ onto $\prod_n \lambda_n$, we get a sequence $\langle h_\xi : \xi < \kappa^+ \rangle$ with the required properties. □

As an application of Theorem 24.8 we give a short proof of Kunen's Theorem 17.7, due to Zapletal [1996].

Assume that $j : V \to M$ is elementary, with critical point κ, and let $\lambda = \lim_n j^n(\kappa)$. As λ is a strong limit cardinal of cofinality ω, let $\langle \lambda_n : n < \omega \rangle$ be an increasing sequence of regular cardinals with limit λ such that $\kappa < \lambda_0$ and that $\prod_n \lambda_n$ has a λ^+-scale $F = \langle f_\xi : \xi < \lambda^+ \rangle$ (modulo finite). Since $j(\lambda) = \lambda$, we have $j(\lambda^+) = \lambda^+$, and $j(F)$ is a λ^+-scale in $\prod_n j(\lambda_n)$.

Since $j``\lambda^+$ is cofinal in $j(\lambda^+) = \lambda^+$, $j``F$ is cofinal in $j(F)$ and thus in $\prod_n j(\lambda_n)$. However, let $g \in \prod_n j(\lambda_n)$ be the function $g(n) = \sup j``\lambda_n$; we have $g(n) < j(\lambda_n)$ because $j(\lambda_n)$ is regular. If $f \in \prod_n \lambda_n$ then $g > j(f)$ pointwise because $j(f) = j``f$. Hence g is an upper bound for $j``F$, a contradiction. □

Toward the pcf theory, we shall now prove several results on ordinal functions and scales. Let I be an ideal on A.

Lemma 24.10. *If $\lambda > 2^{|A|}$ is a regular cardinal then every $<_I$-increasing λ-sequence of ordinal functions on A has an exact upper bound.*

Proof. Let $F = \langle f_\alpha : \alpha < \lambda \rangle$ be $<_I$-increasing. Let M be an elementary submodel of H_ϑ for a sufficiently large ϑ such that $I \in M$, $F \in M$, $|M| = 2^{|A|}$ and $M^{|A|} \subset M$. For every α, let

$$g_\alpha(a) = \text{the least } \beta \in M \text{ such that } \beta \geq f_\alpha(a) \qquad (a \in A).$$

Since $M^{|A|} \subset M$, we have $g_\alpha \in M$, and since $|M| < \lambda$, there exists some $f \in M$ such that $f = g_\alpha$ for λ many α's. Since $\langle f_\alpha \rangle_\alpha$ is increasing and $f \geq_I f_\alpha$ for λ many α's, f is an upper bound of F.

To show that whenever $h <_I f$ then $h <_I f_\alpha$ for some α, it is enough to show this for every $h \in M$. Thus let $h \in M$ be such that $h <_I f$.

Let α be any α such that $f = g_\alpha$. For every $a \in A$ such that $h(a) < g_\alpha(a)$ we necessarily have $h(a) < f_\alpha(a)$ because $h(a) \in M$ and $g_\alpha(a)$ is the least $\beta \in M$ such that $\beta \geq f_\alpha(a)$. Hence $h <_I f_\alpha$. □

If F is a set of ordinal functions on A and g is an upper bound of F, then we say that F is *bounded below* g if it has an upper bound $h <_I g$; F is *cofinal* in g if it is cofinal in $\prod_{a\in A} g(a)$. If $X \in I^+$ then $f <_I g$ on X, etc., means $f <_{I\restriction X} g$ where $I\restriction X$ is the ideal generated by $I \cup \{A - X\}$.

Corollary 24.11. *If $\lambda > 2^{|A|}$ is regular, $F = \langle f_\alpha : \alpha < \lambda\rangle$ is $<_I$-increasing and g is an upper bound of F, then either F is bounded below g, or F is cofinal in g, or $A = X \cup Y$ with $X, Y \in I^+$ such that F is bounded below g on X and is cofinal in g on Y.*

Proof. Let f be an exact upper bound of F and let $X = \{a \in A : f(a) < g(a)\}$. □

Corollary 24.12. *Let $\lambda > 2^{|A|}$ be a regular cardinal, let γ_a, $a \in A$, be limit ordinals, and assume that $\prod_{a\in A} \gamma_a$ is λ-directed in $<_I$. Then either $\prod_{a\in A} \gamma_a$ is λ^+-directed, or has a λ-scale, or $A = X \cup Y$ with $X, Y \in I^+$ such that $\prod_{a\in A} \gamma_a$ has a λ-scale on X and is λ^+-directed on Y.*

Proof. Assume that $\prod_{a\in A} \gamma_a$ is λ-directed but not λ^+-directed, and let $S \subset \prod_{a\in A} \gamma_a$ be such that $|S| = \lambda$ and S is not bounded. Using the λ-directness, we construct an increasing sequence $F = \langle f_\alpha : \alpha < \lambda\rangle$ such that for every $f \in S$, there exists an $\alpha < \lambda$ such that $f <_I f_\alpha$. As F is not bounded, there exists some $Z \in I^+$ such that F is a scale on Z.

Now let $\mathcal{Z}$ be the collection of all $Z \in I^+$ that have a λ-scale, and for each $Z \in \mathcal{Z}$ let $\langle f^Z_\alpha : \alpha < \lambda\rangle$ be a λ-scale on Z. Let $S = \{f^Z_\alpha : \alpha < \lambda, Z \in \mathcal{Z}\}$; since $2^{|A|} = \lambda$, we have $|S| = \lambda$, and we can construct an increasing λ-sequence $F = \langle f_\alpha : \alpha < \lambda\rangle$ such that for every $f \in S$ there is an $\alpha < \lambda$ with $f \le_I f_\alpha$.

Either F is a scale, or $A = X \cup Y$ such that F is bounded on X and cofinal on Y. To complete the proof, we show that $\prod_{a\in A} \gamma_a$ is λ^+-directed; i.e., that for every set of size λ is bounded on X. If not, we repeat the argument above and find a $Z \subset X$ that has a scale. This contradicts the fact that S is bounded on X. □

Definition 24.13. Let $F = \langle f_\alpha : \alpha < \lambda\rangle$, λ regular, be a $<_I$-increasing sequence of ordinal functions on A and let $\gamma < \lambda$ be a regular uncountable cardinal. F is *γ-rapid* if for every $\beta < \lambda$ of cofinality γ there exists a closed unbounded set $C \subset \beta$ such that for every limit ordinal $\alpha < \beta$, $f_\alpha >_I s_{C\cap\alpha}$, where $s_{C\cap\alpha}$ is the pointwise supremum of $\{f_\xi(a) : \xi \in C \cap \alpha\}$:

$$s_{C\cap\alpha}(a) = \sup\{f_\xi(a) : \xi \in C \cap \alpha\} \qquad (a \in A).$$

Lemma 24.14. *Let $F = \langle f_\alpha : \alpha < \lambda\rangle$ be γ-rapid, with $\gamma > |A|$. For each $a \in A$, let $S_a \subset \lambda$ be such that $|S_a| < \gamma$. Then there exists an $\alpha < \lambda$ with the property that for every $h \in \prod_{a\in A} S_a$, if $h >_I f_\alpha$, then h is an upper bound of F.*

Proof. Assume by contradiction that for every $\alpha < \lambda$ there exists an $h \in \prod_{a\in A} S_a$ such that $h >_I f_\alpha$ but h is not an upper bound of F. By induction, we construct a continuous increasing sequence α_ξ, $\xi < \gamma$, and functions $h_\xi \in \prod_{a\in A} S_a$ such that for every ξ, $f_{\alpha_\xi} <_I h_\xi$ and $f_{\alpha_{\xi+1}} \not<_I h_\xi$. Let $\beta = \lim_{\xi\to\gamma} \alpha_\xi$.

As F is γ-rapid, there exists a closed unbounded $C \subset \beta$ such that $f_\alpha >_I s_{C\cap\alpha}$ for every $\alpha \in C$. We may assume that $\alpha_\xi \in C$ for every $\xi < \gamma$ (otherwise replace $\{\alpha_\xi\}_{\xi<\gamma}$ by its intersection with C).

For each $\xi < \gamma$ we have $s_{C\cap\alpha_\xi} <_I f_{\alpha_\xi} <_I h_\xi \not>_I f_{\alpha_{\xi+1}}$ and so there exists some $a_\xi \in A$ such that

$$s_{C\cap\alpha_\xi}(a_\xi) < f_{\alpha_\xi}(a_\xi) < h_\xi(a_\xi) < f_{\alpha_{\xi+1}}(a_\xi).$$

As $\gamma > |A|$, there exist a set $Z \subset \gamma$ of size γ and some $a \in A$ such that $a_\xi = a$ for all $\xi \in Z$. Now if ξ and η are in Z, such that $\xi+1 < \eta$, then $\alpha_{\xi+1} \in C\cap\alpha_\eta$ and we have

$$h_\xi(a) < f_{\alpha_{\xi+1}}(a) \le s_{C\cap\alpha_\eta}(a) < h_\eta(a).$$

This is a contradiction because $|S_a| < \gamma$ while $|Z| = \gamma$. □

Corollary 24.15. *If $F = \langle f_\alpha : \alpha < \lambda\rangle$ is γ-rapid, with $|A| < \gamma < \lambda$, and if f is the least upper bound of F, then $\operatorname{cf} f(a) \ge \gamma$ for I-almost all $a \in A$.*

Proof. Let f be an upper bound of F, and assume that $B = \{a \in A : \operatorname{cf} f(a) < \gamma\} \in I^+$. We shall find an upper bound h of F such that $h <_I f$ on B.

For $a \in B$, let S_a be a cofinal subset of $f(a)$ of size $< \gamma$. By Lemma 24.14 there is an $\alpha < \lambda$ such that for every $h \in \prod_{a\in B} S_a$, $h >_I f_\alpha$ on B implies that h is an upper bound of F on B. Given this α, we consider a function $h \in \prod_{a\in B} S_a$ as follows: If $f_\alpha(a) < f(a)$, let $h(a) \in S_a$ be such that $f_\alpha(a) < h(a) < f(a)$. The function h is an upper bound of F on B, and $h <_I f$ on B. □

Theorem 24.16 (Shelah). *Let κ be a regular uncountable cardinal, and let $I = I_{\rm NS}$ be the nonstationary ideal on κ. Let $\langle\eta_\xi : \xi < \kappa\rangle$ be a continuous increasing sequence with limit η. Then $\prod_{\xi<\kappa} \aleph_{\eta_\xi+1}$ has true cofinality $\aleph_{\eta+1}$ (in $<_I$).*

We shall prove this theorem only under the assumption $2^\kappa < \aleph_\eta$ (we only need the weaker version for the proof of Theorem 24.33). For the general proof, see Burke and Magidor [1990].

Proof. Let $\lambda = \aleph_{\eta+1}$. We wish to find a λ-scale. It is not difficult to see that $\prod_{\xi<\kappa} \aleph_{\eta_\xi+1}$ is λ-directed. By Corollary 24.12 (as we assume $2^\kappa < \lambda$), if there is no λ-scale then there is a stationary set $S \subset \kappa$ such that $\prod_{\xi\in S} \aleph_{\eta_\xi+1}$ is λ^+-directed.

We shall construct a $<_I$-increasing λ-sequence in $\prod_{\xi\in S} \aleph_{\eta_\xi+1}$ that is γ-rapid for all regular $\gamma < \aleph_\eta$. For every limit ordinal $\beta < \lambda$, let $C_\beta \subset \beta$ be

closed unbounded, of size $\operatorname{cf}\beta$. We construct $F = \langle f_\alpha : \alpha < \lambda\rangle$ by induction. Let α be a limit ordinal. For each limit $\beta > \alpha$, let s_β be the pointwise supremum of $\{f_\nu : \nu \in C_\beta \cap \alpha\}$. For eventually all $\xi < \kappa$, $s_\nu(\xi) < \aleph_{\eta_{\xi+1}}$, so $s_\nu \in \prod_{\xi\in S} \aleph_{\eta_\xi+1}$. Since $\prod_{\xi\in S} \aleph_{\eta_\xi+1}$ is λ^+-directed, we can find f_α so that $f_\alpha >_I s_\beta$ on S for all limit $\beta < \lambda$. This guarantees that F is γ-rapid for every regular uncountable $\gamma < \lambda$.

By Lemma 24.10, F has an exact upper bound g, and without loss of generality, $g(\xi) \le \aleph_{\eta_\xi+1}$ for all $\xi \in S$. We claim that $g(\xi) \ge \aleph_{\eta_\xi+1}$ for almost all $\xi \in S$, and hence F is a scale on S, contrary to the assumption on S. If $g(\xi) < \aleph_{\eta_\xi+1}$ for stationary many ξ, then $\operatorname{cf} g(\xi) < \aleph_{\eta_\xi}$, and hence for some $\gamma < \aleph_{\eta+1}$, $\operatorname{cf} g(\xi) < \gamma$ for stationary many ξ. This contradicts Corollary 24.15, as F is γ-rapid for all $\gamma < \lambda$. □

The pcf Theory

Shelah's pcf theory is the theory of *possible cofinalities* of ultraproducts of sets of regular cardinals. Let A be a set of regular cardinals, and let D be an ultrafilter on A. $\prod A = \prod_{a\in A}\{a : a \in A\}$ denotes the product $\{f : \operatorname{dom}(f) = A \text{ and } f(a) \in a\}$; the ultraproduct $\prod A/D$ is linearly ordered, and $\operatorname{cof} D = \operatorname{cof}\prod A/D$ is its cofinality.

Definition 24.17. If A is a set of regular cardinals, then

$$\operatorname{pcf} A = \{\operatorname{cof} D : D \text{ is an ultrafilter on } A\}.$$

The set $\operatorname{pcf} A$ is a set of regular cardinals, includes A (for every $a \in A$ consider the principal ultrafilter given by a), has cardinality at most $2^{2^{|A|}}$ and satisfies $\operatorname{pcf}(A_1 \cup A_2) = \operatorname{pcf} A_1 \cup \operatorname{pcf} A_2$.

We shall investigate the structure of pcf in the next section. In this section we explore the relation between pcf and cardinal arithmetic. Instead of the general theory we concentrate on the special case when $A = \{\aleph_n\}_{n=0}^\infty$. We prove the following theorem:

Theorem 24.18 (Shelah). *If $\aleph_\omega$ is a strong limit cardinal then*

$$\max(\operatorname{pcf}\{\aleph_n\}_{n=0}^\infty) = 2^{\aleph_\omega}.$$

A stronger theorem is true: If $2^{\aleph_0} < \aleph_\omega$ then $\max(\operatorname{pcf}\{\aleph_n\}_{n=0}^\infty) = \aleph_\omega^{\aleph_0}$; again, we refer the reader to Burke and Magidor [1990].

We say that a set of regular cardinals A is an *interval* if it contains every regular λ such that $\min A \le \lambda < \sup A$.

Lemma 24.19. *Let A be an interval of regular cardinals such that $\min A = (2^{|A|})^+$. Then $\operatorname{pcf} A$ is an interval.*

Proof. Let D be an ultrafilter on A and let λ be a regular cardinal such that $\min A \leq \lambda < \operatorname{cof} D$. We shall find an ultrafilter E on A such that $\operatorname{cof} E = \lambda$.

Let $\{f_\alpha : \alpha < \operatorname{cof} D\}$ be a D-increasing sequence in $\prod A$. Since $\lambda > 2^{|A|}$, the sequence has a least upper bound g in $\leq_D$ (by Lemma 24.10). For each $a \in A$ let $h(a) = \operatorname{cf} g(a)$ and let S_a be a cofinal subset of $g(a)$ of order-type $h(a)$. It is easy to see that $\prod_{a\in A} S_a/D$ has an increasing λ-sequence cofinal in g, and hence $\prod_{a\in A} h(a)/D$ has a cofinal sequence $\{h_\alpha : \alpha < \lambda\}$.

For D-almost all a, $h(a) > 2^{|A|}$: This is because the number of functions from A into $2^{|A|}$ is less than λ. Thus we may assume that $h(a) \in A$ for all $a \in A$. Let E be the ultrafilter on A defined by

$$E = \{X \subset A : h^{-1}(X) \in D\}.$$

We now construct, by induction on α, functions g_α, $\alpha < \lambda$, such that the sequence $\{g_\alpha \circ h : \alpha < \lambda\}$ is D-increasing and cofinal in h. Then $\{g_\alpha : \alpha < \lambda\}$ is E-increasing and cofinal in $\prod A/E$. □

Corollary 24.20. *If $\aleph_\omega$ is a strong limit cardinal, then $\operatorname{pcf}\{\aleph_n\}_{n=0}^\infty$ is an interval and $\sup \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty < \aleph_{\aleph_\omega}$.*

Proof. Apply Lemma 24.19 to the interval $A = [(2^{\aleph_0})^+, \aleph_\omega)$, and use $|\operatorname{pcf} A| \leq 2^{2^{\aleph_0}} < \aleph_\omega$. □

Toward the proof of Theorem 24.18, we assume that $\aleph_\omega$ is strong limit and let

$$\lambda = \sup \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty.$$

We shall show that $2^{\aleph_\omega} = \lambda$. Since $\operatorname{cf} 2^{\aleph_\omega} > \aleph_\omega$ (by König's Theorem) and $\lambda < \aleph_{\aleph_\omega}$, it follows that $2^{\aleph_\omega}$ is a successor cardinal, and therefore $2^{\aleph_\omega} = \max(\operatorname{pcf}\{\aleph_n\}_{n=0}^\infty)$.

Lemma 24.21. *There exists a family F of functions in $\prod_{n=0}^\infty \aleph_n$, $|F| = \lambda$, such that for every $g \in \prod_{n=0}^\infty \aleph_n$ there is some $f \in F$ with $g(n) \leq f(n)$ for all n.*

Proof. For every ultrafilter D on ω choose a sequence $\langle f_\alpha^D : \alpha < \operatorname{cof} D\rangle$ that is cofinal in $\prod_{n=0}^\infty \aleph_n/D$, and let F be the set of all $f = \max\{f_{\alpha_1}^{D_1}, \ldots, f_{\alpha_m}^{D_m}\}$ where $\{D_1, \ldots, D_m\}$ is a finite set of ultrafilters and $\{\alpha_1, \ldots, \alpha_m\}$ a finite set of ordinals. Since $\lambda > \aleph_\omega > 2^{2^{\aleph_0}}$, we have $|F| = \lambda$.

Assume, by contradiction, that there is a $g \in \prod_{n=0}^\infty \aleph_n$ that is not majorized by any $f \in F$. Thus if we let, for every D and every α, $X_\alpha^D = \{n : g(n) > f_\alpha^D(n)\}$, then the family $\{X_\alpha^D\}_{\alpha,D}$ has the finite intersection property, and so extends to an ultrafilter U. Then $g <_U f_\alpha^U$ for some α, a contradiction. □

Let us fix such a family F of size λ, and let $k < \omega$ be such that $2^{\aleph_0} \leq \aleph_k$ and $\lambda < \aleph_{\aleph_k}$. Let ϑ be sufficiently large, and consider elementary submodels of

$(H_\vartheta, \in, <)$ where $<$ is some well-ordering of H_ϑ. For every countable subset a of $\aleph_\omega$ we shall construct an elementary chain of models M^a_α, of length ω_k. Each M^a_α will have size $\aleph_k$ and will be such that $M^a_\alpha \supset a \cup \omega_k$.

We choose M^a_0 of size $\aleph_k$ so that $M^a_0 \supset a \cup \omega_k$. If $\alpha < \omega_k$ is a limit ordinal, we let $M^a_\alpha = \bigcup_{\beta<\alpha} M^a_\beta$. Given M^a_α, we find $M^a_{\alpha+1}$ as follows. Let

$$\chi^a_\alpha(n) = \sup(M^a_\alpha \cap \omega_n) \qquad (\text{all } n > k), \tag{24.6}$$

the *characteristic function* of M^a_α. There exists a function $f^a_\alpha \in F$ such that $f^a_\alpha(n) \geq \chi^a_\alpha(n)$ for all $n > k$; let $M^a_{\alpha+1}$ be such that $f^a_\alpha \in M^a_{\alpha+1}$.

Then we let $M^a = \bigcup_{\alpha<\omega_k} M^a_\alpha$, and

$$\chi^a(n) = \sup(M^a \cap \omega_n) \qquad (\text{all } n > k).$$

Lemma 24.22. *If a and b are countable subsets of $\aleph_\omega$ and if $\chi^a = \chi^b$, then $M^a \cap \aleph_\omega = M^b \cap \aleph_\omega$.*

Proof. By induction on n we show that $M^a \cap \aleph_n = M^b \cap \aleph_n$, for all $n \geq k$. This is true for $n = k$; thus assume that this is true for n and prove it for $n+1$. Both $M^a \cap \aleph_{n+1}$ and $M^b \cap \aleph_{n+1}$ contain a closed unbounded subset of the ordinal $\chi^a(n+1) = \chi^b(n+1)$ (of cofinality $\aleph_k$), and so there is a cofinal subset C of this ordinal such that $C \subset M^a$ and $C \subset M^b$. For every $\gamma \geq \omega_n$ in C there is a one-to-one function π that maps ω_n onto γ. If we let π be the $\prec$-least such function in H_ϑ, then π is both in M^a and in M^b. It follows that $\gamma \cap M^a = \gamma \cap M^b$. Consequently, $\omega_{n+1} \cap M^a = \omega_{n+1} \cap M^b$ and the lemma follows. □

We shall complete the proof of Theorem 24.18 by showing that the set $\{\chi^a : a \subset \aleph_\omega \text{ countable}\}$ has size at most λ. Since each M^a has $\aleph_k$ countable subsets it will follow that there are at most λ countable subsets of $\aleph_\omega$, and therefore $2^{\aleph_\omega} = \lambda$.

For each a and each n we have

$$\chi^a(n) = \sup_{\alpha<\omega_k} \chi^a_\alpha(n) = \sup_{\alpha<\omega_k} f^a_\alpha(n).$$

If S is any subset of ω_k of size $\aleph_k$, then $\chi^a(n) = \sup\{f^a_\alpha(n) : \alpha \in S\}$ and so the set $\{f^a_\alpha : \alpha \in S\}$ determines χ^a.

Lemma 24.23. *There exists a family F_λ of λ subsets of λ, each of size $\aleph_k$, such that for every subset $Z \subset \lambda$ of size $\aleph_k$ there exists an $X \in F_\lambda$ such that $X \subset Z$.*

Proof. We prove (by induction on α) that for every ordinal α such that $2^{\aleph_k} \leq \alpha \leq \lambda$ there is a family $F_\alpha \subset [\alpha]^{\aleph_k}$, $|F_\alpha| \leq |\alpha|$ such that for every $Z \in [\alpha]^{\aleph_k}$ there is an $X \in F_\alpha$ such that $X \subset Z$. This is true for $\alpha = 2^{\aleph_k}$. If α is not a cardinal, then F_α can be obtained by a one-to-one transformation from $F_{|\alpha|}$. If α is a cardinal then since $\alpha \leq \lambda < \aleph_{\aleph_k}$, we have $\operatorname{cf}\alpha \neq \aleph_k$, and it follows that $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$ has the required property. □

Now we complete the proof of Theorem 24.18. For each countable subset a of $\aleph_\omega$ let $Z_a = \{f^a_\alpha : \alpha < \omega_k\}$; each Z_a is a subset of F, and $|Z| = \aleph_k$. Apply Lemma 24.23 to the set F (instead of λ) and obtain a family $F_\lambda \subset [F]^{\aleph_k}$ such that for each a there exists some $X \in F_\lambda$ such that $X \subset Z$. Since $|X| = \aleph_k$, X determines χ^a. It follows that $|\{\chi^a : a \subset \aleph_\omega \text{ countable}\}| \le \lambda$. □

The Structure of pcf

Let A be a set of regular cardinals and let $\operatorname{pcf} A$ denote the set of all possible cofinalities of $\prod A$. First we mention some facts about pcf:

(24.7)
(i) $A \subset \operatorname{pcf} A$.
(ii) If $A_1 \subset A_2$ then $\operatorname{pcf} A_1 \subset \operatorname{pcf} A_2$.
(iii) $\operatorname{pcf}(A_1 \cup A_2) = \operatorname{pcf} A_1 \cup \operatorname{pcf} A_2$.
(iv) $|\operatorname{pcf} A| \le 2^{2^{|A|}}$.
(v) $\sup \operatorname{pcf} A \le |\prod A|$.

In Lemma 24.19 we showed:

(vi) If A is an interval and $2^{|A|} < \min A$ then $\operatorname{pcf} A$ is an interval.

This is true in general, under the assumption $|A| < \min A$ (see Shelah [1994]).

In the following Lemma 24.24 we prove

(vii) If $|\operatorname{pcf} A| < \min A$ then $\operatorname{pcf}(\operatorname{pcf} A) = \operatorname{pcf} A$.

Finally, Theorem 24.18 is true in general, and under weaker assumptions; we state this without a proof.

(viii) If A is an interval without a greatest element and $(\min A)^{|A|} < \sup A$, then $(\sup A)^{|A|} = \max \operatorname{pcf} A$.

For proof, see e.g. Burke and Magidor [1990].

Lemma 24.24. *If* $|\operatorname{pcf} A| < \min A$ *then* $\operatorname{pcf}(\operatorname{pcf} A) = \operatorname{pcf} A$.

Proof. Let $B = \operatorname{pcf} A$. For each $\lambda \in B$ choose D_λ on A such that $\operatorname{cof} D_\lambda = \lambda$, and let $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ be cofinal in $\prod A/D_\lambda$. Let $\mu \in \operatorname{pcf} B$; choose D on B with $\operatorname{cof} D = \mu$, and let $\langle g_\alpha : \alpha < \mu\rangle$ be cofinal in $\prod B/D$. Let

$$E = \{X \subset A : \{\lambda \in B : X \in D_\lambda\} \in D\}.$$

E is an ultrafilter on A and we shall show that $\operatorname{cof} E = \mu$, thus proving $\mu \in \operatorname{pcf} A$, and hence $\operatorname{pcf} B = B$.

For every $\alpha < \mu$, let

$$h_\alpha(a) = \sup_{\lambda \in B} f^\lambda_{g_\alpha(\lambda)}(a) \qquad (\text{all } a \in A).$$

Since $\min A > |B|$, we have $h_\alpha(a) < a$ for all $a \in A$. We will show that for each $h \in \prod A$, eventually all h_α are $\geq_E h$. The we can find a subsequence of $\langle h_\alpha : \alpha < \mu \rangle$ that is cofinal in $\prod A/E$.

Let $h \in \prod A$. For each $\lambda \in B$ there exists a $g(\lambda) < \lambda$ such that $h <_{D_\lambda} f^\lambda_{g(\lambda)}$. For eventually all $\alpha < \mu$ we have $g <_D g_\alpha$, and we claim that whenever $g <_D g_\alpha$ then $h <_E h_\alpha$.

Let α be such that $g <_D g_\alpha$. Let $X = \{a \in A : h(a) < h_\alpha(a)\}$. If λ is such that $g(\lambda) < g_\alpha(\lambda)$ then for D_λ-almost all a, $h(a) < f^\lambda_{g(\lambda)}(a) < f^\lambda_{g_\alpha(\lambda)}(a) \leq h_\alpha(a)$ and hence $a \in X$. Thus $X \in D_\lambda$ for D-almost all λ, and so $X \in E$. □

The fundamental theorem of the pcf theory is the following.

Theorem 24.25 (Shelah). *If A is a set of regular cardinals such that $2^{|A|} < \min A$, then there exist sets $B_\lambda \subset A$, $\lambda \in \operatorname{pcf} A$, such that for every $\lambda \in \operatorname{pcf} A$*

(a) $\lambda = \max \operatorname{pcf} B_\lambda$.
(b) $\lambda \notin \operatorname{pcf}(A - B_\lambda)$.
(c) $\prod\{a : a \in B_\lambda\}$ *has a λ-scale* mod J_λ *where J_λ is the ideal generated by the sets B_ν, $\nu < \lambda$.*

(To see that J_λ is an ideal, we observe that if $X \in J_\lambda$ then $X \subset B_{\nu_1} \cup \ldots \cup B_{\nu_k}$, hence $\operatorname{pcf} X \subset \operatorname{pcf} B_{\nu_1} \cup \ldots \cup \operatorname{pcf} B_{\nu_k}$ and so by (a), $\lambda \notin \operatorname{pcf} X$. Hence $X \neq A$.)

The theorem is true under the weaker assumption $|A| < \min A$; see Shelah [1994] or Burke and Magidor [1990].

Note that (a) and (b) can be formulated as follows:

(a) For every ultrafilter D on B_λ, $\operatorname{cof} D \leq \lambda$; and there exists some D on B_λ such that $\operatorname{cof} D = \lambda$.
(b) For every ultrafilter D on A, if $\operatorname{cof} D = \lambda$ then $B_\lambda \in D$.

The sets B_λ, $\lambda \in \operatorname{pcf} A$, are called the *generators* of $\operatorname{pcf} A$. It follows from (a) and (b) that the cofinality of an ultrafilter on A is determined by which generators it contains:

$$\operatorname{cof} D = \text{the least } \lambda \text{ such that } B_\lambda \in D. \tag{24.8}$$

Corollary 24.26. *If $2^{|A|} < \min A$ then $|\operatorname{pcf} A| \leq 2^{|A|}$.*

Proof. The number of generators is at most $2^{|A|}$. □

Corollary 24.27. *If $\aleph_\omega$ is strong limit then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.*

Proof. Corollary 24.26, Corollary 24.20 and Theorem 24.18. □

Corollary 24.28. *If* $2^{|A|} < \min A$ *then* $\operatorname{pcf} A$ *has a greatest element.*

Proof. Assume that $\operatorname{pcf} A$ does not have a greatest element. Then the set $\{A - B_\lambda : \lambda \in \operatorname{pcf} A\}$ has the finite intersection property, and so extends to an ultrafilter D. By (b), $B_{\operatorname{cof} D} \in D$, a contradiction. □

Proof of Theorem 24.25. We shall apply the results on ordinal functions proved earlier in this chapter. If I is an ideal on a set A of regular cardinals then we say that I has a λ-scale if $\prod A$ has a λ-scale in $<_I$; similarly, we say that I is λ-*directed* if $\prod A$ is λ-directed in $\leq_I$.

We construct the generators B_λ by induction, so that for each cardinal $\kappa \leq \sup \operatorname{pcf} A$ the following conditions are satisfied:

(24.9)
- (i) the ideal J_κ generated by $\{B_\lambda : \lambda < \kappa \text{ and } \lambda \in \operatorname{pcf} A\}$ is κ-directed;
- (ii) if $\kappa \notin \operatorname{pcf} A$ then J_κ is κ^+-directed;
- (iii) if $\kappa \in \operatorname{pcf} A$ and κ is not a maximal element of $\operatorname{pcf} A$ then there exists a $B_\kappa \in J_\kappa^+$ such that J_κ has a κ-scale on B_κ and $J_\kappa[B_\kappa]$, the ideal generated by $J_\kappa \cup \{B_\kappa\}$, is a κ^+-directed ideal;
- (iv) if $\kappa = \max(\operatorname{pcf} A)$ then J_κ has a κ-scale on A (and we let $B_\kappa = A$).

If the conditions (24.9) are satisfied, then the sets B_λ satisfy Theorem 24.25:

To prove (a), let $\lambda \in \operatorname{pcf} A$. Choose an ultrafilter D on B_λ that extends the dual filter of J_λ. J_λ has a λ-scale on B_λ, and this scale is also a scale for $<_D$; therefore $\operatorname{cof} D = \lambda$, and so $\lambda \in \operatorname{pcf} B_\lambda$. Also, if D is any ultrafilter on B_λ, then either $D \cap J_\lambda = \emptyset$ in which case $\operatorname{cof} D = \lambda$, or else there is some $\nu < \lambda$ such that $B_\nu \in D$. If ν is the least such ν then D is an ultrafilter on B_ν and $D \cap J_\nu = \emptyset$. Since J_ν has a ν-scale on B_ν, we have $\operatorname{cof} D = \nu$. In either case, $\operatorname{cof} D \leq \lambda$.

To prove (b), let D be an ultrafilter on A such that $B_\lambda \notin D$; we claim that $\operatorname{cof} D \neq \lambda$. Either $D \ni B_\lambda$ for some $\nu < \lambda$ in which case $\operatorname{cof} D < \lambda$, or else $D \cap J_\lambda[B_\lambda] \neq \emptyset$, and since $J_\lambda[B_\lambda]$ is λ^+-directed, D is λ^+-directed, and we have $\operatorname{cof} D > \lambda$.

Finally, (c) follows from (24.9)(iii) and (iv). We prove (24.9) by induction on $\kappa \leq \sup \operatorname{pcf} A$:

(i) If $\kappa \leq \min A$ then $J_\kappa = \{\emptyset\}$ is κ-directed. If κ is a limit cardinal, then $J_\kappa = \bigcup_{\lambda<\kappa} J_\lambda$ and the claim follows easily. If $\kappa = \lambda^+$ then either $\lambda \notin \operatorname{pcf} A$ and $J_\kappa = J_\lambda$ is λ^+-directed by (ii), or $\lambda \in \operatorname{pcf} A$ and $J_\kappa = J_\lambda[B_\lambda]$ is λ^+-directed by (iii).

(ii) Let $\kappa \notin \operatorname{pcf} A$ and $\kappa \geq \min A$; hence $\kappa > 2^{|A|}$. If κ is singular, then it is easy to see that since J_κ is κ-directed, it is κ^+-directed. If κ is regular, assume by contradiction that J_κ is κ-directed but not κ^+-directed. By Corollary 24.12, J_κ has a κ-scale on some $X \in J_\kappa^+$. Let D be any ultrafilter on X such that $D \cap J_\kappa = \emptyset$. Then $\operatorname{cof} D = \kappa$ and so $\kappa \in \operatorname{pcf} A$, a contradiction.

(iii) Let $\kappa \in \operatorname{pcf} A$ be such that $\kappa < \sup \operatorname{pcf} A$. We claim that J_κ is not κ^+-directed and that J_κ does not have a κ-scale on A. Then a B_κ exists by Corollary 24.12. Assume by contradiction that J_κ is κ^+-directed, and let D be any ultrafilter on A. If $D \ni B_\lambda$ for some $\lambda < \kappa$, then $\operatorname{cof} D < \kappa$. Otherwise, $D \cap J_\kappa = \emptyset$ and since J_κ is κ^+-directed, D is κ^+-directed and so $\operatorname{cof} D > \kappa$. In either case $\operatorname{cof} D \neq \kappa$, hence $\kappa \notin \operatorname{pcf} A$, a contradiction.

Now assume that J_κ does have a κ-scale on A. Then for every ultrafilter D on A, either $D \ni B_\lambda$ for some $\lambda < \kappa$, and then $\operatorname{cof} D < \kappa$, or $D \cap J_\kappa = \emptyset$, so D has a κ-scale and $\operatorname{cof} D = \kappa$. Hence $\kappa = \max(\operatorname{pcf} A)$, a contradiction.

(iv) Let $\kappa = \max(\operatorname{pcf} A)$ and again assume, by contradiction, that J_κ does not have a scale on A. Then by Corollary 24.12 there exists a $Y \in J_\kappa^+$ such that $J_\kappa[Y]$ is κ^+-directed. If D is any ultrafilter on A such that $D \cap J_\kappa[Y] = \emptyset$ then $<_D$ is κ^+-directed and so $\operatorname{cof} D > \kappa$. Hence κ is not the maximal element of $\operatorname{pcf} A$, a contradiction. □

The same argument that shows that $\operatorname{pcf} A$ has a greatest element yields the following property of pcf, called *compactness*:

Corollary 24.29. *Let B_λ, $\lambda \in \operatorname{pcf} A$, be generators of $\operatorname{pcf} A$. For every $X \subset A$ there exists a finite set $\{\nu_1, \ldots, \nu_k\} \subset \operatorname{pcf} X$ such that $X \subset B_{\nu_1} \cup \ldots \cup B_{\nu_k}$.*

Proof. Assume the contrary. Then $\{X - B_\nu : \nu \in \operatorname{pcf} X\}$ has the finite intersection property and there exists an ultrafilter D on X such that $B_\nu \notin D$ for all $\nu \in \operatorname{pcf} X$. If $\lambda = \operatorname{cof} D$ then $B_\lambda \in D$ by Theorem 24.25(b), a contradiction. □

We conclude this section with the following improvement of Theorem 24.16:

Corollary 24.30. *Let κ be a regular uncountable cardinal, and let $\aleph_\eta$ be a singular cardinal of cofinality κ such that $2^\kappa < \aleph_\eta$. Then there is a closed unbounded set $C \subset \eta$ such that $\max(\operatorname{pcf}\{\aleph_{\alpha+1} : \alpha \in C\}) = \aleph_{\eta+1}$; $\prod_{\alpha\in C} \aleph_{\alpha+1}$ has true cofinality $\aleph_{\eta+1}$ mod I where I is the ideal of all bounded subsets of C.*

Proof. Let C_0 be any closed unbounded subset of η of order-type κ such that $2^\kappa < \aleph_{\alpha_0}$ where $\alpha_0 = \min C_0$. Let $A_0 = \{\aleph_{\alpha+1} : \alpha \in C_0\}$, let $\lambda = \aleph_{\eta+1}$, and let B_λ be a generator for $\operatorname{pcf} A_0$, for this λ (by Theorem 24.16, $\lambda \in \operatorname{pcf} A_0$). Let $X = \{\alpha \in C_0 : \aleph_{\alpha+1} \in B_\lambda\}$. If D is any ultrafilter on C_0 that extends the closed unbounded filter, then by Theorem 24.16, $\operatorname{cof} \prod_{\alpha\in C_0} \aleph_{\alpha+1}/D = \lambda$, and by Theorem 24.25(b), $X \in D$. Thus X contains a closed unbounded set C. Let $A = \{\aleph_{\alpha+1} : \alpha \in C\}$. By Theorem 24.25(a), $\max(\operatorname{pcf} A) \leq \lambda$, and therefore $= \lambda$.

Now let B_ν, $\nu \leq \lambda$, denote the generators of $\operatorname{pcf} A$. Every B_ν for $\nu < \lambda$ is a bounded subset of A and so the ideal of all bounded subsets of A extends J_λ, the ideal generated by the B_ν, $\nu < \lambda$. Thus $\prod_{\alpha\in C} \aleph_{\alpha+1}/I$ has a λ-scale. □

Transitive Generators and Localization

Let A be a set of regular cardinals with $2^{|A|} < \min A$, let B_λ, $\lambda \in \operatorname{pcf} A$, be generators for $\operatorname{pcf} A$, and let J_κ be, for each $\kappa \leq \max(\operatorname{pcf} A)$, the ideal generated by $\{B_\lambda : \lambda < \kappa\}$. The following shows that the ideals J_κ are independent of the choice of generators for $\operatorname{pcf} A$:

(24.10) For every $X \subset A$, $X \in J_\kappa$ if and only if $\operatorname{cof} D < \kappa$ for every ultrafilter D on X.

To see this, note first that if $X \in J_\kappa$ then $X \subset B_{\nu_1} \cup \ldots \cup B_{\nu_k}$ for some $\nu_1, \ldots, \nu_k < \kappa$, and so $\max(\operatorname{pcf} X) < \kappa$. Conversely, if $X \notin J_\kappa$ then the set $\{X - B_\lambda : \lambda < \kappa\}$ has the finite intersection property, and so there exists an ultrafilter D on X such that $B_\lambda \notin D$ for all $\lambda < \kappa$. By Theorem 24.25(b), $\operatorname{cof} D \geq \kappa$. Each generator B_λ is uniquely determined up to equivalence mod J_λ; if B is any set such that $B \triangle B_\lambda \in J_\lambda$, then B also satisfies (a) and (b) of Theorem 24.25. To see this, note that by (24.10), if $X \triangle Y \in J_\lambda$ then $\operatorname{pcf} X - \lambda = \operatorname{pcf} Y - \lambda$; thus $\max \operatorname{pcf} B = \lambda$ and $\lambda \notin \operatorname{pcf}(A - B)$.

We shall now produce generators for pcf that are *transitive*:

Lemma 24.31 (Transitive Generators). *Let A be a set of regular cardinals such that $A = \operatorname{pcf} A$ and $(2^{|A|})^+ < \min A$. There exist generators B_λ, $\lambda \in A$, for $\operatorname{pcf} A$ with the property*

(24.11) *if $\mu \in B_\lambda$ then $B_\mu \subset B_\lambda$.*

In other words, the relation "$\mu \in B_\lambda$" of μ and λ is transitive. The lemma holds under weaker assumptions on A; see Shelah [1994].

Proof. Let B_λ, $\lambda \in A$, be generators for $\operatorname{pcf} A$. We shall replace each B_λ by an equivalent generator $\overline{B_\lambda}$ so that (24.11) is satisfied.

For each $\lambda \in A$ there exists a sequence $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ of functions in $\prod A$ that is $<_{J_\lambda}$-increasing and is cofinal on B_λ. Moreover, by Lemma 24.10 we may assume that for each λ and each α of cofinality greater than $2^{|A|}$, f^λ_α is an exact upper bound of $\{f^\lambda_\beta : \beta < \alpha\}$.

Let $\kappa = (2^{|A|})^+$. Let ϑ be sufficiently large, and consider elementary submodels of $(H_\vartheta, \in, <)$ where $<$ is some well-ordering of H_ϑ. Consider a continuous elementary chain

$$M_0 \prec M_1 \prec \ldots \prec M_\eta \prec \ldots \prec M_\kappa = M \prec H_\vartheta$$

of models, each of size κ, such that M_0 contains A, each $\lambda \in A$, all subsets of A, each $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$, every function from a subset of A into $A^{<\omega}$, and such that

(24.12) $$\langle M_\xi : \xi \leq \eta \rangle \in M_{\eta+1} \qquad (\text{all } \eta < \kappa).$$

Let χ_η, $\eta \leq \kappa$, be the *characteristic functions* of M_η:

$$\chi_\eta(\lambda) = \sup(M_\eta \cap \lambda) \qquad \text{(for all } \lambda \in A\text{)}, \tag{24.13}$$

and let $\chi = \chi_\kappa$, the characteristic function of M. Each χ_η ($\eta < \kappa$) belongs to $M_{\eta+1}$ and therefore to M. If $\xi < \eta$ then $\chi_\xi(\lambda) < \chi_\eta(\lambda)$ for all $\lambda \in A$, and $\langle \chi_\eta(\lambda) : \eta < \kappa \rangle$ is an increasing continuous sequence with limit $\chi(\lambda) < \lambda$.

We claim that for each $\lambda \in A$, χ is the $<_{J_\lambda}$-exact upper bound of $\langle f^\lambda_\alpha : \alpha \in M \cap \lambda \rangle$ on B_λ and consequently,

$$f^\lambda_{\chi(\lambda)}(\mu) = \chi(\mu) \qquad \text{for } J_\lambda\text{-almost all } \mu \in B_\lambda. \tag{24.14}$$

If $\alpha \in M \cap \lambda$ then $f^\lambda_\alpha \in M$ and so $f^\lambda_\alpha(\mu) < \chi(\mu)$ for all $\mu \in A$. Hence χ is an upper bound of $\langle f^\lambda_\alpha : \alpha \in M \cap \lambda \rangle$. To show that χ is the $<_{J_\lambda}$-exact upper bound on B_λ, it suffices to show that for each $\eta < \kappa$, $\chi_\eta <_{J_\lambda} f^\lambda_\alpha$ on B_λ for some $\alpha \in M \cap \lambda$, since χ is the pointwise supremum of $\{\chi_\eta : \eta < \kappa\}$, and $|A| < \kappa$. Thus let $\eta < \kappa$; there exists an $\alpha < \lambda$ such that $\chi_\eta <_{J_\lambda} f^\lambda_\alpha$ on B_λ, and since M is an elementary submodel, there exists such an α in M.

Since $\operatorname{cf} \chi(\lambda) = \kappa > 2^{|A|}$, $f^\lambda_{\chi(\lambda)}$ is a $<_{J_\lambda}$-exact upper bound of $\{f^\lambda_\alpha : \alpha \in M \cap \lambda\}$ on B_λ, and (24.14) follows.

Now we let, for each $\lambda \in A$,

$$B^*_\lambda = \{\mu \in B_\lambda : f^\lambda_{\chi(\lambda)}(\mu) = \chi(\mu)\}; \tag{24.15}$$

if follows from (24.14) that B^*_λ is J_λ-equivalent to B_λ.

The transitive generators $\overline{B_\lambda}$ are defined as follows:

(24.16) $\nu \in \overline{B_\lambda}$ if and only if there exists a finite increasing sequence (with $k \geq 0$) $\langle \nu_0, \ldots, \nu_k \rangle$ such that $\nu_0 = \nu$, $\nu_k = \lambda$ and $\nu_i \in B^*_{\nu_{i+1}}$ for every $i = 0, \ldots, k-1$.

It is clear that $\overline{B_\lambda}$ is transitive, $B^*_\lambda \subset \overline{B_\lambda}$, and $\lambda = \max \overline{B_\lambda}$. It remains to prove that $\overline{B_\lambda}$ is J_λ-equivalent to B_λ; for that it suffices to show that $\overline{B_\lambda} \in J_{\lambda^+} = J_\lambda[B_\lambda]$.

For each $\nu \in \overline{B_\lambda}$, fix a finite sequence $\varphi(\nu) = \langle \nu_0, \ldots, \nu_k \rangle$ to satisfy (24.16). Note that the function φ on $\overline{B_\lambda}$ belongs to M. Let $\langle g_\alpha : \alpha < \lambda \rangle$ be the λ-sequence of functions in $\prod A$ defined as follows:

If $\nu \notin \overline{B_\lambda}$, we let $g_\alpha(\nu) = 0$. If $\nu \in \overline{B_\lambda}$ then $\varphi(\nu) = \langle \nu_0, \ldots, \nu_k \rangle$ with $\nu_0 = \nu$ and $\nu_k = \lambda$, and we consider the sequence $\langle \beta_0, \ldots, \beta_k \rangle$, where $\beta_i < \nu_i$ for each i, obtained as follows (by descending induction):

$$\begin{aligned} &\beta_k = \alpha, \\ &\beta_i = f^{\nu_{i+1}}_{\beta_{i+1}}(\nu_i) \qquad (i = k-1, \ldots, 0). \end{aligned} \tag{24.17}$$

and let $g_\alpha(\nu) = \beta_0$.

As M is an elementary submodel and $\varphi \in M$, the sequence $\langle g_\alpha : \alpha < \lambda \rangle$ is defined in M. Since J_{λ^+} is λ^+-directed, there exists a function $g \in \prod A$

such that $g_\alpha < g \bmod J_{\lambda^+}$ for every $\alpha < \lambda$. Since $M \prec H_\vartheta$, such a function g exists in M. Since $g \in M$, we have $g(\nu) < \chi(\nu)$ for all ν and therefore $g_\alpha < \chi \bmod J_{\lambda^+}$ for every $\alpha < \lambda$.

Now let $\alpha = \chi(\lambda)$. We shall finish the proof by showing that $g_\alpha(\nu) = \chi(\nu)$ for every $\nu \in \overline{B_\lambda}$. This implies that $\overline{B_\lambda} \in J_{\lambda^+}$.

So let $\nu \in \overline{B_\lambda}$. Let $\langle \nu_0, \ldots, \nu_k \rangle = \varphi(\nu)$, and let $\langle \beta_0, \ldots, \beta_k \rangle$ be the sequence obtained in (24.17) for $\alpha = \chi(\lambda)$. We claim that for each i, $\beta_i = \chi(\nu_i)$, and therefore $g_\alpha(\nu) = \beta_0 = \chi(\nu_0) = \chi(\nu)$.

For each i we have $\nu_i \in B^*_{\nu_{i+1}}$, and so by (24.15), $f^{\nu_{i+1}}_{\chi(\nu_{i+1})}(\nu_i) = \chi(\nu_i)$. For $i = k$, we have $\beta_k = \alpha = \chi(\lambda) = \chi(\nu_k)$, and then for each $i = k-1, \ldots, 0$, we have by (24.17)

$$\beta_i = f^{\nu_{i+1}}_{\beta_{i+1}}(\nu_i) = f^{\nu_{i+1}}_{\chi(\nu_{i+1})}(\nu_i) = \chi(\nu_i). \qquad \square$$

Using transitive generators we now prove the *Localization Lemma*:

Lemma 24.32 (Localization). *Let A be a set of regular cardinals such that $2^{|\operatorname{pcf} A|} < \min A$, let $X \subset \operatorname{pcf} A$ and let $\lambda \in \operatorname{pcf} X$. There exists a set $W \subset X$ such that $|W| \le |A|$ and such that $\lambda \in \operatorname{pcf} W$.*

Again, the Localization Lemma holds under the weaker assumption $|\operatorname{pcf} A| < \min A$.

Proof. First, since $2^{|X|} < \min X$, there exist generators for $\operatorname{pcf} X$, and in particular there exists a set $Y \subset X$ with $\max(\operatorname{pcf} Y) = \lambda$. Let $\overline{A} = \operatorname{pcf} A$. By (24.7)(vii) we have $\operatorname{pcf} \overline{A} = \overline{A}$, and since $2^{|\overline{A}|} < \min A$, we can find transitive generators B_ν, $\nu \in \overline{A}$, for $\operatorname{pcf} \overline{A}$.

For every $\nu \in Y$, let $B^A_\nu = B_\nu \cap A$. Since $Y \subset \operatorname{pcf} A$, there exists an ultrafilter D on A with $\operatorname{cof} D = \nu$, and by Theorem 24.25, $B_\nu \in D$. Hence $\nu \in \operatorname{pcf} B^A_\nu$. Let

$$E = \bigcup \{B^A_\nu : \nu \in Y\}.$$

Since $\nu \in \operatorname{pcf} E$ for every $\nu \in Y$, we have $Y \subset \operatorname{pcf} E$, hence $\operatorname{pcf} Y \subset \operatorname{pcf}\operatorname{pcf} E$, and since (by (24.7)(vii)) $\operatorname{pcf}\operatorname{pcf} E = \operatorname{pcf} E$, we have $\operatorname{pcf} Y \subset \operatorname{pcf} E$. In particular, $\lambda \in \operatorname{pcf} E$.

Since $E \subset A$, there exists a set $W \subset Y$ of size $\le |A|$ such that $E \subset \bigcup \{B^A_\nu : \nu \in W\}$. We shall prove that $\lambda \in \operatorname{pcf} W$.

Assume, by contradiction, that $\lambda \notin \operatorname{pcf} W$. By compactness (Corollary 24.29) there exist $\lambda_1, \ldots, \lambda_n \in \operatorname{pcf} W$ such that $W \subset B_{\lambda_1} \cup \ldots \cup B_{\lambda_n}$, and since $\max \operatorname{pcf} W \le \max \operatorname{pcf} Y = \lambda$, we have $\lambda_i < \lambda$ for all $i = 1, \ldots, n$. Now

$$E \subset \bigcup \{B_\nu : \nu \in W\} \subset \bigcup \{B_\nu : \nu \in B_{\lambda_1}\} \cup \ldots \cup \bigcup \{B_\nu : \nu \in B_{\lambda_n}\},$$

and since, by transitivity (Lemma 24.31), $\bigcup_{\nu \in B_\mu} B_\nu \subset B_\mu$ for every μ, we have

$$E \subset B_{\lambda_1} \cup \ldots \cup B_{\lambda_n}.$$

It follows that $\operatorname{pcf} E \subset \operatorname{pcf}(B_{\lambda_1} \cup \ldots \cup B_{\lambda_n}) = \operatorname{pcf} B_{\lambda_1} \cup \ldots \cup \operatorname{pcf} B_{\lambda_n}$, and so $\max(\operatorname{pcf} E) \le \max\{\lambda_1, \ldots, \lambda_n\} < \lambda$, a contradiction. $\square$

Shelah's Bound on $2^{\aleph_\omega}$

As an application of the pcf theory, we shall now present the following result of Shelah:

Theorem 24.33 (Shelah). *If $\aleph_\omega$ is a strong limit cardinal then $2^{\aleph_\omega} < \aleph_{\omega_4}$.*

Proof. Let us assume that $\aleph_\omega$ is strong limit. We already know, by Corollary 24.27, that $2^{\aleph_\omega} = \max \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty < \aleph_{\aleph_\omega}$. We shall prove that

$$\max \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty < \aleph_{\omega_4}.$$

Let ϑ be the ordinal such that $2^{\aleph_\omega} = \aleph_{\vartheta+1}$; we shall prove that $\vartheta < \omega_4$.

Lemma 24.34. *There exists an ordinal function on $P(\vartheta)$ with the following properties*:

(24.18)
- (i) *If $X \subset Y$ then $F(X) \le F(Y)$.*
- (ii) *For every limit ordinal $\eta < \vartheta$ of uncountable cofinality there is a closed unbounded set $C \subset \eta$ such that $F(C) = \eta$.*
- (iii) *If $X \subset \vartheta$ has order-type ω_1 then there exists some $\gamma \in X$ such that $F(X \cap \gamma) \ge \sup X$.*

Proof. Let $X \subset \vartheta$ and consider the set $A = \{\aleph_{\xi+1} : \xi \in X\}$. As $2^{|A|} = \aleph_k$ for some finite k, $\max(\operatorname{pcf} A)$ exists and is equal to some $\aleph_{\gamma+1}$. We define $F(X) = \gamma$.

It is clear that $X \subset Y$ implies $F(X) \le F(Y)$ and that $F(X) \ge \sup X$.

Property (ii) follows from Corollary 24.30. If $\kappa = \operatorname{cf} \eta$ then $\kappa < \aleph_\omega$ and so $2^\kappa < \aleph_\omega < \aleph_\eta$ and the corollary applies.

Property (iii) is a consequence of the Localization Lemma 24.32: If $X \subset \vartheta$ then $\{\aleph_{\xi+1} : \xi \in X\} \subset \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty$ and since $2^{|\operatorname{pcf}\{\aleph_n\}_n|} \le 2^{2^{\aleph_0}} < \aleph_\omega$, Lemma 24.32 applies (with e.g. $\lambda = \aleph_{\eta+1}$ where $\eta = \sup X$) and X has a countable subset W such that $F(W) \ge \sup X$. □

We complete the proof of Shelah's Theorem by showing that $\vartheta < \omega_4$.

Assume, by contradiction, that $\vartheta \ge \omega_4$. Let $\langle C_\alpha : \alpha \in E^{\aleph_3}_{\aleph_1}\rangle$ be a club-guessing sequence (see Theorem 23.3). Each C_α is a closed unbounded subset of α, and for every closed unbounded $C \subset \omega_3$, the set $\{\alpha \in E^{\aleph_3}_{\aleph_1} : C_\alpha \subset C\}$ is stationary.

Let M_α, $\alpha < \omega_3$, be a continuous elementary chain of models of size $\aleph_3$ that contain the family $\{C_\alpha\}_\alpha$, are closed under F, such that $\langle M_\xi : \xi \le \alpha\rangle \in M_{\alpha+1}$ for each α, and that for each α, $\eta_\alpha = M_\alpha \cap \omega_4$ is an ordinal. Let $\eta : \omega_3 \to \omega_4$ be the continuous function $\eta(\alpha) = \eta_\alpha$. By (24.18)(ii) there is a closed unbounded set $C \subset \omega_3$ such that $F(\eta\text{“}C) = \sup_\alpha \eta_\alpha$. Let $\alpha \in E^{\aleph_3}_{\aleph_1}$ be such that $C_\alpha \subset C$. By (24.18)(iii) there exists a $\beta < \alpha$ such that $F(\eta\text{“}(C_\alpha \cap \beta)) \ge \eta(\alpha)$. Let $X = \eta\text{“}(C_\alpha \cap \beta)$.

Since $C_\alpha \in M_\alpha$ and $\eta{\restriction}\beta \in M_\alpha$, we have $X \in M_\alpha$. Since $X \subset \eta\text{“}C$ we have $F(X) \le F(\eta\text{“}C) < \omega_4$. As M_α is closed under F, we have $F(X) \in M_\alpha$, and since $\omega_4 \cap M_\alpha = \eta(\alpha)$, it follows that $F(X) < \eta(\alpha)$, a contradiction. □

Exercises

24.1. If $\beta < \omega_1$ and if $2^{\aleph_\alpha} \leq \aleph_{\alpha+\beta}$ for a stationary set of α's, then $2^{\aleph_{\omega_1}} \leq \aleph_{\omega_1+\beta}$. [By induction on β: If $\varphi(\alpha) \leq \beta$ on a stationary set, then $\|\varphi\| \leq \beta$.]

24.2. If $\beta < \omega_1$, if $2^{\aleph_1} < \aleph_{\omega_1}$, and if $\aleph_\alpha^{\aleph_0} \leq \aleph_{\alpha+\beta}$ for a stationary set of α's, then $\aleph_{\omega_1}^{\aleph_1} \leq \aleph_{\omega_1+\beta}$.

24.3. If $2^{\aleph_\alpha} \leq \aleph_{\alpha+2}$ holds for all cardinals of cofinality ω, then the same holds for all singular cardinals.

24.4. If $\aleph_1 \leq \operatorname{cf} \aleph_\eta < \aleph_\eta$, if $\beta < \operatorname{cf} \aleph_\eta$, and if $2^{\aleph_\alpha} \leq \aleph_{\alpha+\beta}$ for all $\alpha < \eta$, then $2^{\aleph_\eta} \leq \aleph_{\eta+\beta}$.

24.5. If $2^{\aleph_\alpha} \leq \aleph_{\alpha+\alpha+1}$ for a stationary set of $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} \leq \aleph_{\omega_1+\omega_1+1}$. [If $\varphi(\alpha) = \alpha$ for all $\alpha < \omega_1$, then $\|\varphi\| = \omega_1$.]

24.6. If $2^{\aleph_{\omega_1+\alpha}} < \aleph_{\omega_1+\alpha+\alpha}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1+\omega_1}} < \aleph_{\omega_1+\omega_1+\omega_1}$. [Use the sets $A_\alpha = \omega_{\omega_1+\alpha}$.]

24.7. If $2^{\aleph_1} < \aleph_{\omega_1}$ and if $\aleph_\alpha^{\aleph_0} \leq \aleph_{\alpha+\alpha+1}$ for all $\alpha < \omega_1$, then $\aleph_{\omega_1}^{\aleph_1} \leq \aleph_{\omega_1+\omega_1+1}$.

24.8. If κ is a strong limit cardinal, $\kappa = \aleph_\eta$, and $\operatorname{cf} \kappa \geq \aleph_1$, then $2^\kappa < \aleph_\gamma$, where $\gamma = (|\eta|^{\operatorname{cf} \kappa})^+$.

24.9. If $\aleph_1 \leq \operatorname{cf} \kappa < \kappa$ and if $\lambda^{\operatorname{cf} \kappa} < \kappa$ for all $\lambda < \kappa$, then $\kappa^{\operatorname{cf} \kappa} < \aleph_\gamma$, where $\gamma = (|\eta|^{\operatorname{cf} \kappa})^+$.

The next exercise uses the notation from Chapter 8. Let κ be a regular uncountable cardinal, let $M_0 = \kappa$, $M_{\eta+1} = \operatorname{Tr}(M_\eta)$, $M_\eta = \bigcap_{\nu<\operatorname{cf}\eta} M_{\xi_\nu}$ or $M_\eta = \triangle_{\nu<\kappa} M_{\xi_\nu}$ (if $\operatorname{cf}\eta = \kappa$) as long as M_η is stationary.

24.10. Let f_η, $\eta < \kappa^+$, be the canonical functions on κ. Let $S_\eta = \{\alpha < \kappa : o(\alpha) = f_\eta(\alpha)\}$. Show that $S_\eta = M_\eta - M_{\eta+1} \bmod I_{\mathrm{NS}}$ and that $o(S) = \eta$ for every stationary $S \subset S_\eta$.

The sets S_η are the *canonical stationary sets* (of order η).

24.11. Find a partially ordered set of cofinality $\aleph_\omega$; of cofinality 1, 2, 3, etc.

24.12. The lexicographical ordering $\omega \times \omega_1$ does not have true cofinality.

24.13. Let $I = I_{\mathrm{NS}}$ be the nonstationary ideal on ω_1, let c_γ, $\gamma < \omega_1$, be the constant functions (with value γ) on ω_1, and let $d(\alpha) = \alpha$ be the diagonal function. The function d is a least upper bound, but not an exact upper bound of the set $\{c_\gamma : \gamma < \omega_1\}$, in $<_I$.

Historical Notes

The Galvin-Hajnal Theorem appeared in [1975]. Shelah's investigation leading to the pcf theory started in [1978], and the book [1982] contains the first proof of a bound on $2^{\aleph_\omega}$. In a sequence of papers starting in 1978, Shelah developed the theory of possible cofinalities. A complete presentation is in his book [1994].

There are several papers that give an exposition and/or simplified proofs of Shelah's results; we mention Burke and Magidor [1990] and Jech [1992].

25. Descriptive Set Theory

Descriptive set theory is the study of definable sets of real numbers, in particular projective sets, and is mostly interested in how well behaved these sets are. A prototype of such results is Theorem 11.18 stating that $\mathbf{\Sigma}^1_1$ sets are Lebesgue measurable, have the Baire property, and have the perfect set property. This chapter continues the investigations started in Chapter 11. Throughout, we shall work in set theory ZF + DC (the Principle of Dependent Choice).

The Hierarchy of Projective Sets

Modern descriptive set theory builds on both the classical descriptive set theory and on recursion theory. It has become clear in the 1950's that the topological approach of classical descriptive set theory and the recursion theoretic techniques of logical definability describe the same phenomena. Modern descriptive set theory unified both approaches, as well as the notation. An additional ingredient is in the use of infinite games and determinacy; we shall return to that subject in Part III.

We first reformulate the hierarchy of projective sets in terms of the *lightface* hierarchy Σ^1_n, Π^1_n and Δ^1_n and its *relativization* for real parameters. While we introduce these concepts explicitly for subsets of the Baire space $\mathcal{N} = \omega^\omega$, analogous definitions and results apply to product spaces $\mathcal{N} \times \mathcal{N}$, $\mathcal{N}^r$ as well as the spaces ω, ω^k, $\omega^k \times \mathcal{N}^r$.

Definition 25.1.

(i) A set $A \subset \mathcal{N}$ is Σ^1_1 if there exists a recursive set $R \subset \bigcup_{n=0}^{\infty}(\omega^n \times \omega^n)$ such that for all $x \in \mathcal{N}$,

$$x \in A \quad \text{if and only if} \quad \exists y \in \omega^\omega \, \forall n \in \omega \, R(x\restriction n, y\restriction n). \tag{25.1}$$

(ii) Let $a \in \mathcal{N}$; a set $A \subset \mathcal{N}$ is $\Sigma^1_1(a)$ (Σ^1_1 in a) if there exists a set R recursive in a such that for all $x \in \mathcal{N}$,

$$x \in A \quad \text{if and only if} \quad \exists y \in \omega^\omega \, \forall n \in \omega \, R(x\restriction n, y\restriction n, a\restriction n).$$

(iii) $A \subset \mathcal{N}$ is Π^1_n (in a) if the complement of A is Σ^1_n (in a).
(iv) $A \subset \mathcal{N}$ is Σ^1_{n+1} (in a) if it is the projection of a Π^1_n (in a) subset of $\mathcal{N} \times \mathcal{N}$.
(v) $A \subset \mathcal{N}$ is Δ^1_n (in a) if it is both Σ^1_n and Π^1_n (in a).

A similar lightface hierarchy exists for Borel sets: A set $A \subset \mathcal{N}$ is Σ^0_1 (*recursive open* or *recursively enumerable*) if

$$A = \{x : \exists n \; R(x\restriction n)\} \tag{25.2}$$

for some recursive R, and Π^0_1 (*recursive closed*) if it is the complement of a Σ^0_1 set. Thus Σ^1_1 sets are projections of Π^0_1 sets, and as every open set is Σ^0_1 in some $a \in \mathcal{N}$ (namely an a than codes the corresponding union of basic open intervals), we have

$$\mathbf{\Sigma}^1_1 = \bigcup_{a \in \mathcal{N}} \Sigma^1_1(a),$$

and more generally, every $\mathbf{\Sigma}^1_n$ ($\mathbf{\Pi}^1_n$) set is Σ^1_n (Π^1_n) in some parameter $a \in \mathcal{N}$.

For $n \in \omega$, the lightface hierarchy of Σ^0_n and Π^0_n sets describes the *arithmetical* sets: For instance, a set A is Σ^0_3 if

$$A = \{x \in \mathcal{N} : \exists m_1 \, \forall m_2 \, \exists m_3 \, R(m_1, m_2, x\restriction m_3)\}$$

for some recursive R, etc. Arithmetical sets are exactly those $A \subset \mathcal{N}$ that are definable (without parameters) in the model $(HF, \in)$ of hereditary finite sets.

The following lemma gives a list of closure properties of projective relations on $\mathcal{N}$. We use the logical (rather than set-theoretic) notation for Boolean operations; compare with Lemma 13.10.

Lemma 25.2. *Let $n \geq 1$.*

(i) *If A, B are $\Sigma^1_n(a)$ relations, then so are $\exists x \, A$, $A \wedge B$, $A \vee B$, $\exists m \, A$, $\forall m \, A$.*
(ii) *If A, B are $\Pi^1_n(a)$ relations, then so are $\forall x \, A$, $A \wedge B$, $A \vee B$, $\exists m \, A$, $\forall m \, A$.*
(iii) *If A is $\Sigma^1_n(a)$, then $\neg A$ is Π^1_n; if A is $\Pi^1_n(a)$, then $\neg A$ is Σ^1_n.*
(iv) *If A is $\Pi^1_n(a)$ and B is $\Sigma^1_n(a)$, then $A \to B$ is $\Sigma^1_n(a)$; if A is $\Sigma^1_n(a)$ and B is $\Pi^1_n(a)$, then $A \to B$ is $\Pi^1_n(a)$.*
(v) *If A and B are $\Delta^1_n(a)$, then so are $\neg A$, $A \wedge B$, $A \vee B$, $A \to B$, $A \leftrightarrow B$, $\exists m \, A$, $\forall m \, A$.*

Proof. We prove the lemma for $n = 1$; the general case follows by induction. Moreover, clauses (ii)–(v) follow from (i).

First, let $A \in \Sigma^1_1(a)$ and let us show that $\exists x \, A$ is $\Sigma^1_1(a)$. We have

$$(x, y) \in A \leftrightarrow \exists z \, \forall n \, (x\restriction n, y\restriction n, z\restriction n, n) \in R,$$

where R is recursive in a. Thus

$$y \in \exists x\, A \leftrightarrow \exists x\, \exists z\, \forall n\, (x\restriction n, y\restriction n, z\restriction n, n) \in R.$$

We want to contract the two quantifiers $\exists x\, \exists z$ into one. Let us consider some recursive homeomorphism between $\mathcal{N}$ and $\mathcal{N}^2$, e.g., for $u \in \mathcal{N}$ let u^+ and u^- be

$$u^+(n) = u(2n), \qquad u^-(n) = u(2n+1), \qquad (n \in \boldsymbol{N}).$$

There exists a relation R' recursive in R, such that for all $u, y \in \mathcal{N}$,

$$\forall n\, (u\restriction n, y\restriction n, n) \in R' \quad \text{if and only if} \quad \forall k\, (u^+\restriction k, y\restriction k, u^-\restriction k, k) \in R. \tag{25.3}$$

Namely, if $n = 2k$ (or $n = 2k+1$), we let $(s,t,n) \in R'$ just in case $\mathrm{length}(s) = \mathrm{length}(t) = n$ and

$$(\langle s(0), \ldots, s(2k-2)\rangle, \langle t(0), \ldots, t(k-1)\rangle, \langle s(1), \ldots, s(2k-1)\rangle, k) \in R.$$

Now (25.3) implies that

$$y \in \exists x\, A \leftrightarrow \exists u\, \forall n\, (u\restriction n, y\restriction n, n) \in R',$$

and hence $\exists x\, A$ is $\Sigma^1_1(a)$.

Now let A and B be $\Sigma^1_1(a)$:

$$\begin{aligned} x \in A &\leftrightarrow \exists z\, \forall n\, (x\restriction n, z\restriction n, n) \in R_1, \\ x \in A &\leftrightarrow \exists z\, \forall n\, (x\restriction n, z\restriction n, n) \in R_2 \end{aligned}$$

where both R_1 and R_2 are recursive in a. Note that

$$x \in A \wedge B \leftrightarrow \exists z_1\, \exists z_2\, \forall n\, [(x\restriction n, z_1\restriction n, n) \in R_1 \wedge (x\restriction n, z_2\restriction n, n) \in R_2]$$

and hence, by contraction of $\exists z_1\, \exists z_2$, there is some R, recursive in R_1 and R_2 such that

$$x \in A \wedge B \leftrightarrow \exists z\, \forall n\, (x\restriction n, z\restriction n, n) \in R.$$

Thus $A \wedge B$ is $\Sigma^1_1(a)$.

The following argument shows that the case $A \vee B$ can be reduced to the case $\exists m\, C$. Let us define R as follows ($s, t \in \mathit{Seq}$, $m, n \in \boldsymbol{N}$):

$$\begin{aligned} (s,m,t,n) \in R \leftrightarrow \text{either } & m = 1 \text{ and } (s,t,n) \in R_1 \\ & \text{or } m = 2 \text{ and } (s,t,n) \in R_2. \end{aligned}$$

R is recursive in R_1 and R_2, and

$$\begin{aligned} x \in A \vee B &\leftrightarrow \exists z\, \forall n\, (x\restriction n, z\restriction n, n) \in R_1 \vee \exists z\, \forall n\, (x\restriction n, z\restriction n, n) \in R_2 \\ &\leftrightarrow \exists m\, \exists z\, \forall n\, (x\restriction n, m, z\restriction n, n) \in R \\ &\leftrightarrow x \in \exists m\, C \end{aligned}$$

where C is $\Sigma^1_1(a)$.

The contraction of quantifiers $\exists m\, \exists z$ is easier than the contraction $\exists x\, \exists z$ above. We employ the following recursive homomorphism between $\mathcal{N}$ and $\omega \times \mathcal{N}$: $h(u) = (u(0), u')$, where

$$u'(n) = u(n+1) \qquad (n \in \boldsymbol{N}).$$

If

$$(x, m) \in A \leftrightarrow \exists z\, \forall n\, (x\restriction n, m, z\restriction n, n) \in R,$$

then we leave it to the reader to find a relation R', recursive in R, such that for all $u, x \in \mathcal{N}$,

$$\forall n\, (x\restriction n, u\restriction n, n) \in R' \leftrightarrow \forall k\, (x\restriction k, u(0), u'\restriction k, k) \in R.$$

Then

$$x \in \exists m\, A \leftrightarrow \exists u\, \forall n\, (x\restriction n, u\restriction n, n) \in R'.$$

It remains to show that if A is $\Sigma^1_1(a)$, then $\forall m\, A$ is $\Sigma^1_1(a)$. Let

$$(x, m) \in A \leftrightarrow \exists z\, \forall n\, (x\restriction n, m, z\restriction n, n) \in R$$

where R is recursive in a. Thus

$$x \in \forall m\, A \leftrightarrow \forall m\, \exists z\, \forall n\, (x\restriction n, m, z\restriction n, n) \in R. \tag{25.4}$$

We want to replace the quantifiers $\forall m\, \exists z$ by $\exists u\, \forall m$ and then contract the two quantifiers $\forall m\, \forall n$ into one. Let us consider the pairing function $\Gamma : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$ and the following homeomorphism between $\mathcal{N}$ and $\mathcal{N}^\omega$: For each $u \in \mathcal{N}$, let u_m, $m \in \boldsymbol{N}$, be

$$u_m(n) = u(\Gamma(m, n)) \qquad (m, n \in \boldsymbol{N}).$$

Now we can replace $\forall m\, \exists z$ in (25.4) by $\exists u\, \forall m$ (note that in the forward implication we use the Countable Axiom of Choice):

$$\forall m\, \exists z\, \forall n\, (x\restriction n, m, z\restriction n, n) \in R \leftrightarrow \exists u\, \forall m\, \forall n\, (x\restriction n, m, u_m\restriction n, n) \in R. \tag{25.5}$$

Let $\alpha : \boldsymbol{N} \to \boldsymbol{N}$ and $\beta : \boldsymbol{N} \to \boldsymbol{N}$ be the inverses of the function Γ: If $\Gamma(m, n) = k$, then $m = \alpha(k)$ and $n = \beta(k)$. From (25.4) and (25.5) we get

$$x \in \forall m\, A \leftrightarrow \exists u\, \forall k\, (x\restriction \beta(k), \alpha(k), u_{\alpha(k)}\restriction \beta(k), \beta(k)) \in R. \tag{25.6}$$

Now it suffices to show that there exists a relation $R' \subset Seq^2 \times \boldsymbol{N}$, recursive in R, such that for all $u, x \in \mathcal{N}$,

$$\forall k\, (x\restriction k, u\restriction k, k) \in R' \leftrightarrow \forall k\, (x\restriction \beta(k), \alpha(k), u_{\alpha(k)}\restriction \beta(k), \beta(k)) \in R. \tag{25.7}$$

The relation R' is found in a way similar to the relation R' in (25.3), and we leave the details as an exercise.

Hence $\forall m\, A$ is $\Sigma^1_1(a)$ because by (25.6) and (25.7),

$$x \in \forall m\, A \leftrightarrow \exists u\, \forall k\, (x\restriction k, u\restriction k, k) \in R'.$$

In Lemma 11.8 we proved the existence of a universal $\mathbf{\Sigma}^1_n$ set. An analysis of the proof (and of Lemma 11.2) yields a somewhat finer result: There exists a Σ^1_n set $A \subset \mathcal{N}^2$ (lightface) that is a universal $\mathbf{\Sigma}^1_n$ set.

$\mathbf{\Pi^1_1}$ Sets

We formulate a normal form for Π^1_1 sets in terms of trees. This is based on the idea that analytic sets are projections of closed sets, and that closed sets in $\mathcal{N}$ are represented by sets $[T]$ where T is a sequential tree; cf. (4.6). Let us consider the product space $\mathcal{N}^r$, for an arbitrary integer $r \geq 1$. As in the case $r = 1$, the closed subsets of $\mathcal{N}^r$ can be represented by trees: Let Seq_r denote the set of all r-tuples $(s_1, \ldots, s_r) \in Seq^r$ such that $\text{length}(s_1) = \ldots = \text{length}(s_r)$. A set $T \subset Seq_r$ is an (*r-dimensional sequential*) *tree* if for every $(s_1, \ldots, s_r) \in T$ and each $n \leq \text{length}(s_1)$, $(s_1 \restriction n, \ldots, s_r \restriction n)$ is also in T. Let

$$[T] = \{(a_1, \ldots, a_r) \in \mathcal{N}^r : \forall n\, (a_1 \restriction n, \ldots, a_r \restriction n) \in T\}. \tag{25.8}$$

The set $[T]$ is closed, and every closed set in $\mathcal{N}^r$ has the form (25.8), for some tree T.

We call a sequential tree $T \subset Seq_r$ *well-founded* if $[T] = \emptyset$, i.e., if the reverse inclusion on T is a well-founded relation. T is *ill-founded* if it is not well-founded.

For $T \subset Seq_{r+1}$ and for each $x \in \mathcal{N}$, let

$$T(x) = \{(s_1, \ldots, s_r) \in Seq_r : (x \restriction n, s_1, \ldots, s_r) \in T \text{ where } n = \text{length}\, s_i\}. \tag{25.9}$$

Now if $A \subset \mathcal{N}$ is analytic, there exists a tree $T \subset Seq_2$ such that A is the projection of $[T]$; consequently, for all $x \in \mathcal{N}$ we have

$$x \in A \quad \text{if and only if} \quad T(x) \text{ is ill-founded.} \tag{25.10}$$

More generally, if A is Σ^1_1, let R be recursive such that

$$x \in A \leftrightarrow \exists y \in \mathcal{N}\, \forall n\, R(x \restriction n, y \restriction n)$$

and define $T = \{(t, s) \in Seq_2 : \forall n \leq \text{length}(s)\, R(t \restriction n, s \restriction n)\}$. For all $x \in \mathcal{N}$, we have $T(x) = \{s \in Seq : \forall n \leq \text{length}(s)\, R(x \restriction n, s \restriction n)\}$ and $x \in A$ if and only if $T(x)$ is ill-founded.

Theorem 25.3 (Normal Form for Π^1_1). *A set $A \subset \mathcal{N}$ is Π^1_1 if and only if there exists a recursive mapping $x \mapsto T(x)$ such that each $T(x)$ is a sequential tree, and*

$$x \in A \quad \textit{if and only if} \quad T(x) \textit{ is well-founded.} \tag{25.11}$$ □

Similarly, a relation $A \subset \mathcal{N}^r$ is Π^1_1 if and only if $A = \{\vec{x} : T(\vec{x}) \text{ is well-founded}\}$ where $\langle T(\vec{x}) : \vec{x} \in \mathcal{N}^r \rangle$ is a recursive system of r-dimensional trees.

One consequence of normal forms is that Π^1_1 (and Σ^1_1) relations are absolute for transitive models:

Theorem 25.4 (Mostowski's Absoluteness). *If P is a $\mathbf{\Sigma}^1_1$ property then P is absolute for every transitive model that is adequate for P.*

Proof. "Adequate" here means that the model satisfies enough axioms to know that well-founded trees have a rank function, and contains the parameter in which P is Σ^1_1. The proof is similar to Lemma 13.11.

Let M be a transitive model and let $T \in M$ be a tree such that $P = \{x : T(x) \text{ is ill-founded}\}$. Let $x \in M$. If $M \vDash (T(x)$ is ill-founded) then $T(x)$ is ill-founded. Conversely, if $M \vDash (T(x)$ is well-founded) then $M \vDash (\exists f : T(x) \to Ord$ such that $f(s) < f(t)$ whenever $s \supset t)$ and therefore $T(x)$ is well-founded. □

Trees, Well-Founded Relations and κ-Suslin Sets

Much of modern descriptive set theory depends on a generalization of the Normal Form for Π^1_1 sets. A tree $T \subset Seq_r$ consists of r-tuples of finite sequences. We can also identify T with finite sequences of r-tuples, which enables us to consider a more general concept:

Definition 25.5.

(i) A tree T (on a set X) is a set of finite sequences (in X) closed under initial segments.
(ii) If $s, t \in T$ then $s \leq t$ means $s \supset t$, i.e., t is an initial segment of s.
(iii) If $s \in T$ then $T/s = \{t : s^\frown t \in T\}$.
(iv) If $(T, \leq)$ is well-founded then $\|T\|$ is the height of $\leq$, and for $t \in T$, $\rho_T(t)$ is the rank of t in $\leq$.
(v) $[T] = \{f \in X^\omega : \forall n\, f\restriction n \in T\}$.

If S and T are well-founded trees and if $f : S \to T$ is order-preserving then $\|S\| \leq \|T\|$; this is easily verified by induction on rank. But the converse is also true:

Lemma 25.6. *If S and T are well-founded trees and $\|S\| \leq \|T\|$ then there exists an order-preserving map $f : S \to T$.*

Proof. By induction on $\|T\|$. For each $\langle a\rangle \in S$, $\|S/\langle a\rangle\| < \|S\| \leq \|T\|$ and there exists a $t_a \neq \emptyset$ such that $\|S/\langle a\rangle\| \leq \|T/\langle t_a\rangle\|$. Let $f_a : S/\langle a\rangle \to S/t_a$ be order-preserving. Now define $f : S \to T$ as follows: $f(\emptyset) = \emptyset$, and $f(a^\frown s) = t_a^\frown f_a(s)$ whenever $a^\frown s \in S$. □

We remark that the above proof (as well as the existence of rank), uses the Principle of Dependent Choices. If T is ill-founded, note that for *any* S there exists an order-preserving $f : S \to T$ (into an infinite branch of T). Thus we have

Corollary 25.7. *There exists an order-preserving $f : S \to T$ if and only if either T is ill-founded or $\|S\| \leq \|T\|$.* □

Trees used in descriptive set theory are trees on $\omega \times K$ (or on $\omega^r \times K$) where K is some set, usually well-ordered.

Let $Seq(K)$ be the set of all finite sequences in K. A *tree* on $\omega \times K$ is a set of pairs $(s, h) \in Seq \times Seq(K)$ such that $\text{length}(s) = \text{length}(h)$ and that for each $n \leq \text{length}(s)$, $(s\restriction n, h\restriction n) \in T$. For every $x \in \mathcal{N}$, let

$$T(x) = \{h \in Seq(K) : (x\restriction n, h) \in T \text{ where } n = \text{length}(h)\}. \tag{25.12}$$

$T(x)$ is a tree on K. Further we let

$$\begin{aligned} p[T] &= \{x \in \mathcal{N} : T(x) \text{ is ill-founded}\} \\ &= \{x \in \mathcal{N} : [T(x)] \neq \emptyset\} \\ &= \{x \in \mathcal{N} : (\exists f \in K^\omega)\, \forall n\, (x\restriction n, f\restriction n) \in T\}. \end{aligned}$$

Trees on $\omega^r \times K$ are defined analogously.

Definition 25.8. Let κ be an infinite cardinal. A set $A \subset \mathcal{N}$ is *κ-Suslin* if $A = p[T]$ for some tree on $\omega \times \kappa$.

By the Normal Form Theorem for Π^1_1 sets, every $\mathbf{\Sigma}^1_1$ set is ω-Suslin. In fact if A is $\Sigma^1_1(a)$ then $A = p[T]$ where T is a tree on $\omega \times \omega$ recursive in a. Let us associate with each $x \in \mathcal{N}$ the following binary relation E_x on $\boldsymbol{N}$:

$$m\ E_x\ n \leftrightarrow x(\Gamma(m, n)) = 0 \tag{25.13}$$

where Γ is a (recursive) pairing function of $\boldsymbol{N} \times \boldsymbol{N}$ onto $\boldsymbol{N}$; we say that x *codes* the relation E_x. We define

$$\begin{aligned} &\text{WF} = \{x \in \mathcal{N} : x \text{ codes a well-founded relation}\}, \\ &\text{WO} = \{x \in \mathcal{N} : x \text{ codes a well-ordering on } \boldsymbol{N}\}. \end{aligned} \tag{25.14}$$

Lemma 25.9. *The sets* WF *and* WO *are* Π^1_1.

Proof. We prove in some detail that WF is Π^1_1. E_x is well-founded if and only if there is no $z : \boldsymbol{N} \to \boldsymbol{N}$ such that $z(k+1)\ E_x\ z(k)$ for all k. Thus

$$x \in \text{WF} \leftrightarrow \forall z\, \exists k\, \neg z(k+1)\ E_x\ z(k).$$

In other words, $\text{WF} = \forall z\, A$, where

$$(x, z) \in A \leftrightarrow \exists k\, x(\Gamma(z(k+1), z(k))) \neq 0$$

and it suffices to show that A is arithmetical. But

$$\begin{aligned} (x, z) \in A \leftrightarrow \exists n, m, j, k\, [&i = (z\restriction n)(k+1) \wedge j = (z\restriction n)(k) \wedge \\ &m = \Gamma(i, j) \wedge (x\restriction n)(m) \neq 0]. \end{aligned}$$

To show that WO is Π^1_1 it suffices to verify that the set

$$\text{LO} = \{x : E_x \text{ is a linear ordering of } \boldsymbol{N}\}$$

is arithmetical. Then $\text{WO} = \text{WF} \wedge \text{LO}$ is Π^1_1. □

We show below that neither WF nor WO is a $\mathbf{\Sigma}^1_1$ set; thus neither is a Borel set.

For each $x \in \mathrm{WF}$, let

$$\|x\| = \text{the height of the well-founded relation } E_x \tag{25.15}$$

(see (2.7)). For each x, $\|x\|$ is a countable ordinal (and for each $\alpha < \omega_1$ there is some $x \in \mathrm{WF}$ such that $\|x\| = \alpha$). If $x \in \mathrm{WO}$, then $\|x\|$ is the order-type of the well-ordering E_x.

Lemma 25.10. *For each $\alpha < \omega_1$, the sets*

$$\mathrm{WF}_\alpha = \{x \in \mathrm{WF} : \|x\| \leq \alpha\}, \qquad \mathrm{WO}_\alpha = \{x \in \mathrm{WO} : \|x\| \leq \alpha\}$$

are Borel sets.

Proof. Note that the set $\{(x, n) : n \in \mathrm{field}(E_x)\}$ is arithmetical (and hence Borel). Let us prove the lemma first for WO_α.

For each $\alpha < \omega_1$, let

$$B_\alpha = \{(x, n) : E_x \text{ restricted to } \{m : m\ E_x\ n\} \\ \text{is a well-ordering of order type} \leq \alpha\}.$$

We prove, by induction on $\alpha < \omega_1$, that each B_α is a Borel set. It is easy to see that B_0 is arithmetical. Thus let $\alpha < \omega_1$ and assume that all B_β, $\beta < \alpha$, are Borel. Then $\bigcup_{\beta<\alpha} B_\beta$ is Borel and hence B_α is also Borel because

$$(x, n) \in B_\alpha \leftrightarrow \forall m \Big(m\ E_x\ n \to (x, m) \in \bigcup_{\beta<\alpha} B_\beta\Big).$$

It follows that each WO_α is Borel because

$$x \in \mathrm{WO}_\alpha \leftrightarrow \forall n \Big(n \in \mathrm{field}(E_x) \to (x, n) \in \bigcup_{\beta<\alpha} B_\beta\Big).$$

To handle WF_α, note that the rank function ρ_E can be defined for any binary relation E; namely:

$$\rho_E(u) = \alpha \quad \text{if and only if} \quad \forall v\,(v\ E\ u \to \rho_E(v) \text{ is defined}) \text{ and} \\ \alpha = \sup\{\rho_E(v) + 1 : v\ E\ u\}.$$

For each $\alpha < \omega_1$, let

$$C_\alpha = \{(x, n) : \rho_{E_x}(n) \text{ is defined and} \leq \alpha\}.$$

Again, C_0 is arithmetical, and if we assume that all C_β, $\beta < \alpha$, are Borel, then C_α is also Borel:

$$(x, n) \in C_\alpha \leftrightarrow \forall m \Big(m\ E_x\ n \to (x, m) \in \bigcup_{\beta<\alpha} C_\beta\Big).$$

Hence each C_α is Borel, and it follows that each WF_α is Borel:

$$x \in \mathrm{WF}_\alpha \leftrightarrow \forall n \Big(n \in \mathrm{field}(E_x) \to (x, n) \in \bigcup_{\beta<\alpha} C_\beta\Big). \qquad \square$$

Corollary 25.11. *The sets* $\{x \in \mathrm{WF} : \|x\| = \alpha\}$ *and* $\{x \in \mathrm{WF} : \|x\| < \alpha\}$ *are Borel* (*similarly for* WO).

Proof. $\{x \in \mathrm{WF} : \|x\| < \alpha\} = \bigcup_{\beta<\alpha} \mathrm{WF}_\beta$. □

Theorem 25.12. *If* C *is a* $\mathbf{\Pi}^1_1$ *set, then there exists a continuous function* $f : \mathcal{N} \to \mathcal{N}$ *such that* $C = f_{-1}(\mathrm{WF})$, *and there exists a continuous function* $g : \mathcal{N} \to \mathcal{N}$ *such that* $C = g_{-1}(\mathrm{WO})$.

Proof. We shall give the proof for WF; the proof for WO is similar. Let $T \subset Seq_2$ be such that

$$x \in C \leftrightarrow T(x) \text{ is well-founded.}$$

Let $\{t_0, t_1, \ldots, t_n, \ldots\}$ be an enumeration of the set Seq. For each $x \in \mathcal{N}$, let $y = f(x)$ be the following element of $\mathcal{N}$:

$$y(\Gamma(m,n)) = \begin{cases} 0 & \text{if } t_m, t_n \in T(x), \text{ and } t_m < t_n, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that E_y is isomorphic to $(T(x), <)$, and hence $y \in \mathrm{WF}$ if and only if $T(x)$ is well-founded. Thus $C = f_{-1}(\mathrm{WF})$ and it remains to show only that f is continuous. But it should be obvious from the definitions of $T(x)$ and of $y = f(x)$ that for any finite sequence $s = \langle \varepsilon_0, \ldots, \varepsilon_{k-1} \rangle$, there is $\check{s} \in Seq$ such that if $x \supset \check{s}$ and $y = f(x)$, then $y \restriction k = s$. Hence f is continuous. □

Corollary 25.13. WF *is not* $\mathbf{\Sigma}^1_1$; WO *is not* $\mathbf{\Sigma}^1_1$.

Proof. Otherwise every $\mathbf{\Pi}^1_1$ set would be the inverse image by a continuous function of an analytic set and hence analytic; however, there are $\mathbf{\Pi}^1_1$ sets that are not analytic. □

Corollary 25.14 (Boundedness Lemma). *If* $B \subset \mathrm{WO}$ *is* $\mathbf{\Sigma}^1_1$, *then there is an* $\alpha < \omega_1$ *such that* $\|x\| < \alpha$ *for all* $x \in B$.

Proof. Otherwise we would have

$$\mathrm{WO} = \{x \in \mathcal{N} : \exists z \, (z \in B \wedge \|x\| \leq \|z\|)\}.$$

Hence $\|x\| \leq \|z\|$ for $x, z \in \mathcal{N}$ means: Either $z \notin \mathrm{WO}$ or $\|x\| \leq \|z\|$; this relation is Σ^1_1; see Exercise 25.3. This would mean that WO is Σ^1_1, a contradiction. □

Corollary 25.15. *Every* $\mathbf{\Pi}^1_1$ *set is the union of* $\aleph_1$ *Borel sets.*

Proof. If C is $\mathbf{\Pi}^1_1$, then $C = f_{-1}(\mathrm{WF})$ for some continuous f. But $\mathrm{WF} = \bigcup_{\alpha<\omega_1} \mathrm{WF}_\alpha$, and hence

$$C = \bigcup_{\alpha<\omega_1} f_{-1}(\mathrm{WF}_\alpha).$$

Each $f_{-1}(\mathrm{WF}_\alpha)$ is the inverse image of a Borel set by a continuous function, hence Borel. □

Corollary 25.16. *Assuming the Axiom of Choice, every* $\mathbf{\Pi}^1_1$ *set is either at most countable, or has cardinality* $\aleph_1$*, or cardinality* $2^{\aleph_0}$*.* □

Theorem 25.19 below improves Corollary 25.15 by showing that every $\mathbf{\Sigma}^1_2$ set is the union of $\aleph_1$ Borel sets. The following lemma is the first step toward that theorem.

Lemma 25.17. *Every* $\mathbf{\Sigma}^1_1$ *set is the union of* $\aleph_1$ *Borel sets.*

Proof. Let A be a $\mathbf{\Sigma}^1_1$ set. Let $T \subset Seq_2$ be a tree such that $A = p[T]$. We prove by induction on α that for each $t \in Seq$ and every $\alpha < \omega_1$, the set

$$\{x \in \mathcal{N} : \|T(x)/t\| \leq \alpha\} \tag{25.16}$$

is Borel. Namely, $\{x : \|T(x)/t\| \leq 0\} = \{x : (x\restriction n, t) \notin T\}$ and if $\alpha > 0$, then $\|T(x)/t\| \leq \alpha$ if and only if $\forall n\,(\exists \beta < \alpha)\,\|T(x)/t^\frown n\| \leq \beta$.

Let us define, for each α, the set B_α as follows:

$$x \in B_\alpha \leftrightarrow \neg(\|T(x)\| < \alpha) \wedge \forall t\,(\neg\|T(x)/t\| = \alpha).$$

Since the sets in (25.16) are Borel, it follows that each B_α is Borel. We shall prove that $A = \bigcup_{\alpha<\omega_1} B_\alpha$. First let $x \in A$. Thus $T(x)$ is ill-founded; hence $\|T(x)\| \not< \alpha$ for any α, and it suffices to show that there is an α such that $\|T(x)/t\| \neq \alpha$ for all t. If there is no such α, then for every α there is t such that $\|T(x)/t\| = \alpha$, but there are $\aleph_1$ α's and only $\aleph_0$ t's; a contradiction.

Next let $x \notin A$, and let us show that $x \notin B_\alpha$, for all α. Let $\alpha < \omega_1$ be arbitrary. Since $T(x)$ is well-founded, either $\|T(x)\| < \alpha$ and $x \notin B_a$, or $\|T(x)\| \geq \alpha$ and there exists some $t \in T(x)$ such that $\|T(x)/t\| = \alpha$ and again $x \notin B_\alpha$. □

$\mathbf{\Sigma^1_2}$ Sets

The Normal Form Theorem for Π^1_1 sets provides a tree representation for Σ^1_2 sets:

Theorem 25.18. *Every* $\mathbf{\Sigma}^1_2$ *set is* ω_1*-Suslin. If* A *is* $\Sigma^1_2(a)$ *then* $A = p[T]$ *where* T *is a tree on* $\omega \times \omega_1$ *and* $T \in L[a]$*.*

Proof. Let A be a $\Sigma^1_2(a)$ subset of $\mathcal{N}$. There is a tree $U \subset Seq_3$, recursive in a such that

$$x \in A \leftrightarrow \exists y\,\forall z\,\exists n\,(x\restriction n, y\restriction n, z\restriction n) \notin U.$$

In other words,

$$x \in A \leftrightarrow \exists y\,U(x, y) \text{ is well-founded.}$$

A necessary and sufficient condition for a countable relation to be well-founded is that it admits an order-preserving mapping into ω_1. Thus

$$\begin{aligned} x \in A &\leftrightarrow \exists y\, (\exists f : U(x,y) \to \omega_1) \text{ if } u \subset v \text{ then } f(u) > f(v) \\ &\leftrightarrow \exists y\, (\exists f : \mathit{Seq} \to \omega_1)\, f{\restriction}U(x,y) \text{ is order-preserving.} \end{aligned}$$

Let $\{u_n : n \in \boldsymbol{N}\}$ be a recursive enumeration of the set *Seq* such that for every n, $\text{length}(u_n) \leq n$. If f is a function on (a subset of) $\boldsymbol{N}$, let f^* be the function on (a subset of) *Seq* defined by $f^*(u_n) = f(n)$. Thus

$$x \in A \leftrightarrow \exists y\, (\exists f : \omega \to \omega_1)\, f^*{\restriction}U(x,y) \text{ is order-preserving.} \tag{25.17}$$

Now we define a tree T' on $\omega^2 \times \omega_1$ as follows: If $s, t \in \mathit{Seq}$ and $h \in \mathit{Seq}(\omega_1)$ are all of length n, we let

$$(s,t,h) \in T' \leftrightarrow h^*{\restriction}U_{s,t} \text{ is order-preserving} \tag{25.18}$$

where $U_{s,t} = \{u \in \mathit{Seq} : k = \text{length}\, u \leq n \text{ and } (s{\restriction}k, t{\restriction}k, u) \in U\}$. Clearly, T' is a tree on $\omega^2 \times \omega_1$.

Let $x, y \in \mathcal{N}$. We claim that if $(x{\restriction}n, (y{\restriction}n, h) \in T'$, then $h^*{\restriction}U(x,y)$ is order-preserving. This is because if $u, v \in \text{dom}(h^*) \cap U(x,y)$, then $u = u_i$, $v = u_j$ for some $i, j < n$, hence $\text{length}(u), \text{length}(v) < n$ and hence $u, v \in U_{s,t}$, where $s = x{\restriction}n$, $t = y{\restriction}n$. Thus

$$f \in T'(x,y) \leftrightarrow \forall n\, (f{\restriction}n)^*{\restriction}U(x,y) \text{ is order-preserving.}$$

But clearly a mapping $f : \omega \to \omega_1$ satisfies the right-hand side if and only if $f^*{\restriction}U(x,y)$ is order-preserving. Hence (25.17) and (25.18) give

$$\begin{aligned} x \in A &\leftrightarrow \exists y\, \exists f : \omega \to \omega_1\, f \in T'(x,y) \\ &\leftrightarrow \exists y\, \exists f : \omega \to \omega_1\, \forall n\, (x{\restriction}n, y{\restriction}n, f{\restriction}n) \in T'. \end{aligned}$$

Now we transform T' (on $\omega^2 \times \omega_1$) into a tree T'' (on $\omega \times K$ where $K = \omega \times \omega_1$) such that we replace triples

$$(\langle s(0), \ldots, s(n-1)\rangle, \langle t(0), \ldots, t(n-1), \rangle, \langle h(0), \ldots, h(n-1)\rangle)$$

by pairs

$$(\langle s(0), \ldots, s(n-1)\rangle, \langle (t(0), h(0)), \ldots, (t(n-1), h(n-1))\rangle)$$

and we get

$$x \in A \leftrightarrow (\exists g : \omega \to K)\, \forall n\, (x{\restriction}n, g{\restriction}n) \in T''.$$

Since $K = \omega \times \omega_1$ is in an obvious one-to-one correspondence with ω_1, it is clear that we can find a tree T on $\omega \times \omega_1$ such that

$$x \in A \leftrightarrow (\exists g : \omega \to \omega_1)\, \forall n\, (x{\restriction}n, g{\restriction}n) \in T, \tag{25.19}$$

that is $A = p[T]$. The tree T so obtained is constructible from the tree U, which in turn is constructible from a. □

One consequence of Theorem 25.18 is the following:

Theorem 25.19 (Sierpiński). *Every* $\mathbf{\Sigma}^1_2$ *set is the union of* $\aleph_1$ *Borel sets.*

It follows that in ZFC, every $\mathbf{\Sigma}^1_2$ set has cardinality either at most $\aleph_1$, or $2^{\aleph_0}$.

Proof. Let A be a $\mathbf{\Sigma}^1_2$ set. By Theorem 25.18 there is a tree T on $\omega \times \omega_1$ such that $A = p[T]$. For each $\gamma < \omega_1$ let $T^\gamma = \{(s,h) \in T : h \in Seq(\gamma)\}$. Since every $f : \omega \to \omega_1$ has the range included in some $\gamma < \omega_1$, it is clear that

$$A = \bigcup_{\gamma<\omega_1} p[T^\gamma].$$

For each $\gamma < \omega_1$, the set $p[T^\gamma]$ is analytic (because $p[T^\gamma] = p[\tilde{T}]$ for some $\tilde{T} \subset Seq_2$) and is the union of $\aleph_1$ Borel sets. In fact, Lemma 25.17 gives a uniform decomposition into $\aleph_1$ Borel sets for any $p[U]$ where U is a tree on $\omega \times S$ with S countable. If we let

$$x \in B^\gamma_\alpha \leftrightarrow \neg(\|T^\gamma(x)\| < \alpha) \wedge (\forall t \in Seq(\gamma))(\neg\|T^\gamma(x)/t\| = \alpha)$$

then $A = \bigcup_{\alpha<\omega_1} \bigcup_{\gamma<\omega_1} B^\gamma_\alpha$. □

The main application of Theorem 25.18 is absoluteness of Σ^1_2 (and Π^1_2) relations.

Theorem 25.20 (Shoenfield's Absoluteness Theorem). *Every* $\Sigma^1_2(a)$ *relation and every* $\Pi^1_2(a)$ *relation is absolute for all inner models* M *of* ZF + DC *such that* $a \in M$. *In particular,* $\mathbf{\Sigma}^1_2$ *and* $\mathbf{\Pi}^1_2$ *relations are absolute for* L.

It is clear from the proof that every $\mathbf{\Sigma}^1_2(a)$ relation is absolute for every transitive model M of a finite fragment of ZF + DC such that $\omega_1 \in M$.

Proof. Let $a \in \mathcal{N}$ and let A be a $\Sigma^1_2(a)$ subset of $\mathcal{N}$; let $A = \{x : A(x)\}$ where $A(x)$ is a $\Sigma^1_2(a)$ property. Let M be an inner model of ZF + DC such that $a \in M$. We shall prove that $M \vDash A$ if and only if A holds.

Let $U \subset Seq_3$ be a tree, arithmetical in a, such that for all $x \in \mathcal{N}$,

$$x \in A \leftrightarrow \exists y\, U(x,y) \text{ is well-founded.}$$

Thus for all $x \in \mathcal{N} \cap M$

$$x \in A^M \leftrightarrow (\exists y \in M)\, M \vDash U(x,y) \text{ is well-founded.}$$

However, for all $x, y \in M$, $U(x,y)$ is the same tree in M as in V; and since well-foundedness is absolute, we have

$$x \in A^M \leftrightarrow (\exists y \in M)\, U(x,y) \text{ is well-founded.}$$

Thus, if $x \in A^M$, then $x \in A$, and it suffices to prove that if $x \in A \cap M$ then $x \in A^M$.

We use the tree representation of $\mathbf{\Sigma}^1_2$ sets. Let T be the tree on $\omega \times \omega_1$ constructed in the proof of Theorem 25.18. Hence $T \in L[a]$ and for every $x \in \mathcal{N}$,

$$x \in A \leftrightarrow T(x) \text{ is ill-founded.}$$

Now if $x \in M$ is such that $x \in A$, then $T(x)$ is ill-founded, and by absoluteness of well-foundedness,

$$M \vDash T(x) \text{ is ill-founded.}$$

In other words, there exists a function $g \in M$ from $\mathcal{N}$ into the ordinals such that $\forall n\, (x\restriction n, g\restriction n) \in T$. Now following the proof of Theorem 25.18 backward, from (25.19) to the beginning, and working inside M, one finds a $y \in M$ such that

$$M \vDash U(x, y) \text{ is well-founded.}$$

Hence if $x \in A \cap M$, then $x \in A^M$ and we are done. □

With only notational changes Theorem 25.18 gives a tree representation of subsets of ω (or ω^k) and we have:

Corollary 25.21. *If $A \subset \omega$ is $\Sigma^1_2(a)$ then $A \in L[a]$. In particular, every Σ^1_2 real (and every Π^1_2 real) is constructible.*

The following lemma is an interesting application of Shoenfield's Absoluteness.

Lemma 25.22. *Let S be a set of countable ordinals such that the set $A = \{x \in \mathrm{WO} : \|x\| \in S\}$ is Σ^1_2. Then S is constructible. (And more generally, if A is $\Sigma^1_2(a)$, then $S \in L[a]$.)*

Proof. Let $A(x)$ be the Σ^1_2 property such that $A = \{x : A(x)\}$. For each countable ordinal α, let P_α be the notion of forcing that collapses α; i.e., the elements of P_α are finite sequences of ordinals less than α. Each P_α is constructible; let us consider, in L, the forcing languages associated with the P_α, and the corresponding Boolean-valued models L^{P_α}.

We shall show that for every $\alpha < \omega_1$, α belongs to S if and only if

$$L \vDash \text{every } p \in P_\alpha \text{ forces } \exists x\, (A(x) \wedge \|x\| = \alpha). \tag{25.20}$$

This will show that S is constructible.

In order to prove that $\alpha \in S$ is equivalent to (25.20), let us consider a generic extension N of V in which ω_1^V is countable. Let us argue in N.

The notion of forcing P_α has only countably many constructible dense subsets, and hence for every $p \in P_\alpha$ there exists a $G \subset P_\alpha$ such that G is L-generic and $p \in G$. It follows that for every α, every φ and every $p \in P_\alpha$,

$$L \vDash (p \Vdash \varphi) \quad \text{if and only if} \quad \text{for every } L\text{-generic } G \ni p,\ L[G] \Vdash \varphi. \tag{25.21}$$

Let $\alpha < \omega_1^V$, and let $z \in V$ be such that $\|z\| = \alpha$. Clearly, α belongs to S if and only if V satisfies

$$\exists x\,(A(x) \wedge \|x\| = \|z\|). \tag{25.22}$$

The property (25.22) is Σ^1_2 and by absoluteness, it holds in V if and only if it holds in N.

Let G be an arbitrary L-generic filter on P_α, and let $u \in L[G]$ be such that $\|u\| = \alpha$. Since N satisfies (25.22) if and only if it satisfies the Σ^1_2 property

$$\exists x\,(A(x) \wedge \|x\| = \|u\|), \tag{25.23}$$

it follows that $\alpha \in S$ if and only if $L[G]$ satisfies (25.23). Since an L-generic filter on P_α exists in N, we conclude (still in N), that $\alpha \in S$ is equivalent to:

$$\text{For every } L\text{-generic } G \subset P_\alpha,\ L[G] \vDash \exists x\,(A(x) \wedge \|x\| = \alpha).$$

But in view of (25.21) this last statement is equivalent to (25.20). □

Another application of the tree representation of $\mathbf{\Sigma}^1_2$ sets is the Perfect Set Theorem of Mansfield and Solovay:

Theorem 25.23 (Mansfield-Solovay). *Let A be a $\Sigma^1_2(a)$ set in $\mathcal{N}$. If A contains an element that is not in $L[a]$, then A has a perfect subset.*

The theorem follows from this more general lemma:

Lemma 25.24. *Let T be a tree on $\omega \times K$ and let $A = p[T]$. Either $A \subset L[T]$, or A contains a perfect subset; moreover, in the latter case there is a perfect tree $U \in L[T]$ on ω such that $[U] \subset A$.*

Proof. The proof follows the Cantor-Bendixson argument. If T is a tree on $\omega \times K$, let

$$T' = \{(s,h) \in T : \text{there exist } (s_0,h_0),(s_1,h_1) \in T \text{ such that } s_0 \supset s,\ s_1 \supset s,\ h_0 \supset h,\ h_1 \supset h, \text{ and that } s_0 \text{ and } s_1 \text{ are incompatible}\} \tag{25.24}$$

and then, inductively,

$$T^{(0)} = T, \qquad T^{(\alpha+1)} = (T^{(\alpha)})',$$
$$T^{(\alpha)} = \bigcap_{\beta<\alpha} T^{(\beta)} \quad \text{if } \alpha \text{ is limit.}$$

The definition (25.24) is absolute for all models that contain T, and hence $T^{(\alpha)} \in L[T]$ for all α. Let α be the least ordinal such that $T^{(\alpha+1)} = T^{(\alpha)}$.

Let us assume first that $T^{(\alpha)} = \emptyset$; we shall show that $A \subset L[T]$. Let $x \in A$ be arbitrary. There exists an $f \in K^\omega$ such that $(x,f) \in [T]$. Let $\gamma < \alpha$ be such that $(x,f) \in [T^{(\gamma)}]$ but $(x,f) \notin [T^{(\gamma+1)}]$. Thus there is some

$(s,h) \in T^{(\gamma)}$ such that $s \subset x$, $h \subset f$, and $(s,h) \notin T^{(\gamma+1)}$; this means that for any $(s',h') \in T^{(\gamma)}$, if $s' \supset s$ and $h' \supset h$, then $s' \subset x$. Now it follows that $x \in L[T]$; in $L[T]$, x is the unique $x = \bigcup\{s' \supset s : (s',h') \in T^{(\gamma)}$ for some $h \supset h'\}$.

Now let us assume that $T^{(\alpha)} \neq \emptyset$. The tree $T^{(\gamma)}$ has the property that for every $(s,h) \in T^{(\alpha)}$ there exist two extensions (s_0,h_0) and (s_1,h_1) of (s,h) that are incompatible in the first coordinate. Let us work in $L[T]$. Let (s_0,h_0) and (s_1,h_1) be some elements of $T^{(\alpha)}$ such that s_0 and s_1 are incompatible. Then let $(s_{0,0},h_{0,0})$, $(s_{0,1},h_{0,1})$, $(s_{1,0},h_{1,0})$, and $(s_{1,1},h_{1,1})$ be elements of $T^{(\alpha)}$ such that $s_{i,j} \supset s_i$, $h_{i,j} \supset h_i$ and that the $s_{i,j}$ are incompatible. In this fashion we construct $(s_t,h_t) \in T^{(\alpha)}$ for each 0–1 sequence t. The s_t generate a tree $U = \{s : s \subset s_t$ for some $t\}$. It is clear that U is a perfect three, that $U \in L[T]$, and that $[U] \subset p[T] = A$. □

The following observation establishes a close connection between the projective hierarchy and the Lévy hierarchy of Σ_n properties of hereditarily countable sets:

Lemma 25.25. *A set $A \subset \mathcal{N}$ is Σ^1_2 if and only if it is Σ_1 over $(HC,\in)$.*

Proof. If A is Σ_1 over HC, there exists a Σ_0 formula φ such that

$$x \in A \leftrightarrow HC \vDash \exists u\, \varphi(u,x) \leftrightarrow (\exists u \in HC)\, HC \vDash \varphi[u,x].$$

Since φ is Σ_0, it is absolute for transitive models and we have

$$x \in A \leftrightarrow (\exists \text{ transitive set } M)(\exists u \in M)\, M \vDash \varphi[u,x]$$

(e.g., $M = \mathrm{TC}(\{u,x\})$). By the Principle of Dependent Choices every $\mathrm{TC}(\{u,x\})$ is countable and we have

$$\begin{aligned} x \in A &\leftrightarrow (\exists \text{ countable transitive set } M)(\exists u \in M)\, M \vDash \varphi[u,x] \\ &\leftrightarrow (\exists \text{ well-founded extensional relation } E \text{ on } \omega) \\ &\quad \exists n\, \exists m\, (\pi_E(m) = x \text{ and } (\omega,E) \vDash \varphi[n,m]) \end{aligned}$$

where π_E is the transitive collapse of (ω,E) onto $(M,\in)$. Recalling the definition (25.13) of E_x for $z \in \mathcal{N}$ we have

$$\begin{aligned} x \in A \leftrightarrow (\exists z \in \mathcal{N})(z \in \mathrm{WF} \text{ and } (\omega,E_x) \vDash \text{Extensionality}, \\ \exists n\, \exists m\, (\pi_{E_x}(m) = x \text{ and } (\omega,E_x) \vDash \varphi[n,m])). \end{aligned} \tag{25.25}$$

We shall verify that (25.25) gives a Σ^1_2 definition of A. Since WF is Π^1_1, it suffices to show that the relation "$(\omega,E) \vDash \varphi[n_1,\dots,n_k]$" and "$\pi_E(m) = x$" are arithmetical in E. It is easy to see that $(\omega,E) \vDash \varphi$ is a property

arithmetical in E. As for the transitive collapse, we notice first that if $k \in \boldsymbol{N}$, then

$$\pi_E(m) = k \leftrightarrow \exists \langle r_0, \ldots, r_k \rangle \text{ such that } m = r_k \text{ and } (\omega, E) \vDash r_0 = \emptyset$$
$$\text{and } (\forall i < k)\,(\omega, E) \vDash (r_{i+1} = r_i \cup \{r_i\}).$$

Then for $x \subset \omega$ we have

$$\pi_E(m) = x \leftrightarrow \forall n\,(n\ E\ m \leftrightarrow \pi_E(n) \in x)$$

and a similar formula, arithmetical in E, defines $\pi_E(m) = x$ for $x \in \mathcal{N}$.

Hence $A \in \Sigma^1_2$.

Conversely, if A is a Σ^1_2 set then for some Π^1_1 property P, $A = \{x : \exists y\, P(x, y)\}$. By Mostowski's Absoluteness, $x \in A$ if and only if for some countable transitive model $M \ni x$ adequate for P there exists a $y \in M$ such that $M \vDash P(x, y)$. But this gives a Σ_1 definition of A over $(HC, \in)$. □

As a consequence, Σ^1_{n+1} sets are exactly those that are Σ_n over HC.

Projective Sets and Constructibility

We now compute the complexity of the set of all constructible reals:

Theorem 25.26 (Gödel). *The set of all constructible reals is a Σ^1_2 set. The ordering $<_L$ is a Σ^1_2 relation.*

The field of $<_L$ is $\boldsymbol{R} \cap L$. If all reals are constructible, then $<_L$ is also Π^1_2 (because $x <_L y$ if and only if $y \not\leq_L x$) and hence $<_L$ is then a Δ^1_2 relation.

The theorem easily generalizes to $L[a]$: If $a \in \boldsymbol{R}$ (or $a \subset \omega$ or $a \in \mathcal{N}$), then the set $\boldsymbol{R} \cap L[a]$ is $\Sigma^1_2(a)$; also, the relation "x is constructible from y" is a Σ^1_2 relation.

We proved in Chapter 13 that "x is constructible" and "$x <_L y$" are Σ_1 relations over the model $(HC, \in)$. Thus Theorem 25.26 follows from Lemma 25.25.

The following lemma tells even more than $<_L$ is a Σ^1_2 relation. For any $z \in \mathcal{N}$, let z_m, $m \in \boldsymbol{N}$, be defined by $z_m(n) = z(\Gamma(m, n))$ (the canonical homeomorphism between $\mathcal{N}$ and $\mathcal{N}^\omega$).

Lemma 25.27. *The following relation R on $\mathcal{N}$ is Σ^1_2:*

$$(z, x) \in R \leftrightarrow \{z_n : n \in \boldsymbol{N}\} = \{y : y <_L x\}.$$

Proof. Since the relation $\{z_n : n \in \boldsymbol{N}\} \subset \{y : y <_L x\}$ is clearly Σ^1_2, it suffices to show that

$$\forall y <_L x\, \exists n\,(y = z_n) \tag{25.26}$$

is Σ^1_2. There is a sentence Θ (provable in ZF) such that if M is a transitive model of Θ, then $<_L$ is absolute for M; and if $x \in M$ is constructible, then every $y <_L x$ is in M. Thus (25.26) is equivalent to

$$\exists \text{ countable transitive model } M \text{ that contains } x,\ z, \text{ and all } z_n,\\ \text{and } M \vDash (\Theta \text{ and } \forall y <_L x\, \exists n\, (y = z_n)).$$

This last property is Σ^1_2 by a proof similar to Lemma 25.25. □

Every $\mathbf{\Sigma}^1_1$ set is Lebesgue measurable, has the Baire property and if uncountable, has a perfect subset. The following results show that this is best possible.

Corollary 25.28. *If $V = L$ then there exists a Δ^1_2 set that is not Lebesgue measurable and does not have the Baire property.*

Proof. Let $A = \{(x, y) : x <_L y\}$. For every y, the set $\{x : (x, y) \in A\}$ is countable, hence null and meager, and by Lemmas 11.12 and 11.16, if A is measurable, then it is null; and if it has the Baire property, then it is meager.

Let B be the complement of A in $\boldsymbol{R}^2$, $B = \{(x, y) : y \leq_L x\}$. Again, for every x, the set $\{y : (x, y) \in B\}$ is countable, and hence null if measurable, and meager if has the Baire property.

It clearly follows that A neither is Lebesgue measurable nor has the property of Baire □

Corollary 25.29. *If $V = L$ then there exists an uncountable Σ^1_2 set without a perfect subset.*

Proof. Let

$$x \in A \leftrightarrow x \in \mathrm{WO} \wedge \forall y <_L x\, (\neg \|y\| = \|x\|).$$

The set A is uncountable: A is a subset of WO and for every $\alpha < \omega_1$ there is exactly one x in A such that $\|x\| = \alpha$. Let us show that A is Σ^1_2: Let R be the Σ^1_2 relation from Lemma 25.27; thus

$$x \in A \leftrightarrow x \in \mathrm{WO} \wedge \exists z\, (R(z, x) \wedge \forall n\, (\neg \|z_n\| = \|x\|)),$$

and since $\neg \|z_n\| = \|x\|$ is Π^1_1, A is Σ^1_2.

The set A does not have a perfect subset; in fact, it does not have an uncountable analytic subset. This follows from the Boundedness Lemma: For every analytic set $X \subset A$, the set $\{\|x\| : x \in X\}$ is bounded, and hence countable (because of the definition of A). □

Below (Corollary 25.37) we improve this by showing that in L there exists an uncountable Π^1_1 set without a perfect subset.

By Shoenfield's Absoluteness Theorem, every Σ^1_2 real is constructible. In Part III we show that it is consistent that a nonconstructible Δ^1_3 real exists. In the presence of large cardinals, an example of a nonconstructible Δ^1_3 real is $0^\sharp$:

Lemma 25.30. *If $0^\sharp$ exists then $0^\sharp$ is a Δ^1_3 real, and the singleton $\{0^\sharp\}$ is a Π^1_2 set.*

Proof. We identify $0^\sharp$ with the set of Gödel numbers of the sentences in $0^\sharp$. We claim that the property $\Sigma = 0^\sharp$ is Π_1 over $(HC, \in)$, and therefore Π^1_2. We use the description (18.24) of $0^\sharp$ and note that the quantifiers $\forall\alpha$ can be replaced by $\forall\alpha < \omega_1$, thus making it a Π_1 property over HC.

Thus $\{0^\sharp\}$ is a Π^1_2 set, and

$$n \in 0^\sharp \leftrightarrow \exists z\,(z \in \{0^\sharp\} \text{ and } z(n) = 1) \leftrightarrow \forall z\,(z \in \{0^\sharp\} \to z(n) = 1)$$

shows that $0^\sharp$ is a Δ^1_3 subset of ω. □

Scales and Uniformization

The tree analysis of Σ^1_2 sets can be refined; an analysis of Kondô's proof of the Uniformization Theorem (Theorem 25.36) led Moschovakis to introduce the concept of *scale* that pervades the modern descriptive set theory.

We start with the definition of *norm* and *prewellordering*. While in the present chapter these concepts are applied to $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$ sets, the theory applies to more general collection of definable sets of reals.

Definition 25.31. A *norm* on a set A is an ordinal function φ on A. A *prewellordering* of A is a transitive relation $\preccurlyeq$ such that $a \preccurlyeq b$ or $b \preccurlyeq a$ for all $a, b \in A$, and that $\prec$ is well-founded.

A prewellordering of a set A induces an equivalence relation ($a \preccurlyeq b \wedge b \preccurlyeq a$) and a well-ordering of its equivalence classes. Its rank function is a norm, and conversely, a norm φ defines a prewellordering

$$a \preccurlyeq_\varphi b \quad \text{if and only if} \quad \varphi(a) \le \varphi(b). \tag{25.27}$$

The tree analysis of Π^1_1 and Σ^1_2 sets produces well behaved prewellorderings of Π^1_1 and Σ^1_2 sets:

Theorem 25.32. *For every Π^1_1 set A there exists a norm φ on A with the property that there exist a Π^1_1 relation $P(x, y)$ and a Σ^1_1 relation $Q(x, y)$ such that for every $y \in A$ and all x,*

$$x \in A \text{ and } \varphi(x) \le \varphi(y) \leftrightarrow P(x, y) \leftrightarrow Q(x, y). \tag{25.28}$$

A norm φ with the above property is called a Π^1_1-norm and the statement "every Π^1_1 set has a Π^1_1-norm" is called the *prewellordering property* of Π^1_1.

A relativization of Theorem 25.32 shows that every $\Pi^1_1(a)$ set has a $\Pi^1_1(a)$-norm. A modification of the proof of Theorem 25.32 yields the prewellordering property of Σ^1_2: every Σ^1_2 set has a Σ^1_2 norm, i.e., a norm for which exist a Σ^1_2 P and a Π^1_2 Q that satisfy (25.28) (cf. Exercises 25.5 and 25.6).

Proof. Let A be a Π^1_1 and let T be a recursive tree on $\omega \times \omega$ such that

$$A(x) \leftrightarrow T(x) \text{ is well-founded.}$$

For each $x \in A$ let $\varphi(x) = \|T(x)\|$ be the height of the well-founded tree.

To define the Σ^1_1 relation Q, let

$$Q(x,y) \leftrightarrow \text{there exists an order-preserving function } f : T(x) \to T(y). \tag{25.29}$$

It is not difficult to see that Q is Σ^1_1, and the equivalence in (25.28) follows from Corollary 25.7. For the Π^1_1 relation, let

$$P(x,y) \leftrightarrow \forall s \neq \emptyset \text{ there exists no order-preserving } f : T(y) \to T(x)/s. \tag{25.30}$$

This is Π^1_1 and says that $T(x)$ is well-founded and it is not the case that $\|T(y)\| < \|T(x)\|$. □

The prewellordering property of Π^1_1 implies the *reduction principle* for Π^1_1 and the *separation principle* for Σ^1_1—see Exercises. This in turn implies Suslin's Theorem that every $\mathbf{\Delta}^1_1$ set is Borel.

The prewellordering property has an important strengthening, the *scale property* which we now introduce.

Let A be a Π^1_1 set. Following the proof of Theorem 25.18 we obtain a tree T on $\omega \times \omega_1$ such that $A = p[T]$. In detail, let U be a recursive tree on $\omega \times \omega$ such that

$$x \in A \leftrightarrow U(x) \text{ is well-founded} \leftrightarrow \exists g : U(x) \to \omega_1 \text{ order preserving.}$$

Let $\{u_n : n \in \boldsymbol{N}\}$ be a recursive enumeration of *Seq* such that $\text{length}(u_n) \leq n$, and let T be the tree on $\omega \times \omega_1$ defined by

$$(s,h) \in T \leftrightarrow \forall m,n < \text{length}(s)(\text{if } u_m \supset u_n \text{ and } (s\restriction k, s\restriction u_m) \in U \text{ where } k = \text{length}(u_m), \text{ then } h(m) < h(n)). \tag{25.31}$$

The relevant observation is that not only that $A = p[T]$, i.e.,

$$x \in A \leftrightarrow \exists \text{ a branch } g \text{ in } T(x)$$

but that for every $x \in p[T]$ there exists a (pointwise) *least* branch g in $T(x)$, i.e., for every $f \in p[T]$, $g(n) \leq f(n)$ for all n. To see this, let

$$g_x(n) = \begin{cases} \rho_{T(x)}(u_n) & \text{if } u_n \in U(x), \\ 0 & \text{otherwise.} \end{cases}$$

That g_x is the least branch in $T(x)$ holds because the rank function is the least order-preserving function.

Definition 25.33. A *scale* on a set A is a sequence of norms $\langle \varphi_n : n \in \omega \rangle$ such that: If $\langle x_i : i \in \omega \rangle$ is a sequence of points in A with $\lim_{i\to\infty} x_i = x$ and such that

(25.32) for every n, the sequence $\langle \varphi_n(x_i) : i \in \omega \rangle$ is eventually constant, with value α_n,

then $x \in A$, and for every n, $\varphi_n(x) \leq \alpha_n$.

It is easy to see that every Π^1_1 set A has a scale: Let A be a Π^1_1 set and let T be the tree in (25.31). We have $A = p[T]$ and for each $x \in A$, $T(x)$ has a least branch g_x. Let $\langle \varphi_n : n \in \omega \rangle$ be the sequence of norms on A defined by

$$\varphi_n(x) = g_x(n). \tag{25.33}$$

If $\langle x_i : i \in \omega \rangle$ is a sequence in A with $\lim_{i\to\infty} x_i = x$ that satisfies (25.32) then $\langle \alpha_n : n \in \omega \rangle$ is a branch in $T(x)$ witnessing $x \in p[T]$, and for every n, $g_x(n) \leq \alpha_n$.

The norms defined in (25.33) are Π^1_1-norms; this can be verified as in the proof of Theorem 25.32. To be precise, the scale $\langle \varphi_n : n \in \omega \rangle$ is a Π^1_1-*scale*:

Theorem 25.34. *For every Π^1_1 set A there exists a scale $\langle \varphi_n : n \in \omega \rangle$ on A with the property that there exist a Π^1_1 relation $P(n, x, y)$ and a Σ^1_1 relation $Q(n, x, y)$ such that for every n, every $y \in A$, and all x,*

$$x \in A \text{ and } \varphi_n(x) \leq \varphi_n(y) \leftrightarrow P(n, x, y) \leftrightarrow Q(n, x, y). \tag{25.34}$$ □

The statement "every Π^1_1 set has a Π^1_1-scale" is called the *scale property* of Π^1_1. A relativization shows that every $\Pi^1_1(a)$ set has a $\Pi^1_1(a)$-scale, and a modification of the above construction yields the scale property for Σ^1_2: every $\Sigma^1_2(a)$ set has a $\Sigma^1_2(a)$-scale; cf. Exercises 25.12 and 25.13.

A major application of scales is the *uniformization property*.

Definition 25.35. A set $A \subset \mathcal{N} \times \mathcal{N}$ is *uniformized* by a function F if $\mathrm{dom}(F) = \{x : \exists y\, (x, y) \in A\}$, and $(x, F(x)) \in A$ for all $x \in \mathrm{dom}(F)$.

[Equivalently, $F \subset A$ and $\mathrm{dom}(F) = \mathrm{dom}(A)$.]

Theorem 25.36 (Kondô). *Every Π^1_1 relation $A \subset \mathcal{N} \times \mathcal{N}$ is uniformized by a Π^1_1 function.*

The statement of Theorem 25.36 (the *Uniformization Theorem*) is called the *uniformization property* of Π^1_1. A relativization shows that every $\Pi^1_1(a)$ relation is uniformized by a $\Pi^1_1(a)$ function, and a modification of the proof yields the uniformization property of Σ^1_2; see Exercise 25.15.

Proof. We give a proof of the following statement that easily generalizes to a proof of Kondô's Theorem: If A is a nonempty Π^1_1 subset of $\mathcal{N}$ then there exists an $a \in A$ such that $\{a\}$ is Π^1_1.

Thus let A be a nonempty Π^1_1 subset of $\mathcal{N}$. Given a scale $\langle \varphi_n : n \in \omega \rangle$ on A, we select an element $a \in A$ as follows: We let $A_0 = A$, and for each n let

$$A_{2n+1} = \{x \in A_{2n} : \varphi_n(x) \text{ is least}\},$$
$$A_{2n+2} = \{x \in A_{2n+1} : x(n) \text{ is least}\}.$$

Then $A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots$ and the intersection has at most one element. Definition 25.33 guarantees that the limit a is in A and so $\bigcap_{n=0}^{\infty} A_n = \{a_n\}$.

If the scale $\langle \varphi_n : n \in \omega \rangle$ is Π^1_1 then using (25.34) one verifies that the set $\{a\}$ is Π^1_1. □

Theorem 25.36 can be used to improve the result in Corollary 25.29:

Corollary 25.37. *If $V = L$ then there exists an uncountable Π^1_1 set without a perfect subset.*

Proof. Let A be a Σ^1_2 set without a perfect subset (by 25.29). Now A is the projection of some Π^1_1 set $B \subset \mathcal{N}^2$. By the Uniformization Theorem, B has a Π^1_1 subset f that is a function and has the same projection A. The set f is uncountable; we claim that f does not have a perfect subset. Assume that $P \subset f$ is perfect. The projection of P is an analytic subset of A. Since $P \subset f$, P is itself a function and because P is uncountable, the projection $\mathrm{dom}(P)$ is also uncountable. This is a contradiction since we proved that every analytic subset of A is countable. □

Combining this result with Theorem 25.23 we obtain:

Theorem 25.38. *The following are equivalent:*

(i) *For every $a \subset \omega$, $\aleph_1^{L[a]}$ is countable.*
(ii) *Every uncountable $\mathbf{\Pi}^1_1$ set contains a perfect subset.*
(iii) *Every uncountable $\mathbf{\Sigma}^1_2$ set contains a perfect subset.*

Proof. Obviously, (iii) implies (ii). In order to show that (i) implies (iii), let us assume (i) and let A be an uncountable $\mathbf{\Sigma}^1_2$ set. Let $a \in \mathcal{N}$ be such that $A \in \Sigma^1_2(a)$. Since $\aleph_1^{L[a]}$ is countable, there are only countably many reals in $L[a]$, and hence A has an element that is not in $L[a]$. Thus A contains a perfect subset, by Theorem 25.23.

The remaining implication uses the same argument as Corollaries 25.29 and 25.37. Assume that there exists an $a \subset \omega$ such that $\aleph_1^{L[a]} = \aleph_1$. We claim that there exists an uncountable $\mathbf{\Pi}^1_1$ set without a perfect subset. Let

$$x \in A \leftrightarrow x \in L[a] \wedge x \in \mathrm{WO} \wedge \forall y <_{L[a]} x\, (\neg \|y\| = \|x\|).$$

A is a $\Sigma^1_2(a)$ subset of WO and for all $\alpha < \omega_1$, A has exactly one element x such that $\|x\| = \alpha$. The rest of the proof proceeds as before, and we obtain a $\Pi^1_1(a)$ set of cardinality $\aleph_1$ without a perfect subset. □

Σ^1_2 Well-Orderings and Σ^1_2 Well-Founded Relations

The canonical well-ordering of constructible reals is Σ^1_2, and so if $V = L$ then there exists a Σ^1_2 well-ordering of the set $\boldsymbol{R}$ (and of $\mathcal{N}$). We now prove the converse: If there exists a Σ^1_2 well-ordering of $\boldsymbol{R}$ then all reals are constructible.

Theorem 25.39 (Mansfield). *If $<$ is a Σ^1_2 well-ordering of $\mathcal{N}$ then every real is constructible. More generally, if $<$ is $\Sigma^1_2(a)$ then $\mathcal{N} \subset L[a]$.*

Proof. Let $<$ be a Σ^1_2 well-ordering of $\mathcal{N}$ and let us assume that there is a nonconstructible real. Let $T_0 = Seq(\{0,1\})$, and let $\boldsymbol{C} = [T_0] = \{0,1\}^\omega$ be the Cantor space. Let us consider trees $T \subset T_0$ and functions $f : T \to T_0$ such that $s \subset t$ implies $f(s) \subset f(t)$ and for every $x \in [T]$, $\bigcup_{n=0}^{\infty} f(x\restriction n) \in \boldsymbol{C}$. Every such function induces a continuous function from $[T]$ into $\boldsymbol{C}$, which we denote by f^*.

Lemma 25.40. *If $T \subset T_0$ is a constructible perfect tree and if $f : T \to T_0$ is a constructible function such that f^* is one-to-one, then there exist a constructible perfect tree $U \subset T$ and a constructible $g : U \to T_0$ such that g^* is one-to-one, and $g^*(x) < f^*(x)$ for every $x \in [U]$.*

It suffices to prove this lemma because then we can construct a sequence of trees $T_0 \supset T_1 \supset \ldots \supset T_n \supset \ldots$ and functions $f_0, f_1, \ldots, f_n, \ldots$ where f_0 is the identity such that $f^*_{n+1}(x) < f^*_n(x)$ for all $x \in T_{n+1}$. Since all $[T_n]$ are compact sets, their intersection is nonempty and therefore there exists an x such that $f^*_0(x) > f^*_1(x) > \ldots > f^*_n(x) > \ldots$ contrary to the assumption that $<$ is a well-ordering.

Proof of Lemma 25.40. Let $T \subset T_0$ be a constructible tree and let $f : T \to T_0$ be constructible, such that f^* is one-to-one.

Since T is perfect, there exists a constructible function $h : T \to T_0$ such that $h^* : [T] \to \boldsymbol{C}$ is one-to-one and onto. For each $s \in T_0$, let $\overline{s}$ be the "mirror image" of s, namely if $s = \langle s(0), \ldots, s(k)\rangle$, let $\overline{s} = \langle 1 - s(0), \ldots, 1 - s(k)\rangle$; for $x \in \boldsymbol{C}$, $\overline{x}$ is defined similarly.

We claim that at least one of the sets

$$A = \{x \in [T] : f^*(x) > h^*(x)\}, \qquad B = \{x \in [T] : f^*(x) > \overline{h^*(x)}\}$$

contains a nonconstructible element. Let z be the least nonconstructible element of $\boldsymbol{C}$, and let $x, y \in [T]$ be such that $h^*(x) = z$ and $h^*(y) = \overline{z}$. Then both x and y are nonconstructible and hence $f^*(x) \geq z$ and $f^*(y) \geq z$. Thus either $f^*(x) > z$ or $f^*(y) > z$ and so either A or B contains a nonconstructible element. For instance, assume that A does.

Since $<$ is Σ_2, and T, f, and h are constructible subsets of HF, the set A is $\Sigma^1_2(a)$ for some $a \in L$. By Lemma 25.24 there exists a constructible perfect tree U such that $[U] \subset A$. If we let $g = h\restriction U$, then U and g satisfy the lemma. □

The set WO is Π^1_1 but not Σ^1_1. One consequence of this fact, related to the Boundedness Lemma, is that there is no $\mathbf{\Sigma}^1_1$ well-ordering of the reals, in fact every $\mathbf{\Sigma}^1_1$ well-ordering of a set of reals is countable. A more general statement holds:

Lemma 25.41. *Every $\mathbf{\Sigma}^1_1$ well-founded relation on $\mathcal{N}$ has countable height.*

Proof. Assuming that some $\mathbf{\Sigma}^1_1$ well-founded relation on $\mathcal{N}$ has height $\geq \omega_1$, we reach a contradiction by describing the set WO in a $\mathbf{\Sigma}^1_1$ way.

First consider the special case of well-orderings. Let E be a $\mathbf{\Sigma}^1_1$ well-ordering and let us assume that its order-type is $\geq \omega_1$. Then for every $\alpha < \omega_1$ there is an order-preserving mapping of $(\alpha, <)$ into $(\mathcal{N}, E)$. Conversely, if a countable linearly ordered set $(Q, <)$ can be embedded in $(\mathcal{N}, E)$, then $(Q, <)$ is a well-ordering. Hence let E_x be, for each $x \in \mathcal{N}$, the relation coded by x (see (25.13)), and let LO be the arithmetical set of all x that code a linear ordering of $\boldsymbol{N}$. Then

$$\begin{aligned} (25.35)\quad x \in \mathrm{WO} \leftrightarrow x \in \mathrm{LO} \wedge (\exists f : \omega \to \mathcal{N})\, \forall n\, \forall m \\ (n\ E_x\ m \to (f(n), f(m)) \in E) \\ \leftrightarrow x \in \mathrm{LO} \wedge \exists z \in \mathcal{N}\, \forall n\, \forall m\, (n\ E_x\ m \to (z_n, z_m) \in E), \end{aligned}$$

where for each $z \in \mathcal{N}$ and each n, z_n is the element of $\mathcal{N}$ defined by $z_n(k) = z(\Gamma(n,k))$ for all $k \in \boldsymbol{N}$, where Γ is the pairing function. Now (25.35) gives a $\mathbf{\Sigma}^1_1$ description of WO, a contradiction.

In the general case when E is a $\mathbf{\Sigma}^1_1$ well-founded relation we observe that if α is a countable ordinal less than the height of E, then there exist a countable set $S \subset \mathcal{N}$ and a function f of S onto α such that for every $u \in S$ and every $\beta < f(u)$ there exists a $v \in S$ such that $v\ E\ u$ and $\beta \leq f(v)$ (namely $f(x) = \rho_E(x)$, and the countable set S is constructed with the help of the Principle of Dependent Choices). Conversely, if $(Q, <)$ is a linearly ordered set and if there is a function f from a subset of $\mathcal{N}$ onto Q such that for every $u \in \mathrm{dom}(f)$ and every $q < f(u)$ there is $v \in \mathrm{dom}(f)$ such that $v\ E\ u$ and $q \leq f(v)$, then $(Q, <)$ is a well-ordering. Thus if E has height $\geq \omega_1$, we have

$$\begin{aligned} (25.36)\quad x \in \mathrm{WO} \leftrightarrow x \in \mathrm{LO} \wedge (\exists \text{ countable } S = \{z_n : n \in \boldsymbol{N}\}) \\ (\exists f : S \overset{\text{onto}}{\to} \boldsymbol{N})\, \forall n\, \forall k\, [\text{if } (k, f(z_n)) \in E_x, \text{ then} \\ \exists m \text{ such that } (z_m, z_n) \in E \text{ and} \\ \text{either } k = f(z_m) \text{ or } (k, f(z_m)) \in E_x]. \end{aligned}$$

Again, (25.36) can be written in a $\mathbf{\Sigma}^1_1$ manner, and we get a contradiction. □

The next theorem gives an upper bound on heights of $\mathbf{\Sigma}^1_2$ well-founded relations.

Theorem 25.42 (Martin). *Every $\mathbf{\Sigma}^1_2$ well-founded relation on $\mathcal{N}$ has length $< \omega_2$.*

Note that since every prewellordering is a well-founded relation, the theorem implies that $\boldsymbol{\delta}^1_2 \leq \omega_2$, where

$$\boldsymbol{\delta}^1_2 = \sup\{\alpha : \alpha \text{ is the length of a } \mathbf{\Sigma}^1_2 \text{ prewellordering}\}.$$

Proof. Let $E \subset \mathcal{N} \times \mathcal{N}$ be a $\mathbf{\Sigma}^1_2$ relation. Let T be a tree on $\omega^2 \times \omega_1$ such that for all $x, y \in \mathcal{N}$,

$$(x,y) \in E \leftrightarrow (\exists f : \omega \to \omega_1)\, \forall n\, (x{\restriction}n, y{\restriction}n, f{\restriction}n) \in T. \tag{25.37}$$

As usual, for each $z \in \mathcal{N}$ and each $n \in \boldsymbol{N}$, let $z_n \in \mathcal{N}$ be such that $z_n(k) = z(\Gamma(n,k))$ for all k; similarly, for each $f : \omega \to \omega_1$ and each n, let $f_n : \omega \to \omega_1$ be such that $f_n(k) = f(\Gamma(n,k))$ for all k. (Here Γ is the pairing function.)

Each of the following formulas is equivalent to the statement that the relation E is not well-founded:

$$\begin{aligned}
&\exists x\, \forall m\, (x_{m+1}, x_m) \in E,\\
&\exists x\, \forall m\, \exists f\, \forall n\, (x_{m+1}{\restriction}n, x_m{\restriction}n, f{\restriction}n) \in T,\\
&\exists x\, \exists f\, \forall m\, \forall n\, (x_{m+1}{\restriction}n, x_m{\restriction}n, f_m{\restriction}n) \in T.
\end{aligned}$$

It is easy to construct a tree U on $\omega \times \omega_1$ such that for all $x \in \mathcal{N}$ and all $f : \omega \to \omega_1$,

$$\forall m\, \forall n\, (x_{m+1}{\restriction}n, x_m{\restriction}n, f_m{\restriction}n) \in T \text{ if and only if } \forall k\, (x{\restriction}k, f{\restriction}k) \in U. \tag{25.38}$$

It follows from (25.38) that

$$E \text{ is well-founded if and only if } U \text{ is well-founded.} \tag{25.39}$$

Now let $E \subset \mathcal{N} \times \mathcal{N}$ be a Σ^1_2 well-founded relation; we want to show that its height is $< \omega_2$. Let T be a tree on $\omega^2 \times \omega_1$ such that (25.37) holds for all $x, y \in \mathcal{N}$ and let U be the tree on $\omega \times \omega_1$ constructed from T as above; since E is well-founded, U is well-founded.

Let us consider a generic extension $V[G]$ of the universe in which ω_1^V is countable and $\omega_2^V = \omega_1^{V[G]}$. Let us argue in $V[G]$.

Let E^* be the relation on $\mathcal{N}$ defined by (25.37). First we observe that $E \subset E^*$: If $x, y \in V$, then

$$\begin{aligned}
(x,y) \in E &\leftrightarrow V \vDash T(x,y) \text{ is ill-founded}\\
&\leftrightarrow V[G] \vDash T(x,y) \text{ is ill-founded}\\
&\leftrightarrow (x,y) \in E^*.
\end{aligned}$$

(because well-foundedness is absolute). We notice further that E^* is well-founded: This is because by the construction of U (which is absolute) and

the definition of E^*, $V[G]$ satisfies (25.39), i.e.,

$$E^* \text{ is well-founded if and only if } U \text{ is well-founded.}$$

Hence E^* is well-founded, and $\text{height}(E) \leq \text{height}(E^*)$.

The tree T is a tree on $\omega \times \omega_1^V$ and ω_1^V is a countable ordinal. Since $E^* = p[T]$, it follows that E^* is a $\mathbf{\Sigma}^1_1$ relation. By Lemma 25.41, the height of E^* is countable. It follows that $\text{height}(E) < \omega_1^{V[G]} = \omega_2^V$.

Now we can step back into the ground model and look at the result of the above argument: $\text{height}(E) < \omega_2$. □

Both Theorem 25.42 and Lemma 25.41 are special cases of the more general Kunen-Martin Theorem:

Theorem 25.43. *Let κ be an infinite cardinal. Every κ-Suslin well-founded relation on $\mathcal{N}$ has height $< \kappa^+$.*

Proof. Let $<$ be a κ-Suslin well-founded relation on $\mathcal{N}$. We first associate with $<$ a tree $\mathcal{T}$ on $\mathcal{N}$ as follows:

$$\mathcal{T} = \{\langle x_0, \ldots, x_{n-1}\rangle : x_{n-1} < x_{n-2} < \ldots < x_0\}, \tag{25.40}$$

(and $\langle x\rangle \in \mathcal{T}$ for all $x \in \mathcal{N}$). $\mathcal{T}$ is well-founded and it suffices to prove that the height of $\mathcal{T}$ is $< \kappa^+$.

As $<$ is κ-Suslin, there exists a tree T on $\omega \times \omega \times \kappa$ such that

$$x < y \quad \text{if and only if} \quad \exists f\, (x, y, f) \in [T].$$

Let W be the set of ill sequences (of nodes at the same level of T)

$$w = \langle (s_1, s_0, h_0), \ldots, (s_{i+1}, s_i, h_i), \ldots, (s_k, s_{k-1}, h_{k-1})\rangle$$

with $(s_{i+1}, s_i, h_i) \in T$, and let

$$\begin{aligned} w' \prec w \quad \text{if and only if} \quad & k = \text{length}(w) < \text{length}(w') = k' \\ & \text{length}(s_0) < \text{length}(s'_0), \text{ and} \\ & \forall i < k\; s_i \subset s'_i \text{ and } h_i \subset h'_i. \end{aligned} \tag{25.41}$$

We claim that the relation $\prec$ is well-founded. Otherwise, let $w_n = \langle (s^n_{i+1}, s^n_i, h^n_i) : i < k_n\rangle$ be such that $w_{n+1} \prec w_n$ for all n. For each $i \in \omega$, let $x_i = \bigcup_{n=0}^{\infty} s^n_i$, and $f_i = \bigcup_{n=0}^{\infty} h^n_i$ (these exist by (25.41)). It follows that $(x_{i+1}, x_i, f_i) \in [T]$ for all i, hence $x_{i+1} < x_i$, and therefore $x_0 > x_1 > \ldots > x_i > \ldots$, a contradiction.

The set W has cardinality κ and it suffices to find an order preserving mapping from $\mathcal{T} - \{\emptyset\}$ into $(W, \prec)$. For every pair (x, y) such that $x < y$, the tree $T(x, y)$ on κ is not well-founded and has a branch h; let $h_{x,y}$ be the

leftmost branch of the tree $T(x, y)$. Now let $\pi : \mathcal{T} - \{\emptyset\} \to W$ be as follows: $\pi(\langle x \rangle) = \emptyset$ for every $x \in \mathcal{N}$, and for $k \geq 2$,

$$\pi(\langle x_0, \ldots, x_{k-1} \rangle) = \langle (x_1 \restriction k, x_0 \restriction k, h_{x_1,x_0} \restriction k), \ldots, (x_k \restriction k, x_{k-1} \restriction k, h_{x_k,x_{k-1}} \restriction k) \rangle.$$

As $\pi(\langle x_0, \ldots, x_{k-1}, x_k \rangle) \prec \pi(\langle x_0, \ldots, x_{k-1} \rangle)$, the mapping is order-preserving, completing the proof. □

Borel Codes

Every Borel set of reals is obtained, in fewer than ω_1 steps, from open intervals by taking complements and countable unions. We shall show how this procedure can be coded by a function $c \in \omega^\omega$. We shall define the set BC of *Borel codes* and assign to each $c \in \mathrm{BC}$ a unique Borel set A_c. The code c not only describes the Borel set A_c but also describes the procedure by which the set A_c is constructed from basic open sets.

Let $I_1, I_2, \ldots, I_k, \ldots$ be a recursive enumeration of open intervals with rational endpoints (i.e., the sequence of the pairs of endpoints is rercursive). For each $c \in \mathcal{N}$, let

$$u(c) \quad \text{and} \quad v_i(c) \qquad (i \in \boldsymbol{N}) \tag{25.42}$$

be elements of $\mathcal{N}$ defined as follows: If $d = u(c)$, then $d(n) = c(n + 1)$ for all n; if $d = v_i(c)$, then $d(n) = c(\Gamma(i, n)+1)$ for all n (where Γ is the canonical one-to-one correspondence between $\boldsymbol{N} \times \boldsymbol{N}$ and $\boldsymbol{N}$).

For $0 < \alpha < \omega_1$, we define sets Σ_α and $\Pi_\alpha \subset \mathcal{N}$ as follows:

$$\begin{aligned}
&c \in \Sigma_1 && \text{if } c(0) > 1;\\
&c \in \Pi_\alpha && \text{if either } c \in \Sigma_\beta \cup \Pi_\beta \text{ for some } \beta < \alpha\\
&&& \quad \text{or } c(0) = 0 \text{ and } u(c) \in \Sigma_\alpha;\\
&c \in \Sigma_\alpha \ (\alpha > 1) && \text{if either } c \in \Sigma_\beta \cup \Pi_\beta \text{ for some } \beta < \alpha\\
&&& \quad \text{or } c(0) = 1 \text{ and } v_i(c) \in \textstyle\bigcup_{\beta<\alpha}(\Sigma_\beta \cup \Pi_\beta) \text{ for all } i.
\end{aligned} \tag{25.43}$$

If $c \in \Sigma_\alpha$ (if $c \in \Pi_\alpha$), we call c a $\boldsymbol{\Sigma}^0_\alpha$-*code* (a $\boldsymbol{\Pi}^0_\alpha$-*code*). Let BC, the set of all *Borel codes*, be

$$\mathrm{BC} = \bigcup_{\alpha<\omega_1} \Sigma_\alpha = \bigcup_{\alpha<\omega_1} \Pi_\alpha.$$

For every $c \in \mathrm{BC}$, we define a Borel set A_c as follows (we say that c *codes* A_c):

$$\begin{aligned}
&\text{if } c \in \Sigma_1 && \text{then } A_c = \textstyle\bigcup\{I_n : c(n) = 1\};\\
&\text{if } c \in \Pi_\alpha \text{ and } c(0) = 0 && \text{then } A_c = \boldsymbol{R} - A_{u(c)};\\
&\text{if } c \in \Sigma_\alpha \text{ and } c(0) = 1 && \text{then } A_c = \textstyle\bigcup_{i=0}^{\infty} A_{v_i(c)}.
\end{aligned} \tag{25.44}$$

It is clear that for every $\alpha > 0$, if $c \in \Sigma_\alpha$ (if $c \in \Pi_\alpha$), then $A_c \in \mathbf{\Sigma}^0_\alpha$ ($A_c \in \mathbf{\Pi}^0_\alpha$). Conversely, if B is a $\mathbf{\Sigma}^0_\alpha$ set (a $\mathbf{\Pi}^0_\alpha$ set), then there exists $c \in \Sigma_\alpha$ ($c \in \Pi_\alpha$) such that $B = A_c$. This is proved by induction on α using facts like: If c_i, $i \in \omega$ are elements of $\bigcup_{\beta<\alpha} \Pi_\beta$, then there is $c \in \Sigma_\alpha$ such that $c_i = v_i(c)$ for all $i \in \omega$.

Thus $\{A_c : c \in \mathrm{BC}\}$ is the collection of all Borel sets.

Lemma 25.44. *The set* BC *of all Borel codes is* Π^1_1.

Proof. Let us consider the following relation E on $\mathcal{N}$:

$$x \mathrel{E} y \quad \text{if and only if} \quad \begin{array}{l}\text{either } y(0) = 0 \text{ and } x = u(y),\\ \text{or } y(0) = 1 \text{ and } x = v_i(y) \text{ for some } i \in \omega.\end{array} \tag{25.45}$$

The relation E is arithmetical. If $y \in \Sigma_1$, then y is E-minimal (i.e., $\mathrm{ext}_E(y) = \emptyset$) and vice versa; if $y \in \Pi_\alpha$ and $x \mathrel{E} y$, then $x \in \Sigma_\alpha$, and if $y \in \Sigma_\alpha$ ($\alpha > 1$) and $x \mathrel{E} y$, then $x \in \bigcup_{\beta<\alpha}(\Sigma_\beta \cup \Pi_\beta)$.

We claim that

$$\begin{aligned} y \in \mathrm{BC} &\leftrightarrow E \text{ is well-founded below } y\\ &\leftrightarrow \text{there is no } \langle z_0, z_1, \dots z_n, \dots\rangle \text{ such that } z_0 = y\\ &\quad \text{and that } \forall n\,(z_{n+1} \mathrel{E} z_n).\end{aligned} \tag{25.46}$$

By the remark following (25.45), if $y \in \mathrm{BC}$, then there can be no infinite descending sequence $z_0 = y$, $z_1 \mathrel{E} z_0$, $z_2 \mathrel{E} z_1$, etc. Conversely, if E is well-founded below y, let ρ denote the rank function for E on $\mathrm{ext}_E(y)$. By induction on $\rho(x)$, one can see that every $x \in \mathrm{ext}_E(y)$ is a Borel code, and finally that y is itself a Borel code.

Now (25.46) gives a Π^1_1 definition of the set BC and the lemma follows. □

Lemma 25.45. *The properties* $A_c \subset A_d$, $A_c = A_d$, *and* $A_c = \emptyset$ *are* Π^1_1 *properties of Borel codes.*

Proof. We shall show that there are properties $P, Q \subset \boldsymbol{R} \times \mathcal{N}$ such that P is Π^1_1 and Q is Σ^1_1 and such that for every $c \in \mathrm{BC}$,

$$a \in A_c \leftrightarrow (a, c) \in P \leftrightarrow (a, c) \in Q. \tag{25.47}$$

Then

$$\begin{aligned} A_c \subset A_d &\leftrightarrow c, d \in \mathrm{BC} \wedge \forall a\,((a, c) \in Q \to (a, d) \in P),\\ A_c = A_d &\leftrightarrow c, d \in \mathrm{BC} \wedge A_c \subset A_d \wedge A_d \subset A_c,\\ A_c = \emptyset &\leftrightarrow c \in \mathrm{BC} \wedge \forall a\,(a, c) \notin Q.\end{aligned}$$

To find P and Q, let $x \in \mathcal{N}$ be fixed. Let T be the smallest set $T \subset \mathcal{N}$ such that

$$x \in T, \text{ and if } y \in T \text{ and } z \mathrel{E} y, \text{ then } z \in T. \tag{25.48}$$

The set T is countable. Let $h : T \to \{0,1\}$ be a function with the following property: For all $y \in T$,

$$\begin{aligned}(25.49)\quad &\text{if } y(0) > 1, \text{ then } h(y) = 1 \text{ if and only if}\\ &\qquad\qquad\qquad\qquad \text{for some } n,\ y(n) = 1 \text{ and } a \in I_n;\\ &\text{if } y(0) = 0, \text{ then } h(y) = 1 \text{ if and only if } h(u(y)) = 0;\\ &\text{if } y(0) = 1, \text{ then } h(y) = 1 \text{ if and only if for some } i,\ h(v_i(y)) = 1.\end{aligned}$$

Note that if x is a Borel code then there is a unique smallest countable set $T \subset \mathcal{N}$ with the property (25.48), and a unique function h with the property (25.49); moreover, for every $y \in T$ we have $h(y) = 1$ if and only if $a \in A_y$. Thus we let

$$\begin{aligned}(25.50)\quad &(a,x) \in P \leftrightarrow (\forall \text{ countable } T \subset \mathcal{N})(\forall h : T \to \{0,1\})\\ &\qquad\qquad\qquad\qquad \text{if (25.48) and (25.49) then } h(x) = 1,\end{aligned}$$

and

$$\begin{aligned}(25.51)\quad &(a,x) \in Q \leftrightarrow (\exists \text{ countable } T \subset \mathcal{N})(\exists h : T \to \{0,1\})\\ &\qquad\qquad\qquad\qquad (25.48) \wedge (25.49) \wedge h(x) = 1,\end{aligned}$$

and it is clear that if $c \in \mathrm{BC}$, then $a \in A_c$ if and only if $(a,c) \in P$ if and only if $(a,c) \in Q$.

It is a routine matter to verify that (25.50) can be written in Π^1_1 way and (25.51) in a Σ^1_1 way. (The quantifiers $\forall T$, $\forall h$, and $\exists T$, $\exists h$ are the only ones for which one needs quantifiers over $\mathcal{N}$; note that for instance, $\forall z\,(z\ E\ y \to y \in T)$ in (25.48) can be written as

$$(y(0) = 0 \to u(y) \in T) \wedge (y(0) = 1 \to \forall i\,(v_i(y) \in T)).) \qquad \square$$

We shall now show that certain properties of Borel codes are absolute for transitive models of $\mathrm{ZF} + \mathrm{DC}$. (As usual, full $\mathrm{ZF} + \mathrm{DC}$ is not needed, and the absoluteness holds for adequate transitive models.) If M is a transitive model of $\mathrm{ZF} + \mathrm{DC}$ and $c \in \omega^\omega$ is in M, then because the set BC is Π^1_1, c is a Borel code if and only if $M \vDash c$ is a Borel code. By Lemma 25.45 the properties of the codes $A_c \subset A_d$, $A_c = A_d$, and $A_c = \emptyset$ are Π^1_1 and therefore absolute: $A_c = A_d$ holds if and only if $A_c^M = A_d^M$, etc., where A_c^M denotes the Borel set in M coded by c. Moreover, since $a \in A_c$ is Π^1_1, it follows that $A_c^M = A_c \cap M$ for every Borel code $c \in M$.

Lemma 25.46. *The following properties (of codes) are absolute for all transitive models M of* $\mathrm{ZF} + \mathrm{DC}$:

$$\begin{aligned}&A_e = A_c \cup A_d, \qquad A_e = A_c \cap A_d,\\ &A_e = \boldsymbol{R} - A_c, \qquad A_e = A_c \mathbin{\triangle} A_d, \qquad A_e = \bigcup_{n=0}^{\infty} A_{c_n}\end{aligned}$$

(*we assume that the codes c, d, e are in M, as is the sequence $\langle c_n : n \in \omega\rangle$*).

We say that the operations $\cup$, $\cap$, $-$, $\triangle$, $\bigcup_{n=0}^{\infty}$ on Borel sets with codes in M are absolute for M.

Proof. If $c_0, c_1, \ldots, c_n, \ldots$ is a sequence of Borel codes in M, let $c \in \mathcal{N}$ be such that $c(0) = 1$ and that $v_i(c) = c_i$ for all $i \in \omega$. Clearly, c is a Borel code, $c \in M$, and c codes (both in the universe and in M) the Borel set $\bigcup_{n=0}^{\infty} A_{c_n}$. Hence for any Borel code $e \in M$, we have

$$A_e^M = \bigcup_{n=0}^{\infty} A_{c_n}^M \leftrightarrow A_e^M = A_c^M \leftrightarrow A_e = A_c \leftrightarrow A_e = \bigcup_{n=0}^{\infty} A_{c_n}$$

because $A_e = A_c$ is absolute for M. Thus $A_e = \bigcup_{n=0}^{\infty} A_{c_n}$ is absolute.

An analogous argument shows that $\boldsymbol{R} - A_c$ is absolute, and the rest of the lemma follows easily because the operations $\cap$, and $\triangle$ can be defined from $\cup$ and $-$. □

Exercises

25.1. If $A \subset Seq \times \omega$ is arithmetical then $\{(x, n) : (x|n, n) \in A\}$ is Δ_1^1.

25.2. (i) Every arithmetical relation is Δ_1^1.
(ii) If $A \subset \mathcal{N} \times \mathcal{N}$ is arithmetical then $\exists x\, A$ is Σ_1^1 and $\forall x\, A$ is Π_1^1.

25.3. The set $A = \{(x, z) : z \notin \mathrm{WO} \vee \|x\| \leq \|z\|\}$ is Σ_1^1. Hence for each α, WO_α is $\Sigma_1^1(z)$ for each $z \in \mathrm{WO}$ such that $\|z\| = \alpha$.
[$(x, z) \in A \leftrightarrow z \notin \mathrm{WO} \vee (\exists h : \boldsymbol{N} \to \boldsymbol{N})\, \forall m \forall n\, (m\ E_x\ n \to h(m)\ E_x\ h(n))$.]

25.4. Every Σ_1 sentence is absolute for all inner models; in fact for all transitive models $M \supset L_\vartheta$ where $\vartheta = \omega_1^L$.
[Use Shoenfield's Absoluteness Lemma and Lemma 25.25.]

25.5. Modify the proof of Theorem 25.32 to show that Σ_2^1 has the prewellordering property.

25.6. Prove the prewellordering property of Σ_2^1 from the prewellordering property of Π_1^1.

A collection $\mathcal{C}$ of subsets of $\mathcal{N}$ satisfies the *reduction principle* if for every pair $A, B \in \mathcal{C}$ there are disjoint $A', B' \in \mathcal{C}$ such that $A' \subset A$, $B' \subset B$, and $A' \cup B' = A \cup B$. $\mathcal{C}$ satisfies the *separation principle* if for every pair of disjoint sets $A, B \in \mathcal{C}$ there is a set E such that both E and $\neg E$ are in $\mathcal{C}$, and that $A \subset E$ and $B \subset \neg E$. Lemma 11.11 proves that the collection of all analytic sets satisfies the separation principle.

25.7. The collection of Π_1^1 sets satisfies the reduction principle.
[Let φ and ψ be Π_1^1 norms on the Π_1^1 sets A and B and let $A' = \{x \in A : \psi(x) \not< \varphi(x)\}$ and $B' = \{x \in B : \varphi(x) \not\leq \psi(x)\}$.]

25.8. The collection of Σ_2^1 sets satisfies the reduction principle.

The two exercises above hold also for $\Pi_1^1(a)$ and $\Sigma_2^1(a)$.

25.9. If a collection $\mathcal{C}$ satisfies the reduction principle then the collection $\mathcal{C}^* = \{A : \neg A \in \mathcal{C}\}$ satisfies the separation principle.

[If $A, B \in \mathcal{C}^*$ are disjoint, then $\neg A \cup \neg B = \mathcal{N}^r$ and so if $A', B' \in \mathcal{C}$ are disjoint such that $A' \subset \neg A$, $B' \subset \neg B$ and $A' \cup B' = \neg A \cup \neg B$, then $B' = \neg A'$ and both A' and B' are in $\mathcal{C}^*$.]

Hence the separation principle holds for Σ^1_1 and for Π^1_2 (and $\Sigma^1_1(a)$ and $\Pi^1_2(a)$).

25.10. There is no universal $\mathbf{\Delta}^1_n$ set, for any $n \in \boldsymbol{N}$, i.e., no $D \subset \mathcal{N}^2$ such that D is $\mathbf{\Delta}^1_n$ and that for every $\mathbf{\Delta}^1_n$ set $A \subset \mathcal{N}$ there is $v \in \mathcal{N}$ such that $A = \{x : (x, v) \in D\}$.

[Assume there is such a D and let $A = \{x : (x, x) \notin D\}$.]

25.11. The collection of $\mathbf{\Pi}^1_1$ sets (or $\mathbf{\Sigma}^1_2$ sets) does not satisfy the separation principle.

[The reason is that $\mathbf{\Pi}^1_1$ satisfies the reduction principle ($\mathbf{\Sigma}^1_2$ is similar). Let h be a homeomorphism of $\mathcal{N} \times \mathcal{N}$ onto $\mathcal{N}$, and let $U \subset \mathcal{N}^2$ be a universal $\mathbf{\Pi}^1_1$ set. Let $(x, h(u, v)) \in A$ if and only if $(x, u) \in U$, $(x, h(u, v)) \in B$ if and only if $(x, v) \in V$, and let A', B' be disjoint $\mathbf{\Pi}^1_1$ sets such that $A' \subset A$, $B' \subset B$, and $A' \cup B' = A \cup B$. If there existed $E \in \mathbf{\Delta}^1_1$ such that $A' \subset E$ and $B' \subset \neg E$, then E would be a universal $\mathbf{\Delta}^1_1$ set.]

25.12. Modify the proof of Theorem 25.34 to show that Σ^1_2 has the scale property.

25.13. Prove the scale property of Σ^1_2 from the scale property of Π^1_1.

25.14. Let $\langle \varphi_n : n \in \omega \rangle$ be a scale on A and let T be the tree $\{(s, \langle \alpha_0, \dots, \alpha_{n-1} \rangle) : (\exists x \in A)\, x \restriction n = s \text{ and } \forall i < n\, \alpha_i = \varphi_i(x)\}$. Show that $A = p[T]$ and that for each $x \in A$, $T(x)$ has a least branch.

25.15. Using the scale property of Σ^1_2 prove the uniformization property of Σ^1_2.

Historical Notes

For classical descriptive set theory, see the books of Luzin [1930] and Kuratowski [1966]; the terminology is that of modern descriptive set theory based on the analogy with Kleene's hierarchies ([1955]).

The basic facts on $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ sets are all in Luzin's book [1930] and some are of earlier origin: Lemma 25.10 was in effect proved by Lebesgue in [1905], and Corollary 25.13 and Lemma 25.17 were proved by Luzin and Sierpiński in [1923].

Theorem 25.19 appeared in Sierpiński [1925]. Theorem 25.36 is due to Kondô [1939].

Theorem 25.20 is due to Shoenfield [1961]. Previously, Mostowski had established absoluteness of Σ^1_1 and Π^1_1 predicates (Theorem 25.4). Lemma 25.25: Lévy [1965b].

The tree representation of Σ^1_2 sets is implicit in Shoenfield's proof in [1961]. Lemma 25.22 is due to Kechris and Moschovakis [1972].

Theorem 25.23 is due to Mansfield [1970] and Solovay [1969]. Lemma 25.24 was formulated and first proved by Mansfield.

Theorem 25.26 and corollaries: In his announcement [1938] Gödel stated that the Axiom of Constructibility implies that there exists a nonmeasurable $\mathbf{\Delta}^1_2$ set and an uncountable $\mathbf{\Pi}^1_1$ set without a perfect subset. Gödel did not publish the proof but gave an outline in the second printing (in 1951) of his monograph [1940].

Novikov in [1951] gave a proof of the corollaries (Kuratowski's paper [1948] contains somewhat weaker results) and Addison in [1959] worked out the details of Gödel outline of the proof of the theorem.

Lemma 25.30: Solovay [1967].

For scales and uniformization, see Moschovakis' book [1980]. Moschovakis introduced scales in [1971].

Theorem 25.38: Solovay [1969].

Theorem 25.39: Mansfield [1975].

Theorem 25.42 (as well as the present proof) is due to Martin; and Theorem 25.43 is due to Kunen and Martin, the present proof is Kunen's.

Borel codes are as in Solovay [1970].

The reduction and separation principles were introduced by Kuratowski; they are discussed in detail in Kuratowski's book [1966] and in Addison [1959].

Exercise 25.7: Kuratowski [1936].

26. The Real Line

This chapter deals with some properties of the real line, primarily with questions concerning measure and category. Among others we present the theorem of Solovay establishing the consistency of the statement "every set of reals is Lebesgue measurable."

Random and Cohen reals

Let us consider generic extensions using either the algebra of Borel sets modulo the ideal of null sets or the algebra of Borel sets modulo the ideal of meager sets.

Let $\mathcal{B}$ be the σ-algebra of Borel sets of reals, let I_m and I_c (m for measure, c for category) be the σ-ideals

$$I_m = \{B \in \mathcal{B} : \mu(B) = 0\}, \qquad I_c = \{B \in \mathcal{B} : B \text{ is meager}\}$$

and let

$$\mathcal{B}_m = \mathcal{B}/I_m = \{[B]_m : B \in \mathcal{B}\}, \qquad \mathcal{B}_c = \mathcal{B}/I_c = \{[B]_c : B \in \mathcal{B}\} \tag{26.1}$$

where $[B]_m$ and $[B]_c$ denote equivalence classes mod I_m and mod I_c, respectively. $\mathcal{B}_m$ and $\mathcal{B}_c$ are complete Boolean algebras and if B_n, $n \in \omega$, are Borel sets then (in either $\mathcal{B}_m$ or $\mathcal{B}_c$),

$$\sum_{n=0}^{\infty} [B_n] = \Big[\bigcup_{n=0}^{\infty} B_n\Big].$$

(Also, $-[B] = [\boldsymbol{R} - B]$).

Forcing with $\mathcal{B}_c$ is the same as adjoining a Cohen generic real, see Exercise 26.1.

Let M be a transitive model of $\mathrm{ZF} + \mathrm{DC}$. Let us consider Borel sets in M; let $\mathcal{B}$ denote the collection of all Borel sets in M, and let $\mathcal{B}_m$ and $\mathcal{B}_c$ denote the complete Boolean algebras (26.1) in M.

Let B be a Borel set in M. B has a Borel code $c \in M$, $B = A_c$. Let us denote B^* the Borel set in the universe coded by c. This definition does not depend on the choice of $c \in \mathrm{BC}^M$ because by Lemma 25.45, if $A_c = A_d$, then $A_c^* = A_d^*$. We recall that $B = B^* \cap M$, for every $B \in \mathcal{B}$.

Lemma 26.1. *"A_c is null" and "A_c is meager" are properties absolute for all transitive models of* ZF + DC.

Proof. Let M be a transitive model of ZF + DC. Let μ denote the Lebesgue measure. First we claim that if $c \in M$ is a $\mathbf{\Sigma}^0_1$-code, then $\mu^M(A_c^M) = \mu(A_c)$. Let $k_0, k_1, \ldots, k_n, \ldots$ be all the $k \in \boldsymbol{N}$ such that $c(k) = 1$; thus A_c is the union $\bigcup_{n=0}^{\infty} I_{k_n}$ of open intervals with rational endpoints. For each n, let $X_n = I_{k_n} - (I_{k_0} \cup \ldots \cup I_{k_{n-1}})$; hence $A_c = \bigcup_{n=0}^{\infty} X_n$ and $\mu(A_c) = \sum_{n=0}^{\infty} \mu(X_n)$ is absolute. Hence $\mu^M(A_c^M) = \mu(A_c)$.

A similar argument shows that if $c \in M$ is a $\mathbf{\Pi}^0_1$-code, then $\mu^M(A_c^M) = \mu(A_c)$.

Next we claim that if $c \in M$ is a $\mathbf{\Pi}^0_1$-code, then A_c is nowhere dense if and only if $M \vDash A_c$ is nowhere dense. This is because $d = u(c) \in \Sigma_1$ and it is easily verified (using open rational intervals) that "A_d is dense" is absolute.

Now we are ready to prove the lemma. Let us consider first the property "A_c is null." We use the following properties of Lebesgue measure: (1) X is null if and only if for every $n \in \boldsymbol{N}$, there is an open set $G \supset X$ of measure $\leq 1/n$, and (2) $\mu(X) > 0$ if and only if there is a closed set $F \subset X$ of positive measure.

If $M \vDash A_c$ is null, then M satisfies

$$\forall n \, \exists e \, (e \in \Sigma_1 \text{ and } A_e \supset A_c \text{ and } \mu(A_e) \leq 1/n). \tag{26.2}$$

Since the part $(\ldots)$ of (26.2) is absolute, it is clear that (26.2) holds in V, and hence A_c is null.

If $M \vDash A_c$ is not null, then M satisfies

$$\exists e \, (e \in \Pi_1 \text{ and } A_e \subset A_c \text{ and } \mu(A_e) > 0). \tag{26.3}$$

Again, $(\ldots)$ is absolute, thus (26.3) holds in V and hence A_c is not null.

Finally, we consider the property "A_c is meager." If $M \vDash A_c$ is meager, then M satisfies:

(26.4) There exist $c_n \in \Pi_1$, $n = 0, 1, \ldots$, such that each A_{c_n} is nowhere dense, and $A_c \subset \bigcup_{n=0}^{\infty} A_{c_n}$.

Then (26.4) holds in V and so A_c is meager.

A Borel set B is not meager if and only if there is a nonempty open set G such that $B \triangle G$ is meager. Thus if $M \vDash A_c$ is not meager, then M satisfies

$$\exists d \, \exists e \, (d \in \Sigma_1 \text{ and } A_d \neq \emptyset \text{ and } A_e = A_c \triangle A_d \text{ and } A_e \text{ is meager}). \tag{26.5}$$

Then (26.5) holds in V and hence A_c is meager. □

As before, it is not necessary that the transitive models in Lemma 26.1 satisfy all of ZF. The properties are absolute for all adequate transitive models, in particular for all transitive models of $\mathrm{ZF}^- + \mathrm{DC}$.

Lemma 26.2.

(i) *If G is an M-generic ultrafilter on $\mathcal{B}_m$, then there is a unique real number x_G such that for all $B \in \mathcal{B}$,*

$$x_G \in B^* \leftrightarrow [B]_m \in G. \tag{26.6}$$

The formula (26.6) *determines G and hence $M[G] = M[x_G]$.*

(ii) *If G is an M-generic ultrafilter on $\mathcal{B}_c$, then there is a unique real number x_G such that for all $B \in \mathcal{B}$,*

$$x_G \in B^* \leftrightarrow [B]_c \in G. \tag{26.7}$$

The formula (26.7) *determines G and hence $M[G] = M[x_G]$.*

Definition 26.3. If x is a real number and if $x = x_G$ for some $G \subset \mathcal{B}_m$ generic over M, then x is *random* over M. If $x = x_G$ for some $G \subset \mathcal{B}_c$ generic over M, then x is *Cohen* over M.

Proof. The same proof works for both (i) and (ii); let $[B]$ denote $[B]_m$ in case (i) and $[B]_c$ in case (ii).

First we claim that there is at most one real number x that satisfies

$$x \in B^* \leftrightarrow [B] \in G \qquad (\text{for all } B \in \mathcal{B}). \tag{26.8}$$

If x satisfies (26.8), then x belongs to all B^* such that $[B] \in G$. If $x < y$ are two real numbers, let r be a rational number such that $x < r < y$, and let A be the interval $(r, \infty) = \{z \in \boldsymbol{R} : z > r\}$. Either $[A]$ or $[\boldsymbol{R} - A]$ belongs to G but $x \notin A^*$ and $y \notin (\boldsymbol{R} - A)^*$.

In order to show that there exists a real number x that satisfies (26.8), let

$$x = \sup\{r : r \text{ is a rational number and } [(r, \infty)] \in G\}. \tag{26.9}$$

By the genericity of G, there exists r such that $[(r, \infty)] \notin G$, and hence the supremum (26.9) exists. Note also that $x \notin M$ (by the genericity of G). We shall show that x satisfies (26.8). We shall show, by induction on Borel codes in M, that for every $c \in \mathrm{BC}^M$,

$$x \in A_c^* \leftrightarrow [A_c] \in G. \tag{26.10}$$

First we consider $\mathbf{\Sigma}^0_1$-codes (in M), and let us start with those $c \in \Sigma_1 \cap M$ that code a rational interval, i.e., such that $c(n) = 1$ for exactly one n; then c codes the interval I_n. Let $I_n = (p, q)$. We have

$$\begin{aligned} x \in A_c^* \quad &\text{if and only if} \quad p < x < q \\ &\text{if and only if} \quad p < \sup\{r : [(r, \infty)] \in G\} < q \\ &\text{if and only if} \quad [(p, \infty)] \in G \text{ and } [(q, \infty)] \notin G \\ &\text{if and only if} \quad [(p, q)] \in G \\ &\text{if and only if} \quad [A_c] \in G. \end{aligned}$$

Now if $c \in \Sigma_1$, then $A_c = \bigcup_{n=0}^{\infty} I_{k_n}$, where $\{k_n : n = 0, 1, \ldots\}$ is the set $\{k : c(k) = 1\}$, and we have

$$\begin{array}{lllll} x \in A_c^* & \text{if and only if} & x \in \bigcup_{n=0}^{\infty} I_{k_n}^* & \text{if and only if} & \exists n\,(x \in I_{k_n}^*) \\ & \text{if and only if} & \exists n\,([I_{k_n}] \in G) & \text{if and only if} & \sum_{n=0}^{\infty} [I_{k_n}] \in G \\ & \text{if and only if} & [\bigcup_{n=0}^{\infty} I_{k_n}] \in G & \text{if and only if} & [A_c] \in G. \end{array}$$

Next let $\alpha < \omega_1^M$ and let $c \in \Pi_\alpha \cap M$, and let us assume that (26.10) holds for all $c \in \Sigma_\alpha \cap M$. We may assume that $c(0) = 0$; then $u(c) \in \Sigma_\alpha \cap M$ and $A_{u(c)} = \boldsymbol{R} - A_c$, and we have

$$\begin{array}{lllll} x \in A_c^* & \text{if and only if} & x \notin A_{u(c)}^* & & \\ & \text{if and only if} & [A_{u(c)}] \notin G & \text{if and only if} & [A_c] \in G. \end{array}$$

Finally, the induction step for Σ_α is handled in a way similar to the case for $c \in \Sigma_1$. Thus (26.10) holds for every $c \in \mathrm{BC}^M$, and thus x is the unique real number that satisfies (26.6) (in case of $\mathcal{B}_m$) or (26.7) (in case of $\mathcal{B}_c$). □

The following lemma provides a characterization of random and Cohen reals.

Lemma 26.4. *A real number is random over M if and only if it does not belong to any null Borel set coded in M, and is Cohen over M if and only if it does not belong to any meager Borel set coded in M.*

Hence if $R(M)$ and $C(M)$ denote the sets of all random and all Cohen reals over M, we have

$$\begin{aligned} R(M) &= R^* - \bigcup\{A_c^* : c \in \mathrm{BC}^M \text{ and } A_c^* \text{ is null}\}, \\ C(M) &= R^* - \bigcup\{A_c^* : c \in \mathrm{BC}^M \text{ and } A_c^* \text{ is meager}\}. \end{aligned} \tag{26.11}$$

Note that by Lemma 26.1, A_c is null (in M) if and only if A_c^* is null (in V).

Proof. On the one hand, if x is random over M, let G be an M-generic ultrafilter on $\mathcal{B}_m$ such that $x = x_G$. Then if A_c is null then $[A_c] \notin G$, and by (26.6), $x \notin A_c^*$. Similarly for x that is Cohen over M.

On the other hand, let x be such that $x \notin A_c^*$ whenever A_c is null (and $c \in M$). First we observe that if $[A_c] = [A_d]$ then $A_c \mathbin{\triangle} A_d$ is null, hence $A_c^* \mathbin{\triangle} A_d^*$ is null and it follows that x belongs to A_c^* if and only if x belongs to A_d^*. Let

$$G = \{[A_c] : x \in A_c^*\}. \tag{26.12}$$

It is easy to see that G is a filter on $\mathcal{B}_m$: If $[A_c] \in G$ and $[A_d] \in G$, then $x \in A_c^* \cap A_d^*$ and hence $[A_c \cap A_d] \in G$; similarly, if $[A_c] \leq [A_d]$ and $[A_c] \in G$, then $[A_d] \in G$.

We shall show that G is M-generic. Since $\mathcal{B}_m$ satisfies the c.c.c., it suffices to show that if $\{A_{c_n} : n \in \omega\} \in M$ is such that $\sum_{n=0}^{\infty}[A_{c_n}] \in G$, then some $[A_{c_n}]$ is in G. But this is true because

$$\sum_{n=0}^{\infty}[A_{c_n}] = \Big[\bigcup_{n=0}^{\infty} A_{c_n}\Big] \quad \text{and} \quad \Big(\bigcup_{n=0}^{\infty} A_{c_n}\Big)^* = \bigcup_{n=0}^{\infty} A^*_{c_n}.$$

Finally, we claim that $x = x_G$. But this follows from (26.12), by the genericity of G. Thus a real number x is random over M if and only if $x \notin A_c^*$ for any null Borel set $A_c \in M$.

The proof is entirely similar for Cohen reals. □

Solovay Sets of Reals

Let M be a transitive model of ZFC. Let S be a set of reals. We say that the set S is *Solovay* over M if there is a formula $\varphi(x)$, with parameters in M, such that for all reals x,

$$x \in S \leftrightarrow M[x] \vDash \varphi(x). \tag{26.13}$$

Lemma 26.5. *Let S be a Solovay set of reals over M. There exist Borel sets A and B such that*

$$S \cap R(M) = A \cap R(M) \quad \textit{and} \quad S \cap C(M) = B \cap C(M).$$

Proof. Let us prove the lemma for random reals. Let us consider the forcing language in M associated with $\mathcal{B}_m$. Let $\dot{G}$ be the canonical name for a generic ultrafilter on $\mathcal{B}_m$, and let $\dot{a}$ be the canonical name for a random real; i.e., let $\dot{a}$ be the $\mathcal{B}_m$-valued name defined in $M^{\mathcal{B}_m}$ from $\dot{G}$, by (26.6): $\|\dot{a} = x_G\| = 1$.

Let $\varphi(x)$ be a formula with parameters in M such that (26.13) holds for all x. Let $A_c \in \mathcal{B}$ be such that $[A_c] = \|\varphi(\dot{a})\|$ and let $A = A_c^*$. The set A is a Borel set (in the universe); we claim that for all $x \in R(M)$, x belongs to S if and only if x belongs to A. But if x is random over M, let G be M-generic on $\mathcal{B}_m$ such that $x = x_G$; then $\dot{a}$ is a name for x and we have

$$x \in S \leftrightarrow M[x] \Vdash \varphi(x) \leftrightarrow M[G] \Vdash \varphi(x) \leftrightarrow \|\varphi(\dot{a})\| \in G \leftrightarrow [A_c] \in G \leftrightarrow x \in A_c^*.$$

□

Corollary 26.6. *Let S be a Solovay set of reals over M.*

(i) *If the set of all reals that are not random over M is null, then S is Lebesgue measurable.*
(ii) *If the set of all reals that are not Cohen over M is meager, then S has the property of Baire.*

Proof. Under the assumptions of the corollary, $S \triangle A$ is null and $S \triangle B$ is meager. □

The Lévy Collapse

We review properties of the forcing that collapses uncountable cardinals to $\aleph_0$, and establish the homogeneity of the Lévy collapse.

If λ is an infinite cardinal, let P_λ denote the set of all finite sequences

$$p = \langle p(0), \dots, p(n-1) \rangle \qquad (n \in \omega) \tag{26.14}$$

of ordinals less than λ and let $\mathrm{Col}(\aleph_0, \lambda) = B(P_\lambda)$.

The following lemma provides a characterization of the collapsing algebra:

Lemma 26.7. *Let $(Q, <)$ be a notion of forcing such that $|Q| = \lambda > \aleph_0$ and such that Q collapses λ onto $\aleph_0$, i.e.,*

$$\|\check{\lambda} \text{ is countable}\|_{B(Q)} = 1.$$

Then $B(Q) = \mathrm{Col}(\aleph_0, \lambda)$.

Proof. Without loss of generality we may assume that $(Q, <)$ is a separative partial ordering. We shall find a dense subset of Q isomorphic to P_λ.

Let $B = B(Q)$, and let $\dot{G}$ be the canonical name for the generic filter on Q. Let $\dot{f} \in V^B$ be such that

$$\|\dot{f} \text{ maps } \check{\omega} \text{ onto } \dot{G}\|_B = 1.$$

For each $p \in P_\lambda$, we shall construct $q(p) \in Q$ such that $D = \{q(p) : p \in P_\lambda\}$ is dense in Q and that $p \mapsto q(p)$ is an isomorphism of P_λ onto D. We construct $q(p)$ by induction on the length of p.

If $p = \langle p(0) \rangle$, we construct $q(p)$ as follows: Since Q collapses λ, there exists an antichain $W \subset Q$ of size λ. Moreover, we may find such W of size λ with the additional property that each $w \in W$ *decides* $\dot{f}(0)$, i.e., there is $q_w \in Q$ such that $w \Vdash \dot{f}(0) = \check{q}_w$. Thus let $W_\emptyset$ be a maximal antichain with these properties, $W_\emptyset = \{w_\xi : \xi < \lambda\}$, and for each $p = \langle p(0) \rangle \in P_\lambda$ we let $q(p) = w_\xi$, where $\xi = p(0)$.

Having constructed $q(p)$, where $p = \langle p(0), \dots, p(n-1) \rangle$, we construct $q(p^\frown \xi)$, $\xi < \lambda$, as follows: We let $W_p = \{w_\xi : \xi < \lambda\}$ be a maximal antichain below $q(p)$ such that $|W_p| = \lambda$ and that each $w \in W_p$ decides $\dot{f}(n)$. Then we let $q(p^\frown \xi) = w_\xi$, for all $w_\xi \in W_p$.

The set $D = \{q(p) : p \in P_\lambda\}$ is clearly isomorphic to P_λ. Let us show that D is dense in Q. Let $q \in Q$ be arbitrary. Since $q \Vdash \check{q} \in \dot{G}$, and $q \Vdash \check{q} \in \mathrm{ran}(\dot{f})$, there is $r \leq q$ and $n < \omega$ such that $r \Vdash q = \dot{f}(n)$. Now there is $p \in P_\lambda$ of length $n+1$ such that $q(p)$ is compatible with r; since $q(p)$ decides $\dot{f}(n)$, we necessarily have $q(p) \Vdash \dot{f}(n) = \check{q}$. Therefore, $q(p) \Vdash \check{q} \in \dot{G}$. Since Q is separative, it follows that $q(p) \leq q$. This proves that D is dense in Q. □

Corollary 26.8 (Kripke). *If B is a complete Boolean algebra and $|B| \leq \lambda$ then B embeds as a complete subalgebra of* $\mathrm{Col}(\aleph_0, \lambda)$.

Proof. Let B be a complete Boolean algebra, $|B| \leq \lambda$. The notion of forcing $Q = B^+ \times P_\lambda$ has cardinality λ and collapses λ. By Lemma 26.7, $B(Q) = \mathrm{Col}(\aleph_0, \lambda)$. In other words, $B \oplus \mathrm{Col}(\aleph_0, \lambda)$ is isomorphic to $\mathrm{Col}(\aleph_0, \lambda)$, and so B is isomorphic to a complete subalgebra of $\mathrm{Col}(\aleph_0, \lambda)$. □

Lemma 26.9. *Let B be a complete Boolean algebra, $|B| = \lambda$. Let C be a complete subalgebra of B such that $|C| < \lambda$, and let h_0 be an embedding of C in $\mathrm{Col}(\aleph_0, \lambda)$. Then there exists an embedding h of B in $\mathrm{Col}(\aleph_0, \lambda)$ such that $h(c) = h_0(c)$ for all $c \in C$.*

Proof. Let D be the image of C under the embedding h_0. Let Col be an abbreviation for $\mathrm{Col}(\aleph_0, \lambda)$; let Col^C and Col^D denote, respectively, the $(\aleph_0, \check{\lambda})$-collapsing algebra in the Boolean valued models V^C and V^D.

First, we find an embedding k of B in $C * \mathrm{Col}^C$: Working in V^C, we observe that $\check{\lambda}$ is a cardinal and that $B : C$ is a complete Boolean algebra that collapses $\check{\lambda}$ onto $\check{\aleph}_0$. Also, since $B : C$ is a quotient of $\check{B}$, $B : C$ has cardinality $\check{\lambda}$. Thus by Corollary 26.8 (in V^C), there is an embedding of $B : C$ in Col^C.

It follows that there is an embedding k of $C * (B : C)$ into $C * \mathrm{Col}^C$ such that $k(c) = c$ for all $c \in C$ (and C is considered a complete subalgebra of both those algebras). Since $C * (B : C) = B$, we have $k : B \to C * \mathrm{Col}^C$ such that $k(c) = c$ for all $c \in C$.

Next we find an isomorphism between Col and $D * \mathrm{Col}^D$: Working in V^D, we observe that $\check{\lambda}$ is a cardinal, and that $\mathrm{Col} : D$ collapses $\check{\lambda}$ onto $\check{\aleph}_0$. Also, since the algebra $\mathrm{Col}^\vee$ has a dense subset $\check{P}_\lambda$ of size $\check{\lambda}$, its quotient $\mathrm{Col} : D$ has a dense subset $\dot{Q}$ of size $\check{\lambda}$. Thus by Lemma 26.7 (in V^D), there is an isomorphism between $\mathrm{Col} : D$ and Col^D.

By the same argument as above, we get an isomorphism π between $\mathrm{Col} = D * (\mathrm{Col} : D)$ and $D * \mathrm{Col}^D$ such that $\pi(d) = d$ for all $d \in D$.

Since C and D are isomorphic, there exists an isomorphism $\sigma : C * \mathrm{Col}^C \to D * \mathrm{Col}^D$ such that $\sigma(c) = h_0(c)$ for all $c \in C$. Thus we define $h : B \to \mathrm{Col}$ as follows: $h(b) = \pi^{-1}(\sigma(k(b)))$, for all $b \in B$:

$$\begin{array}{ccc} C * \mathrm{Col}^C & \xrightarrow{\ \sigma\ } & D * \mathrm{Col}^D \\ {\scriptstyle k}\uparrow & & \uparrow{\scriptstyle \pi} \\ B & \xrightarrow{\ h\ } & \mathrm{Col} \end{array}$$

Clearly, h is an embedding of B into Col, and $h(c) = h_0(c)$ for all $c \in C$. □

Corollary 26.10. *Let G be a generic filter on P_λ and let X be a set of ordinals in $V[G]$. Then either $V[X] = V[G]$ or there exists a $V[X]$-generic filter H on P_λ such that $V[X][H] = V[G]$.*

Proof. If λ is uncountable in $V[X]$, then $V[G]$ is a generic extension of $V[X]$ by $\mathrm{Col}^{V[X]}(\aleph_0, \kappa)$, where $\kappa = |\lambda|^{V[X]}$. However, P_κ is isomorphic in $V[X]$

to P_λ. If λ is countable in $V[X]$ and $V[X] \neq V[G]$, then $V[G]$ is a generic extension of $V[X]$ by a countable atomless notion of forcing Q. There is only one atomless complete Boolean algebra with a countable dense subset and so $B(Q)$ is isomorphic (in $V[X]$) to $B(P_\lambda)$. □

We now consider the Lévy collapse $\mathrm{Col}(\aleph_0, <\lambda)$: Let λ be an inaccessible cardinal. The conditions are functions p on finite subsets of $\lambda \times \omega$ such that $p(\alpha, n) < \alpha$ whenever $(\alpha, n) \in \mathrm{dom}(p)$; p is stronger than q if $p \supset q$.

Corollary 26.11 (The Factor Lemma). *Let G be a generic filter on the Lévy collapse P, and let X be a countable set of ordinals in $V[G]$. Then there exists a $V[X]$-generic filter H on P such that $V[X][H] = V[G]$.*

Proof. For each $\nu < \lambda$ we have a decomposition of P into $P_\nu \times P^\nu$ where $P_\nu = \{p \in P : \mathrm{dom}\, p \subset \nu \times \omega\}$ and $P^\nu = \{p \in P : \mathrm{dom}\, p \subset (\lambda - \nu) \times \omega\}$. Note that if ν is an infinite cardinal then $|P_{\nu+1}| = \nu$ and so by Lemma 26.7 $B(P_{\nu+1}) = \mathrm{Col}(\aleph_0, \nu)$.

Let $\nu < \lambda$ be such that $X \in V[G \cap P_{\nu+1}]$. By Corollary 26.10 there exists a $K \subset P_{\nu+1}$ generic over $V[X]$ such that $V[G \cap P_{\nu+1}] = V[X][K]$. Hence $V[G] = V[X][K][G \cap P^{\nu+1}]$; let $H = K \times (G \cap P^{\nu+1})$. □

Theorem 26.12 (The Homogeneity of the Lévy Collapse). *Let $B = \mathrm{Col}(\aleph_0, <\lambda)$. If A and A' are isomorphic complete subalgebras of B such that $|A| = |A'| < |B|$ and if π_0 is an isomorphism between A and A', then there exists an automorphism π of B such that $\pi(a) = \pi_0(a)$ for all $a \in A$.*

Proof. First we construct increasing sequences of complete subalgebras $A_0 \subset A_1 \subset \ldots \subset A_n \subset \ldots$, and $A'_0 \subset A'_1 \subset \ldots \subset A'_n \subset \ldots$, as follows: We let $A_0 = A$ and $A'_0 = A'$. There is ν_1 such that $A'_0 \subset B_{\nu_1}$; we let $A'_1 = B_{\nu_1}$. The embedding π_0^{-1} of A'_0 in B can be extended to an embedding π_1^{-1} of A'_1 in B, and we let $A_1 = \pi_1^{-1}(A'_1)$. Then there is $\nu_2 > \nu_1$ such that $A_1 \subset B_{\nu_2}$; we let $A_2 = B_{\nu_2}$. Then $\pi_1 : A_1 \to B$ extends to some $\pi_2 : A_2 \to B$, and we let $A'_2 = \pi_2(A_2)$. We proceed in this manner.

Clearly, $\bigcup_{n=0}^\infty A_n = \bigcup_{n=0}^\infty A'_n = \bigcup_{n=1}^\infty B_{\nu_n}$, and $\bigcup_{n=0}^\infty \pi_n$ is an automorphism of this Boolean algebra. This automorphism extends to a unique automorphism π_ω of $B_\nu = B(P_\nu)$, where $\nu = \lim_{n\to\infty} \nu_n$.

Now $B = B_\nu \oplus B^\nu$ where $B^\nu = B(P^\nu)$, and the automorphism π_ω of B_ν can be extended to an automorphism π of $B_\nu \oplus B^\nu$ by $\pi(u, v) = (\pi_\omega u, v)$. □

Corollary 26.13. *If u and v are elements of $\mathrm{Col}(\aleph_0, <\lambda)$ such that $u \neq 0, 1$ and $v \neq 0, 1$, then there exists an automorphism π of B such that $\pi(u) = v$.* □

It follows that for any formula φ and all $x_1, \ldots, x_n$, $\|\varphi(\check{x}_1, \ldots, \check{x}_n)\|_B$ is either 1 or 0.

Solovay's Theorem

Theorem 26.14 (Solovay). *Assume that there exits an inaccessible cardinal.*

(i) *There is a model of* ZF + DC *in which all sets of real numbers are Lebesgue measurable and have the property of Baire, and every uncountable set of reals has a perfect subset.*
(ii) *There is a model of* ZFC *in which every projective set of reals is Lebesgue measurable, has the Baire property, and if uncountable, then it contains a perfect subset.*

Let M be a transitive model of ZFC and let κ be an inaccessible cardinal in M. Let B be the Lévy collapse for κ, i.e., $B = B(P)$ where P is the notion of forcing that collapses each $\alpha < \kappa$ onto $\aleph_0$: The conditions are functions p on subsets of $\kappa \times \omega$ such that each $\mathrm{dom}(p)$ is finite, and $p(\alpha, n) < \alpha$ whenever $(\alpha, n) \in \mathrm{dom}(p)$.

Let G be an M-generic ultrafilter on B. We shall show that in $M[G]$ every projective set of reals is Lebesgue measurable, has the property of Baire, and if uncountable, then it contains a perfect subset.

In $M[G]$, let S be the class of all infinite sequences of ordinal numbers, $S = Ord^\omega$, and let $N = HOD(S)$ be the class of all sets hereditarily ordinal definable over S. The class N is a model of ZF; in fact, N is a model of ZF + DC (see Lemma 26.15 below), and we shall show that in N every set of reals is Lebesgue measurable, has the Baire property, and if uncountable, then it contains a perfect subset.

Let s be an infinite sequence of ordinals in $M[G]$, let φ be a formula, and let $X \in M[G]$ be a set such that

$$X = \{x : M[G] \vDash \varphi(x, s)\}. \tag{26.15}$$

X is (in $M[G]$) ordinal definable over $S = Ord^\omega$. Conversely, if $X \in OD(S)$, then for some formula ψ and a finite sequence $\langle s_1, \dots, s_k \rangle$ of elements of S, $X = \{x \in M[G] : \psi(x, \langle s_1, \dots, s_k \rangle)\}$. Then clearly there exist φ and $s \in S$ such that (26.15) holds. Hence the class $OD(S)$ consists of all sets X of the form (26.15)—*sets definable in $M[G]$ from a sequence of ordinals.*

Note that every projective set of reals is definable from a sequence of ordinals: If A is $\Sigma^1_n(a)$ for some $a \in \mathcal{N}$, then A is definable in HC from a, and therefore $A \in OD(S)$.

Lemma 26.15.

(i) *If $f \in M[G]$ is a function on ω with values in N, then $f \in N$.*
(ii) *The model N satisfies the Principle of Dependent Choices.*

Proof. (i) We show that if f is a function from ω into $OD(S)$ then $f \in OD(S)$. By (26.15), $OD(S) = \bigcup\{OD(s) : s \in S\}$; therefore there is a definable

function F on $Ord \times S$ such that for each $s \in S$, the function $F_s(\alpha) = F(\alpha, s)$ maps Ord onto $OD(s)$. Let $f : \omega \to OD(S)$. For each n, we choose α_n and s_n such that $f(n) = F(\alpha_n, s_n)$. Clearly, f is definable from $\langle \alpha_n : n \in \omega \rangle$ and $\langle s_n : n \in \omega \rangle$. It is easy to find a single sequence u of ordinals such that both $\langle \alpha_n : n \in \omega \rangle$ and $\langle s_n : n \in \omega \rangle$ are definable from u. Hence f is definable from u, and so $f \in OD(S)$.

(ii) In N, let ρ be a relation over a nonempty set A such that for every $x \in A$ there is a y such that $y \,\rho\, x$. Since $M[G]$ satisfies the Axiom of Choice, there exists in $M[G]$ a sequence $a_0, a_1, \ldots, a_n, \ldots$ such that $a_{n+1} \,\rho\, a_n$ for all n. However, by part (i) of this lemma, the sequence $\langle a_n : n \in \omega \rangle$ is in N. □

We shall now prove the part of Theorem 26.14 dealing with Lebesgue measure and the Baire property, using Lemma 26.5.

Lemma 26.16. *Let $s \in M[G]$ be an infinite sequence of ordinals. The set of all reals (in $M[G]$) that are not random over $M[s]$ is null; the set of all reals that are not Cohen over $M[s]$ is meager.*

Proof. Since the algebra B is κ-saturated, there exists a subalgebra $D \subset B$ such that $|D| < \kappa$ and $M[s] = M[D \cap G]$. It follows that κ is inaccessible in $M[s]$; and since $\kappa = \aleph_1^{M[G]}$, $M[s]$ has only countably many subsets of ω. Thus there are only countably many Borel codes in $M[s]$; and by (26.11), the complement of the set $R(M[s])$ is the union of countably many null sets and hence null. Similarly, the complement of $C(M[s])$ is meager. □

Lemma 26.17. *Let $X \in M[G]$ be a set of reals that is definable in $M[G]$ from a sequence s of ordinals. Then X is (in $M[G]$) Solovay over $M[s]$.*

Proof. The proof uses the properties of the Lévy collapse discussed above, in particular the Factor Lemma. We shall first prove the following: Given a formula φ, there is a formula $\tilde{\varphi}$ such that for every sequence of ordinals $x \in M[G]$,

$$M[G] \vDash \varphi(x) \quad \text{if and only if} \quad M[x] \vDash \tilde{\varphi}(x). \tag{26.16}$$

The forcing conditions are finite and so the definition of the Lévy collapse P is absolute for all models. We denote M^P the Boolean valued model constructed in M using P and if ψ is a formula and $z \in M^P$, we denote $\|\psi(z)\|^M$ the Boolean value (computed in M using P) of $\psi(z)$. If $a \in M$, then $\check{a} \in M^P$ is the canonical name for a.

Let $\tilde{\varphi}(x)$ be the following formula

$$\|\varphi(\check{x})\|^{M[x]} = 1. \tag{26.17}$$

Let x be a countable sequence of ordinals in $M[G]$; we shall show that $M[G] \vDash \varphi(x)$ if and only if $M[x] \vDash \tilde{\varphi}(x)$. By the Factor Lemma there exists

an $M[x]$-generic filter H on P such that $M[G] = M[x][H]$. Arguing in $M[x]$, we invoke the homogeneity of the Lévy collapse: The Boolean value $b = \|\varphi(\check{x})\|^{M[x]}$ is either 0 or 1. Since H is generic on P over $M[x]$, $\varphi(x)$ is true in $M[x][H]$ if $b = 1$ and false if $b = 0$. Hence $\varphi(x)$ is true in $M[G]$ if and only if $\tilde{\varphi}(x)$ is true in $M[x]$.

For a formula φ with two variables there is a formula $\tilde{\varphi}$ such that for all $x, y \in M[G] \cap Ord^{\omega}$,

$$M[G] \vDash \varphi(x, y) \quad \text{if and only if} \quad M[x, y] \vDash \tilde{\varphi}(x, y).$$

Now let $X \in M[G]$ be a set of reals that is definable in $M[G]$ from a sequence of ordinals s. For some formula φ

$$x \in X \leftrightarrow M[G] \vDash \varphi(x, s)$$

for all reals $x \in M[G]$. Thus we have, for all $x \in \boldsymbol{R}^{M[G]}$,

$$x \in X \leftrightarrow M[s][x] \vDash \tilde{\varphi}(s, x)$$

which shows that X is Solovay over $M[s]$. □

Corollary 26.18. *In $M[G]$ every set of reals definable from a sequence of ordinals (and in particular, every projective set of reals) is Lebesgue measurable and has the property of Baire.*

Proof. This follows from Lemmas 26.5 26.16, and 26.17. □

Corollary 26.19. *In N, every set of reals is Lebesgue measurable and has the property of Baire.*

Proof. Clearly, the model N has the same reals as the model $M[G]$. In particular, N and $M[G]$ have the same Borel codes, and since $A_c^M = A_c^{M[G]} \cap N = A_c^{M[G]}$ for every $c \in \mathrm{BC}^{M[G]}$, the two models have the same Borel sets.

If $X \in N$ is a set of reals, then X is definable in $M[G]$ from a sequence of ordinals and hence $M[G] \vDash (X$ is Lebesgue measurable and has the Baire property). Thus there are (in $M[G]$) Borel sets A, B, H, K such that $X \triangle A \subset H$, $X \triangle B \subset K$, and H is null and K is meager (in $M[G]$). By Lemma 26.1, N satisfies that H is null and K is meager, and hence N satisfies that X is Lebesgue measurable and has the Baire property. □

We shall now finish the proof of Theorem 26.14 by showing that in $M[G]$ every uncountable set of reals definable from a countable sequence of ordinals contains a perfect subset. Then it follows that in N, every uncountable set A of reals has a perfect subset: If A is uncountable in N, then A is uncountable in $M[G]$ (by Lemma 26.15); and since A is definable from a sequence of ordinals, A has a perfect subset F (in $M[G]$); but then $N \vDash F$ is a perfect set.

By Lemma 26.17, every set of reals definable in $M[G]$ from s is Solovay over $M[s]$; thus it suffices to prove that in $M[G]$ every uncountable set of reals, Solovay over $M[s]$, contains a perfect subset. Furthermore, it suffices to give the proof only for sets of reals Solovay over M since the general case (Solovay over $M[s]$) follows from the special case by the Factor Lemma: $M[G] = M[s][H]$ is a generic extension of $M[s]$ by the Lévy collapse. And finally, we can consider subsets of the Cantor space instead of sets of reals.

Thus let A be, in $M[G]$, an uncountable subset of $\{0,1\}^\omega$, and let φ be a formula (with parameters in M) such that for all $x \in \{0,1\}^\omega$ in $M[G]$,

$$x \in A \quad \text{if and only if} \quad M[x] \vDash \varphi(x).$$

Since A is uncountable, there exists an $x \in A$ such that $x \notin M$. There exists (in M) a complete subalgebra $C \subset B$ such that $|C| < \kappa$ and that $x \in M[G \cap C]$. Let us consider the Boolean-valued model M^C and the corresponding forcing relation $\Vdash$. There exists a name $\dot{x} \in M^C$ and a condition $p \in C \cap G$ such that

$$p \Vdash \dot{x} \in \{0,1\}^\omega \text{ and } \dot{x} \notin M \text{ and } (M[\dot{x}] \vDash \varphi(\dot{x})). \tag{26.18}$$

Since $P^M(C)$ is countable in $M[G]$, let D_0, D_1, …, D_n, … be an enumeration (in $M[G]$) of all open dense subsets of C in M.

We shall construct conditions $p_s \in C$, for all finite 0–1 sequences s, as follows:

Let $p_\emptyset \le p$ be such that $p_\emptyset \in D_0$. Given p_s, there exists $n_s \in \omega$ such that p_s does not decide $\dot{x}(n_s)$ (because $p \Vdash \dot{x} \notin M$), and we let $p_{s^\frown 0}$ and $p_{s^\frown 1}$ be such that $p_{s^\frown 0} \Vdash \dot{x}(n_s) = 0$ and $p_{s^\frown 1} \Vdash \dot{x}(n_s) = 1$; moreover, we choose $p_{s^\frown 0}$ and $p_{s^\frown 1}$ so that both are in the open dense set D_k where k is the length of s.

For every $z \in \{0,1\}^\omega$, let $G_z = \{p \in C : p \ge p_s \text{ for some } s \subset z\}$. Clearly, $G_z \cap D_n \ne \emptyset$ for every n, and hence G_z is an M-generic ultrafilter on C. Let $f(z) = \dot{x}^{G_z}$ be the interpretation of $\dot{x}$ by G_z. Since G_z is generic, and by (26.18), we have $f(z) \in A$. Thus f is a function from $\{0,1\}^\omega$ into A.

It follows from the construction of f that f is one-to-one and continuous. Thus $f(\{0,1\}^\omega)$, the one-to-one continuous image of a perfect compact set, is a perfect subset of A. □

Lebesgue Measurability of Σ^1_2 Sets

Lemma 26.5 and its Corollary 26.6 provide the following equivalences:

Theorem 26.20 (Solovay). *Let $a \in \mathcal{N}$.*

(i) *Every $\Sigma^1_2(a)$ set of reals is Lebesgue measurable if and only if almost all reals are random over $L[a]$.*

(ii) *Every $\Sigma^1_2(a)$ set of reals has the Baire property if and only if the set $\{x : x$ is not a Cohen real over $L[a]\}$ is meager.*

Proof. We prove only part (i) as part (ii) is similar.

First we note that every $\Sigma^1_2(a)$ set is Solovay over $L[a]$: Let A be $\Sigma^1_2(a)$, and let $T \in L[a]$ be a tree on $\omega \times \omega_1$ such that for all $x \in \mathcal{N}$,

$$x \in A \quad \text{if and only if} \quad T(x) \text{ is ill-founded.}$$

By absoluteness of well-foundedness we have

$$x \in A \quad \text{if and only if} \quad L[a][x] \vDash T(x) \text{ is ill-founded,}$$

and hence A is Solovay over $L[a]$.

If almost all reals are random over $L[a]$ then every $\Sigma^1_2(a)$ set is Lebesgue measurable by Corollary 26.18.

Thus assume that every $\Sigma^1_2[a]$ set is Lebesgue measurable; we shall prove that the union

$$B = \bigcup\{A_c : c \in \mathrm{BC},\ c \in L[a] \text{ and } A_c \text{ is null}\}$$

of all null Borel sets coded in $L[a]$ is null. Let

$$C(x, c) \leftrightarrow c \in \mathrm{BC} \wedge A_c \text{ is null} \wedge x \in A_c,$$
$$D(x, c) \leftrightarrow C(x, c) \wedge c \in L[a] \wedge \forall d\,(d <_{L[a]} c \to \neg C(x, d)),$$

and for $x, y \in B$,

$$x \preccurlyeq y \leftrightarrow \exists c\, \exists d\, (D(x, c) \wedge D(x, d) \wedge c \leq_{L[a]} d).$$

The set B as well as the relations C, D and $\preccurlyeq$ are $\Sigma^1_2(a)$, and $\preccurlyeq$ is a prewellordering of B. Under the assumption of Lebesgue measurability of $\Sigma^1_2(a)$ sets, B is Lebesgue measurable and $\preccurlyeq$ is a measurable subset of $\mathcal{N} \times \mathcal{N}$.

The order-type of $\mathcal{N}$ in $<_{L[a]}$ is $\omega_1^{L[a]} \leq \omega_1$. Hence for every $y \in B$, the set $\{x : x \preccurlyeq y\}$ is a countable union of null sets and therefore null. Thus $\preccurlyeq$ is a null set, by Fubini's Theorem. By the same argument, the complement of $\preccurlyeq$ in $B \times B$ is null as well, and hence $B \times B$ is null. Therefore B is a null set. □

Corollary 26.21. *If $\omega_1^{L[a]} < \omega_1$, then every $\Sigma^1_2(a)$ set of reals is Lebesgue measurable and has the Baire property.*

Proof. Under the assumption, each $L[a]$ has only countably many reals and hence only countably many Borel codes, and it follows that almost all reals are random over $L[a]$. Similarly for Cohen reals. □

Ramsey Sets of Reals and Mathias Forcing

For an infinite set $A \subset \omega$, let $[A]^\omega$ denote the set of all infinite subsets of A. Let us consider the following partition property for $[\omega]^\omega$: If $S \subset [\omega]^\omega$, we call an infinite set $H \subset \omega$ *homogeneous* for S if either $[H]^\omega \subset S$ or $[H]^\omega \cap S = \emptyset$. A set $S \subset [\omega]^\omega$ is a *Ramsey* set if there exists an infinite homogeneous set H for S.

A consequence of the Axiom of Choice is that not every set $S \subset [\omega]^\omega$ is Ramsey (Exercise 26.3). We prove that the Axiom of Choice is necessary, and that all analytic sets are Ramsey.

Identifying subsets of ω with their characteristic functions, we consider $[\omega]^\omega$ as a G_δ subspace of the Cantor space. We prove the following theorems:

Theorem 26.22 (Galvin-Prikry, Silver). *Every analytic subset of* $[\omega]^\omega$ *is Ramsey.*

Theorem 26.23 (Mathias). *Let* $M[G]$ *and* N *be the models from Theorem* 26.14.

(i) *In* N, *every subset of* $[\omega]^\omega$ *is Ramsey.*
(ii) *In* $M[G]$, *every projective subset of* $[\omega]^\omega$ *is Ramsey.*

The method of proof of both theorems uses a notion of forcing introduced by Mathias, and a topology based on the Mathias forcing.

Definition 26.24 (Mathias Forcing). A forcing condition is a pair (s, A) where s is a finite subset of ω and A is an infinite subset of ω such that $\max s < \min A$. A condition (s, A) is stronger than a condition (t, B) if

(26.19) (i) t is an initial segment of s;
(ii) $A \subset B$;
(iii) $s - t \subset B$.

(Compare this with the Prikry forcing (21.15).) For the rest of this section, (s, A) will denote a Mathias forcing condition.

For $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, let $A \setminus s = A - (\max(s)+1) = \{n \in A : n > k \text{ for all } k \in s\}$, and

$$[s, A]^\omega = \{X \in [\omega]^\omega : s \subset X \text{ and } X \setminus s \subset A\}. \tag{26.20}$$

Note that $[\emptyset, A]^\omega = [A]^\omega$, and $[s, A]^\omega \subset [t, B]^\omega$ if and only if (s, A) is stronger than (t, B).

Definition 26.25. The *Ellentuck topology* on $[\omega]^\omega$ has as basic open sets the sets of the form $[s, A]^\omega$ where $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$.

Note that every open set in the usual topology is open in the Ellentuck topology.

Definition 26.26 (Galvin-Prikry).

(i) A set $S \subset [\omega]^\omega$ is *completely Ramsey* if for every (s, A) there exists an infinite $H \subset A$ such that either $[s, H]^\omega \subset S$ or $[s, H]^\omega \cap S = \emptyset$.
(ii) A set $N \subset [\omega]^\omega$ is *Ramsey null* if for every (s, A) there exists an infinite $H \subset A$ such that $[s, H]^\omega \cap S = \emptyset$.

We first observe that every Ramsey null set is nowhere dense in the Ellentuck topology: S is nowhere dense if and only if for every basic open set there exists a basic open subset disjoint from S, i.e.,

$$\forall (s, A)\, \exists (t, B) < (s, A)\, [t, B]^\omega \cap S = \emptyset.$$

Let S be completely Ramsey, let $\mathrm{int}(S)$ denote the interior of S (in the Ellentuck topology), and let $N = S - \mathrm{int}(S)$. For every (s, A) there exists an $H \subset A$ such that either $[s, H]^\omega \subset S$, and since $[s, H]^\omega$ is open, we have $[s, H]^\omega \subset \mathrm{int}(S)$; or $[s, H]^\omega \cap S = \emptyset$, and in either case $[s, H]^\omega \cap N = \emptyset$. Hence N is Ramsey null, and therefore nowhere dense. It follows that $S = \mathrm{int}(S) \cup N$ has the Baire property.

We shall prove the following (for the Ellentuck topology):

Lemma 26.27.

(i) *A set S is completely Ramsey if and only if it has the Baire property.*
(ii) *A set N is Ramsey null if and only if it is nowhere dense if and only if it is meager.*

Toward the proof of Lemma 26.27, let S be a given subset of $[\omega]^\omega$. Given (s, A) we say that A *accepts* s if $[s, A]^\omega \subset S$; we say that A *rejects* s if no $X \subset A$ accepts s.

Lemma 26.28. *There is an X that accepts or rejects each of its finite subset.*

Proof. Let X_0 be such that X_0 either accepts or rejects $\emptyset$ (if no X accepts $\emptyset$ then $X_0 = \omega$ rejects $\emptyset$). Let a_0 be the least element of X_0. Let $X_1 \subset X$ be such that X_1 either accepts or rejects each subset of $\{a_0\}$. Let a_1 be the least element of $X_1 \setminus \{a_0\}$, and let $X_2 \subset X_1$ be such that X_2 accepts or rejects each subset of $\{a_0, a_1\}$. We continue in this fashion and construct a set $X = \{a_0, a_1, a_2, \ldots\}$. This X accepts or rejects each of its finite subsets. □

Lemma 26.29. *There is a Y that either accepts $\emptyset$ or rejects each of its finite subsets.*

Proof. Let X be as in Lemma 26.28, and assume that it rejects $\emptyset$. We construct $Y = \{a_0, a_1, \ldots\} \subset X$ as follows: Assume we have constructed $a_0, \ldots, a_{n-1}$ such that X rejects each subset of $\{a_0, \ldots, a_{n-1}\}$. For every $s \subset \{a_0, \ldots, a_{n-1}\}$ there are only finitely many $z \in X$ such that X accepts

$s \cup \{z\}$ (otherwise there is an infinite $Z \subset X$ such that X accepts $s \cup \{z\}$ for each $z \in Z$; then Z accepts s and hence X does not reject s). Therefore we can find $a_n \in X \setminus \{a_0, \ldots, a_{n-1}\}$ such that X rejects each subset of $\{a_0, \ldots, a_n\}$. □

Lemma 26.30. *Every open set is Ramsey.*

Proof. Let S be open, and let X be as in Lemma 26.29. If X accepts $\emptyset$ then $[X]^\omega = [\emptyset, X]^\omega \subset S$.

If X rejects each of its finite subsets, we claim that $[X]^\omega \cap S = \emptyset$. Otherwise, there is an infinite $Y \subset X$ such that $Y \in S$. Since S is open, there is an open neighborhood of Y included in S; i.e., there exists a finite $s \subset Y$ such that $[s, Y \setminus s]^\omega \subset S$. Hence Y accepts s, contrary to the assumption that X rejects s. □

Lemma 26.31. *Every open set is completely Ramsey.*

Proof. Let S be open and let (s, A) be arbitrary. Let $f : \omega \to A$ be a one-to-one increasing enumeration of A, and for each $X \in [\omega]^\omega$, let $f^*(X) = s \cup f\text{“}X$. The function f^* is a continuous function from $[\omega]^\omega$ into $[\omega]^\omega$. Let $T = \{X : f^*(X) \in S\}$; T is open and hence Ramsey. If K is a homogeneous set for T, then $H = f\text{“}K$ satisfies either $[s, H]^\omega \subset S$ or $[s, H]^\omega \cap S = \emptyset$. □

Lemma 26.32. *Every nowhere dense set is Ramsey null.*

Proof. Let S be nowhere dense; we may also assume that S is closed. Let (s, A) be arbitrary. By Lemma 26.31 there is an $H \subset A$ such that either $[s, H]^\omega \subset S$ or $[s, H]^\omega \cap S = \emptyset$. But $[s, H]^\omega \subset S$ is impossible since S is nowhere dense. □

Lemma 26.33. *If $S = \bigcup_{n=0}^{\infty} S_n$ and each S_n is Ramsey null then S is Ramsey null.*

Proof. Let (s, A) be arbitrary. We construct an infinite $H = \{a_0, a_1, a_2, \ldots\} \subset A$ as follows: Let $X_0 \subset A$ be such that $[s, X_0]^\omega \cap S_0 = \emptyset$, and let a_0 be the least element of X_0. Find an $X_1 \subset X_0 \setminus \{a_0\}$ such that for every t with $s \subset t \subset s \cup \{a_0\}$, $[t, X_1]^\omega \subset S_1$. Having constructed $X_0 \supset X_1 \supset \ldots \supset X_n$, let $a_n = \min(X_n)$ and find an $X_{n+1} \subset X_n \setminus \{a_0, \ldots, a_n\}$ such that for every t with $s \subset t \subset s \cup \{a_0, \ldots, a_n\}$, $[t, X_{n+1}]^\omega \subset S_{n+1}$. It follows that $[s, H]^\omega \cap S = \emptyset$. □

Proof of Lemma 26.27. By Lemmas 26.32 and 26.33, every meager set is Ramsey null, proving (ii). To prove (i), let S be a set with the Baire property; we have $S = G \triangle M$ where G is open and M is meager. Let (s, A) be arbitrary. By (ii) there is some $X \subset A$ such that $[s, X]^\omega \cap M = \emptyset$. By Lemma 26.31 there is some $H \subset X$ such that either $[s, H]^\omega \subset G$ or $[s, H]^\omega \cap G = \emptyset$. It follows that either $[s, H]^\omega \subset S$ or $[s, H]^\omega \cap S = \emptyset$. □

Proof of Theorem 26.22. Every analytic set (in the usual topology) is the result of the Suslin operation $\mathcal{A}$ applied to closed sets. Every closed set is closed in the Ellentuck topology and therefore has the Baire property (in the Ellentuck topology). It can be proved (as in Theorem 11.18) that the Baire property in the Ellentuck topology is preserved under the operation $\mathcal{A}$. Hence every analytic set is completely Ramsey, by Lemma 26.27(i). □

The combinatorial content of Lemma 26.27 is this property of Mathias forcing (compare with Lemma 21.12):

Lemma 26.34. *Let σ be a sentence of the forcing language and let (s, A) be a condition. Then there exists an infinite set $B \subset A$ such that (s, B) decides σ.*

Proof. Let $Q^+ = \{p : p \Vdash \sigma\}$, $Q^- = \{p : p \Vdash \neg\sigma\}$, $S^+ = \bigcup\{[t, X]^\omega : (t, X) \in Q^+\}$ and $S^- = \bigcup\{[t, X]^\omega : (t, X) \in Q^-\}$. Since the complement of $S^+ \cup S^-$ is nowhere dense, there exists, by Lemma 26.27, an infinite $B \subset A$ such that $[s, B]^\omega \subset S^+$ or $[s, B]^\omega \subset S^-$. We claim that in the former case $(s, B) \Vdash \sigma$ and in the latter case $(s, B) \Vdash \neg\sigma$. This is because for every $(t, X) < (s, B)$ there exists some $(u, Y) < (t, X)$ which is in Q^+ (or Q^-). □

If G is a generic filter on the Mathias forcing (over a ground model M), let x_G be the infinite set

$$x_G = \bigcup\{s : (s, A) \in G \text{ for some } A\}; \tag{26.21}$$

x_G is called a *Mathias real* (over M). The filter G is determined by $x = x_G$, as

$$G = G_x = \{(s, A) : s \subset x \subset s \cup A\}. \tag{26.22}$$

Mathias reals admit the following characterization, analogous to Theorem 21.14:

Theorem 26.35 (Mathias). *Let M be a transitive model of* ZFC. *An infinite set $x \subset \omega$ is a Mathias real over M if and only if for every maximal almost disjoint family $\mathcal{A} \in M$ of subsets of ω, there exists an $X \in \mathcal{A}$ such that $x - X$ is finite.*

Proof. The condition is necessary: If $\mathcal{A}$ is a maximal almost disjoint family then $D = \{(s, A \setminus s) : s \in [\omega]^{<\omega},\ A \in \mathcal{A}\}$ is a predense set of forcing conditions, and it follows that if x is a Mathias real then $G_x \cap D \neq \emptyset$.

For the proof of sufficiency, let D be an open dense set of Mathias forcing conditions (in the ground model). We need a more detailed analysis of Mathias forcing. If $X \subset \omega$ is infinite and $\max s < \min X$ we say that X *captures* (s, D) if for every infinite $Y \subset X$ there exists an initial segment t of Y such that $(s \cup t, X) \in D$.

Lemma 26.36. *For every infinite set $A \subset \omega$ and for every finite $s \subset \omega$ there exists an infinite set $X \subset A \setminus s$ such that X captures (s, D).*

Proof. We construct a sequence $Y_0 \supset Y_1 \supset \ldots \supset Y_n \supset \ldots$ of infinite sets and a sequence $m_0 < m_1 < \ldots < m_n < \ldots$ such that $m_n = \min Y_n$, as follows: Let $Y_0 = A \setminus s$. Given Y_n, we can find $Y_{n+1} \subset Y_n \setminus \{m_n\}$ with the property that for every $t \subset \{m_0, \ldots, m_n\}$, if there exists a $Y \subset Y_n$ such that $(s \cup t, Y) \in D$, then $(s \cup t, Y_{n+1}) \in D$ (we use the fact that D is an open set of conditions).

Let $Y = \{m_0, m_1, \ldots, m_n, \ldots\}$. As the set $U = \bigcup\{[t, S]^\omega : (t, S) \in D\}$ is a dense open subset of $[\omega]^\omega$ (in the Ellentuck topology) it follows from Lemma 26.27(ii) that there exists an infinite set $X \subset Y$ such that $[s, X]^\omega \subset U$. We claim that X captures (s, D).

If $Z \subset X$ is infinite then because $s \cup Z \in U$, there exist an initial segment t of Z and an infinite $S \subset \omega$ such that $(s \cup t, S) \in D$ and $s \cup Z \in [s \cup t, S]^\omega$. It follows that $(s \cup t, Z \setminus t) \in D$, and if $\max t = m_n$, we have $(s \cup t, Y_{n+1}) \in D$. It follows that $(s \cup t, X \setminus t) \in D$. □

Lemma 26.37. *For every infinite $A \subset \omega$ there exists an $X \subset A$ such that for every s, $X \setminus s$ captures (s, D).*

Proof. By Lemma 26.36 there exist sets $X_s \subset A$ such that for each s, X_s captures (s, D). We construct $X_0 \supset X_1 \supset \ldots \supset X_n \supset \ldots$ and $m_0 < m_1 < \ldots < m_n < \ldots$ such that $m_n = \min X_n$, as follows: Let $X_0 = X_\emptyset$. Given X_n, we find an X_{n+1} such that for every s with $\max s = m_n$, X_{n+1} captures (s, D) (here we use the fact that if X captures and $X' \subset X$, then X' also captures). Let $X = \{m_0, m_1, \ldots, m_n, \ldots\}$. It follows that $X \setminus s$ captures (s, D) for every s. □

We now finish the proof of Theorem 26.35. Let $x \subset \omega$ be infinite and assume that for every maximal almost disjoint $\mathcal{A} \in M$ there exists an $X \in \mathcal{A}$ such that $x - X$ is finite. By Lemma 26.37 there exists (in M) a maximal almost disjoint family $\mathcal{A}$ such that for every $X \in \mathcal{A}$ and every s, $X \setminus s$ captures (s, D). Let $X \in \mathcal{A}$ be such that $x - X$ is finite, and let s be an initial segment of x such that $x \subset s \cup X$. As $X \setminus s$ captures (s, D), we have (in M)

$$(26.23) \quad \forall \text{ infinite } Y \subset X \setminus s \ \exists \text{ initial segment } t \subset Y \text{ such that } (s \cup t, X \setminus t) \in D.$$

Consider the set of finite sets $W = \{t \subset X \setminus s : (s \cup t, X \setminus t) \notin D\}$ partially ordered by the relation $t \preccurlyeq t'$ if and only if t' is an initial segment of t. Then (26.23) states that $(W, \prec)$ is well-founded in M. By absoluteness, $(W, \prec)$ is well-founded in any larger universe, and so (26.23) holds in any $V \supset M$. In particular, letting $Y = x \setminus s$, we obtain an initial segment t of $x \setminus s$ such that $(s \cup t, X \setminus t) \in D$, and since $s \cup t$ is an initial segment of x and $x \subset s \cup t \cup X$, the filter G_x from (26.22) meets D. Since D was an arbitrary open dense set in M, G_x is generic, and x is a Mathias real over M. □

Corollary 26.38. *If x is a Mathias real over M and $y \subset x$ is infinite, then y is Mathias over M.*

Proof of Theorem 26.23. Let $M[G]$ be a generic extension of M by the Lévy collapse. We shall prove that every set of reals in $M[G]$ that is definable from a countable sequence of ordinals is Ramsey. Thus let u be a countable sequence of ordinals in $M[G]$ and let $X \in M[G]$ be a subset of $[\omega]^\omega$ definable from u. By Lemma 26.17 X is a Solovay set over $M[u]$ and so for some formula φ,

$$x \in X \quad \text{if and only if} \quad M[u, x] \vDash \varphi(x)$$

for all $x \in [\omega]^\omega \cap M[G]$.

Let us consider the Mathias forcing in $M[u]$, and let $\dot{x}$ be the canonical name for a Mathias generic real. By Lemma 26.34 there exists an infinite set $A \in M[u]$ such that $(\emptyset, A)$ decides $\varphi(\dot{x})$. Assume that $(\emptyset, A) \Vdash \varphi(\dot{x})$ as the other case is similar.

Since $\aleph_1^{M[G]}$ is inaccessible in $M[u]$, there exists a Mathias generic filter in $M[G]$ containing $(\emptyset, A)$; therefore there exists a Mathias real x over $M[u]$ such that $x \subset A$. We complete the proof by verifying that $[x]^\omega \subset X$.

If y is an infinite subset of x then by Corollary 26.38, y is a Mathias real over $M[u]$. Since $y \subset A$ and $(\emptyset, A) \Vdash \varphi(\dot{x})$, we have $M[u][y] \vDash \varphi(y)$ and so $y \in X$. □

Measure and Category

Lebesgue measure and Baire property have been the most thoroughly investigated properties of sets of reals, both in the classical descriptive set theory, and in the modern era of independence results. We shall touch briefly on the subject, with emphasis on the role of Martin's Axiom and combinatorial "cardinal invariants." We start with the following application of Martin's Axiom:

Theorem 26.39 (Martin-Solovay). *If Martin's Axiom holds, then the union of fewer than $2^{\aleph_0}$ null sets is null, and the union of fewer than $2^{\aleph_0}$ meager sets is meager.*

Proof. First we prove that the union of fewer than $2^{\aleph_0}$ null sets is mull. Let $\kappa < 2^{\aleph_0}$ and let A_α, $\alpha < \kappa$, be null sets of reals. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$. In order to prove that A is null, it suffices to find, for each $\varepsilon > 0$, an open set $U \supset A$ such that $\mu(U) \leq \varepsilon$. Let $\varepsilon > 0$.

We apply Martin's Axiom as follows. Let P be the set of all open sets of measure $< \varepsilon$, and let $p \in P$ be stronger than $q \in P$ if $p \supset q$. We claim that the notion of forcing $(P, \supset)$ satisfies the countable chain condition.

It suffices to show that if W is an uncountable subset of P, then there are $p, q \in W$, $p \neq q$, such that $\mu(p \cup q) < \varepsilon$. Let S be the countable set of

all unions of finitely many open intervals with rational endpoints. If $W \subset P$ is uncountable, then there exist an $n \in \boldsymbol{N}$ and an uncountable $Z \subset W$ such that $\mu(p) < \varepsilon - 1/n$ for all $p \in Z$. For each $p \in Z$, let $p^* \in S$ be such that $p^* \subset p$ and $\mu(p - p^*) < 1/n$. Since S is countable, there exist $p, q \in Z$, $p \neq q$, such that $p^* = q^*$. Then $\mu(p \cup q) < \varepsilon$.

For each $\alpha < \kappa$, let $D_\alpha = \{p \in P : A_\alpha \subset p\}$. Each D_α is a dense subset of P: If $p \in P$, then since A_α is null, there exists an open set $q \supset A_\alpha$ such that $\mu(p) + \mu(q) < \varepsilon$, and hence $p \cup q \in D_\alpha$ and $p \cup q \supset p$.

By Martin's Axiom, there exists a filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Let $U = \bigcup\{p : p \in G\}$. It is clear that $A \subset U$ and it remains to show that $\mu(U) \leq \varepsilon$. We use the well-known fact (easy to verify) that if $U = \bigcup\{p : p \in G\}$, then there is a countable $H \subset G$ such that $U = \bigcup\{p : p \in H\}$. Thus if $\mu(U) > \varepsilon$, there exist $p_1, \ldots, p_n \in H$ such that $\mu(p_1 \cup \ldots \cup p_n) > \varepsilon$. But this is impossible: Since G is a filter on P, we have $p \cup q \in G$ whenever $p \in G$ and $q \in G$; thus $p_1 \cup \ldots \cup p_n \in G$ and hence $\mu(p_1 \cup \ldots \cup p_n) < \varepsilon$.

This completes the proof that the union of $< 2^{\aleph_0}$ null sets is null if MA holds.

In order to show that the union of less than $2^{\aleph_0}$ meager sets is meager, it suffices to show that the union of less than $2^{\aleph_0}$ closed nowhere dense sets is meager. The following lemma will complete the proof:

Lemma 26.40. *Assume Martin's Axiom. Let $\kappa < 2^{\aleph_0}$ and let A_α, $\alpha < \kappa$, be closed nowhere dense sets of reals. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$. Then there exists a countable family of dense open sets H_i, $i = 0, 1, 2, \ldots$, such that A is disjoint from $\bigcap_{i=0}^{\infty} H_i$.*

Proof. We apply Martin's Axiom as follows. Let P be the set of all finite sequences of pairs

$$p = \langle (U_0, E_0), (U_1, E_1), \ldots, (U_n, E_n) \rangle$$

such that

(26.24)
- (i) each U_i is the union of finitely many open intervals with rational endpoints;
- (ii) each E_i is a finite subset of κ; and
- (iii) for each i, U_i is disjoint from $\bigcup_{\alpha \in E_i} A_\alpha$.

A condition $p' = \langle (U'_0, E'_0), \ldots, (U'_m, E'_m) \rangle$ is stronger than a condition $p = \langle (U_0, E_0), \ldots, (U_n, E_n) \rangle$ if

(26.25)
- (i) $m \geq n$; and
- (ii) for each $i \leq n$, $U'_i \supset U_i$ and $E'_i \supset E_i$.

It is clear that this notion of forcing satisfies the countable chain condition: If two conditions have the same $U_0, \ldots, U_n$, then they are compatible, and there are only countably many sequences $\langle U_0, \ldots, U_n \rangle$.

Let I_k, $k = 0, 1, 2, \ldots$, be an enumeration of all open intervals with rational endpoints. We let, for each $\alpha < \kappa$ and all $i, k = 0, 1, 2, \ldots$,

$$D_\alpha = \{p : p = \langle (U_0, E_0), \ldots, (U_n, E_n)\rangle \text{ and } \alpha \in E_i \text{ for some } i \le n\},$$
$$E_{i,k} = \{p : p = \langle (U_0, E_0), \ldots, (U_n, E_n)\rangle \text{ and } U_i \cap I_k \neq \emptyset\}. \tag{26.26}$$

Since each A_α, $\alpha < \kappa$, is nowhere dense, it is clear that for all i an k, every condition can be extended to a condition $p \in E_{i,k}$, and hence each $E_{i,k}$ is dense in P. Also, each D_α is dense in P.

By Martin's Axiom, there exists a filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$, and $G \cap E_{i,k} \neq \emptyset$ for all $i, k \in \omega$. For each $i = 0, 1, 2, \ldots$, we let

$$H_i = \bigcup\{U_i : (\exists p \in G)\, p = \langle \ldots, (U_i, E_i), \ldots\rangle\}.$$

Since $E_{i,k}$ is a dense set of conditions, for all k, H_i is a dense open set of reals.

Now if $\alpha < \kappa$, then because D_α is dense, there exists $i \in \omega$ such that H_i is disjoint from A_α, and hence A_α is disjoint from $\bigcap_{i=0}^\infty H_i$. Therefore A is disjoint from $\bigcap_{i=0}^\infty H_i$. □

Corollary 26.41. *If* MA *holds, then both the algebra of Lebesgue measurable sets and the algebra of sets with the Baire property are $2^{\aleph_0}$-complete, and moreover, Lebesgue measure is $2^{\aleph_0}$-additive, i.e., if $\kappa < 2^{\aleph_0}$ and A_α, $\alpha < \kappa$, are pairwise disjoint, then*

$$\mu\Big(\bigcup_{\alpha<\kappa} A_\alpha\Big) = \sum_{\alpha<\kappa} \mu(A_\alpha). \tag{26.27}$$

Proof. We prove by induction on $\kappa < 2^{\aleph_0}$ that if A_α, $\alpha < \kappa$, are Lebesgue measurable, then $A = \bigcup_{\alpha<\kappa} A_\alpha$ is Lebesgue measurable. Given A_α, $\alpha < \kappa$, let $B_\alpha = A_\alpha - \bigcup_{\beta<\alpha} A_\beta$, for each $\alpha < \kappa$. The sets B_α are Lebesgue measurable (by the induction hypothesis), and being pairwise disjoint, all but countably many are null. It follows from the theorem that $A = \bigcup_{\alpha<\kappa} B_\alpha$ is Lebesgue measurable. The same argument proves (26.27) (see also Lemma 10.6), and the property of Baire is analogous. □

Corollary 26.42. *If* MA *holds and if $2^{\aleph_0} > \aleph_1$, then every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable and has the property of Baire.*

Proof. If A is $\Sigma^1_2(a)$, then since $(2^{\aleph_0})^{L[a]} = \aleph_1^{L[a]} \le \aleph_1$, the set of all reals that are not random over $L[a]$ is the union of at most $\aleph_1$ null sets, hence null (by Theorem 26.39). By Theorem 26.20, A is Lebesgue measurable. The Baire property is analogous. □

The proof of Theorem 26.39 yields a slightly better result: It shows that for every $\kappa \le 2^{\aleph_0}$, MA_κ implies the κ-additivity of the ideals of null and meager sets. The study of additivity of measure and category initiated by

Theorem 26.39 developed into an extensive theory that established a detailed relationship between various properties of measure and category. We refer the reader to Bartoszyński's article [∞] in the Handbook of Set Theory.

Definition 26.43. (i) Additivity:

$\mathrm{add}(\mathrm{LM})$ = the least cardinal κ such that the union of some family of κ null sets is not null,

$\mathrm{add}(\mathrm{BP})$ = the least cardinal κ such that the union of some family of κ meager sets is not meager.

(ii) Covering:

$\mathrm{cov}(\mathrm{LM})$ = the least cardinal κ for which $\boldsymbol{R}$ is the union of κ null sets,

$\mathrm{cov}(\mathrm{BP})$ = the least cardinal κ for which $\boldsymbol{R}$ is the union of κ meager sets.

(iii) Uniformity:

$\mathrm{unif}(\mathrm{LM})$ = the least cardinal κ such that there exists a set of cardinality κ that is not null,

$\mathrm{unif}(\mathrm{BP})$ = the least cardinal κ such that there exists a set of cardinality κ that is not meager.

(iv) Cofinality:

$\mathrm{cof}(\mathrm{LM})$ = the least cardinality of a family $\mathcal{F}$ of null sets such that every null set is included in a set from $\mathcal{F}$,

$\mathrm{cof}(\mathrm{BP})$ = the least cardinality of a family $\mathcal{F}$ of meager sets such that every meager set is included in a set from $\mathcal{F}$.

The proof of Theorem 26.39 shows that MA_κ implies $\mathrm{add}(\mathrm{LM}) \geq \kappa$ and $\mathrm{add}(\mathrm{BP}) \geq \kappa$. In a series of results a complete picture of inequalities emerged between these properties. First, it is obvious that add $\leq$ cov $\leq$ cof and add $\leq$ unif $\leq$ cof, both for measure and category (Exercise 26.5). Secondly, two of the inequalities have been known classically; see Exercise 26.7.

Before we proceed we introduce two cardinal invariants that are not only relevant in this context but appear frequently in results in set-theoretic topology. First some notation:

$$\begin{array}{lll} \forall^\infty & \text{means} & \text{for all but finitely many } n \in \omega, \\ \exists^\infty & \text{means} & \text{for infinitely many } n \in \omega. \end{array}$$

A family $F \subset \omega^\omega$ is a *dominating* family if

$$\forall g \in \omega^\omega \, \exists f \in F \, \forall^\infty n \, g(n) < f(n); \tag{26.28}$$

F us an *unbounded* family if

$$\forall g \in \omega^\omega \, \exists f \in F \, \exists^\infty n \, g(n) \leq f(n). \tag{26.29}$$

Definition 26.44. The *dominating* number

$$\mathfrak{d} = \text{the least cardinality of a dominating family;}$$

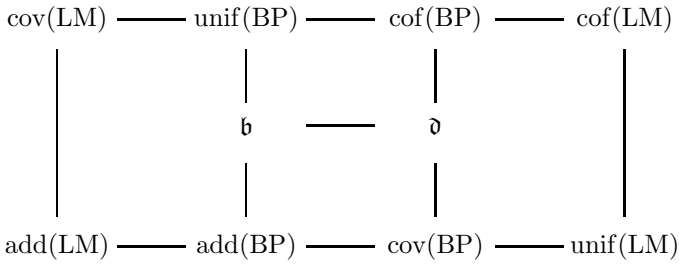
the *bounding* number

$$\mathfrak{b} = \text{the least cardinality of an unbounded family.}$$

It is clear that $\mathfrak{b} \leq \mathfrak{d}$, and $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$. Martin's Axiom implies that $\mathfrak{b} = \mathfrak{d} = \mathfrak{c}$; see Exercise 26.8.

To see the significance of $\mathfrak{b}$ and $\mathfrak{d}$ for Baire category, notice that for each $f \in \omega^\omega$ and each k, the set $\{g \in \omega^\omega : \forall n \geq k\ g(n) < f(n)\}$ is nowhere dense (in the space $\mathcal{N}$). Hence if $g < f$ means $\forall^\infty n\ g(n) < f(n)$, each set $\{g : g < f\}$ is meager, and it follows that $\mathfrak{b} \leq \text{unif(BP)}$ and $\text{cov(BP)} \leq \mathfrak{d}$; see Exercise 26.10.

The relationship between the invariants defined in 26.43 and 26.44 can be illustrated by the following diagram:



The cardinals become larger as one moves right and up. Exercises 26.5, 26.7, and 26.10 give proofs of the easy inequalities. The remaining inequalities are given by these theorems (that we state without proofs):

Theorem 26.45 ((i) Truss, Miller; (ii) Fremlin).

(i) $\text{add(BP)} = \min\{\mathfrak{b}, \text{cov(BP)}\}$.
(ii) $\text{cof(BP)} = \max\{\mathfrak{d}, \text{unif(BP)}\}$.

Theorem 26.46 (Bartoszyński, Raisonnier-Stern).

(i) $\text{add(LM)} \leq \text{add(BP)}$.
(ii) $\text{cof(BP)} \leq \text{cof(LM)}$.

It is no accident that each result is accompanied by a dual version: There is a general theory that explains this duality. (For details, see Bartoszyński's Handbook article.) For instance, consider Theorem 26.46. Both (i) and (ii) can be proved from this general result (see Exercise 26.11):

Theorem 26.47 (Pawlikowski). *Let I_c and I_m be the ideal of all meager sets and the ideal of null sets. There exists a function $\varphi : I_c \to I_m$ with the property that for every family $\mathcal{F} \subset I_m$, if $\bigcup \mathcal{F}$ is null then $\bigcup \varphi^{-1}(\mathcal{F})$ is meager.* □

A significant part of the theory of invariants of measure and category is the characterization of invariants in terms of functions from ω to ω. The cardinals cov(BP) and unif(BP) were so described first by Miller, with the final form due to Bartoszyński:

Theorem 26.48. (i) cov(BP) *is the least cardinality of a family $F \subset \omega^\omega$ such that*

$$\forall g \in \omega^\omega \, \exists f \in F \, \forall^\infty n \, f(n) \neq g(n). \tag{26.30}$$

(ii) unif(BP) *is the least cardinality of a family $F \subset \omega^\omega$ such that*

$$\forall g \in \omega^\omega \, \exists f \in F \, \exists^\infty n \, f(n) = g(n). \tag{26.31}$$

□

For the easy direction of (i) and (ii) see Exercise 26.12.

The key ingredient of Theorem 26.46(i) is the following characterization of add(LM):

Theorem 26.49. add(LM) *is the least cardinality of a family $F \subset \omega^\omega$ such that*

$$\forall \varphi \in S \, \exists f \in F \, \exists^\infty n \, f(n) \notin \varphi(n), \tag{26.32}$$

where S is the set of all functions $\varphi : \omega \to [\omega]^{<\omega}$ such that $|\varphi(n)| = n$ for all n. □

See Exercise 26.13 for the proof of add(LM) $\leq$ cov(BP).

The diagram, along with Theorem 26.45, gives a complete relationship among these invariants. Nothing else can be proved in ZFC, and models have been constructed verifying all independence results based on the diagram.

We conclude this chapter with two of the earliest independence results concerning measure and category. We give an example of a model where $2^{\aleph_0}$ is large and the set $\boldsymbol{R} \cap L$ is not Lebesgue measurable, and another example where $2^{\aleph_0}$ is large and $\boldsymbol{R} \cap L$ does not have the Baire property.

Lemma 26.50. *If there exists a nonconstructible real, then*:

(i) $\boldsymbol{R} \cap L$ *is either null or not Lebesgue measurable.*
(ii) $\boldsymbol{R} \cap L$ *is either meager or does not have the Baire property.*

Proof. (i) Let S be the set of all constructible reals in the unit interval $[0, 1]$. Let a be a nonconstructible real. For each $n > 0$, let $S_n = \{x + (a/n) : x \in S\}$. The sets S_n are pairwise disjoint, $\mu(S_n) = \mu(S)$ for all n and $\bigcup_{n=0}^{\infty} S_n$ is a bounded set. Therefore if S is measurable, then $\mu(S) > 0$ is impossible.

(ii) Let S be the set of all constructible reals. First we prove that $\boldsymbol{R} - S$ is not meager. Let a be a nonconstructible real and let $S_a = \{x + a : x \in S\}$; clearly, $S \cap S_a = \emptyset$. Thus $\boldsymbol{R} = (\boldsymbol{R} - S) \cup (\boldsymbol{R} - S_a) \cup (S \cap S_a) = (\boldsymbol{R} - S) \cup (\boldsymbol{R} - S_a)$, and if $\boldsymbol{R} - S$ were meager, then $\boldsymbol{R} - S_a$ would also be meager, a contradiction.

It follows that for any nonempty interval I, $I - S$ is not meager: For each rational r, let $A_r = \{x + r : x \in I - S\}$; if $I - S$ is meager, then each A_r is meager, and $\boldsymbol{R} - S = \bigcup\{A_r : r \text{ is rational}\}$.

If S has the Baire property, then because $U - S$ is not meager for any nonempty open set U, S is meager. □

Example 26.51 (A model where $2^{\aleph_0} > \aleph_1$ and the set of all constructible reals is not Lebesgue measurable). Let λ be a regular uncountable cardinal and let B be the following measure algebra: Let $(S, \mathcal{F}, m)$ be the product measure space, where S is the product of $\lambda \times \omega$ copies of $\{0, 1\}$, $\mathcal{F}$ is the least σ-complete field of subsets of S containing all $\{t \in S : t(\alpha, n) = 0\}$, and m is the product measure, and let B be the measure algebra $B = \mathcal{F}/\text{sets of measure 0}$.

Let us consider the generic extension of the constructible universe by the measure algebra B. The generic extension $L[G]$ satisfies $2^{\aleph_0} = \lambda$. We shall show that in $L[G]$ the set of all constructible reals is not Lebesgue measurable.

In view of Lemma 26.50, it suffices to show that the set of all constructible reals is not null. Thus assume that it is null and let I_k, $k = 0, 1, 2, \ldots$, be an enumeration (in L) of all intervals with rational endpoints. Let μ denote Lebesgue measure.

Assuming that $L[G] \vDash \mu(\boldsymbol{R} \cap L) = 0$, there is a B-valued name $\dot{X}$ for a subset of ω, and a rational $\varepsilon > 0$ such that the Boolean value

(26.33) $\| \bigcup\{I_k : k \in \dot{X}\}$ contains all constructible reals, and has Lebesgue measure $\leq \varepsilon \|$

is in G. We may assume, without loss of generality, that the Boolean value (26.33) is 1.

For each $k \in \boldsymbol{N}$, let $A_k \in \mathcal{F}$ be such that $\|k \in \dot{X}\| = [A_k]$. Let us consider (in L) the product measure space $(\boldsymbol{R}, \mu) \times (S, \mathcal{F}, m)$ with the product measure $\nu = \mu \times m$. Let $E \subset \boldsymbol{R} \times S$ be the set

$$E = \bigcup_{k=0}^{\infty} (I_k \times A_k).$$

We claim that $\nu(E) \leq \varepsilon$. It suffices to show that $\nu(\bigcup_{k<k_0}(I_k \times A_k)) \leq \varepsilon$ for every k_0. Let $k_0 \in \boldsymbol{N}$. By (26.33), for every condition $a = [A]$ there exist a stronger condition $c = [C]$ and a set $Y \subset k_0$ such that

$$c \Vdash \dot{X} \cap \check{k}_0 = \dot{Y}$$

and that $\mu(\bigcup_{k \in Y} I_k) \leq \varepsilon$. Clearly, $[C] \leq [A_k]$ if $k \in Y$, and $[C] \cdot [A_k] = 0$ if $k \in k_0 - Y$ and hence

$$\nu\Big(\bigcup_{k<k_0} I_k \times (A_k \cap C)\Big) = \mu\Big(\bigcup_{k \in Y} I_k\Big) \cdot m(C) \leq \varepsilon \cdot m(C). \tag{26.34}$$

Thus the set of all $[C]$ for which (26.34) holds is dense in the algebra B and hence

$$\nu\Big(\bigcup_{k<k_0} I_k \times A_k\Big) \leq \varepsilon.$$

Since $\nu(E) \leq \varepsilon$, the complement of E has positive measure and hence there exists, by Fubini's Theorem, a number $x \in \boldsymbol{R}$ such that

$$m(\{t \in S : (x,t) \notin E\}) > 0.$$

It follows that there exists $A \in \mathcal{F}$ of positive measure such that

$$(x,t) \notin E \qquad \text{for all } t \in A. \tag{26.35}$$

We shall show that

$$[A] \Vdash x \notin \bigcup\{I_k : k \in \dot{X}\}, \tag{26.36}$$

completing the proof.

If (26.36) were not true, there would exist some $k \in \boldsymbol{N}$ and some $C \subset A$ of positive measure such that $x \in I_k$ and $[C] \Vdash k \in \dot{X}$. But then $[C] \leq [A_k]$ and hence there is some $t \in C$ such that $(x,t) \in E$, contrary to (26.35). □

Example 26.52 (A model where $2^{\aleph_0} > \aleph_1$ and the set of all constructible reals does not have the property of Baire). Let λ be a regular uncountable ordinal and let P be the notion of forcing that adjoins λ Cohen reals: A condition is a finite 0–1 function whose domain is a subset of λ.

Let us consider the generic extension of the constructible universe by P. In $L[G]$, $2^{\aleph_0} = \lambda$. We shall show that in $L[G]$ the set of all constructible reals does not have the Baire property.

In view of Lemma 26.50, it suffices to show that the set of all constructible reals is not meager. For every $S \subset \lambda$ (in L), let $P_S = \{p \in P : \mathrm{dom}(p) \subset S\}$, and let $G_S = G \cap P_S$.

Lemma 26.53. *If $L[G] \vDash \boldsymbol{R} \cap L$ is meager, then there exists a countable $S \subset \lambda$ (in L) such that $L[G_S] \vDash \boldsymbol{R} \cap L$ is meager.*

Proof. Let I_k, $k \in \boldsymbol{N}$, be an enumeration of all open intervals with rational endpoints. If $L[G] \vDash \boldsymbol{R} \cap L$ is meager, then there exists a sequence $\langle U_n : n \in \boldsymbol{N}\rangle \in L[G]$ such that for every $n \in \boldsymbol{N}$, $L[G] \vDash U_n$ is dense open, and that $\boldsymbol{R} \cap L \subset \bigcap_{n=0}^{\infty}(\boldsymbol{R} - U_n)$. Let $A = \{(n,k) : I_k \subset U_n\}$, and let $\dot{A}$ be a name for A. Since P satisfies the countable chain condition, there is a countable $S \subset \lambda$ such that $\dot{A}$ is P_S-valued. It is easy to verify that if $U'_n = U_n \cap L[G_S]$, then for each $n \in \boldsymbol{N}$, $L[G_S] \vDash U'_n$ is dense open, and that $\boldsymbol{R} \cap L \subset \bigcap_{n=0}^{\infty}(\boldsymbol{R} - U'_n)$. Thus $L[G_S] \vDash \boldsymbol{R} \cap L$ is meager. □

Since P_S is countable, it suffices to prove the following lemma:

Lemma 26.54. *If P is a countable notion of forcing in L and if G is an L-generic filter on P, then $L[G] \vDash \boldsymbol{R} \cap L$ is not meager.*

Proof. It suffices to show that if $\langle U_n : n \in \boldsymbol{N}\rangle$ is (in $L[G]$) a sequence of dense open sets of reals, then there is a constructible real a such that $a \in \bigcap_{n=0}^{\infty} U_n$. Let $\dot{U}_n$ be a name for U_n and let us assume, without less of generality, that every condition forces that each $\dot{U}_n$ is dense open. It is enough to find (in L) a real number x such that for each n and each $p \in P$, there is a $q \leq p$ such that $q \Vdash x \in \dot{U}_n$.

Let t_k, $k = 0, 1, \ldots$, be an enumeration of all pairs $t = (n, p)$ where $n \in \boldsymbol{N}$ and $p \in P$. Let us construct a sequence $I_0 \supset I_1 \supset \ldots \supset I_k \supset \ldots$ of closed bounded intervals as follows: Let $t_k = (n, p)$. Since $p \Vdash \dot{U}_n$ is dense open, there is an open interval $J \subset I_{k-1}$ with rational endpoints such that some $q \leq p$ forces $J \subset \dot{U}_n$. Let $I_k \subset J$. The intersection $\bigcap_{k=0}^{\infty} I_k$ is nonempty; and if x is in it, then for each n and each $p \in P$ there is $q \leq p$ such that $q \Vdash x \in \dot{U}_n$. □

Exercises

26.1. The algebra $\mathcal{B}_c$ is the unique atomless complete Boolean algebra that has a countable dense subset.

[If B is a meager Borel set, then there is a nonempty open set U such that $U \triangle B$ is meager; hence there is a rational interval I such that $[I]_c \leq [B]_c$.]

26.2. Every $\Delta^1_2(a)$ set of reals is Lebesgue measurable if and only if there exists a random real over $L[a]$. Every $\Delta^1_2(a)$ set of reals has the Baire property if and only if there exists a Cohen real over $L[a]$.

[If there are no random reals over $L[a]$ then the prewellordering $\preccurlyeq$ in the proof of Theorem 26.20 is $\Delta^1_2(a)$.

Assume that there is a random real over L, and let A be a Δ^1_2 set, $A = \{x : P(x)\} = \{x : \neg Q(x)\}$ with P and Q being Σ^1_2. In L, force with Borel sets mod measure 0, and let $\dot{r}$ be a name for the random real. Show that the set

$$D = \{p : p \Vdash P^{L[\dot{r}]}(\dot{r}) \text{ or } p \Vdash Q^{L[\dot{r}]}(\dot{r})\}$$

is dense, and let $W \subset D$ be a (countable) maximal antichain. Let

$$Z_P = \bigcup\{A_c : c \in \mathrm{BC} \cap L,\ A^L_c \in W \text{ and } A^L_c \Vdash P^{L[\dot{r}]}(\dot{r})\},$$

and Z_Q similarly. Show that $Z_P - P$ and $Z_Q - Q$ are null and conclude that $Z_P \triangle A$ is null. (For details, see Judah and Shelah [1989] or Theorem 14.6 in Kanamori [1994]).]

26.3. For every infinite $X \subset \omega$, let X^* be a chosen representative of the class of all $Y \subset \omega$ such that $X \triangle Y$ is finite. Show that the set

$$S = \{X \in [\omega]^\omega : |X \triangle X^*| \text{ is even}\}$$

is not Ramsey.

26.4. "Every clopen set is Ramsey" implies Ramsey's Theorem.

26.5. (i) $\mathrm{add(LM)} \le \mathrm{cov(LM)} \le \mathrm{cof(LM)}$, $\mathrm{add(LM)} \le \mathrm{unif(LM)} \le \mathrm{cof(LM)}$.
(ii) $\mathrm{add(BP)} \le \mathrm{cov(BP)} \le \mathrm{cof(BP)}$, $\mathrm{add(BP)} \le \mathrm{unif(BP)} \le \mathrm{cof(BP)}$.

26.6. There exists a decomposition $\boldsymbol{R} = M \cup N$ into a meager set M and a null set N.

26.7. (i) $\mathrm{cov(LM)} \le \mathrm{unif(BP)}$.
(ii) $\mathrm{cov(BP)} \le \mathrm{unif(LM)}$.
[Let $\boldsymbol{R} = M \cup N$ where M is meager and N is null. To prove (i) it suffices to show that if X is a nonmeager set then $\boldsymbol{R} = \bigcup\{N + x : x \in X\}$. By contradiction, assume that some r is not of the form $z + x$ where $z \in N$, and $x \in X$. It follows that $(X - r) \cap \{-z : z \in N\} = \emptyset$, hence $X - r \subset \{-z : z \in M\}$ and so X is meager.]

26.8. MA_κ implies $\mathfrak{b} \ge \kappa$.
[Let $\lambda < \kappa$ and let $\{f_\alpha : \alpha < \lambda\} \subset \omega^\omega$. A forcing condition is a pair (s, E) where s is a finite sequence in ω and F is a finite subset of λ; (s, E) is stronger than (t, F) if $s \supset t$ and $(\forall \alpha \in F)(\forall n \in \mathrm{dom}(s) - \mathrm{dom}(t))\, s(n) > f_\alpha(n)$. This forcing is c.c.c. and every $D_\xi = \{(s, E) : \alpha \in E\}$ is dense. MA_κ produces a function g such that $\forall^\infty n\, f_\alpha(n) < g(n)$ for all $\alpha < \lambda$.]

26.9. $\mathfrak{b} \le \mathrm{cf}(\mathfrak{d})$.
[Find a dominating family $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{d}\}$ such that whenever $\alpha < \beta$ then $\exists^\infty n\, f_\alpha(n) < f_\beta(n)$. If $\{\alpha_\nu : \nu < \mathrm{cf}\,\mathfrak{d}\}$ is cofinal in $\mathfrak{d}$ then $\{f_{\alpha_\nu} : \nu < \mathrm{cf}\,\mathfrak{d}\}$ is an unbounded family.]

26.10. $\mathfrak{b} \le \mathrm{unif(BP)}$ and $\mathrm{cov(BP)} \le \mathfrak{d}$.
[If $F \subset \omega^\omega$ is not meager then F is an unbounded family. If F is a dominating family, then $\omega^\omega = \bigcup_{f \in F}\{g : g < f\}$.]

26.11. Using Theorem 26.47, show that $\mathrm{add(LM)} \le \mathrm{add(BP)}$ and $\mathrm{cof(BP)} \le \mathrm{cof(LM)}$.
[Let $\psi : I_m \to I_c$ be as follows: For each $X \in I_m$ let $\psi(X) = \bigcup\{Z : \varphi(Z) \subset X\}$. If F is a family of fewer than add(LM) meager sets, let X be the null set $\bigcup\{\varphi(Z) : Z \in F\}$. Then $\bigcup F \subset \psi(X)$ is meager. If $F \subset I_m$ generates I_m then $\{\psi(X) : X \in F\}$ generates I_c.]

26.12. (i) If F satisfies (26.30) and has size κ then $\mathcal{N}$ is the union of κ meager sets.
(ii) If F is not meager then it satisfies (26.31).
[For every f, the set $\{g : \forall^\infty n\, f(n) \ne g(n)\}$ is meager.]

26.13. Use Theorems 26.49 and 26.48(i) to verify $\mathrm{add(LM)} \le \mathrm{cov(BP)}$.
[Let $\kappa < \mathrm{add(LM)}$ and let $F \subset \omega^\omega$ be such that $|F| = \kappa$. Let I_n be pairwise disjoint subsets of ω, $|I_n| = n$. Apply (26.32) to the family $\{f' : f \in F\}$, where $f'(n) = f{\restriction}I_n$, to find a φ such that $\forall f \in F\, \forall^\infty n\, f{\restriction}I_n \in \varphi(n)$. Now let $g : \omega \to \omega$ be as follows: If a is the kth element of I_n, let $g(a) = s_k(a)$ where s_k is the kth element of $\varphi(n)$. For every $f \in F$ we have $\exists^\infty n\, g(n) = f(n)$, contradicting (26.30); hence $\kappa < \mathrm{cov(BP)}$.]

A set of reals A has *strong measure* 0 if for every sequence $a_0 \ge a_1 \ge \ldots \ge a_n \ge \ldots$ of positive reals, there exists a sequence of open intervals I_n, $n = 0, 1, \ldots$, such that $\mathrm{length}(I_n) \le a_n$ and $A \subset \bigcup_{n=0}^\infty I_n$. It is clear that every set of strong measure 0 is null, but not every null set has necessarily strong measure 0:

26.14. The Cantor set does not have strong measure 0.
[$\boldsymbol{C}$ cannot be covered by open intervals of lengths $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, ...]

26.15. If $A \subset \boldsymbol{R}$ contains a perfect subset, then it does not have strong measure zero.
[Use the fact that A contains a subset homeomorphic to the Cantor set and that a uniformly continuous image of a set of strong measure 0 has strong measure 0.]

26.16. Martin's Axiom implies that every set $A \subset \boldsymbol{R}$ of size $< 2^{\aleph_0}$ has strong measure 0.
[May assume that $2^{\aleph_0} > \aleph_1$. Consider the forcing notion $(P, <) = (Seq, \supset)$ which adjoins a Cohen generic $x \in \omega^\omega$. Let $a_0 \geq a_1 \geq \ldots \geq a_n \geq \ldots$ be positive reals. Since $|A| < 2^{\aleph_0}$, MA implies that there exists $x \in \omega^\omega$, P-generic over every $L[a, \langle a_n : n \in \omega\rangle]$, $a \in A$. Let r_0, r_1, ..., r_n, ... be an enumeration of all rational numbers; let for each n, I_n be the interval with center $r_{x(n)}$ and diameter a_n. Use the genericity of x to show that $a \in \bigcup_{n=0}^{\infty} I_n$, for each $a \in A$.]

26.17. If $2^{\aleph_0} = \aleph_1$, then there exists an uncountable set $E \subset \boldsymbol{R}$ such that for every nowhere dense set F, $E \cap F$ is at most countable. (E is called a *Luzin set*.) More generally, MA implies that there is a set F of size $2^{\aleph_0}$ whose intersection with every nowhere dense set has size $< 2^{\aleph_0}$.
[Let F_0, F_1, ..., F_α, ..., $\alpha < 2^{\aleph_0}$, be an enumeration of all closed nowhere dense sets. Let $E = \{e_\alpha : \alpha < 2^{\aleph_0}\}$, where for each α, $e_\alpha \notin \bigcup_{\beta<\alpha} F_\beta$. Each e_α exists; the MA case uses Theorem 26.39.]

26.18. Martin's Axiom (and in particular the Continuum Hypothesis) implies that there is an uncountable set of reals of strong measure 0.
[Let E be the set from Exercise 26.17. Let $a_0 \geq a_1 \geq \ldots a_n \geq \ldots$ be given positive reals. For each n, let I_{2n} be the interval of length a_{2n} around the nth rational. The set $U = \bigcup_{n=1}^{\infty} I_{2n}$ is open dense and hence $E - U$ has size $< 2^{\aleph_0}$. By Exercise 26.16 there are intervals I_{2n+1} of length a_{2n+1} such that $E - U \subset \bigcup_{n=1}^{\infty} I_{2n+1}$.]

The smallest cardinality of a set which does not have strong measure zero also admits a combinatorial characterization:

26.19. Let κ be the least cardinality of a bounded family $F \subset \omega^\omega$ that satisfies (26.30). Show that every set $A \subset 2^\omega$ of size $< \kappa$ has strong measure 0.
[Given $\langle \varepsilon_n : n \in \omega\rangle$ let $h \in \omega^\omega$ be such that $1/2^{h(n)} \leq \varepsilon_n$. For each $a \in A$ let $f_a(n) = a{\restriction}h(n)$. The family $\{f_a : a \in A\}$ can be coded as a bounded family. Let $g \in \omega^\omega$ be such that $\forall a \in A\, \exists^\infty n\, f_a(n) = g(n)$; use g to produce the intervals covering A.]

The converse is also true, and κ is the least size of a set that fails to have strong measure zero.

26.20. Martin's Axiom implies that every dense subset of $\mathcal{B}_m$ has size $2^{\aleph_0}$.
[Let $\kappa = 2^{\aleph_0}$. Let x_α, $\alpha < \kappa$, be an enumeration of all reals. MA implies that for every α, $\{x_\beta : \beta \geq \alpha\}$ has positive measure; let K_α be a compact subset of $\{x_\beta : \beta \geq \alpha\}$ such that $\mu(K_\alpha) > 0$. If $\mathcal{B}_m$ has a dense subset of size $< \kappa$, then since κ is regular, there exist a $W \subset \kappa$ of size κ and a set X of positive measure such that $X - K_\alpha$ is null for all $\alpha \in W$. Hence every finite subset of $\{K_\alpha : \alpha \in W\}$ has nonempty intersection and so $\bigcap_{\alpha\in W} K_\alpha$ is nonempty; a contradiction.]

26.21. If $\mathfrak{d} = \mathfrak{c}$ then there exists a p-point.
[Use the proof of Theorem 16.27.]

For subsets of ω, let $X \subset^* Y$ mean that $X - Y$ is finite. A family $\{X_\alpha : \alpha < \kappa\}$ of infinite subsets of ω is a *tower* if $X_\alpha \supset^* X_\beta$ whenever $\alpha < \beta$ and there is no X such that $X_\alpha \supset^* X$ for all $\alpha < \kappa$; let $\mathfrak{t}$ be the least cardinality of a tower.

26.22. $\mathfrak{t} \leq \mathfrak{b}$.
[Let $\kappa < \mathfrak{t}$, and let $F = \{f_\alpha : \alpha < \kappa\} \subset \omega^\omega$. For $X \in [\omega]^\omega$ let g_X be the increasing enumeration of X. Construct a sequence $\langle X_\alpha : \alpha \leq \kappa\rangle$ of infinite sets such that $X_\alpha \supset^* X_\beta$ for $\beta < \alpha$ and such that for every α, $\forall^\infty n\, f_\alpha(n) < g_{X_{\alpha+1}}(n)$. The function g_{X_κ} eventually dominates each $f \in F$.]

Let $\mathfrak{u}$ be the least cardinality of a family of subsets of ω that generates an ultrafilter.

26.23. $\mathfrak{b} \leq \mathfrak{u}$.
[For $X \in [\omega]^\omega$ let g_X be the increasing enumeration of X. For an increasing $f \in \omega^\omega$ let $S_f \subset \omega$ be the union of the intervals $[f^{2n}(0), f^{2n+1}(0))$, $n < \omega$. Show that if an increasing f eventually dominates g_X than both $S \cap X$ and $S - X$ are infinite.]

Historical Notes

The model in which all sets of reals are Lebesgue measurable is due to Solovay [1970], as is the concept of random reals, as well as Lemmas 26.1, 26.2, 26.4, 26.5, the Factor Lemma (Corollary 26.11), 26.16, and Theorem 26.20. Corollary 26.8 is due to Kripke [1967], and Theorem 26.12 is due to Jensen.

Galvin and Prikry proved in [1973] that every Borel set is Ramsey; this was extended by Silver in [1970b] to analytic sets, and Ellentuck [1974] gave the proof of Theorem 26.22 that we reproduce here. Theorem 26.23 is due to Mathias [1977].

Theorem 26.39 is due to Martin and Solovay [1970]. A systematic study of the properties of measure and category from Definition 26.43 was started by Miller in [1981], although the two results in Exercise 26.7 were proved by Rothberger in [1938]. Similarly, there had been various scattered results on what is now known as cardinal invariants (such as $\mathfrak{b}$, $\mathfrak{d}$, etc.) but the first comprehensive account appeared in van Douwen's [1984]. A most recent survey of the results stated here is Bartoszyński's chapter [∞] in the Handbook of Set Theory. Theorem 26.45(i) is due to Truss [1977] and Miller [1981]. Theorem 26.46 was proved independently by Bartoszyński [1984] and Raisonnier and Stern [1985]; Pawlikowski's Theorem 26.47 followed in [1985]. Theorems 26.48 and 26.49 Bartoszyński [1987] and [1984].

Example 26.51 is due to Solovay, and Example 26.52 is due to Vopěnka and Hrbáček [1967].

Exercise 26.2: Judah and Shelah [1989].
Exercise 26.3: Erdős and Rado [1952].
Exercise 26.10: Rothberger [1941].
Exercise 26.13: Bartoszyński [1984].
Strong measure zero sets were introduced by Borel [1919].
Exercise 26.15: Marczewski [1930b].
Exercise 26.16: Kunen.
Exercise 26.17: Luzin [1914].
Exercise 26.18: Sierpiński [1928].

Exercise 26.19: Rothberger [1941].
Exercise 26.20: the argument is due to Erdős.
Exercise 26.22: Rothberger.

Part III

Selected Topics

27. Combinatorial Principles in L

The Fine Structure Theory

In his paper [1972], Ronald Jensen embarked on a detailed analysis of the levels of the constructible hierarchy. The resulting theory, the *fine structure theory*, describes precisely how new sets arise in the construction of L, and has significant applications. Historically, the first application of the fine structure theory was Jensen's proof of $\square_\kappa$ in L, and we shall use that as a motivation for the introduction of fine-structural concepts. We have already described another, later, application of Jensen's theory, the Covering Theorem 18.30. While Magidor's proof presented in Chapter 18 does not use the full force of the fine structure theory, it can serve as a starting point toward the study of fine structure.

We have seen that the constructible hierarchy is Σ_1, in a uniform way, and we have also seen the role played by the condensation arguments. In particular we mention Lemma 18.38, the *Condensation Lemma*, stating that every Σ_1-elementary submodel of L_α is isomorphic to some L_γ, for every infinite ordinal α.

Every L_α (for $\alpha \geq \omega$) has a Σ_1 Skolem function, with a Σ_1 definition independent of α. Precisely, there is a Σ_0 formula Φ such that for every $\alpha \geq \omega$, the (partial) function $h_\alpha : \omega \times L_\alpha \to L_\alpha$ defined by

$$y = h_\alpha(n, x) \leftrightarrow (L_\alpha, \in) \vDash \exists z\, \Phi(n, x, y, z) \tag{27.1}$$

is a Σ_1 *Skolem function* for L_α in the sense that for every $X \subset L_\alpha$,

$$h_\alpha\text{“}(\omega \times X) = H_1^\alpha(X) \tag{27.2}$$

is the Σ_1 Skolem hull of X in L_α. This can be deduced by using the Σ_1 well-ordering $<_L$, as in (18.5). (For details, we refer the reader to Devlin's book [1984], in particular Lemma II.6.5.)

In Chapter 18 we introduced Σ_n Skolem functions for $n > 1$ as well, but mentioned (following Definition 18.40) that a Σ_n Skolem function is not necessarily a Σ_n function. In fact, for $n > 1$ there is no uniform Σ_2 Skolem function (in the sense of (27.1)–(27.2)); for details, see Exercises on pages 106–107 in Devlin's book [1984], or Proposition 2 in Friedman's [1997].

To overcome this obstacle, Jensen introduced an elaborate machinery by which arguments about Σ_n predicates on L_α can be reduced to arguments about Σ_1 predicates on a structure (L_ρ, A) which in some sense describes the Σ_n properties on L_α.

The Principle $\square_\kappa$

We recall (cf. (23.4)) that for an uncountable cardinal κ, a square-sequence is a sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ such that every C_α is closed unbounded in α, $|C_\alpha| < \kappa$ whenever $\mathrm{cf}\,\alpha < \kappa$, and if $\bar\alpha$ is a limit point of C_α then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$. In [1972], Jensen proved that in L, every uncountable cardinal κ has a square-sequence.

Theorem 27.1 (Jensen). *If $V = L$ then $\square_\kappa$ holds for every uncountable cardinal κ.*

To illustrate how the proof of $\square_\kappa$ uses condensation principles, and to introduce the fine structure theory, we shall now outline the construction of C_α in the most important special case.

First we observe that it suffices to define the sets C_α for a closed unbounded set of $\alpha < \kappa^+$; a square-sequence is then easily produced. Thus we consider only the $\alpha \in \mathrm{Lim}(\kappa^+)$ that satisfy

$$\alpha > \kappa \text{ and } L_\alpha \models \forall \gamma < \alpha\, |\gamma| \le \kappa; \tag{27.3}$$

these α's form a closed unbounded set.

As α is a singular limit ordinal, there is a stage of the constructible hierarchy where that is witnessed. Let $\beta = \beta(\alpha) \ge \alpha$ be the least β such that there is a cofinal subset of α of smaller order-type that is definable over L_β. Let $n = n(\alpha)$ be the least positive integer such that there exists such a subset that is Σ_n over L_β (with parameters in L_β).

We outline the construction of C_α for the special case when β is a limit ordinal and $n = 1$. (In general one has to consider also successor β's and $n > 1$—this is where the fine structure comes in.)

Using our assumption on α one proves that there exists a function g, Σ_1 over L_β, that maps κ onto L_β: Firstly, since there exists a $\Sigma_1(L_\beta)$ subset of α that is not in L_β (by minimality of $\beta(\alpha)$), there exists a $\Sigma_1(L_\beta)$ function that maps α onto L_β (Exercise 27.1). Then, using (27.3), one gets a Σ_1 function on κ.

Moreover, we can find such a function g in a canonical way. Since g exists, we have $L_\beta = H_1^\beta(\kappa \cup p)$, the Σ_1-Skolem hull of $\kappa \cup p$ in L_β, where p is some finite subset of L_β, and therefore

$$L_\beta = h_\beta{}^{\text{“}}(\omega \times (\kappa \cup p)), \tag{27.4}$$

where h_β is the canonical Σ_1 Skolem function from (27.1). Disregarding the parameter p (which in general is taken to be the $<_L$-least such p), we obtain

from h_β (uniformly) a Σ_1 function g_β mapping κ onto L_β; using (27.1) we find a Σ_0 formula Ψ such that

$$g_\beta(\nu) = y \leftrightarrow (\exists z \in L_\beta)\, \Psi(\nu, y, z). \tag{27.5}$$

Now we construct C_α as an increasing continuous transfinite sequence $\langle \alpha_\xi : \xi < \vartheta \rangle$ of limit ordinals $< \alpha$ with limit α. Simultaneously, we construct ordinals $\mu_\xi < \beta$ and $\nu_\xi < \kappa$, with $\langle \mu_\xi : \xi < \vartheta \rangle$ increasing and continuous, as follows: Given $\alpha_\xi < \alpha$ and $\mu_\xi < \beta$, we let

$$\nu_\xi = \text{least } \nu \text{ such that } \alpha_\xi < g_\beta(\nu) < \alpha \text{ and } L_{g_\beta(\nu)} \vDash |\alpha_\xi| = \kappa, \tag{27.6}$$

$$\mu_{\xi+1} = \text{least } \mu \text{ such that } \alpha_\xi, \mu_\xi, g_\beta(\nu_\xi) \in H_1^\mu(\kappa) \text{ and } (\exists z \in L_\mu)\ \Psi(\nu_\xi, g_\beta(\nu_\xi), z). \tag{27.7}$$

It follows from the assumptions on α that the least ordinal ϑ such that $\lim_{\xi\to\vartheta} \mu_\xi = \beta(\alpha)$ is the least ordinal with $\lim_{\xi\to\vartheta} \alpha_\xi = \alpha$, producing the set $C_\alpha = \{\alpha_\xi : \xi < \vartheta\}$, with $\vartheta \le \kappa$. The canonical Σ_1 definition (27.5) of g_β is the key to the coherence property (23.4)(ii) of the C_α's. Let $\bar\alpha < \alpha$ be a limit point of C_α, $\bar\alpha = \alpha_\lambda$ where λ is limit. Let $\bar\mu = \mu_\lambda$, and let $L_{\bar\beta}$ be the transitive collapse of $H_1^{\bar\mu}(\kappa)$. Let $e : L_{\bar\beta} \to L_\beta$ be the inverse of the transitive collapse; e is Σ_1-elementary. Using condensation arguments, one proves that $\bar\alpha \subset H_1^{\bar\mu}(\kappa)$ (and therefore $e{\restriction}\bar\alpha$ is the identity), $\bar\beta = \beta(\bar\alpha)$, $e(\bar\alpha) = \alpha$, and finally that the definition of $C_{\bar\alpha} = \{\bar\alpha_\xi : \xi < \lambda\}$ agrees with the definition of C_α up to λ. In other words, $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.

This completes the outline for the special case. When $n(\alpha) = 1$ and $\beta(\alpha)$ is a successor ordinal, it can be shown that $\operatorname{cf}\alpha = \omega$ and this case is sufficiently exceptional to allow to choose C_α a sequence of order-type ω, without limit points. When $n(\alpha) > 1$, the proof requires the machinery of the fine structure theory: the model $(L_{\beta(\alpha)}, \in)$ is replaced by $(L_\rho, \in, A)$ where ρ is $(n-1)$-*projectum* of β, allowing the use of canonical Σ_1-Skolem functions for models $(L_\rho, \in, A)$.

A complete proof of Theorem 27.1 can be found in Jensen's paper [1972] or in Devlin's book [1984]. There have been several attempts at simplification of the proof; among the more recent published proofs we mention Friedman [1997] and Friedman and Koepke [1997]. □

As Jensen pointed out in [1972], his proof of $\square_\kappa$ in L shows that if κ^+ is not Mahlo in L then $\square_\kappa$ holds. As a consequence the consistency strength of the failure of Square is at least that of a Mahlo cardinal. By a result of Solovay (Exercise 27.2), the consistency strength of $\neg\square_{\omega_1}$ is that of a Mahlo cardinal.

We also note a result of Solovay from [1974] that the existence of supercompact cardinals implies the failure of Square (Exercise 27.3).

The Jensen Hierarchy

One of the technical obstacles in the analysis how constructible sets arise in the hierarchy L_α is that the sets L_α are not closed under the formation of ordered pairs. This can be overcome by modifying the constructible hierarchy in an inessential way. The resulting hierarchy J_α has become the preferred tool for studying the fine structure of L and of more general inner models.

Definition 27.2 (Rudimentary Functions).

(i) $F(x_1, \ldots, x_n) = x_i \qquad (i = 1, \ldots, n)$,
$F(x_1, \ldots, x_n) = \{x_i, x_j\} \qquad (i, j = 1, \ldots, n)$,
$F(x_1, \ldots, x_n) = x_i - x_j \qquad (i, j = 1, \ldots, n)$
are rudimentary.

(ii) If G is rudimentary, then so is

$$F(y, x_1, \ldots, x_{n-1}) = \bigcup_{z \in y} g(z, x_1, \ldots, x_{n-1}).$$

(iii) A composition of rudimentary functions is rudimentary.

The *rudimentary closure* of a set X is the smallest $Y \supset X$ closed under all rudimentary functions. If X is transitive then so is its rudimentary closure, and for every transitive set M, let

$$\text{rud}(M) = \text{the rudimentary closure of } M \cup \{M\}. \tag{27.8}$$

It can be shown that for every transitive set M,

$$\text{rud}(M) \cap P(M) = \text{def}(M) \tag{27.9}$$

(compare with Corollary 13.8).

Definition 27.3 (The Jensen Hierarchy).

(i) $J_0 = \emptyset$, $J_{\alpha+1} = \text{rud}(J_\alpha)$,
(ii) $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ if α is a limit ordinal.

Each J_α is transitive, the hierarchy is cumulative, and for each α,

$$J_\alpha \subset V_{\omega\alpha} \quad \text{and} \quad J_\alpha \cap \mathit{Ord} = \omega\alpha.$$

From (27.9) it follows that

$$J_{\alpha+1} \cap P(J_\alpha) = \text{def}(J_\alpha).$$

The exact relationship between the J_α's and the L_α's is not important, but we have

$$J_\alpha = L_\alpha \text{ for all } \alpha \text{ such that } \alpha = \omega\alpha. \tag{27.10}$$

Every J_α is closed under $\{x, y\}$, $\bigcup x$, $x \times y$, and if A is a Σ_0 subset of J_α then $A \cap x \in J_\alpha$ for every $x \in J_\alpha$. This has the effect that

$$\langle J_\xi : \xi < \alpha \rangle$$

is uniformly Σ_1 over J_α, and there is a well-ordering $<_J$ of L such that its restriction to J_α is (uniformly) Σ_1 over J_α. Also, there is a (uniform) Σ_1 function over J_α that maps $\omega\alpha$ onto J_α. Similarly as for the L_α, every J_α has a canonical Σ_1 Skolem function h_α (analogous to (27.1) and (27.2)).

The fine structure theory capitalizes on the fact that the existence of a uniform Σ_1 Skolem function relativizes to models (J_α, A) where A is a one-place predicate as long as

$$A \cap u \in J_\alpha \text{ for all } u \in J_\alpha; \tag{27.11}$$

such models (J_α, A) are called *amenable*. There is a Σ_0 formula Φ of the language $(\in, A)$ such that for every α and every amenable model (J_α, A), the (partial) function $h_{\alpha,A} : \omega \times J_\alpha \to J_\alpha$ defined by

$$y = h_{\alpha,A}(n, x) \leftrightarrow (J_\alpha, \in, A) \vDash \exists z\, \Phi(n, x, y, z) \tag{27.12}$$

is a Σ_1 Skolem function for (J_α, A).

Projecta, Standard Codes and Standard Parameters

Definition 27.4. For $n > 0$, the Σ_n-*projectum* ρ_α^n of α is the smallest ordinal $\rho \leq \alpha$ such that there exists a $\Sigma_n(J_\alpha)$ function f such that $f\text{“}J_\rho = J_\alpha$; for $n = 0$, let $\rho_\alpha^0 = \alpha$.

An argument similar to Exercise 27.1 is used to prove that ρ_α^n is the smallest ρ such that there exists a $\Sigma_n(J_\alpha)$ subset of $\omega\rho$ not in J_α.

The main feature of the fine structure is that a predicate definable over J_α can be reduced to a Σ_1 predicate over an amenable structure (J_ρ, A) where ρ is a projectum of α. For each α and each $n > 0$ there exists a set $A_\alpha^n \subset J_{\rho_\alpha^n}$ that is Σ_n over J_α such that $(J_{\rho_\alpha^n}, A_\alpha^n)$ is amenable, and such that

$$\Sigma_1(J_{\rho_\alpha^n}, A_\alpha^n) = P(J_{\rho_\alpha^n}) \cap \Sigma_{n+1}(J_\alpha). \tag{27.13}$$

For $n = 0$, we let $A_\alpha^0 = \emptyset$. The sets A_α^n are called *standard codes*.

If P is a Σ_{n+1} predicate over J_α, let f be a $\Sigma_n(J_\alpha)$ function that maps $J_{\rho_\alpha^n}$ onto J_α. Then $f_{-1}(P)$ is a $\Sigma_{n+1}(J_\alpha)$ subset of $J_{\rho_\alpha^n}$ and therefore, by (27.13), Σ_1 over the amenable model $(J_{\rho_\alpha^n}, A_\alpha^n)$. This reduction is canonical, as both the standard codes, and the Σ_n functions $f : J_{\rho_\alpha^n} \to J_\alpha$ are canonical. Precisely, we define standard codes along with *standard parameters* p_α^n, by induction on n: $p_\alpha^0 = \emptyset$ and

$$p_\alpha^{n+1} \text{ is the } <_J\text{-least } p \in J_{\rho_\alpha^n} \text{ such that } J_{\rho_\alpha^n} \text{ is the } \Sigma_1\text{-Skolem hull of } J_{\rho_\alpha^{n+1}} \cup p \text{ in } J_{\rho_\alpha^n}; \tag{27.14}$$

(27.15) $A_\alpha^{n+1} = \{(k,x) : (J_{\rho_\alpha^n}, A_\alpha^n) \vDash \varphi_k(x, p_\alpha^{n+1})\}$

where φ_k, $k \in \omega$, is a recursive enumeration of the Σ_1 formulas.

Then a $\Sigma_n(J_\alpha)$ function from $J_{\rho_\alpha^n}$ onto J_α can be produced from the canonical Σ_1 Skolem functions and the standard parameters via (27.14). The fundamental property of standard codes is the following Condensation Lemma:

Lemma 27.5. *Let (J_γ, A) be amenable and let*

$$e : (J_\gamma, A) \to (J_{\rho_\alpha^n}, A_\alpha^n)$$

be a Σ_0-elementary embedding. There exists a unique $\bar\alpha$ such that $\gamma = \rho_{\bar\alpha}^n$ and $A = A_{\bar\alpha}^n$. The embedding e extends to a unique Σ_n-elementary embedding

$$\bar e : J_{\bar\alpha} \to J_\alpha$$

such that $\bar e(p_{\bar\alpha}^i) = p_\alpha^i$ for all $i = 1, \ldots, n$. Moreover, if e is Σ_m-elementary then $\bar e$ is Σ_{n+m}-elementary. □

A detailed account of the fine structure theory can be found in Jensen's paper [1972], or in Devlin's book [1984].

Diamond Principles

Let κ be a regular uncountable cardinal and let E be a stationary subset of κ. $\diamondsuit(E)$, or (more precisely) $\diamondsuit_\kappa(E)$, is the following principle (23.1):

(27.16) There exists a sequence of sets $\langle S_\alpha : \alpha \in E\rangle$ with $S_\alpha \subset \alpha$ such that for every $X \subset \kappa$, the set $\{\alpha \in E : X \cap \alpha = S_\alpha\}$ is a stationary subset of κ.

When $E = \kappa$, $\diamondsuit_\kappa(\kappa)$ is denoted by $\diamondsuit_\kappa$. $\diamondsuit_\kappa$ is a generalization of $\diamondsuit$ from Theorem 13.21, and can be proved under $V = L$ by a similar argument (Exercise 27.4).

Gregory's Theorem 23.2 shows that under GCH, $\diamondsuit_{\kappa^+}$ holds for every successor cardinal κ^+, in fact proving $\diamondsuit(E_\lambda^{\kappa^+})$ whenever $\lambda < \operatorname{cf}\kappa$. This was extended by Shelah in [1979] by showing, under GCH, that $\diamondsuit(E_\lambda^{\kappa^+})$ holds whenever $\lambda \neq \operatorname{cf}\kappa$, and if κ is singular, then GCH and $\square_\kappa$ together imply $\diamondsuit(E_{\operatorname{cf}\kappa}^{\kappa^+})$. See also Devlin [1984], Lemma IV.2.8. For $\kappa = \aleph_1$, GCH yields a weak version of $\diamondsuit$. In [1978], Devlin and Shelah formulate and prove, under the assumption $2^{\aleph_0} < 2^{\aleph_1}$ the following statement:

(27.17) For every $F : \{0,1\}^{<\omega_1} \to \{0,1\}$ there exists a $g \in \{0,1\}^{\omega_1}$ such that for every $f \in \{0,1\}^{\omega_1}$, the set $\{\alpha < \omega_1 : F(f{\restriction}\alpha) = g(\alpha)\}$ is stationary.

(27.17) is a consequence of $\diamondsuit$ and fails under $\mathrm{MA}_{\aleph_1}$.

Trees in L

Let κ be an infinite cardinal. Generalizing Definition 9.12, we have:

Definition 27.6. A κ^+-*Suslin tree* is a tree of height κ^+ such that every branch and every antichain have cardinality at most κ.

The following result generalizes Theorem 15.26:

Theorem 27.7 (Jensen). *If $V = L$ then for every infinite cardinal κ there exists a κ^+-Suslin tree.*

When κ is regular, the proof is a straightforward generalization of the construction of a Suslin tree using $\diamondsuit$: instead we use $\diamondsuit(E^{\kappa^+}_\kappa)$. We construct a tree by induction on levels. At limit levels α of cofinality $< \kappa$ we extend all branches in T_α; since $\kappa^{<\kappa} = \kappa$, the αth level has size κ. If $\operatorname{cf}\alpha = \kappa$ then we use Diamond to destroy potential antichains of size κ^+. Note that since all branches have been extended at lower cofinalities, every $x \in T_\alpha$ has an α-branch in T_α going through x. The proof that the resulting tree is a κ^+-Suslin tree is exactly as in Theorem 15.26.

When κ is singular, this approach does not work as there are κ^+-branches in T_α when $\operatorname{cf}\alpha = \operatorname{cf}\kappa$. By not extending all of them we cannot guarantee that at a later stage β, T_β has β-branches at all. Jensen's proof succeeds by involving not only $\diamondsuit$, but the $\square_\kappa$ principle as well. The proof shows that if $\square_\kappa$ holds and if $\diamondsuit_{\kappa^+}(E)$ for all E, then a κ^+-Suslin tree exists. For a proof, see Devlin [1984], Theorem IV.2.4.

Let us recall (Definition 9.24) that a tree of height ω_1 is a *Kurepa tree* if it has countable levels and at least $\aleph_2$ uncountable branches.

Theorem 27.8 (Solovay). *If $V = L$ then there exists a Kurepa tree.*

Proof. Assume $V = L$. We shall construct a family of subsets of ω_1 that satisfy (9.12).

For each $\alpha < \omega_1$, there is a smallest elementary submodel M of $(L_{\omega_1}, \in)$ such that $\alpha \in M$. Moreover (see Exercise 13.17), $M = L_\gamma$ for some $\gamma < \omega_1$, and we denote γ by $f(\alpha)$:

$$f(\alpha) = \text{the least } \gamma \text{ such that } \alpha \in L_\gamma \prec (L_{\omega_1}, \in). \tag{27.18}$$

Let $\mathcal{F}$ be the following family of subsets of ω_1:

$$\mathcal{F} = \{X \subset \omega_1 : X \cap \alpha \in L_{f(\alpha)} \text{ for every } \alpha < \omega_1\}. \tag{27.19}$$

It is immediately clear that $\{X \cap \alpha : X \in \mathcal{F}\}$ is countable for each $\alpha < \omega_1$; and hence if we show that $|\mathcal{F}| = \aleph_2$, $\mathcal{F}$ will satisfy (9.12).

Assume that $|\mathcal{F}| \leq \aleph_1$. Then $\mathcal{F}$ has an enumeration

$$C = \langle X_\xi : \xi < \omega_1 \rangle \tag{27.20}$$

and any such enumeration is in L_{ω_2}. If we let C be the $<_L$-least such C in L_{ω_2}, then since the function f is a definable element of L_{ω_2} (by the definition (27.18)) and the X_ξ satisfy (27.19) in $(L_{\omega_2}, \in)$, it follows that C is a definable element of $(L_{\omega_2}, \in)$.

Now, we construct an elementary chain of submodels of $(L_{\omega_2}, \in)$:

$$N_0 \prec N_1 \prec \ldots \prec N_\nu \prec \ldots \prec (L_{\omega_2}, \in) \qquad (\nu < \omega_1)$$

as follows: N_0 is the smallest elementary submodel of L_{ω_2}; $N_{\nu+1}$ is the smallest $N \prec L_{\omega_2}$ such that $N_\nu \subset N$ and $N_\nu \in N$; if η is a limit ordinal, then $N_\eta = \bigcup_{\nu<\eta} N_\nu$. Note that each N_ν is countable, and $\omega_1 \cap N_\nu = \alpha_\nu$, for some $\alpha_\nu < \omega_1$ (see Exercise 13.18). Moreover,

$$\langle \alpha_\nu : \nu < \omega_1 \rangle \tag{27.21}$$

is a continuous increasing sequence of countable ordinals.

Now, we let $X = \{\alpha_\nu : \alpha_\nu \notin X_\nu\}$. Obviously, $X \neq X_\xi$ for all $\xi < \omega_1$, and we shall show that X satisfies the condition in (27.19), which will contradict the assumption that (27.20) is an enumeration of *all* elements of $\mathcal{F}$.

We want to show that $X \cap \alpha \in L_{f(\alpha)}$ for all $\alpha < \omega_1$. By induction on α, if α is not a limit point of the sequence (27.21), then let α_ν be the largest $\alpha_\nu < \alpha$. Then either $X \cap \alpha = X \cap \alpha_\nu$ or $X \cap \alpha = (X \cap \alpha_\nu) \cup \{\alpha_\nu\}$; in either case, since $X \cap \alpha_\nu \in L_{f(\alpha_\nu)} \subset L_{f(\alpha)}$ (by the induction hypothesis), we have $X \cap \alpha \in L_{f(\alpha)}$. Thus it suffices to show that $X \cap \alpha_\eta \in L_{f(\alpha_\eta)}$ whenever η is a limit ordinal.

We shall show that

$$\begin{aligned} &\text{(i)}\ \ \langle \alpha_\nu : \nu < \eta \rangle \in L_{f(\alpha_\eta)}; \\ &\text{(ii)}\ \ \langle X_\xi \cap \alpha_\eta : \xi < \alpha_\eta \rangle \in L_{f(\alpha_\eta)}. \end{aligned} \tag{27.22}$$

Since $L_{f(\alpha_\eta)}$ is a model of ZF^-, the set $X \cap \alpha_\eta$ has the following definition in $L_{f(\alpha_\eta)}$:

$$X \cap \alpha = \{\alpha_\nu : \nu < \eta \text{ and } \alpha_\nu \notin X_\nu \cap \alpha_\eta\}.$$

For each $\nu < \omega_1$, let π_ν be the transitive collapse of N_ν. Each N_ν is isomorphic to some $L_{\delta(\nu)}$, and since $\omega_1 \cap N_\nu = \alpha_\nu$, we have $\pi_\nu(\omega_1) = \alpha_\nu$. Since C is a definable element of L_{ω_2}, we have $C \in N_\nu$ for all ν and one can see that $\pi_\nu(C) = \langle X_\xi \cap \alpha_\nu : \xi < \alpha_\nu \rangle$.

Note that α_η is uncountable in $L_{\delta(\eta)}$, while it is countable in $L_{f(\alpha_\eta)}$. It follows that $\delta(\eta) < f(\alpha_\eta)$, and we have $\pi_\eta(C) \in L_{\delta(\eta)} \subset L_{f(\alpha_\eta)}$, which proves (27.22)(ii).

To prove (27.22)(i), let us construct, inside $L_{f(\alpha_\eta)}$ (which is a model of ZF^-), an elementary chain N'_ν, $\nu < \eta$ of submodels of $(L_{\delta(\eta)}, \in)$: N'_0 is the

smallest elementary submodel of $L_{\delta(\eta)}$; $N'_{\nu+1}$ is the smallest $N \prec L_{\delta(\eta)}$ such that $N'_\nu \cup \{N'_\nu\} \subset N$, etc. It is not difficult to show, by induction on $\nu < \eta$, that for each ν, N'_ν is isomorphic to N_ν. Then the transitive collapse of N'_ν is $L_{\delta(\nu)}$, and so $\langle L_{\delta(\nu)} : \nu < \eta\rangle \in L_{f(\alpha_\eta)}$. It follows that $\langle \alpha_\nu : \nu < \eta\rangle \in L_{f(\alpha_\eta)}$, proving (27.22)(i). □

One consequence of the foregoing proof is that a Kurepa tree exists unless $\aleph_2$ is inaccessible in L (Exercise 27.5). This is complemented by the following consistency result:

Theorem 27.9 (Silver [1971c]). *If there exists an inaccessible cardinal then there is a generic extension in which there are no Kurepa trees.*

Proof. Let λ be an inaccessible cardinal. Let $(P, <)$ be the Lévy collapse of λ to $\aleph_2$: forcing conditions are countable functions p on subsets of $\lambda \times \omega_1$ such that $p(\alpha, \xi) < \alpha$ for every $(\alpha, \xi) \in \mathrm{dom}(p)$ and p is stronger than q if $p \supset q$.

$(P, <)$ is $\aleph_0$-closed, and so V and $V[G]$ have the same countable sequences in V. Also, $\aleph_1^{V[G]} = \aleph_1$, and $\aleph_2^{V[G]} = \lambda$.

Lemma 27.10. *If P is an $\aleph_0$-closed notion of forcing and T is an ω_1-tree in the ground model such that every level of T is countable, then T has no new branches in $V[G]$.*

Proof. Assume that T has a branch $b \in V[G]$ that is not in V; since $V[G]$ has no new countable sets, b has length ω_1. There is a name $\dot b$ for b and a condition $p_0 \in G$ such that $p_0 \Vdash \dot b \neq \check a$ for all $a \in V$. We construct, by induction, conditions $p_s < p_0$ and nodes $x_s \in T$ for all finite sequences s of 0's and 1's. Having constructed p_s, we can find two incomparable nodes $x_{s^\frown 0}$ and $x_{s^\frown 1}$ both $> x_s$, and two conditions $p_{s^\frown 0}$ and $p_{s^\frown 1}$, both stronger than p_s such that $p_{s^\frown 0} \Vdash x_{s^\frown 0} \in \dot b$ and $p_{s^\frown 1} \Vdash x_{s^\frown 1} \in \dot b$. Moreover, we can find such $x_{s^\frown 0}$ and $x_{s^\frown 1}$ at the same level of T. Let $\alpha < \omega_1$ be such that all x_s lie below level α in T. For each $f : \omega \to \{0, 1\}$, let p_f be a condition stronger than all $p_{f\restriction n}$, $n \in \omega$. Since $p_0 \Vdash \dot b$ is uncountable, there exist $q < p_f$ and x_f at the αth level of T such that $q \Vdash x_f \in \dot b$. Now it is clear that $x_f \neq x_g$ whenever f and g are distinct 0–1 functions on ω. Thus the αth level of T has at least $2^{\aleph_0}$ elements, contrary to our assumption. □

It follows immediately from the lemma that in $V[G]$, no tree $T \in V$ whose levels are countable can be a Kurepa tree: Since every branch of T in $V[G]$ is in V, T has at most $(2^{\aleph_1})^V$ branches, but $(2^{\aleph_1})^V < \lambda = \aleph_2^{V[G]}$, and so T has (in $V[G]$) fewer than $\aleph_2$ branches.

A similar argument can be used for any tree in $V[G]$, with a slight modification. For each $\alpha < \lambda$, let P_α denote the set of all conditions whose domain is a subset of $\alpha \times \omega_1$; similarly, let $P^\alpha = \{p \in P : \mathrm{dom}(p) \subset (\kappa - \alpha) \times \omega_1\}$. Clearly, P is (isomorphic to) the product $P_\alpha \times P^\alpha$. Let $X \in V[G]$ be a subset of ω_1, and let $\dot X$ be a name of X; since P satisfies the λ-chain condition, there exists for each $\xi < \omega_1$ a set of conditions $W_\xi \subset P$ of size less than λ such

that $\|\xi \in \dot{X}\| = \sum\{p : p \in W_\xi\}$. There exists an $\alpha < \lambda$ such that $W_\xi \subset P_\alpha$, for all $\xi < \omega_1$. It follows that $X \in V[G \cap P_\alpha]$.

Now let $T \in V[G]$ be an ω_1-tree with countable levels. There exists an $\alpha < \lambda$ such that $T \in V[G \cap P_\alpha]$. By the Product Lemma, $G \cap P^\alpha$ is P^α-generic over $V[G \cap P_\alpha]$ and $V[G] = V[G \cap P_\alpha][G \cap P^\alpha]$. Since $V[G \cap P_\alpha]$ and V have the same countable sequences in V, it follows that P^α is $\aleph_0$-closed not only in V, but also $V[G \cap P_\alpha] \vDash P^\alpha$ is $\aleph_0$-closed. Thus Lemma 27.10 applies and every branch of T in $V[G]$ is in $V[G \cap P_\alpha]$. However, $(2^{\aleph_1})^{V[G \cap P_\alpha]} < \lambda = \aleph_2^{V[G]}$, and so T is not a Kurepa tree in $V[G]$. This completes the proof. □

Canonical Functions on ω_1

For ordinal functions on ω_1, let $f < g$ if $\{\xi < \omega_1 : f(\xi) < g(\xi)\}$ contains a closed unbounded set. The rank of f in $<$ is the Galvin-Hajnal norm $\|f\|$; cf. Definition 24.4. By induction on α, the αth *canonical function* f_α is defined (if it exists) as the $<$-least ordinal function greater than each f_β, $\beta < \alpha$. If f_α exists then it is unique up to the equivalence $=_{I_{\rm NS}}$. Lemma 24.5 shows that for every $\alpha < \omega_2$ the αth canonical function exists; see also Exercise 27.6.

It is possible that the constant function ω_1 is the ω_2nd canonical function (see Exercise 27.7) but this is known to have large cardinal consequences; in particular, in L there is a function $f : \omega_1 \to \omega_1$ such that $\|f\| = \omega_2$ (Exercise 27.8).

If canonical functions f_α exist for all α, then the ideal $I_{\rm NS}$ is precipitous (Exercise 27.10) and hence there is an inner model with a measurable cardinal. Conversely, a combination of the method from Jech and Mitchell [1983] with the proof of Theorem 23.10 yields the consistency, relative to a measurable cardinal, of canonical functions for all α.

The following result shows that in L, the ω_2nd canonical function does not exist.

Theorem 27.11 (Hajnal). *If $V = L$ then there is no ω_2nd canonical function on ω_1.*

Proof. Assume $V = L$, and assume that there is an ω_2nd canonical function. This statement can be expressed in L_{ω_2}:

$$(\exists f : \omega_1 \to \omega_1)\, \forall \eta\, (f_\eta < f) \text{ and}$$
$$(\forall \text{ stationary } S)(\forall g <_S f)(\exists \text{ stationary } T \subset S)\, \exists \eta\, g{\restriction}T = f_\eta{\restriction}T.$$

Let γ be the least ordinal such that $(L_\gamma, \in)$ is elementarily equivalent to $(L_{\omega_2}, \in)$. Let f be the ω_2nd canonical function in $(L_\gamma, \in)$ and let $\delta = \omega_1^{L_\gamma}$. We shall find a $\xi < \delta$ such that $(L_\xi, \in) \equiv L_\gamma$, reaching a contradiction.

Consider the generic ultrapower of L_γ by the nonstationary ideal $(I_{\rm NS})^{L_\gamma}$ on $\delta = \omega_1^{L_\gamma}$ (using functions in L_γ). As f is the $\omega_2^{L_\gamma}$nd canonical function, the

ultraproduct $\prod_{\xi<\delta} f(\xi)/G$ has order-type γ, and moreover, the ultraproduct $\mathrm{Ult}_G = \prod_{\xi<\delta} L_{f(\xi)}/G$ is isomorphic to L_γ. Thus if a sentence σ is true in $(L_\gamma, \in)$ then it is forced to be true in Ult_G by every stationary $S \subset \delta$ in L_γ, and so if we let

$$T_\sigma = \{\xi < \delta : L_{f(\xi)} \vDash \sigma\},$$

then (since $f \in L_\gamma$) $T_\sigma \in L_\gamma$ and

$$L_\gamma \vDash T_\sigma \text{ contains a closed unbounded set.}$$

If $\{\sigma_n : n \in \omega\}$ enumerates all sentences of ZF, then $\langle T_{\sigma_n} : n < \omega\rangle \in L_\gamma$, and

$$L_\gamma \vDash \bigcap\{T_{\sigma_n} : n \in \omega \text{ and } T_{\sigma_n} \text{ contains a closed unbounded set}\} \neq \emptyset.$$

If ξ is an element of this intersection, then $L_{f(\xi)} \equiv L_\gamma$. □

The existence of an ω_2nd canonical function is not a large cardinal property, as this consistency result shows:

Theorem 27.12 (Jech-Shelah). *There is a generic extension of L in which the ω_2nd canonical function exists.*

The model is obtained by first adding (by forcing with countable conditions) an increasing sequence $\langle f_\alpha : \alpha \leq \omega_2\rangle$ of ordinal functions from ω_1 into ω_1. Then one uses an iterated forcing, with countable support of length ω_2, that successively destroys all stationary subsets of ω_1 that witness that the sequence $\langle f_\alpha : \alpha \leq \omega_2\rangle$ is not canonical. For details, consult Jech and Shelah [1989].

Exercises

27.1. Let $\alpha \leq \beta$ be limit ordinals and assume that there exists a set $Z \subset \alpha$ that is Σ_1 over L_β but $Z \notin L_\beta$. Then there exists a $\Sigma_1(L_\beta)$ function g such that $g``\alpha = L_\beta$.

[First show that there is a Σ_1 function $g : \alpha \to \beta$ unbounded in β. Let $Z = \{\xi < \alpha : (\exists y \in L_\beta)\, \varphi(\xi, y, p)\}$ where φ is Σ_0, and let $g(\xi)$ be the least η such that $(\exists y \in L_\eta)\, \varphi$.]

27.2. If a Mahlo cardinal λ is Lévy collapsed to $\aleph_2$ (by countable conditions) then $\square_{\omega_1}$ fails in the extension.

27.3. If κ is supercompact then $\square_\lambda$ fails for all $\lambda \geq \kappa$.

27.4. If $V = L$ then $\diamondsuit_\kappa(E)$ holds for every regular uncountable κ and every stationary $E \subset \kappa$.

27.5. If $\aleph_2$ is not inaccessible in L then a Kurepa tree exists.

[There exists an $A \subset \omega_1$ such that $\omega_1^{L[A]} = \omega_1$ and $\omega_2^{L[A]} = \omega_2$; modify Theorem 27.8 to construct a Kurepa tree in $L[A]$.]

27.6. If $\omega_1 \leq \alpha < \omega_2$, and if g is a one-to-one function of ω_1 onto α, let $f(\xi) =$ the order-type of $g``\xi$. Show that f is the αth canonical function.

27.7. If $I_{\rm NS}$ is $\aleph_2$-saturated then the constant function ω_1 is the $\aleph_2$nd canonical function.

27.8. In L, find a function $f : \omega_1 \to \omega_1$ of norm ω_2.
[As in the proof of Theorem 27.8.]

27.9. If $f : \omega_1 \to Ord$ and S is stationary then $\|f\|_S = \alpha$ if S forces $j(f)(\omega_1^V) = \alpha$ in $P(\omega_1)/I_{\rm NS}$.

27.10. If a canonical f_α exists for every α, then $I_{\rm NS}$ is precipitous.

Historical Notes

The fine structure theory was introduced by Jensen in [1972]. The paper gives, among others, proofs of $\square_\kappa$ and of the existence of κ^+-Suslin trees in L. It also formulates a combinatorial principle $\diamondsuit^+$ that implies the existence of a Kurepa tree (abstracting Solovay's proof given here). Silver's model with no Kurepa trees appears in [1971c]. Theorem 27.11 is an unpublished result of András Hajnal from 1976; the model in Theorem 27.11 is from Jech and Shelah [1989].

Exercises 27.2, 27.3: Solovay.

Exercise 27.4: Jensen.

28. More Applications of Forcing

In this chapter we present a selection of forcing constructions related to topics discussed earlier in the book.

A Nonconstructible Δ^1_3 Real

By Shoenfield's Absoluteness Theorem, every Π^1_2 or Σ^1_2 real is constructible; on the other hand $0^\sharp$ is a Δ^1_3 real. We now present a model due to Jensen that produces a nonconstructible Δ^1_3 real by forcing over L.

Theorem 28.1 (Jensen). *There is a generic extension $L[a]$ of L such that a is a Δ^1_3 real.*

The construction is a combination of perfect set forcing and arguments using the $\diamondsuit$-principle. Let us consider *perfect trees* $p \subset Seq(\{0,1\})$, cf. (15.24). The *stem* of a perfect tree p is the maximal $s \in Seq(\{0,1\})$ such that for every $t \in p$, either $t \subset s$ or $s \subset t$. If p is a perfect tree and if $s \in p$, we denote by $p{\restriction}s$ the perfect tree $\{t \in p : t \subset s \text{ or } t \supset s\}$.

Assume that P is a set of perfect trees, partially ordered by $\subset$, such that if $p \in P$ and $s \in p$, then $p{\restriction}s \in P$, and let G be an L-generic filter on P. Then there is a unique $f \in \{0,1\}^\omega$ which is a branch in every $p \in G$; and conversely, $G = \{p \in P : f \text{ is a branch in } p\}$. Therefore $L[G] = L[f]$, and we call f P-generic over L. Note that $f \in \{0,1\}^\omega$ is P-generic over L if and only if for every constructible predense set $X \subset P$, f is a branch in some $p \in X$.

Similarly, a generic filter G on $P \times P$ corresponds to a unique pair (a,b) such that for each $(p,q) \in G$, a is a branch in p and b is a branch in q. A pair (a,b) is $(P \times P)$-generic over L if and only if for every constructible predense set $X \subset P \times P$, there exists a pair $(p,q) \in X$ such that a is a branch in p and b is a branch in q.

In Chapter 15 we used the Fusion Lemma for perfect trees. Let $T = \{T(s) : s \in Seq(\{0,1\})\}$ be a collection of perfect trees such that for every s,

(28.1)
(i) $T(s)$ is a perfect tree whose stem has length $\geq$ length(s).
(ii) $T(s^\frown 0) \subset T(s)$ and $T(s^\frown 1) \subset T(s)$.
(iii) $T(s^\frown 0)$ and $T(s^\frown 1)$ have incompatible stems.

If T satisfies (28.1), we say that T is *fusionable* and we let

$$\mathcal{F}(T) = \bigcap_{n=0}^{\infty} \bigcup_{s\in\{0,1\}^n} T(s). \tag{28.2}$$

For each fusionable T, $p = \mathcal{F}(T)$ (the *fusion* of T) is a perfect tree; and for each s, if t is the stem of $p_s = T(s)$, then $p{\restriction}t$ is stronger than both p and p_s.

We shall not use the set of all perfect trees as the notion of forcing; rather we shall construct a set P of perfect trees with the property that if $p \in P$ and $s \in p$, then $p{\restriction}s \in P$. We shall construct P such that if a is P-generic over L, then a is the only P-generic set in $L[a]$, and such that $\{n \in \boldsymbol{N} : a(n) = 1\}$ is (in $L[a]$) a Δ^1_3 subset of $\boldsymbol{N}$.

We shall construct P as the union of countable sets

$$P_0 \subset P_1 \subset \ldots \subset P_\alpha \subset \ldots \qquad (\alpha < \omega_1)$$

of perfect trees. The construction uses the $\diamondsuit$-principle. There is a $\diamondsuit$-sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ that is Δ_1 over L_{ω_1}; let us fix such a sequence. Also, let us fix a Δ_1 over L_{ω_1} function τ that is a one-to-one mapping of L_{ω_1} onto ω_1.

We shall now construct the sequence $P_0 \subset P_1 \subset \ldots \subset P_\alpha \subset \ldots$:

(28.3) $P_0 =$ the set of all $p_0{\restriction}s$ where p_0 is the full binary tree, $p_0 = Seq(\{0,1\})$, and $s \in p_0$;

$P_\alpha = \bigcup_{\beta<\alpha} P_\beta$ if α is a limit ordinal.

$P_{\alpha+1} = P_\alpha \cup Q_{\alpha+1}$ where $Q_{\alpha+1}$ is a set of perfect trees defined as follows:

Let $P_\alpha = \{p^\alpha_n : n \in \omega\}$; and let us consider the $<_L$-least such enumeration. Let $\mathcal{X}_\alpha$ and $\mathcal{Y}_\alpha$ be the following countable collections of subsets of P_α and and $P_\alpha \times P_\alpha$ respectively:

(28.4) $\mathcal{X}_\alpha$ contains:
- (i) all Q_β, $\beta \le \alpha$,
- (ii) all $X \subset P_\alpha$ such that τ"$X = S_\beta$ for some $\beta \le \alpha$.

$\mathcal{Y}_\alpha$ contains:
- (i) $Q_\beta \times Q_\beta$ for all $\beta \le \alpha$;
- (ii) all $Y \subset P_\alpha \times P_\alpha$ such that τ"$Y = S_\beta$ for some $\beta \le \alpha$.

There exists a family $\{T_n : n \in \omega\}$ of fusionable collections of elements of P_α such that:

(28.5)
- (i) $T_n(\emptyset) = p^\alpha_n$ for all n;
- (ii) for every $X \in \mathcal{X}_\alpha$, if X is predense in P_α, then for every $n \in \boldsymbol{N}$ and every $h \in \boldsymbol{N}$ there is $k \ge h$ such that for each $s \in \{0,1\}^k$, there exists an $x \in X$ such that $T_n(s) \le x$;
- (iii) for every $Y \in \mathcal{Y}_\alpha$, if Y is predense in $P_\alpha \times P_\alpha$, then for every n, every m, and every h there is a $k \ge h$ such that for each $s \in \{0,1\}^k$ and each $t \in \{0,1\}^k$; if either $n \ne m$ or $s \ne t$, then there exists $(x, y) \in Y$ such that $(T_n(s), T_m(t)) \le (x, y)$.

A family $\{T_n : n \in \omega\}$ with properties (28.5)(i)–(iii) is easily constructed because $\mathcal{X}_\alpha$ and $\mathcal{Y}_\alpha$ are countable. We denote $\{T^\alpha_n : n \in \omega\}$ the $<_L$-least such family, and let

$$Q_{\alpha+1} = \{p\restriction s : p = \mathcal{F}(T^\alpha_n) \text{ for some } n, \text{ and } s \in p\}. \tag{28.6}$$

We let $P = \bigcup_{\alpha<\omega_1} P_\alpha$. The following sequence of lemmas will show that if $a \in \{0,1\}^\omega$ is P-generic over L, then in $L[a]$ the set $\{n : a(n) = 1\}$ is Δ^1_3.

Lemma 28.2. *For each α, $Q_{\alpha+1}$ is dense in $P_{\alpha+1}$, and $Q_{\alpha+1} \times Q_{\alpha+1}$ is dense in $P_{\alpha+1} \times P_{\alpha+1}$.*

Proof. It suffices to show that below each $p \in P_\alpha$ there is some $q \in Q_{\alpha+1}$. If $p \in P_\alpha$, the $p = p^\alpha_n$ for some n, and $\mathcal{F}(T_n) \subset T_n(\emptyset) = p$. □

Lemma 28.3. *For each α, if $X \in \mathcal{X}_\alpha$ is predense in P_α, then X is predense in $P_{\alpha+1}$; if $Y \in \mathcal{Y}_\alpha$ is predense in $P_\alpha \times P_\alpha$, then Y is predense in $P_{\alpha+1} \times P_{\alpha+1}$. Consequently, if $X \in \mathcal{X}_\alpha$ is predense in P_α (if $Y \in \mathcal{Y}_\alpha$ is predense in $P_\alpha \times P_\alpha$), then X is predense in P (Y is predense in $P \times P$).*

Proof. Let $X \in \mathcal{X}_\alpha$ be predense in P_α; we have to show that for each $p\restriction n \in Q_{\alpha+1}$ there is a stronger $q \in Q_{\alpha+1}$ such that $q \le x$ for some $x \in X$. Let $p = \mathcal{F}(T_n)$ and let $u \in p$. Let $h = \text{length}(u)$. There is $k \ge h$ such that n and k satisfy (28.5)(ii). There is $s \in \{0,1\}^k$ such that $u \in T_n(s)$; let v be the stem of $T_n(s)$. Then $p\restriction v \le T_n(s)$ and $T_n(s) \le x$ for some $x \in X$.

A similar argument, using (28.5)(iii), shows that if $Y \in \mathcal{Y}_\alpha$ is predense in $P_\alpha \times P_\alpha$, then Y is predense in $P_{\alpha+1} \times P_{\alpha+1}$.

Since the sequences $\mathcal{X}_\alpha$, $\alpha < \omega_1$, and $\mathcal{Y}_\alpha$, $\alpha < \omega_1$, are increasing, it follows by induction that X is predense in every P_β, $\beta < \omega_1$, and hence in P. Similarly for Y. □

Lemma 28.4. *$P \times P$ satisfies the countable chain condition (and hence P also satisfies the countable chain condition).*

Proof. Here we use $\diamondsuit$. Let us assume that $Y \subset P \times P$ is a maximal incompatible set of conditions in $P \times P$ and that Y is uncountable. Since each P_α is countable, it is easy to see that the set of all $\alpha < \omega_1$ such that $\tau(Y \cap (P_\alpha \times P_\alpha)) = \tau(Y) \cap \omega_1$ is closed unbounded (τ is the one-to-one function of L_{ω_1} onto ω_1). Then it is not much more difficult to see that the set of all $\alpha < \omega_1$ such that $Y \cap (P_\alpha \times P_\alpha)$ is a maximal antichain in $P_\alpha \times P_\alpha$, is closed unbounded (compare this argument with the $\diamondsuit$-construction of a Suslin tree in L).

By $\diamondsuit$, there exists an α such that $Y' = Y \cap (P_\alpha \times P_\alpha)$ is predense in $P_\alpha \times P_\alpha$ and that $\tau(Y') = S_\alpha$. Therefore $Y' \in \mathcal{Y}_\alpha$ and by Lemma 28.3, Y' is predense in $P \times P$. It follows that $Y' = Y$. Thus Y is countable. □

Lemma 28.5.

(i) *If $a \in \{0,1\}^\omega$, then a is P-generic over L if and only if for every $\alpha < \omega_1$ there is some $n \in \boldsymbol{N}$ such that a is a branch in $\mathcal{F}(T_n^\alpha)$.*

(ii) *If $a \neq b \in \{0,1\}^\omega$, then (a,b) is $(P \times P)$-generic over L if and only if for every $\alpha < \omega_1$ there exist $n, m \in \boldsymbol{N}$ such that a is a branch in $\mathcal{F}(T_n^\alpha)$ and b is a branch in $\mathcal{F}(T_m^\alpha)$.*

Proof. (i) Let a be P-generic and let $\alpha < \omega_1$. Since $Q_{\alpha+1}$ is dense in $P_{\alpha+1}$ and because $Q_{\alpha+1} \in \mathcal{X}_{\alpha+1}$, $Q_{\alpha+1}$ is predense in P. By the genericity of a, there exists a $q \in Q_{\alpha+1}$ such that a is a branch in q. But $q = p{\restriction}s$ where $p = \mathcal{F}(T_n^\alpha)$ for some n and $s \in p$, and clearly a is a branch in p.

Conversely, let us assume that the condition is satisfied. Let $X \subset P$ be a maximal antichain; we wish to show that a is a branch in some $x \in X$. By Lemma 28.4, X is countable, and there is an α such that $X \in \mathcal{X}_\alpha$. Let $n \in \boldsymbol{N}$ be such that a is a branch in $\mathcal{F}(T_n^\alpha)$.

By (28.5)(ii), there is $k \in \boldsymbol{N}$ such that each $T_n(s)$, $s \in \{0,1\}^k$, is stronger than some $x \in X$. Since a is a branch in $\mathcal{F}(T_n)$, it is clear that there is a unique $s \in \{0,1\}^k$ such that a is a branch in $T_n(s)$. But if $x \in X$ is such that $T_n(s) \subset x$, then a is also a branch in x.

(ii) The proof that the condition is necessary is analogous to (i). Thus let us assume that the condition is satisfied and let $Y \subset P \times P$ be a maximal antichain; we want to find $(x,y) \in Y$ such that a is a branch in x and b is a branch in y. Again, there is α such that $Y \in \mathcal{Y}_\alpha$. Let n and $m \in \boldsymbol{N}$ be such that a is a branch in $\mathcal{F}(T_n^\alpha)$ and that b is in $\mathcal{F}(T_m^\alpha)$.

Let $h \in \boldsymbol{N}$ be such that $a{\restriction}h \neq b{\restriction}h$; by (28.5)(iii), there is some $k \in \boldsymbol{N}$ such that for each $s, t \in \{0,1\}^k$, if either $n \neq m$ or $s \neq t$, then $(T_n(s), T_m(t)) \leq (x,y)$ for some $(x,y) \in Y$. There is a unique pair s, t such that a is a branch in x and b is a branch in y where (x,y) is some element of Y such that $(T_n(s), T_m(t)) \leq (x,y)$. □

Corollary 28.6. *If a and b are P-generic over L and $a \neq b$, then (a,b) is $(P \times P)$-generic over L.* □

Corollary 28.7. *If a is P-generic over L, then $L[a] \vDash a$ is the only P-generic over L.*

Proof. If $a \neq b$ and if both a and b are P-generic over L, then by the Product Lemma, b is a P-generic over $L[a]$ and hence $b \notin L[a]$. □

Lemma 28.8. *The set $H = \{a : a$ is P-generic over $L\}$ is Π_1 over HC.*

Proof. It follows from the construction of P that the function $\alpha \mapsto \langle T_n^\alpha : n \in \omega\rangle$ is Δ_1 over $L_{\omega_1^L}$. Since $L_{\omega_1^L}$ is a Σ_1 set over HC, the function is Δ_1 over HC. By Lemma 28.5,

$$a \in H \leftrightarrow (\forall \alpha < \omega_1^L)(\exists n \in \omega)\, a \text{ is a branch in } \mathcal{F}(T_n^\alpha)$$

and hence H is Π_1 over HC. □

Corollary 28.9. *If a is a P-generic over L and $A = \{n \in \boldsymbol{N} : a(n) = 1\}$, then $L[a] \vDash A$ is a Δ^1_3 subset of $\boldsymbol{N}$.*

Proof. We have in $L[a]$

$$n \in A \leftrightarrow (\exists a \in \mathcal{N})(a \in H \text{ and } a(n) = 1) \leftrightarrow (\forall a \in \mathcal{N})(a \in H \to a(n) = 1).$$

Since H is Π_1 over HC, it is a Π^1_2 subset of $\mathcal{N}$. It follows that A is Δ^1_3. □

Namba Forcing

By Jensen's Covering Theorem if λ is a regular cardinal in L and V is a generic extension of L, then either $\operatorname{cf}\lambda = |\lambda|$ or $\lambda < \aleph_2$. In other words, the only nontrivial change of cofinality is to make $|\lambda| = \aleph_1$ and $\operatorname{cf}\lambda = \omega$. The following model, due to Namba, does exactly that:

Theorem 28.10 (Namba). *Assume* CH. *There is a generic extension $V[G]$ such that $\aleph_1^{V[G]} = \aleph_1$ and $\operatorname{cf}^{V[G]}(\omega_2^V) = \omega$.*

Proof. Let S be the set of all finite sequences of ordinals less than ω_2, $S = \omega_2^{<\omega}$. A *tree* is a set $T \subset S$ such that if $t \in T$ and $s = t{\restriction}n$ for some n, then $s \in T$. A nonempty tree $T \subset S$ is *perfect* if every $t \in T$ has $\aleph_2$ extensions $s \supset t$ in T. (Note that then every $t \in T$ has $\aleph_2$ *incompatible* extensions in T.) In analogy with perfect sets in the Baire space, we have a Cantor-Bendixson analysis of trees $T \subset S$: Let

$$T' = \{t \in T : t \text{ has } \aleph_2 \text{ extensions in } T\}$$

and let $T_0 = T$, $T_{\alpha+1} = T'_\alpha$, $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$ if α is limit. Let $\theta < \omega_3$ be the least θ such that $T'_\theta = T_\theta$. Then T_θ is either empty or perfect.

If T has no perfect $\bar{T} \subset T$, then the above procedure leads to $T_\theta = \emptyset$, and we can associate with each $t \in T$ an ordinal number

$$h_T(t) = \text{the least } \alpha \text{ such that } t \notin T_{\alpha+1}. \tag{28.7}$$

If $s \subset t$, then $h_T(s) \geq h_T(t)$, and for every $t \in T$,

$$|\{s \in T : t \subset s \text{ and } h_T(s) = h_T(t)\}| < \aleph_2. \tag{28.8}$$

Now let us describe the notion of forcing.

Let P be the set of all perfect trees $T \subset S$, partially ordered by inclusion. We shall show that in the generic extension, ω_2 has cofinality ω and ω_1 is preserved.

If G is a generic set of conditions, we define in $V[G]$ a function $f : \omega \to \omega_2^V$ as follows:

$$f(n) = \alpha \leftrightarrow \forall T \in G\ \exists s \in T \text{ such that } s(n) = \alpha.$$

An easy argument using genericity of G shows that $f(n)$ is uniquely defined for each n and that the function f maps ω cofinally into ω_2^V.

We shall prove now that ω_1 is preserved in the extension by showing that every $f : \omega \to \{0,1\}$ in $V[G]$ is in the ground model. Thus let T be a condition, and let $\dot{f}$ be a name such that

$$T \Vdash \dot{f} \text{ is a function from } \omega \text{ into } \{0,1\}.$$

We shall find a stronger condition that decides each $\dot{f}(n)$; that is, we shall find a function $g : \omega \to \{0,1\}$ such that some condition stronger than T forces $\dot{f}(n) = g(n)$, for all n.

We proceed as follows. By induction on length of s, we construct, for each $s \in S$, conditions T_s and numbers $\alpha_s \in \{0,1\}$ such that:

(28.9)
(i) if $s_1 \subset s_2$, then $T_{s_1} \supset T_{s_2}$,
(ii) if $\text{length}(s) = n$, then $T_s \Vdash \dot{f}(n) = \alpha_s$,
(iii) for every n, the conditions T_s, $s \in \omega_2^n$, are mutually incompatible, and moreover, there are mutually incompatible sequences $t_s \in S$, $s \in \omega_2^n$, such that for each $s \in \omega_2^n$, $t_s \in T_s$ and for all $t \in T_s$, either $t \subset t_s$ or $t_s \subset t$.

The "moreover" clause in (iii) is stronger than incompatibility of the conditions and implies that any condition stronger than $\bigcup_{s\in\omega_2^n} T_s$ is compatible with some T_s, $s \in \omega_2^n$.

The construction of conditions satisfying (28.9) is straightforward: We let $t_\emptyset = \emptyset$ and $T_\emptyset \subset T$ be any condition that decides $\dot{f}(0)$: $T_\emptyset \Vdash \dot{f}(0) = \alpha_\emptyset$. Having defined T_s, t_s, and α_s for $s \in \omega_2^n$ we first pick $\aleph_2$ incompatible extensions $t_{s^\frown i}$, $i < \omega_2$, of t_s in T_s, and then find $T_{s^\frown i} \subset T_s$ and $\alpha_{s^\frown i}$, $i < \omega_2$, such that $T_{s^\frown i} \Vdash \dot{f}(n+1) = \alpha_{s^\frown i}$ and that each $t \in T_{s^\frown i}$ is compatible with $t_{s^\frown i}$. Note that if $s_1 \subset s_2$, then $t_{s_1} \subset t_{s_2}$.

For any function $g : \omega \to \{0,1\}$, we define a tree $T(g) \subset S$ (not necessarily a perfect tree) as follows: If $\bar{\beta} = \langle \beta_0, \ldots, \beta_n \rangle$ is a finite sequence of zeros and ones, we let

$$T(\bar{\beta}) = \bigcup \{T_s : s \in \omega_2^n \text{ and } \langle \beta_0, \ldots, \beta_n \rangle = \langle \alpha_\emptyset, \ldots, \alpha_{s\restriction k}, \ldots, \alpha_s \rangle\} \tag{28.10}$$

and

$$T(g) = \bigcap_{n=1}^{\infty} T(g\restriction n). \tag{28.11}$$

Each $T(\bar{\beta})$ is a condition (a perfect tree) and by the remark following (28.9), we have

$$T(\bar{\beta}) \Vdash \dot{f}(k) = \beta_k \qquad (k = 0, \ \ldots, \ n).$$

Thus if we show that there is at least one $g : \omega \to \{0,1\}$ such that the tree $T(g)$ contains a perfect subtree, our proof will be complete.

Lemma 28.11. *There exists some $g : \omega \to \{0,1\}$ such that $T(g)$ contains a perfect subtree.*

Proof. Let us assume that no $T(g)$ has a perfect subtree. Then by (28.8) there exists, for each $g : \omega \to \{0,1\}$, a function $h_g : T(g) \to \omega_3$ such that $h_g(s) \geq h_g(t)$ whenever $s \subset t$, and that for each $t \in T(g)$, there are at most $\aleph_1$ elements $s \supset t$ in $T(g)$ such that $h_g(s) = h_g(t)$.

By induction, we construct a sequence $s_0 \subset s_1 \subset \ldots \subset s_n \subset \ldots$ such that for all n, $s_n \in \omega_2^n$. At stage n we consider the node t_{s_n} of T_{s_n}. Since there are only $\aleph_1$ functions $g : \omega \to \{0,1\}$, there exists an $i < \omega_2$ such that $h_g(t_{s_n^\frown i}) < h_g(t_{s_n})$ for all g for which $h_g(t_{s_n^\frown i})$ is defined. We let $s_{n+1} = s_n^\frown i$.

Given the sequence s_n, $n = 0, 1, \ldots$, we consider the function $g(n) = \alpha_{s_n}$, $n < \omega$. By (28.10) and (28.11), each t_{s_n} belongs to $T(g)$, and so $h_g(t_{s_n})$ is defined for all n. However, then the sequence $h_g(t_{s_0}) > h_g(t_{s_1}) > \ldots$ of ordinals is descending, a contradiction. □

A Cohen Real Adds a Suslin Tree

We proved earlier that Suslin trees exist in L, and that adding generically a subset of ω_1 with countable conditions adds a Suslin tree. It turns out that adding a Cohen real also adds a Suslin tree. This result is due to Shelah; the following proof is due to Todorčević.

Theorem 28.12 (Shelah). *If r is a Cohen real over V then in $V[r]$ there exists a Suslin tree.*

Proof. We start with an alternative construction of an Aronszajn tree, a modification of the construction in Theorem 9.16.

Lemma 28.13. *There exists an ω_1-sequence of functions $\langle e_\alpha : \alpha < \omega_1 \rangle$ such that*

(28.12) (i) *e_α is a one-to-one function from α into ω, for each $\alpha < \omega_1$;*
(ii) *for all $\alpha < \beta < \omega_1$, $e_\alpha(\xi) = e_\beta(\xi)$ for all but finitely many $\xi < \alpha$.*

Proof. Exercise 28.1 (or see Kunen [1980], Theorem II.5.9). □

The set $\{e_\alpha \restriction \beta : \alpha, \beta \in \omega_1\}$ ordered by inclusion is a tree. Since every node at level α is a finite change of e_α, all levels are countable; there are no uncountable branches and so the tree is an Aronszajn tree (Exercise 28.2).

For any function $r : \omega \to \omega$, consider the tree

$$T_r = \{r \circ (e_\alpha \restriction \beta) : \alpha, \beta \in \omega_1\}; \tag{28.13}$$

again, T_r is an ω_1-tree whose all levels are countable (but need not be Aronszajn in general). We prove Theorem 28.12 by showing that if $\langle e_\alpha : \alpha < \omega_1 \rangle$ is, in V, a sequence that satisfies (28.12) and if r is a Cohen real over V, then in $V[G]$, T_r is a Suslin tree.

We show that T_r has no uncountable antichains; this, and an easy argument using genericity of r, also shows that T_r has no uncountable branches. If T_r has an uncountable antichain then, because every uncountable subset of ω_1 in $V[r]$ has an uncountable subset in V (Exercise 28.3), there exist in V, an uncountable set $W \subset \omega_1$ and a function $\langle \alpha(\beta) : \beta \in W \rangle$ such that

$$\{r \circ (e_{\alpha(\beta)} \restriction \beta) : \beta \in W\} \tag{28.14}$$

is an antichain.

For each $\beta \in W$, let $t_\beta = e_{\alpha(\beta)} \restriction \beta$, and let p be a Cohen forcing condition; we shall find a stronger condition q and $\beta_1, \beta_2 \in W$ that forces that $\dot{r} \circ t_{\beta_1}$ and $\dot{r} \circ t_{\beta_2}$ are compatible functions; therefore no condition forces that (28.14) is an antichain in T_r.

Let $p = \langle p(0), \ldots, p(n-1) \rangle$. For each $\beta \in W$, let X_β be the finite set $\{\xi < \beta : t_\beta(\xi) < n\}$. By the Δ-Lemma (Theorem 9.18) there exist a finite set $S \subset \omega_1$ and an uncountable $Z \subset W$ such that when $\beta_1, \beta_2 \in Z$, then $X_{\beta_1} \cap X_{\beta_2} = S$ and that $t_{\beta_1} \restriction S = t_{\beta_2} \restriction S$.

Now let $\beta_1 < \beta_2$ be two elements of Z. We claim that there exists a condition $q \supset p$ such that $q \circ (t_{\beta_2} \restriction \beta_1) = q \circ t_{\beta_1}$ (q obliterates the disagreement). Such a condition q forces $\dot{r} \circ t_{\beta_1} \subset \dot{r} \circ t_{\beta_2}$.

To construct q, let m be greater than $t_{\beta_i}(\xi)$, $i = 1, 2$, for each $\xi < \beta_1$ such that $t_{\beta_1}(\xi) \neq t_{\beta_2}(\xi)$. Let k be such that $n \leq k < m$. If there exist $\xi, \eta < \beta_1$ such that $t_{\beta_2}(\eta) = k$ and $t_{\beta_1}(\eta) = t_{\beta_2}(\xi)$, let $l = t_{\beta_1}(\xi)$ and let $q(k) = p(l)$. More generally, let $f = t_{\beta_1}^{-1} \circ t_{\beta_2}$ and let f^i, $i < \omega$, denote the i-th iterate of f. If there exist $\xi, \eta < \beta_1$ such that $t_{\beta_2}(\eta) = k$ and $\eta = f^i(\xi)$ for some i, let $l = t_{\beta_1}(\xi)$ and let $q(k) = p(l)$. Otherwise, let $q(k) = 0$. Verify that q obliterates the disagreement. □

Consistency of Borel's Conjecture

A set X of real numbers has strong measure zero if for every sequence $\langle \varepsilon_n : n < \omega \rangle$ of positive real numbers there is a sequence $\langle I_n : n < \omega \rangle$ of intervals with $\text{length}(I_n) \leq \varepsilon_n$ such that $X \subset \bigcup_{n=0}^{\infty} I_n$.

Borel's Conjecture. *All strong measure zero sets are countable.*

Borel's Conjecture fails under CH—see Exercise 26.18. The following theorem shows that it is consistent with ZFC:

Theorem 28.14 (Laver). *Assuming* GCH *there is a generic extension* $V[G]$ *in which* $2^{\aleph_0} = \aleph_2$ *and Borel's Conjecture holds.*

Laver's proof uses the countable support iteration (of length ω_2) of a forcing notion that adds a *Laver real*. We shall now describe this forcing. (Subsequently, Laver proved that an iteration of Mathias forcing also yields Borel's Conjecture).

Definition 28.15. A tree $p \subset Seq$ is a *Laver tree* if it has a *stem*, i.e., a maximal node $s_p \in p$ such that $s_p \subset t$ or $t \subset s_p$ for all $t \in p$, and

(28.15) $\forall t \in p$ if $t \supset s_p$ then the set $S^p(t) = \{a \in \omega : t^\frown a \in p\}$ is infinite.

Laver forcing has as forcing conditions Laver trees, partially ordered by inclusion. If G is a generic set of Laver trees, let

$$f = \bigcup\{s_p : p \in G\}; \tag{28.16}$$

the function $f : \omega \to \omega$ is a *Laver real*. Since

$$G = \{p : s_p \subset f \text{ and } \forall n \geq |s|\ f(n) \in S^p(f{\restriction}n)\}$$

we have $V[G] = V[f]$.

Consider a canonical enumeration of Seq in which s appears before t if $s \subset t$ and $s^\frown a$ appears before $s^\frown(a+1)$. If p is a Laver tree, then the part of p above the stem is isomorphic to Seq and we have an enumeration $s^p_0 = s_p$, $s^p_1, \ldots, s^p_n, \ldots$ of $\{t \in p : t \supset s_p\}$, for every Laver tree p. Let

$$q \leq_n p \text{ if } q \leq p \text{ and } s^p_i = s^q_i \text{ for all } i = 0, \ldots, n \tag{28.17}$$

(in particular $q \leq_0 p$ means that $q \leq p$ and p and q have the same stem). A *fusion sequence* is a sequence of Laver trees such that

$$p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots \geq_n \ldots.$$

Lemma 28.16. *If $\{p_n\}^\infty_{n=0}$ is a fusion sequence then $p = \bigcap^\infty_{n=0} p_n$ is a Laver tree (the* fusion *of $\{p_n\}^\infty_{n=0}$), and $p \leq_n p_n$ for all n.*

Proof. Let s_0 bet the stem of p_0. Then s_0 is the stem of p, and the set $S^p(s_0) = \bigcap_n S^{p_n}(s_0)$ is infinite. For every $a \in S^p(s_0)$, the set $S^p(s_0{}^\frown a) = \bigcap_n S^{p_n}(s_0{}^\frown a)$ is infinite, and so on. □

If p is a Laver tree and $s \in p$, then $p{\restriction}s$ is the Laver tree $\{t \in p : t \subset s$ or $t \supset s\}$. Let p be a Laver tree and let $n \geq 0$. For each $i \leq n$, let p_i be the tree with stem s^p_i that is the union of all $p{\restriction}(s^p_i{}^\frown a)$ where $a \in S^p(s^p_i)$ and $s^p_i{}^\frown a$ is not one of the s^p_j, $j \leq n$. The trees $p_0, \ldots, p_n$ (the *n-components* of p) form a maximal set of incompatible subtrees of p.

Let $q_0, \ldots, q_n$ be the Laver trees such that $q_i \leq_0 p_i$ for all $i = 0, \ldots, n$. The *amalgamation* of $\{q_0, \ldots, q_n\}$ into p is the tree

$$r = q_0 \cup \ldots \cup q_n; \tag{28.18}$$

we have $r \leq_n p$.

Lemma 28.17. *If $p \Vdash \dot{X} : \omega \to V$ then there exists a $q \leq_0 p$ and a countable A such that $q \Vdash \dot{X} \subset A$.*

Proof. Let $\{u_n\}_n$ be a sequence of natural numbers such that each number appears infinitely often. We shall construct a fusion sequence $\{p_n\}_n$ with $p_0 = p$, and finite sets A_n so that the fusion forces $\dot{X} \subset \bigcup_n A_n$. At stage n, let $p^0, \ldots, p^n$ be the n-components of the Laver tree p_n. For each $i = 0, \ldots, n$ if there exist a condition $q_i \leq_0 p^i$ and some a^i_n such that

$$q_i \Vdash \dot{X}(u_n) = a^i_n \tag{28.19}$$

we choose such q_i and a^i_n (otherwise let $q_i = p^i$). Let A_n be the collection of the a^i_n, and let p_{n+1} be the amalgamation of $\{q_0, \ldots, q_n\}$ into p_n. We have $p_{n+1} \leq_n p_n$.

Let p_∞ be the fusion of $\{p_n\}_{n=0}^\infty$ and let $A = \bigcup_{n=0}^\infty A_n$. We have $p_\infty \leq_0 p$; to prove that $p_\infty \Vdash \dot{X} \subset A$, let $q \leq p_\infty$ and let $u \in \omega$. Let $\bar{q} \leq q$ and a be such such that $\bar{q} \Vdash \dot{X}(n) = a$. Let n be large enough so that $u = u_n$ and that the stem of $\bar{q}$ is in the set $\{s_0^{p_n}, \ldots, s_n^{p_n}\}$, say $s = s_i^{p_n}$.

Let p^i be the ith n-component of p_n. As $\bar{q} \cap p^i \leq_0 p^i$ and decides $\dot{X}(u_n)$, we have chosen $a^i_n = a$ at that stage, and therefore $a \in A$, and $\bar{q} \Vdash \dot{X}(u) \in A$. Hence $p_\infty \Vdash \dot{X} \subset A$. □

Corollary 28.18. *The Laver forcing preserves* $\aleph_1$. □

The following property of the Laver forcing is reminiscent of Prikry and Mathias forcings:

Lemma 28.19. *Let* $p \Vdash \varphi_1 \vee \ldots \vee \varphi_k$. *Then there exists some* $q \leq_0 p$ *such that*

$$\exists i \leq k \; q \Vdash \varphi_i. \tag{28.20}$$

Proof. Assume to the contrary that the lemma fails. Let s be the stem of p; there are only finitely many $a \in S^p(s)$ such that some $q \leq_0 p{\restriction}(s^\frown a)$ satisfies (28.20). By removing the part of p above these finitely many nodes we obtain $p_1 \leq_0 p$. For every $s^\frown a \in p_1$ there are only finitely many $b \in S^p(s^\frown a)$ such that $\exists q \leq_0 p_1{\restriction}(s^\frown a^\frown b)$ with property (28.20). By removing all such b's (and the nodes above them) we get $p_2 \leq_1 p_1$. Continuing in this way we construct a fusion sequence $p \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots$ and $r = \bigcap_{n=0}^\infty p_n$. If $t \in r$, then there is no $q \leq_0 r{\restriction}t$ with property (28.20). But then no $q \leq r$ forces $\exists i \leq k \; \varphi_i$, a contradiction. □

The main idea of Laver's proof is the following property of the Laver forcing. It shows that forcing with Laver trees kills uncountable strong measure zero sets.

Lemma 28.20. *Let* G *be a generic set for the Laver forcing. Every set of reals in the ground model that has strong measure zero in* $V[G]$ *is countable in* $V[G]$.

We begin by proving two technical lemmas:

Lemma 28.21. *Let p be a Laver tree with stem s and let $\dot{x}$ be a name for a real in $[0,1]$. Then there exist a condition $q \leq_0 p$ and a real u such that for every $\varepsilon > 0$,*

$$q\restriction(s^\frown a) \Vdash |\dot{x} - u| < \varepsilon$$

for all but finitely many $a \in S^q(s)$.

Proof. Let $\{t_n\}_n$ be an enumeration of $\{s^\frown a : a \in S^p(s)\}$. For each n we find, by Lemma 28.19, a condition $q_n \leq_0 p\restriction t_n$ and an interval $J_n = [\frac{m}{n}, \frac{m+1}{n}]$ such that $q_n \Vdash \dot{x} \in J_n$. There is a sequence $\langle k_n : n < \omega\rangle$ so that the J_{k_n} form a decreasing sequence converging to a unique real u. Let $q = \bigcup_{n=0}^\infty q_{k_n}$. □

Lemma 28.22. *Let p be a condition with stem s and let $\langle \dot{x}_n : n < \omega\rangle$ be a sequence of names for reals. Then there exist a condition $q \leq_0 p$ and a set of reals $\{u_t : t \in q,\ t \supset s\}$ such that for every $\varepsilon > 0$ and every $t \in q$, $t \supset s$, for all but finitely many $a \in S^q(t)$,*

$$q\restriction(t^\frown a) \Vdash |\dot{x}_k - u_t| < \varepsilon$$

where $k = \mathrm{length}(t) - \mathrm{length}(s)$.

Proof. Using Lemma 28.21 we get $p_1 \leq_0 p$ and u_s. For every immediate successor t of s in p_1, we get $q_t \leq_0 p_1\restriction t$ and u_t, and let $p_2 = \bigcup_t q_t$. By repeating this argument, we build a fusion sequence $p \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots$, and let $q = \bigcap_{n=0}^\infty p_n$. □

Proof of Lemma 28.20. Let f be the Laver real, and let

$$\varepsilon_n = 1/f(n). \tag{28.21}$$

We shall show that if $X \in V$ is uncountable, then for some n, the sequence $\langle \varepsilon_k : k \geq n\rangle$ witness that X does not have strong measure zero.

Thus let $X \in V$ be a subset of $[0,1]$ and let p be such that $p \Vdash X$ has strong measure zero. Let s be the stem of p of length n. Let $\langle \dot{x}_k : k \geq n\rangle$ be a sequence of names of reals, and for each $k \geq n$ let $\dot{I}_k$ be the interval of length $\dot{\varepsilon}_k$ centered at $\dot{x}_k$. Let us assume that $p \Vdash X \subset \bigcup_{k\geq n} \dot{I}_k$. We shall find a stronger condition that forces that X is countable.

Let $q \leq_0 p$ and $\{u_t : t \in q,\ t \supset s\}$ be a condition and a countable set of reals obtained in Lemma 28.22. We will show that $q \Vdash X \subset \{u_t\}_t$.

Let $v \notin \{u_t : t \in q,\ t \supset s\}$; we shall find some $r \leq q$ such that $r \Vdash v \notin \dot{I}_k$, for all $k \geq n$. We construct r by induction on the levels of q; at stage $k \geq n$ we ensure that $r \Vdash v \notin \dot{I}_k$.

We describe the construction for $k = n$; this can be repeated for all $k \geq n$. Let $\varepsilon = |v - u_s|/2$. For all but finitely many $a \in S^q(s)$, $q\restriction(s^\frown a) \Vdash |\dot{x}_n - u_s| < \varepsilon$. Since $q\restriction(s^\frown a) \Vdash \dot{\varepsilon}_n = 1/\dot{f}(n) = 1/a$, we have $q\restriction(s^\frown a) \Vdash |\dot{x}_n - v| > \dot{\varepsilon}_n$, or $v \notin \dot{I}_n$, for all but finitely many a. Thus we ensure $r \Vdash v \notin \dot{I}_n$ by removing finitely many successors of s. □

Laver's model for the consistency of Borel's Conjecture is obtained by iteration with countable support of length ω_2. At each stage of the iteration, one adds a Laver real by forcing with Laver trees. If the ground model satisfies GCH, then the iteration preserves cardinals and cofinalities, makes $2^{\aleph_0} = \aleph_2$, and the resulting model satisfies Borel's Conjecture.

We state the relevant properties of Laver's model without proof:

Firstly, for every countable set X of ordinals in $V[G]$ there is a set $Y \in V$, countable in V, such that $X \subset Y$. This is the analog of Lemma 28.17 (see Lemma 6(iii) of Laver [1976]) and implies that $\aleph_1$ is preserved by the iteration. In Chapter 31 we prove a more general result, showing that this property is preserved by countable support iteration of *proper* forcing.

Secondly, the iteration satisfies the $\aleph_2$-chain condition (Lemma 10(ii) of Laver [1976]). This can be proved as in Exercise 16.20, or Lemma 23.11, by first showing that for every $\alpha < \omega_2$, the Laver iteration of length α has a dense subset of cardinality $\aleph_1$. Again, this is a general property of countable support iteration of proper forcing, when at each stage, the βth iterate $\dot{Q}_\beta$ has cardinality $\aleph_1$.

The key property of Laver's iteration is that there are no uncountable strong measure zero sets in $V[G]$. If X is a set of reals of size $\aleph_1$ in $V[G]$, then because of the $\aleph_2$-chain condition, X appears at some stage $V[G_\alpha]$, and by forcing a Laver real, one makes X not to have strong measure zero in $V[G_{\alpha+1}]$. However, one has to show that X fails to have strong measure zero in $V[G]$, not just in $V[G_{\alpha+1}]$. The main technical lemma (Laver's Lemma 15) proves that, and is analogous to Lemma 28.20, working with iteration of Laver forcing rather than with Laver trees only.

In his paper [1983] Baumgartner gives the consistency proof of Borel's Conjecture using the countable support iteration of Mathias forcing. His Theorem 7.1 shows that the iteration of either Laver or Mathias forcing preserves $\aleph_1$, and if CH holds in the ground model then iteration of length ω_2 satisfies the $\aleph_2$-chain condition. He also gives a detailed proof of Borel's Conjecture in the iteration of Mathias forcing.

κ^+-Aronszajn Trees

Theorem 9.16 states that there exists an Aronszajn tree, i.e., a tree of length ω_1 with countable levels and no branch of length ω_1. In Chapter 9 we also defined what it means for an infinite cardinal κ to have the *tree property*: Every tree of height κ and levels of size $< \kappa$ has a branch of length κ. When κ is inaccessible then the tree property is equivalent to weak compactness.

Let κ^+ be a successor cardinal. A tree of height κ^+ is a κ^+-*Aronszajn* tree if its levels have size at most κ and it has no branch of length κ^+. When κ is singular, the tree property of κ^+ is related to large cardinals; we shall

now address the case when κ is regular. We discuss the case of $\aleph_2$ as it easily generalizes to any successor of a regular. The construction in Theorem 9.16 generalizes to $\aleph_2$ under the assumption that $2^{\aleph_0} = \aleph_1$ (see Exercises 28.5 and 28.6). It follows that an $\aleph_2$-Aronszajn tree exists unless there is a weakly compact cardinal in L:

Theorem 28.23 (Silver). *If there exists no $\aleph_2$-Aronszajn tree then $\aleph_2$ is a weakly compact cardinal in L.*

Proof. If $\aleph_2$ is a successor cardinal in L, then there exists some $A \subset \omega_1$ such that $\aleph_1^{L[A]} = \aleph_1$ and $\aleph_2^{L[A]} = \aleph_2$. In $L[A]$, $2^{\aleph_0} = \aleph_1$ holds and therefore there exists a special $\aleph_2$-Aronszajn tree T. But then T is a special $\aleph_2$-Aronszajn tree in V. Thus if there are no $\aleph_2$-Aronszajn trees, $\aleph_2$ is inaccessible in L.

To show that $\lambda = \aleph_2$ is weakly compact in L if λ has the tree property, let $B \in L$ be (in L) a λ-complete algebra of subsets of λ and $|B| = \lambda$. We shall find a λ-complete nonprincipal ultrafilter U on B with $U \in L$. (Then, by the argument in Lemma 10.18, it follows that λ is weakly compact in L.)

Let $\alpha < (\lambda^+)^L$ be a limit ordinal such that $B \in L_\alpha$ and $L_\alpha \vDash |B| = \lambda$. Let $\{X_\xi : \xi < \lambda\}$ be an enumeration, in L, of $P(\lambda) \cap L_\alpha$, and let T be the set of all constructible functions $f \in \{0,1\}^{<\lambda}$ such that

$$\left|\bigcap\{X_\xi : f(\xi) = 1\} \cap \bigcap\{\lambda - X_\xi : f(\xi) = 0\}\right| = \lambda.$$

Since λ is inaccessible in L, T is a λ-tree with levels of size $< \lambda$.

Since λ has the tree property, T has a branch of length λ, a function $F : \lambda \to \{0,1\}$ such that $F{\restriction}\nu \in T$ for all $\nu < \lambda$. If we let $D = \{X_\xi : F(\xi) = 1\}$ then D is (in V) a λ-complete nonprincipal ultrafilter on $P(\lambda) \cap L_\alpha$. Let $\mathrm{Ult} = \mathrm{Ult}_D L_\alpha$ be the ultrapower of L_α by D (using functions in L_α), let L_β be its transitive collapse and let $j : L_\alpha \to L_\beta$ be the corresponding elementary embedding.

If $e \in L_\alpha$ is an enumeration of B, $e : \lambda \to B$, then $E = j(e) \in L_\beta$ and $U = \{e(\xi) : \lambda \in E(\xi)\}$ is a constructible λ-complete nonprincipal ultrafilter on B. □

The following theorem shows that it is consistent (relative to a weakly compact cardinal) that there exist no $\aleph_2$-Aronszajn trees.

Theorem 28.24 (Mitchell). *If κ is a weakly compact cardinal then there is a generic extension in which $\kappa = \aleph_2$, $2^{\aleph_0} = \aleph_2$, and there exists no $\aleph_2$-Aronszajn tree.*

The model is obtained by a two-stage iteration $P * \dot{Q}$. The forcing $P = P_\kappa$ adds κ Cohen reals to the ground model; let $G = G_\kappa$ be generic on P; for each $\alpha < \kappa$, let P_α be the forcing for adding α Cohen reals, and let $G_\alpha = G \cap P_\alpha$.

In $V[G]$, consider the forcing conditions q for adding κ Cohen subsets of ω_1: q is a 0–1 function on a countable subset of κ. Let Q be the set of all

such q that satisfy, in addition, the requirement that

$$q\restriction\alpha \in V[G_\alpha] \qquad (\text{all } \alpha < \kappa).$$

This amounts to forcing with pairs (p, q) where $p \in P$ and q is a countable function on a subset of κ with values $q(\alpha) \in B(P_\alpha)$ (then if G is generic on P, we have $\bar{q} \in Q$ where $\bar{q}(\alpha) = 1$ if $q(\alpha) \in G$ and $\bar{q}(\alpha) = 0$ if $q(\alpha) \notin G$).

We list some properties of $P * \dot{Q}$ which are not difficult to verify. Let G be generic on P and let H be $V[G]$-generic on Q.

First, every countable set of ordinals in $V[G][H]$ is in $V[G]$. Hence $\aleph_1$ is preserved.

Second, every cardinal between $\aleph_1$ and κ is collapsed: If $\aleph_1 \leq \delta < \kappa$, let t be the following function on ω_1:

$$t(\alpha) = \{n \in \omega : \exists f \in H\, f(\delta + \omega \cdot \alpha + n) = 1\}.$$

The function maps ω_1 onto $P(\omega)^{V[G_{\delta+\omega_1}]}$ which has cardinality δ.

Third, $P * \dot{Q}$ satisfies the κ-chain condition. This is proved similarly to the κ-chain condition of the Lévy collapse.

Finally, it is clear that $2^{\aleph_0} = \kappa$ in $V[G][H]$. The main technical lemma (Lemma 3.8 of Mitchell [1972/73]) asserts the following: For $\alpha < \kappa$ let $Q_\alpha = \{q \in Q : \mathrm{dom}(q) \subset \alpha\}$, and $H_\alpha = H \cap Q_\alpha$. If $\gamma < \kappa$ is a regular uncountable cardinal and if $t \in V[G][H]$ is an ordinal function on γ such that $t\restriction\alpha \in V[G_\gamma][H_\gamma]$ for all $\alpha < \gamma$, then $t \in V[G_\gamma][H_\gamma]$.

Now one shows that κ has the tree property in $V[G][H]$ as follows: Let $B = B(P * \dot{Q})$ and let $\dot{T}$ be a B-name for a binary relation on κ that is in $V[G][H]$ a tree of height κ with levels of size $< \kappa$. There is a closed unbounded set $C \subset \kappa$ such that if $\gamma \in C$ is an inaccessible cardinal then $B_\gamma = B(P_\gamma * \dot{Q}_\gamma)$ is a complete Boolean subalgebra of $B(P * \dot{Q})$ and that $\dot{T} \cap (\gamma \times \gamma)$ is a B_γ-valued name for $\dot{T}\restriction\gamma$, the first γ levels of $\dot{T}$.

To show that $\dot{T}$ has a branch of length κ, assume that it has none; that this is so in V^B is a Π^1_1 sentence true in $(\kappa, B, \dot{T})$ and since κ is Π^1_1-indescribable, the same is true in V^{B_γ}: $\dot{T}\restriction\gamma$ has no branch of length γ in V^{B_γ}. But any node in the γth level of T produces an ordinal function on γ whose initial segments are in V^{B_γ}; by the technical lemma alluded to above, the function itself is in V^{B_γ}, and is a branch in $\dot{T}\restriction\gamma$. A contradiction. □

A related result is the following theorem that we state without proof:

Theorem 28.25 (Laver-Shelah). *If there exists a weakly compact cardinal then there exists a generic extension in which* $2^{\aleph_0} = \aleph_1$ *and there exists no* $\aleph_2$*-Suslin tree.* □

(In the Laver-Shelah model, $2^{\aleph_1}$ is greater than $\aleph_2$.)

Exercises

28.1. Find $\langle e_\alpha : \alpha < \omega_1 \rangle$ such that each $e_\alpha : \alpha \to \omega$ is one-to-one and if $\alpha < \beta$ then e_α and $e_\beta \restriction \alpha$ differ at only finitely many places.

[Construct the e_α by induction on α, such that for every α, $\omega - \mathrm{ran}(e_\alpha)$ is infinite.]

28.2. Given $\langle e_\alpha : \alpha < \omega_1 \rangle$ as in Exercise 28.1, show that the set $\langle e_\alpha \restriction \beta : \alpha, \beta \in \omega_1 \rangle$ is an Aronszajn tree.

28.3. If r is a Cohen real over V, then for every uncountable $X \subset \omega_1$ in $V[r]$ there exists an uncountable $Y \subset X$ in V.

[The notion of forcing is countable.]

28.4. A Laver real eventually dominates every $g : \omega \to \omega$ in V.

28.5. If $2^{\aleph_0} = \aleph_1$, then there exists an $\aleph_2$-Aronszajn tree.

[Imitate the proof of Theorem 9.16. Let Q be the lexicographically ordered set $\omega_1^{<\omega}$; every $\alpha < \omega_2$ embeds in any interval of Q. Construct T using bounded increasing sequences in Q of length $< \omega_2$. At limit steps of cofinality ω extend all branches that represent bounded sequences in Q; here we use $2^{\aleph_0} = \aleph_1$.]

28.6. The tree constructed in Exercise 28.5 is special, i.e., the union of $\aleph_1$ antichains.

[Compare with Exercises 9.8 and 9.9.]

Historical Notes

The construction of a nonconstructible Δ^1_3 real in Theorem 28.1 is as in Jensen [1970]. Namba's forcing appeared in Namba [1971]; in [1976] Bukovský obtains the same result by a somewhat different forcing construction. The result that adding a Cohen real adds a Suslin tree is due to Shelah [1984]; the present proof is due to Todorčević [1987] (for details see Bagaria [1994]).

The consistency of Borel's Conjecture is due to Laver [1976].

For the construction of a κ^+-Aronszajn tree if $2^{<\kappa} = \kappa$, see Specker [1949]. The consistency proof of the tree property for $\aleph_2$, as well as the proof of Silver's Theorem 28.23 appeared in Mitchell [1972/73]. Theorem 28.25 is in Laver-Shelah [1981].

29. More Combinatorial Set Theory

Ramsey Theory

Ramsey's Theorem 9.1 has been generalized in many ways, giving rise to an area of combinatorial mathematics known as Ramsey theory. In this section we present three results involving combinatorics of infinite sets. For a complete account of Ramsey theory we refer the reader to the book [1980] of Graham, Rothschild and Spencer.

Theorem 29.1 (Hindman). *If $\boldsymbol{N}$ is partitioned into finitely many pieces then one of the pieces A contains an infinite set H such that $a_1+\ldots+a_n \in A$ whenever $a_1, \ldots, a_n$ are distinct members of H.*

For the proof, we introduce the concept of an *idempotent ultrafilter*. If U and V are ultrafilters on $\boldsymbol{N}$, let

$$U + V = \{X \subset \boldsymbol{N} : \{m \in \boldsymbol{N} : X - m \in V\} \in U\} \tag{29.1}$$

where $X-m = \{n : m+n \in X\}$. See Exercises 29.1 and 29.2 for an alternative characterization. In the proof we use the following lemma due to S. Glazer:

Lemma 29.2. *There exists a nonprincipal ultrafilter U on $\boldsymbol{N}$ such that $U + U = U$.*

An ultrafilter U such that $U + U = U$ is *idempotent*. While Glazer's Lemma can be proved directly, it can be deduced from a more general result on topological semigroups. Let $\beta\boldsymbol{N}$ be the space of all ultrafilters on $\boldsymbol{N}$, the Stone-Čech compactification on $\boldsymbol{N}$. The operation $U + V$ on $\beta\boldsymbol{N}$ is a continuous function of U for any fixed V, thus making $(\beta\boldsymbol{N}, +)$ a left-topological semigroup. It can be shown that every compact left-topological semigroup has an idempotent element (Exercises 29.3 and 29.4).

Proof of Theorem 29.1. Given a partition of $\boldsymbol{N}$ into finitely many pieces, let U be an idempotent ultrafilter, and let A be a piece of the partition such that $A \in U$. We construct a sequence $A = A_0 \supset A_1 \supset A_2 \supset \ldots$ with $A_k \in U$ and $a_0 < a_1 < a_2 < \ldots$ as follows: Let $a_0 \in A_0$. Given $A_k \in U$ and a_{k-1}, we find $a_k > a_{k-1}$ such that $a_k \in A_k$ and that $A_k - a_k \in U$ (since $\{n : A_k - n \in U\} \in U$). Let $A_{k+1} = A_k \cap (A_k - a_k)$.

Now let $H = \{a_k : k < \omega\}$. To verify that all finite sums from H are in A, one shows, by induction on n, that if $k_1 > \ldots > k_n$ then $a_{k_1} + \ldots + a_{k_n} \in A_{k_n}$. □

A similar technique can be used to give a proof of the following classical theorem in Ramsey Theory. An *arithmetic progression* (of length k) is a finite set of the form

$$\{n, n+d, n+2d, \ldots, n+(k-1)d\} \tag{29.2}$$

where d is a positive integer.

Theorem 29.3 (van der Waerden). *If $\boldsymbol{N}$ is partitioned into finitely many pieces then one of the pieces contains arbitrarily long arithmetic progressions.*

For the proof of Theorem 29.3, we fix an integer $k \geq 1$ and consider the space $(\beta\boldsymbol{N})^k$. Let I be the set of all arithmetic progressions of length k, and let $\bar{I}$ be its closure in $(\beta\boldsymbol{N})^k$. Arguments similar to those in Exercise 29.3 can be used to show that if $R \subset \beta\boldsymbol{N}$ is any minimal right ideal and $U \in R$, then $\langle U, \ldots, U\rangle \in \bar{I}$. For details, we refer reader to Todorčević's book [1997].

Now to prove the theorem, let U be a nonprincipal ultrafilter on $\boldsymbol{N}$ that belongs to some minimal right ideal on $\beta\boldsymbol{N}$. Let A be the piece of the given partition such that $A \in U$, and let $A^* = \{V \in \beta\boldsymbol{N} : A \in V\}$. If $k \geq 1$ is any integer, let I and $\bar{I}$ be as above. Since $\langle U, \ldots, U\rangle \in \bar{I}$, it follows that $(A^* \times \ldots \times A^*) \cap \bar{I}$ is nonempty, and hence $I \cap (A \times \ldots \times A) \neq \emptyset$. Therefore A contains an arithmetic progression of length k.

The third result, which we state without a proof, is the Hales-Jewett Theorem. Let Σ be a finite set, called *alphabet*, and let W be the set of all *words* on Σ, the set of all finite sequences in Σ. Let v be a *variable*, an object not in Σ. The set V of all *variable words* over Σ is the set of all words on $\Sigma \cup \{v\}$ in which v occurs. An *instance* of a variable word $x \in V$ is the result of substituting some $a \in \Sigma$ for v in x.

Theorem 29.4 (Hales-Jewett). *If W is partitioned into finitely many pieces then there is a variable word $x \in V$ whose all instances lie in the same piece of the partition.*

We refer the reader to Todorčević's book for a proof using topological semigroups.

Gaps in ω^ω

Consider the partial order on ω^ω by eventual domination: $f < g$ if and only if $f(n) < g(n)$ for all but finitely many n.

Definition 29.5. Let κ and λ be regular cardinals. A (κ, λ)-*gap* in ω^ω is a pair of transfinite sequences $\langle f_\alpha : \alpha < \kappa \rangle$ and $\langle g_\beta : \beta < \lambda \rangle$ in ω^ω such that

(29.3)
(i) $f_{\alpha_1} < f_{\alpha_2}$ if $\alpha_1 < \alpha_2$,
(ii) $g_{\beta_1} > g_{\beta_2}$ if $\beta_1 < \beta_2$,
(iii) $f_\alpha < g_\beta$ for all $\alpha < \kappa$ and $\beta < \lambda$,
(iv) there is no h between $\{f_\alpha\}_\alpha$ and $\{g_\beta\}_\beta$, i.e., no h such that $f_\alpha < h < g_\beta$ for all α and β.

We shall prove a classical theorem of Hausdorff stating that (ω_1, ω_1)-gaps exist.

First we prove that (ω, ω)-gaps do not exist:

Lemma 29.6. *If $f_0 < f_1 < \ldots < f_n < \ldots < g_m < \ldots < g_1 < g_0$, then there exists an h between $\{f_n\}_n$ and $\{g_m\}_m$.*

Proof. For each k there exists an n_k such that for all $n \geq n_k$, $m_k(n) = \max\{f_0(n), \ldots, f_k(n)\} \leq \min\{g_0(n), \ldots, g_k(n)\} = M_k(n)$. Choose such n_k's so that $n_0 < n_1 < \ldots < n_k < \ldots$, and let h be a function such that $m_k(n) \leq h(n) \leq M_k(n)$ when $n_k \leq n < n_{k+1}$. □

Another easily seen fact is that a (κ, λ)-gap exists if and only if a (λ, κ)-gap exists (Exercise 29.5). That some gaps exists follows from Zorn's Lemma. In fact, there exists an $(\omega, \mathfrak{b})$-gap, see Exercise 29.6 (and there are no (ω, λ)-gaps for $\lambda < \mathfrak{b}$, see Exercise 29.7).

Apart from Hausdorff's Theorem 29.7, the existence of specific (κ, λ)-gaps is unprovable: For instance, $(\mathfrak{c}, \mathfrak{c})$-gaps may or may not exist, depending on the model. A detailed account of known consistency results on gaps can be found in Scheepers [1993].

Theorem 29.7 (Hausdorff). *There exists an (ω_1, ω_1)-gap in ω^ω.*

Proof. We construct an increasing $\langle f_\alpha : \alpha < \omega_1 \rangle$ and a decreasing $\langle g_\beta : \beta < \omega_1 \rangle$ such that

(29.4)
(i) for all α and β, $\lim_{n\to\infty} g_\beta(n) - f_\alpha(n) = \infty$,
(ii) for every $\alpha < \omega_1$ and every $n \in \omega$, there are only finitely many $\beta < \alpha$ such that $\forall k \geq n\ f_\alpha(k) < g_\beta(k)$.

Let us show first that (29.4) guarantees that $\{f_\alpha\}_\alpha$, $\{g_\beta\}_\beta$ is a gap. Assume that $h \in \omega^\omega$ is between $\{f_\alpha\}_\alpha$, $\{g_\beta\}_\beta$. Then there exist an uncountable $Z \subset \omega_1$ and some $n \in \omega$ such that for all $k \geq n$ and $h(k) < g_\alpha(k)$ for all $k \geq n$. Thus $f_\alpha(k) < g_\beta(k)$ for all $\alpha, \beta \in Z$ and all $k \geq n$. Now if α is the ωth element of Z, the set $\{\beta < \alpha : \forall k \geq n\ f_\alpha(k) < g_\beta(k)\}$ is infinite, contradicting (29.4)(ii).

We construct f_α and g_α by induction on α. Let $f_0(n) = 0$ and $g_0(n) = n$ for all n. Let $\gamma < \omega_1$. If f_α and g_β satisfy (29.4) for all $\alpha, \beta \leq \gamma$, then it is easy to find $f_{\gamma+1}$ and $g_{\gamma+1}$ such that (29.4) remains true.

Thus let γ be a limit ordinal and assume that $\{f_\alpha\}_{\alpha<\gamma}$ $\{g_\beta\}_{\beta<\gamma}$ satisfy (29.4). Let us use the following terminology: If $f < g_\beta$ for all $\beta < \gamma$ and if $C \subset \gamma$, we say that f is *near* C if for every n the set $\{\beta \in C : \forall n \geq k\ f(k) < g_\beta(k)\}$ is finite. Note that if $f < f'$ and f is near C then f' is near C.

We wish to find f and g such that $f_\alpha < f < g < g_\beta$ for all $\alpha, \beta < \gamma$ (and $\lim_n(g(n) - f(n)) = \infty$) and that f is near γ. Let h be some function such that $f_\alpha < h < g_\beta$ for all $\alpha, \beta < \gamma$; such an h exists by Lemma 29.6. As each f_α is near α, it follows that h is near α for all $\alpha < \gamma$. It now suffices to find some $f > h$ such that $f < g_\beta$ for all $\beta < \gamma$ and such that f is near α. Then g is easily found. For each n let $C_n = \{\beta < \gamma : \forall k \geq n\, h(k) < g_\beta(k)\}$. Clearly, $C_0 \subset C_1 \subset \ldots \subset C_n \subset \ldots$. As long as all C_n are finite, h is near γ and we are done. Thus assume that the C_α are eventually infinite.

We construct inductively a sequence $h = h_0 < h_1 < \ldots < h_n < \ldots$ of functions below $\{g_\beta\}_{\beta<\gamma}$ such that for each n, h_{n+1} is near C_n. Then if f is any function between $\{h_n\}_{n<\omega}$ and $\{g_\beta\}_{\beta<\gamma}$ and $f(n) \geq h(n)$ for all n, then for each n, f is near the set $\{\beta < \gamma : \forall k \geq n\, f(k) < g_\beta(k)\} \subset C_n$ and hence f is near γ.

Let $n \geq 0$. If C_n is finite, any h_{n+1} is near C_n; thus assume that C_n is infinite. Since for each $\alpha < \gamma$, the set $C_n \cap \alpha$ is finite (because h is near α), the order-type of C_n is ω, and C_n is cofinal in γ. Let $\beta_0 < \beta_1 < \ldots < \beta_i < \ldots$ be the enumeration of C_n. It suffices to find $h_{n+1} > h_n$ such that $h_{n+1} < g_{\beta_i}$ for all i, and that for every m,

$$\{i < \omega : \forall k \geq m\, h_{n+1}(k) < g_{\beta_i}(k)\} \text{ is finite.} \tag{29.5}$$

Let $m_0 < m_1 < \ldots < m_i < \ldots$ be such that for every i,

$$h_n(k) < g_{\beta_i}(k) < g_{\beta_{i-1}}(k) < \ldots < g_{\beta_0}(k) \text{ for all } k \geq m_i.$$

Then the function h_{n+1} defined by

$$h_{n+1}(k) = \begin{cases} h_n(k) & \text{if } k < m_0, \\ g_{\beta_i}(k) & \text{if } m_i \leq k < m_{i+1} \end{cases}$$

satisfies (29.5) and hence is near C_n. □

The Open Coloring Axiom

We shall now discuss the axiom OCA (Open Coloring Axiom) that has a number of applications in combinatorial set theory. Let X be a set of reals (or $X \subset \mathcal{N}$, or $X \subset P(\omega)$, etc.) and let $K \subset [X]^2$. We say that K is *open* if the set $\{(x, y) : \{x, y\} \in K\}$ is an open set in the space $X \times X$. The Open Coloring Axiom (OCA) states:

(29.6) Let X be a subset of $\boldsymbol{R}$. For any partition $[X]^2 = K_0 \cup K_1$ with K_0 open, either there is an uncountable $Y \subset X$ such that $[Y]^2 \subset K_0$, or there exist sets H_n, $n \in \omega$, such that $X = \bigcup_{n=0}^{\infty} H_n$ and $[H_n]^2 \subset K_1$ for all n.

The axiom OCA is consistent with ZFC; we discuss this in Chapter 31. It should be noted that its dual version is false; Exercise 29.9. OCA has a number of consequences; see Todorčević [1989]. (One example is Exercise 29.10.) The most notable is the following result:

Theorem 29.8 (Todorčević). *If* OCA *holds then* $\mathfrak{b} = \aleph_2$.

First we show that under OCA, $\mathfrak{b} > \omega_1$.

Lemma 29.9. *Assume* OCA. *Then every subset of* ω^ω *of size* $\aleph_1$ *is bounded.*

Proof. In order to show that every subset $X \subset \omega^\omega$ of size $\aleph_1$ is bounded, it is clearly enough to show this for every increasing $X = \{f_\alpha\}_{\alpha<\omega_1}$ (i.e., $f_\alpha < f_\beta$ if $\alpha < \beta$), and assume that each f_α is an increasing function from ω to ω. Let $X = \{f_\alpha\}_\alpha$ be such and let $[X]^2 = K_0 \cup K_1$ where K_0 consists of all $\{f_\alpha, f_\beta\}$ with $\alpha < \beta$ such that $f_\alpha(k) > f_\beta(k)$ for some k.

First assume that $X = \bigcup_{n=0}^\infty H_n$ and $[H_n]^2 \subset K_1$ for all n. Then for some n, H_n is uncountable, and if $\alpha < \beta$ are such that $f_\alpha, f_\beta \in H_n$ then $f_\alpha(k) \leq f_\beta(k)$ for all k, and $f_\alpha(k) < f_\beta(k)$ for some k. Then if we let $S_\alpha = \{(m,k) : m \leq f_\alpha(k)\}$, we have an ω_1-chain of subsets of $\omega\times\omega$, a contradiction.

Thus, assuming OCA, there is an uncountable $Y \subset X$ such that $[Y^2] \subset K_0$. We claim that Y is bounded (and it follows that X is bounded). To prove the claim, assume that Y is not bounded and let $\{g_\alpha : \alpha < \omega_1\}$ be the increasing enumeration of Y. For each $t \in \omega^{<\omega}$ that is an initial segment of some $g \in Y$, choose α_t such that $t \subset g_{\alpha_t}$. Then let $\gamma > \sup_t \alpha_t$, and let k_0 be such that for uncountably many β, $g_\gamma(k) < g_\beta(k)$ for all $k \geq k_0$. Thus there is an uncountable $Z \subset \omega_1 - \gamma$ such that $g_\gamma(k) < g_\beta(k)$ for all $\beta \in Z$ and all $k \geq k_0$ and that $g_{\beta_1}{\restriction}k_0 = g_{\beta_2}{\restriction}k_0$ whenever $\beta_1, \beta_2 \in Z$.

Now let $m \geq k_0$ be the least m such that the set $\{g_\beta(m) : \beta \in Z\}$ is infinite (m exists because $\{g_\beta\}_{\beta\in Z}$ is not bounded). There exist some $t \in \omega^m$ and some $W \subset Z$ such that $g_\beta{\restriction}m = t$ for all all $\beta \in W$ and $\{g_\beta(m) : \beta \in W\}$ is infinite.

Let $\alpha = \alpha_t$; since $\alpha < \gamma$, there exists a $k_1 \geq m$ such that $g_\alpha(k) < g_\gamma(k)$ for all $k \geq k_1$. Let $\beta \in W$ be such that $g_\beta(m) \geq g_\alpha(k_1)$. Since $g_\beta{\restriction}m = t = g_\alpha{\restriction}m$, and since g_α and g_β are increasing, we have $g_\alpha(k) \leq g_\beta(k)$ for all $k \leq k_1$. But for $k \geq k_1$ we have $g_\alpha(k) \leq g_\gamma(k) < g_\beta(k)$; hence $g_\alpha(k) \leq g_\beta(k)$ for all k. This contradicts the assumption that $\{g_\alpha, g_\beta\} \in K_0$. Hence Y is bounded, and so X is bounded. □

Toward the proof of $\mathfrak{b} \leq \aleph_2$ we prove the following result on gaps:

Lemma 29.10. *Assume* OCA. *There is no* (κ,λ)*-gap in* ω^ω *such that* κ *and* λ *are regular uncountable, and* $\kappa > \omega_1$.

Proof. Let $\kappa \geq \lambda$ be regular uncountable with $\kappa > \omega_1$, and assume that $\{f_\alpha\}_{\alpha<\kappa}$, $\{g_\beta\}_{\beta<\lambda}$ is a gap. In order to define an open partition, we first modify the gap. For each $\alpha < \kappa$ there exists an m_α such that for λ many β's,

$f_\alpha(k) < g_\beta(k)$ for all $k \geq m_\alpha$; for κ many α's, this m_α is the same. Therefore there is a gap for which $m_\alpha = 0$ for all $\alpha < \kappa$, and we assume that the given gap is such. For each $\alpha < \kappa$, let $S_\alpha = \{\beta < \lambda : \forall k\, f_\alpha(k) < g_\beta(k)\}$; $|S_\alpha| = \lambda$.

Let $X = \{(f_\alpha, g_\beta) : \alpha < \kappa \text{ and } \beta \in S_\alpha\}$, a subspace of $\mathcal{N} \times \mathcal{N}$. Consider the partition $[X]^2 = K_0 \cup K_1$ where $\{(f_\alpha, g_\beta), (f_\gamma, g_\delta)\} \in K_0$ when for some k, either $f_\alpha(k) > g_\delta(k)$ or $f_\gamma(k) > g_\beta(k)$. Because of the additional assumption on the gap, K_0 is open.

First assume that $X = \bigcup_{n=0}^\infty H_n$ with $[H_n]^2 \subset K_1$ for each n. Since κ and λ are uncountable, there exist $A \subset \kappa$ of size κ and for each $\alpha < \kappa$ some $T_\alpha \subset S_\alpha$ of size λ such that all (f_α, g_β) with $\alpha \in A$ and $\beta \in T_\alpha$ are in the same H_n. Since $[H_n]^2 \subset K_1$, we have $\forall k\, f_\alpha(k) < g_\delta(k)$ whenever $\alpha, \gamma \in A$ and $\delta \in T_\gamma$. Thus fix $\gamma \in A$ and let $B = T_\gamma$. A is cofinal in κ, B is cofinal in λ, and if $\alpha \in A$ and $\beta \in B$ then $\forall k\, f_\alpha(k) < g_\beta(k)$. But then the function h defined by $g(k) = \min_{\beta \in B} g_\beta(k)$ is between $\{f_\alpha\}_\alpha$ and $\{g_\beta\}_\beta$, a contradiction.

Next assume that there exists an uncountable $Y \subset X$ such that $[Y]^2 \subset K_0$. If (f_α, g_β) and (f_γ, g_δ) are distinct elements of Y, then because $\beta \in S_\alpha$, $\delta \in S_\gamma$ and $\{(f_\alpha, g_\beta), (f_\gamma, g_\delta)\} \in K_0$, we have $\alpha \neq \gamma$ and $\delta \neq \beta$; thus Y is one-to-one. Therefore there exist increasing ω_1-sequences $\langle \alpha_\nu : \nu < \omega_1 \rangle$ and $\langle \beta_\nu : \nu < \omega_1 \rangle$ such that $\{(f_{\alpha_\nu}, g_{\beta_\nu}) : \nu < \omega_1\} \subset Y$.

Now since $\kappa > \omega_1$, let $h = f_\delta$ where $\delta > \sup_\nu \alpha_\nu$. The function h is between $\{f_{\alpha_\nu}\}_\nu$ and $\{g_{\beta_\nu}\}_\nu$. Now we can find an uncountable $Z \subset \omega_1$ and some m such that for all $\nu, \eta \in Z$, $f_{\alpha_\nu}(k) < h(k) < g_{\beta_\eta}(k)$ for all $k \geq m$, and $f_{\alpha_\nu} \restriction m = f_{\alpha_\eta} \restriction m$, $g_{\beta_\nu} \restriction m = g_{\beta_\eta} \restriction m$. Since $\beta_\nu \in S_{\alpha_\nu}$ for each ν, it follows that $f_{\alpha_\nu}(k) < g_{\beta_\eta}(k)$ for all k, contrary to the assumption that $\{(f_{\alpha_\nu}, g_{\beta_\nu}), (f_{\alpha_\eta}, g_{\beta_\eta})\} \in K_0$. □

Proof of Theorem 29.8. Assuming $\mathfrak{b} > \omega_2$, we shall construct an (ω_2, λ)-gap with λ regular uncountable. Then OCA and Lemma 29.10 complete the proof.

Let $\langle f_\alpha : \alpha < \omega_2 \rangle$ be an increasing sequence of increasing functions. Since $\mathfrak{b} > \omega_2$, there exists some g_0 such that $g_0 > f_\alpha$ for all α. Let $\langle g_\beta : \beta < \vartheta \rangle$ be a maximal decreasing sequence of functions such that $g_\beta > f_\alpha$ for all α. At successor stages we can let $g_{\beta+1}(k) = g_\beta(k) - 1$ and so ϑ is a limit ordinal. We complete the proof by showing that $\operatorname{cf} \vartheta > \omega$.

Thus assume that $\vartheta = \lim_n \beta_n$. Given $\alpha < \omega_2$ let $m_\alpha(0) < m_\alpha(1) < \ldots < m_\alpha(n) < \ldots$ be such that for all $i = 0, \ldots, n$, $f_\alpha(k) < g_{\alpha_i}(k)$ for all $k \geq m_\alpha(n)$. Since $\mathfrak{b} > \omega_2$, there exists a function h such that $h > m_\alpha$ for all α. Now if we let $g(n) = \min_{i \leq n} g_{\alpha_i}(h(n))$, then $g > f_\alpha$ for all α and $g < g_{\alpha_n}$ for all n, contrary to the maximality of ϑ. □

Almost Disjoint Subsets of ω_1

Let κ be a regular uncountable cardinal, and let X and Y be unbounded subsets of κ. The sets X and Y are almost disjoint if $|X \cap Y| < \kappa$ (cf. Definition 9.20). Similarly, two functions f and g on κ are *almost disjoint* if for

some $\gamma < \kappa$, $f(\alpha) \neq g(\alpha)$ for all $\alpha > \gamma$ (cf. Definition 9.22). Unlike in the case $\kappa = \omega$, it is a nontrivial question how large a set of almost disjoint sets of functions can be; clearly, the maximal size of an almost disjoint family of subsets of κ is equal to the maximal size of an almost disjoint family of functions from κ to κ.

For simplicity, we consider the case $\kappa = \omega_1$. This can be generalized to any regular uncountable κ.

First, there exists an almost disjoint family of size $\aleph_2$ (Lemma 9.23), and if $2^{\aleph_0} = \aleph_1$ then there exists one of size $2^{\aleph_1}$. We shall prove the following:

Theorem 29.11. *If $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$ then there exists an almost disjoint family of $2^{\aleph_1}$ uncountable subsets of ω_1.*

Compare this with Theorem 22.16: If I is the ideal of bounded subsets of ω_1, then Theorem 29.11 states that $\mathrm{sat}(I) = (2^{\aleph_1})^+$; by Theorem 22.16, $\mathrm{sat}(I) \geq 2^{\aleph_1}$. As $\mathrm{sat}(I)$ is regular (by Theorem 7.15), the following lemma implies the theorem:

Lemma 29.12. *Assume $2^{\aleph_0} < \aleph_{\omega_1}$. If κ is a regular cardinal such that $2^{\aleph_0} < \kappa \leq 2^{\aleph_1}$, then there exists a family of κ almost disjoint functions from ω_1 into ω_1.*

Proof. Let $\mathcal{F}$ be a family of almost disjoint functions on ω_1; we call $\mathcal{F}$ a *branching family* if whenever $f, g \in \mathcal{F}$ and α is such that $f(\alpha) = g(\alpha)$, then $f(\xi) = g(\xi)$ for all $\xi \leq \alpha$.

For each $X \subset \omega_1$, let $f_X = \langle X \cap \alpha : \alpha < \omega_1 \rangle$. The family $\mathcal{F} = \{f_X : X \in P(\omega_1)\}$ is a branching family of functions on ω_1, $|\mathcal{F}| = 2^{\aleph_1}$; and for each $\alpha < \omega_1$, the functions in $\mathcal{F}$ take values in $P(\alpha)$. Thus there exists a branching family of $2^{\aleph_1}$ functions from ω_1 into $2^{\aleph_0}$.

Let κ be a regular cardinal such that $2^{\aleph_0} < \kappa \leq 2^{\aleph_1}$. We shall show that for every $\aleph_\gamma$ such that $\aleph_1 < \aleph_\gamma \leq 2^{\aleph_0}$, if there is a branching family of κ functions from ω_1 into ω_γ, then there is a branching family of κ functions from ω_1 into some $\omega_\delta < \omega_\gamma$. Then the lemma clearly follows.

First let $\aleph_\gamma = \aleph_{\delta+1}$ where $\aleph_1 \leq \aleph_\delta$, and let $\mathcal{F}$ be a branching family of κ functions from ω_1 into $\omega_{\delta+1}$. Each $f \in \mathcal{F}$ is bounded below $\omega_{\delta+1}$ (because $\omega_{\delta+1} > \omega_1$), and because κ is regular and $\kappa > \omega_{\delta+1}$, there exists $\alpha < \omega_{\delta+1}$ such that $\mathrm{ran}(f) \subset \alpha$ for κ functions in $\mathcal{F}$. Thus there exists a branching family of κ functions from ω_1 into α; and since $|\alpha| \leq \aleph_\delta$, there is also a branching family of κ functions from ω_1 into ω_δ.

If $\aleph_\gamma$ is a limit cardinal, then $\mathrm{cf}(\omega_\gamma) = \omega$ because $\aleph_\gamma < \aleph_{\omega_1}$. Let $\mathcal{F}$ be a branching family of κ functions from ω_1 into ω_γ. For each $f \in \mathcal{F}$ there exists an ordinal $\eta_f < \omega_\gamma$ such that $f(\alpha) < \eta_f$ for uncountably many α's. Since κ is a regular cardinal and $\kappa > \aleph_\gamma$, there exists $\aleph_\delta$ such that $\aleph_1 \leq \aleph_\delta < \aleph_\gamma$, and a family $\mathcal{G} \subset \mathcal{F}$ of size κ such that for every $f \in \mathcal{G}$, $f(\alpha) < \omega_\delta$ for uncountably many α's.

For each $\alpha < \omega_1$, let $S_\alpha = \{f(\alpha) : f \in \mathcal{G}\}$. Since $\mathcal{G}$ is a branching family and $\mathcal{G} \subset \prod_{\alpha<\omega_1} S_\alpha$, it suffices to show that $|S_\alpha| \leq \aleph_\delta$ for all $\alpha < \omega_1$. Thus let $\alpha < \omega_1$. We define a function $t : S_\alpha \to \omega_1 \times \omega_\delta$ as follows: For each $x \in S_\alpha$, we first pick some $f \in \mathcal{G}$ such that $x = f(\alpha)$. Then there exists some $\xi > \alpha$ such $f(\xi) < \omega_\delta$, and we let

$$t(x) = (\xi, f(\xi)).$$

We shall now complete the proof by showing that the function t is one-to-one, and hence $|S_\alpha| \leq \aleph_\delta$. Let $x, y \in S_\alpha$ be such $t(x) = t(y)$. Let $\xi > \alpha$ and $f, g \in \mathcal{G}$ be such that $x = f(\alpha)$, $y = g(\alpha)$, and $t(x) = t(y) = (\xi, f(\xi)) = (\xi, g(\xi))$. Since $\mathcal{G}$ is a branching family and $f(\xi) = g(\xi)$, we have $f(\alpha) = g(\alpha)$ and hence $x = y$. □

The assumption $2^{\aleph_0} < 2^{\aleph_1}$ in Theorem 29.11 is necessary; see Exercise 29.11.

Functions from ω_1 into ω

Consider the set ω^{ω_1} of all functions from ω_1 into ω, partially ordered by eventual domination:

$$f < g \quad \text{if and only if} \quad \exists\gamma\, \forall\alpha \geq \gamma\, f(\alpha) < g(\alpha). \tag{29.7}$$

Let $\mathrm{cof}(\omega^{\omega_1})$ be the smallest size of a cofinal family $\mathcal{F} \subset \omega^{\omega_1}$, i.e., for every g there exists some $f \in \mathcal{F}$ such that $g < f$. It is an open problem whether $\mathrm{cof}(\omega^{\omega_1}) < 2^{\aleph_1}$ is possible.

Theorem 29.13.

(i) *If* $\mathrm{cof}(\omega^{\omega_1}) < 2^{\aleph_1}$ *then* $2^{\aleph_0} \geq \aleph_3$.
(ii) *If* $2^{\aleph_0} < 2^{\aleph_1}$ *and* $2^{\aleph_0} < \aleph_{\omega_1}$ *then* $\mathrm{cof}(\omega^{\omega_1}) = 2^{\aleph_1}$.

The theorem is a consequence of this lemma:

Lemma 29.14. *If there exist* $2^{\aleph_1}$ *almost disjoint functions from* ω_1 *into* ω_2 *then* $\mathrm{cof}(\omega^{\omega_1}) = 2^{\aleph_1}$.

Then Theorem 29.13 follows: If $2^{\aleph_0} \leq \aleph_2$ then use Exercise 29.12(ii); for (ii), use Theorem 29.11.

Toward the proof of Lemma 29.14, let I be an ideal on a set S. We say that two functions f, g on S are *I-disjoint* if $\{x \in S : f(x) = g(x)\} \in I$. If I and J are ideals on S and T, then $I \times J$ is the ideal on $S \times T$

$$X \in I \times J \quad \text{if and only if} \quad \{x \in S : \{y \in T : (x, y) \in X\} \notin J\} \in I. \tag{29.8}$$

Lemma 29.15. *There exists a σ-ideal I on ω_1 such that there exist $\aleph_2$ I-disjoint functions from ω_1 into ω.*

Proof. We find such an I on $\omega_1 \times \omega_1$: Let $I = I_0 \times I_0$ where I_0 is the σ-ideal of all countable subsets of ω_1 (each $X \in I$ is included in the union of ω vertical lines and the set under the graph of a function from ω_1 into ω_1). Let $\{g_\alpha : \alpha < \omega_2\}$ be a family of $\aleph_2$ almost disjoint functions from ω_1 into ω_1 (cf. Lemma 9.23), and $\{f_\beta : \beta < \omega_1\}$ a family of $\aleph_1$ almost disjoint functions from ω_1 into ω (Exercise 29.12(i)). For $\alpha < \omega_2$, let $h_\alpha(\xi, \eta) = f_{g_\alpha(\xi)}(\eta)$, for all $(\xi, \eta) \in \omega_1 \times \omega_1$. It is easy to verify that h_α, $\alpha < \omega_2$, are I-disjoint functions from ω_1 into ω. □

Lemma 29.16. *If there exist $2^{\aleph_1}$ almost disjoint functions from ω_1 into ω_2 then there exists a σ-ideal J on ω_1 such that there are $2^{\aleph_1}$ J-disjoint functions from ω_1 into ω.*

Proof. We find such a J on $\omega_1 \times \omega_1$: Let $J = I_0 \times I$ where I_0 is the ideal of countable sets and I is the ideal given by Lemma 29.15. Let $\{g_\alpha : \alpha < 2^{\aleph_1}\}$ be a family of almost disjoint functions from ω_1 into ω_2, and $\{f_\beta : \beta < \omega_2\}$ a family of I-disjoint functions from ω_1 into ω. For $\alpha < \omega_2$, let $h_\alpha(\xi, \eta) = f_{g_\alpha(\xi)}(\eta)$, for all $(\xi, \eta) \in \omega_1 \times \omega_1$. The functions h_α, $\alpha < 2^{\aleph_1}$, are J-disjoint. □

Proof of Lemma 29.14. By Lemma 29.16 there exist a σ-ideal J on ω_1 and a family $\mathcal{H} = \{h_\alpha : \alpha < 2^{\aleph_1}\}$ of J-disjoint functions from ω_1 into ω. Let $\mathcal{F}$ be a cofinal family in ω^{ω_1} such that $|\mathcal{F}| < 2^{\aleph_1}$. There exists an $f \in \mathcal{F}$ that eventually dominates infinitely many h_α; then let $A \subset 2^{\aleph_1}$ be a countable infinite set such that $h_\alpha < f$ for all $\alpha \in A$. The set $\{\xi < \omega_1 : h_\alpha(\xi) = h_\beta(\xi)$ for some distinct $\alpha, \beta \in A\}$ is the union of countably many sets in J, hence belongs to J, and hence its complement is uncountable. Thus for uncountably many $\xi < \omega_1$, the set $\{h_\alpha(\xi) : \alpha \in A\}$ is an infinite subset of ω. This contradicts the fact that there exists a $\gamma < \omega_1$ such that for all $\xi \geq \gamma$, $h_\alpha(\xi) < f(\xi)$ for all $\alpha \in A$. □

Exercises

29.1. If U is an ultrafilter on $\boldsymbol{N}$ and φ a formula, let $(Un)\,\varphi$ be an abbreviation for $\{n : \varphi(n)\} \in U$. Then $(U + V)k\,\varphi(k)$ if and only if $(Un)(Vm)\,\varphi(m + n)$.

29.2. If $\{x_k\}_{k=0}^{\infty}$ is a sequence of real numbers then $\lim_{U+V} x_k = \lim_U y_m$ where $y_m = \lim_V x_{m+n}$.

29.3. Let S be a minimal closed subsemigroup of a compact left-topological semigroup and let $u \in S$. Then $u + u = u$.

[$S + u$ is a continuous image of S, hence closed and $S + u = S$. Then $\{v \in S : v + u = u\} \subset S$ is closed and hence equals S; $u + u = u$ follows.]

29.4. $\beta\boldsymbol{N} - \boldsymbol{N}$ contains an idempotent element.

[By Zorn's Lemma and by compactness, $\beta\boldsymbol{N} - \boldsymbol{N}$ has a nonempty minimal closed subsemigroup.]

29.5. If a (κ, λ)-gap exists then a (λ, κ)-gap exists.
[Given $\{f_\alpha\}_\alpha$ and $\{g_\beta\}_\beta$, consider $\{g_0 - g_\beta\}_\beta$ and $\{g_0 - f_\alpha\}_\alpha$.]

29.6. There exists an $(\omega, \mathfrak{b})$-gap.
[Take constant functions as the ω-part of the gap. Then the $\mathfrak{b}$-part of the gap can be constructed in the family M of monotone unbounded functions. For $f \in M$ let $\varphi(f) = g$ in M be defined by $g(n) = \min\{k : f(k) \geq n\}$ and let $M' = \varphi\text{“}M$. $\varphi : (M, >) \to (M', <)$ is an order isomorphism and $(M', <)$ is cofinal in ω^ω while $(M, >)$ is cofinal in the family of all functions $f \in \omega^\omega$ which are above the constant functions ordered by $>$.]

29.7. There are no (ω, λ)-gaps for $\lambda < \mathfrak{b}$.
[The constant functions in the preceding exercise can be replaced by any $<$-increasing ω-sequence of functions which shows that $\mathfrak{b}$ is the minimal cardinal κ such that there exists an (ω, κ)-gap.]

29.8. $\mathcal{N}$ is the union of an increasing ω_1-sequence of G_δ sets.
[Let $\{f_\alpha\}_\alpha$, $\{g_\beta\}_\beta$ be an (ω_1, ω_1)-gap and let A_α be the complement of $\{h \in \mathcal{N} : f_\alpha < h < g_\alpha\}$.]

29.9. Let X be the set of all increasing transfinite sequences of rationals (a subspace of $P(\boldsymbol{Q})$), and let K_0 be the set of all $\{s, t\}$ such that $s \subset t$ or $t \subset s$. The set K_0 is closed and has no uncountable homogeneous subset. Show that there are no H_n with $[H_n]^2 \subset X - K_0$ such that $X = \bigcup_{n=0}^{\infty} H_n$.
[Let H_n be such that $[H_n]^2 \subset X - K_0$. Construct $q_0 > q_1 > \ldots$ and $t_0 \subset t_1 \subset \ldots$ such that $\sup t_n < q_n$, and if possible, $t_n \in H_n$. Then $t = \bigcup_n t_n$ is not a member of any H_n.]

29.10. Assuming OCA, every uncountable subset of $P(\omega)$ contains an uncountable chain or antichain.
[$\{A, B\} \in K_0$ if and only if A and B are incomparable.]

29.11. It is consistent that the ideal of countable subsets of ω_1 is ω_3-saturated while $2^{\aleph_1}$ is large.
[Adjoin κ Cohen reals to a model of GCH. Assume that $\{A_i : i < \omega_3\}$ are almost disjoint. For each pair i, j there is a $\gamma_{i,j}$ such that $A_i \cap A_j \subset \gamma_{i,j}$ is forced by all conditions. By the Erdős-Rado Theorem (in the ground model), there exists a subfamily of $\{A_i\}_i$ of size $\aleph_2$ for which $\gamma_{i,j}$ is the same γ. This gives (in $V[G]$) a family of $\aleph_2$ disjoint subsets of γ, a contradiction.]

29.12. (i) There exist $\aleph_1$ almost disjoint functions from ω_1 into ω.
(ii) There exist $2^{\aleph_1}$ almost disjoint functions from ω_1 into $2^{\aleph_0}$.
[(i) For $\xi \leq \alpha < \omega_1$, let $f_\xi(\alpha) = \xi$; this gives $\aleph_1$ almost disjoint functions in $\prod_{\alpha<\omega_1} \alpha$.
(ii) For $X \subset \omega_1$, let $f_X(\alpha) = X \cap \alpha$; this gives $2^{\aleph_1}$ almost disjoint functions in $\prod_{\alpha<\omega_1} P(\alpha)$.]

29.13. If there is a family $\mathcal{F}$ of $\aleph_2$ almost disjoint functions $f : \omega_1 \to \omega$ then Chang's Conjecture fails.
[Consider a model $\mathfrak{A}$ with the universe $\mathcal{F} \cup \omega_1$ and the designated predicate ω_1. If $(\mathcal{G} \cup B, B) \prec \mathfrak{A}$ with $|\mathcal{G}| = \aleph_1$ and $|B| = \aleph_0$, then $B \subset \alpha$ for some $\alpha < \omega_1$. Show that $f(\alpha) \neq g(\alpha)$ for all $f, g \in \mathcal{G}$, a contradiction.]

29.14. Assume that there exists a cofinal $\mathcal{F} \subset \omega^{\omega_1}$ such that the set of all initial segments of all $f \in \mathcal{F}$ has size $\aleph_1$. Then $2^{\aleph_0} = \aleph_1$.

[Let $\langle t_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all the initial segments, and $\operatorname{dom} t_\alpha \leq \alpha$. By induction on $\alpha < \omega_1$, we construct closed subsets $K_{n,\alpha}$ of $[0,1]$ $(n \in \omega)$ such that $K_{0,\alpha} \subset K_{1,\alpha} \subset \ldots \subset K_{n,\alpha} \subset \ldots$, and the union is $[0,1]$. At stage α, consider the set $K = \bigcap_{\xi \in \operatorname{dom} t_\alpha} K_{t_\alpha(\xi),\xi}$. If K is countable, let $K_{n,\alpha} = [0,1]$ for all n; if K is uncountable, choose a limit point x of K and let $K_{n,\alpha} = \{x\} \cup \{y : |x-y| \geq 1/n\}$. Now if $f \in \mathcal{F}$ then $K_f = \bigcap_\alpha K_{f(\alpha),\alpha}$ is countable, and there exists some $\alpha_f < \omega_1$ such that $K_f = \bigcap_{\alpha<\alpha_f} K_{f(\alpha),\alpha}$. It follows that

$$\begin{aligned}[0,1] &= \bigcap_{\alpha<\omega_1} \bigcup_{n=0}^{\infty} K_{n,\alpha} = \bigcup_{f:\omega_1\to\omega} \bigcap_{\alpha<\omega_1} K_{f(\alpha),\alpha} \\ &= \bigcup_{f\in\mathcal{F}} \bigcap_{\alpha<\omega_1} K_{f(\alpha),\alpha} = \bigcup_{f\in\mathcal{F}} \bigcap_{\alpha<\alpha_f} K_{f(\alpha),\alpha} = \bigcup_{\gamma<\omega_1} \bigcap_{\alpha\in\operatorname{dom} t_\gamma} K_{t_\gamma(\alpha),\alpha},\end{aligned}$$

which is a union of $\aleph_1$ countable sets.]

Historical Notes

Hindman's Theorem appeared in [1974]. The present proof is due to Glazer and can be found e.g. in the book [1980] by Graham et al. The book also contains van der Waerden's Theorem and its generalizations. The topological proof presented here is as in Todorčević's book [1997]. For the Hales-Jewett Theorem, see Hales and Jewett [1963].

Hausdorff's Theorem appeared in Hausdorff [1909]. We follow the construction presented in Scheepers [1993], which gives a comprehensive account of the subject of gaps. The Open Coloring Axiom was isolated by Todorčević in [1989]; related partition axioms were previously introduced by Abraham, Rubin and Shelah in [1985]. Theorem 29.8 is due to Todorčević [1989].

Theorem 29.11 as well as Exercise 29.11 are results of Baumgartner [1976]; for almost disjoint functions see Jech and Prikry [1979]. Part (i) of Theorem 29.13 is due to Galvin. For part (ii) and Lemma 29.14, see Jech and Prikry [1984].

Exercise 29.3 is attributed to R. Ellis; see Todorčević [1997].

Exercises 29.6 and 29.7: Rothberger [1941].

Exercise 29.8: Hausdorff [1936a].

Exercise 29.9: Todorčević.

Exercise 29.10: Abraham, Rubin and Shelah [1985]; Baumgartner [1980].

Exercise 29.13: Silver.

Exercise 29.14: Gödel.

30. Complete Boolean Algebras

Measure Algebras

A complete Boolean algebra B is a *measure algebra* if it carries a (strictly positive probabilistic) *measure*, i.e., a real-valued function m on B that satisfies (22.1) (or cf. Definition 30.2 below). In Chapters 26 and 22 we looked at two examples of measure algebras: The algebra B_m of (26.1), and the more general product measure algebra defined in (22.3). We present below a theorem that states that this measure algebra is essentially the only measure algebra that exists.

Throughout this section, we consider measure algebras, and for simplicity assume that all the measure algebras under consideration are atomless. Note that every measure algebra satisfies the countable chain condition, and consequently, all questions of completeness can be reduced to σ-completeness.

If G is a subset of a measure algebra B, we say that G *σ-generates* B if B is the smallest σ-subalgebra containing G. The *weight* of B is the least size of $G \subset B$ that σ-generates B. B is *homogeneous* if each $B\restriction u$ (with $u \neq 0$) has the same weight. Note that every measure algebra is the direct sum of ω many homogeneous measure algebras.

The result that we shall prove in this section is the following:

Theorem 30.1 (Maharam). *Every infinite homogeneous measure algebra is the unique measure algebra of its weight.*

If A and B are infinite homogeneous measure algebras of the same weight and if μ and ν are strictly positive probabilistic measures on A and B, then there exists an isomorphism f between A and B such that $\nu(f(a)) = \mu(a)$ for all $a \in A$.

We begin by introducing some terminology and presenting two lemmas that are standard techniques of measure theory.

Definition 30.2. Let B be a complete Boolean algebra. A *measure* on B is a real-valued function μ on B that satisfies

(i) $\mu(0) = 0$,
(ii) $\mu(a) \geq 0$ for all $a \in A$,

(iii) for all pairwise disjoint a_n, $n = 0, 1, \ldots,$

$$\mu(\sum_{n=0}^{\infty} a_n) = \sum_{n=0}^{\infty} (a_n).$$

A measure μ is *strictly positive* if

(iv) $\mu(a) > 0$ for all $a \neq 0$,

and *probabilistic*, if also

(v) $\mu(1) = 1$.

Finally, a function μ that satisfies (i) and (iii) is called a *signed measure*.

Lemma 30.3. *If ν is a signed measure on B that satisfies c.c.c. then there exists an $a \in B$ such that $\nu(x) \geq 0$ for all $x \leq a$ and $\nu(x) \leq 0$ for all $x \leq -a$.*

Proof. First we claim that when $\nu(a) > 0$ then there exists some $b \leq a$ such that

$$\nu(b) > 0, \text{ and } \nu(x) \geq 0 \text{ for all } x \leq b. \tag{30.1}$$

If (30.1) fails then for every $b \leq a$, $b \neq 0$, there exists an $x \leq a$, $x \neq 0$, with $\nu(x) \leq 0$. Thus let W be a maximal antichain below a such that $\nu(x) \leq 0$ for every $x \in W$. Then $\sum W = a$ and we have $\nu(a) \leq 0$, a contradiction.

Now let Z be a maximal antichain such that (30.1) holds for every $b \in Z$. If $\nu(a) \leq 0$ for all $a \in B$ then the lemma holds trivially. Otherwise, Z is nonempty, and let $a = \sum Z$. This a satisfies the lemma. □

Lemma 30.4. *Let μ and ν be measures on B and let $a \in B$ be such that $\nu(a) > 0$. Then there exist a $b \leq a$, $b \neq 0$, and a number $\varepsilon > 0$ such that $\nu(x) \geq \varepsilon \cdot \mu(x)$ for all $x \leq b$.*

Proof. Let $\varepsilon > 0$ be such that $\nu(a) > \varepsilon \cdot \mu(a)$ and consider the signed measure $\nu - \varepsilon\mu$ on $B\restriction a$. By Lemma 30.3 there exists a $b \leq a$ such that $(\nu - \varepsilon\mu)(x) \geq 0$ for all $x \leq b$, and $(\nu - \varepsilon\mu)(x) \leq 0$ for all $x \leq a - b$. Since $(\nu - \varepsilon\mu)(b) \geq (\nu - \varepsilon\mu)(a) > 0$, we have $b \neq 0$. □

The next lemma is due to Fremlin:

Lemma 30.5 (Fremlin [1989]). *Let A be a measure algebra and let μ be a strictly positive measure on A. Let B be a complete subalgebra of A and let ν be a measure on B such that $\nu(b) \leq \mu(b)$ for all $b \in B$. Assume that*

$$A\restriction a \neq \{a \cdot b : b \in B\} \text{ for every } a \in A^+. \tag{30.2}$$

Then there exists some $a \in A$ such that

$$\nu(b) = \mu(a \cdot b) \text{ for all } b \in B. \tag{30.3}$$

Proof. For each $a \in A$, let ν_a denote the measure on B defined by (30.3): $\nu_a(b) = \mu(a \cdot b)$. We first prove the following consequence of (30.2): For every $a \in A^+$ and every $\varepsilon > 0$ there exists a $c \in (A\restriction a)^+$ such that $\nu_c(b) \leq \varepsilon \cdot \nu_a(b)$ for all $b \in B$.

It is enough to prove this claim for $\varepsilon = \frac{1}{2}$, as the general case follows by a repeated application of the special case.

Thus let $a \in A^+$. By (30.2) there exists some $d < a$ such that $d \neq a \cdot b$ for every $b \in B$. Consider the signed measure $\frac{1}{2}\nu_a - \nu_d$ on B. By Lemma 30.3 there exists some $b \in B$ such that $\nu_d(x) \leq \frac{1}{2}\nu_a(x)$ for all $x \in B\restriction b$ and $\nu_d(x) \geq \frac{1}{2}\nu_a(x)$ for all $x \in B\restriction(-b)$.

If $b \cdot d > 0$, we let $c = b \cdot d$, and we have $\nu_c(x) \leq \frac{1}{2}\nu_a(x)$ for all $x \in B$.

If $b \cdot d = 0$ then $d \leq a - b$, and we let $c = (a - b) \cdot (a - d)$. Since $d \neq a - b$ (by (30.2)), we have $c \neq 0$. For all $x \in B$, $\nu_c(x) \leq \nu_a(x) - \nu_d(x) \leq \frac{1}{2}\nu_a(x)$.

This proves the claim for $\varepsilon = \frac{1}{2}$ and the general case follows. To prove the lemma, let $a \in A$ be a maximal (in the partial order $\leq$ on A) element such that $\nu_a(b) \leq \nu(b)$ for all $b \in B$. We finish the proof by showing that $\nu_a = \nu$.

By contradiction, assume that there exists some $b_1 \in B$ such that $\nu_a(b_1) < \nu(b_1)$. By Lemma 30.4 there exist some $b_2 \leq b_1$, $b_2 \neq 0$, and $\varepsilon > 0$ such that $(\nu - \nu_a)(x) \geq \varepsilon\mu(x)$ for all $x \in B\restriction b_2$. Note that $b_2 \not\leq a$, since otherwise we would have $\nu_a(b_2) = \mu(b_2) \geq \nu(b_2)$.

Now we apply the earlier claim to $b_2 - a$, and get some $c \leq b_2 - a$, $c \neq 0$, such that $\nu_c(x) \leq \varepsilon\nu_{b_2-a}(x) \leq \nu(x) - \nu_a(x)$ for all $x \in B$. Since $c \cdot a = 0$, we have $\nu_{a+c} = \nu_a + \nu_c \leq \nu$, contradicting the maximality of a. □

Lemma 30.5 allows one to extend partial measure-preserving isomorphisms between homogeneous measure algebras. If μ and ν are probabilistic measures on measure algebras A and B, then an isomorphism f of A onto B is *measure-preserving* if $\nu(f(a)) = \mu(a)$ for all $a \in A$.

Lemma 30.6. *Let A_1 and A_2 be homogeneous measure algebras, both of the same weight κ, and let μ_1 and μ_2 be probabilistic measures on A_1 and A_2. Let B_1 and B_2 be complete subalgebras of A_1 and A_2, let f be a measure-preserving isomorphism of B_1 onto B_2, and assume that B_1 is σ-generated by fewer than κ generators. Then for every $a_1 \in A_1$ there exist an $a_2 \in A_2$ and a measure-preserving isomorphism $g \supset f$ of $\langle B_1 \cup \{a_1\}\rangle$, the subalgebra generated by $B_1 \cup \{a_1\}$, onto $\langle B_2 \cup \{a_2\}\rangle$.*

Proof. First we note that since every $A_1\restriction a$ has weight κ, the subalgebra B_1 satisfies (30.2); similarly for A_2 and B_2. Let $a_1 \in A_1$; if we let $\nu(f(b)) = \mu_1(a_1 \cdot b)$ for every $b \in B_1$, then ν is a measure on B_2 with $\nu \leq \mu_2$. By Lemma 30.5 there exists some $a_2 \in A_2$ such that $\nu(f(b)) = \mu_2(a_2 \cdot f(b))$ for every $b \in B_1$.

The algebra $\langle B_1 \cup \{a_1\}\rangle$ consists of all elements of the form $b \cdot a_1 + c \cdot (-a_1)$ where $b, c \in B_1$. Thus we let

$$g(b \cdot a_1 + (c - a_1)) = f(b) \cdot a_2 + (f(c) - a_2). \tag{30.4}$$

We have to verify that g is well-defined. If $b \in B_1$ and $b \leq a_1$ then $\mu_1(b) = \mu_1(a_1 \cdot b)$, and we have $\mu_2(f(b)) = \mu_1(b) = \mu_1(a_1 \cdot b) = \nu(f(b)) = \mu_2(a_2 \cdot f(b))$, and so $f(b) \leq a_2$. It follows that $b \cdot a_1 = b' \cdot a_1$ implies $f(b) \cdot a_2 = f(b') \cdot a_2$. Similarly, one proves that if $c \in B_1$ and $c \leq -a_1$ then $f(c) \leq -a_2$, and therefore $c - a_1 = c' - a_1$ implies $f(c) - a_2 = f(c') - a_2$. Thus g is well-defined.

Since $\mu_2(f(b) \cdot a_2 + (f(c) - a_2)) = \mu_1(b \cdot a_1 + (c - a_1))$, g is measure-preserving, and a one-to-one homomorphism of $\langle B_1 \cup \{a_1\}\rangle$ onto $\langle B_2 \cup \{a_2\}\rangle$. □

Proof of Theorem 30.1. The construction proceeds by induction. Let A and B be homogeneous measure algebras of weight κ and let μ and ν be probabilistic measures on A and B. Let $\{a_\alpha : \alpha < \kappa\}$ and $\{b_\alpha : \alpha < \kappa\}$ be generators of A and B. Inductively, we construct $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$ and $B_0 \subset B_1 \subset \ldots \subset B_\alpha \subset \ldots$ and measure-preserving isomorphisms $f_0 \subset f_1 \subset \ldots \subset f_\alpha \subset \ldots$ such that for every α, A_α is a complete subalgebra of A of weight $< \kappa$ and $a_\alpha \in A_\alpha$, similarly for B_α, and $f_\alpha(A_\alpha) = B_\alpha$.

At successor stages we apply Lemma 30.6 to either $\langle A_\alpha \cup \{a_{\alpha+1}\}\rangle$ or $\langle B_\alpha \cup \{b_{\alpha+1}\}\rangle$. At a limit stage α, we consider the algebras $\tilde{A}_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and $\tilde{B}_\alpha = \bigcup_{\beta<\alpha} B_\beta$. These are subalgebras of A and B, not necessarily complete. However, the completion A_α of $\tilde{A}_\alpha$ can be described as follows: The elements of A_α are limits of convergent countable sequences in A_α (see Exercise 30.1). The measure-preserving isomorphism $\tilde{f} = \bigcup_{\beta<\alpha} f_\beta$ between $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$ extends to a unique measure-preserving isomorphism between A_α and the completion B_α of $\tilde{B}_\alpha$ (use Exercise 30.2). □

Cohen Algebras

Let κ be an infinite cardinal. We consider the notion of forcing P_κ that adds κ Cohen reals: conditions in P_κ are finite 0–1 functions with domain $\subset \kappa$. Let

$$\boldsymbol{C}_\kappa = B(P_\kappa) \tag{30.5}$$

denote the complete Boolean algebra corresponding to P_κ. Throughout this section, $\overline{B}$ denotes the completion of a Boolean algebra B.

Definition 30.7. A Boolean algebra B is a *Cohen algebra* if $\overline{B} = \boldsymbol{C}_\kappa$ for some infinite cardinal κ.

In Theorem 30.10 below we give a combinatorial characterization of Cohen algebras.

Definition 30.8. A subalgebra A of a Boolean algebra B is a *regular subalgebra*,

$$A \leq_{\rm reg} B,$$

if for any $X \subset A$, if $\sum^A X$ exists then $\sum^A X = \sum^B X$.

The following is easily established:

Lemma 30.9. *The following are equivalent*:

(i) $A \leq_{\rm reg} B$.
(ii) *Every maximal antichain in* A *is maximal in* B.
(iii) *For every* $b \in B^+$ *there exists an* $a \in A^+$ *such that for every* $x \in A^+$, *if* $x \leq a$ *then* $x \cdot b \neq 0$. □

See Exercises 30.3–30.9 for further properties of $\leq_{\rm reg}$.

If A is a subalgebra of B and $b \in B$, then the *projection* of b to A, $\mathrm{pr}^A(b)$, is the smallest element $a \in A$, if it exists, such that $b \leq a$. (Similarly, $\mathrm{pr}_A(b)$ is the greatest $a \in A$ such that $a \leq b$.)

The *density* of a Boolean algebra B is the least size of a dense subset of B. B has *uniform density* if for every $a \in B^+$, $B\restriction a$ has the same density.

If X is a subset of a Boolean algebra B, we denote

$$\langle X\rangle = \text{the subalgebra generated by } X, \tag{30.6}$$

and if A is a subalgebra of B and $b_1, \dots, b_n \in B$,

$$A(b_1, \dots, b_n) = \langle A \cup \{b_1, \dots, b_n\}\rangle. \tag{30.7}$$

Theorem 30.10. *Let* B *be an infinite Boolean algebra of uniform density.* B *is a Cohen algebra if and only if the set* $\{A \in [B]^\omega : A \leq_{\rm reg} B\}$ *contains a closed unbounded set* C *with the property*

$$\text{if } A_1, A_2 \in C \text{ then } \langle A_1 \cup A_2\rangle \in C. \tag{30.8}$$

If B is countable, the condition is trivially satisfied as $C = \{B\}$ is a closed unbounded subset of $[B]^\omega$.

First we prove the forward direction of the theorem: If B is a dense subalgebra of $\boldsymbol{C}_\kappa$, then B has the property stated in Theorem 30.10. (In particular, $\boldsymbol{C}_\kappa$ itself has the property.) Let B be a dense subalgebra of $\boldsymbol{C}_\kappa$. For every $S \subset \kappa$, consider the forcing P_S consisting of finite 0–1 functions with domain $\subset S$, and let $\boldsymbol{C}_S = B(P_S)$. Note that $\boldsymbol{C}_S \leq_{\rm reg} \boldsymbol{C}_\kappa$.

Now let C be the set of all countable subalgebras A of B with the property that there exists a countable $S \subset \kappa$ such that

$$A \text{ is dense in } B \cap \boldsymbol{C}_S \text{ and } B \cap \boldsymbol{C}_S \text{ is dense in } \boldsymbol{C}_S. \tag{30.9}$$

The following lemma will establish the forward direction.

Lemma 30.11. *The set* C *is closed unbounded in* $[B]^\omega$, *satisfies* (30.8), *and every* $A \in C$ *is a regular subalgebra of* B.

Proof. Let $A \in C$ and let S be a countable subset of κ such that (30.9) holds. Since $B \cap \boldsymbol{C}_S$ is dense in $\boldsymbol{C}_S$ and $\boldsymbol{C}_S \leq_{\rm reg} \boldsymbol{C}_\kappa$, we have $B \cap \boldsymbol{C}_S \leq_{\rm reg} \boldsymbol{C}_\kappa$, and since B is dense in $\boldsymbol{C}_\kappa$, we have $B \cap \boldsymbol{C}_S \leq_{\rm reg} B$. As A is dense in $B \cap \boldsymbol{C}_S$, it follows that $A \leq_{\rm reg} B$. To see that C is unbounded, note that there are arbitrarily large countable sets S such that $B \cap \boldsymbol{C}_S$ is dense in $\boldsymbol{C}_S$ (because $\boldsymbol{C}_\kappa$ has the countable chain condition). Thus for any $a \in B$ we can find a countable S and a countable algebra $A \subset B$ such that $a \in A$, that A is dense in $B \cap \boldsymbol{C}_S$ and $B \cap \boldsymbol{C}_S$ is dense in $\boldsymbol{C}_S$.

To show that C is closed, let $\{A_n\}_{n=0}^\infty$ be an increasing chain in C and let $A = \bigcup_{n=0}^\infty A_n$; let $\{S_n\}_{n=0}^\infty$ be witnesses for $A_n \in C$. The sets S_n form a chain, and if we let $S = \bigcup_{n=0}^\infty S_n$, it follows that A is dense in $B \cap \boldsymbol{C}_S$ and $B \cap \boldsymbol{C}_S$ is dense in $\boldsymbol{C}_S$.

Now we verify (30.8); we shall show that if A_1 is dense in $\boldsymbol{C}_{S_1}$ and A_2 is dense in $\boldsymbol{C}_{S_2}$ then $A = \langle A_1 \cup A_2 \rangle$ is dense in $\boldsymbol{C}_S$ where $S = S_1 \cup S_2$. Let $b \in \boldsymbol{C}_S^+$; we shall find $a_1 \in A_1$ and $a_2 \in A_2$ such that $0 \neq a_1 \cdot a_2 \leq b$.

Let $p \in P_S$ be such that $p \leq b$, and let $p_1 = p{\restriction}S_1$, $p_2 = p{\restriction}S_2$. First we find some $a_1 \in A_1^+$ such that $a_1 \leq p_1$ and then some $q_1 \in P_{S_1}$ such that $q_1 \leq a_1$. Let $q_2 = p_2{\restriction}(S_2 - S_1) \cup (q_1{\restriction}S_2)$; we have $q_2 \in P_{S_2}$. Now we find some $a_2 \in A_2^+$ such that $a_2 \leq q_2$. It remains to show that $a_1 \cdot a_2 \neq 0$: There exists some $r_2 \in P_{S_2}$ with $r_2 \leq a_2$, and then $r_2 \cup (q_1{\restriction}(S_1 - S_2)) \in P_S$ is below both a_1 and a_2. □

For the opposite direction, let B be an infinite Boolean algebra of uniform density κ and let C be a closed unbounded set of countable regular subalgebras of B that satisfies (30.8). First we note that B satisfies the countable chain condition: See Exercise 30.10. Let

$$\mathcal{S} = \{\langle \bigcup X \rangle : X \subset C\}. \tag{30.10}$$

We claim that every $A \in \mathcal{S}$ is a regular subalgebra of B. Let $A = \langle \bigcup X \rangle$ and let W be a maximal antichain in A; we verify that W is maximal in B. As W is countable, we have $W \subset \langle \bigcup Y \rangle$ for some countable $Y \subset X$. Since C is closed unbounded and satisfies (30.8) it follows that $A_0 = \langle \bigcup Y \rangle \in C$ and hence $A_0 \leq_{\rm reg} B$. Since W is a maximal antichain in A_0 and $A_0 \leq_{\rm reg} B$, W is maximal in B.

A set $G \subset B$ is *independent* if

$$\pm x_1 \cdot \pm x_2 \cdot \ldots \cdot \pm x_n \neq 0$$

for all distinct $x_1, \ldots, x_n \in G$. If G is independent then $\langle G \rangle = {\rm Fr}_G$ is the unique *free* algebra over G; note that the completion of ${\rm Fr}_G$ is $\boldsymbol{C}_G$. Our goal is to find an independent $G \subset \overline{B}$ such that $\langle G \rangle$ is dense in $\overline{B}$.

Let A be a subalgebra of a Boolean algebra D. An element $u \in D$ is *independent over* A if $a \cdot u \neq 0 \neq a - u$ for all $a \in A^+$.

Lemma 30.12. *Let D be a complete Boolean algebra of uniform density and let A be a complete subalgebra of D of smaller density. For every $v \in D$ there exists some $u \in D$ independent over A such that $v \in A(u)$.*

Proof. Exercises 30.11 and 30.12. □

Let $\{d_\alpha : \alpha < \kappa\}$ be a dense subset of B. If A_1 and A_2 are subalgebras of $\overline{B}$ we say that A_1 and A_2 are *co-dense* if for every $a_1 \in A_1^+$ there exists some $a_2 \in A_2^+$ with $a_2 \leq a_1$, and for every $a_2 \in A_2^+$ there exists some $a_1 \in A_1^+$ with $a_1 \leq a_2$.

We construct, by induction on $\alpha < \kappa$, two continuous chains $G_0 \subset G_1 \subset \ldots \subset G_\alpha \subset \ldots$ and $B_0 \subset B_1 \subset \ldots \subset B_\alpha \subset \ldots$ such that

(30.11)
(i) $B_\alpha \in \mathcal{S}$,
(ii) $A_\alpha = \langle G_\alpha \rangle$ and B_α are co-dense,
(iii) $d_\alpha \in B_{\alpha+1}$,
(iv) $G_{\alpha+1} - G_\alpha$ is countable,
(v) G_α is an independent subset of $\overline{B}$.

This will prove that B is a Cohen algebra, because by (iii), $\bigcup_\alpha B_\alpha$ is dense in B, hence $\bigcup_\alpha A_\alpha$ is dense in $\overline{B}$, and by (v), $\bigcup_\alpha A_\alpha$ is the free algebra Fr_G (where $G = \bigcup_\alpha G_\alpha$).

At limit stages, we let $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ and $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. To construct $G_{\alpha+1}$ and $B_{\alpha+1}$, we proceed as follows: Since A_α is dense in $\overline{B_\alpha}$, $\overline{A_\alpha} = \overline{B_\alpha}$ is a complete subalgebra of $\overline{B}$. Moreover, if $u_1, \ldots, u_n \in \overline{B}$ then $\overline{A_\alpha}(u_1, \ldots, u_n)$ is a complete subalgebra of $\overline{B}$.

Since $|A_\alpha| < \kappa$, we find, by Lemma 30.12, for every $b \in \overline{B}$ some $u \in \overline{B}$ independent over $\overline{A_\alpha}$ such that $b \in \overline{A_\alpha}(u)$. More generally, if $b, u_1, \ldots, u_n \in \overline{B}$, then there exists some u independent over $\overline{A_\alpha}(u_1, \ldots, u_n)$ such that $b \in \overline{A_\alpha}(u_1, \ldots, u_n, u)$.

Given $u \in \overline{B}$, there exists a countable set $\{b_n\}_{n=0}^\infty \subset B$ such that $\sum_{n=0}^\infty b_n = u$. Then there exists some $X \in C$ such that $\{b_n\}_n \subset X$ and so $\langle B_\alpha \cup X \rangle$ is dense in $\overline{A_\alpha}(u)$. Therefore there exist a countable set $\{u_n\}_{n=0}^\infty \subset \overline{B}$ and some $B_{\alpha+1} \in \mathcal{S}$ such that $d_\alpha \in B_{\alpha+1}$, that $G_{\alpha+1} = G_\alpha \cup \{u_n\}_{n=0}^\infty$ is independent and that $A_{\alpha+1} = \langle G_{\alpha+1} \rangle$ and $B_{\alpha+1}$ are co-dense. □

The following property is a natural weakening of the characterization of Cohen algebras in Theorem 30.10:

Definition 30.13. An infinite Boolean algebra B of uniform density is *semi-Cohen* if $[B]^\omega$ has a closed unbounded set of countable regular subalgebras.

An immediate consequence of the definition is that if B is semi-Cohen and $|B| \leq \aleph_1$ then B is a Cohen algebra. This is because $[B]^\omega$ has a closed unbounded subset that is a chain, and therefore satisfies (30.8).

The important feature of semi-Cohen algebras is that the property is hereditary:

Theorem 30.14. *If B is a semi-Cohen algebra and if A is a regular subalgebra of B of uniform density then A is semi-Cohen.*

Proof. $[B]^\omega$ has a closed unbounded subset of regular subalgebras of B. Since $A \leq_{\mathrm{reg}} B$, there exists for every $b \in B^+$ some $a \in A^+$ such that there is no $x \in A^+$ with $x \leq a - b$. Let $F : B^+ \to A^+$ be a function that to each $b \in B^+$ assigns such an $a \in A^+$. Let $C \subset [B]^\omega$ be a closed unbounded set of regular subalgebras closed under F.

If $X \in C$ then $A \cap X \leq_{\mathrm{reg}} X$ because X is closed under F. Every maximal antichain in $A \cap X$ is maximal in X, hence in B (because $X \leq_{\mathrm{reg}} B$), hence in A. Therefore $A \cap X \leq_{\mathrm{reg}} A$.

There is a closed unbounded set $D \subset [A]^\omega$ such that $D \subset \{X \cap A : X \in C\}$; D witnesses that A is semi-Cohen. □

Corollary 30.15. *If B is semi-Cohen and has density $\aleph_1$ then B is a Cohen algebra.*

Proof. B has a dense subalgebra A of size $\aleph_1$. By Theorem 30.14, A is also semi-Cohen, and hence Cohen. But $\overline{A} = \overline{B}$, and hence B is Cohen. □

Corollary 30.16. *Every complete subalgebra of $\boldsymbol{C}_\kappa$ of uniform density $\aleph_1$ is isomorphic to $\boldsymbol{C}_{\omega_1}$.* □

The property of being semi-Cohen is also preserved by completion. This can be proved using the following lemma:

Lemma 30.17. *A Boolean algebra B of uniform density is semi-Cohen if and only if B is Cohen in V^P, where P is the collapse of $|B|$ onto $\aleph_1$ with countable conditions.*

Proof. As $|B| = \aleph_1$ in V^P, it suffices to show that B is semi-Cohen if and only if it is semi-Cohen in V^P.

As P does not add new countable sets, $[B]^\omega$ remains the same in V^P. By property (iii) of Lemma 30.9, the relation $\leq_{\mathrm{reg}}$ is absolute. Let S be the set of all regular subalgebras of B. If S contains a closed unbounded set C then C is closed unbounded in V^P. Conversely, if S does not contain a closed unbounded set, then it does not contain one in V^P. □

Corollary 30.18. *B is semi-Cohen if and only if its completion is semi-Cohen.*

Proof. Let B be semi-Cohen and let $A = \overline{B}$. Let P be the ω-closed collapse of $|A|$ to $\aleph_1$. In V^P, A has a dense subalgebra B that is a Cohen algebra, hence A itself is Cohen. Therefore A is semi-Cohen.

The converse follows from Theorem 30.14. □

Not every semi-Cohen algebra is a Cohen algebra, and Corollary 30.16 does not extend to density $\aleph_2$. Koppelberg and Shelah gave an example of a complete subalgebra of C_{ω_2} of (uniform density $\aleph_2$) that is not isomorphic

to C_{ω_2}. Another example, due to Zapletal, is the forcing that adds $\aleph_2$ eventually different reals: Let

$$P = \{z : z \text{ is a finite function with } \mathrm{dom}(z) \subset \omega_2 \text{ and } \mathrm{ran}(z) \subset \omega^{<\omega}\},$$

and let $z \leq w$ if z is a coordinate-wise extension of w and for $\alpha \neq \beta$ in $\mathrm{dom}(w)$, if $n \in \mathrm{dom}(z(\alpha) - w(\alpha))$ and $n \in \mathrm{dom}(z(\beta))$, then $z(\alpha)(n) \neq z(\beta)(n)$.

If $B = B(P)$ then B can be embedded in $\boldsymbol{C}_{\omega_2}$ but is not isomorphic to $\boldsymbol{C}_{\omega_2}$. We omit the proof.

Suslin Algebras

Definition 30.19. A *Suslin algebra* is a complete atomless Boolean algebra that is ω-distributive and satisfies the countable chain condition.

If T is a normal Suslin tree, and P_T is the forcing with T upside down, then $B(P_T)$ is a Suslin algebra. Conversely, if B is a Suslin algebra of density $\aleph_1$ then $B = B(P_T)$ for some Suslin tree T; in general, if B is a Suslin algebra then B has a complete subalgebra B_T such that $B_T = B(P_T)$ for some Suslin tree T.

Theorem 30.20. *If B is a Suslin algebra then $|B| \leq 2^{\aleph_1}$.*

Proof. Let $\kappa = 2^{\aleph_1}$. Assume that there is a Suslin algebra B such that $|B| > \kappa$. We shall reach a contradiction.

Without loss of generality we assume that $|B{\restriction}u| > \kappa$ for all $u \in B^+$. We shall construct a κ-sequence

$$B_0 \subset B_1 \subset \ldots \subset B_\alpha \subset \ldots \qquad (\alpha < \kappa) \tag{30.12}$$

of complete subalgebras of B, each of size $\leq \kappa$. If $D \subset B$ and $|D| \leq \kappa$, then there are $\kappa^{\aleph_1} = \kappa$ D-valued names for relations on ω_1; thus for every such D let $\dot{R}^D_\gamma$, $\gamma < \kappa$, be a fixed enumeration of all such names. Let $\alpha \mapsto (\beta_\alpha, \gamma_\alpha)$ be the canonical mapping of κ onto $\kappa \times \kappa$; we recall $\beta_\alpha \leq \alpha$ for all α.

The sequence (30.12) is constructed as follows: We let $B_0 = \{0, 1\}$; if α is a limit ordinal, then B_α is the complete subalgebra of B generated by $\bigcup_{\nu<\alpha} B_\nu$. If $|B_\nu| \leq \kappa$ for each $\nu < \alpha$, then $|B_\alpha| < \kappa$. At successor steps, we construct $B_{\alpha+1}$ as follows: Let $D = B_{\beta_\alpha}$ and let $\dot{R} = \dot{R}^D_{\gamma_\alpha}$. If

$$\|(\omega_1, \dot{R}) \text{ is a Suslin tree}\|_{B_\alpha} = 1, \tag{30.13}$$

if $\dot{C} \in V^{B_\alpha}$ is the Suslin algebra (in V^{B_α}) corresponding to the Suslin tree and if $B_\alpha * \dot{C}$ is (isomorphic to) a complete subalgebra of B, then we let $B_{\alpha+1} = B_\alpha * \dot{C}$. Otherwise, we let $B_{\alpha+1} = B_\alpha$. In either case, if $|B_\alpha| \leq \kappa$, then $|B_{\alpha+1}| \leq \kappa$.

Now let B_κ be the complete subalgebra of B generated by $\bigcup_{\alpha<\kappa} B_\alpha$. Clearly, $|B_\kappa| \leq \kappa$. Let $\dot{A} \in V^{B_\kappa}$ be the complete Boolean algebra $B : B_\kappa$ (in V^{B_κ}). Since both B and B_κ satisfy the c.c.c., we have

$$\|\dot{A} \text{ satisfies the c.c.c.}\|_{B_\kappa} = 1.$$

Similarly, since both B and B_κ are ω-distributive, we have

$$\|\dot{A} \text{ is } \omega\text{-distributive}\|_{B_\kappa} = 1.$$

We have assumed that $|B\restriction u| > \kappa$ for all $u \neq 0$, and we also have $|B_\kappa| \leq \kappa$. Thus

$$\| |\dot{A}| > \kappa\|_{B_\kappa} = 1$$

and consequently

$$\|\dot{A} \text{ is not atomic}\|_{B_\kappa} = 1.$$

Now we work inside V^{B_κ}; There exists a $\dot{T} \subset \dot{A}$ such that $(\dot{T}, \geq_{\dot{A}})$ is a normal Suslin tree; let $\dot{B}_T \subset \dot{A}$ be the Suslin algebra, subalgebra of $\dot{A}$, generated by $\dot{T}$. Let $\dot{R}$ be a binary relation on ω_1 isomorphic to $\dot{T}$.

The name $\dot{R}$ is B_κ-valued; and since B_κ satisfies the countable chain condition, $\dot{R}$ involves at most $\aleph_1$ elements of B_κ. Since $\operatorname{cf}\kappa > \aleph_1$, there exists a $\beta < \kappa$ such that $\dot{R} \in V^{B_\beta}$; furthermore, let $\gamma < \kappa$ be such that $\dot{R}$ is the γth B_β-valued binary relation on ω_1, $\dot{R} = \dot{R}^{B_\beta}_\gamma$.

Let $\alpha < \kappa$ be such that $\beta = \beta_\alpha$ and $\gamma = \gamma_\alpha$. Since

$$\|(\omega_1, \dot{R}) \text{ is a Suslin tree}\|_{B_\kappa} = 1,$$

it follows that

$$\|(\omega_1, \dot{R}) \text{ is a Suslin tree}\|_{B_\alpha} = 1.$$

If $\dot{C}$ denotes the corresponding Suslin algebra in V^{B_α}, we have

$$B_\alpha * \dot{C} \subset B_\kappa * \dot{B}_T \subset B_\kappa * \dot{A} = B$$

and it follows that $B_{\alpha+1} = B_\alpha * \dot{C}$. However, forcing with a Suslin tree destroys its Suslinity, and we have

$$\|(\omega_1, \dot{R}) \text{ is not a Suslin tree}\|_{B_{\alpha+1}} = 1,$$

a contradiction. □

Suslin algebras of size $2^{\aleph_1}$ can be constructed by forcing (cf. Jech [1973b]), or in L (an unpublished result of Laver).

Simple Algebras

Definition 30.21. A complete Boolean algebra B is *simple* if it is atomless and if it has no proper atomless complete subalgebra.

The problem of existence of simple algebras originated in forcing and was first discussed by McAloon in [1971]. It is clear that a simple algebra is *minimal*, i.e., when forcing with it, there is no intermediate model between the ground model and the generic extension. Minimality, when formulated in Boolean-algebraic terms, is the following property:

(30.14) If A is a complete atomless subalgebra of B then there exists a partition W such that $A\restriction w = B\restriction w$ for all $w \in W$.

(An example of a minimal algebra is $B(P)$ where P is the Sacks forcing.)

Simple algebras, in addition to being minimal, are *rigid*, i.e., have no nontrivial automorphisms (Exercise 30.13). It turns out that the conjunction of these two properties also implies that the algebra is simple (Exercise 30.14). Thus we have:

Theorem 30.22. *An atomless complete Boolean algebra is simple if and only if it is rigid and minimal.* □

An example of a rigid and minimal algebra is B_P where P is Jensen's forcing from Theorem 28.1 that produces a minimal Δ^1_3 real. B_P is minimal because the generic real has minimal degree of constructibility, and rigid because it is definable. It follows that if $V = L$ then a simple complete Boolean algebra exists.

In L, one can also construct Suslin algebras that are simple (see Exercises 30.15 and 30.16 for the construction of a rigid Suslin algebra).

Simple complete Boolean algebras have been constructed in ZFC; we refer the reader to Jech-Shelah's papers [1996] and [2001]. The former constructs a countably generated simple algebra and uses a modification of the Sacks forcing to produce a minimal definable real. The latter construction is somewhat less complicated and yields forcing that produces a minimal definable uncountable set.

Infinite Games on Boolean Algebras

Infinite games have many applications in set theory, particularly in descriptive set theory, and we shall investigate these methods in some detail in the chapter on Axiom of Determinacy. In the present section we look into some properties of complete Boolean algebras, and of forcing, that are formulated in terms of infinite games.

Let B be a Boolean algebra, and let $\mathcal{G}$ be the following infinite game between two players I and II: I chooses a nonzero element $a_0 \in B$ and then II chooses some $b_0 \in B^+$ such that $b_0 \leq a_0$. Then I plays (chooses) $a_1 \leq b_0$ and II plays $b_1 \leq a_1$ (both $\neq 0$). The game continues, with I's moves $a_n \in B^+$, $n < \omega$, and II's moves $b_n \in B^+$, $n < \omega$, such that

$$a_0 \geq b_0 \geq a_1 \geq b_1 \geq \ldots \geq a_n \geq b_n \geq \ldots . \tag{30.15}$$

Player I wins the game if $\prod_{n=0}^{\infty} a_n = 0$; player II wins otherwise: if the chain (30.15) has a nonzero lower bound. A *strategy* for player I is a function $\sigma : B^{<\omega} \to B$; it is a *winning strategy* if I wins every play (30.15) in which I follows σ, i.e., for each n, $a_n = \sigma(\langle b_0, \ldots, b_{n-1}\rangle)$. A (winning) strategy for II is defined similarly. If player I has a winning strategy then II does not, and vice versa, and in general, neither player need have a winning strategy.

Lemma 30.23. *Player I has a winning strategy in $\mathcal{G}$ if and only if B is not ω-distributive.*

Proof. Let σ be a winning strategy for I. Let $a_0 = \sigma(\langle\rangle)$; we shall find partitions W_n of a_0 without a common refinement. Let $W_0 = \{a_0\}$. Having constructed $W_0, \ldots, W_n$, consider all finite sequences $a_0 \geq b_0 \geq \ldots \geq a_n \geq b_n$ where the a_n's are chosen by σ and $a_k \in W_k$ for all $k \leq n$. Let W_n be a maximal antichain whose members are elements $a_{n+1} = \sigma(\langle b_0, \ldots, b_n\rangle)$ where $a_0 \geq b_0 \geq \ldots \geq a_n \geq b_n$ is as described. The W_n's are partitions of a_0 and do not have a common refinement.

Conversely, if B is not ω-distributive, there exist some a_0 and open dense sets D_n below a_0 such that $\bigcap_{n=0}^{\infty} D_n = \emptyset$. We define $\sigma(\langle\rangle) = a_0$, and if $a_0 \geq b_0 \geq \ldots \geq a_n \geq b_n$ is such that the a_n's are chosen by σ, let $\sigma(\langle b_0, \ldots, b_n\rangle)$ be some element of D_n below b_n. The function σ is a winning strategy for I. □

Let P be a separative notion of forcing, and consider the infinite game $\mathcal{G}$ in which players I and II take turns to play a descending chain $a_0 \geq b_0 \geq \ldots \geq a_n \geq b_n \geq \ldots$ in P. I wins if and only if the chain does not have a lower bound. It is easy to see that either player has a winning strategy in this game if and only if the same player has a winning strategy in $\mathcal{G}$ played on $B(P)$ (Exercise 30.17).

Definition 30.24. A separative notion of forcing P (a Boolean algebra B) is *strategically ω-closed* if player II has a winning strategy in the game $\mathcal{G}$.

Being strategically ω-closed is a hereditary property. If B is strategically ω-closed and if A is a regular subalgebra of B then also A is strategically ω-closed (Exercise 30.18).

It is obvious that if P is ω-closed then player II has a winning strategy in $\mathcal{G}$. Hence if B has a dense ω-closed subset, then B is strategically ω-closed. The converse is true for small algebras:

Theorem 30.25 (Foreman). *If B has density $\aleph_1$ and is strategically ω-closed, then it has a dense ω-closed subset.*

Proof. Let $\{d_\alpha : \alpha < \omega_1\}$ be a dense set in B. By induction on α, we find partitions W_α of 1 such that W_β refines W_α if $\alpha < \beta$, and every W_α has some $w \leq d_\alpha$; at limit stages, we use ω-distributivity of B. Let $T = \bigcup_{\alpha<\omega_1} W_\alpha$; T is dense in B and is a tree. Let σ be a winning strategy for II in the game $\mathcal{G}$ on T. We shall find a dense subset P of T that is ω-closed.

If $t \in T$, we call $p = \langle a_0, b_0, \ldots, a_n, b_n\rangle$ a *partial play above t* if the b_k's are played by σ and $b_n > t$. We claim:

(30.16) $(\forall t \in T)\,(\exists t^* < t)$ if p is a partial play above t^* and if $u > t^*$ then there is a partial play $q \supset p$ above t^* with last move b such that $u > b > t^*$.

To prove the claim, we construct $t_0 > t_1 > \ldots > t_n > \ldots$ such that $t_0 = t$ and that for every n and every partial play p above t_n, if $u > t_n$ then there exist some $q \supset p$ with last move b and some m such that $u > b > t_m$. This is possible because there are only countably many such p's and u's. Therefore there exists a play $\langle a_0, b_0, \ldots, a_n, b_n, \ldots\rangle$ that is played according to σ and that is cofinal in $\langle t_n\rangle_{n=0}^\infty$. As σ is a winning strategy, $\langle t_n\rangle_{n=0}^\infty$ has a lower bound, let t^* be a maximal lower bound (it exists because T is a tree). This proves (30.16). Now let

$$P = \{s \in T : \text{for some descending chain } \{t_n\}_{n=0}^\infty,\ s \text{ is a maximal lower bound of } \{t_n^*\}_{n=0}^\infty\}.$$

The set P is ω-closed: Given $s_0 > s_1 > \ldots$ in P, find $\{t_n\}_{n=0}^\infty$ such that $t_0^* > s_0 > t_1^* > \ldots$. The chain $\{t_n^*\}_{n=0}^\infty$ has a lower bound (because there exists a cofinal play by σ) and its maximal lower bound is in P.

The set P is dense in T: Given $t \in T$, let $\{t_n\}_{n=0}^\infty$ be the chain t, t^*, $t^{**}, \ldots$. There exists a cofinal σ-play, and so $\{t_n\}_{n=0}^\infty$ has a lower bound. The maximal lower bound is in P. □

As a corollary, we get the following characterization of strategically ω-closed forcings:

Corollary 30.26. *B is strategically ω-closed if and only if B is a regular subalgebra of some algebra that has an ω-closed dense subset.*

Proof. Sufficiency follows from Exercise 30.18. Thus assume that B is strategically ω-closed and let σ be a winning strategy for II. Let P be the collapse with countable conditions of $|B|$ to $\aleph_1$. In V^P, σ is still a winning strategy, and by Theorem 30.25, B has an ω-closed dense subset $\dot{E}$. Let $A = B(P \times B^+)$; B is a regular subalgebra of A. Let $D = \{(p, b) : p \Vdash b \in \dot{E}\}$; D is dense in A. D is ω-closed: Let $\{(p_n, b_n)\}_n$ be descending and let $p = \bigcup_n p_n$. Then p forces that $\{b_n\}_n$ is descending, and there is a $b \in B^+$ such that $p \Vdash (b \in \dot{E}$ and $b \leq b_n$ for all $n)$. Hence (p, b) is a lower bound of $\{(p_n, b_n)\}_n$. □

Foreman's Theorem does not extend to $\aleph_2$: It is consistent that there is a strategically ω-closed complete Boolean algebra of density $\aleph_2$ that does not have an ω-closed dense subset (Jech and Shelah [1996]).

There are many other infinite games that can be used to define properties of forcing and Boolean algebras, see Jech [1984]. We'll show in Chapter 31 that proper forcing admits such characterization. See also Exercise 30.19.

Exercises

Let B be a σ-complete Boolean algebra. If $\{a_n\}_{n<\omega}$ is a sequence in B, let $\limsup_n a_n = \prod_{n=0}^{\infty} \sum_{k\geq n} a_n$ and $\liminf_n a_n = \sum_{n=0}^{\infty} \prod_{k\geq n} a_n$. If $\limsup_n a_n = \liminf_n a_n = a$, we say that $\{a_n\}_{n<\omega}$ *converges*, and let $\lim_n a_n = a$.

30.1. If A is a subalgebra a measure algebra B then the complete subalgebra of B σ-generated by A consists of all limits of convergent seequences in A.

30.2. If μ is a measure on a measure algebra B and if $a = \lim_n a_n$, then $\mu(a) = \lim_n \mu(a_n)$.

30.3. If A is a finite subalgebra of B then $A \leq_{\rm reg} B$.

30.4. If $A \leq_{\rm reg} B$ and $B \leq_{\rm reg} C$ then $A \leq_{\rm reg} C$.

30.5. If A is a subalgebra of B, B is a subalgebra of C, and $A \leq_{\rm reg} C$ then $A \leq_{\rm reg} B$.

30.6. If A is a dense subalgebra of B then $A \leq_{\rm reg} B$.

30.7. $A \leq_{\rm reg} B$ if and and only if $\overline{A} \leq_{\rm reg} \overline{B}$.

30.8. If A and B are complete then $A \leq_{\rm reg} B$ if and only if A is a complete subalgebra of B.

30.9. If $\mathrm{pr}^A(b)$ exists for all $b \in B$, then $A \leq_{\rm reg} B$.

30.10. If $\{A \in [B]^\omega : A \leq_{\rm reg} B\}$ is stationary, then B has the countable chain condition.

[Let W be a maximal antichain and consider the model $M = (B, \leq, W)$. There exists an elementary submodel A of M such that $A \leq_{\rm reg} B$. $W \cap A$ is maximal in A, therefore in B, and hence $W = W \cap A$.]

30.11 (Vladimirov's Lemma). Let D be a complete Boolean algebra of uniform density and A a complete subalgebra of smaller density. Then there exists an element $u \in D$ independent over A.

[Let $X = \{x \in D^+ :$ there is no $a \in A^+$ with no $a \in A^+$ with $a \leq x\}$; X is dense. Let $Y = \{\mathrm{pr}^A(x) : x \in X\}$; Y is dense. Let $W \subset Y$ be a maximal antichain, and let $Z \subset X$ be such that $W = \{\mathrm{pr}^A(z) : z \in Z\}$. Let $u = \sum Z$. If $a \in A^+$, let $z \in Z$ be such that $a \cdot \mathrm{pr}^A(z) \neq 0$; we also have $a \cdot (\mathrm{pr}^A(z) - z) \neq 0$. Hence u is independent over A.]

30.12. Under same assumptions, for every $v \in D - A$ there exists some $u \in D$ independent over A such that $v \in A(u)$.

[Let $z = \mathrm{pr}_A(v) + -\mathrm{pr}^A(v)$. If $z = 0$, let $u = v$. Otherwise, apply Exercise 30.11 to $D\restriction z$, to get some $w \leq z$ independent over $A\restriction z$. Then let $u = w + (v - z)$. We have $v \in A(u)$ since $v = \mathrm{pr}_A(v) + u \cdot \mathrm{pr}^A(v)$; also, u is independent over A.]

30.13. Every simple complete Boolean algebra is rigid.

[Let π be a nontrivial automorphism. There exist disjoint a and b such that $\pi(a) = b$. Each x has a decomposition $x = a \cdot x + b \cdot x + y$; let A be the complete subalgebra $\{x : b \cdot x = \pi(a \cdot x)\}$. A is atomless and $a \notin A$.]

30.14. Every rigid minimal complete Boolean algebra is simple.

[Let B be minimal and A a complete atomless subalgebra such that $A \neq B$. There exists a $z \notin A$ such that $A\restriction z = B\restriction z$. Let $u_1 = z - \mathrm{pr}_A(z)$, $v_1 = \mathrm{pr}^A(z) - z$. Let $0 \neq v \leq v_1$ be such that $A\restriction v = B\restriction v$, and let $u = u_1 \cdot \mathrm{pr}^A(v)$. For all $a \in A$ let $\pi(a \cdot u) = a \cdot v$; show that π is an automorphism between $B\restriction u$ and $B\restriction v$. Then π extends to a nontrivial automorphism of B.]

30.15. Let T be a normal Suslin tree and let B_T be the corresponding Suslin algebra. If π is an automorphism of B_T then there is a closed unbounded set $C \subset \omega_1$ such that $\pi\restriction T^C$ is an automorphism of T^C, where $T^C = \{t \in T : o(t) \in C\}$.

30.16. If $V = L$ then there exists a Suslin tree T such that B_T is rigid.

[Use $\diamondsuit$ and Exercise 30.15 to destroy all potential automorphisms of B_T.]

30.17. Player I (player II) has a winning strategy in $\mathcal{G}$ played on P if and only if the same player has one in $\mathcal{G}$ on $B(P)$.

30.18. If a complete Boolean algebra B is strategically ω-closed and if A is a complete subalgebra of B then A is strategically ω-closed.

[Let σ be a winning strategy on B; then the following σ_A is a winning strategy on A: When I plays a_0, let $b_0 = \sigma(a_0)$ and let $\sigma_A(a_0) = \mathrm{pr}^A(b_0)$. When I plays $a_1 \leq \sigma_A(a_0)$, let $b_1 = \sigma(\langle a_0, a_1 \cdot b_0\rangle)$ and $\sigma_A(\langle a_0, a_1\rangle) = \mathrm{pr}^A(b_1)$. And so on.]

30.19. Let B be a Boolean algebra of uniform density. Consider the infinite game on B in which two players select elements $a_0, b_0, \ldots, a_n, b_n, \ldots$ and II wins if and only if the set $\{a_n, b_n\}_{n=0}^{\infty}$ generates a regular subalgebra of B. Show that II has a winning strategy if and only if B is semi-Cohen.

[If σ is a winning strategy then the set C of all countable subalgebras closed under σ is a closed unbounded set of regular subalgebras; the converse is similar.]

Historical Notes

Maharam's Theorem 30.1 appeared in [1942]; the present proof is based on Fremlin's article [1989]. Theorem 30.10 appeared in Balcar, Jech and Zapletal [1997] improving a similar earlier result of Koppelberg [1993]. The [1997] investigates semi-Cohen algebra, the concept introduced by Fuchino and Jech. Corollary 30.16: Koppelberg. Theorem 30.20 is due to Solovay.

Rigid minimal algebras were studied by McAloon in [1971]. Constructions of a simple complete Boolean algebra in ZFC appeared in Jech and Shelah [1996] and [2001].

The game $\mathcal{G}$ on a Boolean algebra was introduced in Jech [1978]; this and similar games were studied in Jech [1984]. Foreman's Theorem 30.25 appeared in [1983].

Exercises 30.11 and 30.12: Vladimirov [1969].

Exercises 30.13 and 30.14: McAloon [1971].

31. Proper Forcing

Definition and Examples

Proper forcing was introduced by S. Shelah who isolated properness as the property of forcing that is common to many standard examples of forcing notions and that is preserved under countable support iteration.

Definition 31.1. A notion of forcing $(P, <)$ is *proper* if for every uncountable cardinal λ, every stationary subset of $[\lambda]^\omega$ remains stationary in the generic extension.

Properness is a generalization of both the countable chain condition and of being ω-closed. The following two lemmas are the analogs of Lemma 22.25 and Lemma 23.7 (for $\kappa = \aleph_1$):

Lemma 31.2. *If P satisfies the countable chain condition then for every uncountable λ, every closed unbounded set $C \subset [\lambda]^\omega$ in $V[G]$ has a subset $D \in V$ that is closed unbounded in V. Hence every stationary set $S \subset [\lambda]^\omega$ remains stationary in $V[G]$.*

Proof. Let $p \Vdash \dot{C}$ is closed unbounded; let $\dot{F}$ be a name for a function from $\lambda^{<\omega}$ into λ such that $p \Vdash C_{\dot{F}} \subset \dot{C}$ (where $C_{\dot{F}}$ is the set of all closure points of $\dot{F}$—see Theorem 8.28). Let $f : \lambda^{<\omega} \to [\lambda]^\omega$ be the function

$$f(e) = \{\alpha \in \lambda : \|\dot{F}(e) = \alpha\| \neq 0\}.$$

$f(e)$ is countable because P satisfies the countable chain condition. Let $D = C_f$.

Since $p \Vdash \dot{F}(e) \in f(e)$, if x is closed under f then $p \Vdash \dot{F}(e) \in x$, and so $p \Vdash D \subset \dot{C}$. □

Lemma 31.3. *If P is ω-closed then every stationary set $S \subset [\lambda]^\omega$ remains stationary in $V[G]$.*

Proof. Let $p \Vdash \dot{F} : \lambda^{<\omega} \to \lambda$. We shall find a condition $q \leq p$ and some $x \in S$ such that $q \Vdash \dot{F}(x^{<\omega}) \subset x$.

Consider the model $(H_\kappa, \in, (P, <), p, \dot{F}, \Vdash)$ where $\kappa \geq \lambda$ is sufficiently large. Let C be the closed unbounded set in $[H_\lambda]^\omega$ of all countable elementary

submodels of the model. By Theorem 8.27 there exists some $N \in C$ such that $N \cap \lambda \in S$. Let $x = N \cap \lambda$.

Enumerate $x^{<\omega} = \langle e_n : n < \omega \rangle$ and construct a sequence of conditions $p = p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots$ such that for each n there exists an $\alpha_n \in N \cap \lambda$ such that $p_n \Vdash \dot{F}(e_n) = \alpha_n$ (by elementarity). Let q be a lower bound for $\{p_n\}_n$. Then $q \Vdash \dot{F}(x^{<\omega}) \subset x$. □

Proper forcing does not collapse $\aleph_1$. In fact, an easy argument shows that a stronger property is true:

Lemma 31.4. *If P is proper then every countable set of ordinals in $V[G]$ is included in a set in V that is countable in V.*

Proof. Let X be a countable set of ordinals in $V[G]$ and let λ be uncountable in V such that $X \subset \lambda$. The set $([\lambda]^\omega)^V$ remains stationary in $V[G]$ and therefore meets the set $\{A \in [\lambda]^\omega : A \supset X\}$, which is a closed unbounded set in $V[G]$. Thus $X \subset A$ for some $A \in ([\lambda]^\omega)^V$. □

We shall now formulate a technical condition that is equivalent to properness of a forcing notion, and that will be used to prove that properness is preserved under countable support iteration. We refer the reader to the exercises for other equivalents of properness.

Let $(P, <)$ be a fixed notion of forcing. We say that λ is *sufficiently large* if λ is a cardinal and $\lambda > 2^{|P|}$. A *model* M is an elementary submodel of $(H_\lambda, \in, <, \ldots)$ where H_λ is the collection of all sets hereditarily of cardinality less than λ, $<$ is some unspecified well-ordering of H_λ (to allow for inductive constructions), and the structure of H_λ contains all the relevant parameters; in particular, M contains $(P, <)$.

Definition 31.5. A condition q is (M, P)-*generic* if for every maximal antichain $A \in M$, the set $A \cap M$ is predense below q.

The following lemma (the proof is a routine exercise) illuminates the concept of (M, P)-genericity:

Lemma 31.6. *Let λ be sufficiently large, let $M \prec H_\lambda$ be such that $P \in M$, and let $q \in P$. The following are equivalent*:

(i) *q is (M, P)-generic.*
(ii) *If $\dot{\alpha} \in M$ is an ordinal name then $q \Vdash \dot{\alpha} \in M$, i.e.,*

$$\forall r \leq q \; \exists s \leq r \; \exists \beta \in M \; s \Vdash \dot{\alpha} = \beta.$$

(iii) *$q \Vdash \dot{G} \cap M$ is a filter on P generic over M.* □

Theorem 31.7. *A forcing notion P is proper if and only if for all sufficiently large λ there is a closed unbounded set C of elementary submodels $M \prec (H_\lambda, \ldots)$ such that*

$$\forall p \in M \; \exists q \leq p \; (q \text{ is } (M, P)\text{-generic}). \tag{31.1}$$

Proof. First we show that the condition is necessary. Let P be proper and let λ be sufficiently large. Toward a contradiction assume that the set of all models $M \prec H_\lambda$ for which (31.1) fails is stationary. By normality there exist a stationary set $S \subset [H_\lambda]^\omega$ and a condition $p \in P$ such that for every $q \leq p$ and every $M \in S$, q is not (M, P)-generic.

Now let $V[G]$ be a generic extension with $G \ni p$, and let us argue in $V[G]$. Every maximal antichain A below p (in V) meets G in a unique condition q_A. Let

$$C = \{M \prec (H_\lambda)^V : \text{if } A \in M \text{ then } q_A \in M\};$$

C is closed unbounded. Since S remains stationary in $V[G]$, there exists some $M \in S \cap C$.

For each $A \in M$ we have $\sum(A \cap M) \in G$ (because $q_A \in G$), and by genericity, $\prod_{A \in M} \sum(A \cap M) \in G$. Let $q \leq \prod_{A \in M} \sum(A \cap M)$. Then $q \leq p$ and q is (M, P)-generic, contradicting $M \in S$.

Now we prove that the condition is sufficient. Let P be a forcing notion that satisfies the condition of the theorem; we shall prove that P preserves stationary sets. Let λ be an uncountable cardinal and let $S \subset [\lambda]^\omega$ be stationary. Let $\dot{F}$ be a name for a function $\dot{F} : \lambda^{<\omega} \to \lambda$ and $p \in P$. We shall find a $q \leq p$ and $x \in S$ such that $q \Vdash x$ is closed under $\dot{F}$.

Let $\mu \geq \lambda$ be sufficiently large. By the assumption there exists a closed unbounded set $C \subset [H_\mu]^\omega$ such that (31.1) holds for every $M \in C$. By Theorem 8.27, $\{M \cap \lambda : N \in C\}$ contains a closed unbounded set in $[\lambda]^\omega$ and hence there exists some $M \in C$ with $M \cap \lambda \in S$.

Let $q \leq p$ be (M, P)-generic. We finish the proof by showing that $q \Vdash M \cap \lambda$ is closed under $\dot{F}$. Let $e \in (M \cap \lambda)^{<\omega}$; we shall show that $q \Vdash \dot{F}(e) \in M$. There is $A \in M$ such that A is a maximal antichain below p and every $w \in A$ decides $\dot{F}(e)$. Now if $r \leq q$ forces $\dot{F}(e) = \alpha$, then because $A \cap M$ is predense below q, r is compatible with some $w \in A \cap M$ and so $w \Vdash \dot{F}(e) = \alpha$. Since α is definable from w, $\dot{F}$, and e, we have $\alpha \in M$. □

Another characterization of properness is formulated in terms of infinite games.

Definition 31.8. Let P be a forcing notion and let $p \in P$. The *proper game* (for P, below p) is played as follows: I plays P-names $\dot{\alpha}_n$ for ordinal numbers, and II plays ordinal numbers β_n. Player II wins if there exists a $q \leq p$ such that

$$q \Vdash \forall n \, \exists k \, \dot{\alpha}_n = \beta_k. \tag{31.2}$$

Theorem 31.9. *A forcing notion P is proper if and only if for every $p \in P$, II has a winning strategy for the proper game.*

Proof. Exercise 31.3. □

We shall now present some examples of proper forcing. The following concept is due to J. Baumgartner:

Definition 31.10. A notion of forcing $(P, <)$ satisfies *Axiom A* if there is a collection $\{\leq_n\}_{n=0}^{\infty}$ of partial orderings of P such that $p \leq_0 q$ implies $p \leq q$ and for every n, $p \leq_{n+1} q$ implies $p \leq_n q$, and

(i) if $\langle p_n : n \in \omega \rangle$ is a sequence such that $p_0 \geq_0 p_1 \geq_1 \ldots \geq_{n-1} p_n \geq_n \ldots$ then there is a q such that $q \leq_n p_n$ for all n;
(ii) for every $p \in P$, for every n and for every ordinal name $\dot{\alpha}$ there exist a $q \leq_n p$ and a countable set B such that $q \Vdash \dot{\alpha} \in B$.

Lemma 31.11. *If P satisfies Axiom A then P is proper.*

Proof. Let P satisfy Axiom A and let $p \in P$. The following is a winning strategy for II in the game from Exercise 31.2: When I plays $\dot{\alpha}_n$, let II find a condition $p_n \leq_{n-1} p_{n-1}$ (with $p_0 \leq p$) and a countable set B_n such that $p_n \Vdash \dot{\alpha}_n \in B_n$. If q is a lower bound for $\{p_n\}_{n=0}^{\infty}$ then q witnesses that II wins the game. □

Example 31.12. Every ω-closed forcing satisfies Axiom A.
Let $p \leq_n q$ if and only if $p \leq q$, for all n. □

Example 31.13. Every c.c.c. forcing satisfies Axiom A.
Let $p \leq_n q$ if and only if $p = q$, for all $n > 0$. □

Example 31.14. The notions of forcing that add a Sacks real, a Mathias real or a Laver real satisfy Axiom A.

For Sacks reals, see (15.26). For Laver forcing, see (28.17); Mathias forcing is similar. □

In Exercises 31.5 and 31.6 we give Baumgartner's example of proper forcing that does not satisfy Axiom A.

Iteration of Proper Forcing

It is obvious that a two-step iteration of proper forcing is proper: If P preserves stationary sets and in V^P, $\dot{Q}$ preserves stationary sets then $P * \dot{Q}$ preserves stationary sets. What is more important however is that properness is preserved under countable support iteration. The present section is devoted to the proof of this.

Theorem 31.15 (Shelah). *If P_α is a countable support iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$ such that every $\dot{Q}_\beta$ is a proper forcing notion in $V^{P_\alpha \restriction \beta}$, then P_α is proper.*

Toward the proof of Theorem 31.15 we first observe that the properness condition in Theorem 31.7 can be somewhat simplified:

Lemma 31.16. *P is proper if and only if for every $p \in P$, every sufficiently large λ and every countable $M \prec (H_\lambda, \in, <)$ containing P and p, there exists a $q \leq p$ that is (M, P)-generic.*

Proof. Let P be proper and $p \in P$. Let $\mu = 2^{|P|}$ and $\lambda > \mu$; we recall that $<$ is a well-ordering of H_λ. By Theorem 31.7 the set of all countable elementary submodels of H_μ with property (31.1) contains a closed unbounded set, and so it contains C_F for some function $F : H_\mu^{<\omega} \to H_\mu$. If F is the least such function in H_λ then every $M \prec (H_\lambda, \in, <)$ is closed under F and so $M \cap H_\mu$ satisfies (31.1). Hence every such M with $P, p \in M$ satisfies the condition of the lemma. □

In order to prove that an iteration P_α is proper, we wish to show that if λ is sufficiently large and $M \prec H_\lambda$ contains P_α then for every $p \in P_\alpha \cap M$ there is some (M, P_α)-generic $q \in P_\alpha$ such that $q \Vdash_\alpha p \in \dot{G}$. We prove this by induction; the main point is that the inductive condition is somewhat stronger:

Lemma 31.17. *Let P_α be a countable support iteration of proper forcing notions. Let λ be sufficiently large and let $M \prec (H_\lambda, \in, <)$ be countable, with $P_\alpha \in M$. For every $\gamma \in \alpha \cap M$, every $q_0 \in P_\gamma = P_\alpha \restriction \gamma$ that is (M, P_γ)-generic, and every $\dot{p} \in V^{P_\gamma}$ such that*

$$q_0 \Vdash_\gamma \dot{p} \in (P_\alpha \cap M) \text{ and } \dot{p} \restriction \gamma \in \dot{G}_\gamma \tag{31.3}$$

there exists an (M, P_α)-generic condition $q \in P_\alpha$ such that $q \restriction \gamma = q_0$ and $q \Vdash_\alpha \dot{p} \in \dot{G}_\alpha$.

$\dot{G}_\alpha$ and $\dot{G}_\gamma$ are the canonical names for generic filters on P_α and P_γ respectively. Letting $\gamma = 0$ (and q_0 the trivial condition 1 in $P_0 = \{1\}$), we get the desired result.

Lemma 31.17 is proved by induction on α. In order to handle the successor stages we need first to prove the special case $\alpha = 2$, $\gamma = 1$; then the inductive step from α to $\alpha + 1$ is a routine modification of the special case:

Lemma 31.18. *Let P be proper, let $\dot{Q} \in V^P$ be such that $\Vdash_P \dot{Q}$ is proper and let $R = P * \dot{Q}$. Let $M \prec H_\lambda$ be countable, with $R \in M$. For every (M, P)-generic $q_0 \in P$ and every $\dot{p} \in V^P$ such that*

$$q_0 \Vdash_P \dot{p} \in (M \cap R) \text{ and } \dot{p}_0 \in \dot{G}_P$$

(where $\dot{p}$ is a name for $(\dot{p}_0, \dot{p}_1)$ and $\dot{G}_P$ is generic on P) there is some $\dot{q}_1 \in V^P$ such that $(q_0, \dot{q}_1)$ is (M, R)-generic and $(q_0, \dot{q}_1) \Vdash_R \dot{p} \in \dot{G}_R$.

Proof. To find the name $\dot{q}_1$, let G be a generic filter on P containing q_0. Let $p = \dot{p}^G$ and $q = \dot{Q}^G$; then $p \in M \cap R$ and $p = (p_0, \dot{p}_1)$ with $p_0 \in G$. Since $\dot{p}_1 \in M$, we have $p_1 \in M[G] \cap Q$, and since Q is proper, there exists (in $V[G]$)

a stronger condition q_1 that is $(M[G], Q)$-generic. (Here we use the fact that $M[G] \prec H_\lambda^{V[G]}$ which we leave as an exercise.) This describes $\dot{q}_1$.

That $(q_0, \dot{q}_1)$ is (M, R)-generic follows from q_0 being (M, P)-generic and $q_0 \Vdash \dot{q}_1$ is $(M[\dot{G}_P], \dot{Q})$-generic (this is routine). Also, since $q_0 \Vdash \dot{p}_0 \in \dot{G}_P$ and $q_0 \Vdash \dot{q}_1 \leq \dot{p}_1$, we conclude that $(q_0, \dot{q}_1) \Vdash_R \dot{p} \in \dot{G}_R$. □

Proof of Lemma 31.17. We assume that α is a limit ordinal; hence $\alpha \cap M$ is a countable set of ordinals without a maximal element. Let $\langle \gamma_n : n \in \omega \rangle$ be an increasing set of ordinals in M with $\gamma_0 = \gamma$, cofinal in $\alpha \cap M$. Let $\{D_n : n \in \omega\}$ be an enumeration of al dense subsets of P_α that are in M. Let $q_0 \in P_{\gamma_0}$ be (M, P_{γ_0})-generic and let $\dot{p}$ be a $V^{P_{\gamma_0}}$-name such that (31.3) holds. We shall find a (M, P_α)-generic condition $q \in P_\alpha$ such that $q{\restriction}\gamma_0 = q_0$ and $q \Vdash_\alpha \dot{p} \in \dot{G}_\alpha$.

We construct q as the limit of conditions $q_n \in P_{\gamma_n}$ such that $q_{n+1}{\restriction}\gamma_n = q_n$, and such that each q_n is (M, P_{γ_n})-generic.

Along with the q_n we construct P_{γ_n}-names $\dot{p}_n$ such that $\dot{p}_0 = \dot{p}$ and that for each n, q_n forces

$$
\begin{aligned}
(31.4) \qquad &\text{(i)} \ \dot{p}_n \in (\dot{P}_\alpha \cap M), \\
&\text{(ii)} \ \dot{p}_n \leq \dot{p}_{n-1}, \\
&\text{(iii)} \ \dot{p}_n \in D_{n-1}, \\
&\text{(iv)} \ \dot{p}_n{\restriction}\gamma_n \in \dot{G}_{\gamma_n}.
\end{aligned}
$$

Assume that q_n and $\dot{p}_n$ have been constructed. To find $\dot{p}_{n+1}$, let G be a P_{γ_n}-generic filter such that $q_n \in G$, and let $p_n = \dot{p}_n^G$. We have $p_n \in P_\alpha \cap M$ and $p_n{\restriction}\gamma_n \in G$. Since q_n is (M, P_{γ_n})-generic and $D_n \in M$, we can find a condition $p_{n+1} \leq p_n$ in $D_n \cap M$ such that $p_{n+1}{\restriction}\gamma_n \in G$. This describes the $P_{\gamma_{n+1}}$-name $\dot{p}_{n+1}$. Now we apply the inductive condition to γ_{n+1} (in place of α) and γ_n (in place of γ), for q_n and $\dot{p}_{n+1}{\restriction}\gamma_{n+1}$; we obtain a $q_{n+1} \in P_{\gamma_n+1}$ that forces (31.4) (with n replaced by $n+1$).

Now we let q be the limit of the q_n. Clearly, $q \in P_\alpha$ and $q{\restriction}\gamma_0 = q_0$. We complete the proof by showing that for every n, $q \Vdash_\alpha \dot{p}_n \in \dot{G}_\alpha$. This implies not only that $q \Vdash_\alpha \dot{p} \in \dot{G}_\alpha$, but also that q is (M, P_α)-generic, because $q \Vdash \dot{p}_n \in (D_{n-1} \cap M)$.

To verify that $q \Vdash_\alpha \dot{p}_n \in \dot{G}_\alpha$, let G be a generic filter on P_α and let $p_n = \dot{p}_n^G$. We have $p_n \in M$ and $p_n{\restriction}\gamma_k \in G_{\gamma_k} \cap M$ for all $k \geq n$. Thus if we let $\delta = \sup(\alpha \cap M)$, we have $p_n{\restriction}\delta \in G_\delta$. Since $p_n \in M$, its support is included in M and therefore $p_n{\restriction}\delta = p_n$. It follows that $p_n \in G$. □

A significant consequence of Theorem 31.15 is that countable support iteration of proper forcing preserves $\aleph_1$. As for cardinals above $\aleph_1$, one often needs additional assumptions on the iterates $\dot{Q}_\beta$ to calculate the chain condition. The easiest case was already stated in Exercise 16.20: If P is a countable support iteration of length $\kappa \geq \aleph_2$ such that each $P{\restriction}\beta$, $\beta < \kappa$, has a dense subset of size $< \kappa$, then P satisfies the κ-chain condition. In particular, iteration of length ω_2 with each $P{\restriction}\beta$ having a dense set of size $\aleph_1$, satisfies the $\aleph_2$-chain condition, and all cardinals are preserved.

A somewhat better result is the following which we state without a proof. For a proof, see Abraham's paper [∞] in the Handbook of Set Theory. [Shelah's book [1998] contains more general chain condition theorems.]

Theorem 31.19. *Assume* CH. *If P is a countable support iteration of length $\kappa \leq \omega_2$ of proper forcings $\dot{Q}_\beta$ of size $\aleph_1$, then P satisfies the $\aleph_2$-chain condition.* □

The Proper Forcing Axiom

When we replace the countable chain condition in Martin's Axiom $\mathrm{MA}_{\aleph_1}$ by properness we obtain a more powerful statement, the *Proper Forcing Axiom* (PFA):

Definition 31.20 (Proper Forcing Axiom (PFA)). If $(P, <)$ is a proper notion of forcing and if $\mathcal{D}$ is a collection of $\aleph_1$ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.

It turns out that PFA implies that $2^{\aleph_0} = \aleph_2$, and therefore PFA is a generalization of Martin's Axiom MA. Unlike MA, consistency of PFA requires large cardinals: It follows from the results stated later in this chapter that at least a Woodin cardinal is necessary. The consistency proof given below uses a supercompact cardinal.

Theorem 31.21. *If there exists a supercompact cardinal then there is a generic model that satisfies* PFA.

Proof. The proof follows loosely the proof of the consistency of MA. Let κ be a supercompact cardinal. The model is obtained by countable support iteration of length κ. Each notion of forcing used in the iteration is proper and has size $< \kappa$, thus both $\aleph_1$ and all cardinals $\geq \kappa$ are preserved. Cardinals between $\aleph_1$ and κ are collapsed and so κ becomes $\aleph_2$, and the model satisfies $2^{\aleph_0} = \aleph_2$.

In order to show that the resulting model satisfies PFA, we use a Laver function (see Theorem 20.21); this makes it possible to handle all potential proper forcing notions in κ steps.

Let $f : \kappa \to V_\kappa$ be a Laver function. We construct a countable support iteration P_κ of $\{\dot{Q}_\alpha : \alpha < \kappa\}$ as follows: At stage α, if $f(\alpha)$ is a pair $(\dot{P}, \dot{\mathcal{D}})$ of P_α-names such that $\dot{P}$ is a proper forcing notion and $\mathcal{D}$ is a γ-sequence of dense subsets of $\dot{P}$ for some $\gamma < \kappa$, we let $\dot{Q}_\alpha = \dot{P}$; otherwise, $\dot{Q}_\alpha$ is the trivial forcing.

Let G be a generic filter on P_κ, the countable support iteration of $\{\dot{Q}_\alpha : \alpha < \kappa\}$. Since each $\dot{Q}_\alpha$ is proper, P_κ is proper and therefore $\aleph_1$ is preserved. Each P_α (the iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$) has size less than κ (because $f(\alpha) \in V_\alpha$) and so P_κ has the κ-chain condition; hence all cardinals $\geq \kappa$ are preserved.

Lemma 31.22. *In $V[G]$, if P is proper and $\mathcal{D} = \{D_\alpha : \alpha < \gamma\}$, with $\gamma < \kappa$, is a family of dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.*

This lemma will complete the proof of the theorem: For every $\gamma < \kappa$, let P be the forcing that collapses γ onto ω_1 with countable conditions, and for $\alpha < \gamma$ let $D_\alpha = \{p \in P : \alpha \in \mathrm{ran}(p)\}$. By Lemma 31.22, there exists a collapsing map of γ onto ω_1. Thus $\kappa = \aleph_2$ in $V[G]$. Now Lemma 31.22 implies that $V[G]$ satisfies PFA. Moreover, $2^{\aleph_0} = \aleph_2$ in $V[G]$: On the one hand, PFA implies $\mathrm{MA}_{\aleph_1}$ and so $2^{\aleph_0} > \aleph_1$, and on the other hand, $2^{\aleph_0} \leq \kappa$ because $|P_\kappa| = \kappa$.

Proof of Lemma 31.22. Let $\dot{P}$ and $\dot{\mathcal{D}}$ be P_κ-names for P and $\mathcal{D}$. Let $\lambda > 2^{2^{|P|}}$ be sufficiently large; we may also assume that $P \subset \lambda$. Since f is a Laver function, there exists an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $M^\lambda \subset M$, and $(jf)(\kappa) = (\dot{P}, \dot{\mathcal{D}})$.

P is a proper forcing in $V[G]$. This is witnessed by some closed unbounded set $C \subset [H_\eta]^\omega$ of countable models for some η with $2^{|P|} < \eta < \lambda$. Since $M^\lambda \subset M$ and P_κ has the κ-chain condition, $V[G]$ satisfies that $M[G]^\lambda \subset M[G]$, and therefore C is closed unbounded in $M[G]$. Therefore P is proper in the model $M[G]$.

Now consider the forcing notion $j(P_\kappa)$ in M. It is a countable support iteration of length $j(\kappa)$ using the Laver function $j(f)$. Since $j{\restriction}V_\kappa$ is the identity, we have $j(P_\kappa){\restriction}\kappa = P_\kappa$. As $(jf)(\kappa) = (\dot{P}, \dot{\mathcal{D}})$ and P is proper in $M[G]$, it follows that $(j\dot{Q})_\kappa = \dot{P}$. Hence

$$j(P_\kappa) = P_\kappa * \dot{P} * \dot{R}$$

for some $\dot{R}$.

Let $H * K$ be a $V[G]$-generic ultrafilter on $\dot{P} * \dot{R}$. In $V[G * H * K]$ we extend the elementary embedding $j : V \to M$ to an elementary embedding $j^* : V[G] \to M[G * H * K]$ as follows: For every P_κ-name $\dot{x}$, let

$$j^*(\dot{x}^G) = j(\dot{x})^{G*H*K}.$$

The definition of j^* does not depend on the choice of the name $\dot{x}$, since $\|\dot{x} = \dot{y}\| \in G$ implies $\|j(\dot{x}) = j(\dot{y})\| \in G * H * K$ (because $j(p) = p$ for every $p \in P_\kappa$). Similarly, $\|\varphi(\dot{x})\| \in G$ implies $\|\varphi(j(\dot{x}))\| \in G * H * K$, and so j^* is elementary. Clearly, j^* extends j.

The filter H on P is $V[G]$-generic and thus meets every D_α, $\alpha < \gamma$. Let $E = \{j(p) : p \in H\}$. Since $j{\restriction}\lambda \in M$, the set E is in $M[G * H * K]$, and generates a filter on $j^*(P)$ that is $j^*(\mathcal{D})$-generic. Thus

$$M[G * H * K] \vDash \text{there exists a } j^*(\mathcal{D})\text{-generic filter on } j^*(P)$$

and since $j^* : V[G] \to M[G * H * K]$ is elementary, there exists in $V[G]$ a $\mathcal{D}$-generic filter on P. □

Applications of PFA

Our first goal is to outline the proof of the following theorem:

Theorem 31.23 (Todorčević). PFA *implies* $2^{\aleph_0} = \aleph_2$.

As the first step we show that the Open Coloring Axiom (29.6) is a consequence of PFA. If $[X]^2 = K_0 \cup K_1$ with K_0 open, let us call $Z \subset X$ 0-*homogeneous* if $[Z]^2 \subset K_0$ and 1-*homogeneous* if $[Z]^2 \subset K_1$. It is clear that the closure of a 1-homogeneous set is also 1-homogeneous, and so in (29.6) we can further assume that the sets H_n are closed.

The proof of OCA from PFA uses the following technical lemma that we state without proof:

Lemma 31.24 (Todorčević). *Assume* $2^{\aleph_0} = \aleph_1$. *Let* $X \subset \boldsymbol{R}$ *and* $[X]^2 = K_0 \cup K_1$ *with* K_0 *open, and assume that* X *is not the union of countably many closed* 1*-homogeneous sets. Then there exists an uncountable* $Y \subset X$ *such that in any uncountable set* $W \subset \{p \in [Y]^{<\omega} : p$ *is* 0*-homogeneous*$\}$ *there exist* $p \neq q$ *such that* $p \cup q$ *is* 0*-homogeneous.*

Proof. See Theorem 4.4 of Todorčević [1989]. (To apply the theorem, let $F(x)$ be the closure of $\{y \in X : x < y$ and $\{x, y\} \in K_1\}$.) □

Theorem 31.25. PFA *implies* OCA.

Proof. Let $X \subset \boldsymbol{R}$ and let $[X]^2 = K_0 \cup K_1$ with K_0 open, and assume that X is not the union of countably many closed 1-homogeneous sets. We shall use PFA to find an uncountable 0-homogeneous set.

Let P be the forcing (15.2) that adds a subset of ω_1 with countable conditions. By Exercise 15.14, V^P satisfies $2^{\aleph_0} = \aleph_1$. Since P does not add new reals, it does not add new closed sets of reals and so in V^P, X is not the union of countably many closed 1-homogeneous sets.

By Lemma 31.24 there exists an uncountable $\dot{Y} \in V^P$ such that if we let $\dot{Q} = \{p \in [\dot{Y}]^{<\omega} : p$ is 0-homogeneous$\}$ (and p is stronger than q if $p \supset q$) then the forcing notion $\dot{Q}$ satisfies the countable chain condition. Hence $P * \dot{Q}$ is proper.

Let $\langle y_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of $\dot{Y}$ in V^P. For each $\alpha < \omega_1$, the set $D_\alpha = \{(p, q) \in P \times \dot{Q} : p \Vdash y_\alpha \in q\}$ is a dense set in $P * \dot{Q}$. Let $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$. By PFA there exists a $\mathcal{D}$-generic filter G on $P * \dot{Q}$, and then the set $Y = \bigcup\{q : (p, q) \in G\}$ is an uncountable 0-homogeneous set. □

Theorem 31.25 appears in Todorčević [1989]. Its proof does not require the full force of PFA. What we used is a weaker statement that is obtained by replacing "proper notion of forcing" in Definition 31.20 by "Axiom A forcing of cardinality $\leq 2^{\aleph_0}$." This axiom is weaker than PFA (and stronger than $\mathrm{MA}_{\aleph_1}$) and is consistent relative to ZFC+"there exists a weakly compact cardinal" (see Baumgartner [1984]).

The consistency of related partition axioms was first established in Abraham, Rubin and Shelah [1985].

By Theorems 31.25 and 29.8, PFA implies OCA which implies $\mathfrak{b} = \aleph_2$. Thus to complete the proof of Theorem 31.23 it is enough to show that PFA implies $\mathfrak{b} = 2^{\aleph_0}$. We shall use another technical lemma of Todorčević that we state without a proof.

Let $\kappa \leq 2^{\aleph_0}$ be a regular uncountable cardinal and let $F : [\kappa]^2 \to \omega$ be a partition. Let P be the forcing with countable conditions that adds a subset of ω_1. In V^P, $|\kappa| = \aleph_1$ and $\operatorname{cf} \kappa = \omega_1$; let $\dot{C} \in V^P$ be a closed unbounded subset of κ of order-type ω_1, consisting of limit ordinals. For every n and k let $\dot{R}^k_n$ be the forcing where conditions are finite k-homogeneous (for F) subsets of $\dot{C} + n = \{\alpha + n : \alpha \in \dot{C}\}$. $\dot{R}^k_n$ adds a k-homogeneous subset $\dot{G}^k_n$ of $\dot{C} + n$. (In general, $\dot{R}^k_n$ need not satisfy the countable chain condition.) Let $\dot{Q}^k_n$ be the product of ω copies of $\dot{R}^k_n$, and for every real $r \in \omega^\omega$, let $\dot{Q}_r = \dot{Q}_r(\dot{C})$ be the product of $\dot{Q}^{r(n)}_n$, $n < \omega$.

Lemma 31.26. *There exists a partition $F : [\mathfrak{b}]^2 \to \omega$ such that in V^P, for every $\dot{C}$ as above and every $r \in \omega^\omega$, $\dot{Q}_r(\dot{C})$ satisfies the countable chain condition.*

Proof. See Bekkali [1991], page 49. The partition F is obtained by using oscillating real numbers, cf. Chapter 1 of Todorčević [1989]. □

Lemma 31.27. PFA *implies* $\mathfrak{b} = 2^{\aleph_0}$.

Proof. Let $F : [\mathfrak{b}]^2 \to \omega$ be as in Lemma 31.26. Let P be the ω-closed forcing that adds a subset of ω_1, and let $\dot{C} \in V^P$ be a closed unbounded subset of $\mathfrak{b}$, of order-type ω_1.

Let $r \in \omega^\omega$. The forcing $P * \dot{Q}_r(\dot{C})$ is proper and we apply PFA to obtain a sufficiently generic filter $G \times \prod_n \prod_i G_{n,i}$. Let $C(r) = C = \dot{C}^G$; C is a closed unbounded subset of some $\delta(r) = \delta < \mathfrak{b}$, $\operatorname{cf} \delta = \omega_1$, and for each n, each $G_{n,i}$ is an $r(n)$-homogeneous subset of $C + n$. Let $C_{n,i} = G_{n,i} - n$; by genericity, we have $C = \bigcup_{i<\omega} C_{n,i}$ for each n, and

$$r(n) = k \quad \text{if and only if} \quad \forall i\ \forall \alpha, \beta \in C_{n,i}\ F(\alpha + n, \beta + n) = k. \tag{31.5}$$

We claim that if $r \neq s$ then $\delta(r) \neq \delta(s)$.

Let n be such that $r(n) \neq s(n)$. Assuming that $\delta(r) = \delta(s) = \delta$, the set $C(r) \cap C(s)$ is closed unbounded in δ, and we can find i and j such that $C_{n,i}(r) \cap C_{n,j}(s)$ is unbounded. Let $\alpha < \beta$ be in this unbounded set; then by (31.5), $F(\alpha + \beta, \beta + n) = r(n) = r(s)$, a contradiction.

Thus we have produced a one-to-one mapping of ω^ω into $\mathfrak{b}$. □

The next theorem establishes the consistency strength of PFA (see the discussion following the proof):

Theorem 31.28 (Todorčević). PFA *implies that $\square_\kappa$ fails for every uncountable cardinal κ.*

Proof. Let κ be an uncountable cardinal, assume that $\square_\kappa$ holds, and let $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ be a square-sequence (cf. (23.4)).

Let T be the tree whose nodes are limit ordinals below κ^+, and $\beta \prec \alpha$ if $\beta \in \mathrm{Lim}(C_\alpha)$. Since $\langle C_\alpha\rangle_\alpha$ is a square-sequence, T has no κ^+-branch.

Let λ be sufficiently large, and consider countable elementary submodels M of H_λ such that $\langle C_\alpha : \alpha < \kappa^+\rangle \in M$; let $\delta_M = \sum(M \cap \kappa^+)$. An *elementary chain* is a sequence $\langle M_\alpha : \alpha < \omega_1\rangle$ of elementary submodels of H_λ such that $M_\alpha \subset M_\beta$ and $M_\alpha \in M_\beta$ whenever $\alpha < \beta$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ if α is a limit ordinal. If E is a finite subset of ω_1 then an E-chain is $\langle M_\alpha : \alpha \in E\rangle$ such that each M_α is an elementary submodel of H_λ, and $M_\alpha \cup \{M_\alpha\} \subset M_\beta$ for $\alpha < \beta$ in E.

We now define a forcing notion P as follows: A condition $p \in P$ is a pair $(\langle N_\alpha : \alpha \in E\rangle, f)$ where

(31.6)
- (i) E is a finite subset of ω_1 and $\langle N_\alpha : \alpha \in E\rangle$ is an E-chain such that there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ such that $N_\alpha = M_\alpha$ for all $\alpha \in E$,
- (ii) f is a function from $\{\delta_{N_\alpha} : \alpha \in E\}$ into ω such that $f(\gamma) \neq f(\delta)$ whenever $\gamma \prec \delta$.

A condition q is stronger than p if $p = q{\restriction}E$.

Note that (i) resembles the forcing that adds a closed unbounded set with finite conditions, and (ii) resembles the forcing that specializes an Aronszajn tree.

Lemma 31.29. *P is proper.*

Proof. We omit the proof, as it is similar to the proof of properness in Exercise 31.5 (and using the fact that $(T, \prec)$ has no κ^+-branch). □

Now we use PFA to reach a contradiction. Let G be a sufficiently generic filter on P. The filter G yields an elementary chain $\langle N_\alpha : \alpha < \omega_1\rangle$ and a closed unbounded set $\{\delta_\alpha : \alpha < \omega_1\}$ (where $\delta_\alpha = \delta_{N_\alpha}$) with supremum γ. There is a closed unbounded set $C \subset \omega_1$ such that for all $\alpha \in C$, δ_α is a limit point of C_γ. Since $\delta_\alpha \prec \delta_\beta$ whenever $\alpha < \beta \in C$, it follows that $\{\delta_\alpha : \alpha \in C\}$ is an ω_1-chain in T.

On the other hand, the filter G yields a specializing function on $\{\delta_\alpha : \alpha < \omega_1\}$, that is a function F with values in ω such that $F(\delta_\alpha) \neq F(\delta_\beta)$ whenever $\delta_\alpha \prec \delta_\beta$. A contradiction. □

The proof of Theorem 31.28 has been modified by Magidor to show that under PFA, even a weak version of $\square$ fails (we shall discuss these versions of $\square$ in Chapter 38). It has been proved by Schimmerling that the failure of those principles imply an inner model for a Woodin cardinal. Thus we have:

Theorem 31.30 (Schimmerling). *If* PFA *holds then there exists an inner model of "there exists a Woodin cardinal."* □

Martin's Axiom $\mathrm{MA}_{\aleph_1}$ implies that there are no Suslin trees, and moreover, that every Aronszajn tree is special. PFA implies a stronger result. If T is a normal ω_1-tree and $C \subset \omega_1$ a closed unbounded set, then $T{\restriction}C$ is the tree $\{t \in T : o(t) \in C\}$. Two trees T_1 and T_2 are *club-isomorphic* if there exists a closed unbounded C such that $T_1{\restriction}C$ and $T_2{\restriction}C$ are isomorphic.

Theorem 31.31. *If* PFA *holds then any two normal Aronszajn trees are club-isomorphic.*

Proof. Let T_1 and T_2 be two normal Aronszajn trees. Consider the forcing with finite conditions (E, f) such that

(31.7)
(i) E is a finite subset of ω_1,
(ii) $\mathrm{dom}(f)$ is a subtree of $T_1{\restriction}E$ in which every branch has size $|E|$; similarly for $\mathrm{ran}(f) \subset T_2{\restriction}E$,
(iii) f us an isomorphism.

We omit the proof that P is proper and refer the reader to Todorčević [1984], Theorem 5.10.

A sufficiently generic filter on P yields an uncountable set A and an isomorphism between $T_1{\restriction}A$ and $T_2{\restriction}A$, which easily extends to $T_1{\restriction}C$ where C is the closure of A. □

We present one more consequence of PFA, due to J. Baumgartner (compare with Theorem 28.24):

Theorem 31.32. *If* PFA *holds then there are no* $\aleph_2$*-Aronszajn trees.*

Proof. Assume that T is an $\aleph_2$-Aronszajn tree. Let P be the forcing that adds a subset of ω_1 with countable conditions. Since $2^{\aleph_0} = \aleph_2$, P collapses ω_2 and so there is in V^P a closed unbounded subset $\dot{C}$ of ω_2, of order-type ω_1. The tree T has no new branches (this is proved as in Lemma 27.10, because the levels of T have size $\aleph_1 < 2^{\aleph_0}$). Thus $\dot{U} = T{\restriction}\dot{C}$ is in V^P an ω_1-tree with no ω_1-branches.

Now let $\dot{Q} \in V^P$ be the specializing forcing for $\dot{U}$, as in Theorem 16.17. $\dot{Q}$ satisfies the countable chain condition, and so $P * \dot{Q}$ is proper.

Let G be a sufficiently generic filter on $P * \dot{Q}$. It yields a closed unbounded subset C of some $\gamma < \omega_2$, a tree $U = T{\restriction}C$, and a specializing function $f : U \to \omega$. This is a contradiction, since a special tree has no ω_1-branches, while every $t \in T$ at level γ produces an ω_1-branch in U. □

Exercises

31.1. If P is strategically ω-closed then P is proper.

The following two exercises present equivalent versions of the proper game:

31.2. Let $p \in P$. Player II has a winning strategy in the proper game if and only if II has a winning strategy in the game where I plays ordinal names $\dot\alpha_n$ and II plays countable sets of ordinals B_n, and II wins if some $q \leq p$ forces $\forall n\ \exists k\ \dot\alpha_n \in B_k$.

31.3. P is proper if and only if for every $p \in P$, II has a winning strategy in the following game: At move n, I plays a maximal antichain A_n and II responds by playing countable sets $B_0^n \subset A_0$, ..., $B_n^n \subset A_n$. II wins if for some $q \leq p$, $\forall n$ $\bigcup_{k=n}^\infty B_n^k$ is predense below q.

[In the forward direction, let λ be sufficiently large and let C be a closed unbounded set of models $M \prec H_\lambda$ that satisfy (31.1) and $p \in M$. The following is a winning strategy for II: When I plays A_n, let II choose some $M_n \in C$ such that $M_n \supset M_{n-1}$ and $A_n \in M_n$, and let $B_k^n = A_k \cap M_n$, $k = 0, \ldots, n$. Let $M = \bigcup_{n=0}^\infty M_n$ and let $q \leq p$ be (M, P)-generic. Since $A_n \cap M = \bigcup_{k=0}^\infty B_n^k$, II wins.

Conversely, let σ be a winning strategy for II, and let λ be sufficiently large with $\sigma \in H_\lambda$. Show that for every $M \prec H_\lambda$ such that $P, p, \sigma \in M$ there is some (M, P)-generic $q \leq p$ (by playing a game in which I plays successively all maximal antichains $A \in M$). Let C_p be the closed unbounded set of all such M; the diagonal intersection $\triangle_p C_p$ witnesses that P is proper.]

31.4. If P satisfies Axiom A and $p \in P$ then II has a winning strategy in the following game (more difficult for player II than the proper game): I plays ordinal names $\dot\alpha_n$ and II plays countable sets of ordinals B_n; II wins if some $q \leq p$ forces $\forall n\ \dot\alpha_n \in B_n$.

Adding a closed unbounded set with finite conditions: A condition $p \in P$ is a finite function with $\mathrm{dom}(p) \subset \omega_1$, $\mathrm{ran}(p) \subset \omega_1$ such that there exists a normal function $f : \omega_1 \to \omega_1$ with $f \supset p$. A condition q is stronger than p if $q \supset p$. If G is generic then $f_G = \bigcup\{p : p \in G\}$ is a normal function. Note that if $\alpha = \omega^\beta$ (an *indecomposable* ordinal) and $p \subset \alpha \times \omega$ is a condition then $p \cup \{(\alpha, \alpha)\}$ is also a condition.

31.5. Let P be as above and let $p \in P$. Then II has a winning strategy in the game from Exercise 31.3. Hence P is proper.

[When I plays A_n, II finds some indecomposable $\alpha_n > \alpha_{n-1}$ such that for all $k \leq n$

$$(\forall \beta < \alpha)(\exists \gamma < \alpha)(\forall p \subset \gamma \times \gamma)(\exists q \subset \gamma \times \gamma)\ q \in A_k$$

(and q is compatible with p), and plays $B_k^n = \{p \in A_k : p \subset \alpha_n \times \alpha_n\}$.]

31.6. Let P be as above. Then I has a winning strategy in the game from Exercise 31.4. Hence P does not satisfy Axiom A.

[Let $\dot f$ be the name for f_G. At move n, player I chooses an indecomposable ordinal α_n greater than B_{n-1} and plays $\dot f(\alpha_n)$.]

31.7. Let P be the ω-closed forcing for collapsing ω_2 to ω_1 with countable conditions. There exists a set $\mathcal{D}$ of $\aleph_2$ dense sets for which there is no $\mathcal{D}$-generic filter.

[For $\alpha < \omega_2$, let $D_\alpha = \{p \in P : \alpha \in \mathrm{ran}(p)\}$.]

PFA$^+$ is the following statement: If P is proper, if $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ are dense sets and if $\Vdash \dot S \subset \omega_1$ is stationary, then there exists a $\mathcal{D}$-generic filter G such that $\dot S^G$ is stationary (where $\dot S^G = \{\alpha : \exists p \in G\ p \Vdash \alpha \in \dot S\}$).

31.8. PFA^+ is consistent relative to a supercompact cardinal.
[Modify the proof of Theorem 31.21.]

31.9. PFA^+ implies that for every regular $\kappa \geq \omega_2$, every stationary set $A \subset E^\kappa_\omega$ reflects at some γ of cofinality ω_1.
[Let $A \subset E^\kappa_\omega$ be stationary. Let P consist of closed countable subsets of κ, ordered by end-extension. P is ω-closed and adds a closed unbounded subset $\dot{C} \subset \kappa$ of order-type ω_1. A remains stationary and so $A \cap \dot{C}$ is a stationary subset of $\dot{C}$; let $\dot{S} = f_{-1}(A \cap \dot{C})$ where f is the isomorphism between ω_1 and $\dot{C}$. If G is sufficiently generic such that $\dot{S}^G$ is stationary, then $A \cap \dot{C}^G$ is stationary in $\gamma = \sup \dot{C}^G$.]

PFA^- is the statement: If P is proper such that $|P| \leq \aleph_1$ and if $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ are dense then there exists a $\mathcal{D}$-generic filter. In [1982] Shelah proves that PFA^- is consistent relative to ZFC only.

31.10. PFA^- implies that any two normal Aronszajn trees are club-isomorphic.
[The forcing in (31.7) has size $\aleph_1$.]

Historical Notes

Proper forcing was introduced by Shelah, cf. [1982] and [1998]. The iteration Theorem 31.15 is due to Shelah; our treatment follows Abraham's article [∞]. The proper game was formulated independently by Shelah and C. Gray.

Proper Forcing Axiom was introduced by Baumgartner [1984]; earlier (Baumgartner [1983]) he introduced Axiom A. Theorem 31.21 is due to Baumgartner.

Theorem 31.23: Todorčević [1989], see also Bekkali [1991]. (The claim in Veličković [1992] to this result cannot be substantiated.)

Theorem 31.25: Todorčević [1989]; Abraham, Rubin and Shelah [1985].

Theorem 31.28: Todorčević [1984].

Theorem 31.31: Abraham and Shelah [1985].

Theorem 31.32: Baumgartner [1984].

The forcing for adding a closed unbounded set with finite conditions is due to Baumgartner [1983].

Exercises 31.8, 31.9: Baumgartner [1984].

Exercise 31.10: Abraham and Shelah [1985].

32. More Descriptive Set Theory

$\mathbf{\Pi}^1_1$ Equivalence Relations

Theorem 32.1 (Silver). *If E is a $\mathbf{\Pi}^1_1$ equivalence relation on $\mathcal{N}$ then either E has at most $\aleph_0$ equivalence classes or there exits a perfect set of mutually inequivalent reals.*

Thus every $\mathbf{\Pi}^1_1$ (and in particular) Borel equivalence relation has either at most countably many or $2^{\aleph_0}$ equivalence classes. This can be viewed as a generalization of the perfect set property for analytic sets (Theorem 11.18); cf. Exercise 32.1. The theorem does not extend to $\mathbf{\Sigma}^1_1$, as there exists a Σ^1_1 equivalence relation with exactly $\aleph_1$ equivalence classes (Exercise 32.2).

We present a proof of Theorem 32.1 that is due to Leo Harrington. We start with an easy lemma.

Lemma 32.2. *Let E be a meager equivalence relation on $\mathcal{N}$. Then there exist a perfect set of inequivalent reals.*

Proof. Let $\{D_n\}_n$ be dense open sets in $\mathcal{N}\times\mathcal{N}$ such that $\mathcal{N}^2 - E \supset \bigcap_{n=0}^{\infty} D_n$. We construct a binary tree of finite sequences $\{u_s : s \in \mathit{Seq}(\{0,1\})\} \subset \mathit{Seq}$ such that for every n, if $|s| = |t| = n$ and $s \neq t$, then $O(u_s) \times O(u_t) \subset D_n$.

This is done by induction on the length of s. If the u_s have been defined for all $s \in \{0,1\}^n$, we consider successively all possible pairs $(s^\frown i, t^\frown j)$, and using the density of D_{n+1}, successively extend each $u_{s^\frown i}$ until $O(u_{s^\frown i}) \times O(u_{t^\frown j}) \subset D_{n+1}$ for all $s, t \in \{0,1\}^n$ and $i, j = 0, 1$.

For each $f \in \{0,1\}^\omega$ let a_f be the unique member of $\bigcap_{n=0}^{\infty} O(u_{f\restriction n})$. The set $\{a_f : f \in \{0,1\}^\omega\}$ is perfect, and if $f \neq g$ then $(a_f, a_g) \notin E$. □

We shall use a version of Lemma 32.2 for a different topology on $\mathcal{N}\times\mathcal{N}$. Toward the proof let us recall some basic facts about the property of Baire. In particular, Lemmas 11.16 and 11.17 as well as the fact that the σ-algebra of sets with the Baire property is closed under the Suslin operation $\mathcal{A}$, remain true in every second countable space (i.e., space that has a countable basis).

We shall prove Silver's Theorem for (lightface) Π^1_1 equivalence relations; the proof relativizes to $\Pi^1_1(a)$ for every real parameter a.

Definition 32.3. The Σ^1_1-*topology* on $\mathcal{N}$ is the topology with basic open sets being all the Σ_1 subsets of $\mathcal{N}$.

The Σ^1_1-topology has a countable base and is larger than the standard topology, as every basic open set $O(s)$ in $\mathcal{N}$ is Σ^0_1.

Lemma 32.4. *The Σ^1_1-topology satisfies the Baire Category Theorem.*

Proof. Exercise 32.3. □

Lemma 32.5. *If X is comeager in the Σ^1_1-topology then for every nonempty Σ^1_1 subset A of $\mathcal{N} \times \mathcal{N}$, $A \cap (X \times X) \neq \emptyset$.*

Proof. The lemma states that $X \times X$ is dense in the Σ^1_1-topology on $\mathcal{N} \times \mathcal{N}$ (which is larger than the product of the Σ^1_1-topology). If D is a dense open set in the Σ^1_1-topology then $D \times \mathcal{N}$ is dense open in the Σ^1_1-topology on $\mathcal{N}^2$: This is because if $A \neq \emptyset$ is a Σ^1_1 subset of $\mathcal{N}^2$ then its projection is Σ^1_1 and hence meets D.

Let $X \supset \bigcap_{n=0}^{\infty} D_n$ where each D_n is dense open. Then $X \times X \supset \bigcap_{n=0}^{\infty}(D_n \times \mathcal{N}) \cap \bigcap_{n=0}^{\infty}(\mathcal{N} \times D_n)$, and the latter set is dense, by the Baire Category Theorem applied to the Σ^1_1-topology on $\mathcal{N} \times \mathcal{N}$. □

Given a Π^1_1 equivalence relation E on $\mathcal{N}$, consider the set that is the complement of the union of all Σ^1_1 sets contained in some equivalence class:

$$H = \{a \in \mathcal{N} : \text{for every } \Sigma^1_1 \text{ set } U, \text{ if } a \in U \text{ then there is a } b \in U \text{ with } (a,b) \notin E\}. \tag{32.1}$$

Note that if H is empty then every equivalence class is the union of Σ^1_1 sets and therefore there are at most $\aleph_0$ equivalence classes. We shall prove that if $H \neq \emptyset$ then there exists a perfect set of inequivalent reals.

Lemma 32.6. *H is a Σ^1_1 set.*

Proof. First note that if an equivalence class A of E contains a nonempty Σ^1_1 set U then A is Π^1_1:

$$x \in A \leftrightarrow \forall y\, (y \in U \rightarrow x \; E \; y).$$

Then by the separation principle there exists a Δ^1_1 set V such that $U \subset V \subset A$. It follows that

$$H = \{a : \text{for every } \Delta^1_1 \text{ set } U, \text{ if } a \in U \text{ then } \exists b \in U \text{ with } (a,b) \notin E\}. \tag{32.2}$$

The quantification "for every Δ^1_1 set" in (32.2) can be replaced by "for every Borel code for a Δ^1_1 set" and since we are dealing only with lightface Δ^1_1 sets, this can be replaced by a number quantifier $\forall n$. Similarly, "$a \in U$" and "$b \in U$" are Δ^1_1 properties, and it follows that H is Σ^1_1. □

Lemma 32.7. *For every $a \in \mathcal{N}$, $E_a \cap H$ is meager in the Σ^1_1-topology, where $E_a = \{b : (a,b) \in E\}$.*

Proof. If $H = \emptyset$ then there is nothing to prove; thus assume $H \neq \emptyset$. The set E_a is Π^1_1 and therefore has the Baire property in the Σ^1_1-topology. If $E_a \cap H$ is not meager then there exists a nonempty Σ^1_1 set U such that $E_a \cap U$ is comeager in U. As $U \subset H$, $U \times U$ is not contained in E and so $U^2 - E$ is nonempty; hence we have (by Lemma 32.5) $(U^2 - E) \cap (E_a \cap U)^2 \neq \emptyset$. In other words there exist $b, c \in U$ such that $a\ E\ b$, $a\ E\ c$ and $(b, c) \notin E$, a contradiction. □

Lemma 32.8. *$E \cap (H \times H)$ is meager (in the product of the Σ^1_1-topology).*

Proof. By Lemma 32.7 and Lemma 11.16. □

Proof of Theorem 32.1. If H is empty then E has at most $\aleph_0$ equivalence classes. If $H \neq \emptyset$ then H is Σ^1_1 and therefore a basic open set in the Σ^1_1-topology. By Lemma 32.8 $E \cap (H \times H)$ is meager in the product of the Σ^1_1-topology. The rest of the proof (which we omit) is a combination of the construction in the proof of Lemma 32.2 and the construction in Exercise 32.3: One can produce a perfect set $\{a_f : f \in \{0,1\}^\omega\} \subset H^2$ such that $(a_f, a_g) \notin E$ whenever $f \neq g$. □

$\mathbf{\Sigma^1_1}$ Equivalence Relations

Theorem 32.9. *If E is a $\mathbf{\Sigma}^1_1$ equivalence relation on $\mathcal{N}$ then either E has at most $\aleph_1$ equivalence classes or there exists a perfect set of mutually inequivalent reals.*

This theorem, due to J. Burgess, extends Silver's Theorem and uses it in the proof. Note that Exercise 32.2 makes it best possible.

Proof. Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation. There exists a tree T on $(\omega \times \omega) \times \omega$ such that for all $a, b \in \mathcal{N}$

$$a\ E\ b \leftrightarrow T(a,b) \text{ is ill-founded.} \tag{32.3}$$

We define, for each $\alpha < \omega_1$, a relation E^α on $\mathcal{N}$ as follows:

$$a\ E^\alpha\ b \leftrightarrow \text{not } (\|T(a,b)\| < \alpha). \tag{32.4}$$

It is clear that each E^α is a Borel relation, $E^\alpha \supset E^\beta$ if $\alpha < \beta$, $E^\alpha = \bigcap_{\beta<\alpha} E^\beta$ if α is limit, and $E = \bigcap_{\alpha<\omega_1} E^\alpha$. Moreover, each E^α is reflexive as $E^\alpha \supset E$.

Lemma 32.10. *There is a closed unbounded set $C \subset \omega_1$ such that for each $\alpha \in C$, E^α is an equivalence relation.*

Proof. If $T(x,y)$ is well-founded then so is $T(y,x)$ (by the symmetry of E) and so for every $\alpha < \omega_1$ the set $\{T(y,x) : \|T(x,y)\| < \alpha\}$ is a set of well-founded trees. The set is $\mathbf{\Sigma}^1_1$ and so, by the Boundedness Lemma there is a countable ordinal $f(\alpha)$ such that $\|T(y,x)\| < f(\alpha)$ whenever $\|T(x,y)\| < \alpha$. Let γ be a closure point of f, i.e., if $\alpha < \gamma$ then $f(\alpha) < \gamma$. Let $a, b \in \mathcal{N}$. If $(b,a) \notin E^\gamma$, or $\|T(b,a)\| < \gamma$, then $\|T(a,b)\| < \gamma$, or $(a,b) \notin E^\gamma$ and so E^γ is symmetric.

Similarly, there is a function $g : \omega_1 \to \omega_1$ such that if γ is a closure point of g then for all $a, b, c \in \mathcal{N}$, if $(a,c) \notin E^\gamma$ then either $(a,b) \notin E^\gamma$ or $(b,c) \notin E^\gamma$. Let C be the set of all closure points of both f and g. □

Now assume that E has more than $\aleph_1$ equivalence classes. We shall prove that there exists a perfect set of E-inequivalent reals.

Let $V[G]$ be a generic extension of V that collapses $\aleph_1$ and makes $\aleph_2^V = \aleph_1^{V[G]}$. Let $\tilde{E}$ denote the relation defined in $V[G]$ by (32.3), and for each $\alpha < \omega_1^{V[G]}$ let $\tilde{E}^\alpha$ be defined by (32.4). $\tilde{E}$ is $\mathbf{\Sigma}^1_1$, and each $\tilde{E}^\alpha$ is Borel. By absoluteness, $\tilde{E} \cap V = E$ and $\tilde{E}$ is an equivalence relation, $\tilde{E}^\alpha \cap V = E^\alpha$ for each $\alpha < \omega_1^V$, and if E^α is an equivalence relation then so is $\tilde{E}^\alpha$. Since $\tilde{E}^{\omega_1^V} = \bigcap_{\alpha < \omega_1^V} \tilde{E}^\alpha$, it is a Borel equivalence relation. We assume that E has, in V, a set X of size $\aleph_2$ of inequivalent reals. If $x, y \in X$ and $x, y \notin E$ then $(x,y) \notin E^\alpha$ for some $\alpha < \omega_1^V$. Hence X is a set of $\tilde{E}^{\omega_1^V}$-inequivalent reals, and X is uncountable in $V[G]$.

By Silver's Theorem, $\tilde{E}^{\omega_1^V}$ has a perfect set of inequivalent reals. These reals are $\tilde{E}$-inequivalent and so

(32.5) $\qquad V[G] \vDash$ there is a perfect set of $\tilde{E}$-inequivalent reals.

However, the statement in (32.5) true in $V[G]$ is clearly $\mathbf{\Sigma}^1_2$ and so by Shoenfield's Absoluteness Theorem, it holds in V. Therefore in V, there exists a perfect set of E-inequivalent reals. □

Constructible Reals and Perfect Sets

We recall (Lemma 26.50) that if there exists a nonconstructible real then the set $\boldsymbol{R} \cap L$ is Lebesgue measurable only if it is null, and has the property of Baire only if it is meager. The following theorem proves a similar result for perfect sets.

Theorem 32.11. *If there exists a nonconstructible real then the set $\boldsymbol{R} \cap L$ does not have a perfect subset.*

Proof. As a first step we show that $\boldsymbol{R} \cap L$ does not have a *superperfect* subset. A tree $T \subset Seq$ is *superperfect* if for every $t \in T$ there exists an $s \supset t$ in T such that $s^\frown k \in T$ for infinitely many $k \in \omega$. (We call s an *ω-splitting node* of T.) A nonempty set $P \subset \mathcal{N}$ is *superperfect* if $P = [T]$ for some superperfect tree T.

Lemma 32.12. *If $\mathcal{N} \cap L$ has a superperfect subset then every real is constructible.*

Proof. Instead of $\mathcal{N}$, consider the space $[\omega]^\omega$ of increasing sequences of natural numbers. Let x, y, z be distinct elements of $[\omega]^\omega$ and let

$$O(x,y,z) = \{n \in \omega : z(n-1) \leq x(n-1),\ z(n-1) \leq y(n-1) \text{ and } z(n) > x(n),\ z(n) > y(n)\}. \tag{32.6}$$

If $O(x,y,z)$ is infinite, let $\langle n_k : k \in \omega\rangle$ be its increasing enumeration and let

$$o(x,y,z) = \{k : x(n_k) \leq y(n_k)\}. \tag{32.7}$$

Now assume that $P = [T]$ is a superperfect subset of $[\omega]^\omega$ such that every $x \in P$ is constructible. We shall prove that every real is constructible as follows: Let $A \subset \omega$ be arbitrary; we shall find $x, y, z \in [T]$ such that $o(x,y,z)$ (is defined and) is equal to A. Then A is constructible, as the definition (32.7) is absolute for L.

Thus let $A \subset \omega$ be arbitrary. We find $x, y, z \in [T]$ by constructing inductively their initial segments. We construct sequences $x_0 \subset x_1 \subset \ldots \subset x_k \subset \ldots$, $y_0 \subset y_1 \subset \ldots \subset y_k \subset \ldots$, and $z_0 \subset z_1 \subset \ldots \subset z_k \subset \ldots$ of ω-splitting nodes of T such that for each k, $n_k = |z_k|$ is the kth element of $O(x,y,z)$, and $k \in o(x,y,z)$ if and only if $k \in A$. Inductively, we arrange $l_k = |x_k| > n_k$ and $m_k = |y_k| > n_k$, as well as $z_k(n_k - 1) \leq x_k(n_k - 1)$ and $z_k(n_k - 1) \leq y_k(n_k - 1)$.

We omit the initial stage of the induction as it is similar to the induction step: At stage $k+1$ we find an integer i greater than $x_k(l_k - 1)$ and $y_k(m_k - 1)$ such that $z_k{}^\frown i \in T$. Then we let $z_{k+1} \supset z_k$ be an ω-splitting node above such that $n_{k+1} = |z_{k+1}|$ is greater than l_k and m_k. Now if $k+1 \in A$, let $j > z_{k+1}(n_{k+1} - 1)$ be such that $x_k{}^\frown j \in T$, and let $x_{k+1} \supset x_k{}^\frown j$ be an ω-splitting node such that $l_{k+1} = |x_{k+1}| > n_{k+1}$. Then let $h > x_{k+1}(l_{k+1} - 1)$ be such that $y_k{}^\frown h \in T$ and let $y_{k+1} \supset y_k{}^\frown h$ be an ω-splitting node such that $m_{k+1} = |y_{k+1}| \geq l_{k+1}$. If $k+1 \notin A$, we reverse the construction of x_{k+1} and y_{k+1}. Since x_{k+1}, y_{k+1}, and z_{k+1} are all increasing it follows that n_{k+1} is the least $n > n_k$ that belongs to $O(x,y,z)$, and the construction guarantees that $x(n_{k+1}) \leq y(n_{k+1})$ if and only if $k+1 \in A$. □

Now we complete the proof of the theorem. If $\boldsymbol{R} \cap L$ is countable then the theorem is true trivially, so assume that $\aleph_1^L = \aleph_1$. If X is a countable subset of $\boldsymbol{R} \cap L$, then given a constructible enumeration $\langle a_\alpha : \alpha < \omega_1\rangle$ of $\boldsymbol{R} \cap L$, we have $X \subset \{a_\alpha : \alpha < \gamma\}$ for some $\gamma < \omega_1$, and so there exists a constructible $Y \subset \boldsymbol{R} \cap L$ such that $X \subset Y$ and $|Y|^L = \aleph_0$.

Let P be a perfect subset of the Cantor space and assume that $P \subset \{0,1\}^\omega \cap L$. Applying the preceding argument to a countable dense subset $X \subset P$, we obtain a constructible countable set $D \in L$ that is a dense subset of $\boldsymbol{C} = \{0,1\}^\omega$ and that $D \cap P$ is dense in P. Let $\boldsymbol{C} - D = \mathcal{X}$.

The space $\mathcal{X}$ is homeomorphic to the irrationals which in turn is homeomorphic to $\mathcal{N}$ and $\mathcal{N}$ is homeomorphic to $[\omega]^\omega$. Thus there exists a homeomorphism h between $\mathcal{X}$ and $[\omega]^\omega$; moreover h is coded in L because $D \in L$. The set $P - D$, a closed subset of $\mathcal{X}$, contains no compact subset with nonempty interior, and therefore the set $h(P - D)$ has the same property in $[\omega]^\omega$; it follows that $h(P - D)$ is superperfect. Hence $h(P - D)$ is a superperfect subset of $[\omega]^\omega \cap L$, contradicting Lemma 32.12. □

Projective Sets and Large Cardinals

One of the successes of modern set theory has been the discovery of the close relationship between the hierarchy of definable sets of reals and the hierarchy of large cardinals. We shall elaborate on this relationship in subsequent chapters. In the present section we apply the large cardinal theory to $\mathbf{\Sigma}^1_3$ sets.

By Theorem 25.38, the perfect set property for $\mathbf{\Sigma}^1_2$ sets is equivalent to the large cardinal assumption

(32.8) $\qquad \aleph_1$ is inaccessible in $L[a]$, for every $a \in \boldsymbol{R}$

(see Exercise 32.4). The statement (32.8) also implies that every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable and has the Baire property.

By Solovay's Theorem 26.14, inaccessibility is sufficient for the consistency of Lebesgue measurability and the Baire property of all projective sets. The following theorem shows that the assumption is necessary for Lebesgue measurability, while by another result of Shelah, the Baire property for all projective sets is consistent relative to ZFC only:

Theorem 32.13 (Shelah [1984]). *If every $\mathbf{\Sigma}^1_3$ set of reals is Lebesgue measurable then $\aleph_1$ is an inaccessible cardinal in L.* □

We shall outline a result that shows that under a suitable strengthening of (32.8), every $\mathbf{\Sigma}^1_3$ set is Lebesgue measurable, has the Baire property, and has the perfect set property. The key is a tree representation of $\mathbf{\Sigma}^1_3$ sets in the presence of a measurable cardinal.

Theorem 32.14 (Martin and Solovay [1969], Mansfield [1971]). *If there exists a measurable cardinal then for every $\mathbf{\Sigma}^1_3$ set A there exists a tree T on $\omega \times \lambda$ (for some λ) such that $A = p[T]$.*

Proof. Let κ be a measurable cardinal and let U be a normal measure on κ. For each n, let U_n be the ultrafilter $\{X \subset \kappa^n : X \supset [Z]^n \text{ for some } Z \in U\}$, and let $j_n = i_{n,n+1}$ be the canonical elementary embedding $i_{n,n+1} : \mathrm{Ult}_{U_n}(V) \to \mathrm{Ult}_{U_{n+1}}(V)$.

Let $A \subset \mathcal{N}$ be a Σ^1_3 set. A can be expressed as

$$x \in A \leftrightarrow \exists y \forall z\, R(x, y, z) \text{ is ill-founded} \tag{32.9}$$

where R is a recursive function, $R(x, y, z)$ is, for each x, y, z, a linear order of ω and $R(x, y, z)$ restricted to $n = \{0, \ldots, n-1\}$ depends only on $x\restriction n$, $y\restriction n$, $z\restriction n$. Let $\pi : Seq_3 \to \omega$ be defined so that $\pi(x\restriction n, y\restriction n, z\restriction n)$ is the position of $n-1$ in the order $R(x, y, z)\restriction n$. Let $\{s_k\}_{k=0}^{\infty}$ be an enumeration of Seq. We let $\alpha = i_{0,\omega}(\kappa)$, and define

$$(x\restriction n, y\restriction n, \langle \beta_0, \ldots, \beta_{n-1}\rangle) \in T \leftrightarrow j_{\pi(x\restriction l, y\restriction l, s_k)}(\beta_k) > \beta_i \tag{32.10}$$

for every $i = 0, \ldots, n-2$, where $l = \mathrm{length}(s_i)$ and $s_k = s_i\restriction l$.

We leave to the reader to verify that $x \in A$ if and only if $T(x)$, a tree on $\omega \times \alpha$, is ill-founded. For details we refer to Kanamori's book [1994], Chapter 15. □

A careful analysis of the tree representation in Theorem 32.14 shows that the assumption can be weakened to

$$\text{for every } a \in \boldsymbol{R},\ a^\sharp \text{ exists} \tag{32.11}$$

and the tree T can be constructed on $\omega \times \omega_2$ (see Kanamori [1994] for details). Thus one obtains:

Theorem 32.15 (Martin). *If for every $a \in \boldsymbol{R}$, $a^\sharp$ exists, then every $\boldsymbol{\Sigma}^1_3$ set is ω_2-Suslin, and hence a union of $\aleph_2$ Borel sets.* □

The following theorem establishes good behavior of $\boldsymbol{\Sigma}^1_3$ sets under a large cardinal assumption:

Theorem 32.16 (Magidor [1980]). *Let us assume that there exists a measurable cardinal, and that ω_1 carries a precipitous ideal. Then every $\boldsymbol{\Sigma}^1_3$ set is Lebesgue measurable, has the Baire property, and is either countable or contains a perfect subset.*

Proof. Let A be a $\boldsymbol{\Sigma}^1_3$ set and let $A = p[T]$ where T is the tree defined in the proof of Theorem 32.14. We shall prove that under the given assumptions,

$$\boldsymbol{R} \cap L[T] \text{ is countable.} \tag{32.12}$$

Then the statements on Lebesgue measurability and the Baire property can be derived as the corresponding result (Theorem 26.20 and Corollary 26.21) for $\boldsymbol{\Sigma}^1_2$ sets: Using absoluteness, one can show that

$$A = \{x \in \boldsymbol{R} : L[T] \vDash \varphi(x)\}$$

for some formula φ, and apply Corollary 26.6. The perfect set property is derived by using Lemma 25.24.

Let I be a precipitous ideal on ω_1, and let $M = \mathrm{Ult}_G(V)$ be the generic ultrapower by I, that is by a generic ultrafilter G obtained by forcing P consisting of I-positive sets. As I is precipitous, M is well-founded and we identify it with a transitive class $M \subset V[G]$. Let $i : V \to M$ be the corresponding elementary embedding. We shall prove

$$i(T) = T. \tag{32.13}$$

This will suffice, as (32.13) implies (32.12), as follows: Assume that $a_\xi, \xi < \omega_1$, are uncountably many (distinct) reals in $L[T]$. The function $\langle a_\xi : \xi < \omega_1 \rangle$ represents a real $a \in \mathrm{Ult}_G$, and since each $a_\xi \in L[T]$, we have $a \in L[i(T)] = L[T]$; hence $a \in V$. But then $i(a) = a$, and so $a = a_\xi$ for G-almost all ξ. This is a contradiction since G is nonprincipal.

Toward the proof of (32.13), let κ, U, U_n, and j_n be as in the proof of Theorem 32.14.

If γ is an inaccessible cardinal then γ is still inaccessible in $V[G]$ and it follows that $i(\gamma) = \gamma$. In particular $i(\kappa) = \kappa$. Let $\overline{U}$ be the filter in $V[G]$ generated by U; similarly $\overline{U_n}$.

Lemma 32.17. *$i(U) = \overline{U} \cap M$, $i(U_n) = \overline{U_n} \cap M$.*

Proof. It suffices to show that $i(U) \subset \overline{U} \cap M$; if $X \in i(U)$ we want a $W \in U$ such that $X \supset W$. X is represented by $\langle X_\xi : \xi < \omega_1 \rangle$, so let $Y = \bigcap_{\xi < \omega_1} X_\xi$; we have $Y \in U$ and $i(Y) \subset X$. Now if $W = \{\gamma \in Y : \gamma \text{ is inaccessible}\}$ we have $W \in U$ and $W = i\text{“}Y \subset i(Y) \subset X$. □

Lemma 32.18. *Let $h \in V[G]$ be a function $h : \kappa \to V$. Then there exists a function $H \in V$ such that $h(\alpha) = H(\alpha)$ a.e.* mod U. *Similarly for $h : \kappa^n \to V$ (and U_n).*

Proof. For each $\alpha < \kappa$ there is a maximal antichain W_α in P and a set $\{x^\alpha_p : p \in W_\alpha\}$ such that $p \Vdash \dot{h}(\alpha) = x^\alpha_p$. Let W be such that $W_\alpha = W$ for U-almost all α, and let p be the unique $p \in G \cap W$. Now let $H(\alpha) = x^\alpha_p$, for all $\alpha < \kappa$. □

Lemma 32.19. *Let $f \in V$ be a function $f : \kappa \to Ord$. Then there exists a function $g \in M$ such that $f(\alpha) = g(\alpha)$ a.e.* mod U. *Similarly for $f : \kappa^n \to Ord$.*

Proof. Every ordinal β is represented in M by some $h_\beta : \omega_1 \to Ord$, $h_\beta \in V$. For each $\alpha < \kappa$, pick (in $V[G]$) some $h_{f(\alpha)} : \omega_1 \to Ord$ that represents $f(\alpha)$ and let $h(\alpha) = h_{f(\alpha)}$. By Lemma 32.18 there is some $H \in V$ such that $H(\alpha) = h_{f(\alpha)}$ a.e.; let $A \in U$ be a set of inaccessibles such that $H(\alpha) = h_{f(\alpha)}$ for all $\alpha \in A$.

For each $\xi < \omega_1$, let g_ξ (a function on κ, in V) be defined by $g_\xi(\alpha) = (H(\alpha))(\xi)$, and let $G(\xi) = g_\xi$. G is in V and represents in M some function g.

For each $\alpha \in A$, $i(\alpha) = \alpha$, and $g(\alpha)$ is represented by the function that sends ξ to $(H(\alpha))(\xi)$, but since $(H(\alpha))(\xi) = h_{f(\alpha)}(\xi)$ for all ξ, $g(\alpha)$ is represented by $h_{f(\alpha)}$. It follows that $g(\alpha) = f(\alpha)$. □

As a consequence of Lemmas 32.17, 32.18, and 32.19, if a function $f \in V$ represents an ordinal in $\mathrm{Ult}_{U_n}(V)$ then there is an U_n-equivalent function $g \in M$ that represents the same ordinal in $\mathrm{Ult}_{i(U_n)}(M)$. Consequently,

$$(i(j_n))(\alpha) = j_n(\alpha) \text{ for all } \alpha \tag{32.14}$$

(where $i(j_n) = \bigcup_{\gamma \in Ord} i(j \restriction V_\gamma)$). Using (32.14) and the definition of T, one can now verify that $i(T) = T$. □

The existence of a measurable cardinal alone is not sufficient in Theorem 32.16. If $V = L[U]$ then there exists an uncountable Σ^1_3 set that is not Lebesgue measurable, does not have the Baire property, and does not contain a perfect subset:

Theorem 32.20 (Silver [1971a]). *$\boldsymbol{R} \cap L[U]$ is a Σ^1_3 set. The ordering $<_{L[U]}$ of $\boldsymbol{R}$ is a Σ^1_3 relation.* □

Also, the analog of Lemma 25.27 holds for $<_{L[U]}$, and the arguments for Σ^1_2 sets and L can be adopted for Σ^1_3 and $L[U]$.

Universally Baire sets

Definition 32.21. A set $A \subset \boldsymbol{R}$ is *universally Baire* if for any compact Hausdorff space X and any continuous function $f : X \to \boldsymbol{R}$, the set $f_{-1}(A)$ has the property of Baire in X.

The set of all universally Baire sets is a σ-algebra and is closed under operation $\mathcal{A}$. Thus every $\mathbf{\Sigma}^1_1$ set is universally Baire. We show below that every universally Baire set is Lebesgue measurable and that the statement that every $\mathbf{\Delta}^1_2$ set is universally Baire has consistency strength between inaccessible and Mahlo cardinals. The assumption that every projective (or even every $\mathbf{\Sigma}^1_2$ set) is universally Baire is considerably stronger; we refer to Feng, Magidor and Woodin [1992] for details.

Theorem 32.22. *A set $A \subset \boldsymbol{R}$ is universally Baire if and only if for every notion of forcing P there exist trees T and S on $\omega \times \lambda$ (where $\lambda = 2^{|P|}$) such that*

$$A = p[T], \qquad \boldsymbol{R} - A = p[S] \tag{32.15}$$

and for every generic filter G on P,

$$V[G] \vDash p[T] \cup p[S] = \boldsymbol{R} \text{ and } p[T] \cap p[S] = \emptyset. \tag{32.16}$$

□

We omit the proof of this equivalence. We remark that in the definition the space X can be replaced by the generalized Cantor space λ^ω (for all λ), and in the theorem, the forcing notion P can be replaced by $\mathrm{Col}(\lambda) = \mathrm{Col}(\omega, \lambda)$, the collapse of λ with finite conditions.

Corollary 32.23. *Every universally Baire set is Lebesgue measurable.*

Proof. Let $A \subset \boldsymbol{R}$ be universally Baire. Let B be the measure algebra, and let T and S be trees on $\omega \times \lambda$ such that $A = p[T]$, $\boldsymbol{R} - A = p[S]$, and (32.16) holds for every generic ultrafilter G on B.

Let $\dot{a}$ be the canonical name for a random real and let B be a Borel set such that $\|\dot{a} \in p[T]\| = [B]$. We will show that $A \triangle B$ has measure 0, and thus A is measurable.

Let M be a countable elementary submodel of H_κ where H_κ is sufficiently large. We claim that for every random real x over M,

$$x \in p[T] \leftrightarrow x \in B. \tag{32.17}$$

If $x \in B$ then $B \Vdash \dot{a} \in p[T]$ and hence $M \vDash (B \Vdash \dot{a} \in p[T])$. Thus $M[x] \vDash x \in p[T]$, and so $x \in p[T]$. If $x \notin B$ then $-B \Vdash \dot{a} \in p[S]$ and $M[x] \vDash x \in p[S]$, and hence $x \notin p[T]$, proving (32.17).

Since M is countable, almost all reals are random over M, and therefore $A \triangle B$ is null. □

Theorem 32.24. *The following are equivalent*:

(i) *Every* $\boldsymbol{\Delta}^1_2$ *set is universally Baire.*
(ii) V *is* $\boldsymbol{\Sigma}^1_3$*-absolute with respect to every generic extension.*

The statement (ii) states precisely: If P is a forcing notion and $\varphi(x_1, \ldots, x_n)$ a Π^1_3 formula then for all reals $a_1, \ldots, a_n$,

$$\varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad \Vdash_P \varphi(a_1, \ldots, a_n).$$

Its consistency strength is between inaccessible and Mahlo: It is the existence of an inaccessible cardinal κ such that $V_\kappa \prec_{\Sigma_2} V$; see Exercises 32.6, 32.7, 32.8.

Proof. First assume $\boldsymbol{\Sigma}^1_3$-absoluteness for generic extensions and let A be a $\boldsymbol{\Delta}^1_2$ set. We have

$$x \in A \leftrightarrow \exists y\, \varphi(x, y) \leftrightarrow x \in p[T] \tag{32.18}$$

and

$$x \in A \leftrightarrow \exists y\, \psi(x, y) \leftrightarrow x \in p[S] \tag{32.19}$$

where φ and ψ are $\boldsymbol{\Pi}^1_1$ and T and S are trees on $\omega \times \omega_1$.

If $V[G]$ is a generic extension then the second equivalences in (32.18) and (32.19) hold in $V[G]$, by $\boldsymbol{\Sigma}^1_2$-absoluteness. Since $p[T] \cup p[S] = \boldsymbol{R}$ is a $\boldsymbol{\Pi}^1_3$ statement (namely $\forall x\, (\exists y\, \varphi \vee \exists y\, \psi)$), and $p[T] \cap p[S] = \emptyset$ is $\boldsymbol{\Pi}^1_2$, $\boldsymbol{\Sigma}^1_3$-absoluteness gives (32.16).

Now assume that every $\mathbf{\Delta}^1_2$ set is universally Baire and prove the generic $\mathbf{\Sigma}^1_3$-absoluteness. It is enough to prove it for $P = \mathrm{Col}(\lambda)$, as every V^P embeds in $V^{\mathrm{Col}(\lambda)}$ for sufficiently large λ.

Let φ be a $\mathbf{\Sigma}^1_1$ formula (with a parameter in $\boldsymbol{R}^V$) and assume that $V[G] \vDash \exists x\, \forall y\, \varphi(x, y)$. Let $\dot{x} \in V^{\mathrm{Col}(\lambda)}$ be such that $\Vdash \forall y\, \varphi(\dot{x}, y)$. There is a function $f : \lambda^\omega \to \omega^\omega$ such that f is continuous on a comeager G_δ set such that for every generic collapse $G \in \lambda^\omega$, $f(G) = \dot{x}^G$.

Toward a contradiction, assume that $V \vDash \forall x\, \exists y\, \neg\varphi(x, y)$. By Kondô's Uniformization Theorem, there exists a $\mathbf{\Pi}^1_1$ function g such that $\forall x\, \neg\varphi(x, g(x))$. We claim that the function $g \circ f$ is continuous on a comeager set in λ^ω. For each $s \in Seq$, $g_{-1}(O(s))$ is a $\mathbf{\Delta}^1_2$ set, therefore universally Baire, and so there exists an open set D_s such that $B_s = D_s \mathbin{\triangle} f_{-1}(g_{-1}(O(s)))$ is meager. Let $A = \lambda^\omega - \bigcup\{B_s : s \in Seq\}$; A is comeager and $g \circ f$ is continuous on A.

We may assume that $A = \bigcap_{n=0}^\infty D_n$, with each D_n dense open. For $x \in A$, let $F(x) = (f(x), g(f(x)))$; F is continuous on A.

Let T be a tree on $\omega \times \omega \times \omega$ such that $p[T]$ is the $\mathbf{\Sigma}^1_1$ set $\{(x, y) : \varphi(x, y)\}$. Since $V[G] \vDash \varphi(f(G), g(f(G)))$ and $G \in \bigcap_{n=0}^\infty D_n$, we have that $T(F(G))$ is ill-founded. In other words, $V[G]$ satisfies

$$\exists x\, T(F(x)) \text{ is ill-founded.} \tag{32.20}$$

The statement (32.20) can be expressed as "T^* is ill-founded" where T^* is the tree

$$\begin{aligned}(\sigma, s, t, u) \in T^* \leftrightarrow (s, t, u) \in T,\ &\mathrm{length}(\sigma) = \mathrm{length}(s) = n \\ &\text{and } \exists \tau\, O(\sigma^\frown\tau) \subset \textstyle\bigcap_{i \le n} D_i \text{ and} \\ &\qquad F\text{“}(O(\sigma^\frown\tau) - \textstyle\bigcup_{n=0}^\infty (\lambda^\omega - D_n)) \subset O(s, t)\end{aligned}$$

(where $O(\sigma^\frown\tau)$ and $O(s, t)$ are basic open sets in λ^ω and $\omega^\omega \times \omega^\omega$).

By absoluteness, T^* is ill-founded in V, and so (32.20) holds in V. In other words, for some $x \in A$ we have $\varphi(f(x), g(f(x)))$, a contradiction. □

Exercises

32.1. Let $A \subset \mathcal{N}$ be $\mathbf{\Sigma}^1_1$. The equivalence relation on $\mathcal{N}$ whose equivalence classes are the singletons $\{a\}$ where $a \in A$, and the complement of A, is $\mathbf{\Pi}^1_1$. If A is uncountable then it has a perfect subset.

32.2. The relation "$\|a\| = \|b\|$ or $a, b \notin \mathrm{WO}$" is $\mathbf{\Sigma}^1_1$ and has $\aleph_1$ equivalence classes.

32.3. Let $\{D_n\}_{n=1}^\infty$ be dense open sets in the Σ^1_1-topology and let B be a nonempty Σ^1_1 set. Then $B \cap \bigcap_{n=1}^\infty D_n \neq \emptyset$.

[Let $B = p[T_0]$ for some recursive tree T on $\omega \times \omega$. By induction on n, construct recursive trees T_n, a finite sequence s_n of length n, finite sequences t^i_n $(0 \le i \le n)$ of length n such that $\emptyset = s_0 \subset \ldots \subset s_n$ and $t^i_0 \subset \ldots \subset t^i_n$, and $\bigcap_{i=0}^n \{x : x \supset s_n \text{ and } (\exists y \supset t^i_n)\,(x, y) \in [T_i]\} \neq \emptyset$, and for all $1 \le i \le n$, $\{x : x \supset s_n \text{ and } (\exists y \supset t^i_n)\,(x, y) \in [T_i]\} \subset D_i$.]

32.4. If $a \in \boldsymbol{R}$ and $\aleph_1$ is a successor cardinal in $L[a]$, then for some $b \in \boldsymbol{R}$, $\aleph_1^{L[a,b]} = \aleph_1$.
[If $\aleph_1 = (\kappa^+)^{L[a]}$, let $b \subset \omega$ code the countable ordinal κ.]

32.5. Every universally Baire set is Ramsey.
[Use Mathias forcing.]

32.6. $\boldsymbol{\Sigma}^1_3$-absoluteness for generic extensions implies that $\aleph_1$ is inaccessible in each $L[a]$, $a \in \boldsymbol{R}$.
["$\omega_1^{L[a]}$ is countable" is $\Sigma^1_3(a)$.]

32.7. Generic $\boldsymbol{\Sigma}^1_3$-absoluteness implies that $L_\kappa \prec_{\Sigma_2} L$ where $\kappa = \aleph_1$.

32.8. If $V_\kappa \prec_{\Sigma_2} V$ and κ is inaccessible, let P be the Lévy collapse bellow κ. Show that V^P satisfies the generic $\boldsymbol{\Sigma}^1_3$-absoluteness.
[If $\dot{Q} \in V^P$, and φ is Σ^1_3, then $V^P \vDash \varphi$ if and only if $V^{P*\dot{Q}} \vDash \varphi$.]

Historical Notes

Theorem 32.1 on $\boldsymbol{\Pi}^1_1$ equivalence relations is due to Silver [1980]. The present proof is Harrington's as presented in Kechris and Martin [1980]. Lemma 32.5 is due to Louveau. Silver's Theorem was extended to Theorem 32.9 by J. Burgess in [1978].

Theorem 32.11 is due to Groszek and Slaman [1998]. The proof presented here is from Veličković and Woodin [1998].

Theorem 32.13: Shelah [1984]. The tree representation of $\boldsymbol{\Sigma}^1_3$ sets is implicit in Martin and Solovay [1969] and described in Mansfield [1971]. Theorem 32.16 is due to Magidor [1980].

Universally Baire sets are investigated in Feng, Magidor, and Woodin [1992].

33. Determinacy

With each subset A of ω^ω we associate the following game G_A, played by two players I and II. First I chooses a natural number a_0, then II chooses a natural number b_0, then I chooses a_1, then II chooses b_1, and so on. The game ends after ω steps; if the resulting sequence $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ is in A, then I wins, otherwise II wins.

A *strategy* (for I or II) is a rule that tells the player what move to make depending on the previous moves of both players. A strategy is a *winning strategy* if the player who follows it always wins. The game G_A is *determined* if one of the players has a winning strategy.

The Axiom of Determinacy (AD) states that for every $A \subset \omega^\omega$, the game G_A is determined.

Determinacy and Choice

First some definitions: Let $A \subset \omega^\omega$ be given and let G_A denote the corresponding game. A *play* is a sequence $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in \omega^\omega$; for each n, a_n is the nth *move of player* I and b_n is the nth *move of player* II. A *strategy* for I is a function σ with values in ω whose domain consists of finite sequences $s \in Seq$ of even length. Player I plays $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ *by the strategy* σ if $a_0 = \sigma(\emptyset)$, $a_1 = \sigma(\langle a_0, b_0\rangle)$, $a_2 = \sigma(\langle a_0, b_0, a_1, b_1\rangle)$, and so on; it is clear that if I plays by σ, then the play is determined by σ and the sequence $b = \langle b_n : n \in \omega\rangle$. We denote the play by $\sigma * b$. A strategy σ is a *winning strategy* for I if

$$\{\sigma * b : b \in \mathcal{N}\} \subset A,$$

in other words, if all plays that I plays by σ are in A. Similarly, a strategy for II is a function τ with values in ω, defined on finite sequences $s \in Seq$ of odd length. If $a \in \mathcal{N}$ and if τ is a strategy for II, then $a * \tau$ denotes the play in which I plays a and II plays by τ. A strategy τ for II is a winning strategy if

$$\{a * \tau : a \in \mathcal{N}\} \subset \mathcal{N} - A.$$

We sometimes consider games G_A whose moves are not natural numbers but elements of an arbitrary set S. A play is then a sequence $p \in S^\omega$, and the

result of the game depends on whether $p \in A$ or $p \notin A$ (here A is a subset of S^ω). The other relevant notions are defined accordingly.

Since the number of strategies is $2^{\aleph_0}$, an easy diagonal argument shows that the Axiom of Choice is incompatible with the Axiom of Determinacy:

Lemma 33.1. *Assuming the Axiom of Choice, there exists $A \subset \omega^\omega$ such that the game G_A is not determined.*

Proof. Let $\{\sigma_\alpha : \alpha < 2^{\aleph_0}\}$ and $\{\tau_\alpha : \alpha < 2^{\aleph_0}\}$ enumerate all strategies for I and all strategies for II. We construct sets $X = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ and $Y = \{y_\alpha : \alpha < 2^{\aleph_0}\}$, subsets of $\mathcal{N}$, as follows: Given $\{x_\xi : \xi < \alpha\}$ and $\{y_\xi : \xi < \alpha\}$, let us choose some y_α such that $y_\alpha = \sigma_\alpha * b$ for some b and $y_\alpha \notin \{x_\xi : \xi < \alpha\}$ (such y_α exist because the set $\{\sigma_\alpha * b : b \in \mathcal{N}\}$ has size $2^{\aleph_0}$); similarly, let us choose x_α such that $x_\alpha = a * \tau_\alpha$ for some a and $x_\alpha \notin \{y_\xi : \xi \leq \alpha\}$. It is clear that the sets X and Y are disjoint, that for each α there is b such that $\sigma_\alpha * b \notin X$, and there is a such that $a * \tau_\alpha \in X$. Thus neither I nor II has a winning strategy in the game G_X, and hence G_X is not determined. □

In contrast with this lemma, the Axiom of Determinacy implies a weak form of the Axiom of Choice:

Lemma 33.2. *The Axiom of Determinacy implies that every countable family of nonempty sets of real numbers has a choice function.*

Proof. We prove that if $\mathcal{X} = \{X_n : n \in \omega\}$ is a family of nonempty subsets of $\mathcal{N}$, then there exists f on $\mathcal{X}$ such that $f(X_n) \in X_n$ for all n. Let us consider the following game: If I plays $\langle a_0, a_1, a_2, \ldots\rangle$ and and II plays $\langle b_0, b_1, b_2, \ldots\rangle$, then II wins if and only if $b \in X_{a_0}$. It is clear that I does not have a winning strategy: Once I plays a_0, the player II finds some $b \in X_{a_0}$, plays $b = \langle b_0, b_1, b_2, \ldots\rangle$ and wins. Hence II has a winning strategy τ, and we can define f on $\mathcal{X}$ as follows: $f(X_n) = \tau * \langle n, 0, 0, 0, \ldots\rangle$. □

As we show below, Determinacy has desirable consequences for sets of reals: AD implies that every set of reals is Lebesgue measurable, has the Baire property and the perfect set property. Thus it is natural to postulate that Determinacy holds to the extent it does not contradict the Axiom of Choice. The appropriate postulate turns out to be that AD holds in the model $L(\boldsymbol{R})$, and therefore all sets of reals definable from a real parameter are determined. This implies, in particular, that the game G_A is determined for every projective set—*Projective Determinacy* (PD). It has been established that both $\mathrm{AD}^{L(\boldsymbol{R})}$ and PD are large cardinal axioms; we shall elaborate on this later in this chapter.

Throughout the rest of the present chapter we work in ZF + the Principle of Dependent Choices.

Some Consequences of AD

We shall now prove that under the assumption of Determinacy, sets of real numbers are well behaved.

Theorem 33.3. *Assume the Axiom of Determinacy. Then*:

(i) *Every set of reals is Lebesgue measurable.*
(ii) *Every set of reals has the property of Baire.*
(iii) *Every uncountable set of reals contains a perfect subset.*

Proof. (i) It suffices to prove the following lemma:

Lemma 33.4. *Assuming* AD, *let S be a set of reals such that every measurable $Z \subset S$ is null. Then S is null.*

It is easy to see that Lemma 33.4 implies that every set X is Lebesgue measurable: Let $A \supset X$ be a measurable set with the property that every measurable $Z \subset A - X$ is null. Then $A - X$ is null and hence X is measurable.

Proof. Thus let S be a set of reals with the property

(33.1) if $Z \subset S$ is Lebesgue measurable, then Z is null;

we shall use AD to show that S is null. It is clear that we can restrict ourselves to subsets of the unit interval; thus assume that $S \subset [0,1]$. In order to show that S is null, it suffices to show that the outer measure $\mu^*(S)$ is less than any $\varepsilon > 0$. Thus let ε be a fixed positive real number.

33.5. The Covering Game. Given S and ε, let us set up a game as follows: If $\langle a_0, a_1, a_2, \ldots \rangle$ is a sequence of 0's and 1's, let a be the real number

$$a = \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}}. \tag{33.2}$$

For each $n \in \omega$, let G_k^n, $k = 0, 1, 2, \ldots$, be an enumeration of the set K_n of all sets G such that

(33.3) (i) G is a union of finitely many intervals with rational endpoints;
(ii) $\mu(G) \leq \varepsilon / 2^{2(n+1)}$.

The rules of the game are that player I tries to play a real number $a \in S$, and player II tries to cover the real a by the union $\bigcup_{n=0}^{\infty} H_n$ such that $H_n \in K_n$ for all n. More precisely, a play $\langle a_0, b_0, a_1, b_1, \ldots \rangle$ is won by player I if

(i) $a_n = 0$ or 1, for all n;
(ii) $a \in S$; and
(iii) $a \notin \bigcup_{n=0}^{\infty} G_{b_n}^n$.

We claim that player I does not have a winning strategy in the game. To show this, notice that if σ is a winning strategy for I, then the function f that to each $b = \langle b_0, b_1, b_2, \ldots\rangle \in \mathcal{N}$ assigns the real number $a = f(b)$ such that $\langle a_0, b_0, a_1, b_1, \ldots\rangle = \sigma * b$ is continuous and hence the set $Z = f(\mathcal{N})$ is analytic and hence measurable. Moreover $Z \subset S$, and therefore Z is null. Now a null set can be covered by a countable union $\bigcup_{n=0}^{\infty} H_n$ such that $H_n \in K_n$ for all n, and therefore, if II plays $\langle b_0, b_1, b_2, \ldots\rangle$ where $G^n_{b_n} = H_n$ and I plays by σ, then II wins. Thus σ cannot be a winning strategy for I.

Assuming AD, the covering game is determined, and therefore player II has a winning strategy. Let τ be such a strategy. For each finite sequence $s = \langle a_0, \ldots, a_n\rangle$ of 0's and 1's, let $G_s \in K_n$ be the set $G^n_{b_n}$, where $\langle b_0, \ldots, b_n\rangle$ are the moves that II plays by τ in response to $a_0, \ldots, a_n$. Since τ is a winning strategy, every $a \in S$ is in the set $\bigcup\{G_s : s \subset a\}$ and hence

$$S \subset \bigcup\{G_s : s \in \mathit{Seq}(\{0,1\})\} = \bigcup_{n=1}^{\infty} \bigcup_{s\in\{0,1\}^n} G_s. \tag{33.4}$$

Now for every $n \geq 1$, if $s \in \{0,1\}^n$, then $\mu(G_s) \leq \varepsilon/2^{2n}$ and hence

$$\mu\big(\textstyle\bigcup_{s\in\{0,1\}^n}\big) \leq \frac{\varepsilon}{2^{2n}} \cdot 2^n = \frac{\varepsilon}{2^n}.$$

It follows that $\mu(\bigcup_{n=1}^{\infty} \bigcup_{s\in\{0,1\}^n} G_s) \leq \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$ and thus $\mu^*(S) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, S is null. This completes the proof. □

(ii) Next we consider the property of Baire:

33.6. The Banach-Mazur Game. Let X be a subset of the Baire space $\mathcal{N}$, and let us consider the following game: Player I plays a finite sequence $s_0 \in \mathit{Seq}$; then II plays a proper extension $t_0 \supset s_0$; then I plays $s_1 \supset t_0$, etc.:

$$s_0 \subset t_0 \subset s_1 \subset t_1 \subset \ldots. \tag{33.5}$$

The sequence (33.5) converges to some $x \in \mathcal{N}$. If $x \in A$, then I wins, and otherwise II wins.

First we verify that this game can be reformulated as a game G_A of the kind introduced at the beginning of the section (i.e., when the moves are natural numbers). Let u_k, $k \in \boldsymbol{N}$, be an enumeration of the set Seq. If $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ is a sequence of numbers, then consider the sequence

$$u_{a_0},\, u_{b_0},\, u_{a_1},\, u_{b_1},\, \ldots \tag{33.6}$$

and let A be the set of all $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in \mathcal{N}$ such that: Either there is n such that

$$u_{a_0} \subset u_{b_0} \subset \ldots u_{a_n} \not\subset u_{b_n}$$

or the sequence (33.6) is increasing and converges to some $x \in X$. It is clear I wins the Banach-Mazur game if and only if I wins the game G_A.

Thus if AD holds, the game is determined, for every $X \subset \mathcal{N}$. We will use this to show that every $X \subset \mathcal{N}$ has the Baire property.

Lemma 33.7. *Player II has a winning strategy in the Banach-Mazur game if and only if X is meager.*

Proof. Let Y be the complement of X in $\mathcal{N}$. For each $s \in Seq$, $O(s)$ denotes the basic open set $\{x \in \mathcal{N} : s \subset x\}$.

(a) If X is a meager set, then there exist open dense sets G_n, $n \in \boldsymbol{N}$, such that $Y \supset \bigcap_{n=0}^{\infty} G_n$. It is easy to find a winning strategy τ for II: If I plays s_0, let $t_0 = \tau(\langle s_0 \rangle)$ be some $t_0 \supset s_0$ such that $U_{t_0} \subset G_0$; such t_0 exists because G_0 is dense. Then if I plays $s_1 \supset t_0$, let $t_1 = \tau(\langle s_0, t_0, s_1 \rangle)$ be some $t_1 \supset s_1$ such that $U_{t_1} \subset G_1$, and so on. It is clear that every such play $s_0 \subset t_0 \subset s_1 \subset \ldots$ converges to $x \in \bigcap_{n=0}^{\infty} G_n$, and hence τ is a winning strategy for II.

(b) Conversely, assume that II has a winning strategy τ. A *correct position* is a finite sequence $\langle s_0, t_0, \ldots, s_n, t_n \rangle$ such that $s_0 \subset t_0 \subset \ldots \subset t_n$ and $t_0 = \tau(\langle s_0 \rangle)$, $t_1 = \tau(\langle s_0, t_0, s_1 \rangle)$, etc. We shall first prove the following claim: Let $x \in \mathcal{N}$ and assume that for every correct position $p = \langle s_0, \ldots, t_n \rangle$ with $t_n \subset x$ there exists $s \supset t_n$ such that $\tau(p^\frown s) \subset x$. Then $x \in Y$.

To prove the claim, let x satisfy the condition. To begin, there exists s_0 such that $\tau(\langle s_0 \rangle) \subset x$; let $t_0 = \tau(\langle s_0 \rangle)$. Then there exists $s_1 \supset t_0$ such that $t_1 = \tau(\langle s_0, t_0, s_1 \rangle) \subset x$; then there is $s_2 \supset t_1$ such that $\tau(\langle s_0, t_0, s_1, t_1, s_2 \rangle) \subset x$; and so on. The sequence $s_0 \subset t_0 \subset s_1 \subset t_1 \subset \ldots$ converges to x and is a play in which II plays by τ. Hence $x \in Y$.

For every correct position $p = \langle s_0, \ldots, t_n \rangle$, let

$$F_p = \{x \in \mathcal{N} : t_n \subset x \text{ and } (\forall s \supset t_n)\, \tau(p^\frown s) \not\subset x\}.$$

By the claim, for every $x \in X$ there is a correct position p such that $x \in F_p$; in other words,

$$X \subset \bigcup \{F_p : p \text{ is a correct position}\}.$$

It is easy to see that for each p, $O(t_n) - F_p$ is on open dense set in $O(t_n)$; hence F_p is a closed nowhere dense set. The number of correct positions is countable and hence X is meager. □

Corollary 33.8. *Let $X \subset \mathcal{N}$. Player I has a winning strategy in the Banach-Mazur game if and only if for some $s \in Seq$, $O(s) - X$ is meager.*

Proof. Note that I has a winning strategy if and only if there exists $s \in Seq$ (the first move of I) such that player II has a winning strategy in the following game: I plays $t_0 \supset s$, II plays $s_0 \supset t_0$, I plays $t_1 \supset s_0$, etc.; and I wins if $t_0 \subset s_0 \subset t_1 \subset \ldots$ converges to $x \in U_s - X$. By Lemma 33.7, II has a winning strategy in this game if and only if $O(s) - X$ is meager. □

Now part (ii) of Theorem 33.3 follows. If $X \subset \mathcal{N}$, then since the Banach-Mazur game is determined, either X is meager or for some $s \in Seq$, $O(s) - X$ is meager. Thus let $X \subset \mathcal{N}$ be arbitrary. If X is meager, then X has the Baire property. If X is nonmeager, then let $G = \bigcup \{O(s) : O(s) - X$ is

meager}. Clearly, $G - X$ is meager, and $X - G$ must be meager too because otherwise there would exist some s such that $O(s) - (X - G)$ is meager, which contradicts the definition of G. It follows that X has the Baire property.

(iii) We will use AD to prove that every uncountable set in the Cantor space $\boldsymbol{C} = \{0,1\}^\omega$ has a perfect subset. We consider the following game:

33.9. The Perfect Set Game. Let X be a subset of $\{0,1\}^\omega$. The game is defined as follows: Player I plays a sequence $s_0 \in \mathit{Seq}(\{0,1\})$ of 0's and 1's (possibly the empty sequence), then player II plays $n_0 \in \{0,1\}$, then I plays $s_1 \in \mathit{Seq}(\{0,1\})$, and so on. Let $x = s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \ldots$. Player I wins if $x \in X$, and II wins if $x \notin X$.

The game can be reformulated as a game G_A, for some $A \subset \omega^\omega$.

Lemma 33.10. *Let $X \subset \boldsymbol{C}$. If II has a winning strategy in the perfect set game, then X is countable.*

Proof. Let τ be a winning strategy for II. A *correct position* is a finite sequence $\langle s_0, n_0, \ldots, s_k, n_k\rangle$ such that $n_0 = \tau(\langle s_0\rangle)$, $n_1 = \tau(\langle s_0, n_0, s_1\rangle)$, etc. By the same argument as in Lemma 33.7, we get the following claim: Let $x \in \{0,1\}^\omega$ and assume that for every correct position $p = \langle s_0, \ldots, n_k\rangle$ if $s_0{}^\frown n_0{}^\frown \ldots {}^\frown n_k \subset x$, then there exists an $s \in \mathit{Seq}(\{0,1\})$ such that $s_0{}^\frown n_0{}^\frown \ldots {}^\frown n_k{}^\frown s{}^\frown \tau(p{}^\frown s) \subset x$. Then $x \notin X$.

It follows that $X \subset \bigcup\{F_p : p \text{ is a correct position}\}$, where

$$F_p = \{x \in \boldsymbol{C} : s_0{}^\frown \ldots {}^\frown n_k \subset x \text{ and } \forall s\, (s{}^\frown \ldots {}^\frown n_k{}^\frown s{}^\frown \tau(p{}^\frown s) \not\subset x)\}.$$

The lemma will follow if we show that each F_p has exactly one element $x \in \boldsymbol{C}$. This element x is uniquely determined as follows (because each $x(m)$ is either 0 or 1); first, for some $l \in \boldsymbol{N}$, $\langle x(0), \ldots, x(l-1)\rangle = s_0{}^\frown n_0{}^\frown \ldots {}^\frown n_k$; then $x(l) = 1 - \tau(p{}^\frown \emptyset)$, $x(l+1) = 1 - \tau(p{}^\frown \langle x(l)\rangle)$, $x(l+2) = 1 - \tau(p{}^\frown \langle x(l), x(l+1)\rangle)$, and so on. □

Now part (iii) of Theorem 33.3 follows. If $X \subset \boldsymbol{C}$ is uncountable, then II does not have a winning strategy; and since the game is determined, I has a winning strategy σ. For each $x = \langle n_0, n_1, \ldots\rangle \in \boldsymbol{C}$, let $F(x) \in \boldsymbol{C}$ denote the 0–1 sequence

$$s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \ldots$$

where $s_0 = \sigma(\emptyset)$, $s_1 = \sigma(\langle s_0{}^\frown n_0\rangle)$, $s_2 = \sigma(\langle s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1\rangle)$, etc. The function f is continuous and one-to-one, and hence $f(\boldsymbol{C})$ is a perfect set. But $X \supset f(\boldsymbol{C})$ and hence X has a perfect subset. □

We proved earlier that if $\aleph_1 = \aleph_1^{L[a]}$ for some $a \subset \omega$, then there is an uncountable set without a perfect subset. Thus we have:

Corollary 33.11. *If* AD *holds, then $\aleph_1$ is inaccessible in $L[a]$, for every $a \subset \omega$.* □

AD and Large Cardinals

To illustrate the relationship between the Axiom of Determinacy and the theory of large cardinals, we show that AD implies that $\aleph_1$ and $\aleph_2$ are measurable cardinals.

Theorem 33.12 (Solovay). *The Axiom of Determinacy implies that*:

(i) $\aleph_1$ *is a measurable cardinal, and moreover, the closed unbounded filter on* $\aleph_1$ *is an ultrafilter.*
(ii) $\aleph_2$ *is a measurable cardinal.*

Proof. (i) We first show that AD implies that ω_1 is measurable. We already know that ω_1 is inaccessible in every $L[a]$, $a \subset \omega$.

Let us consider the following partial ordering of the Baire space:

$$x \preccurlyeq y \quad \text{if and only if} \quad x \in L[y] \tag{33.7}$$

and the corresponding equivalence relation

$$x \equiv y \quad \text{if and only if} \quad x \preccurlyeq y \text{ and } y \preccurlyeq x. \tag{33.8}$$

We say that $A \subset \mathcal{N}$ is $\equiv$-*closed* if ($x \in A$ and $y \equiv x$) implies $y \in A$. Note that the collection $\mathcal{B}$ of all $\equiv$-closed sets in $\mathcal{N}$ is a complete Boolean algebra.

If $x_0 \in \mathcal{N}$, then we let

$$\text{cone}(x_0) = \{x \in \mathcal{N} : x_0 \preccurlyeq x\} = \{x : x_0 \in L[x]\} \tag{33.9}$$

and call $\text{cone}(x_0)$ a *cone*. Clearly, every cone is $\equiv$-closed. Let

$$\mathcal{F} = \{A \in \mathcal{B} : A \text{ contains a cone}\}.$$

We claim that $\mathcal{F}$ is a σ-complete filter on $\mathcal{B}$. Let $A_0, A_1, \ldots, A_n, \ldots$ be elements of $\mathcal{F}$. For each n, we choose $x_n \in \mathcal{N}$ such that $A_n \supset \text{cone}(x_n)$. Let $x \in \mathcal{N}$ be defined as follows: $x(\langle n, m\rangle) = x_n(m)$ for all $n, m \in \boldsymbol{N}$ (where $\langle\ \rangle$ is a pairing function). It is clear that for each n, $x_n \in L[x]$ and hence $\text{cone}(x) \subset \text{cone}(x_n) \subset A_n$. Thus $\bigcap_{n=0}^{\infty} A_n$ is in $\mathcal{F}$.

Lemma 33.13. AD *implies that for every* $\equiv$*-closed* $A \subset \mathcal{N}$, *either* A *or its complement contains a cone. Hence* $\mathcal{F}$ *is a* σ*-complete ultrafilter on* $\mathcal{B}$.

Proof. We show that if I has a winning strategy in the game G_A, then A contains a cone (and similarly, if II has a winning strategy, then $\mathcal{N} - A \in \mathcal{F}$). Let σ be a winning strategy for I. It suffices to show that A contains the cone $\{x \in \mathcal{N} : \sigma \in L[x]\}$.

Let $x \in \mathcal{N}$ be such that $\sigma \in L[x]$. Then $a = \sigma * x$ is in A because σ is a winning strategy. Clearly, $x \in L[a]$, and because $\sigma \in L[x]$, we also have $a \in L[x]$ and hence $x \equiv a$. Since A is $\equiv$-closed, we have $x \in A$. □

Now we can use AD to find a nonprincipal σ-complete ultrafilter U on ω_1. For each $x \in \mathcal{N}$, let $f(x) = \aleph_1^{L[x]}$; $f(x)$ is a countable ordinal. Note that if $x \equiv y$, then $f(x) = f(y)$, and hence for every $X \subset \omega_1$, the set $f_{-1}(X) \subset \mathcal{N}$ is $\equiv$-closed. Let

$$U = \{X \subset \omega_1 : f_{-1}(X) \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is a σ-complete ultrafilter on $\mathcal{B}$, U is a σ-complete ultrafilter on ω_1. It remains to show that U is nonprincipal. But for every $\alpha < \omega_1$, if $x \in \mathcal{N}$ is such that $\aleph_1^{L[x]} = \alpha$, then there is $y \succcurlyeq x$ such that $\aleph_1^{L[y]} > \alpha$; and hence $f_{-1}(\{\alpha\}) \notin \mathcal{F}$.

Thus AD implies that ω_1 is a measurable cardinal.

Lemma 33.14. *Assume* AD. *Then for every $S \subset \omega_1$, the set $\{x \in \mathrm{WO} : \|x\| \in S\}$ is $\mathbf{\Pi}^1_1$. Consequently, there is some $a \subset \omega$ such that $S \in L[a]$.*

Proof. If $x \in \mathcal{N}$, then for each $n \in \boldsymbol{N}$ we let $x_n \in \mathcal{N}$ be such that $x_n(m) = x(\langle n, m\rangle)$ for all $m \in \boldsymbol{N}$. We consider the following game:

33.15. The Solovay Game. Let $S \subset \omega_1$. Player I plays $a = \langle a(0), a(1), \ldots\rangle$, and II plays $b = \langle b(0), b(1), \ldots\rangle$. If $a \notin \mathrm{WO}$, then II wins; if $a \in \mathrm{WO}$, then II wins if

$$\{\alpha \in S : \alpha \leq \|a\|\} \subset \{\|b_n\| : n \in \omega\} \subset S.$$

We claim that I does not have a winning strategy in the Solovay game. Let σ be a winning strategy for I; for each $b \in \mathcal{N}$, let $f(b)$ be the $a \in \mathcal{N}$ such that $\langle a(0), b(0), a(1), b(1), \ldots\rangle = \sigma * b$. The set $f(\mathcal{N})$ is a $\mathbf{\Sigma}^1_1$ subset of WO, and by the Boundedness Lemma, there is an $\alpha < \omega_1$ such that $\|f(b)\| < \alpha$ for all $b \in \mathcal{N}$. Hence let $b \in \mathcal{N}$ be such that $\{\|b_n\| : n \in \omega\} = S \cap \alpha$. Then $\sigma * b$ is a play won by player II, and hence σ cannot be a winning strategy for I.

Now the lemma follows: Let $S \subset \omega_1$. By AD, player II has a winning strategy τ in the Solovay game. For each a, let $g(a)$ be the $b \in \mathcal{N}$ such that $\langle a(0), b(0), \ldots\rangle = a * \tau$. It follows that for each $a \in \mathrm{WO}$,

$$\|a\| \in S \quad \text{if and only if} \quad \exists n\, \|a\| = \|(g(a))_n\|$$

and consequently the set $\{x \in \mathrm{WO} : \|x\| \in S\}$ is $\mathbf{\Pi}^1_1$. By Lemma 25.22, $S \in L[a]$ for some $a \subset \omega$. □

We can now complete the proof of (i). If $X \subset \omega_1$, then $X \in L[a]$ for some $a \subset \omega$. Since $\aleph_1$ is a measurable cardinal, $a^\sharp$ exists, and it follows that either X or $\omega_1 - X$ contains a closed unbounded subset. Thus the closed unbounded filter on ω_1 is an ultrafilter.

By the Countable Axiom of Choice, the closed unbounded filter is σ-complete, and we therefore conclude (as we work in ZF + the Principle of Dependent Choices) that AD implies that the closed unbounded filter on ω_1 is the unique σ-complete normal ultrafilter on ω_1.

(ii) We shall now show that, assuming AD, $\aleph_2$ is a measurable cardinal. For each $x \in \mathcal{N}$, let $f(x)$ denote the successor cardinal of the (real) cardinal $\aleph_1$ in $L[x]$:

$$f(x) = ((\aleph_1)^+)^{L[x]}.$$

If $x \subset \omega$, then because $x^\sharp$ exists, $f(x)$ is an ordinal less than $\aleph_2$. Moreover, if $x \equiv y$, then $f(x) = f(y)$, and hence $f_{-1}(X)$ is $\equiv$-closed for each $X \subset \omega_2$. Thus let us define an ultrafilter U on ω_2 as follows:

$$U = \{X \subset \omega_2 : f_{-1}(X) \in \mathcal{F}\}.$$

We wish to show that U is an $\aleph_2$-complete nonprincipal ultrafilter on ω_2. Since $\mathcal{F}$ is σ-complete, U is σ-complete. It is also easy to see that U is nonprincipal: If $\alpha < \omega_2$ and $f(x) = \alpha$, then there exists an $S \subset \omega_1$ such that α is not a cardinal in $L[S]$; then by Lemma 33.14 there is a $y \in \mathcal{N}$ such that $x \in L[y]$ and $f(y) > \alpha$. Hence $f_{-1}(\{\alpha\})$ does not contain a cone.

It remains to show that U is $\aleph_2$-complete. Since U is σ-complete, it suffices to show that if

$$X_0 \supset X_1 \supset \ldots \supset X_\alpha \supset \ldots \qquad (\alpha < \omega_1) \tag{33.10}$$

is a descending sequence of subsets of ω_2 such that each $f_{-1}(X)$ contains a cone then $f_{-1}(\bigcap_{\alpha<\omega_1} X_\alpha)$ contains a cone.

Let us consider such a sequence (33.10), and let $X = \bigcap_{\alpha<\omega_1} X_\alpha$. We shall use the following game: Player I plays $a = \langle a(0), a(1), \ldots\rangle$, and II plays $b = \langle b(0), b(1), \ldots\rangle$. If $a \notin$ WO, then I loses; if $a \in$ WO and $\|a\| = \alpha$, then II wins if $\mathrm{cone}(b) \subset f_{-1}(X_\alpha)$.

We claim that I does not have a winning strategy in this game: If σ is a winning strategy for I, then the set of all $a \in \mathcal{N}$ that I plays by σ against all possible $b \in \mathcal{N}$, is a $\mathbf{\Sigma}^1_1$ subset of WO and hence there is α such that $\|a\| < \alpha$ for all these a's. Now II can beat I simply by playing some $b \in \mathcal{N}$ such that $\mathrm{cone}(b) \subset f_{-1}(X_\alpha)$.

Thus II has a winning strategy τ, and we intend to show that $f_{-1}(X)$ contains the cone $\{x \in \mathcal{N} : \tau \in L[x]\}$. Let $\alpha < \omega_1$ and let $x \in \mathcal{N}$ be such that $\tau \in L[x]$; we want to show that $f(x) \in X_\alpha$.

Let P_α be the notion of forcing that collapses α onto ω: The conditions are finite sequences of ordinals less than α. Since $\aleph_1$ is inaccessible in $L[x]$, $L[x]$ has only countably many subsets of P_α, and therefore there exists an $L[x]$-generic filter G on P_α. Let $a \in$ WO be such that $\|a\| = \alpha$ and let $L[a] = L[G]$ and let $y \in \mathcal{N}$ be such that $L[y] = L[x][G] = L[x][a]$.

Since G is generic on P_α over $L[x]$, all cardinals in $L[x]$ greater than α are preserved in $L[x][G]$. In particular, $(\aleph_1^+)^{L[x]}$ is preserved and hence $f(y) = f(x)$.

Now if I plays $a = \langle a(0), a(1), \ldots\rangle$ and if II plays against a by his winning strategy τ, II produces $b = \langle b(0), b(1), \ldots\rangle$ such that $\mathrm{cone}(b) \subset f_{-1}(X_\alpha)$. But since $b \in L[\tau, a]$ and $\tau \in L[x]$, we have $b \in L[x, a] = L[y]$ and therefore

$y \in \text{cone}(b)$. It follows that $f(y) \in X_\alpha$; and because $f(x) = f(y)$, we have $f(x) \in X_\alpha$, as we wanted to prove.

This completes the proof of Theorem 33.12. □

It turns out that under Determinacy there exist many measurable cardinals. Of particular interest have been the *projective* ordinals $\boldsymbol{\delta}^1_n$. By definition

$$\boldsymbol{\delta}^1_n = \sup\{\xi : \xi \text{ is the length of a } \boldsymbol{\Delta}^1_n \text{ prewellordering of } \mathcal{N}\}.$$

By the results in Chapter 25, $\boldsymbol{\delta}^1_1 = \omega_1$ and $\boldsymbol{\delta}^1_2 \leq \omega_2$. It has been established (under AD) that all the $\boldsymbol{\delta}^1_n$ are measurable cardinals, along with other properties, such as $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$. The size of each $\boldsymbol{\delta}^1_{2n+1}$ has now been calculated exactly; in particular, $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$ and $\boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$. The analysis of the $\boldsymbol{\delta}^1_n$'s depends heavily on calculations of length of ultrapowers by measures on projective ordinals.

An important ordinal (isolated by Moschovakis) is

$$\Theta = \sup\{\xi : \xi \text{ is the length of a prewellordering of } \mathcal{N}\}.$$

AD implies that $\Theta = \aleph_\Theta$, and if in addition $V = L(\boldsymbol{R})$ then Θ is a regular cardinal (Solovay). Θ is the limit of measurable cardinals (Kechris and Woodin), and for every $\lambda < \Theta$, there exists a normal ultrafilter on $[\lambda]^\omega$ (Solovay). As for the consistency strength of AD, we have:

Theorem 33.16 (Woodin). *Assume* AD *and* $V = L(\boldsymbol{R})$. *Then there exists an inner model with infinitely many Woodin cardinals.* □

Theorem 33.16 is optimal, as the existence of infinitely many Woodin cardinals is equiconsistent with AD; see Theorem 33.26. (We remark that the proof of Theorem 33.16 uses the following result: If AD and $V = L(\boldsymbol{R})$, then Θ is a Woodin cardinal in the model *HOD*.)

Projective Determinacy

In this section we address the question how strong is the determinacy assumption when restricted to games that have a simple enough definition. In particular, we turn our attention to the game G_A where $A \subset \mathcal{N}$ is a projective set.

When A is open (or closed) then G_A is determined:

Lemma 33.17. *If $A \subset \mathcal{N}$ is an open set, then G_A is determined.*

Proof. Player I plays $\langle a_0, a_1, \ldots\rangle$, player II plays $\langle b_0, b_1, \ldots\rangle$, and I wins if $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in A$. Let us assume that player I does not have a winning strategy, and let us show that II has a winning strategy. The strategy for II is as follows: When I plays a_0, then because I does not have a winning strategy,

there exists b_0 such the position $\langle a_0, b_0\rangle$ is not yet lost for II. That is, I does not have a winning strategy in the game $G_A^{\langle a_0,b_0\rangle}$ that starts at $\langle a_0, b_0\rangle$, in which I plays $\langle a_1, a_2, \ldots\rangle$ and II plays $\langle b_1, b_2, \ldots\rangle$, and in which I wins when $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in A$.

Let II plays such b_0. When I plays a_1, then again, because II is not yet lost at $\langle a_0, b_0\rangle$, there exists b_1 such that II is not yet lost at $\langle a_0, b_0, a_1, b_1\rangle$. Let II play such b_1. And so on. We claim that this strategy for II is a winning strategy.

Let $x = \langle a_0, b_0, a_1, b_1, \ldots\rangle$ be a play which II plays by the above strategy; We want to show that $x \notin A$. If $x \in A$, then because A is open, there is $s = \langle a_0, b_0, \ldots, a_n, b_n\rangle \subset x$ such that $O(s) \subset A$. But then it is clear that II is lost at s; a contradiction. □

The same argument (interchanging the players) would show that every closed game is determined. Or, we can show that every closed game is determined as follows: If A is closed, then I has a winning strategy in G_A if and only if there is $a_0 \in \boldsymbol{N}$ such that II does not have a winning strategy in the open game $G_A^{a_0}$ in which II make a first move b_0, then I plays a_1, etc., and II wins if $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ is in the open set $\mathcal{N} - A$. Since $G_A^{a_0}$ is determined for all $a_0 \in \boldsymbol{N}$, G_A is determined.

One of the major results in descriptive set theory is Martin's proof that for every Borel set A the game G_A is determined:

Theorem 33.18 (Martin [1975]). *All Borel games are determined.* □

We shall not give a proof. It can be found either in Martin's paper [1975], or in the survey article [1980] by Kechris and Martin; furthermore, Martin gives a simplification of his proof in [1985].

Analytic Determinacy, i.e., determinacy of all analytic games, is already a large cardinal assumption:

Theorem 33.19. *Let $a \in \mathcal{N}$. Every $\Sigma^1_1(a)$ game is determined if and only if $a^\sharp$ exists.*

Thus Analytic Determinacy is equivalent to the statement

$$a^\sharp \text{ exists for all } a \in \mathcal{N}. \tag{33.11}$$

The proof of Analytic Determinacy from (33.11) is due to Martin [1969/70]. The necessity of (33.11) is a result of Harrington [1978]. We omit Harrington's proof and prove a corollary of Martin's result. We note however that the proof of the corollary can be converted into a proof of the "if" part of Theorem 33.19 without much difficulty.

Corollary 33.20. *If there exists a measurable cardinal, then all analytic games are determined.*

Proof. Let κ be a measurable cardinal and let $A \subset \mathcal{N}$ be an analytic set. We want to show that the game G_A is determined.

Let us use the tree representation of analytic sets. There is a tree $T \subset Seq_2$ such that for all $x \in \mathcal{N}$,

$$x \in A \leftrightarrow T(x) \text{ is ill-founded.}$$

Let $\preccurlyeq$ be the linear ordering of the set Seq that extends the partial ordering $\supset$: If $s, t \in Seq$, then $s \preccurlyeq t$ either if $s \supset t$, or if s and t are incompatible, and $s(n) < t(n)$ where n is the least n such that $s(n) \neq t(n)$. Thus

$$x \in A \leftrightarrow T(x) \text{ is not well-ordered by } \preccurlyeq.$$

We also recall that $T(x) = \{t : (x{\restriction}n, t) \in T \text{ for some } n\}$ and that the first n levels of $T(x)$ depend only on $x{\restriction}n$. For $s \in Seq$, we let $T_s = \{t : (u, t) \in T \text{ for some } u \subset s\}$; then $T_{x{\restriction}n}$ is exactly the first n levels of the tree $T(x)$. We need some further notation. Let $t_0, t_1, \ldots, t_n, \ldots$ be an enumeration of the set Seq. If $s \in Seq$ is a sequence of length $2n$, let K_s be the finite set $\{t_0, \ldots, t_{n-1}\} \cap T_s$ and let $k_s = |K_s|$.

We shall now define an auxiliary game G^*: Player I plays natural numbers $a_0, a_1, a_2, \ldots$, and player II plays pairs (b_0, h_0), (b_1, h_1), (b_2, h_2), $\ldots$ where $b_0, b_1, b_2, \ldots$ are natural numbers, and for each n, h_n is an order-preserving mapping from $(K_s, \preccurlyeq)$ into κ where $s = \langle a_0, b_0, \ldots, a_n, b_n\rangle$ such that $h_0 \subset h_1 \subset h_2 \subset \ldots \subset h_n \subset \ldots$. If player II is able to follow these rules throughout the game, then he wins. Otherwise, I wins.

It is clear that the game G^* is determined: If I does not have a winning strategy, then he cannot prevent II from following the rules and thus II has a winning strategy, namely his each move is to reach a position in which I does not have a winning strategy. (The argument is the same as in the proof of determinacy of open games; in fact, G^* is an open game in a suitable topology.)

If II wins a play in the game G^*, then he has constructed an order-preserving mapping $h = \bigcup_{n=0}^{\infty} h_n$ of $(T(x), \preccurlyeq)$ into κ, where $x = \langle a_0, b_0, a_1, b_1, \ldots\rangle$; hence $\preccurlyeq$ well-orders $T(x)$ and so $x \notin A$. Thus we can view the game G^* as a variant of G_A, but more difficult for player II: II tries to make sure that $x \notin A$, and in addition, he tries to construct an embedding of $(T(x), \preccurlyeq)$ in κ. Hence it is fairly obvious that if II has a winning strategy in the game G^*, then II has a winning strategy in G_A: If τ^* is a winning strategy for II in G^*, let τ be as follows. When I plays a_0, let $\tau(\langle a_0\rangle) = b_0$ where $(b_0, h_0) = \tau^*(\langle a_0\rangle)$; then when I plays a_1, let $\tau(\langle a_0, b_0, a_1\rangle) = b_1$ where $(b_1, h_1) = \tau^*(\langle a_0, (b_0, h_0), a_1\rangle)$; etc.

Since G^* is determined, it suffices to prove the following lemma in order to show that G_A is determined:

Lemma 33.21. *If I has a winning strategy in G^*, then I has a winning strategy in G_A.*

Proof. Let σ^* be a winning strategy for I in G^*. After $2n+2$ moves, the players have produced a sequence $s = \langle a_0, b_0, \ldots, a_n, b_n\rangle$, and II has constructed order-preserving functions $h_0 \subset \ldots \subset h_n$; the strategy σ^* then tells player I what to play next. Let E be the range of h_n; E is a finite subset of κ, and in fact its size is k_s. We observe that there is one and only one way II could have constructed $h_0, \ldots, h_n$ so that $\mathrm{ran}(h_n) = E$; the reason is that h_n is the unique order-preserving one-to-one function between $(K_s, \preccurlyeq)$ and E. Thus σ^* depends (as long as II plays correctly) only on $s \in Seq$ and the finite set $E \subset \kappa$.

For each $s \in Seq$ of even length, let F_s be the following function from $[\kappa]^{k_s}$ into ω:

$$F_s(E) = \sigma^*(s, E). \tag{33.12}$$

Each F_s is a partition of $[\kappa]^{k_s}$ into ω pieces; and because κ is measurable, there exists a set $H \subset \kappa$ of size κ homogeneous for each F_s. Let us denote by $\sigma(s)$ the unique value of $F_s(E)$ for $E \in [H]^{k_s}$.

We shall complete the proof by showing that σ is a winning strategy for I in the game G_A. Let $x = \langle a_0, b_0, a_1, b_1, \ldots\rangle$ be a play in which I plays by σ. We shall show that $x \in A$.

Assume that on the contrary, $x \notin A$. Then $(T(x), \preccurlyeq)$ is well-ordered and has order-type $< \omega_1$. Since H is uncountable, there exists an embedding h of $(T(x), \preccurlyeq)$ into H. Let us consider the following play of the game G^*: I plays a_0. Then II plays (b_0, h_0) where h_0 is the restriction of h to $K_{\langle a_0, b_0\rangle}$. Then I plays a_1 and II plays (b_1, h_1) where h_1 is the restriction of h to $K_{\langle a_0, b_0, a_1, b_1\rangle}$. And so on.

We show that in this play, player I plays by the strategy σ^*. Clearly, $a_0 = \sigma(\emptyset) = \sigma^*(\emptyset, \emptyset)$. Then $a_1 = \sigma(\langle a_0, b_0\rangle)$, and by the definition of σ it is clear that $\sigma(\langle a_0, b_0\rangle) = \sigma^*(\langle a_0, b_0\rangle, h(K_{\langle a_0, b_0\rangle}))$ and therefore a_1 is a move according to σ^*. And so on: All the moves $a_0, a_1, \ldots, a_n, \ldots$ are by σ^*.

This is a contradiction because σ^* is a winning strategy for I in G^*, but the play we described is won by player II. It follows that $x \in A$ and hence σ is a winning strategy for I in the game G_A. □

This completes the proof of $\mathbf{\Sigma}^1_1$ Determinacy assuming a measurable cardinal. This assumption can be weakened to the assumption that $a^\sharp$ exists for all $a \subset \omega$. The above proof is then modified as follows: We play the auxiliary game as before; κ is an uncountable cardinal. The definition of the auxiliary game is absolute for the model $L[T]$, and it follows that either I or II has a winning strategy for G^*, which si in $L[T]$. In particular, in Lemma 33.21, we may take $\sigma^* \in L[T]$. Then the collection $\{F_s : s \in Seq\}$, where F_s is defined by (33.12), is in $L[T]$, and an indiscernibility argument shows that there is an uncountable set $H \subset \kappa$ of indiscernibles for $L[T]$ such that each F_s has the same value for all $E \in [H]^{k_s}$. The rest of the proof is the same.

Determinacy of all projective games is considerably stronger than Analytic Determinacy: $\mathbf{\Delta}^1_2$ Determinacy yields an inner model with a Woodin

cardinal, and for every n, $\mathbf{\Delta}^1_{n+1}$ Determinacy yields an inner model with n Woodin cardinals.

The proof of Theorem 33.3 shows that Projective Determinacy implies that every projective set of reals is Lebesgue measurable, has the Baire property, and if uncountable, contains a perfect subset. The most important consequence of PD for the structure of projective sets of reals is the existence of scales. The following is a generalization of (25.28) and (25.34):

Definition 33.22. A Π^1_n-*norm* on a Π^1_n set A is a norm φ on A with the property that there exist a Π^1_n relation $P(x, y)$ and a Σ^1_n relation $Q(x, y)$ such that for every $y \in A$ and all x,

$$x \in A \text{ and } \varphi(x) \leq \varphi(y) \leftrightarrow P(x, y) \leftrightarrow Q(x, y).$$

A Σ^1_n-*norm* on a Σ^1_n set A is defined similarly, as is a Π^1_n-*scale* on a Π^1_n set (or a Σ^1_n-*scale* on a Σ^1_n set).

We say that the class Π^1_n has the *prewellordering property* (the *scale property*) if every Π^1_n set has a Π^1_n-norm (a Π^1_n-scale). Π^1_n has the *uniformization property* if every Π^1_n relation on $\mathcal{N} \times \mathcal{N}$ is uniformized by a Π^1_n function. Similarly for Σ^1_n.

Theorem 33.23 (Moschovakis [1971]). *Assume Projective Determinacy. Then the following classes have the scale property (for every $a \in \mathcal{N}$):*

$$\Pi^1_1(a),\ \Sigma^1_2(a),\ \Pi^1_3(a),\ \Sigma^1_4(a),\ \ldots,\ \Pi^1_{2n+1}(a),\ \Sigma^1_{2n+2}(a),\ \ldots.$$

Corollary 33.24. *Assume* PD. *The classes $\Pi^1_{2n+1}(a)$ and $\Sigma^1_{2n+2}(a)$ have the prewellordering property and the uniformization property and satisfy the reduction principle*; *the classes $\Sigma^1_{2n+1}(a)$ and $\Pi^1_{2n+2}(a)$ satisfy the separation principle.*

The scale property generalizes the prewellordering property, and implies uniformization (using the proof of Kondô's Theorem 25.26; cf. Exercise 33.4). The prewellordering property implies the reduction principle (as in Exercise 25.7; see Exercise 33.5), which in turn implies the separation principle for the dual class (cf. Exercise 25.9).

Moreover, since reduction holds for Π^1_{2n+1} and Σ^1_{2n+2}, separation fails for these classes (see Exercise 25.11 and Exercise 33.6). Hence reduction, prewellordering and scale properties fail for the dual classes Σ^1_1, Π^1_2, Σ^1_3, ...

Instead of proving Theorem 33.23 we shall prove the weaker statement: Assuming PD, every Π^1_{2n+1} and every Σ^1_{2n+2} have the prewellordering property. The full result is proved by a similar, somewhat more complicated, method.

In Chapter 25 we proved that every Π^1_1 set has a Π^1_1-norm and that every Σ^1_2 set has a Σ^1_2-norm. The latter statement is easily derived from the former (Exercise 25.6). The same proof shows that if Π^1_{2n+1} has the prewellordering property then so does Σ^1_{2n+2} (Exercise 33.7). Thus it suffices to prove the following:

Lemma 33.25. *Assume that every $\mathbf{\Delta}^1_{2n}$ game is determined, and that every Σ^1_{2n} set has a Σ^1_{2n}-norm. Then every Π^1_{2n+1} set has a Π^1_{2n+1}-norm.*

Proof. Assume that the hypotheses hold and let B be a Π^1_{2n+1} set

$$x \in B \leftrightarrow \forall u\, (x, u) \in A$$

where A is Σ^1_{2n}. Let ψ be a Σ^1_{2n}-norm on A. For $x, y \in \mathcal{N}$ consider the game $G(x, y)$ where I plays $a(0)$, $a(1)$, $\ldots$, $a(k)$, $\ldots$ and II plays $b(0)$, $b(1)$, $\ldots$, $b(k)$, $\ldots$ and II wins if $(y, b) \notin A$, or $(x, a) \in A$ and $\psi(x, a) \leq \psi(y, b)$. The game $G(x, y)$ is determined: If $y \notin B$ then II can win by playing b such that $(y, b) \notin A$; if $y \in B$ then

$$\text{II wins } G(x, y) \leftrightarrow P(x, a, y, b) \leftrightarrow Q(x, a, y, b)$$

and so the payoff set is $\mathbf{\Delta}^1_{2n}$ and hence determined.

For $x, y \in B$, define

$$x \preccurlyeq y \leftrightarrow \text{II has a winning strategy in } G(x, y).$$

We will show that $\preccurlyeq$ is a prewellordering of B and the corresponding norm is a Π^1_{2n+1}-norm.

Clearly, $x \preccurlyeq x$ for every $x \in B$ (II wins by copying I's moves).

To check that $\preccurlyeq$ is transitive, let $x \preccurlyeq y$ and $y \preccurlyeq z$. Thus II has winning strategies both in $G(x, y)$ and $G(y, z)$. We describe a winning strategy for II in $G(x, z)$: Let $k \geq 0$. When I plays $a(k)$ in $G(x, z)$, consider this the kth move in $G(x, y)$ and apply the strategy in $G(x, y)$ to respond $b(k)$. Consider $b(k)$ to be the kth move of I in $G(y, z)$ and apply the strategy in $G(y, z)$ to respond $c(k)$. This $c(k)$ is then the kth move of II in $G(x, z)$. It is clear that II wins.

Now assume that $x, y \in B$ and $x \not\preccurlyeq y$. Then I has a winning strategy in $G(x, y)$ (because II does not); we describe a winning strategy for II in $G(y, x)$ so that $y \preccurlyeq x$: Let $k \geq 0$. When I plays $a(k)$ in $G(y, x)$, let $b(k)$ be the move by I's winning strategy in $G(x, y)$ (responding to II's $a(k-1)$). Let II play $b(k)$ in $G(y, x)$. As I wins in $G(x, y)$, we have $\psi(x, a) > \psi(y, b)$, and so II wins.

To verify that $\preccurlyeq$ is well-founded, we assume to the contrary that $x_0 \succ x_1 \succ \ldots \succ x_n \succ \ldots$ is a descending chain, that I has a winning strategy in each of the games $G(x_i, x_{i+1})$. Let $a_0(0)$, $a_1(0)$, $\ldots$, $a_i(0)$, $\ldots$ be the first moves of I by the winning strategies in the games $G(x_i, x_{i+1})$, and for each $k \geq 1$, let $a_0(k)$, $a_1(k)$, $\ldots$, $a_i(k)$, $\ldots$ be I's moves responding to $a_1(k-1)$, $a_2(k-1)$, $\ldots$, $a_{i+1}(k-1)$, $\ldots$ II's moves in these games. Since I wins all these games, we have $\psi(x_0, a_0) > \psi(x_1, a_1) > \ldots > \psi(x_i, a_i) > \ldots$, a contradiction.

Finally, for every $y \in B$,

$$x \in B \text{ and } x \preccurlyeq y \leftrightarrow \exists \tau\, \forall a\, (x, a) \leq_\psi (y, a * \tau) \leftrightarrow \forall \sigma\, \exists b\, (x, \sigma * b) \leq_\psi (y, b)$$

(where σ and τ denote strategies for I and II) and since ψ is a Σ^1_{2n}-norm on A, it follows that the norm associated with $\preccurlyeq$ is a Π^1_{2n+1}-norm on B. $\square$

Consistency of AD

The following theorem confirms what has been expected since the early 1970's: Determinacy is a large cardinal axiom:

Theorem 33.26 (Martin-Steel-Woodin). *If there exist infinitely many Woodin cardinals and a measurable cardinal above them, then the Axiom of Determinacy holds in* $L(\boldsymbol{R})$. □

In the rest of this chapter we shall outline some ideas on which this result is based. But first we state two related results:

Theorem 33.27 (Woodin). *The following are equiconsistent*:

(i) ZFC + "*There exist infinitely many Woodin cardinals.*"
(ii) ZF + AD. □

Theorem 33.28 (Martin-Steel). *Let* $n \in \boldsymbol{N}$. *If there exist* n *Woodin cardinals with a measurable cardinal above them then every* $\boldsymbol{\Pi}^1_{n+1}$ *game is determined.* □

The crucial concept in these proofs is that of a *homogeneous tree*.

Following the terminology and notation of Chapter 25, and specifically Definition 25.8, let K be a set and let T be a tree on $\omega \times K$ (or more generally, on $\omega^r \times K$). For $s \in Seq$ let

$$T_s = \{t : (s,t) \in T\}. \tag{33.13}$$

In the present context, a *measure* is a σ-complete ultrafilter, not necessarily nonprincipal.

Definition 33.29. A tree T on $\omega \times K$ is *homogeneous* if there are measures μ_s, $s \in Seq$, such that μ_s is a measure on T_s and:

(i) If t extends s then $\pi_{s,t}(\mu_t) = \mu_s$ where $\pi_{s,t}$ is the natural projection map from T_t to T_s.
(ii) If $x \in p[T]$ then the direct limit of the ultrapowers by $\{\mu_{x\restriction n} : n \in \omega\}$ is well-founded.

(See Exercise 33.8 for an explicit formulation of (ii).)

A tree T is κ-*homogeneous* (where κ is a regular uncountable cardinal) if the measures μ_s are all κ-complete. A set $A \subset \mathcal{N}$ is (κ-)*homogeneously Suslin* if $A = p[T]$ for some (κ-)homogeneous tree T.

Homogeneous trees are an abstraction of Martin's proof of $\boldsymbol{\Pi}^1_1$ Determinacy from a measurable cardinal. First, an analysis of Martin's proof shows the following:

Lemma 33.30. *If* $A \subset \mathcal{N}$ *is* $\boldsymbol{\Pi}^1_1$ *and* κ *is a measurable cardinal then* A *is* κ*-homogeneously Suslin.*

Proof. Exercise 33.10. □

Martin's proof essentially uses this (Exercise 33.11):

Lemma 33.31. *If $A \subset \mathcal{N}$ is homogeneously Suslin then A is determined.* □

A related concept is a *weakly homogeneous tree*:

Definition 33.32. A tree T on $\omega \times K$ is *weakly homogeneous* if there are measures $\mu_{s,t}$, where $s, t \in Seq$ and $\text{length}(s) = \text{length}(t)$, such that $\mu_{s,t}$ is a measure on T_s and

(i) If $\bar{s} \supset s$ and $\bar{t} \supset t$ then $\pi_{s,\bar{s}}(\mu_{\bar{s},\bar{t}}) = \mu_{s,t}$.
(ii) If $x \in p[T]$ then there exists a $y \in \mathcal{N}$ such that the direct limit of the ultrapowers by $\{\mu_{x\restriction n, y\restriction n} : n \in \omega\}$ is well-founded.

A tree T is *κ-weakly homogeneous* if the $\mu_{s,t}$ are κ-complete. A set A is *(κ-)weakly homogeneously Suslin* if $A = p[T]$ for some (κ-)weakly homogeneous tree T.

It is not difficult to show that a set $A \subset \mathcal{N}$ is κ-weakly homogeneously Suslin if and only if it is a projection of a homogeneously Suslin set $B \subset \mathcal{N} \times \mathcal{N}$ (Exercises 33.12 and 33.13).

Theorem 33.26 follows, via Lemma 33.31, from the following two deep results:

Theorem 33.33 (Woodin [1988]). *If there exist infinitely many Woodin cardinals with a measurable cardinal above, then every subset of $\mathcal{N}$ in $L(\boldsymbol{R})$ is δ^+-weakly homogeneously Suslin, for some Woodin cardinal δ.* □

Theorem 33.34 (Martin and Steel [1988]). *If $A \subset \mathcal{N}$ is δ^+-weakly homogeneously Suslin, where δ is a Woodin cardinal, then $\mathcal{N} - A$ is homogeneously Suslin.* □

We shall return to Theorem 33.33 in a later chapter. As for Theorem 33.34, assume that $A = p[T]$ where T is weakly homogeneous. Then one constructs a tree $\tilde{T}$ such that $\mathcal{N} - A = p[\tilde{T}]$ in a manner similar to the tree representation for $\boldsymbol{\Pi}^1_2$ sets in Theorem 32.14. The heart of the argument in Martin-Steel's proof is to show that $\tilde{T}$ is a homogeneous tree.

Exercises

33.1. (i) The function $f(b) = \sigma * b$ is a one-to-one continuous function.
(ii) The set $\{\sigma * b : b \in \mathcal{N}\}$ contains a perfect subset.

33.2. I has a winning strategy in the perfect set game if and only if X has a perfect subset. II has a winning strategy if and only X is at most countable.

33.3. Let $n > 0$. If G_A is determined for every $\mathbf{\Sigma}^1_n$ set A, then G_A is determined for every $\mathbf{\Pi}^1_n$ set, and vice versa.

33.4. If every Π^1_{2n+1} set has a Π^1_{2n+1}-scale then every Π^1_{2n+1} relation is uniformized by a Π^1_{2n+1} function.

33.5. If every Π^1_{2n+1} set has a Π^1_{2n+1}-norm then Π^1_{2n+1} satisfies the reduction principle.

33.6. If Π^1_{2n+1} satisfies the reduction principle then it does not satisfy the separation principle.

33.7. If every Π^1_{2n+1} set has a Π^1_{2n+1}-norm (has a Π^1_{2n+1}-scale) then every Σ^1_{2n+2} set has a Σ^1_{2n+2}-norm (has a Σ^1_{2n+2}-scale).

33.8. Property (ii) in Definition 33.29 is equivalent to this: If $x \in p[T]$ and A_1, A_2, ... are such that $\mu_{x\restriction n}(A_n) = 1$, then there exists an $f \in K^\omega$ such that $(x, f) \in [T]$ and $f\restriction n \in A_n$ for all n.

33.9. Every closed set is homogeneously Suslin.
[T is on $\omega \times \omega$ and each μ_s is principal.]

33.10. Let κ be a measurable cardinal. If A is $\mathbf{\Pi}^1_1$ then there is a κ-homogeneous tree T on $\omega \times \kappa$ such that $A = p[T]$.
[As A is $\mathbf{\Pi}^1_1$ there are linear orders $<_s$, $s \in Seq$, such that $<_s$ orders $\{0, \dots, n-1\}$ where $n = \text{length}(s)$, $<_t$ extends $<_s$ if $s \subset t$, and such that $A = \{x : <_x \text{ is a well-ordering}\}$ where $<_x$ is the limit of the $<_{x\restriction n}$. Let T be the tree on $\omega \times \kappa$ such that $[T] = \{(x, f) : f \text{ is order-preserving from } (\omega, <_x) \text{ into } (\kappa, <)\}$. Let U be a normal measure on κ and let for s of length n, let μ_s on T_s be induced by U_n (on $[\kappa]^n$).]

33.11. If $A = p[T]$ and T is a homogeneous tree then the game G_A is determined.
[Use an auxiliary game G^* as in the proof of Corollary 33.20.]

33.12. If $B \subset \mathcal{N}^2$ is weakly homogeneously Suslin then so is the projection of B.

33.13. If T is a weakly homogeneous tree on $\omega \times K$ then there exists a homogeneous tree U on $(\omega \times \omega) \times K$ such that $p[T]$ is the projection of $p[U]$.

33.14. Let T be a homogeneous tree on $(\omega \times \omega) \times K$, and let $T' = \{(s, (t, u)) : ((s, t), u) \in T\}$. Then T' is a weakly homogeneous tree on $\omega \times (\omega \times K)$.

Historical Notes

Infinite games were first considered in the 1930. Mazur described an infinite game and conjectured its connection to Baire category, which was then proved by Banach.

In [1953] Gale and Stewart investigated infinite games in general and proved that the Axiom of Choice implies that there exist undetermined games and that open games are determined.

In [1962] Mycielski and Steinhaus proposed an axiom and called it the Axiom of Determinateness (AD). In [1963/64, 1966] Mycielski gave a comprehensive account of consequences of AD and related open problems.

Theorem 33.3(i) is due to Mycielski and Świerczkowski [1964]; the present proof (and the covering game) is due to Harrington. Theorem 33.3(iii) is due to Morton Davis [1964].

Following Solovay's discovery that AD implies that $\aleph_1$ is a measurable cardinal, attention has been turned to the relation between Determinacy and large cardinals. There have been numerous results in this direction, and a vast of literature exists on the subject. The reader can find an excellent account of current research on AD in Kanamori's book [1994]; a comprehensive treatment of the subject is expected to appear in the near future (Woodin et al. [∞]).

Theorem 33.12 is due to Solovay; the present proof of measurability of $\aleph_1$ (Lemma 33.13) is due to Martin [1968].

Projective ordinals $\boldsymbol{\delta}^1_n$ as well as the cardinal Θ were introduced by Moschovakis [1970] and studied extensively by Kechris [1974, 1978]. The calculation of the size of the $\boldsymbol{\delta}^1_5$ was accomplished by Steve Jackson, cf. [1988, 1999]. For the results on Θ, see e.g. Kechris [1985].

Theorem 33.18 (Borel Determinacy) is due to Martin [1975]; see also Martin [1985] and Kechris and Martin [1980].

Theorem 33.19: In [1969/70], Martin proved that analytic games are determined if $a^\sharp$ exists for all $a \in \mathcal{N}$; the converse was proved by Harrington in [1978].

Moschovakis' Theorem 33.23, cf. [1971], is the culmination of applications of Projective Determinacy to classical descriptive set theory: among others, see Blackwell [1967], Addison and Moschovakis [1968] and Martin [1968]. For a comprehensive survey, see Kechris and Moschovakis [1978].

Consistency of AD follows from the results of Martin, Steel and Woodin, cf. Martin and Steel [1988, 1989] and Woodin [1988].

Homogeneous trees are implicit in Martin and Solovay [1969] and in Martin [1969/70]. They were explicitly isolated by Kechris [1981]. Weakly homogeneous trees figured prominently in Woodin [1988].

34. Supercompact Cardinals and the Real Line

In this chapter we present results showing the effect of very large cardinals (such as supercompact) on the structure of sets of real numbers. In earlier chapters we showed that if $\aleph_1$ is inaccessible in every $L[x]$ (where $x \in \boldsymbol{R}$) then all $\mathbf{\Sigma}^1_2$ sets of reals are Lebesgue measurable, have the Baire property, and the perfect set property. If $x^\sharp$ exists for all $x \in \boldsymbol{R}$ then every $\mathbf{\Pi}^1_1$ game is determined. Thus already the existence of moderately large cardinals (such as measurable) has an effect on regularity of projective sets (but recall that—by Silver's Theorem 32.20—measurability is still weak to influence $\mathbf{\Sigma}^1_3$ sets, as measurable cardinals are consistent with a Σ^1_3 well-ordering of $\boldsymbol{R}$). It follows from the results presented below that if a supercompact cardinal exists, then all sets of reals in $L(\boldsymbol{R})$ have the regularity properties mentioned above.

Woodin Cardinals

As we mentioned in the last chapter (Theorem 33.27), the consistency strength of Determinacy is below a supercompact cardinal; the appropriate large cardinal concept (a Woodin cardinal) was isolated in the course of investigations leading to the proof of AD. Let us elaborate on the definition (Definition 20.31) of Woodin cardinals: Let κ and $\lambda \geq \kappa$ be cardinals, and let A be an arbitrary set. We say that κ is λ-*strong for* A if there exists an elementary embedding $j : V \to M$ with critical point κ such that

(34.1) (i) $j(\kappa) > \lambda$,
(ii) $V_\lambda \subset M$,
(iii) $A \cap V_\lambda = j(A) \cap V_\lambda$.

Hence κ is λ-strong if it is λ-strong for $\emptyset$, and by definition, δ is a Woodin cardinal if for every $A \subset V_\delta$ there are arbitrarily large $\kappa < \delta$ that are λ-strong for A for all $\lambda < \delta$. We now present a different definition of Woodin cardinals and show that it is equivalent to Definition 20.31.

Definition 34.1. A cardinal δ is a *Woodin cardinal* if for every function $f : \delta \to \delta$ there exists a $\kappa < \delta$ with $f\text{“}\kappa \subset \kappa$, and an elementary embedding $j : V \to M$ with critical point κ such that $V_{j(f)(\alpha)} \subset M$.

Every supercompact cardinal is Woodin, and is a limit of Woodin cardinals (Exercise 34.1). An immediate consequence of Definition 34.1 is that every Woodin cardinal is a Mahlo cardinal, and in fact has a stationary set of measurable cardinals. The following lemma proves the equivalence of Definitions 20.31 and 34.1:

Lemma 34.2. *The following are equivalent*:

(i) *For every $A \subset V_\delta$ there exists a $\kappa < \delta$ that is λ-strong for A for all $\lambda < \delta$.*
(ii) *For every $A \subset V_\delta$ the set of all $\kappa < \delta$ that are λ-strong for A for all $\lambda < \delta$ is stationary.*
(iii) *For every $f : \delta \to \delta$ there exists a $\kappa < \delta$ with $f\text{“}\kappa \subset \kappa$, and an elementary embedding $j : V \to M$ with critical point κ such that $V_{j(f)(\alpha)} \subset M$.*
(iv) *For every $f : \delta \to \delta$ there exists a $\kappa < \delta$ with $f\text{“}\kappa \subset \kappa$, and an extender $E \in V_\kappa$ with critical point κ such that $j_E(f)(\kappa) = f(\kappa)$ and $V_{f(\kappa)} \subset \mathrm{Ult}_E$.*

Proof. It suffices to show that (i) implies (iv) and that (iii) implies (ii).

Assume that (i) holds, and let $f : \delta \to \delta$. By (i) there exists a $\kappa < \delta$ that is λ-strong for A for all $\lambda < \delta$. Let $\lambda < \delta$ be sufficiently large, and let E be an extender with critical point κ such that $V_{f(\kappa)} \subset \mathrm{Ult}_E$ and $f \cap V_\lambda = j_E(f) \cap V_\lambda$; such an extender exists in V_δ. Clearly, $f\text{“}\kappa \subset \kappa$, and since λ is sufficiently large, we have $j_E(f)(\kappa) = f(\kappa)$. Therefore (iv) holds.

Now assume that (iii) holds; let $A \subset V_\delta$ and let $C \subset \delta$ be a closed unbounded set. To prove (ii) we need a $\kappa \in C$ that is λ-strong for A for all $\lambda < \delta$. For each $\alpha < \delta$ let $f(\alpha)$ be a limit ordinal $\beta \in C$ such that if there exists a $\lambda < \delta$ such that α is not λ-strong for A, then such a λ exists below β. By (iii) there exists some $\kappa < \delta$ with $f\text{“}\kappa \subset \kappa$, and an elementary $j : V \to M$ with critical point κ such that $V_{j(f)(\kappa)} \subset M$. Since $f\text{“}\kappa \subset \kappa$, $C \cap \kappa$ is closed unbounded in κ and hence $\kappa \in j(C)$. By elementarity it suffices to show that

$$M \vDash \kappa \text{ is } \lambda\text{-strong for } j(A) \text{ for all } \lambda < j(\delta). \tag{34.2}$$

Assume that (34.2) fails, and let λ be the least λ such that in M, κ is not λ-strong for $j(A)$. By definition of f we have $\lambda < j(f)(\kappa)$. Let E be the (κ, λ)-extender derived from j. It is routine to verify that $V_\lambda \subset \mathrm{Ult}_E$ and since $\mathrm{Ult}_E = \{(jf)(a) : f \in V,\ a \in V_\lambda\} \subset M$, it follows that $E \in M$. In M, E is a (κ, λ)-extender, and $V_\lambda^M = V_\lambda \subset \mathrm{Ult}_E(M)$.

We complete the proof by showing that

$$j(A) \cap V_\lambda^M = j_E^M(j(A)) \cap V_\lambda^M. \tag{34.3}$$

Each of the following equalities is easily verified: $j(A) \cap V_{j(\kappa)}^M = j(A \cap V_\kappa) = j_E(A \cap V_\kappa) = j_E^M(A \cap V_\kappa) = j_E^M(j(A) \cap V_\kappa) = j_E^M(j(A)) \cap j_E^M(V_\kappa)$, and (34.3) holds because $\lambda < j_E^M(\kappa)$ and $V_\lambda^M = V_\lambda \subset \mathrm{Ult}_E^M$. Therefore (ii) holds. □

Property (iv) in Lemma 34.2 is Π^1_1 over V_δ and so the least Woodin cardinal is Π^1_1-describable and therefore not weakly compact.

Semiproper Forcing

A property of forcing somewhat weaker than properness, called *semiproperness*, have been of considerable use in the theory of large cardinals. We shall investigate it in some detail in Chapter 37; at this point we introduce semiproper forcing, prove basic properties and use it in an application involving $L(\boldsymbol{R})$ and supercompact cardinals.

Modifying condition (ii) in Lemma 31.6 and the characterization of properness in Theorem 31.7 we obtain the following: Let P be a notion of forcing and let λ be sufficiently large. Let M be a countable elementary submodel of $(H_\lambda, \in, <)$. A condition q is (M, P)-*semigeneric* if for every name $\dot\alpha \in M$ such that $\Vdash \dot\alpha$ is a countable ordinal,

$$q \Vdash \exists \beta \in M\, \dot\alpha = \beta. \tag{34.4}$$

Definition 34.3. A notion of forcing P is *semiproper* if for every sufficiently large λ there is a closed unbounded set in $[H_\lambda]^\omega$ of countable elementary submodels such that

$$\forall p \in M\ \exists q \leq p\ q \text{ is } (M, P)\text{-semigeneric.}$$

Thus semiproperness is a weaker property than properness: Definition 34.3 is obtained by replacing arbitrary ordinal names in Lemma 31.6 by names for countable ordinals. While the condition in Theorem 31.7 is equivalent to preservation of stationary sets in $[\lambda]^\omega$, only the second part of the proof of Theorem 31.7 remains valid for semiproper forcing, and we get:

Theorem 34.4. *If P is semiproper then every stationary set $S \subset \omega_1$ remains stationary in V^P.* □

If P is semiproper and $\dot Q$ is semiproper in V^P, then $P * \dot Q$ is semiproper. Semiproperness is generally not preserved under countable support iteration; the proof of Theorem 31.15 does not generalize to iterations of semiproper forcing. (The reason is that a semiproper forcing may change the cofinality of a regular uncountable cardinal to ω: It is not necessarily the case that a countable set of ordinals in $V[G]$ is included in a set that is countable in V.) When the iteration adds no new countable sets, however, the proof of Theorem 31.15 does go through for semiproper forcing, and we have:

Lemma 34.5. *If P is a countable support iteration of semiproper forcing notions and if P is ω-distributive, then P is semiproper.* □

In Chapter 37 we shall deal with iterations of semiproper forcings.

The Model $L(\boldsymbol{R})$

We shall now show that if there exists a supercompact cardinal then every set of reals in $L(\boldsymbol{R})$ is Lebesgue measurable, has the Baire property and the perfect set property. In fact, the reals in $L(\boldsymbol{R})$ behave exactly as the reals in Solovay's model in Theorem 26.14(i). The regularity of sets of reals in $L(\boldsymbol{R})$ follows of course from Theorem 33.26, and we shall outline later in this chapter the methods that lead to the proof of that theorem. We shall prove the following:

Theorem 34.6 (Woodin). *Let κ be a superstrong cardinal and let $V[G]$ be the generic extension of V by the Lévy collapse $\mathrm{Col}(\aleph_0, <\kappa)$. Then there exists an elementary embedding*

$$j : L(\boldsymbol{R}) \to (L(\boldsymbol{R}))^{V[G]}.$$

(For superstrong cardinals, see Exercise 34.2.)

Corollary 34.7. *If there exists a superstrong cardinal then every set of reals in $L(\boldsymbol{R})$ is Lebesgue measurable, has the Baire property, and has the perfect set property. In particular, there is no projective well-ordering of $\boldsymbol{R}$.* □

The main result used in the proof of Theorem 34.6 is the following result on saturated ideals:

Theorem 34.8. *If κ is a superstrong cardinal then there exists an ω-distributive κ-c.c. notion of forcing P such that in V^P, $\kappa = \aleph_2$ and there exists a normal $\aleph_2$-saturated ideal on ω_1.*

Let us show how Theorem 34.8 implies Theorem 34.6:

Proof of Theorem 34.6. Let P be the notion of forcing from Theorem 34.8, and let M be the generic extension of V by P. In M, let I be a normal $\aleph_2$-saturated ideal on ω_1, and let Q be the notion of forcing $P(\omega_1)/I$. Q yields an M-generic M-ultrafilter G on ω_1; let $N = \mathrm{Ult}_G(M)$ be the generic ultrapower. If $j : M \to N$ is the generic elementary embedding then (by the results proved in Chapter 22), ω_1 is the critical point, $j(\omega_1) = \omega_2^M = \kappa$, and $(P(\omega_1))^N = (P(\omega_1))^{M[G]}$. Hence $\boldsymbol{R}^N = \boldsymbol{R}^{M[G]}$, and since $\boldsymbol{R}^M = \boldsymbol{R}$, j yields an elementary embedding

$$j : L(\boldsymbol{R}) \to (L(\boldsymbol{R}))^{V^{P*Q}}. \tag{34.5}$$

Let $B = B(P * Q)$. Since P satisfies the κ-chain condition and I is κ-saturated in V^P, B satisfies the κ-chain condition. Since P collapses all cardinals below κ to ω_1, and Q collapses ω_1 (because $\aleph_1^N = j(\aleph_1) = \aleph_2^M$),

B makes $\kappa = \aleph_1$. Every complete subalgebra of B generated by fewer than κ elements has size less than κ (by weak compactness of κ), and hence we have

(34.6) (i) B is κ-c.c.,
(ii) $B = \bigcup_{\alpha<\kappa} B_\alpha$ where $|B_\alpha| < \kappa$, and $B_\alpha <_{\rm reg} B_\beta$ for all $\alpha < \beta < \kappa$,
(iii) every $\gamma < \kappa$ is countable in V^{B_α} for some $\alpha < \kappa$.

It follows from (34.6) that B is isomorphic to the Lévy collapse $\mathrm{Col}(\omega, <\kappa)$ (see Exercise 34.5), and Theorem 34.6 now follows from (34.5). □

Proof of Theorem 34.8. The notion of forcing P is a countable support iteration of length κ, where κ is a superstrong cardinal. The goal is to build a model V^P in which for some stationary set $A \subset \omega_1$, the nonstationary ideal estricted to A, i.e., $I_{\rm NS}\restriction A$, is $\aleph_2$-saturated. We shall first describe the iterates.

Let us fix a set $A \subset \omega_1$ such that both A and $\omega_1 - A$ are stationary. Let $\{A_i : i \in W\}$ be a maximal almost disjoint collection of stationary subsets of A (in this context, *almost disjoint* means that $A_i \cap A_k$ is nonstationary). If $|W| > \aleph_1$, consider the following notion of forcing Q_W: First let Q be the forcing that collapses $|W|$ to $\aleph_1$ with countable conditions. In V^Q, let $\dot S = \sum_{i\in W} A_i$, and let $P_{\dot S}$ be the forcing (from Theorem 23.8) that shoots a closed unbounded set through $(\omega_1 - A) \cup \dot S$. Let $Q_W = Q * P_{\dot S}$. Equivalently, let Q_W be the set of all pairs (q, p) such that

(34.7) (i) $q : \gamma + 1 \to W$ for some $\gamma < \omega_1$, and
(ii) $p \subset \omega_1$ is a closed countable set such that $\alpha \in p \cap A$ implies $\alpha \in \bigcup_{\xi<\alpha} A_{q(\xi)}$.

A condition (q', p') is stronger than (q, p) if $q' \supset q$ and p' is an end-extension of p.

The forcing Q_W preserves stationary subsets of ω_1 (Exercise 34.6) but is not necessarily semiproper. If W is not maximal then Q_W makes it maximal, and preserves all stationary subsets of $\omega_1 - A$ and of all A_i (Exercise 34.7). Note also that if A is a nonstationary set then the forcing Q_W as defined in (34.7) has a dense subset that is countably closed.

The effect of Q_W is that in the generic extension, $\sum_{i\in W} A_i = A \pmod{I_{\rm NS}}$ and $|W| = \aleph_1$. (In the intermediate extension by Q there could exist a new stationary subset of A almost disjoint from each A_i, but it is destroyed by $P_{\dot S}$, and in V^{Q_W}, $\sum_{i\in W} A_i \cup (\omega_1 - A)$ contains a closed unbounded set.)

Now we define a countable support iteration P_α; and then we let $P = P_\kappa$. Using some book-keeping device (standard in forcing iterations), at stage α we consider (in V^{P_α}) a maximal almost disjoint collection $\{A_i : i \in W\}$ of stationary subsets of A such that $|W| > \aleph_1$. If Q_W is semiproper, we let $\dot Q_\alpha = Q_W$; otherwise we let $\dot Q_\alpha$ be the collapse with countable conditions of $2^{\aleph_2}$ to $\aleph_1$.

Thus P_α is a countable support iteration of semiproper forcing notions. The role of the set A is to guarantee that P_α is ω-distributive. To show that,

consider the generic extension $V[G]$ obtained by shooting a closed unbounded set through $\kappa - A$. In $V[G]$, each $\dot{Q}_\alpha$ has a countably close dense subset, and so P_α is a countable support iteration of countably closed forcing notions. Hence P_α is ω-distributive in $V[G]$, and therefore in V.

Thus by Lemma 34.5, each P_α is semiproper. Since κ is inaccessible and P_κ is the direct limit of small forcing notions, P_κ satisfies the κ-chain condition. Since at cofinally many stages $\dot{Q}_\alpha$ collapses $2^{\aleph_2}$ (of V^{P_α}) to $\aleph_1$, κ becomes $\aleph_2$ in the model V^P. The model V^P has no new countable sets of ordinals, and every stationary subset of ω_1 remains stationary. Moreover, if $S \in V^P$ is a subset of ω_1 and is stationary in some V^{P_α}, then it remains stationary: This is because V^P is a semiproper forcing extension of V^{P_α}—the iteration from α to κ is a countable support iteration of semiproper forcings in V^{P_α}, and is ω-distributive (in V^{P_α}).

We shall now prove that in V^P, the ideal $I_{\rm NS}\restriction A$ is $\aleph_2$-saturated. Let G be a generic filter on P_κ and assume that in $V[G]$ there exists a maximal almost disjoint family of stationary subsets of A, such that $|W| > \aleph_1$ (hence $|W| = \aleph_2^{V[G]} = \kappa$). Let W be such a family, and assume further that W is chosen by our book-keeping to be the family considered at stage κ of the iteration.

Let $j : V \to M$ be an elementary embedding with critical point κ such that $V_{j(\kappa)} \subset M$. For all $\alpha < j(\kappa)$, $(P_\alpha)^M = P_\alpha$; $(P_{j(\kappa)})^M = j(P_\kappa)$ is the direct limit of the P_α while $P_{j(\kappa)}$ is the direct or inverse limit, depending on $\operatorname{cf} j(\kappa)$ in V. Let H be such that $G * H$ is a generic filter on $P_{j(\kappa)}$. Let $\tilde{H} = \bigcup_{\kappa \le \alpha < j(\kappa)} H\restriction\alpha$; $G * \tilde{H}$ is an M-generic filter on $j(P_\kappa)$, and $j : V \to M$ extends (in $V[G * H]$) to an elementary embedding $j : V[G] \to M[G * \tilde{H}]$.

One more remark before we proceed. If $X \subset \omega_1$ is a stationary set in $M[G][\tilde{H}]$ then it is stationary in $V[G][H]$. This is because $X \in M[G][H\restriction\alpha]$ for some $\alpha < j(\kappa)$, hence X is stationary in $V[G][H\restriction\alpha]$, and $V[G][H]$ is a semiproper forcing extension of $V[G][H\restriction\alpha]$.

Lemma 34.9. *The forcing notion Q_W is semiproper in $V[G]$.*

This will complete the proof: If Q_W is semiproper then $Q_\kappa = Q_W$. It follows that $A = \sum W$, after forcing with Q_κ, hence in $M[G][\tilde{H}]$. This is a contradiction, since $j(W)$ is an almost disjoint family of stationary subsets of A, and $W \subset j(W)$ and $W \neq j(W)$, since $|W| = \kappa$ in $V[G]$.

Proof. Assume that Q_W is not semiproper. Let $N = (H_{\kappa^+})^{V[G]}$; there is a $p \in Q_W$ such that the set

$$S = \{M \in [N]^\omega : p \in M \text{ and no } q \le p \text{ is } (M, Q_W)\text{-semigeneric}\} \tag{34.8}$$

is stationary. Since Q_W is not semiproper, the forcing $\dot{Q}_\kappa$ is the collapse (with countable conditions) of 2^κ to $\aleph_1$. Let G_κ be a generic filter on $\dot{Q}_\kappa$; since $\dot{Q}_\kappa$ is ω-closed, S remains stationary in $V[G][G_\kappa]$. Since $S \in M[G]$, S is in $M[G][G_\kappa]$ a stationary subset of $[N]^\omega$ and N has cardinality $\aleph_1$. Let π

be (in $M[G][G_\kappa]$) a one-to-one correspondence between N and ω_1, and let $\tilde{S} = \omega_1 \cap \pi``S$. $\tilde{S}$ is, in $M[G][G_\kappa]$, a stationary subset of ω_1.

Now work in $V[G][H]$ and consider the forcing notion $Q_{j(W)} = j(Q_W)$ and the condition $p \in Q_W$ from (34.8). By Exercise 34.8, $j(p) = p$ forces that $j(S)$ is nonstationary. In the generic extension, $j(S)$ is a nonstationary subset of $[j(N)]^\omega$, and hence $j``S$ is a nonstationary subset of $[j``N]^\omega$ and therefore S is a nonstationary subset of $[N]^\omega$. It follows that (in V[G][H])

$$p \Vdash_{Q_{j(W)}} \tilde{S} \text{ is a nonstationary subset of } \omega_1. \tag{34.9}$$

The set $\tilde{S}$ is stationary in $M[G][G_\kappa]$ and therefore in $M[G][\tilde{H}]$ (which is a semiproper forcing extension of $M[G][G_\kappa]$). The family $j(W)$ is, in $M[G][\tilde{H}]$, a maximal almost disjoint family of stationary subsets of A and therefore intersects either $\omega_1 - A$ or some $E \in W$ in a stationary set; for instance let $E \in j(W)$ be such that $\tilde{S} \cap E$ is stationary. Thus $\tilde{S} \cap E$ is stationary in $V[G][H]$, and (by Exercise 34.7), remains stationary after forcing (over $V[G][H]$) with $Q_{j(W)}$. This contradicts (34.9). □

Stationary Tower Forcing

We shall describe a forcing notion, due to Hugh Woodin, that is used, among other applications, to generalize Theorem 34.6 and prove Theorem 33.33.

Definition 34.10 (Stationary Tower Forcing). Let κ be an inaccessible cardinal. The forcing notion $Q = Q_{<\kappa}$ consists of conditions (V_α, S) where $\alpha < \kappa$ and S is a stationary subset of $[V_\alpha]^\omega$. A condition (V_β, T) is stronger than (V_α, S) if $\alpha \leq \beta$ and $T \restriction V_\alpha \subset S$.

Equivalently, $(V_\beta, T) \leq (V_\alpha, S)$ if $\alpha \leq \beta$ and $T \subset S^{V_\beta}$ where S^{V_β} is the lifting of S to $[V_\beta]^\omega$; see Theorem 8.27. The forcing $Q_{<\kappa}$ is not separative: Two conditions (V_α, S) and (V_β, T) are equivalent if and only if for some (all) $\gamma \geq \alpha, \beta$, $S^{V_\gamma} \simeq T^{V_\gamma}$ mod the nonstationary ideal on $[V_\gamma]^\omega$.

If (V_α, S) is a condition, V_α is determined by S ($V_\alpha = \bigcup S$), so we can abuse the notation by calling S a condition in $Q_{<\kappa}$; we say that V_α is the support of S.

If G is a generic filter then for each $\alpha < \kappa$, $G \cap [V_\alpha]^\omega$ is a normal ultrafilter extending the closed unbounded filter. In $V[G]$, we define a *generic ultrapower* $\mathrm{Ult}_G(V)$ as follows: Consider formulas $f \in V$ defined on some V_α, $\alpha < \kappa$, and let, for f on V_α and g on V_β,

$$f =_G g \text{ if for some } S \in G \text{ with support} \geq \alpha, \beta,\ f(x \cap V_\alpha) = g(x \cap V_\beta) \text{ for all } x \in S; \tag{34.10}$$

$f \in_G g$ is defined similarly. Below we prove that if κ is a Woodin cardinal then $\mathrm{Ult}_G(V)$ is well-founded.

The following definition was inspired by the earlier sections of this chapter, in particular Exercise 34.9: Let $M \subset N$ be countable models; we say that N *end-extends* M if for all $u \in M$, $u \cap N = u \cap M$.

Definition 34.11. Let A be a dense set of conditions in $Q_{<\kappa}$. A is *semiproper* if for all sufficiently large λ there is a closed unbounded set in $[H_\lambda]^\omega$ of countable elementary submodels M such that for some countable $N \prec H_\lambda$,

(34.11) (i) $M \subset N$ and N end-extends $M \cap V_\kappa$,
(ii) $\exists S \in A \cap N$ with support V_α such that $N \cap V_\alpha \in S$.

The definition has equivalent variants:

Lemma 34.12. *Each of the following two properties is equivalent to semiproperness of* A:

(i) *There is a closed unbounded set of countable* $M \prec V_{\kappa+1}$ *such that some countable* $N \prec V_{\kappa+1}$ *satisfies* (34.11).
(ii) *For all sufficiently large* λ, *for every countable* $M \prec H_\lambda$ *such that* $A \in M$ *there is a countable* $N \prec H_\lambda$ *that satisfies* (34.11).

Proof. For the nontrivial implication (i) $\Rightarrow$ (ii) see Exercise 34.10. □

The following is the key lemma. If δ is a Woodin cardinal and A is a dense subset of $Q_{<\delta}$ then for a closed unbounded set of $\kappa < \delta$, $A \cap Q_{<\kappa}$ is dense in $Q_{<\kappa}$, and for a stationary set of κ, κ is λ-strong for A for all $\lambda < \delta$.

Lemma 34.13. *Let* $\kappa < \delta$ *be such that* $A \cap Q_{<\kappa}$ *is dense in* $Q_{<\kappa}$ *and that* κ *is* λ*-strong for* A *for all* $\lambda < \delta$. *Then* $A \cap Q_{<\kappa}$ *is semiproper in* $Q_{<\kappa}$.

Proof. Toward a contradiction, assume that the set

$$S = \{M \prec V_{\kappa+1} : \text{there is no } N \prec V_{\kappa+1} \text{ such that (34.11) holds}\}$$

is stationary. Let $\lambda > \kappa + 1$ ($\lambda < \delta$) be such that $(V_\lambda, \in) \prec (V_\delta, \in)$. Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$, $V_\lambda \in M$ and $j(A) \cap V_\lambda = A \cap V_\lambda$. We have $S \in M$, $S \in j(Q_{<\kappa})$, and $M \models j(A)$ is dense in $j(Q_{<\kappa})$, and so there exists a $T < S$ such that $T \in j(A) \cap V_\lambda = A \cap V_\lambda$. Note that $T < S$ means that for every $z \in T$, $z \cap V_{\kappa+1} \in S$.

Let V_α be the support of T. We shall find a countable $x \prec V_{\kappa+1}$, a countable $y \prec j(V_\alpha)$, and a countable $z \prec V_\alpha$ such that $y \cap V_\alpha = z \in T$, $z \cap V_{\kappa+1} = x$, and $T \in y$. Then y end-extends $j(x \cap V_\kappa) = x \cap V_\kappa$, $T \in j(A) \cap y$, and $y \cap V_\alpha \in T$. This implies (by (34.11)) that $j(x) \notin j(S)$, but $z \in T < S$ implies that $x \in S$, a contradiction.

To find x, y, and z, let $F : V_\alpha^{<\omega} \to V_\alpha$ be the function $F(e \cup \{f\}) = j(f)(T, e)$ (if defined and $\in V_\alpha$; e is a finite subset of V_α and $f \in V_\alpha$ is a function), and let $z \in T$ be closed under F. Let $y = \{(jf)(T, e) : f \in z \text{ and } e \in z^{<\omega}\}$ and $x = z \cap V_{\kappa+1}$. We have $y \prec j(V_\alpha)$, $y \cap V_\alpha = z$ and $T \in y$, as desired. □

Lemma 34.13 is used to prove the following theorem on the stationary tower forcing:

Theorem 34.14 (Woodin, [1988]). *Let δ be a Woodin cardinal and let $Q_{<\delta}$ be the stationary tower forcing. Let G be a generic filter on $Q_{<\delta}$, and let $j : V \to \mathrm{Ult}_G$ be the canonical elementary embedding into the generic ultrapower. Then*

(i) Ult_G *is well-founded.*
(ii) $j(\omega_1) = \delta$.
(iii) *In $V[G]$, the model Ult_G is closed under $<\delta$-sequences.*

We sketch the proof of (i) and refer the reader to Woodin [1988] for the details of the complete proof. (Woodin's paper states the theorem for a supercompact cardinal but the proof can be easily adapted. See also Woodin [1999], Theorem 2.36.)

Proof. (i) If A is a dense set and N is a countable model, we say that N *captures* A if (34.11)(ii) holds. First we claim that if $A \subset Q_{<\kappa}$ is semiproper then for every condition $p \in Q_{<\kappa}$ there is a stronger condition q such that every $N \in q$ captures A. This is proved by showing that the set

$$q = \{N \prec V_{\kappa+1} : N \cap V_\alpha \in p \text{ and } N \text{ captures } A\}$$

(where V_α is the support of p) is stationary. To show this, let $F : V_{\kappa+1}^{<\omega} \to V_{\kappa+1}$ and let $M \prec H_\lambda$ for some λ be such that $A \in M$, $F \in M$, and $M \cap V_\alpha \in p$. Let $N \supset M$ be such that N end-extends $M \cap V_\kappa$ and captures A. Then $N \cap V_{\kappa+1} \in q$ and is closed under F.

One proves similarly that if A_n, $n < \omega$, are semiproper then for every p there exists a $q < p$ such that every $N \in q$ captures every A_n.

Now let $\langle \dot{f}_n : n < \omega \rangle$ be a sequence of names of functions in the generic ultrapower, names for a descending sequence of ordinals. For each n there is a dense set A_n such that for each $S \in A_n$ there is an ordinal function f_n^S on S such that $S \Vdash \dot{f}_n = f_n^S$. Let $\kappa < \delta$ be such that each $A_n \cap Q_{<\kappa}$ is semiproper, and let $p \in G$ be such that every $N \in p$ captures each A_n.

Now we define, for each $n < \omega$, a function f_n on p as follows: If $N \in p$, let $f_n(N) = f_n^S(N)$ where $S \in A_n$ (with support V_α) is such that $N \cap V_\alpha \in S$. The functions f_n are defined for almost all (mod I_{NS}) $N \in p$, and $f_{n+1}(N) < f_n(N)$ for all n, producing a descending sequence of ordinals.

(iii) is proved similarly; one can show that if A_α, $\alpha < \gamma$, with $\gamma < \delta$ are semiproper then for every p there exists a $q < p$ such that every $N \in q$ captures A_α for all $\alpha \in N$.

(ii) follows by showing that δ remains a regular cardinal in $V[G]$ and that every $\alpha < \delta$ is collapsed to ω. While the proof of regularity of δ is similar to the proof of (iii), the proof that α becomes countable is a consequence of the following fact that is easy to verify: If $S \in Q_{<\delta}$ has support V_α then

$$S \Vdash j\text{“}V_\alpha \in j(S).$$

□

Weakly Homogeneous Trees

Let δ be a Woodin cardinal. By Theorem 34.14 there exists a generic elementary embedding $j : V \to M$ such that $\boldsymbol{R}^M = \boldsymbol{R}^{V[G]}$ and $j(\omega_1) = \delta$; G is a generic filter on $Q = Q_{<\delta}$. Consider the following forcing notion P in $V[G]$: A forcing condition p is a V-generic filter on the Lévy collapse $\mathrm{Col}(\omega, <\lambda)$ for some $\lambda < \delta$; p is stronger than q if $p \supset q$. The forcing P does not add reals and if $H \subset P$ is $V[G]$-generic then H is a V-generic filter on $\mathrm{Col}(\omega, <\delta)$. Under additional assumptions on δ, such as that δ is also a limit of Woodin cardinals, every countably generated subalgebra of $Q_{<\delta}$ has cardinality less than δ, and $\boldsymbol{R}^{V[G]} = \boldsymbol{R}^{V[H]}$. Hence there exists an elementary embedding $j : L(\boldsymbol{R}) \to L(\boldsymbol{R})^{\mathrm{Col}(\omega,<\delta)}$ and consequently, the sets of reals in $L(\boldsymbol{R})$ have the regularity properties stated in Corollary 34.7.

The above argument yields a stronger result:

Corollary 34.15. *If δ is a Woodin cardinal and a limit of Woodin cardinals, if P is a forcing notion such that $|P| < \delta$, and if G is a generic filter on P, then the model $L(\boldsymbol{R})^{V[G]}$ is elementarily equivalent to $L(\boldsymbol{R})$.*

Proof. As δ remains a Woodin cardinal in $V[G]$, we can find a V-generic filter H on $\mathrm{Col}(\omega, <\delta)$ such that $V[G] \subset V[H]$ and $V[H]$ is a $\mathrm{Col}(\omega, <\delta)$-generic extension of $V[G]$, and elementary embeddings $j : L(\boldsymbol{R}) \to L(\boldsymbol{R})^{V[H]}$ and $k : L(\boldsymbol{R})^{V[G]} \to L(\boldsymbol{R})^{V[H]}$. □

This property of Woodin cardinals (that the theory of $L(\boldsymbol{R})$ is unchanged by small forcing) has been exploited by Woodin to prove the following theorem. In [1988] these results are stated under the assumption that a supercompact cardinal exists, but Woodin subsequently proved the theorem under the assumption stated below. The assumption is close to optimal as ω Woodin cardinals do not suffice; compare also with Theorem 35.20. The proof of (ii) uses the result in (i), and is a restatement of Theorem 33.33, establishing Determinacy in $L(\boldsymbol{R})$.

Theorem 34.16 (Woodin, [1988]). *Assume that there exist infinitely many Woodin cardinals with a measurable cardinal above them. Let λ be the supremum of the first ω Woodin cardinals.*

(i) *For every set $A \subset \boldsymbol{R}$ in $L(\boldsymbol{R})$ there exist trees T and S such that*

$$A = p[T], \qquad \boldsymbol{R} - A = p[S]$$

and for every forcing P such that $|P| < \lambda$, if $G \subset P$ is generic then

$$V[G] \vDash p[T] \cup p[S] = \boldsymbol{R} \text{ and } p[T] \cap p[S] = \emptyset.$$

(ii) *Every set $A \subset \boldsymbol{R}$ in $L(\boldsymbol{R})$ is κ-weakly homogeneously Suslin, for all $\kappa < \lambda$.* □

Exercises

34.1. Let κ be a supercompact cardinal. Then

(i) κ is a Woodin cardinal, and
(ii) there is a normal measure on κ such that almost all $\delta < \kappa$ are Woodin.

A cardinal κ is *superstrong* if there exists an elementary embedding $j : V \to M$ with critical point κ such that $V_{j(\kappa)} \subset M$.

34.2. If κ is supercompact then there exists a normal measure on κ such that almost all $\alpha < \kappa$ are superstrong cardinals.

34.3. If κ is superstrong then κ is a Woodin cardinal, and there exists a normal measure on κ such that almost all $\delta < \kappa$ are Woodin cardinals.

34.4. P is semiproper if and only if for every $p \in P$, player II has a winning strategy in the following game: I plays names $\dot\alpha_n$ for countable ordinals, II plays β_n, and II wins if $\exists q \le p\ q \Vdash \forall n\, \exists k\, (\dot\alpha_n = \beta_k)$.

34.5. Let B be an atomless complete Boolean algebra that satisfies (34.6). Then $B = \mathrm{Col}(\omega, {<}\kappa)$.
[Similar to Theorem 26.12.]

34.6. Q_W preserves stationary subsets of ω_1.
[If $T \subset A$ is stationary then $T \cap A_i$ is stationary for some i, and remains stationary in V^Q. Hence $T \cap A_i \cap \dot S$ is stationary in V^Q, and then use Exercise 23.6.]

34.7. Let W be a family of stationary subsets of $A \subset \omega_1$, and let Q_W be defined as in (34.7). If S is a stationary subset of some $A_i \in W$ or a stationary subset of $\omega_1 - A$, then S remains stationary. Also, $A = \bigcup W$ in V^{Q_W}.
[As in Exercises 23.6 and 34.6.]

34.8. Let N be a transitive model, $N \supset \omega_1$, let $P \in N$ and $p \in P$. Then p forces that the set

$$\{M \in [N]^\omega : M \prec N \text{ and } \exists q \le p\ q \text{ is } (M, P)\text{-semigeneric}\}$$

contains a closed unbounded set.
[Let $C = \{M :$ if $\dot\alpha \in M$ and $\dot\alpha^G < \omega_1$ then $\dot\alpha^G \in M\}$.]

Let W be a family of stationary subsets of ω_1 and let Q_W be defined as in (34.7) (i.e., $A = \omega_1$).

34.9. Q_W is semiproper if and only if for all sufficiently large λ there is a closed unbounded set of $M \prec H_\lambda$ such that exists an $N \prec H_\lambda$ with $M \subset N$ and $\omega_1 \cap M = \omega_1 \cap N$, and for some $S \in W \cap N$, $\omega_1 \cap N \in S$.

34.10. Show that (i) implies (ii) in Lemma 34.12.
[By (i) let $F \in H_\lambda$ be such that $F : (V_{\kappa+1})^{<\omega} \to V_{\kappa+1}$ and that for every $M \prec V_{\kappa+1}$ closed under F there is some $N \prec V_{\kappa+1}$ such that (34.11) holds. Now if $M \prec H_\lambda$ and $A \in M$, there exists such an F in M, and so $\tilde M = M \cap V_{\kappa+1}$ is closed under F. Let $\tilde M \subset \tilde N \prec V_{\kappa+1}$ be so that (34.11) holds for $\tilde N$. Then let $N = \{f(e) : f \in M$ and $e \in (\tilde N \cap V_\kappa)^{<\omega}\}$. Verify that $M \subset N \prec H_\lambda$ and (34.11) holds for N.]

Historical Notes

Woodin cardinals were introduced by Woodin (both Definitions 20.31 and 34.1). Strong and superstrong cardinals were considered by Mitchell [1979a], Dodd and Jensen (Dodd [1982]) and Baldwin [1986], in their study of inner models.

Semiproper forcing was introduced by Shelah and was investigated extensively by Foreman, Magidor, and Shelah in [1988]. For Theorem 34.6, see Shelah and Woodin [1990]. The proof of Theorem 34.8 was inspired by the work of Foreman, Magidor, and Shelah on Martin's Maximum.

Stationary tower forcing and its applications (Theorem 34.14, Corollary 34.15, and Theorem 34.16) are due to Woodin.

35. Inner Models for Large Cardinals

This chapter is an introduction to the highly technical theory of inner models for large cardinals. We present the fundamental concepts and ideas of the theory and state, mostly without a proof (or giving an outline of a proof) some significant results.

There are two major themes in the theory of inner models. One is that with a given large cardinal property one can associate a minimal inner model of ZFC for that property. An example is the model $L[U]$ for a measurable cardinal. The other is the construction of *core models* for large cardinals. These generalize the Dodd-Jensen core model K that we describe in some detail. K is an inner model of ZFC that satisfies GCH and either the Covering Theorem holds for K (see below) or $L[U]$ exists.

Definition 35.1. Let M be an inner model of ZFC. We say that the *Covering Theorem* holds for M if for every uncountable set X of ordinals there exists some $Y \in M$ such that $|Y| = |X|$.

If the Covering Theorem holds for some inner model M that satisfies GCH then (exactly as in Corollaries 18.31–18.33) the Singular Cardinal Hypothesis holds, every singular cardinal is singular in M, and $(\kappa^+)^M = \kappa^+$ for every singular cardinal κ.

A theory of core models has been developed for large cardinals up to a Woodin cardinal. While the Covering Theorem does not hold beyond K, a generalized core model possesses the following feature: If there exists no inner model for a large cardinal with a given property then the core model M for such a property is "close to" the universe V; typically, $(\kappa^+)^M = \kappa^+$ for every singular strong limit cardinal. This feature makes core models a tool for gauging the consistency strength of set-theoretical conjectures.

As an example, Dodd-Jensen's Covering Theorem for K gives a lower bound for the consistency of the failure of SCH: If SCH fails then the Covering Theorem for K fails and therefore there exists an inner model for a measurable cardinal.

The Core Model

The origin of the core model theory was the construction, by Dodd and Jensen, of the core model K. The Dodd-Jensen core model ("the core model up to a measurable cardinal") is an inner model K that contains much of the large cardinal structure without the existence of measurable cardinals. Its main features are:

(1) K has a definable well-ordering, satisfies GCH and combinatorial principles such as $\square$.
(2) There exists a nontrivial elementary embedding $j : K \to K$ if and only if $L[U]$ exists.
(3) If $L[U]$ does not exist then the Covering Theorem holds for K.

If $L[U]$ exists then K has a simple definition:

Definition 35.2. Assume that $L[U]$ exists. The *core model* is the inner model

$$K = \bigcap_{\alpha \in Ord} \mathrm{Ult}_U^{(\alpha)}(L[U]).$$

It is easy to verify that K is an inner model. Only the lower parts of the iterated ultrapowers matter; see Exercise 35.1.

Central to the theory of inner models is the "internal" definition of K. The main idea underlying the theory, including the generalizations of K, is that the core model is approximated by transitive models (sets), so called *mice*. Mice are the building blocks of K, as much as the models L_α are for L.

The preferred hierarchy for the fine structure of L is the Jensen hierarchy J_α. For K, we use its relativization J_α^A for the language $\{\in, A\}$ where A is a unary predicate: We modify Definition 27.2 (of rudimentary functions) by adding the function $F(x) = x \cap A$ to obtain functions *rudimentary in* A, and let, for any set A,

(35.1) $\mathrm{rud}_A(M) =$ the closure of $M \cup \{M\}$ under functions rudimentary in A.

Definition 35.3. $J_0^A = \emptyset$, $J_{\alpha+1}^A = \mathrm{rud}_A(J_\alpha^A)$, $J_\alpha^A = \bigcup_{\beta<\alpha} J_\beta$ if α is a limit ordinal.

It follows that $L[A] = \bigcup_{\alpha \in Ord} J_\alpha^A$. See Exercise 35.2 for some properties of the relativized Jensen hierarchy. Each J_α^A is a transitive set and we abuse the notation by using J_α^A to denote also the model $(J_\alpha^A, \in, A \cap J_\alpha^A)$.

Definition 35.4. A *mouse* is a transitive model $M = J_\alpha^U$ such that

(i) U is a normal κ-complete iterable M-ultrafilter on some $\kappa < \alpha$,
(ii) all iterated ultrapowers of J_α^U by U are well-founded,
(iii) $M = H_1^M(\gamma \cup p)$ (the Σ_1 Skolem hull) for some $\gamma < \kappa$ and some finite $p \subset \alpha$.

More specifically, M is a mouse at κ.

Some remarks about the definition: Iterability is the condition (19.17) that makes possible iterating the ultrapower. In Chapter 19 we assumed that M is a model of ZF^- which is not the case for mice (the requirement (iii) precludes it; see Exercise 35.3). Łoś's Theorem is not true in general in ultrapowers of mice, only for Σ_0 formulas (actually for Σ_1 formulas as we discuss below). One uses the fine structure to overcome this difficulty. Finally, for (ii) it is sufficient that the ω_1st iterate is well-founded.

Definition 35.5. $K = L\{M : M \text{ is a mouse}\}$.

Below we outline a proof of the following theorem:

Theorem 35.6 (Dodd-Jensen).

(i) *K is an inner model of* ZFC *and has a Σ_2 well-ordering.*
(ii) *K satisfies* GCH.
(iii) *$\boldsymbol{R}^K$ has a Σ^1_3 well-ordering.*
(iv) *$K^K = K$, and $K^{V[G]} = K$ for every generic extension.*
(v) *If $L[U]$ exists then $K = \bigcap_{\alpha\in Ord} \mathrm{Ult}^{(\alpha)}_U L[U]$.*
(vi) *In K, $L[U]$ does not exist.*
(vii) *If $0^\sharp$ does not exist then $K = L$. If $0^\sharp$ exists then $0^\sharp \in K$. More generally, for every $x \in K$, if $x^\sharp$ exists then $x^\sharp \in K$.* □

We now outline the basic theory of mice and techniques used in the core model theory. First we state a special case of (vii):

Lemma 35.7. *A mouse exists if and only if $0^\sharp$ exists.*

Proof. If a mouse exists at κ, then the iterates $\kappa^{(\alpha)}$ are indiscernibles for L.

Conversely, let $0^\sharp$ exist and let i_α be the Silver indiscernibles. For each α, let $j_\alpha : L \to L$ be the unique elementary embedding with critical point i_α such that $j_\alpha(i_\alpha) = i_{\alpha+1}$; Let U_α be the corresponding L-ultrafilter. Using indiscernibility, one shows that $j_{U_\alpha} = j_\alpha$. Each U_α is iterable and all iterates $\mathrm{Ult}^{(\beta)}_{U_\alpha}(L)$ are well-founded.

Now consider $\kappa = i_0$ and $U = U_0$, and let $M = J^U_{\kappa+1}$. One proves that $U \subset J^U_{\kappa+1} \subset L$, and $\mathrm{Ult}^{(\alpha)}_U(J^U_{\kappa+1}) = J^{U_\alpha}_{i_\alpha+1}$. Finally, one verifies that $J^U_{\kappa+1} = H^M_1(\emptyset)$, and hence M is a mouse. □

Instrumental in the core model theory is the *comparison* of mice, a Σ_2 well-ordering of the class of all mice obtained by comparing the transfinite iterates of mice.

For every regular uncountable cardinal λ, let $\mathcal{C}_\lambda$ denote the closed unbounded filter on λ. Let $M = J^U_\alpha$ be a mouse at κ, and let λ be a regular

cardinal greater than κ^+. Then (as in Chapter 19), the λth iterate of M has the form

$$\mathrm{Ult}_U^\lambda(M) = J_\beta^{\mathcal{C}_\lambda}. \tag{35.2}$$

Clearly, $J_\beta^{\mathcal{C}_\lambda}$ is constructible from M, but M is also constructible from $J_\beta^{\mathcal{C}_\lambda}$: M is isomorphic to the Σ_1 Skolem hull of $\gamma \cup i_{0,\lambda}(p)$ in $J_\beta^{\mathcal{C}_\lambda}$.

For a given mouse M, γ and p are fixed to be least possible such that $M = H_1^M(\gamma \cup p)$, in the following sense: γ is the least such γ, and then p is the least p in the lexicographic ordering of finite descending sequences of ordinals.

Definition 35.8. Let $M = J_\alpha^U = H_1^M(\gamma \cup p)$ and $M' = J_{\alpha'}^{U'} = H_1^{M'}(\gamma' \cup p')$ be mice, and let λ be (any) sufficiently large regular cardinal. Let $i_{0,\lambda} : M \to J_\beta^{\mathcal{C}_\lambda}$ and $i'_{0,\lambda} : M' \to J_{\beta'}^{\mathcal{C}_\lambda}$ be the iterated ultrapowers, with $q = i_{0,\lambda}(p)$ and $q' = i'_{0,\lambda}(p')$. We define $M < M'$ as follows:

(i) either $\beta < \beta'$,
(ii) or $\beta = \beta'$ and $\gamma < \gamma'$,
(iii) or $\beta = \beta'$ and $\gamma = \gamma'$, and $q < q'$ (in the descending lexicographic ordering).

Lemma 35.9. *$<$ is a well-ordering of mice, and if $M \leq M'$ then $M \in L[M']$.*

Proof. If $\beta < \beta'$ then $J_\beta^{\mathcal{C}_\lambda} \in J_{\beta'}^{\mathcal{C}_\lambda}$, and $M \in L[J_\beta^{\mathcal{C}_\lambda}]$. □

An analysis of the complexity of $<$ reveals that it is a Σ_2 relation (and that $<$ on $\boldsymbol{R}^K$ is Σ_3^1). We recall that the constructible hierarchy is Σ_1; the added complexity in K is caused by the condition that every iterated ultrapower of a mouse is well-founded.

Being a mouse is absolute for transitive models of ZF, and so is the well-ordering of mice. Thus if M is an inner model then $K^M = L\{N : N$ is a mouse and $N \in M\}$, and $K^K = K$. Since the well-ordering of mice is definable in K, K is a model of ZFC.

If $V[G]$ is a generic extension of V (by a set forcing) then for all sufficiently large regular cardinals λ, the closed unbounded filter $\mathcal{C}_\lambda$ on λ in $V[G]$ is generated by the closed unbounded filter in V. Hence $J_\beta^{\mathcal{C}_\lambda}$ is the same in $V[G]$ as in V, and so every mouse in $V[G]$ is in V. Hence $K^{V[G]} = K$.

If $L[U]$ exists then every mouse is in $L[\mathcal{C}_\lambda]$ for some λ; but $L[\mathcal{C}_\lambda] = \mathrm{Ult}_U^{(\lambda)} L[U]$. Hence $K \subset \bigcap_{\alpha \in Ord} \mathrm{Ult}^{(\alpha)} L[U]$. If x is a set of ordinals in $\bigcap_\alpha \mathrm{Ult}^{(\alpha)}$ then for some $\lambda > \sup x$, $x \in L[U^{(\lambda)}]$. Hence there exists a mouse $M \prec_{\Sigma_1} J_{\lambda^+}^{U^{(\lambda)}}$ such that $x \in M$. Therefore $K = \bigcap_{\alpha \in Ord} \mathrm{Ult}^{(\alpha)} L[U]$. The latter model has no submodel with a measurable cardinal and so neither does K.

One important feature of K is the following, which we state without a proof:

Lemma 35.10. *If mice exist then $K = \bigcup\{M : M$ is a mouse$\}$.* □

Proofs in the core model theory such as the proof of Lemma 35.10 involve iterations of mice. One of the difficulties is that since mice do not satisfy ZF^-, the resulting embeddings are not fully elementary. It is easy to verify that $i_{0,1} : M \to \mathrm{Ult}_U M$ is Σ_0-elementary, and an additional argument shows that $i_{0,1}$ is Σ_1-elementary; similarly for $i_{\alpha,\beta}$. While the finite iterates of a mouse are mice, arbitrary iterates are not. See Exercises 35.5–35.7.

The proof of GCH in K resembles somewhat Silver's proof of GCH in $L[U]$. Instrumental in the proof (and proofs of combinatorial principles in K) are condensation arguments, similar to Lemmas 18.38 and 27.5. These proofs use heavily the fine structure of K (including projecta, standard codes and parameters), reducing arguments about Σ_n to Σ_1.

The fine structure of K makes it possible to generalize combinatorial properties such as $\diamondsuit$ and $\square$ from L to K.

There is an alternative way of developing the theory of K and proving the main theorems. This method, due to Magidor, uses the closed unbounded filter directly. Instead of using mice, K can be defined using Definition 35.12 below (this definition is equivalent to the Dodd-Jensen definition).

Definition 35.11. The closed unbounded filter $\mathcal{C}_\kappa$ on κ *survives at* β if for every n and $f : [\kappa]^n \to \{0,1\}$ with $f \in L_{\beta+1}[\mathcal{C}_\kappa]$ there is a set $C \in \mathcal{C}_\kappa$ homogeneous for f.

Definition 35.12. A set x belongs to K if and only if for some $\kappa > \mathrm{rank}(x)$ and some β, $x \in L_\beta[\mathcal{C}_\kappa]$ and $\mathcal{C}_\kappa$ survives at β.

If $\mathcal{C}_\kappa$ survives at β then it survives at all $\beta' < \beta$. If it survives at all β then $L[\mathcal{C}_\kappa]$ is the inner model for one measurable cardinal. $\mathcal{C}_\kappa$ survives vacuously at every $\beta < \kappa$, and survives at κ if and only if $0^\sharp$ exists (Exercise 35.8).

If $L[U]$ exists then for every sufficiently large regular κ, $L[\mathcal{C}_\kappa] = L[U^{(\kappa)}]$, and so $\bigcup\{L_\beta[\mathcal{C}_\kappa] \cap V_\kappa : \mathcal{C}_\kappa \text{ survives at } \beta\} = \bigcup\{L[U^{(\kappa)}] \cap V_\kappa : \kappa > \omega \text{ regular}\} = K$.

The Covering Theorem for K

The two main results on the core model are that unless $L[U]$ exists, K is rigid and the Covering Theorem holds for K:

Theorem 35.13 (Dodd-Jensen). *The following are equivalent*:

(i) *$L[U]$ exists.*
(ii) *There exists a nontrivial elementary embedding $j : K \to K$.* □

Theorem 35.14 (Dodd-Jensen's Covering Theorem for K). *If $L[U]$ does not exist, then for every uncountable set X of ordinals there exists a set $Y \supset X$ in K such that $|Y| = |X|$.* □

If $L[U]$ exists then the ultrapower by U yields an elementary embedding $j : K \to K$. The proof of the converse shows somewhat more: If $j : K \to M$ is elementary, then necessarily $M = K$ (and $L[U]$ exists). The proofs of both theorems use the fine structure of K, but a great deal of the fine structure can be eliminated when using Magidor's approach.

Since K is a model of ZFC, Corollaries 18.31, 18.32 and 18.33 all remain true when L is replaced by K:

Corollary 35.15. *If $L[U]$ does not exist then every singular cardinal is singular in K, $(\kappa^+)^K = \kappa^+$ for every singular κ, and the Singular Cardinal Hypothesis holds.* □

The Covering Theorem for $L[U]$

By Prikry's Theorem 21.10 there is a generic extension of $L[U]$ in which the measurable cardinal κ of $L[U]$ remains a cardinal while $\operatorname{cf} \kappa = \omega$. It follows that the Covering Theorem for $L[U]$ fails. However, it turns out that the existence of a Prikry sequence is the only obstacle to the Covering Theorem:

Theorem 35.16 (Dodd-Jensen's Covering Theorem for $L[U]$). *Assume that there is an inner model with a measurable cardinal, let κ be the least such cardinal and let U be a measure on κ in $L[U]$. Then*

(i) *either $0^\dagger$ exists, or*
(ii) *the Covering Theorem holds for $L[U]$, or*
(iii) *there exists an ω-sequence $S \subset \kappa$ Prikry generic over $L[U]$, such that the Covering Theorem holds for $L[U][S]$.* □

Note that by Theorem 21.14, $L[U][S] = L[S]$.

The Core Model for Sequences of Measures

The theory of K has been generalized by W. Mitchell who constructed a core model K^m for sequences of measures (the "core model up to $o(\kappa) = \kappa^{++}$"). In analogy with K,

(i) K^m has a definable well-ordering, satisfies GCH and $\square$.
(ii) There exists a nontrivial $j : K^m \to K^m$ if and only if there is an inner model for a measurable cardinal κ with $o(\kappa) = \kappa^{++}$.
(iii) If there is no model for $o(\kappa) = \kappa^{++}$ then a "weak" covering theorem holds for K^m.

Mitchell's core model is the union of mice where a mouse is an appropriate generalization of the Dodd-Jensen mouse. The main result on K^m is as follows:

Theorem 35.17 (Mitchell).

(i) *K^m is a model of* ZFC + GCH.
(ii) *K^m has a Σ_2 well-ordering and $\square$ holds; $\boldsymbol{R} \cap K^m$ has a Σ^1_3 well-ordering.*

If there exists no inner model for $o(\kappa) = \kappa^{++}$, then:

(iii) *If U is a normal iterable K^m-ultrafilter with $\mathrm{Ult}_U(K^m)$ well-founded then $U \in K^m$.*
(iv) *If $j : K^m \to M$ is a nontrivial elementary embedding then j is an iterated ultrapower using measures in K^m. (Hence there is no nontrivial $j : K^m \to K^m$.)*
(v) *If κ is a singular strong limit cardinal then $(\kappa^+)^{K^m} = \kappa^+$.* $\square$

Clause (v) is often called "the Weak Covering Theorem."

Theorem 35.17 is a useful tool for obtaining lower bounds for the consistency strength. As an example, we present the following application:

Corollary 35.18 (Mitchell). *Assume that κ is a measurable cardinal and $2^\kappa > \kappa^+$. Then there is an inner model with a measurable λ of order λ^{++}.*

Proof. If there is no such model then (iii) and (iv) hold. Let D be a normal measure on κ and $j_D : V \to M = \mathrm{Ult}_D(V)$; let $j = j_D{\restriction}K^m : K^m \to N$. By (iv), j is an iterated ultrapower, $j = i_{0,\vartheta} : K^m \to \mathrm{Ult}^{(\vartheta)} = N$, by measures in K^m. Let N_ν, $\nu \leq \vartheta$, be the iterates; $N_0 = K^m$ and $N_\vartheta = N$. If $\nu < \vartheta$ is a limit ordinal then there exist $\xi_\nu < \nu$ and $U_\nu \in N_{\xi_\nu}$ such that $N_{\nu+1} = \mathrm{Ult}_{i_{\xi_\nu,\nu}(U_\nu)}(N_\nu)$. Since $o(\kappa) < \kappa^{++} \leq 2^\kappa \leq \vartheta$, there is a stationary set $S \subset \kappa^{++}$ of ordinals of cofinality ω such that $\xi_\nu = \xi$ and $U_\nu = U$ are constant for $\nu \in S$. Let $\nu \in S$ be a limit point of S, let $\langle \nu_n : n < \omega \rangle$ be cofinal in $S \cap \nu$, and let κ_n be the critical point of $i_{\nu_n,\nu}$, for each n. The sequence $\langle \kappa_n : n < \omega \rangle$ generates the measure $i_{\xi,\nu}(U)$ and belongs to M, hence $i_{\xi,\nu}(U) \in M$. By (iii), $i_{\xi,\nu}(U) \in (K^m)^M = N_\vartheta$ but this is impossible since $i_{\xi,\nu}(U) \notin N_{\nu+1}$. $\square$

This, combined with a theorem of Gitik [1989] shows that the existence of a measurable cardinal κ such that $2^\kappa > \kappa^+$ is equiconsistent with the existence of a measurable cardinal κ of Mitchell order κ^{++}.

Another result of Gitik (cf. [1989] and [1991]) shows that the consistency strength of the failure of SCH is exactly a measurable cardinal κ with $o(\kappa) = \kappa^{++}$.

Up to a Strong Cardinal

The current core model theory employs sequences of extenders rather than sequences of measures. This not only enables one to generalize the theory to

large cardinals beyond measurable but also has some technical advantages even in the case of K^m.

Strong cardinals (see Chapter 20) were introduced by Dodd and Jensen who also provided their characterization in terms of extenders. They also constructed an inner model of the form $L[\mathcal{E}]$ where $\mathcal{E}$ is a (transfinite) sequence of extenders such that

(i) $L[\mathcal{E}]$ is a model of ZFC,
(ii) in $L[\mathcal{E}]$, $\mathcal{E}$ witnesses that there exists a strong cardinal,
(iii) $L[\mathcal{E}]$ satisfies GCH, $\square$, and has a Σ^1_3 well-ordering of the reals.

They also introduced a real, the *sharp for a strong cardinal*, that exists if and only if there exists a nontrivial elementary embedding $j : L[\mathcal{E}] \to L[\mathcal{E}]$.

The theory of core models up to a strong cardinal uses mice of the form $J^{\mathcal{E}}_\alpha$ where $\mathcal{E}$ is a sequence of $J^{\mathcal{E}}_\alpha$-extenders. The crucial fact that makes the generalization of the Dodd-Jensen theory possible is that one uses sequences of *non-overlapping* extenders. (Two extenders overlap if one is a (κ, λ)-extender, the other a (κ', λ')-extender and $\kappa \leq \kappa' < \lambda$.) This fact allows the comparison of mice by iteration, and while the generalization is far from routine, one obtains a result similar to those for K and K^m.

Theorem 35.19. *There exists an inner model K^{strong} such that*:

(i) *K^{strong} is a model of* ZFC + GCH.
(ii) *K^{strong} has a Σ_2 well-ordering and $\square$ holds; $\boldsymbol{R} \cap K^{\text{strong}}$ has a Σ^1_3 well-ordering.*

If there exists no inner model for a strong cardinal then:

(iii) *If $j : K^{\text{strong}} \to M$ is a nontrivial elementary embedding then j is an iterated ultrapower by extenders in K^{strong}. (Hence there is no nontrivial $j : K^{\text{strong}} \to K^{\text{strong}}$.)*
(iv) *If κ is a singular strong limit cardinal then $(\kappa^+)^{K^{\text{strong}}} = \kappa^+$.* $\square$

Inner Models for Woodin Cardinals

Inner models for very large cardinals employ a new method of comparison of mice. Due to the presence of overlapping extenders, a "linear" iteration of mice does not work and a new technique has been developed—the theory of *iteration trees*. Iteration trees were introduced by Martin and Steel, who used the technique to construct inner models for Woodin cardinals

Theorem 35.20 (Martin-Steel). *If there are n Woodin cardinals then there is an inner model that has n Woodin cardinals, and its reals have a Σ^1_{n+2} well-ordering.* $\square$

The Σ^1_{n+2} result is best possible: If there are n Woodin cardinals with a measurable cardinal above them then Π^1_{n+1} determinacy holds (Theorem 33.26) and so $\boldsymbol{R}$ does not have a Σ^1_{n+2} well-ordering.

The fine structure for iteration trees was developed further by Mitchell and Steel who constructed an inner model for a Woodin cardinal that satisfies GCH. Then Steel constructed a core model up to a Woodin cardinal, under an additional assumption of a measurable cardinal above.

Let Ω be a measurable cardinal. *Steel's core model* K^{steel} is an inner model of V_Ω and if V_Ω has no inner model with a Woodin cardinal then K^{steel} is both rigid and satisfies the Weak Covering Theorem:

Theorem 35.21. *Let Ω be a measurable cardinal.*

(i) $(K^{\text{steel}})^{V_\Omega[G]} = K^{\text{steel}}$, *for every generic extension of V^Ω (by forcing in V_Ω).*

If V_Ω has no inner model with a Woodin cardinal then:

(ii) *There is no nontrivial elementary embedding $j : K^{\text{steel}} \to K^{\text{steel}}$.*
(iii) *For every singular cardinal $\lambda < \Omega$, $(\lambda^+)^{K^{\text{steel}}} = \lambda^+$.* □

The following is an application of Steel's core model:

Theorem 35.22 (Steel). *If $\aleph_1$ carries an $\aleph_2$-saturated ideal, and if there exists a measurable cardinal, then there exists an inner model with a Woodin cardinal.* □

This is (almost) best possible, as Shelah proved that if κ is a Woodin cardinal then there is a generic extension in which $\kappa = \omega_2$ and NS_{ω_1} is ω_2-saturated.

Exercises

35.1. Assume that $L[U]$ exists; then $K = \bigcup_{\alpha \in Ord}(\text{Ult}_U^{(\alpha)}(L[U]) \cap V_{\kappa^{(\alpha)}})$.

35.2. (i) There is a $\Sigma_1(J^A_\alpha)$ map of ω_α onto J^A_α.
(ii) $\langle J^A_\xi : \xi < \alpha\rangle$ is $\Sigma_1(J^A_\alpha)$.
(iii) J^A_α has a $\Sigma_1(J^A_\alpha)$ well-ordering.
(iv) The relation $J^A_\alpha \vDash \varphi$ is $\Sigma_1(J^A_\alpha)$.

35.3. If M is a mouse then $\rho^1_M < \kappa$, where ρ^1_M, the Σ_1-projectum of M, is the smallest $\rho \leq \alpha$ such that there exists a $\Sigma_1(M)$ function with $f\text{“}\omega\rho = J^U_\rho$.

35.4. Assume that $0^\sharp$ exists, let $a \in L[0^\sharp]$ be a real Cohen generic over L and let $M = L[a]$. Then $M \subset K$ and so $K \cap M = M$, while $K^M = L$.

Let $M = J^M_\alpha = H_1(\gamma \cup p)$ be a mouse. Let $i_{0,\xi} : M \to M_\xi = \text{Ult}_U^{(\xi)}(M)$.

35.5. Łoś's Theorem holds in $\text{Ult}_U(M)$ for Σ_0 formulas.

35.6. $i_{0,1}$ is a cofinal embedding of M into M_1 and therefore Σ_1-elementary. $i_{\xi,\eta} : M_\xi \to M_\eta$ is Σ_1-elementary.

35.7. $M_1 = H_1(\gamma \cup i_{0,1}(p) \cup \{\kappa\})$, $M_n = H_1(\gamma \cup i_{0,n}(p) \cup \{\kappa^{(0)}, \dots, \kappa^{(n-1)}\})$, $M_\xi = H_1(\gamma \cup i_{0,\xi}(p) \cup \{\kappa^{(\nu)} : \nu < \xi\})$.

35.8. For every regular $\kappa > \omega$, $\mathcal{C}_\kappa$ survives at κ if and only if $0^\sharp$ exists.

Historical Notes

The core model K was introduced by Dodd and Jensen in [1981, 1982a, 1982b]; see also Dodd [1982]. Theorem 35.6 is proved in [1981], Theorem 35.13 and 35.14 in [1982a], and the proof of $\diamondsuit$ and $\square$ in K is due to Welch. An overview of K, with some proofs, can be found in Mitchell [1979b] and Dodd [1983]. Magidor's approach is described in Magidor [1990] and in Kanamori's forthcoming book [∞].

The core model K^m for sequences of measures was introduced by Mitchell. Theorem 35.17 was stated in Mitchell [1984]. Its proof has never been published but a detailed sketch will appear in Mitchell [∞b] (in the forthcoming Handbook of Set Theory). Mitchell's article [∞b] and its companion [∞a] give an excellent introduction to the inner model theory, as does the more expository Mitchell [1994].

The inner model $L[\mathcal{E}]$ for a strong cardinal appeared in Dodd [1982]. The definition of K^{strong} is given explicitly in Koepke [1989] where Theorem 35.19 is stated (a proof has not been published).

Iteration trees are introduced in Martin and Steel [1994] where Theorem 35.20 is proved. Fine structure for iteration trees is developed in Mitchell and Steel [1994] obtaining an inner model with a Woodin cardinal and GCH. Steel's core model is constructed in Steel [1996], proving Theorem 35.21(i), (ii) and Theorem 35.22. Theorem 35.21(iii) is proved in Mitchell et al. [1997].

An overview of these (and of more recent results) is given in Löwe and Steel [1999].

36. Forcing and Large Cardinals

In this chapter we continue to develop the techniques introduced in Chapter 21. We shall describe several applications of forcing that use various large cardinal assumptions.

Violating GCH at a Measurable Cardinal

By Silver's Theorem 21.4 it is consistent, relative to a supercompact cardinal, that GCH can fail at a measurable cardinal. This, combined with Prikry forcing, shows further that the Singular Cardinal Hypothesis is unprovable. The consistency strength of both statements has been proved to be exactly $o(\kappa) = \kappa^{++}$:

Theorem 36.1 (Gitik). *The following are equiconsistent*:

(i) *There exists a measurable cardinal* κ *such that* $2^\kappa > \kappa^+$.
(ii) *There exists a strong limit singular cardinal* κ *such that* $2^\kappa > \kappa^+$.
(iii) *There exists a measurable cardinal* κ *of Mitchell order* κ^{++}.

As proved in Chapter 21 (Corollary 21.13), the consistency of (ii) follows from the consistency of (i) by Prikry forcing. The necessity of (iii) for the consistency of "not SCH" was proved by Gitik, by a combination of the pcf theory and Mitchell's inner model for sequences of measures. We omit the proof.

As for the consistency of (i) using $o(\kappa) = \kappa^{++}$, this improvement of Silver's Theorem 21.4 is a combination of an intermediate forcing result of Woodin which we outline below, and an additional forcing argument of Gitik that we also omit. □

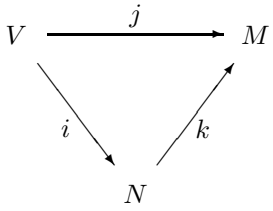
Theorem 36.2 (Woodin). *Assume* GCH *and assume that there exists an elementary embedding* $j : V \to M$ *with critical point* κ *such that* $M^\kappa \subset M$ *and that there exists a function* $f : \kappa \to \kappa$ *with* $j(f)(\kappa) = \kappa^{++}$. *Then there is a generic extension in which* κ *is a measurable cardinal and* $2^\kappa > \kappa^+$.

The assumption of Theorem 36.2 is easily seen to follow from κ being $(\kappa + 2)$-strong (Exercise 36.1). By Gitik, the statement holds in some generic extension of the canonical inner model for $o(\kappa) = \kappa^{++}$.

Proof. We outline the proof, which follows loosely the proof of Silver's Theorem 21.4. However, since the assumption is considerably weaker than supercompactness, more delicate arguments are needed.

We may assume that $j = j_E$ and $M = \mathrm{Ult}_E$, where E is a (κ, κ^{++})-extender (Exercise 36.2). Let U be the ultrafilter $U = \{X \subset \kappa : \kappa \in j(X)\}$ and consider the commutative diagram

(36.1)



where $N = \mathrm{Ult}_U(V)$ and $i : V \to N$ is the corresponding elementary embedding. Let

$$\lambda = i(f)(\kappa) \quad \text{and} \quad \mu = \mathrm{crit}(k). \tag{36.2}$$

We have the following inequalities:

$$\kappa^+ = (\kappa^+)^N = (\kappa^+)^M < \mu \leq \lambda < i(\kappa) < \kappa^{++} < j(\kappa) < \kappa^{+++}. \tag{36.3}$$

Moreover,

$$M = \{k(t)(a) : a \subset \kappa^{++} \text{ finite}, \, t \in N \text{ and } t : [\lambda]^{|a|} \to N\} \tag{36.4}$$

(see Exercise 36.3).

For cardinals α and β, let $\mathrm{Add}(\alpha, \beta)$ denote the notion of forcing that adds β subsets of α, cf. (15.3). The model for $2^\kappa > \kappa^+$ is constructed in two stages: The first stage is (as in Silver's proof) an iteration of length $\kappa + 1$, with Easton support. The final model is then obtained by a forcing extension of this model.

Let $P = P_{\kappa+1}$ be the Easton support iteration of $\dot{Q}_\alpha$, where for each $\alpha < \kappa$, $\dot{Q}_\alpha = \{1\}$ unless α is inaccessible and a closure point of the function f, in which case $\dot{Q}_\alpha = \mathrm{Add}(\alpha, f(\alpha))$ (in V^{P_α} where P_α is the αth iterate). For $\alpha = \kappa$, $\dot{Q}_\kappa = (\mathrm{Add}(\kappa, \kappa^{++}))^{V^{P_\kappa}}$. Let G be a generic filter on P; we have $V[G] = V[G_\kappa][H_\kappa]$ where G_κ is V-generic on P_κ and H_κ is $V[G_\kappa]$-generic on $Q_\kappa = \mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]}$.

We recall some of the facts established in the proof of Theorem 21.4: P_κ is κ-c.c. forcing notion of cardinality κ, κ remains inaccessible in $V[G]$, and $V[G]$ satisfies $2^\kappa = \kappa^{++}$. Since κ is in M a closure point of $j(f)$, $\dot{Q}_\kappa = (\dot{Q}_\kappa)^M$, and so $P = (j(P))_{\kappa+1}$.

As for $i : V \to N$, we use the fact that $N^\kappa \subset N$ and that P_κ is κ-c.c. to conclude (as in Lemma 21.9) that in $V[G_\kappa]$, $(N[G_\kappa])^\kappa \subset N[G_\kappa]$. Also, $(Q_\kappa)^{N[G_\kappa]} = \mathrm{Add}(\kappa, \lambda)^{V[G_\kappa]}$.

Using $k(P_\kappa) = P_\kappa$ and the fact that $k(p) = p$ for every $p \in P_\kappa$, we extend $k : N \to M$ to an embedding $k : N[G_\kappa] \to M[G_\kappa]$. Now consider the forcing Q_κ in $N[G_\kappa]$, and let

$$h_\kappa = k_{-1}(H_\kappa) = \{p \in \mathrm{Add}(\kappa, \lambda)^{V[G_\kappa]} : k(p) \in H_\kappa\}. \tag{36.5}$$

As $Q_\kappa^{N[G_\kappa]}$ has, in $N[G_\kappa]$, the κ^+-chain condition, and $\mathrm{crit}(k) > \kappa^+$, h_κ is $Q_\kappa^{N[G_\kappa]}$-generic over $N[G_\kappa]$ (see Exercise 36.4). Moreover, $G_\kappa * h_\kappa$ is V-generic on $(i(P))_{\kappa+1}$ and in $V[G_\kappa][h_\kappa]$, $(N[G_\kappa][h_\kappa])^\kappa \subset N[G_\kappa][h_\kappa]$ (Exercise 36.5). Note also that (because $\lambda < \kappa^{++}$), $V[G_\kappa][h_\kappa]$ satisfies $2^\kappa = \kappa^+$. It follows that, in $V[G_\kappa][h_\kappa]$, k can be extended to an embedding $k : N[G_\kappa][h_\kappa] \to M[G_\kappa][H_\kappa]$.

Let $\dot{R} \in N$ be the name for the iteration after stage $\kappa + 1$:

$$P_\kappa * \dot{Q}_\kappa^N * \dot{R} = i(P_\kappa) \tag{36.6}$$

and let $R \in N[G_\kappa][h_\kappa]$ be the interpretation of $\dot{R}$ by $G_\kappa * h_\kappa$. In $N[G_\kappa][h_\kappa]$, R is an $i(\kappa)$-c.c. forcing of cardinality $i(\kappa)$, and because the least α for which the αth iterate is nontrivial is above λ, R is λ-closed.

Using the fact that R is λ-closed and that the number of antichains of R in $N[G_\kappa][h_\kappa]$ is small in $V[G_\kappa][h_\kappa]$ we conclude that there exists in $V[G_\kappa][h_\kappa]$ an R-generic filter H over $N[G_\kappa][h_\kappa]$ (Exercise 36.6).

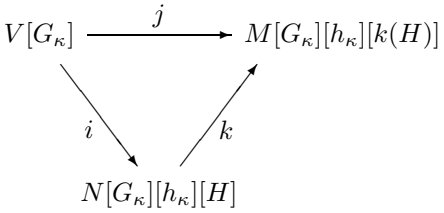
Now define $k(H)$ as follows (in $V[G_\kappa][h_\kappa]$):

$$k(H) = \{q \in k(R) : \exists p \in H\, k(p) \le q\}. \tag{36.7}$$

We claim that $k(H)$ is an $M[G_\kappa][h_\kappa]$-generic filter on $k(R)$. We omit the proof (but see Exercise 36.7).

As $p \in H$ implies $k(p) \in k(H)$ for every $p \in R$, k can be extended, in $V[G_\kappa][h_\kappa]$, to an embedding $k : N[G_\kappa][h_\kappa][H] \to M[G_\kappa][h_\kappa][k(H)]$. It follows that i and j can be extended (in $V[G]$), so that we have the following commutative diagram:

(36.8)



Now we describe the second stage of the construction, namely a generic extension of $V[G] = V[G_\kappa * H_\kappa]$, and in this extension, an elementary embedding that extends j. We force over $V[G]$ with the partial order $Q = i(Q_\kappa)$. Since $i(Q_\kappa)$ is $<i(\kappa)$-closed in $N[G_\kappa][h_\kappa][H]$, it follows from Exercise 36.5 that Q is κ-closed in $V[G_\kappa][h_\kappa]$. However, as the model $V[G] = V[G_\kappa][H_\kappa]$ is a generic extension of $V[G_\kappa][h_\kappa]$ by a κ^+-c.c. forcing $\mathrm{Add}(\kappa, \lambda)$, Q is κ-distributive in $V[G]$ (Exercise 36.8).

We also claim that in $V[G]$, Q is κ^{++}-c.c. To prove the claim, let $\tilde{Q}$ be, in $V[G_\kappa]$, the full support product of κ copies of Q_κ. Note that $\tilde{Q}$ is κ^{++}-c.c. in $V[G_\kappa]$, and since $\tilde{Q} \simeq Q_\kappa \times \tilde{Q}$, $\tilde{Q}$ is κ^{++}-c.c. in $V[G_\kappa][H_\kappa] = V[G]$. Since conditions in $Q = i(Q_\kappa)$ have the form $i(g)(\kappa)$ where $g : \kappa \to Q_\kappa$, an antichain in Q yields an antichain in $\tilde{Q}$, proving the claim.

Hence forcing with Q preserves κ^+ and κ^{++}, and so $2^\kappa = \kappa^{++}$ holds in the extension. Let K be a Q-generic filter over $V[G]$. The final step is to find in $V[G][K]$ a generic $j(K)$ over $M[G][k(H)]$ such that the embedding j from (36.8) extends to an embedding $j : V[G][K] \to M[G][k(H)][j(K)]$. This step, which we omit, first applies k to K and produces a generic X such that j extends to $j : V[G] \to M[G][k(H)][X]$, and then applies j to K and produces a generic Y such that j extends to $j : V[G][K] \to M[G][k(H)][X][Y]$. Details can be found in Gitik's paper [1989].

As this final step is performed inside $V[G][K]$, it follows that in $V[G][K]$, κ is measurable. □

The Singular Cardinal Problem

By Corollary 21.13, the negation of the Singular Cardinal Hypothesis is consistent relative to large cardinals, and its consistency strength is determined by Theorem 36.1. These results belong to a wide area of theorems and conjectures known collectively as the *Singular Cardinal Problem*. Unlike the behaviour of the continuum function on regular cardinals, which by Easton's Theorem can be quite arbitrary, the values of the continuum function at singular cardinals are subject to three kinds of constraint:

(1) By Silver's Theorems 8.12 and 8.13, the value of 2^κ for a singular cardinal κ of uncountable cofinality depends on the continuum function below κ.
(2) The Galvin-Hajnal Theorem 24.1 and Shelah's results in the pcf theory give upper bounds for the value of 2^κ when κ is a strong limit singular cardinal such that $\kappa < \aleph_\kappa$.
(3) Jensen's Covering Theorem 18.30 and the subsequent theory of core models shows that the consistency of the failure of SCH requires large cardinal assumptions.

There is a large body of forcing constructions that, using large cardinals, yield models with various behaviour of the continuum function subject to the above mentioned constraints. There is, however, no comprehensive solution of the Singular Cardinal Problem analogous to Easton's Theorem.

Below we list some of the advances in this area:

Theorem 36.3 (Magidor [1977a], [1977b]).

(i) *If there exists a supercompact cardinal then there is a generic extension in which* $2^{\aleph_n} < \aleph_\omega$ *for all* $n < \omega$ *and* $2^{\aleph_\omega} = \aleph_{\omega+2}$.

(ii) *If there exist $\kappa < \lambda$ with κ supercompact and λ huge then there exists a generic extension in which $2^{\aleph_n} = \aleph_{n+1}$ for all $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.* □

Theorem 36.4 (Woodin, Gitik [1989]). *If there exists a measurable cardinal κ of Mitchell order κ^{++}, then there exists a generic extension in which* GCH *holds below $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.* □

Theorem 36.5 (Magidor [1977a], Shelah [1983], Gitik [∞]). *Assume that there exists a supercompact cardinal.*

(i) *There is a generic extension in which* GCH *holds below $\aleph_\omega$, and $2^{\aleph_\omega} = \aleph_{\omega+\alpha+1}$, where α is any prescribed countable ordinal.*
(ii) *There is a generic extension in which $\aleph_{\omega_1}$ is strong limit and $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+\alpha+1}$ for any prescribed ordinal $\alpha < \omega_2$.*
(iii) *There is a generic extension in which* GCH *holds below the least fixed point of the aleph function $\kappa = \aleph_\kappa$ while 2^κ is any arbitrarily large prescribed successor cardinal.* □

Theorem 36.6 (Woodin, Cummings [1992]).

(i) *If there exists a supercompact cardinal, then there is a generic extension in which $2^\kappa = \kappa^{++}$ for each cardinal κ.*
(ii) *If there exists a strong cardinal, then there is a generic extension in which $2^\kappa = \kappa^+$ for each successor cardinal and $2^\kappa = \kappa^{++}$ for each limit cardinal.* □

There are additional results on the failure of SCH by Gitik, Shelah and others. The main open problem in this area is the following:

Problem 36.7. *Is it consistent that $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$?*

Compare this with Shelah's Theorem 24.33.

Violating SCH at $\aleph_\omega$

We shall now outline the Woodin-Gitik modification of Magidor's technique for getting a model in which $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$. First we describe the preparation forcing (which replaces Magidor's use of a supercompact cardinal):

Lemma 36.8. *Assume that there exists a measurable cardinal κ with $o(\kappa) = \kappa^{++}$.*

(i) *There is a model V that satisfies* GCH *and*

$$\exists j : V \to M,\ \mathrm{crit}(j) = \kappa,\ M^\kappa \subset M \text{ and } (\kappa^{++})^M = \kappa^{++}. \tag{36.9}$$

(ii) *There exists a model V with a measurable cardinal κ such that $2^\kappa = \kappa^{++}$ and a normal measure U on κ with $N = \mathrm{Ult}_U(V)$, and there exists a set $G \in V$ which is an N-generic filter on $\mathrm{Col}^N((\kappa^{+++})^N, <j_U(\kappa))$ (a Lévy collapse in N).*

Proof. (i) We outline how to get a model that satisfies (36.9). First one uses Gitik's forcing from [1989] to get a model of GCH that has a (κ, κ^{++})-extender E and a function $f : \kappa \to \kappa$ such that $j(f)(\kappa) = \kappa^{++}$ (the assumption of Theorem 36.2). Note that in M (where $j : V \to M$), κ^{++} is an inaccessible cardinal. To obtain (36.9), one uses the iteration of length κ, with Easton support, of Lévy collapses $\mathrm{Col}(\alpha^+, <f(\alpha))$, followed by the Lévy collapse $\mathrm{Col}(\kappa^+, <\kappa^{++})$. In this generic extension, the embedding $j : V \to M$ can be extended to an embedding that satisfies (36.9). (For details, see Gitik [1989].)

(ii) Starting with $j : V \to M$ that satisfies (36.9), we do Woodin's construction described in the proof of Theorem 36.2, except that by (36.9), we may assume that $f(\alpha) = \alpha^{++}$ for all α. In the resulting model, κ is measurable, $2^\kappa = \kappa^{++}$, and if U is the measure $\{X : \kappa \in j(X)\}$ given by the extended embedding j, then U has the required property. (Again, details are in Gitik's [1989].) □

For the rest of this section we assume that κ is a measurable cardinal with $2^\kappa = \kappa^{++}$, U is a normal measure on κ, $N = \mathrm{Ult}_U(V)$ and $j = j_U : V \to N$, and $G \in V$ is an N-generic filter on $\mathrm{Col}^N((\kappa^{+++})^N, <j(\kappa))$, the Lévy collapse in N. Magidor's forcing conditions are as follows:

(36.10) A forcing condition has the form $p = (\kappa_0, f_0, \kappa_1, f_1, \ldots, \kappa_{n-1}, f_{n-1}, A, F)$ where
- (i) $\kappa_0 < \kappa_1 < \ldots < \kappa_{n-1}$ are inaccessible cardinals $< \kappa$,
- (ii) $f_i \in \mathrm{Col}(\kappa_i^{+++}, <\kappa_i)$, for $i < n-1$ and $f_{n-1} \in \mathrm{Col}(\kappa_{n-1}^{+++}, <\kappa)$,
- (iii) $A \in U$,
- (iv) F is a function on A and $F(\alpha) \in \mathrm{Col}(\alpha^{+++}, <\kappa)$ for all $\alpha \in A$,
- (v) $[F]_U$, the element of $\mathrm{Col}((\kappa^{+++})^N, <j(\kappa))$ represented by F, belongs to G.

(36.11) A condition $p' = (\kappa'_0, f'_0, \ldots, \kappa'_{m-1}, f'_{m-1}, A', F')$ is stronger than p if
- (i) $m \geq n$,
- (ii) $\kappa'_i = \kappa_i$ for all $i < n$ and $\kappa'_i \in A$ for all i, $n \leq i < m$,
- (iii) $f'_i \supset f_i$ for all $i < n$ and $f'_i \supset F(\kappa'_i)$ for all i, $n \leq i < m$,
- (iv) $A' \subset A$,
- (v) $F'(\alpha) \supset F(\alpha)$ for all $\alpha \in A'$.

This forcing produces a Prikry sequence $\langle \kappa_n : n < \omega \rangle$ cofinal in κ. A consequence of (36.10)(v) is that the forcing satisfies the κ^+-chain condition (Exercise 36.9) and so (if κ is preserved) $2^\kappa = \kappa^{++}$ in the generic extension. The crucial property of this forcing is that the cardinals κ_n are preserved,

and $2^{\kappa_n} < \kappa_{n+1}$. Since all but finitely many cardinals between κ_n and κ_{n+1} are collapsed, there remain exactly ω cardinals between κ_0 and κ. Thus we can follow this generic extension by a Lévy collapse $\mathrm{Col}(\aleph_0, <\kappa_0)$ and in the resulting model we have $\kappa = \aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$ with $\aleph_\omega$ strong limit.

The key to preservation of the cardinals κ_n is an analog of Prikry's Lemma 21.12:

Lemma 36.9. *Let σ be a sentence of the forcing language and let $p = (\kappa_0, f_0, \ldots, \kappa_{n-1}, f_{n-1}, A, F)$ be a condition. Then there exists a stronger condition $p' = (\kappa_0, f'_0, \ldots, \kappa_{n-1}, f'_{n-1}, A', F')$ (with the same n) that satisfies σ.* □

We omit the proof of Lemma 36.9 as well as its application and refer the reader to Magidor [1977a] and Gitik [1989] for the details.

Radin Forcing

As a consequence of Jensen's Covering Theorem, and more generally of the inner model theory, large cardinals are necessary for any nontrivial change of cofinality (with the exception of Namba forcing, Theorem 28.10). Prikry forcing is the prime example of forcing that changes cofinality. In this section we describe its generalizations.

The first example, due to Magidor, generalizes Prikry forcing to change the cofinality of a large cardinal κ to a given regular cardinal $\lambda < \kappa$ while preserving κ as a cardinal:

Let λ be a regular cardinal and let $\kappa > \lambda$ be a measurable cardinal such that $o(\kappa) = \lambda$. Using an inner model for $o(\kappa) = \lambda$, we may assume that there exists a sequence

$$U_0 < U_1 < \ldots < U_\alpha < \ldots \qquad (\alpha < \lambda) \tag{36.12}$$

of normal measures on κ, ordered by the Mitchell order. For every $\alpha < \beta < \lambda$, let $f^\beta_\alpha : \kappa \to V_\kappa$ be the function that represents U_α in Ult_{U_β}, i.e., $[f^\beta_\alpha]_{U_\beta} = U_\alpha$.

(36.13) A forcing condition is a pair (g, G) such that
- (i) g is an increasing function from a finite subset of λ into κ,
- (ii) G is a function on λ such that $G(\alpha) \subset \kappa$ for all $\alpha < \lambda$,
- (iii) if $\alpha > \max(\mathrm{dom}\, g)$ then $G(\alpha) \in U_\alpha$,
- (iv) if $\alpha < \max(\mathrm{dom}\, g)$ and β is the least $\beta \in \mathrm{dom}(g)$ above α, then $G(\alpha) \in f^\beta_\alpha(g(\beta))$.

The finite function g plays the role of the finite sequence in the Prikry forcing. The function G plays the role of the measure one set: Clause (iii) is an obvious generalization while (iv) states that $G(\alpha)$ has measure one with

respect to the measure $f^\beta_\alpha(g(\beta))$ on $g(\beta)$, which reflects the properties of the measure U_α in the ultrapower by U_β.

(36.14) A condition (h, H) is stronger than (g, G) if
(i) $h \supset g$,
(ii) for every α, $H(\alpha) \subset G(\alpha)$,
(iii) for every $\alpha \in \mathrm{dom}(h) - \mathrm{dom}(g)$, $h(\alpha) \in G(\alpha)$.

Magidor's forcing (36.13) is a generalization of Prikry forcing. A generic filter yields a cofinal λ-sequence in κ (Exercise 36.10). Similarly to the Prikry forcing, (36.13) has a κ^+-chain condition (Exercise 36.11). The crucial property of Magidor's forcing is that it preserves cardinals. The proof is a generalization of the proof for the Prikry forcing and the following is the key lemma:

Lemma 36.10. *Let σ be a sentence of the forcing language and let (g, G) be a condition. Then there exists a stronger condition (g, H) (with the same g) that decides σ.* □

Magidor's forcing changes the cofinality of κ to $\mathrm{cf}\,\kappa = \lambda$, under the assumption that $o(\kappa) = \lambda$. If λ is uncountable then by Mitchell [1984] this assumption is necessary.

Radin forcing generalizes Prikry forcing further and uses objects called *measure sequences*. Let $j : V \to M$ be an elementary embedding with critical point κ. Let us define a sequence $\langle u(\alpha) : \alpha < \vartheta\rangle$ as follows:

$$\begin{aligned} &u(0) = \kappa, \\ &u(\alpha) = \{X \subset V_\kappa : u{\restriction}\alpha \in j(X)\} \qquad (\alpha > 0). \end{aligned} \tag{36.15}$$

The sequence $\langle u(\alpha)\rangle_\alpha$ is defined for all α for which $u{\restriction}\alpha \in M$. Thus the length ϑ depends on the strength of the embedding j. For example, if $j = j_U$ is the ultrapower embedding by a normal measure U on κ then $\lambda = 2$,

$$u(1) = \{X \in V_\kappa : \{\alpha : \langle\alpha\rangle \in X\} \in U\}$$

is a measure on V_κ concentrating on 1-sequences $\langle\alpha\rangle$, $\alpha < \kappa$, and $\langle u(0), u(1)\rangle \notin M = \mathrm{Ult}_U$. As long as $u(\alpha)$ is defined, $u(\alpha)$ is a measure on V_κ concentrating on α-sequences.

We define measure sequences as sequences obtained by (36.15) from elementary embeddings, but since we want the measures in measure sequences to concentrate on measure sequences, the definition is as follows.

Definition 36.11 (Measure Sequences). Let

$$\begin{aligned} \mathrm{MS}_0 &= \text{the class of all } u{\restriction}\alpha \text{ where } u \text{ is as in (36.15) for some} \\ &\qquad \text{elementary } j : V \to M, \\ \mathrm{MS}_{n+1} &= \{u \in \mathrm{MS}_n : (\forall\alpha > 0)\, \mathrm{MS}_n \cap V_{u(0)} \in u(\alpha)\}, \\ \mathrm{MS} &= \textstyle\bigcap_{n=0}^\infty \mathrm{MS}_n = \text{the class of all measure sequences.} \end{aligned}$$

Since all the measures are σ-complete it follows that for every measure sequence u and every $0 < \alpha < \text{length}(u)$, $u{\restriction}\alpha \in \text{MS}$ and $\text{MS} \cap V_{u(0)} \in u(\alpha)$. Clearly, some large cardinal assumption is necessary for the existence of nontrivial measure sequences. Under the assumption of a strong cardinal, there exist long measure sequences (Exercise 36.12).

Let U be a measure sequence of length at least 2, and let $\kappa = U(0)$. We associate with U the *Radin forcing* for U, R_U:

(36.16) A forcing condition $p \in R_U$ is a finite sequence

$$\langle (u_0, A_0), \ldots, (u_n, A_n) \rangle$$

such that $u_n = U$, and that, letting $\kappa_i = u_i(0)$ for $i = 0, \ldots, n$,
- (i) for each $i = 0, \ldots, n$, $u_i \in \text{MS}$, $A_i \subset \text{MS}$ and $A_i \in u_i(\alpha)$ for every $0 < \alpha < \text{length}(u_i)$,
- (ii) for each $i = 0, \ldots, n-1$, $(u_i, A_i) \in V_{\kappa_{i+1}}$.

(Thus $\kappa_0 < \kappa_1 < \ldots < \kappa_{n-1} < \kappa_n = \kappa$. These ordinals will produce a cofinal sequence in κ, a generalization of a Prikry sequence.)

(36.17) A forcing condition $p = \langle (u_0, A_0), \ldots, (u_n, A_n) \rangle$ is stronger than $q = \langle (v_0, B_0), \ldots, (v_m, B_m) \rangle$ if
- (i) $n \geq m$,
- (ii) $\{u_0, \ldots, u_n\} \supset \{v_0, \ldots, v_m\}$,
- (iii) for each $j = 0, \ldots, m$, if $u_i = v_j$ then $A_i \subset B_j$,
- (iv) for each i such that $u_i \notin \{v_0, \ldots, v_n\}$ if v_j is the first v_j such that $u_i(0) < v_j(0)$, then $u_i \in B_j$ and $A_i \subset B_j$.

If U is a measure sequence of length 2, then R_U is more or less the Prikry forcing (Exercise 36.13); if U has length 3, R_U produces a cofinal sequence of order type ω^2 (Exercise 36.14).

A generic filter G on R_U produces a set

(36.18) $D_G = \{u : \exists p \in G\ p = \langle (u_i, A_i) : i \leq n \rangle \text{ and } u = u_i \text{ for some } i < n\}$.

As in Prikry forcing, one proves that $V[D_G] = V[G]$. Let

(36.19) $$C_G = \{u(0) : u \in D_G\}.$$

It is not difficult to show:

Lemma 36.12. *C_G is a closed unbounded subset of κ.*

Proof. Exercise 36.15. □

When $\text{length}(U) < \kappa$, Radin forcing is similar to Magidor's forcing (36.13), see Exercise 36.16.

The analog of Prikry's Lemma 21.12 holds for Radin's forcing as well:

Lemma 36.13. *Let σ be a sentence of the forcing language and let $p = \langle (u_i, A_i) : i \leq n \rangle$ be a condition. Then there exists a stronger condition $q = \langle (u_i, B_i) : i \leq n \rangle$ (with the same $\{u_i : i \leq n\}$) that decides σ.* □

As a consequence, all cardinals are preserved in the generic extension.

Radin's forcing is more flexible than Magidor's forcing, and under suitable large cardinal assumptions, κ retains its regularity, or even its large cardinal properties:

Lemma 36.14. *If $\mathrm{cf}(\mathrm{length}(U)) > \kappa$ then κ remains regular in the forcing extension by R_U.* □

Lemma 36.15. *Let $j : V \to M$ and let $U \in \mathrm{MS}$ be defined from j as in* (36.15).

(i) *If j witnesses that κ is $(\kappa + 2)$-strong then κ remains measurable in V^{R_U}.*
(ii) *If j witnesses that κ is λ-supercompact then κ remains λ-supercompact.*

□

Among applications of Radin forcing is the following theorem:

Theorem 36.16 (Mitchell). *If $\exists \kappa\, o(\kappa) = \kappa^{++}$ is consistent then so is* ZF + DC + "*the closed unbounded forcing on $\aleph_1$ is an ultrafilter.*" □

Stationary Tower Forcing

We now describe the general version of the stationary tower forcing (Definition 34.10) which can be used, among others, to change cofinalities in a way that is not possible without very large cardinals.

Let A be an uncountable set. A set $S \subset P(A)$ is *stationary* in $P(A)$ if for every $F : [A]^{<\omega} \to A$, S contains a closure point of F, i.e., a set $X \subset A$ such that $F(e) \in X$ for all $e \in [X]^{<\omega}$. As in Theorem 8.27, projections and liftings of stationary sets are stationary. Also, the analog of Theorem 8.24 holds. For the relation to stationary sets in $P_\kappa(\lambda)$ see Exercise 36.17.

Definition 36.17 (Stationary Tower Forcing). Let δ be a Woodin cardinal. The forcing notion $P = P_{<\delta}$ consists of conditions (A, S) where $A \in V_\delta$ is uncountable and S is stationary in $P(A)$. (B, T) is stronger than (A, S) if $B \supset A$ and $T{\restriction}A \subset S$.

If G is a generic filter on $P_{<\delta}$ then we form the generic ultrapower $\mathrm{Ult}_G(V)$ as in (34.10). The general form of Theorem 34.14 is as follows:

Theorem 36.18 (Woodin [1988]). *Let δ be a Woodin cardinal. If G is generic on $P_{<\delta}$ then the generic ultrapower $\mathrm{Ult}_G(V)$ is well-founded and the model $\mathrm{Ult}_G(V)$ is closed under $< \delta$-sequences.* □

Forcing with $P_{<\delta}$ gives more flexibility than forcing with $Q_{<\delta}$ (from Definition 34.10). In a typical application, one can collapse a successor of a singular cardinal and give it any prescribed cofinality, see Example 36.19 below. In fact, the consistency strength of this is exactly that of a Woodin cardinal.

Example 36.19 (Woodin). Assume that $\aleph_\omega$ be strong limit, and let S be the following stationary set in $P(V_{\aleph_{\omega+1}})$:

$$S = \{X \in [V_{\aleph_{\omega+1}}]^{\aleph_\omega} : X \cap \aleph_{\omega+1} \in \aleph_{\omega+1} \text{ and } \mathrm{cf}(X \cap \aleph_{\omega+1}) = \aleph_3\}.$$

Let G be a generic filter on $P_{<\delta}$ such that $(V_{\aleph_{\omega+1}}, S) \in G$, let $M = \mathrm{Ult}_G(V)$ and let $j : V \to M$ be the generic ultrapower embedding. Then $\mathrm{crit}(j) = \aleph_{\omega+1}$ and $\mathrm{cf}^M \aleph_{\omega+1} = \aleph_3$ (Exercise 36.20). As $P^{V[G]}(\omega_n) = P^M(\omega_n) = P(\omega_n)$ for all n, the forcing $P_{<\delta}$ below $(V_{\aleph_{\omega+1}}, S)$ changes the cofinality of $\aleph_{\omega+1}$ to $\aleph_3$ while preserving $\aleph_\omega$. □

Exercises

36.1. Assume that κ is $(\kappa+2)$-strong, and $j : V \to M$ with critical point κ be such that $V_{\kappa+2} \subset M$. Then the function $f(\alpha) = \alpha^{++}$ $(\alpha < \kappa)$ satisfies $j(f)(\kappa) = \kappa^{++}$.

36.2. Let $j : V \to M$ and $f : \kappa \to \kappa$ be as in Theorem 36.2, and let E be the (κ, κ^{++})-extender derived from j. Then $j_E(f)(\kappa) = \kappa^{++}$ and $(\mathrm{Ult}_E)^\kappa \subset \mathrm{Ult}_E$.

36.3. Prove (36.4).
[Use the fact that $j = j_E$.]

36.4. The filter h_κ is $Q_\kappa^{N[G_\kappa]}$-generic over $N[G_\kappa]$.
[Use the $\mathrm{crit}(k)$-chain condition.]

36.5. The filter $G_\kappa * h_\kappa$ is $(i(P))_{\kappa+1}$-generic over V, and in $V[G_\kappa * h_\kappa]$,

$$(N[G_\kappa * h_\kappa])^\kappa \subset N[G_\kappa * h_\kappa].$$

[Use that $(i(P))_{\kappa+1}$ is κ^+-c.c.]

36.6. In $V[G_\kappa * h_\kappa]$ there exists an $N[G_\kappa * h_\kappa]$-generic filter on R.
[Use the fact that the number of antichains to meet is small, to build R.]

36.7. $k(H)$ is generic over $M[G_\kappa][h_\kappa]$.
[Use (36.4), or rather the corresponding description of $M[G_\kappa][h_\kappa]$. If D is an open dense set in $k(R)$, let $D = k(t)(a)$, where for each $x \in [\lambda]^{|a|}$, $t(x)$ is an open dense subset of R. Then use the fact that $\bigcap_x t(x)$ is open dense to show that $k(H)$ meets D.]

36.8. If Q is κ-closed then it remains κ-distributive in every κ^+-c.c. forcing extension.
[Let P be κ^+-c.c. Show that $\Vdash_Q P$ is κ^+-c.c., that the generics for P and Q are mutually generic, and that V^P and $V^{P*Q} = V^{Q*P}$ have the same κ-sequences of ordinals.]

36.9. The forcing (36.10) satisfies the κ^+-chain condition.
[Use (iii) and (v).]

36.10. If G is generic for the Magidor forcing (36.13) then in $V[G]$, κ has a cofinal subset of order-type λ.

36.11. The forcing (36.13) has the κ^+-chain condition.

36.12. Let κ be a $(\kappa+2)$-strong cardinal. Then there exists a measure sequence u of length $\vartheta \geq \kappa^{++}$.

36.13. Let $U \in \mathrm{MS}$ have length 2, $U = \langle \kappa, u(1) \rangle$. A condition $p \in R_U$ has the form $\langle (\alpha_0, \emptyset), \ldots, (\alpha_{n-1}, \emptyset), (u(1), A) \rangle$. Compare with the Prikry forcing.

36.14. Analyze R_U when $\mathrm{length}(U) = 3$.

36.15. Prove that C_G is a closed unbounded subset of κ.

36.16. Assume that $\mathrm{length}(U) = \lambda$ is a regular uncountable cardinal, $\lambda < \kappa$. Show that the order-type of C_G is λ.

36.17. Stationary sets in $P_\kappa(\lambda)$ are exactly the sets of the form $S \restriction \{X \in P_\kappa(\lambda) : X \cap \kappa \in \kappa\}$ where S is stationary in $P(\lambda)$.

36.18. Let $M = \mathrm{Ult}_G(V)$ where G is generic on $P_{<\delta}$, and let $j : V \to M$ be the generic ultrapower embedding. For each $(A, S) \in P_{<\delta}$, $(A, S) \Vdash j\text{“}A \in j(S)$.

36.19. Each $\alpha \leq \delta$ is represented in $\mathrm{Ult}_G(V)$ by the function $f_\alpha(x) = x \cap \alpha$.

36.20. In Example 36.19, $\mathrm{crit}(j) = \aleph_{\omega+1}$ and $\mathrm{cf}^M \aleph_{\omega+1} = \aleph_3$.

Historical Notes

Following the (unpublished) Theorem 36.2 of Woodin, Gitik proved in [1989] that $o(\kappa) = \kappa^{++}$ suffices for the consistency of a measurable cardinal κ with $2^\kappa = \kappa^{++}$, as well as for a model of "$\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$." In [1991], Gitik showed that the assumption $o(\kappa) = \kappa^{++}$ is necessary for the negation of SCH.

Methods for violating GCH at $\aleph_\omega$ were originated by Magidor in [1977a, 1977b]. Woodin (unpublished) improved the method by using a $(\kappa+2)$-strong cardinal, and Gitik [1989] obtained the result from $o(\kappa) = \kappa^{++}$.

In [1983], Shelah improved Magidor's method in the direction of getting an arbitrary countable gap between $\aleph_\omega$ and $2^{\aleph_\omega}$. In [1992], Gitik and Magidor introduced a novel method for blowing up the power of 2^κ for singular cardinals, leading to results that give the precise consistency strength (e.g., the large cardinal assumptions for Theorem 36.5 are considerably weaker than supercompactness).

In [1991], Foreman and Woodin constructed a model in which GCH fails everywhere; this was then improved by Woodin to Theorem 36.6(i), and Cummings followed with 36.6(ii).

Magidor's forcing for changing cofinality appeared in [1978]. For Radin's forcing, see Radin [1982]. Our presentation is based on improvements by Mitchell [1982], Woodin (unpublished) and Cummings [1992]. Theorem 36.16 is due to Mitchell [1982].

Stationary tower forcing is due to Woodin.

37. Martin's Maximum

This chapter is devoted to a generalization of the Proper Forcing Axiom. The stronger axiom is obtained by replacing "proper notion of forcing" in Definition 31.20 by "stationary set preserving notion of forcing." A notion of forcing P is *stationary set preserving* if every stationary set $S \subset \omega_1$ remains stationary in V^P.

Definition 37.1 (Martin's Maximum (MM)). If $(P, <)$ is a stationary set preserving notion of forcing and if $\mathcal{D}$ is a collection of $\aleph_1$ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.

As every proper notion of forcing is stationary set preserving, MM is a strengthening of PFA which in turn is a strengthening of MA. The axiom MM has been dubbed "Martin's Maximum" as it is ostensibly the strongest possible generalization of Martin's Axiom: If P is not stationary set preserving then the corresponding axiom for P is false (Exercise 37.1).

Below we establish the consistency of Martin's Maximum, and present some applications.

RCS iteration of semiproper forcing

The proof of the consistency of MM is modeled after the consistency proof of either MA or PFA: By iterated forcing one obtains a generic extension in which every stationary set preserving P satisfies the statement of MM.

The straightforward approach, iterating stationary set preserving forcings, does not work: If $g : \omega_1 \to \omega_1$ dominates the canonical functions f_η, $\eta < \omega_2$, (mod $I_{\rm NS}$) then there is a stationary preserving forcing notion P_g that produces a function $f < g$ mod $I_{\rm NS}$ (and still above the f_η) (Exercise 37.2). An ω-iteration of such forcing collapses ω_1.

It turns out that semiproper forcing can be iterated, yielding the consistency of the following principle:

Definition 37.2 (Semiproper Forcing Axiom (SPFA)). If $(P, <)$ is a semiproper notion of forcing and if $\mathcal{D}$ is a collection of $\aleph_1$ dense subsets of P then there exists a $\mathcal{D}$-generic filter on P.

Clearly, MM implies SPFA and SPFA implies PFA. In the next section we show that SPFA is in fact equivalent to MM and so for the consistency of MM it is enough to construct a model of SPFA.

While under special circumstances semiproperness may by preserved under countable support iteration (see Lemma 34.5), in general this is not the case. The reason is that a semiproper forcing notion may change the cofinality of ordinals from uncountable to countable. (An example of such forcing is the Prikry forcing, see Exercise 37.7.)

The iteration applicable to semiproper forcings is the *revised countable support* (RCS) iteration. Informally, a support of a condition is not just a countable set, but even a *name* for a countable set.

Definition 37.3. Let $\alpha \geq 1$. A forcing notion P_α is an RCS (revised countable support) iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$ if it is an iteration (cf. Definition 16.29) consisting of all α-sequences p that satisfy

(37.1) for every $q \leq p$ there exist a $\gamma < \alpha$ and an $r \leq_\gamma q{\restriction}\gamma$ such that $r \Vdash_\gamma \operatorname{cf}\alpha = \omega$ or $\forall \beta \geq \gamma\ p{\restriction}[\gamma,\beta) \Vdash_{P_{\gamma,\beta}} p(\beta) = 1$.

In (37.1), q ranges over elements of the inverse limit of the $\dot{Q}_\beta$, cf. (16.12), and $P_{\gamma,\beta}$ is the restriction of the inverse limit to the interval $[\gamma,\beta) = \{\xi : \gamma \leq \xi < \beta\}$.

The main result on RCS iterations is that they preserve semiproperness:

Theorem 37.4 (Shelah). *If P_α is an* RCS *iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$ such that every $\dot{Q}_\beta$ is a semiproper forcing notion in $V^{P_\alpha{\restriction}\beta}$ then P_α is semiproper.*

Theorem 37.4 can be proved along the lines of the proof of Theorem 31.15. We shall outline the proof of a special case of Theorem 37.4 (Proposition 37.8 below) which suffices for the consistency proof of SPFA. (To be precise, Shelah proved Theorem 37.4 for a more complicated definition of RCS iteration; the current Definition 37.3 is based on simplifications by Schlindwein and Donder).

A two-step iteration of semiproper forcings is semiproper, cf. Exercise 37.8. The proof of Theorem 37.4 proceeds by induction, showing

(37.2) for all $\gamma < \beta \leq \alpha$, $\Vdash_\gamma B_\beta : B_\gamma$ is semiproper;

here $B_\beta = B(P_\beta)$, and $B_\beta : B_\gamma$ is the complete Boolean algebra in V^{P_γ} such that $B_\gamma * (B_\beta : B_\gamma) = B_\beta$ (see Exercise 16.4). One property of RCS that is used in the proof is that $B_\beta : B_\gamma$ is (in V^{P_γ}) an RCS iteration (Exercise 37.9).

The following three lemmas, special cases of Theorem 37.4, can be proved in a similar way as Theorem 31.15:

Lemma 37.5. *Let P_ω be the inverse limit iteration of semiproper forcings $\{\dot{Q}_n : n < \omega\}$. Then P_ω is semiproper.* □

Lemma 37.6. *Let P_{ω_1} be a countable support iteration such that for all $\gamma < \beta < \alpha$, $\Vdash_\gamma B_\beta : B_\gamma$ is semiproper. Then P_{ω_1} is semiproper.* □

Lemma 37.7. *Let λ be a regular uncountable cardinal. Assume that*

(i) *P_λ is a direct limit,*
(ii) *for every $\alpha < \lambda$ of cofinality ω, P_α is the inverse limit,*
(iii) *for all $\gamma < \beta < \lambda$, $\Vdash_\gamma B_\beta : B_\gamma$ is semiproper,*
(iv) *P_λ satisfies the λ-chain condition.*

Then P_λ is semiproper. □

We shall now prove a version of Theorem 37.4 that will be used in the consistency proof of MM:

Proposition 37.8. *If P_α is an* RCS *iteration of semiproper forcings $\{\dot{Q}_\beta : \beta < \alpha\}$ such that for every $\beta < \alpha$, $\Vdash_{\beta+1} |P_\beta| \leq \aleph_1$, then P_α is semiproper.*

Proof. We proceed by induction, proving (37.2). As successor stages present no problem, let α be a limit ordinal. By the induction hypothesis, for every $\gamma < \beta < \alpha$, $\Vdash_\gamma B_\beta : B_\gamma$ is semiproper; we shall prove that P_α is semiproper, and (37.2) for α then follows by Exercise 37.8.

Case I. Let $p \in P$ and $\gamma < \alpha$ be such that $p{\restriction}\gamma \Vdash \operatorname{cf} \alpha = \omega$. We will show that $P{\restriction}p$ is semiproper. In this case, $p{\restriction}\gamma$ forces that $B_\alpha : B_\gamma$ is the inverse limit, and in fact, an inverse limit iteration of length ω of semiproper forcings, hence semiproper by Lemma 37.5. It follows that $P{\restriction}p$ is semiproper.

Case II. Let $p \in P$ be such that $\forall \gamma < \alpha\ p{\restriction}\gamma \Vdash \operatorname{cf} \alpha > \omega$, and let $\gamma < \alpha$ be such that $p{\restriction}\gamma \Vdash \operatorname{cf} \alpha = \omega_1$. Again, we will show that $P{\restriction}p$ is semiproper. In this case, $p{\restriction}\gamma$ forces that $B_\alpha : B_\gamma$ is a direct limit iteration of length ω_1 of semiproper forcings, and hence semiproper by Lemma 37.6. Therefore $P{\restriction}p$ is semiproper.

Case III. Let $p \in P$ be such that $\forall \gamma < \alpha\ p{\restriction}\gamma \Vdash \operatorname{cf} \alpha > \aleph_1$; we will show that in this case too, $P{\restriction}p$ is semiproper. This will complete the proof that P is semiproper.

Without loss of generality, assume that $p = 1$, and since P_α is in this case a direct limit of the P_β, it is a direct limit of the P_{β_i}, $i < \operatorname{cf} \alpha$ (where $\alpha = \lim_{i<\operatorname{cf}\alpha} \beta_i$), so we can assume that α is a regular cardinal. For every $\gamma < \alpha$, since $\Vdash_{\gamma+1} |P_\gamma| \leq \aleph_1 < \operatorname{cf} \alpha$, we have $|P_\gamma| < \alpha$. Also since $\Vdash_{\gamma+1} \operatorname{cf} \alpha > \aleph_1$, there is a stationary set of $\beta < \alpha$ (those for which $\forall \gamma \Vdash_\gamma \operatorname{cf} \beta \geq \aleph_1$) at which P_β is a direct limit. By Theorem 16.30, P_α satisfies the α-chain condition. Therefore P is semiproper by Lemma 37.7. □

Consistency of MM

Theorem 37.9 (Foreman, Magidor and Shelah). *If there exists a supercompact cardinal then there is a generic model that satisfies* MM.

Following the proof of Theorem 31.21, we construct a model that satisfies SPFA. Instead of proper forcings, we iterated semiproper forcings, and use the RCS iteration. At each stage α of the iteration, in addition to using the notion of forcing presented by the Laver function, we also collapse (with countable conditions) the cardinal $|P_\alpha|$ to $\aleph_1$. By Proposition 37.8, such iteration is semiproper. An argument similar to the one in the proof of Theorem 31.21 shows that the iteration up to a supercompact cardinal yields a model in which SPFA holds.

The consistency of MM then follows from this result:

Theorem 37.10 (Shelah). SPFA *implies that every stationary set preserving notion of forcing is semiproper. Therefore* SPFA *implies* MM.

Proof. Let X be a set of countable elementary submodels of $H_\lambda = (H_\lambda, \in, <)$. We denote $X^\perp$ the set

$$X^\perp = \{M \in [H_\lambda]^\omega : M \prec H_\lambda \text{ and } N \notin X \text{ for every countable } N \text{ that satisfies } M \prec N \prec H_\lambda \text{ and } N \cap \omega_1 = M \cap \omega_1\}. \tag{37.3}$$

As in Chapter 31, we call an *elementary chain* (of length $\vartheta \leq \omega_1$) a sequence $\langle M_\alpha : \alpha < \vartheta\rangle$ of countable elementary submodels of $(H_\lambda, \in, <)$ such that $M_\alpha \subset M_\beta$ and $M_\alpha \in M_\beta$ if $\alpha < \beta$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ if α is a limit ordinal. (Note that $\alpha \subset M_\alpha$ for every α.)

Lemma 37.11. *Assume* SPFA, *and let $\omega_1 \leq \kappa < \lambda$ with λ regular and sufficiently large. Let $Y \subset [H_\kappa]^\omega$ be stationary, and let $X = \{M \in [H_\lambda]^\omega : M \cap H_\kappa \in Y\}$ be the lifting of Y to H_λ. There exists an elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of submodels of $(H_\lambda, \in, <)$ such that $M_\alpha \in X \cup X^\perp$ for every α.*

Proof. Let P be the notion of forcing that shoots an elementary chain through $X \cup X^\perp$: Conditions are elementary chains $\langle M_\alpha : \alpha \leq \gamma\rangle$ in $X \cup X^\perp$ where γ is a countable ordinal; a stronger condition is an extension. We shall prove that P is semiproper; then, using SPFA applied to the dense sets $D_\xi = \{\langle M_\alpha : \alpha \leq \gamma\rangle : \gamma \geq \xi\}$ (cf. Exercise 37.10), we obtain an elementary chain of length ω_1 in $X \cup X^\perp$.

To show that P is semiproper, let $\mu > \lambda$ be sufficiently large, let $M \prec (H_\mu, \in, <)$ be countable, with $P \in M$, and let $p \in P \cap M$. It suffices (cf. Exercise 37.6) to find a $q \leq p$ that is (M, P)-semigeneric.

Claim 37.12. *There exists a countable N, $M \prec N \prec H_\mu$ such that $N \cap \omega_1 = M \cap \omega_1$ and $N \cap H_\lambda \in X \cup X^\perp$.*

Proof. If $M \cap H_\lambda \in X^\perp$ let $N = M$. Otherwise, there exists a countable $N' \prec H_\lambda$ such that $M \cap H_\lambda \subset N'$, $N' \cap \omega_1 = M \cap \omega_1$ and $N' \in X$. Let N be the Skolem hull of $M \cup (N' \cap H_\kappa)$ in $(H_\mu, \in, <)$. We claim that $N \cap H_\kappa = N' \cap H_\kappa$; hence $N \cap \omega_1 = M \cap \omega_1$ and $N \cap H_\kappa \in Y$, hence $N \cap H_\lambda \in X$.

The equality $N \cap H_\kappa = N' \cap H_\kappa$ holds because $N \cap H_\kappa \subset N'$: notice that for every Skolem function h for H_μ, $h \cap H_\kappa \in M \cap H_\lambda \subset N'$. □

Continuing the proof of Lemma 37.11, let N be as in Claim 37.12. We can find a decreasing sequence of conditions $p_n \in N$ with $p_0 = p$ such that $p_n = \langle M_\alpha : \alpha \leq \gamma_n \rangle$, such that every name for a countable ordinal in N is decided by some p_n (as an ordinal in N) and $\bigcup_{n=0}^\infty \bigcup_{\alpha \leq \gamma_n} M_\alpha = N \cap H_\lambda$. Let $\gamma = \bigcup_{n=0}^\infty \gamma_n$ and $M_\gamma = N \cap H_\lambda$. Since $N \cap H_\lambda \in X \cup X^\perp$, $q = \langle M_\alpha : \alpha \leq \gamma \rangle$ is a condition, and is (N, P)-semigeneric. Since $M \subset N$ and $M \cap \omega_1 = N \cap \omega_1$, q is (M, P)-semigeneric. □

Now we finish the proof of Theorem 37.10. Assuming SPFA, let Q be a stationary set preserving notion of forcing that is not semiproper. Let κ be sufficiently large (so that all Q-names for countable ordinals are in H_κ). Since Q is not semiproper, there exists some $p \in Q$ such that the set $Y = \{M \prec H_\kappa : \text{there exists no } (M, Q)\text{-semigeneric } q \leq p\}$ is stationary. Let $\lambda > \kappa$ be regular and let X be the lifting of Y to H_λ; since κ is sufficiently large, $X = \{M \prec H_\lambda : \text{there is no } (M, Q)\text{-semigeneric } q \leq p\}$. We may assume that $p = 1$ is the trivial condition.

By Lemma 37.11 there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ in $X \cup X^\perp$. We claim that the set $S = \{\alpha < \omega_1 : M_\alpha \in X\}$ is nonstationary. Assume that S is stationary and let G be a generic filter on Q. Since Q is stationary set preserving, S is stationary in $V[G]$. Let $\dot\delta_\xi$, $\xi < \omega_1$, enumerate all the names in $\bigcup_{\alpha < \omega_1} M_\alpha$ for countable ordinals. In $V[G]$, let

$$C = \{\alpha < \omega_1 : M_\alpha \cap \omega_1 = \alpha \text{ and } (\forall \xi < \alpha)[\dot\delta_\xi \in M_\alpha \text{ and } \dot\delta_\xi^G < \alpha]\}.$$

The set C is closed unbounded, and if $\alpha \in C$ then there exists some $q \in G$ such that for every $\dot\delta_\xi \in M_\alpha$, $q \Vdash (\exists \beta \in M_\alpha)\ \dot\delta_\xi = \beta$; therefore q is (M_α, Q)-semigeneric. Therefore S is nonstationary in $V[G]$, and hence in V.

Thus there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ in $X^\perp$. Let $\mu > \lambda$ be sufficiently large; we shall finish the proof by showing that for every countable $M \prec (H_\mu, \in, <, Q, \langle M_\alpha : \alpha < \omega_1 \rangle)$, for every $p \in M$ there exists an (M, Q)-semigeneric $q \leq p$.

Let M be such; if $\delta = M \cap \omega_1$, then $M \cap H_\lambda \supset M_\delta$ and $\delta = M_\delta \cap \omega_1$, and since $M_\delta \in X^\perp$ we have $M \cap H_\lambda \notin X$ and we are done. □

Applications of MM

The first application deals with cardinal arithmetic. Since MM implies PFA, it follows (by Theorem 31.23) that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. It turns out that from MM one can prove much more, including the Singular Cardinal Hypothesis:

Theorem 37.13 (Foreman, Magidor and Shelah). MM *implies that for every regular* $\kappa \geq \aleph_2$, $\kappa^{\aleph_1} = \kappa$.

Corollary 37.14. MM *implies* $2^{\aleph_0} = \aleph_2$. □

Corollary 37.15. MM *implies* SCH.

Proof. For every cardinal λ of cofinality ω, if $2^{\aleph_0} < \lambda$ then $\lambda^{\aleph_0} \leq (\lambda^+)^{\aleph_1} = \lambda^+$, and SCH follows by Silver's Theorem 8.13. □

Proof of Theorem 37.13. Let A_α, $\alpha < \kappa$, be disjoint stationary subsets of $E^\kappa_\omega = \{\xi < \kappa : \operatorname{cf} \xi = \omega\}$. We shall prove the following claim that implies the theorem: For every increasing $f : \omega_1 \to \kappa$ there exists an ordinal $\gamma_f < \kappa$ of cofinality ω_1 such that

$$\forall \alpha < \kappa \ \alpha \in \operatorname{ran}(f) \text{ if and only if } A_\alpha \cap \gamma_f \text{ is stationary.} \tag{37.4}$$

(It follows that $f \neq g$ implies $\gamma_f \neq \gamma_g$.)

Thus let $f : \omega_1 \to \kappa$ be an increasing function, and let S_α, $\alpha < \omega_1$, be disjoint stationary subsets of ω_1 such that $\bigcup_\alpha S_\alpha = \omega_1$ and that for every stationary S there exists an α such that $S \cap S_\alpha$ is stationary. We shall use MM to find a continuous increasing function $F : \omega_1 \to \kappa$ such that for every $\delta < \omega_1$, if $\delta \in S_\alpha$ then $F(\delta) \in A_{f(\alpha)}$.

Then if we let $\gamma_f = \sup_{\delta<\omega_1} F(\delta)$, $\gamma_f \cap \bigcup_{\alpha<\omega_1} A_{f(\alpha)}$ contains a closed unbounded set $\{F(\delta) : \delta < \omega_1\}$, and (37.4) holds.

Let P be the following notion of forcing: A condition is a continuous increasing function $p = \langle p(\delta) : \delta \leq \gamma\rangle$ where $\gamma < \omega_1$ such that for every $\delta \leq \gamma$, if $\delta \in S_\alpha$ then $p(\delta) \in A_{f(\alpha)}$. A stronger condition is an extension. We will show that P is stationary set preserving and that for every $\alpha < \omega_1$ the set $D_\alpha = \{p \in P : \alpha \in \operatorname{dom}(p)\}$ is dense. Then MM applied to the sets D_α produces the desired function F.

We prove the second claim first, by induction on α. Let α be a limit ordinal, and assume that all D_β, $\beta < \alpha$, are dense; let $p \in P$. Let γ be such that $\alpha \in S_\gamma$. Let λ be sufficiently large, and let $M \prec H_\lambda$ be a countable model with $P, p, \alpha \in M$ such that $\eta = \sup(M \cap \kappa) \in A_\gamma$ (M exists because A_γ is stationary). Let $\langle\alpha_n\rangle_n$ be an increasing sequence with limit α, and let $\langle\eta_n\rangle_n$ be an increasing sequence with limit η. We construct a sequence of conditions $p = p_0 \subset p_1 \subset \ldots \subset p_n \subset \ldots$, each $p_n \in M$, as follows: Given $p_n \in M$, $D_{\alpha_n} \in M$ is dense and so there exists a $p_{n+1} \in M$ such that $p_{n+1} \supset p_n$, $\alpha_n \in \operatorname{dom}(p_{n+1})$ and $p_{n+1}(\alpha_n + 1) \geq \eta_n$. The function $q = \bigcup_{n=0}^\infty p_n \cup \{(\alpha, \eta)\}$ is a condition, proving that D_α is dense.

Now we complete the proof by showing that P is stationary set preserving. Let S be a stationary subset of ω_1, let $p \in P$ and let $\dot C$ be a name for a closed unbounded set. We shall find a $q \leq p$ and some $\delta \in S$ such that $q \Vdash \delta \in \dot C$.

Let α be such that $S \cap S_\alpha$ is stationary. Let λ be sufficiently large and let $M \prec H_\lambda$ be a countable model with $P, p, \dot C \in M$ such that $\eta = \sup(M \cap \kappa) \in$

A_α and $\delta = M \cap \omega_1 \in S \cap S_\alpha$ (see Exercise 37.11). Let $\langle \alpha_n \rangle_n$ be an increasing sequence with limit δ and let $\langle \eta_n \rangle_n$ be an increasing sequence with limit η. As before, we find a sequence of conditions $p = p_0 \subset \ldots \subset p_n \subset \ldots$ in M such that $\alpha_n \in \mathrm{dom}(p_{n+1})$, $p_{n+1}(\alpha_n + 1) \geq \eta_n$, and such that for some $\beta_n \geq \alpha_n$ in M, $p_{n+1} \Vdash \beta_n \in \dot{C}$. The function $q = \bigcup_{n=0}^{\infty} p_n \cup \{(\delta, \eta)\}$ is a condition and since $\delta = \lim_n \beta_n$, we have $q \Vdash \delta \in \dot{C}$. □

Another important application of MM is the saturation of the nonstationary ideal on $\aleph_1$:

Theorem 37.16 (Foreman, Magidor and Shelah). MM *implies that the nonstationary ideal on* $\aleph_1$ *is* $\aleph_2$*-saturated.*

Proof. Assume MM and let $\{A_i : i \in W\}$ be a maximal almost disjoint collection of stationary subsets of ω_1. We shall find a set $Z \subset W$ of size $\leq \aleph_1$ such that $\sum_{i \in Z} A_i$ contains a closed unbounded set. That will prove that I_{NS} is $\aleph_2$-saturated.

Let P be the set of all pairs (q, p) such that

(37.5) (i) $q : \gamma + 1 \to W$ for some $\gamma < \omega_1$, and
(ii) $p \subset \omega_1$ is a closed countable set such that $\alpha \in p$ implies $\alpha \in \bigcup_{\xi < \alpha} A_{q(\xi)}$.

A condition (q', p') is stronger than (q, p) if $q' \supset q$ and p' is an end-extension of p. (See also (34.7).)

P can be viewed as a two-step iteration $Q * P_{\dot{S}}$ where Q collapses $|W|$ to $\aleph_1$ with countable conditions and $P_{\dot{S}}$ shoots a closed unbounded set through $\dot{S} = \sum_{i \in W} A_i$. P is stationary set preserving: If $A \subset \omega_1$ is stationary then for some $i \in W$, $A \cap A_i$ is stationary and remains stationary in V^Q. Hence $A \cap A_i \cap \dot{S}$ is stationary and remains stationary in V^P. Hence A is stationary in V^P. See Exercises 34.6 and 23.6.

For each $\alpha < \omega_1$, let $D_\alpha = \{(q, p) \in P : \alpha \leq \max(p)\}$. Each D_α is dense and so by MM there is a filter G on P that meets all the D_α. Let

$$F = \bigcup \{q : (q, p) \in G \text{ for some } p\}, \quad C = \bigcup \{p : (q, p) \in G \text{ for some } q\}.$$

The set C is closed unbounded, and is equal to the set $\{\alpha : (\exists \xi < \alpha)\ \alpha \in A_{F(\xi)}\} = \sum_{i \in \mathrm{ran}(F)} A_i$. □

Reflection Principles

An important consequence of MM are reflection principles. These combinatorial principles imply some major consequences of MM.

Definition 37.17 (Reflection Principle (RP)). For every regular $\lambda \geq \aleph_2$ the following holds:

RP(λ) If S is a stationary set in $[\lambda]^\omega$ then for every $X \subset \lambda$ of cardinality $\aleph_1$ there exists a $Y \subset \lambda$ of cardinality $\aleph_1$ such that $X \subset Y$ and that $S \cap [Y]^\omega$ is stationary in $[Y]^\omega$.

RP follows from Martin's Maximum, see Theorems 37.21 and 37.23 below. One consequence of RP is that every stationary set preserving notion of forcing is semiproper (Exercise 37.13). This in turn implies that $I_{\rm NS}$ on ω_1 is precipitous (Foreman, Magidor and Shelah [1988], Theorem 26) and is therefore a large cardinal property.

Theorem 37.18 (Todorčević). RP(ω_2) *implies that* $2^{\aleph_0} \leq \aleph_2$.

Proof. One can show that RP(λ) implies a stronger version of RP(λ), namely that $S \cap [Y]^\omega$ is stationary for stationary many $Y \in [\lambda]^{\aleph_1}$ (Exercise 37.14). If $\omega_1 \leq \alpha < \omega_2$, let C_α be a closed unbounded subset of $[\alpha]^\omega$ of order-type ω_1, and let $D = \bigcup_{\omega_1 \leq \alpha < \omega_2} C_\alpha$. Since $|C_\alpha| = \aleph_1$ for each α, we have $|D| = \aleph_2$. By RP(ω_2), D contains a closed unbounded set: Otherwise, if $S = [\omega_2]^\omega - D$ is stationary, there exists an $\alpha \geq \omega_1$ such that $S \cap [\alpha]^\omega$ is stationary, a contradiction.

By a theorem of Baumgartner and Taylor ([1982], Theorem 3.2(a)), every closed unbounded subset of $[\omega_2]^\omega$ has size at least $2^{\aleph_0}$. Therefore $2^{\aleph_0} \leq \aleph_2$. □

RP(ω_2) is not very strong; its consistency follows from a weakly compact cardinal (a modification of Theorem 23.23).

Definition 37.19. A set $S \subset [\lambda]^\omega$ is *projective stationary* if for every stationary set $T \subset \omega_1$, the set $\{X \in S : X \cap \omega_1 \in T\}$ is stationary.

(Equivalently, for every closed unbounded $C \subset [\lambda]^\omega$, the projection $(S \cap C)\restriction \omega_1$ contains a closed unbounded set.)

Definition 37.20 (Strong Reflection Principle SRP). For every regular $\lambda \geq \aleph_2$, the following holds:

SRP(λ) If S is projective stationary in $[H_\lambda]^\omega$ then there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable models such that $M_\alpha \in S$ for all α.

Theorem 37.21. MM *implies* SRP.

Proof. Let $S \subset [H_\kappa]^\omega$ be projective stationary. Let P be the forcing notion that shoots an elementary chain through S: Conditions are elementary chains $\langle M_\alpha : \alpha \leq \gamma \rangle$ where $\gamma < \omega_1$ and $M_\alpha \in S$ for all $\alpha \leq \gamma$. We will show that P is stationary set preserving; then if G is a filter on P that meets $\{p \in P : \alpha \in \mathrm{dom}(p)\}$ for each $\alpha < \omega_1$, $\bigcup G$ is an elementary chain in S.

Let $T \subset \omega_1$ be stationary, let $\dot{C}$ be a P-name for a closed unbounded set, and let $p \in P$. We shall find a $q \leq p$ and a $\delta \in T$ such that $q \Vdash \delta \in \dot{C}$. Let λ be sufficiently large and let $M \prec (H_\lambda, \in, <, P, \dot{C}, S, T, p)$ be a countable model such that $M \cap H_\kappa \in S$ and $\delta = M \cap \omega_1 \in T$. Let $p = p_0 \geq \ldots \geq p_n \geq \ldots$ be conditions in M such that for every open dense set $D \in M$, $p_n \in D$ for some n. If $p_n = \langle M_\alpha : \alpha \leq \gamma_n \rangle$ then $\delta = \lim_n \gamma_n$ and $M \cap H_\kappa = \bigcup_{n<\omega} M_{\gamma_n}$. If we let $q = \bigcup_{n<\omega} p_n \cup \{(\delta, M \cap H_\kappa)\}$, then q is a condition and $q \Vdash \delta \in \dot{C}$. □

Theorem 37.22. SRP *implies that the nonstationary ideal on* ω_1 *is* $\aleph_2$*-saturated.*

Proof. Assume SRP and let W be a maximal antichain of stationary subsets of ω_1. We will show that $|W| \leq \aleph_1$. Consider the set

$$S = \{M \in [H_{\omega_2}]^\omega : M \prec H_{\omega_2},\ W \in M \text{ and } \exists A \in W \cap M\ (M \cap \omega_1 \in A)\}.$$

We claim that S is projective stationary. Let $T \subset \omega_1$ be stationary and let $A \in W$ be such that $T \cap A$ is stationary. Let C be a closed unbounded set in $[H_{\omega_2}]^\omega$. Then there exists a model $M \in C$ such that $M \cap \omega_1 \in A \cap T$; hence S is projective stationary. By SRP there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ such that $M_\alpha \in S$ for all α. Let $M = \bigcup_{\alpha<\omega_1} M_\alpha$; we shall finish the proof by showing that $W \subset M$.

Let $A \in W$ and assume that $A \notin M$. Let N be the Skolem hull of $M \cup \{A\}$ and for each α, let N_α be the Skolem hull of $M_\alpha \cup \{A\}$. Let C be the closed unbounded set of all $\alpha < \omega_1$ such that $M_\alpha \cap \omega_1 = N_\alpha \cap \omega_1 = \alpha$, and let $\alpha \in C \cap A$. Since $M_\alpha \in S$, there exists some $B \in W \cap M_\alpha$ such that $\alpha \in B$. As $A \cap B$ is nonstationary and $A, B \in N_\alpha$, there exists a closed unbounded set $D \in N_\alpha$ such that $A \cap B \cap D = \emptyset$. This is a contradiction, since $\alpha = N_\alpha \cap \omega_1 \in D$, and also $\alpha \in A$ and $\alpha \in B$. □

Theorem 37.23. *For every regular* $\lambda \geq \omega_2$, SRP(λ) *implies* RP(λ).

Proof. Assuming SRP(λ) we prove a stronger version of RP(λ):

(37.6) If S is a stationary set in $[H_\lambda]^\omega$ then there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ such that $\{\alpha : M_\alpha \in S\}$ is stationary.

Let $S \subset [H_\lambda]^\omega$ be stationary. By Exercise 37.19 (since $I_{\rm NS}$ is $\aleph_2$-saturated by SRP(ω_2)), there exists a stationary $A \subset \omega_1$ such that for every stationary $B \subset A$, the set $\{M \in S : M \cap \omega_1 \in B\}$ is stationary. Therefore the set $\{M : M \in S \text{ or } M \cap \omega_1 \notin A\}$ is projective stationary, and by SRP(λ) contains an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$. It follows that $M_\alpha \in S$ for every $\alpha \in A$. □

We mention two other consequences of SRP: the Singular Cardinal Hypothesis (Todorčević, Exercise 37.20) and $2^{\aleph_0} = \aleph_2$ (Woodin [1999], Theorem 9.82, proves that SRP(ω_2) implies $\boldsymbol{\delta}^1_2 = \aleph_2$).

Forcing Axioms

Martin's Maximum (as well as MA and PFA) are principles that postulate the existence of sufficiently generic filter on every forcing notion of a given kind. In general, let $\mathcal{P}$ be a class of forcing notions.

Definition 37.24 (MA($\mathcal{P}$)). If P is a forcing notion in $\mathcal{P}$ and if $\{D_\alpha : \alpha < \omega_1\}$ are dense (or predense) subsets of P, then there exists a filter G on P that meets all the D_α.

Thus MA(c.c.c.) is $\mathrm{MA}_{\aleph_1}$, MA(proper) is PFA and MA(semiproper) = MA(stationary set preserving) is SPFA = MM. A useful strengthening of a given forcing axiom is the following:

Definition 37.25 (MA$^+$($\mathcal{P}$)). If P is a forcing notion in $\mathcal{P}$, if $\{D_\alpha : \alpha < \omega_1\}$ are dense (or predense) subsets of P and if $\dot{S}$ is a name of a subset of ω_1 such that $\Vdash \dot{S}$ is stationary, then there exists a filter G on P that meets all the D_α, and $\dot{S}^G = \{\alpha : \exists p \in G\ p \Vdash \alpha \in \dot{S}\}$ is stationary.

MM$^+$ is stronger than MM; its consistency follows by a modification of the proof of Theorem 37.9. While $\mathrm{MA}^+_{\aleph_1}$ is equivalent to $\mathrm{MA}_{\aleph_1}$ (Exercise 37.22), MA$^+$($\mathcal{P}$) is generally stronger than MA($\mathcal{P}$). A useful special case is MA$^+$(ω-closed). Among others, it implies the Reflection Principle RP and is therefore a large cardinal axiom. Moreover, it implies (37.6) (Exercise 37.23) and hence SCH. The following theorem shows that MA$^+$(ω-closed) follows from MM:

Theorem 37.26 (Shelah). MM *implies* MA$^+$(ω-closed).

Proof. Assume MM and let P be ω-closed, $\mathcal{D}$ a family of $\aleph_1$ dense subsets of P and $\dot{S}$ a P-name for a stationary set. Let $\{A_i : i \in W\}$ be a maximal antichain of those stationary sets for which $\Vdash_P A_i \cap \dot{S}$ is nonstationary. By MM, $|W| \le \aleph_1$. Let $A = \sum_{i\in W} A_i$ be the diagonal union and let $T = \omega_1 - A$. We have $\Vdash_P \dot{S} - T$ is nonstationary (hence T is stationary) and for every stationary $X \subset T$ there exists some $p \in P$ such that $p \Vdash \dot{S} \cap X$ is stationary.

Let Q be the countable support product of ω_1 copies of P; let $Q_\alpha = P$ and $\dot{S}_\alpha = \dot{S}$. For every stationary $X \subset T$ and every $q \in Q$ there exist some $q' \le q$ and $\alpha < \omega_1$ such that $q' \Vdash \dot{S}_\alpha \cap X$ is stationary. It follows that for every stationary $X \subset T$, $\Vdash_Q X \cap \sum_{\alpha<\omega_1} \dot{S}_\alpha$ is stationary. In V^Q, let $\dot{R}$ be the forcing notion that shoots a closed unbounded set $\dot{C}$ through $A \cup \sum_{\alpha<\omega_1} \dot{S}_\alpha$ (with countable conditions). It follows that $Q * \dot{R}$ preserves stationary sets.

By MM there exists a filter $G \times H$ on $Q * \dot{R}$ such that each $G_\alpha = G \restriction Q_\alpha$ is $\mathcal{D}$-generic, that for all α and β, G meets $\{q \in Q : q \text{ decides } \alpha \in \dot{S}_\beta\}$, and that for each α, $G \times H$ meets $\{(q, r) : \max(r) \ge \alpha\}$. Then $C = \dot{C}^{G\times H}$ is a closed unbounded set and $A \cup \sum_{\alpha<\omega_1} \dot{S}^G_\alpha \supset C$. Therefore there exists some α such that $\dot{S}^{G_\alpha}_\alpha$ is stationary, and MA$^+$(ω-closed) follows. □

While PFA is a large cardinal axiom and implies that $2^{\aleph_0} = \aleph_2$ there are weaker versions that do not need large cardinals, and are consistent with $\mathfrak{c} > \aleph_2$: For instance, there is a class $\mathcal{P}$ of proper forcings that includes, among others, the forcings for adding Cohen reals, Sacks reals, Mathias reals and Laver reals and MA($\mathcal{P}$) is consistent (relative to ZFC) with $2^{\aleph_0} > \aleph_2$ (Groszek and Jech [1991]).

Finally, forcing axioms can be further modified by restricting the size of predense sets that the filter should meet. If only D_α of size $\leq \aleph_1$ are involved, these are known as *bounded* forcing axioms:

Definition 37.27 (Bounded MA($\mathcal{P}$)). If P is a forcing notion in $\mathcal{P}$ and if $\{D_\alpha : \alpha < \omega_1\}$ are predense subsets of P such that $|D_\alpha| \leq \aleph_1$ for all α, then there exists a filter G on P that meets all the D_α.

The consistency strength of Bounded PFA is below a Mahlo cardinal (Goldstern and Shelah [1995]). An interesting equivalence for Bounded MM was proved by Bagaria:

Theorem 37.28 (Bagaria [2000]). *Bounded* MM *holds if and only if for every stationary set preserving forcing notion P,*

$$(H_{\omega_2}, \in) \prec_{\Sigma_1} (H_{\omega_2}, \in)^{V^P}. \qquad \square$$

Exercises

37.1. Let P be a notion of forcing such that for some stationary $S \subset \omega_1$, $\Vdash_P S$ is nonstationary. Then there exist $\aleph_1$ dense sets such that no filter G on P meets them all.

[Let $\dot{C}$ be a closed unbounded set in V^P such that $\Vdash_P S \cap \dot{C} = \emptyset$. For each $\alpha < \omega_1$, let $D_\alpha = \{p : (\exists \beta \geq \alpha)\ p \Vdash \beta \in \dot{C}\}$ and $E_\alpha = \{p :$ either $p \Vdash \alpha \in \dot{C}$ or $\exists \gamma < \alpha$ such that $p \Vdash \xi \notin \dot{C}$ for all ξ between γ and $\alpha\}$. If G meets all the D_α and E_α, let $C = \{\alpha : \exists p \in G\ p \Vdash \alpha \in \dot{C}\}$. Show that C is closed unbounded and so $S \cap C \neq \emptyset$; a contradiction.]

37.2. Let $f_\eta : \omega_1 \to \omega_1$, $\eta < \omega_2$, be the canonical ordinal functions, and let $g : \omega_1 \to \omega_1$ be such that $g > f_\eta$ mod $I_{\rm NS}$ for all η. A forcing condition in P_g is $(h, c, \{c_\eta : \eta \in A\})$ where $h : \alpha + 1 \to \omega_1$ for some $\alpha < \omega_1$, c and c_η are closed subsets of $\alpha + 1$, $A \subset \omega_2$ is countable, and $h < g$ on c, $h > f_\eta$ on c_η. The c's in a stronger condition are end-extensions. Show that P_g is stationary preserving.

[Shelah [1982], p. 255.]

Let $|A| \geq \aleph_2$. A set $C \subset [A]^\omega$ is *locally closed unbounded* if for closed unbounded many $X \in [A]^{\aleph_1}$, $C \cap [X]^\omega$ contains a closed unbounded set in $[X]^\omega$.

37.3. The filter of locally closed unbounded sets is a normal filter and extends the closed unbounded filter on $[A]^\omega$.

37.4. A notion of forcing P is stationary set preserving if and only if for every sufficiently large λ there is a locally closed unbounded set in $[H_\lambda]^\omega$ of countable elementary submodels such that $\forall p \in M\ \exists q \leq p\ q$ is (M, P)-semigeneric. (Compare with Definition 34.3).

[Feng and Jech [1989], Theorem 2.1.]

37.5. P is stationary set preserving if and only if for every $p \in P$ and every set X of names for countable ordinals such that $|X| = \aleph_1$, player II has a winning strategy in the following game: I plays $\dot\alpha_n \in X$, II plays β_n, and II wins if $\exists q \leq p\ q \Vdash \forall n\ \exists k\ (\dot\alpha_n = \beta_k)$. (Compare with Theorem 31.9 and Exercise 34.4.)

[Feng and Jech [1989], Theorem 2.1.]

37.6. P is semiproper if and only if for every $p \in P$, every sufficiently large λ and every countable $M \prec (H_\lambda, \in, <)$ containing P and p, there exists a $q \leq p$ that is (M, P)-semigeneric.

[As in Lemma 31.16.]

37.7. Show that the Prikry forcing is semiproper.

[Use Exercise 34.4.]

37.8. If P is semiproper and $\Vdash_P \dot Q$ is semiproper then $P * \dot Q$ is semiproper.

[As in Lemma 31.18, or use the semiproper game from Exercise 34.4.]

37.9. Let P_α be an RCS iteration, let $\gamma < \alpha$, and let $\dot P_\alpha^{(\gamma)}$ be an RCS iteration, in V^{P_γ}, of $\{\dot Q_\beta : \gamma \leq \beta < \alpha\}$. Then $V^{P_\alpha} = V^{P_\gamma * \dot P_\alpha^{(\gamma)}}$.

37.10. For every stationary $S \subset [H_\lambda]^\omega$ and every $\gamma < \omega_1$ there exists an elementary chain $\langle M_\alpha : \alpha \leq \gamma\rangle$ such that $M_\alpha \in S$ for all $\alpha \leq \gamma$.

[It suffices to show that such a chain exists in some V^P where P collapses H_λ with countable conditions. In V^P, consider an elementary chain with limit H_λ and apply Exercise 8.5.]

37.11. Let $S \subset \omega_1$ and $T \subset E^\kappa_\omega$ be stationary and let λ be sufficiently large. Then there exists a countable $M \prec H_\lambda$ such that $M \cap \omega_1 \in S$ and $\sup(M \cap \kappa) \in T$.

[There exists $N \prec H_\lambda$ of size $\aleph_1$ such that $\omega_1 \subset N$ and $\eta = \sup(N \cap \kappa) \in T$ (because T is stationary). Then (because S is stationary) there exists a countable $M \prec N$ with $\sup(M \cap \kappa) = \eta$ and $M \cap \omega_1 \in S$.]

37.12. MM implies that for every regular $\kappa \geq \omega_2$, every stationary $A \subset E^\kappa_\omega$ contains a closed set of order-type ω_1. (Compare with Exercise 8.5.)

[Let P be the set of all continuous increasing $\langle p(\alpha) : \alpha \leq \gamma\rangle$, $\gamma < \omega_1$, in A.]

37.13. RP implies that every stationary set preserving P is semiproper.

[Foreman, Magidor and Shelah [1988], Proposition 14.]

37.14. RP(λ) implies that for every stationary $S \subset [\lambda]^\omega$, the set $\{Y \subset \lambda : |Y| = \aleph_1$ and $S \cap [Y]^\omega$ is stationary$\}$ is stationary in $[\lambda]^{\aleph_1}$.

[Feng and Jech [1989], Theorem 3.1, (3) implies (2).]

37.15. RP(κ) implies that every stationary $A \subset E^\kappa_\omega$ reflects at some γ of cofinality ω_1. (Compare with Exercise 31.9.)

37.16. Let $\aleph_1 < \kappa < \lambda$.

(i) If $S \subset [\lambda]^\omega$ is projective stationary then $S{\restriction}\kappa$ is projective stationary.

(ii) If $S \subset [\kappa]^\omega$ is projective stationary then the lifting of S to λ is projective stationary.

37.17. Let $\kappa < \lambda$, $Y \subset [H_\kappa]^\omega$, let X be the lifting of Y to H_λ. Show that $X \cup X^\perp$ is projective stationary.

[Feng and Jech [1998], Claim 1.2. (Or, modify the proof of Claim 37.12.)]

37.18. SRP implies that for every regular $\kappa \geq \omega_2$, every stationary $A \subset E^\kappa_\omega$ contains a closed set of order-type ω_1.

[Apply SRP(κ) to the set $\{M : A \in M \text{ and } \sup(M \cap \kappa) \in M\}$.]

37.19. If $I_{\rm NS}$ is $\aleph_2$-saturated then for every stationary $S \subset [\lambda]^\omega$ there exists a stationary $A \subset \omega_1$ such that for every stationary $B \subset A$, the set $\{X \in S : X \cap \omega_1 \in B\}$ is stationary.

[For every stationary $A \subset \omega_1$, let $S_A = \{X \in S : X \cap \omega_1 \in A\}$, and let $W = \{A_\xi : \xi < \vartheta\}$, $\vartheta \leq \omega_1$, be a maximal antichain of stationary sets A such that S_A is not stationary. For each ξ let C_ξ be closed unbounded in $[\lambda]^\omega$ such that $S_{A_\xi} \cap C_\xi = \emptyset$. Let $A = \triangle_\xi(\omega_1 - A_\xi)$ and $C = \triangle_\xi C_\xi = \{X : (\forall \alpha \in X \cap \vartheta)\ X \in C_\alpha\}$. Since $C \cap S$ is stationary, A is stationary. A is as desired.]

37.20. SRP(κ) implies that $\kappa^{\aleph_1} = \kappa$.

[Let A_α ($\alpha < \kappa$), $f : \omega_1 \to \kappa$ and S_α ($\alpha < \omega_1$) be as in the proof of Theorem 37.13, and prove (37.4). The set $\{M : (\forall \alpha \in M \cap \omega_1)$ if $M \cap \omega_1 \in S_\alpha$ then $\sup(M \cap \kappa) \in A_{f(\alpha)}\}$ is projective stationary.]

37.21. SRP holds if and only if for all $\kappa < \lambda$ regular uncountable, if $S \subset [\kappa]^\omega$ is stationary then there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ such that $M_\alpha \cap \kappa \in S$ for every α for which there exists a countable M such that $M_\alpha \subset M \prec H_\lambda$, $M \cap \omega_1 = M_\alpha \cap \omega_1$, and $M \cap \kappa \in S$. (In other words, if T is the lifting of S to H_λ then $T \cup T^\perp$ contains an elementary chain. This reflection principle is due to Todorčević, see Bekkali [1991], p. 57.)

[Feng and Jech [1998], Theorem 1.2.]

37.22. $\mathrm{MA}_{\aleph_1}$ implies $\mathrm{MA}^+_{\aleph_1}$.

[Let P be c.c.c., $|\mathcal{D}| = \aleph_1$ (dense sets), and $\dot S$ a P-name for a stationary set. Let Q be the (finite support) product of ω copies of P; let $Q_n = P$ and $\dot S_n = \dot S$. Let $T = \{\alpha : \exists p \in P\ p \Vdash \alpha \in \dot S\}$ and show that $\Vdash_Q T = \bigcup_{n<\omega} \dot S_n$. Apply $\mathrm{MA}_{\aleph_1}$ to Q which is c.c.c. Let G be a filter on Q such that each $G_n = G \restriction Q_n$ is $\mathcal{D}$-generic, and that for every α and every n, G meets $\{q : q$ decides $\alpha \in \dot S_n\}$. Then $T = \bigcup_{n<\omega} \dot S_n^G$ and therefore there exist some n such that $\dot S_n^{G_n}$ is stationary.]

37.23. $\mathrm{MA}^+(\omega\text{-closed})$ implies that for every stationary $S \subset [H_\lambda]^\omega$ there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ such that $\{\alpha : M_\alpha \in S\}$ is stationary.

[Apply $\mathrm{MA}^+(\omega\text{-closed})$ to the ω-closed forcing that produces a generic chain $\langle M^G_\alpha : \alpha < \omega_1\rangle$ such that $\bigcup_{\alpha<\omega_1} M^G_\alpha = (H_\lambda)^V$ (the conditions being countable chains) and the canonical name for the stationary set $S_G = \{\alpha : M^G_\alpha \in S\}$.]

Historical Notes

Martin's Maximum was formulated by Foreman, Magidor and Shelah [1988]. The consistency proof (Theorem 37.9) as well as the major applications (Theorem 37.13, Theorem 37.16 and Definition 37.17) are in that paper. The method of RCS iteration

and Theorem 37.4 are due to Shelah and appear in [1982]. The (simplified) definition presented here follows Fuchs [1992] and Schlindwein [1993].

Theorem 37.10 was proved by Shelah in [1987].

Todorčević proved (in unpublished notes) that RP implies $2^{\aleph_0} \leq \aleph_2$; he also formulated a strong reflection principle (see Exercise 37.21) and used it to prove Theorems 37.21, 37.22 and 37.23, as well as SCH. The present version of SRP (Definition 37.20) is due to Feng and Jech [1998]; so is the equivalence in Exercise 37.21.

$\mathrm{MA}^+(\omega\text{-closed})$ and MM^+ are discussed in Foreman, Magidor and Shelah [1988]. Theorem 37.26 was proved by Shelah in [1987].

Exercises 37.1, 37.12 and 37.23: Foreman, Magidor and Shelah [1988].

Exercise 37.3: Feng and Jech [1989].

Exercises 37.10 and 37.19: Feng and Jech [1998].

Exercises 37.18 and 37.20: Todorčević.

Exercises 37.22: Baumgartner.

38. More on Stationary Sets

Stationary sets play a central role in several areas of set theory. In this final chapter we address some of the issues dealing with stationary sets.

The Nonstationary Ideal on $\aleph_1$

The question of considerable interest is whether the ideal $I_{\rm NS}$ on $\aleph_1$ can be $\aleph_2$-saturated. By Theorem 37.16, the saturation of $I_{\rm NS}$ follows from MM, and thus is consistent relative to a supercompact cardinal. This can be improved:

Theorem 38.1 (Shelah). *If there exists a Woodin cardinal then there is a generic model in which the nonstationary ideal on $\aleph_1$ is $\aleph_2$-saturated.*

Proof (sketch). The model is constructed by an RCS iteration (up to a Woodin cardinal), as in the proof of Theorem 37.9, iterating the forcings described in (37.5), for those maximal antichains for which the forcing (37.5) is semiproper. An argument similar to the one used in the proof of Theorem 34.8 shows that in the resulting model, $I_{\rm NS}$ is saturated. □

Combining this result with Steel's Theorem 35.21, it follows that the consistency strength of the saturation of $I_{\rm NS}$ is approximately that of the existence of a Woodin cardinal.

In contrast to that, the consistency strength of the precipitousness of $I_{\rm NS}$ is only that of the existence of a measurable cardinal (Theorems 22.33 and 23.10).

A σ-complete ideal I on ω_1 is *ω_1-dense* if the Boolean algebra $P(\omega_1)/I$ has a dense subset of size $\aleph_1$. Clearly, every (nontrivial) ω_1-dense ideal is $\aleph_2$-saturated. The following result (that we state without proof) shows that the consistency strength of "$I_{\rm NS}$ is ω_1-dense" is exactly the existence of infinitely many Woodin cardinals.

Theorem 38.2 (Woodin). *The following are equiconsistent*:

(i) *$I_{\rm NS}$ is ω_1-dense.*
(ii) AD *holds in $L(\boldsymbol{R})$.* □

The saturation of the nonstationary ideal implies (almost) that the Continuum Hypothesis fails:

Theorem 38.3 (Woodin). *If $I_{\rm NS}$ is $\aleph_2$-saturated and if there exists a measurable cardinal, then $\boldsymbol{\delta}^1_2 = \aleph_2$ (and hence $2^{\aleph_0} \geq \aleph_2$).* □

Note that the construction in the proof of Theorem 34.8 yields a model in which for some stationary set A, the ideal $I_{\rm NS}\restriction A$ is $\aleph_2$-saturated and the Continuum Hypothesis holds.

Saturation and Precipitousness

By Theorem 23.17, the nonstationary ideal on κ is not κ^+-saturated, for any $\kappa \geq \aleph_2$. The proof of Theorem 23.17 yields a somewhat stronger result: If κ and λ are regular cardinals such that $\lambda^+ < \kappa$, then $I_{\rm NS}\restriction E^\kappa_\lambda$ is not κ^+-saturated. Theorem 38.4 below shows that the saturation of $I_{\rm NS}\restriction {\rm Reg}$ is consistent (and not particularly strong). It remains open whether for a regular uncountable cardinal κ, $I_{\rm NS}\restriction E^{\kappa^+}_\kappa$ can be κ^{++}-saturated.

Let κ be a regular cardinal and let $\alpha < \kappa^+$. The cardinal κ is *α-Mahlo* if the order of κ (as defined in Chapter 8) is at least $\kappa + \alpha$. (Thus 0-Mahlo means weakly inaccessible, 1-Mahlo means weakly Mahlo, etc.)

Theorem 38.4.

(i) *Let κ be an α-Mahlo cardinal, with $0 < \alpha < \kappa^+$. If $I_{\rm NS}\restriction {\rm Reg}$ is κ^+-saturated then κ is a measurable cardinal of Mitchell order at least α in the model K^m.*

(ii) *Let κ be a measurable cardinal of Mitchell order α, with $0 < \alpha < \kappa^+$. There is a generic model in which κ is α-Mahlo and $I_{\rm NS}\restriction {\rm Reg}$ is κ^+-saturated.*

Proof. Cf. Jech and Woodin [1985]. For (i), see Exercise 38.1. □

By Theorem 23.10, the existence of a measurable cardinal is sufficient for the construction of a generic model in which the ideal $I_{\rm NS}$ on ω_1 is precipitous. The construction generalizes to obtain the precipitousness of $I_{\rm NS}\restriction E^{\kappa^+}_\kappa$, for every regular cardinal κ. For the precipitousness of the entire ideal $I_{\rm NS}$ on $\kappa \geq \aleph_2$, more than measurability is needed. For instance:

Theorem 38.5. *The following are equiconsistent*:

(i) *$I_{\rm NS}$ on $\aleph_2$ is precipitous.*

(ii) *There exists a measurable cardinal of Mitchell order 2.*

Proof. Cf. Gitik [1984]. For the lower bound, see Exercise 38.2. □

The consistency strength of the precipitousness of $I_{\rm NS}$ on $\kappa \geq \aleph_3$ is more than $o(\kappa) = \kappa^+$. In [1997], Gitik calculated the exact strength for successors or regulars, and nearly optimal lower and upper bounds for inaccessible κ (in both cases, it is the Mitchell order between κ^+ and κ^{++}). For successors of singulars the consistency strength is in the region of Woodin cardinals.

Reflection

Let $\kappa \geq \aleph_2$ be a regular cardinal. A stationary set $S \subset \kappa$ reflects at $\alpha < \kappa$ if $S \cap \alpha$ is stationary in α (see Definition 23.5). We shall now discuss briefly to what extent can stationary sets reflect.

First we consider the property "every stationary set $S \subset \kappa$ reflects (at some $\alpha < \kappa$)." This implies that κ is either (weakly) inaccessible or the successor of a singular cardinal, because if $\kappa = \lambda^+$ with λ regular, the set $E^{\lambda^+}_{\lambda}$ does not reflect (see Exercise 23.4). Let κ be an inaccessible cardinal. If κ is weakly compact then every stationary $S \subset \kappa$ reflects (Corollary 17.20). If $V = L$ then the converse is true as well: If every stationary set reflects then κ is weakly compact (Jensen [1972], Theorem 6.1).

Following Mekler and Shelah [1989], let us call κ a *reflecting cardinal* if there exists a normal ideal I on κ such that for every $X \in I^+$, $\{\alpha \in \kappa : X \text{ reflects at } \alpha\} \in I^+$. Every weakly compact cardinal is reflecting, and since being a reflecting cardinal is a Π^1_1 property (see Exercise 38.3), every weakly compact cardinal is a limit of reflecting cardinals.

Theorem 38.6. *The following are equiconsistent*:

(i) *There exists a cardinal κ such that every stationary $S \subset \kappa$ reflects.*
(ii) *There exists a reflecting cardinal.*

Proof. Mekler and Shelah [1989]. □

A cardinal κ is *greatly Mahlo* if κ is α-Mahlo for every $\alpha < \kappa^+$. If $V = L$ then every reflecting cardinal is greatly Mahlo and a limit of greatly Mahlo cardinals (Mekler and Shelah [1989]). Thus the consistency strength of "every stationary set reflects" is strictly between "greatly Mahlo" and "weakly compact."

Now let κ be a successor of a singular cardinal λ. The property "every stationary set $S \subset \lambda^+$ reflects" is a very large cardinal property. On the one hand there is this consistency result:

Theorem 38.7. *If there exist infinitely many supercompact cardinals, then there is a generic model in which every stationary set $S \subset \aleph_{\omega+1}$ reflects.*

Proof. Magidor [1982]. □

On the other hand, $\square_\lambda$ implies that there exists a stationary subset of λ^+ that does not reflect (Exercise 38.5). As $\square_\lambda$ holds in the core model K^{strong} then if $(\lambda^+)^{K^{\text{strong}}} = \lambda^+$, $\square_\lambda$ holds in V as well (with the same square sequence) and one concludes (by Theorem 35.19) that if λ is a strong limit singular cardinal and every $S \subset \lambda$ reflects then there exists an inner model for a strong cardinal. This has been extended by Schimmerling and others to show that the consistency strength of this reflection property is more than the existence of a Woodin cardinal.

Now consider the question of what is the largest possible extent of reflection. Let us recall (Definition 8.18) that $S < T$ means that S reflects at almost all $\alpha \in T$. If $S < T$ then $o(S) < o(T)$ and one may ask whether it is possible that $S < T$ holds whenever $o(S) < o(T)$. This is possible for $\kappa = \aleph_2$: By Magidor's Theorem 23.23 it is consistent that every stationary $S \subset E^{\omega_2}_{\omega}$ reflects at almost all α of cofinality ω_1.

For $\kappa > \aleph_2$ it is impossible that $S < T$ whenever $o(S) < o(T)$. If $\mu < \lambda$ are regular cardinals such that $\lambda^+ < \kappa$ then there exist stationary sets $S \subset E^{\kappa}_{\mu}$ and $A \subset E^{\kappa}_{\lambda}$ such that S does not reflect at any $\alpha \in A$ (Exercise 38.7). Thus let us restrict ourselves to reflection at regular cardinals.

Definition 38.8. A weakly inaccessible cardinal κ satisfies *full reflection* if for every stationary set $S \subset \kappa$ and every stationary set $T \subset \kappa$ of regular cardinals, $o(S) < o(T)$ implies $S < T$.

Obviously, the property is meaningful only if κ is at least a (weakly) Mahlo cardinal. The consistency strength of full reflection for cardinals in the Mahlo hierarchy has been established by Jech and Shelah. For instance:

Theorem 38.9. *The following are equiconsistent, for every $n \geq 1$:*

(i) *There exists an n-Mahlo cardinal that satisfies full reflection.*
(ii) *There exists a Π^1_n-indescribable cardinal.*

Proof. Jech and Shelah [1993]. See also Exercises 38.8 and 38.9. □

If κ is a large cardinal such as measurable, strong, or supercompact then there is a generic extension in which κ remains measurable (strong, supercompact) and in addition satisfies full reflection (Gitik and Witzany [1996]).

Stationary Sets in $P_\kappa(\lambda)$

By Theorem 8.28, the closed unbounded filter on $[\lambda]^\omega$ is generated by the sets $C_F = \{x \in [\lambda]^\omega : x \text{ is closed under } F\}$ where $F : [\lambda]^{<\omega} \to \lambda$. Thus in many applications one considers the space $[H_\lambda]^\omega$ and stationary sets are those $S \subset [H_\lambda]^\omega$ such that for every model $(H_\lambda, \in, \ldots)$ there exists an $M \in S$ with $M \prec (H_\lambda, \in, \ldots)$.

When $\kappa > \aleph_1$, the sets C_F do not generate the closed unbounded filter on $P_\kappa(\lambda)$ as the set $\{x \in P_\kappa(\lambda) : |x| \geq \aleph_1\}$ is closed unbounded and does not include any C_F (which contains a countable set). A generalization of Theorem 8.28 yields the following description of the closed unbounded filter: it is the filter generated by the sets C_F and the set $\{x \in P_\kappa(\lambda) : x \cap \kappa \in \kappa\}$ (Exercise 38.10; see also Exercises 8.18, 8.19 and 36.17). For more on this subject, see Exercises 38.11 and 38.12.

By Lemma 31.3, ω-closed forcing preserves stationary sets in $[\lambda]^\omega$. This does not generalize to $P_\kappa(\lambda)$ for $\kappa > \aleph_1$, as $<\kappa$-closed forcing may destroy

stationary sets in $P_\kappa(\lambda)$. The following concept is relevant to this problem and has other applications:

A model $M \prec H_\lambda$ is *internally approachable* if there exists an elementary chain $\langle M_\alpha : \alpha < \gamma\rangle$ with $M = \bigcup_{\alpha<\gamma} M_\alpha$ such that for every $\beta < \gamma$, $\langle M_\alpha : \alpha < \beta\rangle \in M$. In $P_\kappa(H_\lambda)$, let IA denote the set of all internally approachable M. The set IA is stationary, and if $\kappa = \aleph_1$ then IA contains a closed unbounded set (since every countable M is internally approachable). A stationary set S is preserved by $<\kappa$-closed forcing if and only if $S \cap IA$ is stationary (Exercises 38.13 and 38.14).

By Theorem 8.10, every stationary subset of κ can be partitioned into κ disjoint stationary sets. The situation is more complicated for $P_\kappa(\lambda)$. Since $|P_\kappa(\lambda)| = \lambda^{<\kappa}$ we may ask whether stationary sets in $P_\kappa(\lambda)$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets. This is generally not provable, for two reasons. One is cardinal arithmetic and the other are large cardinals.

The cardinal arithmetic reason is that there may exist stationary, or even closed unbounded, sets of size λ while $\lambda^{<\kappa} > \lambda$. For instance, there exists a stationary set $S \subset [\omega_2]^\omega$ of size $\aleph_2$ (Exercise 38.15), or a closed unbounded set $C \subset [\omega_4]^{\omega_2}$ of cardinality $\aleph_4^{\aleph_1}$ (Exercise 38.16).

A generalization of Solovay's proof of Theorem 8.10 gives that every stationary set in $P_\kappa(\lambda)$ can be partitioned into κ disjoint stationary sets (Exercise 38.18). This is best possible as Gitik [1985] constructs a model, using a supercompact cardinal, in which there is a stationary set $S \subset P_\kappa(\kappa^+)$ that cannot be partitioned into κ^+ disjoint stationary sets.

In view of this discussion, the following is best possible:

Theorem 38.10. *Let κ be regular uncountable and $\lambda \geq \kappa$.*

(i) *$P_\kappa(\lambda)$ can be partitioned into λ disjoint stationary sets.*
(ii) *If κ is a successor cardinal then every stationary subset of $P_\kappa(\lambda)$ can be partitioned into λ disjoint stationary sets.*
(iii) *If $0^\sharp$ does not exist then every stationary subset of $P_\kappa(\lambda)$ can be partitioned into λ disjoint stationary sets.*
(iv) *If* GCH *holds then $P_\kappa(\lambda)$ can be partitioned into $\lambda^{<\kappa}$ stationary sets, and if moreover $0^\sharp$ does not exist then every stationary subset of $P_\kappa(\lambda)$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

Proof. Let us consider the following set

$$E = \{x \in P_\kappa(\lambda) : |x \cap \kappa| = |x|\}. \tag{38.1}$$

It is easy to see that E is stationary and that if κ is a successor cardinal then E contains a closed unbounded subset (Exercise 38.19). The question whether, for an inaccessible κ, the complement of E is stationary, involves large cardinals; cf. the following lemma and Exercise 38.20.

Lemma 38.11. *If $\{x \in P_\kappa(\lambda) : |x \cap \kappa| < |x|\}$ is stationary then $0^\sharp$ exists.*

Proof. By the assumption there exists a model $M \in P_\kappa(L_\lambda)$ such that $\kappa \in M \prec L_\lambda$, $\kappa_M = M \cap \kappa \in \kappa$, and $\kappa_M < |M|$. Let L_α be the transitive collapse of M. Thus there is an elementary embedding $j : L_\alpha \to L_\lambda$ with critical point κ_M, and since $\kappa_M < |\alpha|$, $0^\sharp$ exists. □

Consequently, the following two lemmas will complete the proof of Theorem 38.10:

Lemma 38.12. *Every stationary subset of the set E can be partitioned into λ disjoint stationary sets.*

Lemma 38.13. *If* GCH *holds and if* $\operatorname{cf}\lambda < \kappa$ *then every stationary set in $P_\kappa(\lambda)$ can be partitioned into λ^+ disjoint stationary sets.*

Proof of Lemma 38.12. Assume that $\lambda > \kappa$ (if $\lambda = \kappa$ we have Theorem 8.10). Let S be a stationary set in $P_\kappa(\lambda)$ such that $|x \cap \kappa| = |x|$ for every $x \in S$. For each $x \in S$, let $f_x : x \to x \cap \kappa$ be one-to-one. For each $\alpha < \lambda$, let $g_\alpha(x) = f_x(\alpha)$, for all $x \in S$ with $\alpha \in x$. There exists a stationary set S_α such that g_α is constant on S_α, with value $\gamma_\alpha < \kappa$.

Now if μ is any regular cardinal with $\kappa < \mu \le \lambda$, there exists a $\gamma < \kappa$ such that $\gamma_\alpha = \gamma$ for μ many α's. The corresponding sets S_α are pairwise disjoint stationary subsets of S. Thus for every regular cardinal $\mu \le \lambda$, every stationary subset of E has μ pairwise disjoint stationary subsets. It follows easily that every $S \subset E$ can be partitioned into λ disjoint stationary sets. □

Proof of Lemma 38.13. Assume GCH and let $\lambda > \kappa$ be such that $\operatorname{cf}\lambda < \kappa$. First we note that $|P_\kappa(\lambda)| = \lambda^+$, and that every unbounded (and therefore every stationary) subset of $|P_\kappa(\lambda)|$ has size λ^+: If Y is unbounded then $P_\kappa(\lambda) = \bigcup_{x \in Y} P(x)$ and the assertion follows.

Let $\langle f_\alpha : \alpha < \lambda^+ \rangle$ enumerate the set of all functions $f_\alpha : [\lambda]^{<\omega} \to P_\kappa(\lambda)$ such that each function appears cofinally often. By Lemma 8.26, for every closed unbounded set C and every $\gamma < \lambda^+$ there exists an $\alpha > \gamma$ such that $C \supset C(f_\alpha) = \{x : f(e) \subset x \text{ whenever } e \subset x\}$.

Now let S be a stationary set in $P_\kappa(\lambda)$. By induction on $\alpha < \lambda^+$ we construct one-to-one sequences $\langle x^\alpha_\xi : \xi < \alpha \rangle$ such that $\{x^\alpha_\xi : \xi < \alpha\} \subset S \cap C(f_\alpha)$, and that $\{x^\alpha_\xi : \xi < \alpha\}$ and $\{x^\beta_\xi : \xi < \beta\}$ are disjoint whenever $\alpha \ne \beta$. If we let, for each $\xi < \lambda^+$, $S_\xi = \{x^\alpha_\xi : \xi < \alpha < \lambda^+\}$, the sets S_ξ, $\xi < \lambda^+$, are pairwise disjoint, and we complete the proof by showing that each S_ξ is stationary.

If C is closed unbounded, then $C \supset C(f_\alpha)$ for some $\alpha > \xi$, and since $x^\alpha_\xi \in S_\xi \cap C(f_\alpha)$, we have $S_\xi \cap C$ nonempty. □

By Theorem 23.17 the nonstationary ideal on κ is not κ^+-saturated, for any $\kappa \ge \aleph_2$. A similar result is true for the nonstationary ideal on $P_\kappa(\lambda)$:

Theorem 38.14. *If κ is a regular uncountable cardinal and $\lambda > \kappa$ then the nonstationary ideal on $P_\kappa(\lambda)$ is not λ^+-saturated.* □

The result follows easily from Theorem 23.17 when λ is regular: Let $\kappa < \lambda$ be regular uncountable. The proof of Theorem 23.17 shows that there are almost disjoint stationary sets $A_\xi \subset \lambda$, $\xi < \lambda^+$, such that $\operatorname{cf}\alpha < \kappa$ for all $\alpha \in A_\xi$ and all $\xi < \lambda^+$. For each ξ let $S_\xi = \{x \in P_\kappa(\lambda) : \sup x \in A_\xi\}$. Then S_ξ, $\xi < \lambda^+$, are stationary sets with $S_\xi \cap S_\eta$ nonstationary if $\xi \neq \eta$ (Exercise 38.21).

When λ is singular, the result is a combination of several cases, depending on κ and $\operatorname{cf}\lambda$. The nonsaturation of $I_{\rm NS}$ on $[\lambda]^\omega$ for singular λ is an application of the concept of mutually stationary sets that we shall briefly describe in the next section (see Corollary 38.17).

Mutually Stationary Sets

The following definition, due to Foreman and Magidor, exploits the fact that if κ is a regular cardinal and $\lambda > \kappa$ then a set $S \subset \kappa$ is stationary if and only if for every model $\mathfrak{A} = \langle H_\lambda, \in, \ldots\rangle$ there exists some $M \prec \mathfrak{A}$ such that $\sup(M \cap \kappa) \in S$; i.e., if and only if the set $\{M \in P(H_\lambda) : \sup(M \cap \kappa) \in S\}$ is stationary in $P(H_\lambda)$.

Definition 38.15. Let A be a set of regular cardinals and let $\lambda = \sup A$. The sets S_κ, $\kappa \in A$, where $S_\kappa \subset \kappa$ for each $\kappa \in A$, are *mutually stationary* if the set $\{M : \sup(M \cap \kappa) \in S_\kappa$ for every $\kappa \in M\}$ is stationary in $P(H_\lambda)$.

Not much is known about mutual stationarity beyond the following theorem:

Theorem 38.16 (Foreman-Magidor). *Let A be a set of regular cardinals with $\lambda = \sup A$. If for each κ, S_κ is a stationary subset of κ such that $\operatorname{cf}\alpha = \omega$ for every $\alpha \in S_\kappa$, then the S_κ are mutually stationary. For every $\mathfrak{A} = \langle H_\lambda, \in, \ldots\rangle$ there exists a countable $M \prec \mathfrak{A}$ such that $\sup(M \cap x) \in S_\kappa$ for every $\kappa \in M$.*

Proof. Foreman and Magidor [2001]. □

One consequence of this result is that the nonstationary ideal on $[\lambda]^\omega$ is not λ^+-saturated when λ is singular:

Corollary 38.17. *If λ is a limit cardinal then there exist stationary stationary sets S_ξ, $\xi < \lambda^{\operatorname{cf}\lambda}$, in $[\lambda]^\omega$ such that $S_\xi \cap S_\eta$ is nonstationary whenever $\xi \neq \eta$.*

Proof. Let $\mu = \operatorname{cf}\lambda$ and let $A = \{\kappa_\alpha : \alpha < \mu\}$ be a set of regular cardinals with limit λ. For each $\alpha < \mu$, let $\{S^\alpha_\beta : \beta < \kappa_\alpha\}$ be a partition of $E^{\kappa_\alpha}_\omega$ into κ_α disjoint stationary sets. For each function $f \in \prod_{\alpha<\mu}\kappa_\alpha$, let $S_f = \{M \in [\lambda]^\omega : \sup(M \cap \kappa_\alpha) \in S^\alpha_{f(\alpha)}$ for all $\alpha \in M\}$. The sets S_f are stationary in $[\lambda]^\omega$ and if $f \neq g$ then for any α with $f(\alpha) \neq g(\alpha)$, the closed unbounded set $\{M : \alpha \in M\}$ is disjoint from $S_f \cap S_g$. □

Weak Squares

The theory of inner models shows that in the absence of very large cardinals, Jensen's principle $\square_\kappa$ holds whenever κ is a singular cardinal. In this last section we take a look at some weaker versions of Square.

Definition 38.18. Let κ be an uncountable cardinal, and let ν be a cardinal, $1 \leq \nu \leq \kappa$; $\square_{\kappa,\nu}$ is as follows:

(38.2) $(\square_{\kappa,\nu})$ There exists a sequence $\langle \mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ such that
- (i) $\mathcal{C}_\alpha$ is nonempty and $|\mathcal{C}_\alpha| \leq \nu$, and each $C \in \mathcal{C}_\alpha$ is a closed unbounded subset of α;
- (ii) if $C \in \mathcal{C}_\alpha$ and $\beta \in \mathrm{Lim}(C)$ then $C \cap \beta \in \mathcal{C}_\beta$;
- (iii) if $\mathrm{cf}\,\alpha < \kappa$ then $|C| < \kappa$ for every $C \in \mathcal{C}_\alpha$.

The principle $\square_{\kappa,<\nu}$ is defined similarly, replacing $|\mathcal{C}_\alpha| \leq \nu$ by $|\mathcal{C}_\alpha| < \nu$.

The weakest principle of these, $\square_{\kappa,\kappa}$, is also denoted by $\square^*_\kappa$ and is called *Weak Square*. By Jensen [1972], $\square^*_\kappa$ is equivalent to the existence of a special Aronszajn κ^+-tree, and therefore, if κ is regular, $\square^*_\kappa$ follows from $2^\kappa = \kappa^+$. The main interest in the principles $\square_{\kappa,\nu}$ and $\square_{\kappa,<\nu}$ is in the case when κ is a singular cardinal. The failure of $\square^*_\kappa$ for κ singular (which, as mentioned below, entails a Woodin cardinal) is consistent: In [1979] Shelah proved the consistency, relative to a supercompact cardinal, of the negation of $\square^*_{\aleph_\omega}$.

The failure of Weak Square for singular κ has the consistency strength of (roughly) at least one Woodin cardinal: If there is a measurable cardinal and there is no inner model for the Woodin cardinal, then $\square_{\kappa,\mathrm{cf}\,\kappa}$ holds for every strong limit singular cardinal. This follows from results of Mitchell, Schimmerling and Steel; cf. Schimmerling [1995].

Exercise 38.5 shows that if $\square_\kappa$ holds then κ^+ has a nonreflecting stationary set. The proof is easily modified to show that $\square_{\kappa,<\omega}$ suffices, see Exercise 38.23. (In contrast, $\square_{\aleph_\omega,\omega}$ is consistent with "every stationary subset of $\aleph_{\omega+1}$ reflects;" cf. Cummings, Foreman and Magidor [2001].)

The proof of Theorem 31.28 can be modified to show that PFA implies the negation of $\square_{\kappa,\omega_1}$ for every $\kappa \geq \omega_1$; see Exercise 38.24. This, and the afore mentioned results on $\square_{\kappa,\mathrm{cf}\,\kappa}$ and Woodin cardinals yields Schimmerling's Theorem 31.30.

As a final application of weak squares we mention the following; for simplicity, let $\kappa = \aleph_\omega$. By the pcf theory there exists a scale $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in $\prod_{n\in A} \aleph_n$ (mod finite) for some $A \subset \omega$. If $\square_{\aleph_\omega,\aleph_m}$ holds for some m, then there exists a scale $\langle g_\alpha : \alpha < \aleph_{\omega+1}\rangle$ with this property:

(38.3) For every $\alpha < \aleph_{\omega+1}$ such that $\mathrm{cf}\,\alpha > \omega$ there exists a closed unbounded set $C \subset \alpha$ and some k such that for all $\beta < \gamma$ in C and all $n \geq k$ in A, $g_\beta(n) < g_\gamma(n)$.

(See Exercise 38.25.) Such scales were used in Cummings, Foreman and Magidor [2001] to prove (negative) results on stationary reflection.

Exercises

38.1. If κ is α-Mahlo and $I_{\rm NS}\restriction \mathrm{Reg}$ is κ^+-saturated, then κ has Mitchell order α in K^m.

[Use Theorem 35.16 and generic ultrapowers. Find an almost disjoint collection W of stationary subsets of Reg such that the length of $<$ restricted to W is at least α, and that the dual of $I_{\rm NS}\restriction S$ restricted to K^m is a normal measure $U_S \in K^m$. Then show that $S < T$ implies $K \vDash U_S < U_T$. For details, see Jech [1984]. (The symbol $<$ is used both for the hierarchy of stationary sets and for the Mitchell ordering.)]

38.2. If $I_{\rm NS}$ on $\aleph_2$ is precipitous then $\aleph_2$ is a measurable cardinal of Mitchell order 2 in K^m.

[Use Theorem 35.16 and generic ultrapowers, or see Gitik [1984].]

38.3. Let I_0 be the normal ideal generated by the sets that do not reflect, let $I_{\alpha+1}$ be the smallest normal ideal extending I_α that contains every X such that $\{\beta : X \text{ does not reflect at } \beta\} \in I_\alpha$, and let $I_\alpha = \bigcup_{\beta<\alpha} I_\alpha$ if α is limit. Then κ is a reflecting cardinal if and only if $\kappa \notin \bigcup_{\alpha<\kappa^+} I_\alpha$.

38.4. If κ is a reflecting cardinal then κ is a reflecting cardinal in L.

38.5. If $\square_\lambda$ holds then for every stationary set $S \subset \lambda^+$ there exists a stationary $T \subset S$ that does not reflect.

[Let $\langle C_\alpha : \alpha < \lambda^+\rangle$ be a square sequence, and let $f(\alpha)$ be the order-type of C_α. There is a stationary set $T \subset S$ on which f is constant. Show that T does not reflect (as in Lemma 23.6).]

38.6. If κ is supercompact and $\nu < \kappa < \lambda$ (regular cardinals) then every stationary $S \subset E^\lambda_\nu$ reflects.

[Compare with Exercise 27.3.]

38.7. Let $\mu < \lambda < \kappa$ be regular with $\lambda^+ < \kappa$. There exist stationary set $S \subset E^\kappa_\mu$ and $A \subset E^\kappa_\lambda$ such that S does not reflect at any $\alpha \in A$.

[As in Exercise 23.12.]

The Π^1_n *filter* on κ is the filter F^1_n generated by the sets $\{\alpha < \kappa : V_\alpha \vDash \varphi\}$ where φ is a Π^1_n formula true in V_κ; the Π^1_n ideal I^1_n is the dual ideal. κ is Π^1_n-indescribable if and only if the Π^1_n filter is a filter, i.e., $\kappa \notin I^1_n$.

38.8. Let κ be a Mahlo cardinal, let E_0 be the set of all inaccessible non Mahlo cardinals and assume that every stationary set $S \subset \kappa$ of singular cardinals reflects at almost all $\alpha \in E_0$. If $A \in L$ is a subset of κ such that $A \in I^1_1$ in L, then $A \cap E_0$ is nonstationary.

[Jeh and Shelah [1993], Lemma 2.1.]

As a consequence, if κ is Mahlo and satisfies full reflection, then κ is Π^1_1-indescribable in L. The following generalization implies that if κ is n-Mahlo and satisfies full reflection, then κ is Π^1_n-indescribable in L:

38.9. Let κ be an n-Mahlo cardinal that satisfies full reflection and let E_{n-1} be the set of all $\alpha < \kappa$ that are $(n-1)$-Mahlo but not n-Mahlo. If $A \in L$ is a subset of κ such that $A \in I^1_n$ in L, then $A \cap E_{n-1}$ is nonstationary.

38.10. For every closed unbounded set C in $P_\kappa(\lambda)$ there exists a function $F : [\lambda]^{<\omega} \to \lambda$ such that $C \supset \{x : x \cap \kappa \in \kappa \text{ and } F\text{“}[x]^{<\omega} \subset x\}$.

Let $[\lambda]^\nu = \{x \in P_{\nu^+}(\lambda) : |x| = \nu\}$. A set $C \subset [\lambda]^\nu$ is *strongly closed unbounded* if $C = C_F \cap [\lambda]^\nu$ for some $F : [\lambda]^{<\omega} \to \lambda$.

38.11. If the set $\{x \in [\lambda]^\nu : x \supset \nu\}$ contains a strongly closed unbounded set then every closed unbounded set $C \subset [\lambda]^\nu$ contains a strongly closed unbounded set.

38.12. The following are equivalent:

(i) The closed unbounded filter on $[\omega_2]^{\aleph_1}$ is generated by strongly closed unbounded sets.
(ii) Chang's Conjecture.

38.13. If $S \subset IA$ is stationary and if P is $<\kappa$-closed, then S is stationary in V^P.

38.14. Let P be the forcing that collapses $|H_\lambda|$ to κ (with conditions of size $< \kappa$). In V^P, the set $(IA)^V$ contains a closed unbounded set.

38.15. There exists a stationary set $S \subset [\omega_2]^\omega$ of size $\aleph_2$.

[For each $\alpha < \omega_2$, let $f : \alpha \to \omega_1$ be one-to-one. If $\alpha < \omega_2$ and $\xi < \omega_1$, let $X_{\alpha,\xi} = \{\beta < \alpha : f_\alpha(\beta) < \xi\}$. Let $S = \{X_{\alpha,\xi} : \alpha < \omega_2, \xi < \omega_1\}$.]

38.16. There exists a closed unbounded set $C \subset [\omega_4]^{\omega_2}$ of size $\aleph_4^{\aleph_1}$.

[Baumgartner [1991], Corollary 3.5.]

38.17. If X_α, $\alpha < \lambda$, are stationary sets in $P_\kappa(\lambda)$ such that $X_\alpha \cap X_\beta$ is nonstationary for all $\alpha \neq \beta$, then there exist pairwise disjoint stationary sets Y_α with $Y_\alpha \subset X_\alpha$ for all $\alpha < \lambda$.

[$Y_\alpha = X_\alpha \cap \{x : \alpha \in x$ and $\forall \beta \in x$ if $\beta \neq \alpha$ then $x \notin X_\beta\}$.]

38.18. For every stationary set $S \subset P_\kappa(\lambda)$ the ideal $I_{\rm NS} \restriction S$ is not κ-saturated.

[Gitik [1985], p. 893.]

38.19. Let $E = \{x \in P_\kappa(\lambda) : |x \cap \kappa| = |x|\}$.

(i) E is stationary.
(ii) If κ is a successor cardinal then E contains a closed unbounded subset.

38.20. If κ is supercompact and $\lambda > \kappa$ then the set $\{x \in P_\kappa(\lambda) : |x \cap \kappa| < |x|\}$ is stationary.

38.21. Let $\kappa < \lambda$ be regular uncountable and let $A \subset \lambda$ be such that $\operatorname{cf} \alpha < \kappa$ for all $\alpha \in A$. A is stationary if and only if $\{x \in P_\kappa(\lambda) : \sup x \in A\}$ is stationary in $P_\kappa(\lambda)$.

38.22. If κ is supercompact then for all $\lambda \geq \kappa$, $\square_{\lambda,<\kappa}$ fails.

38.23. If $\square_{\kappa,<\omega}$ holds then for every stationary $S \subset \kappa^+$ there exists a stationary $T \subset S$ that does not reflect.

[Let $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ be a $\square_{\kappa,<\omega}$ sequence and let $f(\alpha) = \{\text{o.t.}(C) : C \in \mathcal{C}_\alpha\}$. Then proceed as in Exercise 38.5.]

38.24. PFA implies that $\square_{\kappa,\omega_1}$ fails for every uncountable cardinal κ.

[Let $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ be a $\square_{\kappa,\omega_1}$ sequence, and let T be the tree of all (α, A) with $A \in \mathcal{C}_\alpha$, ordered by $(\alpha, A) \prec (\beta, B)$ if and only if $\alpha \in \operatorname{Lim}(B)$ and $A = B \cap \alpha$. Let $P = \{p \subset \kappa^+ : p$ is closed and countable$\}$, ordered by end-extension. In V^P, $T \restriction \bigcup \dot{G}$ has no ω_1-branch. Let $\dot{Q}$ be the c.c.c. forcing that specializes $T \restriction \bigcup \dot{G}$. Applying PFA to $P * \dot{Q}$ leads to a contradiction as in Theorem 31.28. For details, see Schimmerling [1995].]

38.25. Prove (38.3) using $\square_{\aleph_\omega,\aleph_m}$.

[Let $\langle \mathcal{C}_\alpha : \alpha < \aleph_{\omega+1}\rangle$ be a $\square_{\aleph_\omega,\aleph_m}$ sequence, and assume that $m < n$ for all $n \in A$. For a limit γ, let g_γ be such that $g_\gamma > g_\alpha$ for all $\alpha < \gamma$, $g_\gamma(n) > f_\gamma(n)$ for all $n \in A$, and such that for all $n \in A$, $g_\gamma(n) > \sup\{\sup_{\beta\in C} g_\beta(n) : C \in \mathcal{C}_\gamma, |C| < \aleph_n\}$.]

Historical Notes

The first consistency proof for the saturation of $I_{\rm NS}$ was obtained by Steel and van Wesep [1982], forcing over a model of ZF + $\mathrm{AD}_{\boldsymbol{R}}$ + "Θ is regular" ($\mathrm{AD}_{\boldsymbol{R}}$ is the determinacy of games where moves are real numbers). Following the proof of Theorem 37.16 (by Foreman, Magidor and Shelah), Shelah obtained the consistency from the existence of a Woodin cardinal (Theorem 38.1).

Theorem 38.2 will appear in the forthcoming book on AD, cf. Woodin et al. [∞]. Theorem 38.3 is proved in Woodin [1999].

Theorem 38.4: Jech and Woodin [1985].

Theorem 38.5: Gitik [1984].

Theorem 38.6: Mekler and Shelah [1989].

Theorem 38.9: Jech and Shelah [1993].

Theorem 38.10 is a combination of several results, including Jech [1972/73], Matsubara [1987], [1988] and [1990], Di Prisco and Baumgartner.

Lemma 38.11: Baumgartner.

Lemma 38.12: Di Prisco.

Lemma 38.13: Matsubara.

Theorem 38.14 is a combination of several results, including Gitik and Shelah [1997], Baumgartner and Taylor [1982], Donder and Matet [1993], Burke and Matsubara [1999] and Foreman and Magidor [2001].

Mutually stationary sets are investigated in Foreman and Magidor [2001]. For weak squares, see Schimmerling [1995] and Cummings, Foreman and Magidor [2001].

Exercises 38.3 and 38.4: Mekler and Shelah [1989].

Exercises 38.5: Jensen.

Exercises 38.6 and 38.22: Solovay.

Exercises 38.7: Shelah.

Exercises 38.8 and 38.9: Jech and Shelah [1993].

Exercises 38.10: Kueker.

Exercises 38.11, 38.12, 38.13 and 38.14: Foreman, Magidor and Shelah [1988].

Exercises 38.15, 38.16, 38.19 and 38.20: Baumgartner.

Exercises 38.24: Magidor.

Exercises 38.25: Cummings, Foreman and Magidor [2001].

Bibliography

U. Abraham

[∞] Proper forcing. *In* "Handbook of Set Theory" (A. Kanamori, M. Foreman, et al., eds.).

U. Abraham, M. Rubin, and S. Shelah

[1985] On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types. *Ann. Pure Appl. Logic* **29** (1985), no. 2, 123–206. MR 87d:03132.

U. Abraham and S. Shelah

[1985] Isomorphism types of Aronszajn trees. *Israel J. Math.* **50** (1985), no. 1-2, 75–113. MR 86i:03063.

J. W. Addison Jr.

[1959] Separation principles in the hierarchies of classical and effective descriptive set theory. *Fund. Math.* **46** (1959), 123–135. MR 24 #A1209.

J. W. Addison Jr. and Y. N. Moschovakis

[1968] Some consequences of the axiom of definable determinateness. *Proc. Natl. Acad. Sci. U.S.A.* **59** (1968), 708–712. MR 36 #4979.

P. S. Aleksandrov

[1916] Sur la puissance des ensembles mesurables B. *C. R. Acad. Sci. Paris* **162** (1916), 323–325.

P. S. Aleksandrov and P. S. Urysohn

[1929] Mémoire sur les espaces topologiques compacts. *Verh. Akad. Wetensch. Amsterdam* **14** (1929), 1–96.

J. Bagaria

[1994] Fragments of Martin's axiom and $\mathbf{\Delta}^1_3$ sets of reals. *Ann. Pure Appl. Logic* **69** (1994), no. 1, 1–25.

[2000] Bounded forcing axioms as principles of generic absoluteness. *Arch. Math. Logic* **39** (2000), no. 6, 393–401. MR 2001i:03103.

R. Baire

[1899] Sur les fonctions de variables réelles. *Annali di Math. (3)* **3** (1899), 1–123.

B. Balcar, T. J. Jech, and J. Zapletal

[1997] Semi-Cohen Boolean algebras. *Ann. Pure Appl. Logic* **87** (1997), no. 3, 187–208. MR 99a:03050.

S. Baldwin

[1986] Between strong and superstrong. *J. Symbolic Logic* **51** (1986), no. 3, 547–559. MR 87m:03072.

S. Banach
[1930] Über additive Massfunktionen in abstrakten Mengen. *Fund. Math.* **15** (1930), 97–101.

S. Banach and K. Kuratowski
[1929] Sur une géneralisation du problème de la mesure. *Fund. Math.* **14** (1929), 127–131.

T. Bartoszyński
[1984] Additivity of measure implies additivity of category. *Trans. Amer. Math. Soc.* **281** (1984), no. 1, 209–213. MR 85b:03083.
[1987] Combinatorial aspects of measure and category. *Fund. Math.* **127** (1987), no. 3, 225–239. MR 88m:04001.
[∞] Invariants of measure and category. *In* "Handbook of Set Theory" (A. Kanamori, M. Foreman, et al., eds.).

J. E. Baumgartner
[1976] Almost disjoint sets, the dense set problem and the partition calculus. *Ann. Math. Logic* **9** (1976), no. 4, 401–439. MR 53 #5299.
[1980] Chains and antichains in $\mathcal{P}(\omega)$. *J. Symbolic Logic* **45** (1980), no. 1, 85–92. MR 81d:03054.
[1983] Iterated forcing. *In* "Surveys in Set Theory" (A. R. D. Mathias, ed.). *London Math. Soc. Lecture Note Ser.* **87**, Cambridge University Press, Cambridge-New York, 1983, 1–59. MR 87c:03099; the entire collection MR 86m:03005.
[1984] Applications of the Proper Forcing Axiom. *In* "Handbook of Set-Theoretic Topology" (K. Kunen and J. E. Vaughan, eds.). North-Holland Publishing Co., Amsterdam-New York, 1984, 913–959. MR 86g:03084; the entire collection MR 85k:54001.
[1991] On the size of closed unbounded sets. *Ann. Pure Appl. Logic* **54** (1991), no. 3, 195–227. MR 92m:03072.

J. E. Baumgartner and A. Hajnal
[1973] A proof (involving Martin's axiom) of a partition relation. *Fund. Math.* **78** (1973), no. 3, 193–203. MR 47 #8310.

J. E. Baumgartner, A. Hajnal, and A. Máté
[1975] Weak saturation properties of ideals. *In* "Infinite and Finite Sets" (A. Hajnal et al., eds.), Colloq., Keszthely, 1973; Dedicated to P. Erdős on his 60th birthday, Vol. I. *Coll. Math. Soc. János Bolyai* **10**, North-Holland Publishing Co., Amsterdam, 1975, pp. 137–158. MR 51 #5317; the entire collection MR 50 #12526.

J. E. Baumgartner, L. A. Harrington, and E. M. Kleinberg
[1976] Adding a closed unbounded set. *J. Symbolic Logic* **41** (1976), no. 2, 481–482. MR 55 #7782.

J. E. Baumgartner, J. I. Malitz, and W. N. Reinhardt
[1970] Embedding trees in the rationals. *Proc. Natl. Acad. Sci. U.S.A.* **67** (1970), 1748–1753. MR 47 #3172.

J. E. Baumgartner and K. L. Prikry
[1976] On a theorem of Silver. *Discrete Math.* **14** (1976), no. 1, 17–21. MR 52 #7908.
[1977] Singular cardinals and the generalized continuum hypothesis. *Amer. Math. Monthly* **84** (1977), no. 2, 108–113. MR 55 #2575.

J. E. Baumgartner and A. D. Taylor
[1982] Saturation properties of ideals in generic extensions. I. *Trans. Amer. Math. Soc.* **270** (1982), no. 2, 557–574. MR 83k:03040a.

J. E. Baumgartner, A. D. Taylor, and S. Wagon
[1977] On splitting stationary subsets of large cardinals. *J. Symbolic Logic* **42** (1977), no. 2, 203–214. MR 58 #21619.

M. Bekkali
[1991] "Topics in Set Theory." *Lecture Notes in Mathematics* **1476**, Springer-Verlag, Berlin, 1991. MR 92m:03070.

I. Bendixson
[1883] Quelques théorèmes de la théorie des ensembles de points. *Acta Math.* **2** (1883), 415–429.

P. Bernays
[1937] A system of axiomatic set theory. I. *J. Symbolic Logic* **2** (1937), 65–67. Zbl. 019.29403.
[1942] A system of axiomatic set theory. III. Infinity and enumerability. Analysis. *J. Symbolic Logic* **7** (1942), 65–89. MR 3,290c.

D. H. Blackwell
[1967] Infinite games and analytic sets. *Proc. Natl. Acad. Sci. U.S.A.* **58** (1967), 1836–1837. MR 36 #4518.

A. Blass
[1977] A model without ultrafilters. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **25** (1977), no. 4, 329–331. MR 57 #16070.

G. Bloch
[1953] Sur les ensembles stationnaires de nombres ordinaux et les suites distinguées de fonctions régressives. *C. R. Acad. Sci. Paris* **236** (1953), 265–268. MR 14,733c.

G. Boolos
[1970] On the semantics of the constructible levels. *Z. Math. Logik Grundlagen Math.* **16** (1970), 139–148. MR 42 #4391.

D. Booth
[1970/71] Ultrafilters on a countable set. *Ann. Math. Logic* **2** (1970/71), no. 1, 1–24. MR 43 #3104.

É. Borel
[1898] "Leçons sur la Théorie des Fonctions." Gauthier-Villars, Paris, 1898.
[1905] "Leçons sur les Fonctions de Variables Réelles." Gauthier-Villars, Paris, 1905.
[1919] Sur la classification des ensembles de mesure nulle. *Bull. Soc. Math. France* **47** (1919), 97–125.

L. Bukovský
[1965] The continuum problem and the powers of alephs. *Comment. Math. Univ. Carolinae* **6** (1965), 181–197. MR 32 #1129.
[1973] Changing cofinality of a measurable cardinal. (An alternative proof.). *Comment. Math. Univ. Carolinae* **14** (1973), 689–698. MR 49 #2383.
[1976] Changing cofinality of $\aleph_2$. *In* "Set Theory and Hierarchy Theory" (W. Marek, M. Srebrny, and A. Zarach, eds.), Proc. Second Conf., Bierutowice, 1975. *Lecture Notes in Mathematics* **537**, Springer, Berlin, 1976, pp. 37–49. MR 55 #5441; the entire collection MR 54 #2412.
[1977] Iterated ultrapower and Prikry's forcing. *Comment. Math. Univ. Carolinae* **18** (1977), no. 1, 77–85. MR 56 #5295.

J. P. Burgess
[1978] Equivalences generated by families of Borel sets. *Proc. Amer. Math. Soc.* **69** (1978), no. 2, 323–326. MR 57 #16084.

D. R. Burke and Y. Matsubara
[1999] The extent of strength in the club filters. *Israel J. Math.* **114** (1999), 253–263. MR 2001h:03081.

M. R. Burke and M. Magidor
[1990] Shelah's pcf theory and its applications. *Ann. Pure Appl. Logic* **50** (1990), no. 3, 207–254. MR 92f:03053.

G. Cantor
[1874] Über eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen. *J. f. Math.* **77** (1874), 258–262.
[1878] Ein Bertrag zur Mannigfaltigkeitslehre. *J. f. Math.* **84** (1878), 242–258.
[1932] "Gesammelte Abhandlungen Mathematischen und Philosophischen Inhalts. Mit Erläuternden Anmerkungen Sowie mit Ergänzungen aus dem Briefwechsel Cantor-Dedekind. Herausgegeben. von Ernst Zermelo. Nebst einem Lebenslauf Cantors von Adolf Fraenkel." Julius Springer VII, Berlin, 1932. Zbl. 004.05401; Georg Olms Verlagsbuchhandlung, Hildesheim, 1962, MR 26 #6024.

D. M. Carr
[1982] The minimal normal filter on $P_\kappa\lambda$. *Proc. Amer. Math. Soc.* **86** (1982), no. 2, 316–320. MR 84h:03109.

C. C. Chang and H. J. Keisler
[1973] "Model Theory." *Stud. Logic Foundations Math.* **73**, North-Holland Publishing Co., Amsterdam-London, 1973. MR 53 #12927.

P. J. Cohen
[1963] The independence of the continuum hypothesis. *Proc. Natl. Acad. Sci. U.S.A.* **50** (1963), 1143–1148. MR 28 #1118.
[1964] The independence of the continuum hypothesis. II. *Proc. Natl. Acad. Sci. U.S.A.* **51** (1964), 105–110. MR 28 #2962.
[1966] "Set Theory and the Continuum Hypothesis." W. A. Benjamin, Inc., New York-Amsterdam, 1966. MR 38 #999.

W. W. Comfort and A. W. Hager
[1972] Cardinality of $\mathfrak{k}$-complete Boolean algebras. *Pacific J. Math.* **40** (1972), 541–545. MR 46 #7112.

J. Cummings
[1992] A model in which GCH holds at successors but fails at limits. *Trans. Amer. Math. Soc.* **329** (1992), no. 1, 1–39. MR 92h:03076.

J. Cummings, M. Foreman, and M. Magidor
[2001] Squares, scales and stationary reflection. *J. Math. Logic* **1** (2001), no. 1, 35–98.

M. Davis
[1964] Infinite games of perfect information. *In* "Advances in Game Theory" (M. Dresher, L. S. Shapley, and A. W. Tucker, eds.). *Ann. of Math. Studies*, No. 52, Princeton University Press, Princeton, N.J., 1964, 85–101. MR 30 #965; the entire collection MR 28 #4952.

R. Dedekind
[1872] "Stetigkeit und Irrationale Zahlen." Braunschweig, 1872.

P. Dehornoy
[1975] Solution d'une conjecture de Bukovsky. *C. R. Acad. Sci. Paris Sér. A-B* **281** (1975), no. 20, Ai, A821–A824. MR 53 #12947.
[1978] Iterated ultrapowers and Prikry forcing. *Ann. Math. Logic* **15** (1978), no. 2, 109–160. MR 80k:03057.

K. J. Devlin
[1984] "Constructibility." *Perspectives in Mathematical Logic*, Springer-Verlag, Berlin-New York, 1984. MR 85k:03001.

K. J. Devlin and R. B. Jensen
[1975] Marginalia to a theorem of Silver. *In* "⊨ISILC Logic Conference" (G. H. Müller, A. Oberschelp, and K. Potthoff, eds.), Proc. Internat. Summer Inst. and Logic Colloq., Kiel, 1974. *Lecture Notes in Mathematics* **499**, Springer, Berlin, 1975, pp. 115–142. MR 58 #235; the entire collection MR 52 #10365.

K. J. Devlin and S. Shelah
[1978] A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. *Israel J. Math.* **29** (1978), no. 2-3, 239–247. MR #57 9537.

A. J. Dodd
[1982] "The Core Model." *London Math. Soc. Lecture Note Ser.* **61**, Cambridge University Press, Cambridge-New York, 1982. MR 84a:03062.
[1983] Core models. *J. Symbolic Logic* **48** (1983), no. 1, 78–90. MR 84g:03081.

A. J. Dodd and R. B. Jensen
[1981] The core model. *Ann. Math. Logic* **20** (1981), no. 1, 43–75. MR 82i:03063.
[1982a] The covering lemma for K. *Ann. Math. Logic* **22** (1982), no. 1, 1–30. MR 83i:03082a.
[1982b] The covering lemma for $L[U]$. *Ann. Math. Logic* **22** (1982), no. 2, 127–135. MR 83i:03082b.

H.-D. Donder and P. Matet
[1993] Two cardinal versions of diamond. *Israel J. Math.* **83** (1993), no. 1-2, 1–43. MR 94k:03065.

E. K. van Douwen
[1984] The integers and topology. *In* "Handbook of Set-Theoretic Topology" (K. Kunen and J. E. Vaughan, eds.). North-Holland Publishing Co., Amsterdam-New York, 1984, 111–167. MR 87f:54008; the entire collection MR 85k:54001.

B. Dushnik and E. W. Miller
[1941] Partially ordered sets. *Amer. J. Math.* **63** (1941), 600–610. MR 3,73a.

W. B. Easton
[1970] Powers of regular cardinals. *Ann. Math. Logic* **1** (1970), 139–178. MR 42 #4392.

A. Ehrenfeucht and A. Mostowski
[1956] Models of axiomatic theories admitting automorphisms. *Fund. Math.* **43** (1956), 50–68. MR 18,863a.

E. Ellentuck
[1974] A new proof that analytic sets are Ramsey. *J. Symbolic Logic* **39** (1974), 163–165. MR 50 #1887.

R. Engelking and M. Karłowicz
[1965] Some theorems of set theory and their topological consequences. *Fund. Math.* **57** (1965), 275–285. MR 33 #4880.

P. Erdős and A. Hajnal
[1962] Some remarks concerning our paper "On the structure of set mappings". Non-existence of a two-valued σ-measure, for the first uncountable inaccessible cardinal. *Acta Math. Acad. Sci. Hungar.* **13** (1962), 223–226. MR 25 #5001.

[1966] On a problem of B. Jónsson. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **14** (1966), 19–23. MR 35 #64.

[1971] Unsolved problems in set theory. *In* "Axiomatic Set Theory" (D. S. Scott, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1971, pp. 17–48. MR 43 #6101; the entire collection MR 43 #38.

P. Erdős, A. Hajnal, and E. C. Milner

[1968] On sets of almost disjoint subsets of a set. *Acta Math. Acad. Sci. Hungar.* **19** (1968), 209–218. MR 37 #84.

P. Erdős, A. Hajnal, and R. Rado

[1965] Partitions relations for cardinal numbers. *Acta Math. Acad. Sci. Hungar.* **16** (1965), 93–196. MR 34 #2475.

P. Erdős and R. Rado

[1952] Combinatorial theorems on classifications of subsets of a given set. *Proc. London Math. Soc. (3)* **2** (1952), 417–439. MR 16,455d.

[1956] A partition calculus in set theory. *Bull. Amer. Math. Soc.* **62** (1956), 427–489. MR 18,458a.

[1960] Intersection theorems for systems of sets. *J. London Math. Soc.* **35** (1960), 85–90. MR 22 #2554.

P. Erdős and A. Tarski

[1943] On families of mutually exclusive sets. *Ann. of Math. (2)* **44** (1943), 315–329. MR 4,269b.

[1961] On some problems involving inaccessible cardinals. *In* "Essays on the Foundations of Mathematics" (Y. Bar-Hillel et al., eds.). Magnes Press, Hebrew Univ., Jerusalem, 1961, 50–82. MR 29 #4695; the entire collection MR 28 #3918.

S. Feferman

[1964/65] Some applications of the notions of forcing and generic sets. *Fund. Math.* **56** (1964/65), 352–345. MR 31 #1193.

S. Feferman and A. Lévy

[1963] Independence results in set theory by Cohen's method II. *Notices Amer. Math. Soc.* **10** (1963), 593. Abstract.

Q. Feng and T. J. Jech

[1989] Local clubs, reflection, and preserving stationary sets. *Proc. London Math. Soc. (3)* **58** (1989), no. 2, 237–257. MR 90a:03072.

[1998] Projective stationary sets and a strong reflection principle. *J. London Math. Soc. (2)* **58** (1998), no. 2, 271–283. MR 2000b:03166.

Q. Feng, M. Magidor, and W. H. Woodin

[1992] Universally Baire sets of reals. *In* "Set Theory of the Continuum" (H. Judah, W. Just, and W. H. Woodin, eds.), Berkeley, CA, 1989. *Mathematical Sciences Research Institute Publications* **26**, Springer-Verlag, New York, 1992, pp. 203–242. MR 94g:03095; the entire collection MR 94b:03009.

G. M. Fichtenholz and L. V. Kantorovitch

[1935] Sur les opérations linéaires dans l'espace des fonctions bornées. *Studia Math.* **5** (1935), 69–98. Zbl. 013.06502.

G. Fodor

[1956] Eine Bemerkung zur Theorie der regressiven Funktionen. *Acta Sci. Math. Szeged* **17** (1956), 139–142. MR 18,551d.

M. Foreman
[1983] Games played on Boolean algebras. *J. Symbolic Logic* **48** (1983), no. 3, 714–723. MR 85h:03064.

M. Foreman and M. Magidor
[2001] Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $P_\kappa(\lambda)$. *Acta Math.* **186** (2001), no. 2, 271–300. MR 02g:03094.

M. Foreman, M. Magidor, and S. Shelah
[1988] Martin's maximum, saturated ideals and nonregular ultrafilters. I. *Ann. of Math. (2)* **127** (1988), no. 1, 1–47. MR 89f:03043.

M. Foreman and W. H. Woodin
[1991] The generalized continuum hypothesis can fail everywhere. *Ann. of Math. (2)* **133** (1991), no. 1, 1–35. MR 91k:03130.

A. A. Fraenkel
[1922a] Über den Begriff "definit" und die Unabhängigkeit des Auswahlaxioms. *S.-B. Berlin. Math. Ges.* (1922), 250–273.
[1922b] Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre. *Math. Ann.* **86** (1922), 230–237.

A. A. Fraenkel, Y. Bar-Hillel, and A. Lévy
[1973] "Foundations of Set Theory." *Stud. Logic Foundations Math.* **67**, North-Holland Publishing Co., Amsterdam-London, second revised edition, 1973. MR 49 #10546.

D. H. Fremlin
[1989] Measure algebras. *In* "Handbook of Boolean algebras, Vol. 3" (J. D. Monk and R. Bonnet, eds.). North-Holland Publishing Co., Amsterdam, 1989, 877–980. MR 90k:06004.

H. Friedman
[1974] On closed sets of ordinals. *Proc. Amer. Math. Soc.* **43** (1974), 190–192. MR 48 #5863.

S. D. Friedman
[1997] The Σ^* approach to the fine structure of L. *In* "European Summer Meeting of the Association for Symbolic Logic" (A. Kanamori et al., eds.), Haifa, 1995. *Fund. Math.* **154** (1997), no. 2, 133–158. MR 98k:03107; the entire collection MR 98e:03002.

S. D. Friedman and P. G. Koepke
[1997] An elementary approach to the fine structure of L. *Bull. Symbolic Logic* **3** (1997), no. 4, 453–468. MR 99i:03062.

Z. Frolík
[1968] Fixed points of maps of βN. *Bull. Amer. Math. Soc.* **74** (1968), 187–191. MR 36 #5897.

U. Fuchs
[1992] Donder's version of revised countable support. Preprint. *Freie Universität Berlin* (1992).

V. I. Fukson
[1971] The homogeneous Suslin continuum. *Dokl. Akad. Nauk SSSR* **197** (1971), 292–294. (Russian). MR 43 #3109.

H. Gaifman
[1964] Measurable cardinals and constructible sets. *Notices Amer. Math. Soc.* **11** (1964), 771. Abstract.

[1967] Uniform extension operators for models and their applications. *In* "Sets, Models and Recursion Theory" (J. N. Crossley, ed.), Proc. Summer School Math. Logic and Tenth Logic Colloq., Leicester, 1965. North-Holland Publishing Co., Amsterdam, 1967, pp. 122–155. MR 36 #3640; the entire collection MR 36 #24.

[1974] Elementary embeddings of models of set-theory and certain subtheories. *In* "Axiomatic Set Theory" (T. J. Jech, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1974, pp. 33–101. MR 51 #12523; the entire collection MR 50 #6840.

D. Gale and F. M. Stewart

[1953] Infinite games with perfect information. *In* "Contributions to the Theory of Games," Vol. 2. *Ann. of Math. Studies*, No. 28, Princeton University Press, Princeton, N.J., 1953, 245–266. MR 14,999b.

F. Galvin and A. Hajnal

[1975] Inequalities for cardinal powers. *Ann. of Math. (2)* **101** (1975), 491–498. MR 51 #12535.

F. Galvin and K. L. Prikry

[1973] Borel sets and Ramsey's theorem. *J. Symbolic Logic* **38** (1973), 193–198. MR 49 #2399.

M. Gitik

[1984] The nonstationary ideal on $\aleph_2$. *Israel J. Math.* **48** (1984), no. 4, 257–288. MR 86i:03061.

[1985] Nonsplitting subset of $P_\kappa(\kappa^+)$. *J. Symbolic Logic* **50** (1985), no. 4, 881–894. MR 87g:03054.

[1989] The negation of the singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$. *Ann. Pure Appl. Logic* **43** (1989), no. 3, 209–234. MR 90h:03037.

[1991] The strength of the failure of the singular cardinal hypothesis. *Ann. Pure Appl. Logic* **51** (1991), no. 3, 215–240. MR 92f:03060.

[1995] Some results on the nonstationary ideal. *Israel J. Math.* **92** (1995), no. 1-3, 61–112. MR 96k:03108.

[1997] Some results on the nonstationary ideal. II. *Israel J. Math.* **99** (1997), no. 1-3, 175–188. MR 98k:03063.

[∞] No bound for the first fixed point.

M. Gitik and M. Magidor

[1992] The singular cardinal hypothesis revisited. *In* "Set Theory of the Continuum" (H. Judah, W. Just, and W. H. Woodin, eds.), Berkeley, CA, 1989. *Mathematical Sciences Research Institute Publications* **26**, Springer-Verlag, New York, 1992, pp. 243–279. MR 95c:03131; the entire collection MR 94b:03009.

M. Gitik and S. Shelah

[1989] On certain indestructibility of strong cardinals and a question of Hajnal. *Arch. Math. Logic* **28** (1989), no. 1, 35–42. MR 90e:03063.

[1997] Less saturated ideals. *Proc. Amer. Math. Soc.* **125** (1997), no. 5, 1523–1530. MR 97g:03055.

M. Gitik and J. Witzany

[1996] Consistency strength of the axiom of full reflection at large cardinals. *Israel J. Math.* **93** (1996), 113–124. MR 97m:03082.

K. Gödel

[1931] Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme. I. *Monatsh. Math. Phys.* **38** (1931), 173–198. Zbl. 002.00101.

[1938] The consistency of the axiom of choice and of the generalized continuum hypothesis. *Proc. Natl. Acad. Sci. U.S.A.* **24** (1938), 556–557. Zbl. 020.29701.

[1939] Consistency-proof for the generalized continuum hypothesis. *Proc. Natl. Acad. Sci. U.S.A.* **25** (1939), 220–224. Zbl. 021.00102.

[1940] "The Consistency of the Continuum Hypothesis." *Ann. of Math. Studies*, No. 3, Princeton University Press, Princeton, N.J., 1940. MR 2,66c.

[1965] Remarks before the Princeton bicentennial conference on problems in mathematics. (1946). *In* "The Undecidable. Basic Papers on Undecidable Propositions, Unsolvable Problems and Computable Functions" (M. Davis, ed.). Raven Press, Hewlett, New York, 1965, 84–88. MR 32 #7412.

M. Goldstern and S. Shelah

[1995] The bounded proper forcing axiom. *J. Symbolic Logic* **60** (1995), no. 1, 58–73. MR 96g:03083.

R. L. Graham, B. L. Rothschild, and J. H. Spencer

[1980] "Ramsey Theory." *Wiley Interscience Series in Discrete Mathematics*, John Wiley & Sons, Inc., New York, 1980. MR 82b:05001.

J. Gregory

[1976] Higher Souslin trees and the generalized continuum hypothesis. *J. Symbolic Logic* **41** (1976), no. 3, 663–671. MR 58 #5208.

M. J. Groszek and T. J. Jech

[1991] Generalized iteration of forcing. *Trans. Amer. Math. Soc.* **324** (1991), no. 1, 1–26. MR 91f:03107.

M. J. Groszek and T. A. Slaman

[1998] A basis theorem for perfect sets. *Bull. Symbolic Logic* **4** (1998), no. 2, 204–209. MR 99c:03072.

A. Hajnal

[1956] On a consistency theorem connected with the generalized continuum problem. *Z. Math. Logik Grundlagen Math.* **2** (1956), 131–136. MR 19,1031d.

[1964] Remarks on the theorem of W. P. Hanf. *Fund. Math.* **54** (1964), 109–113. MR 28 #3945.

A. W. Hales and R. I. Jewett

[1963] Regularity and positional games. *Trans. Amer. Math. Soc.* **106** (1963), 222–229. MR 26 #1265.

P. R. Halmos

[1950] "Measure Theory." *University Series in Higher Mathematics*, D. van Nostrand Co., Inc., New York, N. Y., 1950. MR 11,504d.

J. D. Halpern and A. Lévy

[1971] The Boolean prime ideal theorem does not imply the axiom of choice. *In* "Axiomatic Set Theory" (D. S. Scott, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1971, pp. 83–134. MR 44 #1557; the entire collection MR 43 #38.

W. P. Hanf

[1963/64a] Incompactness in languages with infinitely long expressions. *Fund. Math.* **53** (1963/64), 309–324. MR 28 #3943.

[1963/64b] On a problem of Erdős and Tarski. *Fund. Math.* **53** (1963/64), 325–334. MR 28 #3944.

W. P. Hanf and D. S. Scott

[1961] Classifying inaccessible cardinals. *Notices Amer. Math. Soc.* **8** (1961), 445. Abstract.

L. A. Harrington

[1978] Analytic determinacy and $0^\sharp$. *J. Symbolic Logic* **43** (1978), no. 4, 685–693. MR 80b:03065.

L. A. Harrington and S. Shelah

[1985] Some exact equiconsistency results in set theory. *Notre Dame J. Formal Logic* **26** (1985), no. 2, 178–188. MR 86g:03079.

F. Hausdorff

[1904] Der Potenzbegriff in der Mengenlehre. *Jahresber. Deutsch. Math.-Verein* **13** (1904), 569–571.

[1909] Die Graduierung nach dem Endverlauf. *Abh. d. König. Sächs. Gesellschaft d. Wiss. (Math.–Phys. Kl.)* **31** (1909), 296–334.

[1914] "Grundzüge der Mengenlehre." Veit & Co., Leipzig, 1914.

[1916] Die Mächtigkeit der Borelschen Mengen. *Math. Ann.* **77** (1916), 430–437.

[1936a] Summen von $\aleph_1$ Mengen. *Fund. Math.* **26** (1936), 241–255. Zbl. 014.05402.

[1936b] Über zwei Sätze von G. Fichtenholz und L. Kantorovitch. *Studia Math.* **6** (1936), 18–19. Zbl. 015.39501.

S. H. Hechler

[1974] On the existence of certain cofinal subsets of ${}^\omega\omega$. *In* "Axiomatic Set Theory" (T. J. Jech, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1974, pp. 155–173. MR 50 #12716; the entire collection MR 50 #6840.

G. Hessenberg

[1906] "Grundbegriffe der Mengenlehre." Vandenhoeck and Ruprecht, Göttingen, 1906.

N. Hindman

[1974] Finite sums from sequences within cells of a partition of N. *J. Combinatorial Theory Ser. A* **17** (1974), 1–11. MR 50 #2067.

S. Jackson

[1988] AD and the projective ordinals. *In* "Cabal Seminar 81–85" (A. S. Kechris, D. A. Martin, and J. R. Steel, eds.), Proc. Caltech-UCLA Logic Sem., 1981–85. *Lecture Notes in Mathematics* **1333**, Springer-Verlag, Berlin-New York, 1988, pp. 117–220. MR 89k:03055; the entire collection MR 89c:03005.

[1999] A computation of $\boldsymbol{\delta}^1_5$. *Mem. Amer. Math. Soc.* **140** (1999), no. 670. MR 2000d:03104.

T. J. Jech

[1967] Non-provability of Souslin's hypothesis. *Comment. Math. Univ. Carolinae* **8** (1967), 291–305. MR 35 #6564.

[1968] ω_1 can be measurable. *Israel J. Math.* **6** (1968), 363–367. MR 39 #5353.

[1971a] "Lectures in Set Theory, with Particular Emphasis on the Method of Forcing." *Lecture Notes in Mathematics* **217**, Springer-Verlag, Berlin-New York, 1971. MR 48 #105.

[1971b] The closed unbounded filter over $P_\kappa(\lambda)$. *Notices Amer. Math. Soc.* **18** (1971), 663. Abstract.

[1972/73] Some combinatorial problems concerning uncountable cardinals. *Ann. Math. Logic* **5** (1972/73), 165–198. MR 48 #3744.

[1973a] Properties of the gimel function and a classification of singular cardinals. Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, I. *Fund. Math.* **81** (1973), no. 1, 57–64. MR 52 #10424.

[1973b] "The Axiom of Choice." *Stud. Logic Foundations Math.* **75**, North-Holland Publishing Co., Amsterdam-London, 1973. MR 53 #139.

[1976] An infinite game played with stationary sets. *Notices Amer. Math. Soc.* **23** (1976), A-286. Abstract.

[1978] A game theoretic property of Boolean algebras. *In* "Logic Colloquium '77" (A. Macintyre et al., eds.), Proc. Conf., Wrocław, 1977. *Stud. Logic Foundations Math.* **96**, North-Holland, Amsterdam-New York, 1978, pp. 135–144. MR 80c:90184; the entire collection MR 80a:03003.

[1984] Stationary subsets of inaccessible cardinals. *In* "Axiomatic Set Theory" (J. E. Baumgartner, D. A. Martin, and S. Shelah, eds.), Boulder, Colo., 1983. *Contemp. Math.* **31**, Amer. Math. Soc., Providence, R.I., 1984, pp. 115–142. MR 86c:03044; the entire collection MR 85g:03004.

[1992] Singular cardinal problem: Shelah's theorem on $2^{\aleph_\omega}$. *Bull. London Math. Soc.* **24** (1992), no. 2, 127–139. MR 93a:03050.

[∞] Stationary sets. *In* "Handbook of Set Theory" (A. Kanamori, M. Foreman, et al., eds.).

T. J. Jech, M. Magidor, W. J. Mitchell, and K. L. Prikry

[1980] Precipitous ideals. *J. Symbolic Logic* **45** (1980), no. 1, 1–8. MR 81h:03097.

T. J. Jech and W. J. Mitchell

[1983] Some examples of precipitous ideals. *Ann. Pure Appl. Logic* **24** (1983), no. 2, 131–151. MR 85k:03033.

T. J. Jech and K. L. Prikry

[1976] On ideals of sets and the power set operation. *Bull. Amer. Math. Soc.* **82** (1976), no. 4, 593–595. MR 58 #21618.

[1979] Ideals over uncountable sets: Applications of almost disjoint functions and generic ultrapowers. *Mem. Amer. Math. Soc.* **18** (1979), no. 214. MR 80f:03059.

[1984] Cofinality of the partial ordering of functions from ω_1 into ω under eventual domination. *Math. Proc. Cambridge Philos. Soc.* **95** (1984), no. 1, 25–32. MR 85b:03084.

T. J. Jech and S. Shelah

[1993] Full reflection of stationary sets at regular cardinals. *Amer. J. Math.* **115** (1993), no. 2, 435–453. MR 94m:03080.

[1996] A complete Boolean algebra that has no proper atomless complete sublagebra. *J. Algebra* **182** (1996), no. 3, 748–755. MR 97j:03109.

[2001] Simple complete Boolean algebras. *Proc. Amer. Math. Soc.* **129** (2001), no. 2, 543–549. MR 2001e:03083.

T. J. Jech and A. Sochor

[1966a] On Θ-model of the set theory. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **14** (1966), 297–303. MR 34 #2441.

[1966b] Applications of the Θ-model. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **14** (1966), 351–355. MR 37 #3918.

T. J. Jech and W. H. Woodin

[1985] Saturation of the closed unbounded filter on the set of regular cardinals. *Trans. Amer. Math. Soc.* **292** (1985), no. 1, 345–356. MR 87d:03139.

R. B. Jensen

[1965] An embedding theorem for countable ZF models. *Notices Amer. Math. Soc.* **12** (1965), 720. Abstract.

[1968] Souslin's hypothesis is incompatible with $V = L$. *Notices Amer. Math. Soc.* **15** (1968), 935. Abstract.

[1969] Automorphism properties of Souslin continua. *Notices Amer. Math. Soc.* **16** (1969), 576. Abstract.

[1970] Definable sets of minimal degree. *In* "Mathematical Logic and Foundations of Set Theory" (Y. Bar-Hillel, ed.), Proc. Internat. Colloq., Jerusalem, 1968. North-Holland Publishing Co., Amsterdam, 1970, pp. 122–128. MR 46 #5130; the entire collection MR 42 #1643.

[1972] The fine structure of the constructible hierarchy. *Ann. Math. Logic* **4** (1972), 229–308. MR 46 #8834.

H. Judah and S. Shelah

[1989] Δ^1_2-sets of reals. *Ann. Pure Appl. Logic* **42** (1989), no. 3, 207–223. MR 90f:03081.

A. Kanamori

[1976] Weakly normal filters and irregular ultrafilters. *Trans. Amer. Math. Soc.* **220** (1976), 393–399. MR 58 #240.

[1994] "The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings." *Perspectives in Mathematical Logic*, Springer-Verlag, Berlin, 1994. MR 96k:03125.

[∞] "The Higher Infinite," Vol. 2, in preparation.

A. Kanamori and M. Magidor

[1978] The evolution of large cardinal axioms in set theory. *In* "Higher Set Theory" (G. H. Müller and D. S. Scott, eds.), Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977. *Lecture Notes in Mathematics* **669**, Springer, Berlin, 1978, pp. 99–275. MR 80b:03083; the entire collection MR 80a:03005.

C. Karp

[1967] A proof of the relative consistency of the continuum hypothesis. *In* "Sets, Models and Recursion Theory" (J. N. Crossley, ed.), Proc. Summer School Math. Logic and Tenth Logic Colloq., Leicester, 1965. North-Holland Publishing Co., Amsterdam, 1967, pp. 1–32. MR 36 #1320; the entire collection MR 36 #24.

A. S. Kechris

[1974] On projective ordinals. *J. Symbolic Logic* **39** (1974), 269–282. MR 53 #2684.

[1978] AD and projective ordinals. *In* "Cabal Seminar 76–77" (A. S. Kechris and Y. N. Moschovakis, eds.), Proc. Caltech-UCLA Logic Sem., 1976–77. *Lecture Notes in Mathematics* **689**, Springer, Berlin-New York, 1978, pp. 91–132. MR 80j:03069; the entire collection MR 80b:03004.

[1981] Homogeneous trees and projective scales. *In* "Cabal Seminar 77–79" (A. S. Kechris, D. A. Martin, and Y. N. Moschovakis, eds.), Proc. Caltech-UCLA Logic Sem., 1977–79. *Lecture Notes in Mathematics* **839**, Springer, Berlin, 1981, pp. 33–73. MR 82j:03060; the entire collection MR 80b:03002.

[1985] Determinacy and the structure of $L(\boldsymbol{R})$. *In* "Recursion Theory" (A. Nerode and R. A. Shore, eds.), Ithaca, N.Y., 1982. *Proc. Sympos. Pure Math.* **42**, Amer. Math. Soc., Providence, R.I., 1985, pp. 271–283. MR 87d:03142; the entire collection MR 86f:03004.

A. S. Kechris and D. A. Martin

[1980] Infinite games and effective descriptive set theory. *In* "Analytic Sets" (C. A. Rogers and J. J. E., eds.), Lectures delivered at a Conference held at University College, University of London, London, July 16–29, 1978. Academic Press, London-New York, 1980, pp. 403–470. MR 82m:03063.

A. S. Kechris and Y. N. Moschovakis

[1972] Two theorems about projective sets. *Israel J. Math.* **12** (1972), 391–399. MR 48 #1900.

[1978] Notes on the theory of scales. *In* "Cabal Seminar 76–77" (A. S. Kechris and Y. N. Moschovakis, eds.), Proc. Caltech-UCLA Logic Sem., 1976–77. *Lecture Notes in Mathematics* **689**, Springer, Berlin-New York, 1978, pp. 1–53. MR 83b:03059; the entire collection MR 80b:03004.

H. J. Keisler

[1962] Some applications of the theory of models to set theory. *In* "Logic, Methodology and Philosophy of Science" (E. Nagel et al., eds.), Proc. 1960 Internat. Congr. Stanford Univ. Press, Stanford, Calif., 1962, pp. 80–86. MR 32 #7418; the entire collection MR 29 #3347.

H. J. Keisler and A. Tarski

[1963/64] From accessible to inaccessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones. *Fund. Math.* **53** (1963/64), 225–308. MR 29 #3385.

J. Ketonen

[1972/73] Strong compactness and other cardinal sins. *Ann. Math. Logic* **5** (1972/73), 47–76. MR 57 #9549.

[1973] Ultrafilters over measurable cardinals. *Fund. Math.* **77** (1973), no. 3, 257–269. MR 48 #8237.

[1976] On the existence of P-points in the Stone-Čech compactification of integers. *Fund. Math.* **92** (1976), no. 2, 91–94. MR 55 #6363.

S. C. Kleene

[1955] Hierarchies of number-theoretic predicates. *Bull. Amer. Math. Soc.* **61** (1955), 193–213. MR 17,4f.

P. G. Koepke

[1989] An introduction to extenders and core models for extender sequences. *In* "Logic Colloquium '87" (H. D. Ebbinghaus et al., eds.), Granada, 1987. *Stud. Logic Foundations Math.* **129**, North-Holland, Amsterdam-New York, 1989, pp. 137–182. MR 90j:03094; the entire collection 90d:03004.

M. Kondô

[1939] Sur l'uniformisation des complémentaires analytiques et les ensembles projectifs de la seconde classe. *Japan. J. Math.* **15** (1939), 197–230. Zbl. 022.12301.

D. König

[1927] Über eine Schlussweise aus dem Endlichen ins Undendliche. *Acta Litt. Acad. Sci. Hung. (Szeged)* **3** (1927), 121–130.

J. König

[1905] Zum Kontinuumproblem. *Math. Ann.* **60** (1905), 177–180.

S. Koppelberg

[1989] "Handbook of Boolean Algebras," Vol. 1, North-Holland Publishing Co., Amsterdam, 1989. Edited by J. Donald Monk and Robert Bonnet. MR 90k:06002.

[1993] Characterizations of Cohen algebras. *In* "Papers on General Topology and Applications" (S. Andima et al., eds.), Madison, WI, 1991. *Ann. New York Acad. Sci.* **704**, New York Academy of Sciences, New York, 1993, pp. 222–237. MR 95e:06035; the entire collection MR 94m:54001.

S. Kripke

[1967] An extension of a theorem of Gaifman-Hales-Solovay. *Fund. Math.* **61** (1967), 29–32. MR 36 #3693.

D. W. Kueker

[1972] Löwenheim-Skolem and interpolation theorems in infinitary languages. *Bull. Amer. Math. Soc.* **78** (1972), 211–215. MR 45 #36.

[1977] Countable approximations and Löwenheim-Skolem theorems. *Ann. Math. Logic* **11** (1977), no. 1, 57–103. MR 56 #15406.

K. Kunen

[1970] Some applications of iterated ultrapowers in set theory. *Ann. Math. Logic* **1** (1970), 179–227. MR 43 #3080.

[1971a] Elementary embeddings and infinitary combinatorics. *J. Symbolic Logic* **36** (1971), 407–413. MR 47 #40.

[1971b] On the GCH at measurable cardinals. *In* "Logic Colloquium '69" (R. O. Gandy, ed.), Proc. Summer School and Colloq., Manchester, 1969. North-Holland Publishing Co., Amsterdam, 1971, pp. 107–110. MR 43 #3081; the entire collection MR 42 #7479.

[1978] Saturated ideals. *J. Symbolic Logic* **43** (1978), no. 1, 65–76. MR 80a:03068.

[1980] "Set theory, An introduction to independence proofs." North-Holland Publishing Co., Amsterdam, 1980. MR 82f:03001.

K. Kunen and J. B. Paris

[1970/71] Boolean extensions and measurable cardinals. *Ann. Math. Logic* **2** (1970/71), no. 4, 359–377. MR 43 #3114.

K. Kuratowski

[1922] Une méthode d'élimination des nombres transfinis des raisonnements mathématiques. *Fund. Math.* **3** (1922), 76–108.

[1936] Sur les théorèmes de séparation dans la Theórie des ensembles. *Fund. Math.* **26** (1936), 183–191. Zbl. 014.05404.

[1948] Ensembles projectifs et ensembles singuliers. *Fund. Math.* **35** (1948), 131–140. MR 10,358d.

[1966] "Topology," Vol. 1, Academic Press, New York-London, new, rewised and augmented edition, 1966. MR 36 #840.

K. Kuratowski and S. Ulam

[1932] Quelques propriétés topologiques du produit combinatoire. *Fund. Math.* **19** (1932), 247–251. Zbl. 005.18301.

Đ. Kurepa

[1935] Ensembles ordonnés et ramifiés. *Publ. Math. Univ. Belgrade* **4** (1935), 1–138. Zbl. 014.39401.

R. Laver

[1976] On the consistency of Borel's conjecture. *Acta Math.* **137** (1976), no. 3-4, 151–169. MR 54 #10019.

[1978] Making the supercompactness of κ indestructible under κ-directed closed forcing. *Israel J. Math.* **29** (1978), no. 4, 385–388. MR 57 #12226.

R. Laver and S. Shelah

[1981] The $\aleph_2$-Souslin hypothesis. *Trans. Amer. Math. Soc.* **264** (1981), no. 2, 411–417. MR 82e:03049.

M. H. Lebesgue

[1902] "Intégrale, Longuer, Aire." Thèse, Paris, 1902.

[1905] Sur les fonctions représentables analytiquement. *J. de Math.* **1** (1905), Sér 6, 139–216.

A. Lévy

[1957] Indépendance conditionnelle de $V = L$ et d'axiomes qui se rattachent au système de M. Gödel. *C. R. Acad. Sci. Paris* **245** (1957), 1582–1583. MR 19,626b.

[1960a] A generalization of Gödel's notion of constructibility. *J. Symbolic Logic* **25** (1960), 147–155. MR 26 #37.

[1960b] Axiom schemata of strong infinity in axiomatic set theory. *Pacific J. Math.* **10** (1960), 223–238. MR 23 #A1522.

[1965a] A hierarchy of formulas in set theory. *Mem. Amer. Math. Soc.*, No. 57 (1965), 76 pp. MR 32 #7399.

[1965b] Definability in axiomatic set theory. I. *In* "Logic, Methodology and Philosophy of Science" (Y. Bar-Hillel, ed.), Proc. 1964 Internat. Congr. North-Holland Publishing Co., Amsterdam, 1965, pp. 127–151. MR 34 #5653; the entire collection MR 34 #2426.

[1970] Definability in axiomatic set theory. II. *In* "Mathematical Logic and Foundations of Set Theory" (Y. Bar-Hillel, ed.), Proc. Internat. Colloq., Jerusalem, 1968. North-Holland Publishing Co., Amsterdam, 1970, pp. 129–145. MR 42 #2936; the entire collection MR 42 #1643.

A. Lévy and R. M. Solovay

[1967] Measurable cardinals and the continuum hypothesis. *Israel J. Math.* **5** (1967), 234–248. MR 37 #57.

J. Łoś

[1955] Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres. *In* "Mathematical Interpretation of Formal Systems" (T. Skolem et al., eds.). North-Holland Publishing Co., Amsterdam, 1955, 98–113. MR 17,700d.

B. Löwe and J. R. Steel

[1999] An introduction to core model theory. *In* "Sets and Proofs" (S. B. Cooper and J. K. Truss, eds.), Leeds, 1997. *London Math. Soc. Lecture Note Ser.* **258**, Cambridge University Press, Cambridge, 1999, pp. 103–157. MR 2000k:03101; the entire collection 2000e:03005.

N. N. Luzin

[1914] Sur un problème de M. Baire. *C. R. Acad. Sci. Paris* **158** (1914), 1258–1261.

[1917] Sur la classification de M. Baire. *C. R. Acad. Sci. Paris* **164** (1917), 91–94.

[1925] Sur les ensembles projectifs de M. Henri Lebesgue. *C. R. Acad. Sci. Paris* **180** (1925), 1572–1574.

[1927a] Remarques sur les ensembles projectifs. *C. R. Acad. Sci. Paris* **185** (1927), 835–837.

[1927b] Sur les ensembles analytiques. *Fund. Math.* **10** (1927), 1–95.

[1930] "Leçons sur les Ensembles Analytiques." Gauthier-Villars, Paris, 1930.

N. N. Luzin and W. Sierpiński

[1923] Sur un ensemble non mesurable B. *J. de Math.* **2** (1923), Sér 9, 53–72.

M. Magidor

[1971a] There are many normal ultrafilters corresponding to a supercompact cardinal. *Israel J. Math.* **9** (1971), 186–192. MR 50 #110.

[1971b] On the role of supercompact and extendible cardinals in logic. *Israel J. Math.* **10** (1971), 147–157. MR 45 #4966.

[1976] How large is the first strongly compact cardinal? or A study on identity crises. *Ann. Math. Logic* **10** (1976), no. 1, 33–57. MR 55 #2578.

[1977a] On the singular cardinals problem. I. *Israel J. Math.* **28** (1977), no. 1-2, 1–31. MR 58 #10449a.

[1977b] On the singular cardinals problem. II. *Ann. of Math. (2)* **106** (1977), no. 3, 514–547. MR 58 #10449b.

[1978] Changing cofinality of cardinals. *Fund. Math.* **99** (1978), no. 1, 61–71. MR 57 #5754.

[1980] Precipitous ideals and Σ^1_4 sets. *Israel J. Math.* **35** (1980), no. 1-2, 109–134. MR 81k:03048.

[1982] Reflecting stationary sets. *J. Symbolic Logic* **47** (1982), no. 4, 755–771. MR 84f:03046.

[1990] Representing sets of ordinals as countable unions of sets in the core model. *Trans. Amer. Math. Soc.* **317** (1990), no. 1, 91–126. MR 90d:03108.

D. Maharam

[1942] On homogeneous measure algebras. *Proc. Natl. Acad. Sci. U.S.A.* **28** (1942), 108–111. MR 4,12a.

P. Mahlo

[1911] Über lineare transfinite Mengen. *Berichte König. Sächs. Ges. Wiss. Leipzig* **63** (1911), 187–225.

D. C. Makinson

[1969] On the number of ultrafilters of an infinite Boolean algebra. *Z. Math. Logik Grundlagen Math.* **15** (1969), 121–122. MR 39 #6798.

R. B. Mansfield

[1970] Perfect subsets of definable sets of real numbers. *Pacific J. Math.* **35** (1970), 451–457. MR 43 #6100.

[1971] A Souslin operation for Π^1_2. *Israel J. Math.* **9** (1971), 367–379. MR 45 #6629.

[1975] The non-existence of Σ^1_2 well-orderings of the Cantor set. *Fund. Math.* **86** (1975), no. 3, 279–282. MR 52 #7903.

E. Marczewski

[1930a] O mierzalności i warunku Baire a. *In* "Kongres Matematyków Krajów Slowiańskich." Warszawa, 1930, pp. 297–303.

[1930b] Sur une hypothèse de M. Borel. *Fund. Math.* **15** (1930), 126–127.

D. A. Martin

[1968] The axiom of determinateness and reduction principles in the analytic hierarchy. *Bull. Amer. Math. Soc.* **74** (1968), 687–689. MR 37 #2607.

[1969/70] Measurable cardinals and analytic games. *Fund. Math.* **66** (1969/70), 287–291. MR 41 #3283.

[1975] Borel determinacy. *Ann. of Math. (2)* **102** (1975), no. 2, 363–371. MR 53 #7785.

[1985] A purely inductive proof of Borel determinacy. *In* "Recursion Theory" (A. Nerode and R. A. Shore, eds.), Ithaca, N.Y., 1982. *Proc. Sympos. Pure Math.* **42**, Amer. Math. Soc., Providence, R.I., 1985, pp. 303–308. MR 87d:03125; the entire collection MR 86f:03004.

D. A. Martin and R. M. Solovay

[1969] A basis theorem for Σ^1_3 sets of reals. *Ann. of Math. (2)* **89** (1969), 138–159. MR 41 #53.

[1970] Internal Cohen extensions. *Ann. Math. Logic* **2** (1970), no. 2, 143–178. MR 42 #5787.

D. A. Martin and J. R. Steel

[1988] Projective determinacy. *Proc. Natl. Acad. Sci. U.S.A.* **85** (1988), no. 18, 6582–6586. MR 89m:03041.

[1989] A proof of projective determinacy. *J. Amer. Math. Soc.* **2** (1989), no. 1, 71–125. MR 89m:03042.

[1994] Iteration trees. *J. Amer. Math. Soc.* **7** (1994), no. 1, 1–73. MR 94f:03062.

A. R. D. Mathias

[1973] On sequences generic in the sense of Prikry. *J. Austral. Math. Soc.* **15** (1973), 409–414. MR 48 #10809.

[1977] Happy families. *Ann. Math. Logic* **12** (1977), no. 1, 59–111. MR 58 #10462.

Y. Matsubara

[1987] Menas' conjecture and generic ultrapowers. *Ann. Pure Appl. Logic* **36** (1987), no. 3, 225–234. MR 88m:04002.

[1988] Splitting $P_\kappa\lambda$ into stationary subsets. *J. Symbolic Logic* **53** (1988), no. 2, 385–389. MR 90d:03094.

[1990] Consistency of Menas' conjecture. *J. Math. Soc. Japan* **42** (1990), no. 2, 259–263. MR 91c:03039.

K. McAloon

[1971] Les algèbres de Boole rigides et minimales. *C. R. Acad. Sci. Paris Sér. A-B* **272** (1971), A89–A91. MR 42 #7491.

A. H. Mekler and S. Shelah

[1989] The consistency strength of "every stationary set reflects." *Israel J. Math.* **67** (1989), no. 3, 353–366. MR 91b:03090.

T. K. Menas

[1974/75] On strong compactness and supercompactness. *Ann. Math. Logic* **7** (1974/75), 327–359. MR 50 #9589.

[1976] Consistency results concerning supercompactness. *Trans. Amer. Math. Soc.* **223** (1976), 61–91. MR 58 #27488.

A. W. Miller

[1981] Some properties of measure and category. *Trans. Amer. Math. Soc.* **266** (1981), no. 1, 93–114. MR 84e:03058a.

E. C. Milner and R. Rado

[1965] The pigeon-hole principle for ordinal numbers. *Proc. London Math. Soc. (3)* **15** (1965), 750–768. MR 32 #7419.

D. Mirimanov

[1917] Les antinomies de Russell et de Burali-Forti et le problème fondamental de la théorie des ensembles. *L'Enseignment Math.* **19** (1917), 37–52.

W. J. Mitchell

[1972/73] Aronszajn trees and the independence of the transfer property. *Ann. Math. Logic* **5** (1972/73), 21–46. MR 47 #1612.

[1974] Sets constructible from sequences of ultrafilters. *J. Symbolic Logic* **39** (1974), 57–66. MR 49 #8863.

[1979a] Hypermeasurable cardinals. *In* "Logic Colloquium '78" (M. Boffa et al., eds.), Mons, 1978. *Stud. Logic Foundations Math.* **97**, North-Holland Publishing Co., Amsterdam-New York, 1979, pp. 303–316. MR 82j:03067; the entire collection MR 81a:03001.

[1979b] Ramsey cardinals and constructibility. *J. Symbolic Logic* **44** (1979), no. 2, 260–266. MR 80f:03063.

[1982] How weak is a closed unbounded ultrafilter? *In* "Logic Colloquium '80" (D. van Dalen et al., eds.), Prague, 1980. *Stud. Logic Foundations Math.* **108**, North-Holland Publishing Co., Amsterdam-New York, 1982, pp. 209–230. MR 84f:03047; the entire collection 83i:03003.

[1983] Sets constructed from sequences of measures: Revisited. *J. Symbolic Logic* **48** (1983), no. 3, 600–609. MR 85j:03052.

[1984] The core model for sequences of measures. I. *Math. Proc. Cambridge Philos. Soc.* **95** (1984), no. 2, 229–260. MR 85i:03163.

[1994] The core model up to a Woodin cardinal. *In* "Logic, Methodology and Philosophy of Science, IX" (D. Prawitz et al., eds.), Uppsala, 1991. *Stud. Logic Foundations Math.* **134**, North-Holland, Amsterdam, 1994, pp. 157–175. MR 96g:03091; the entire collection MR 96a:03004.

[∞a] An introduction to inner models. *In* "Handbook of Set Theory" (A. Kanamori, M. Foreman, et al., eds.).

[∞b] The covering lemma. *In* "Handbook of Set Theory" (A. Kanamori, M. Foreman, et al., eds.).

W. J. Mitchell, E. Schimmerling, and J. R. Steel

[1997] The covering lemma up to a Woodin cardinal. *Ann. Pure Appl. Logic* **84** (1997), no. 2, 219–255. MR 98b:03067.

W. J. Mitchell and J. R. Steel

[1994] "Fine Structure and Iteration Trees." *Lecture Notes in Logic* **3**, Springer-Verlag, Berlin, 1994. MR 95m:03099.

J. D. Monk and D. S. Scott

[1963/64] Additions to some results of Erdős and Tarski. *Fund. Math.* **53** (1963/64), 335–343. MR 29 #3386.

R. M. Montague

[1955] Well-founded relations; generalizations of principle of induction and recursion. *Bull. Amer. Math. Soc.* **61** (1955), 442. Abstract.

[1961] Fraenkel's addition to the axioms of Zermelo. *In* "Essays on the Foundations of Mathematics" (Y. Bar-Hillel et al., eds.). Magnes Press, Hebrew Univ., Jerusalem, 1961, 91–114. MR 29 #1139; the entire collection MR 28 #3918.

R. M. Montague and R. L. Vaught

[1959] Natural models of set theories. *Fund. Math.* **47** (1959), 219–242. MR 23 #A1520.

C. F. Morgenstern

[1977] *Compact Extensions of $L^{<\omega}$ and Topics in Set Theory.* PhD thesis, University of Colorado, 1977.

Y. N. Moschovakis

[1970] Determinacy and prewellorderings of the continuum. *In* "Mathematical Logic and Foundations of Set Theory" (Y. Bar-Hillel, ed.), Proc. Internat. Colloq., Jerusalem, 1968. North-Holland Publishing Co., Amsterdam, 1970, pp. 24–62. MR 43 #6082; the entire collection MR 42 #1643.

[1971] Uniformization in a playful universe. *Bull. Amer. Math. Soc.* **77** (1971), 731–736. MR 44 #2609.

[1980] "Descriptive Set Theory." *Stud. Logic Foundations Math.* **100**, North-Holland Publishing Co., Amsterdam-New York, 1980. MR 82e:03002.

A. Mostowski

[1939] Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip. *Fund. Math.* **32** (1939), 201–252. Zbl. 022.12004.

[1949] An undecidable arithmetical statement. *Fund. Math.* **36** (1949), 143–164. MR 12,2d.

J. Mycielski

[1963/64] On the axiom of determinateness. *Fund. Math.* **53** (1963/64), 205–224. MR 28 #4991.

[1966] On the axiom of determinateness. II. *Fund. Math.* **59** (1966), 203–212. MR 35 #1489.

J. Mycielski and H. Steinhaus

[1962] A mathematical axiom contradicting the axiom of choice. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **10** (1962), 1–3. MR 25 #3850.

J. Mycielski and S. Świerczkowski

[1964] On the Lebesgue measurability and the axiom of determinateness. *Fund. Math.* **54** (1964), 67–71. MR 28 #4992.

J. R. Myhill and D. S. Scott

[1971] Ordinal definability. *In* "Axiomatic Set Theory" (D. S. Scott, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1971, pp. 271–278. MR 43 #7318; the entire collection MR 43 #38.

K. Namba

[1971] Independence proof of (ω, ω_α)-distributive law in complete Boolean algebras. *Comment. Math. Univ. St. Paul.* **19** (1971), 1–12. MR 45 #6602.

J. von Neumann

[1925] Eine Axiomatisierung der Mengenlehre. *J. f. Math.* **154** (1925), 219–240.

P. S. Novikov

[1951] On the consistency of some propositions of the descriptive theory of sets. *Tr. Mat. Inst. Steklov.* **38** (1951), 279–316. (Russian). MR 14,234a. English translation: *Amer. Math. Soc. Transl. (2)*, **29**, 51–89, 1963. MR 27 #3537.

J. B. Paris

[1969] *Boolean Extensions and Large Cardinals.* PhD thesis, Manchester University, 1969.

J. Pawlikowski

[1985] Lebesgue measurability implies Baire property. *Bull. Sci. Math. (2)* **109** (1985), no. 3, 321–324. MR 87h:03076.

R. S. Pierce
[1958] A note on complete Boolean algebras. *Proc. Amer. Math. Soc.* **9** (1958), 892–896. MR 21 #1280.

D. Pincus
[1971] Support structures for the axiom of choice. *J. Symbolic Logic* **36** (1971), 28–38. MR 44 #61.
[1972] Zermelo-Fraenkel consistency results by Fraenkel-Mostowski methods. *J. Symbolic Logic* **37** (1972), 721–743. MR 49 #2374.

B. Pospíšil
[1937] Remark on bicompact spaces. *Ann. of Math. (2)* **38** (1937), 845–846. Zbl. 017.42901.

W. C. Powell
[1972] Almost huge cardinals and Vopenka's principle. *Notices Amer. Math. Soc.* **19** (1972), A-616. Abstract.

K. L. Prikry
[1970] Changing measurable into accessible cardinals. *Dissertationes Math. Rozprawy Mat.* **68** (1970), 55 pp. MR 41 #6685.
[1975] Ideal and powers of cardinals. *Bull. Amer. Math. Soc.* **81** (1975), no. 5, 907–909. MR 51 #10100.

L. B. Radin
[1982] Adding closed cofinal sequences to large cardinals. *Ann. Math. Logic* **22** (1982), no. 3, 243–261. MR 83m:03062.

J. Raisonnier and J. Stern
[1985] The strength of measurability hypotheses. *Israel J. Math.* **50** (1985), no. 4, 337–349. MR 87d:03129.

F. P. Ramsey
[1929/30] On a problem of formal logic. *Proc. London Math. Soc.* **30** (1929/30), 264–286.

W. N. Reinhardt
[1974] Remarks on reflection principles, large cardinals, and elementary embeddings. *In* "Axiomatic Set Theory" (T. J. Jech, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1974, pp. 189–205. MR 53 #5302; the entire collection MR 50 #6840.

W. N. Reinhardt and J. H. Silver
[1965] On some problems of Erdős and Hajnal. *Notices Amer. Math. Soc.* **12** (1965), 723. Abstract.

F. Rothberger
[1938] Eine Äquivalenz zwischen der Kontinuumhypothese und der Existenz der Lusinschen und Sierpińskischen Mengen. *Fund. Math.* **30** (1938), 215–217. Zbl. 018.39401.
[1941] Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C. *Proc. Cambridge Philos. Soc.* **37** (1941), 109–126. MR 2,352a.

F. Rowbottom
[1971] Some strong axioms of infinity incompatible with the axiom of constructibility. *Ann. Math. Logic* **3** (1971), no. 1, 1–44. MR 48 #1928.

M. E. Rudin
[1966] Types of ultrafilters. *In* "Topology Seminar" (R. H. Bing and R. J. Bean, eds.), Wisconsin, 1965. *Ann. of Math. Studies*, No. 60, Princeton

University Press, Princeton, N.J., 1966, pp. 147–151. MR 35 #7284; the entire collection MR 34 #1974.

[1971] Partial orders on the types in βN. *Trans. Amer. Math. Soc.* **155** (1971), 353–362. MR 42 #8459.

W. Rudin

[1956] Homogeneity problems in the theory of Čech compactifications. *Duke Math. J.* **23** (1956), 409–419. MR 18,324d.

G. E. Sacks

[1971] Forcing with perfect closed sets. *In* "Axiomatic Set Theory" (D. S. Scott, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1971, pp. 331–355. MR 43 #1827; the entire collection MR 43 #38.

M. Scheepers

[1993] Gaps in ω^ω. *In* "Set Theory of the Reals" (H. Judah, ed.), Ramat Gan, 1991. *Israel Math. Conf. Proc.* **6**, Bar-Ilan Univ., Ramat Gan, distributed by the Amer. Math. Soc., Providence, RI, 1993, pp. 439–561. MR 95a:03061; the entire collection MR 94b:03010.

E. Schimmerling

[1995] Combinatorial principles in the core model for one Woodin cardinal. *Ann. Pure Appl. Logic* **74** (1995), no. 2, 153–201. MR 96f:03041.

C. Schlindwein

[1993] Simplified RCS iterations. *Arch. Math. Logic* **32** (1993), no. 5, 341–349. MR 94f:03061.

D. S. Scott

[1961] Measurable cardinals and constructible sets. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **9** (1961), 521–524. MR 26 #1263.

[1967] A proof of the independence of the continuum hypothesis. *Math. Systems Theory* **1** (1967), 89–111. MR 36 #1321.

N. A. Shanin

[1946] A theorem from the general theory of sets. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **53** (1946), 399–400. MR 8,333i.

S. Shelah

[1978] Jonsson algebras in successor cardinals. *Israel J. Math.* **30** (1978), no. 1-2, 57–64. MR 58 21572.

[1979] On successors of singular cardinals. *In* "Logic Colloquium '78" (M. Boffa et al., eds.), Mons, 1978. *Stud. Logic Foundations Math.* **97**, North-Holland Publishing Co., Amsterdam-New York, 1979, pp. 357–380. MR 82d:03079; the entire collection MR 81a:03001.

[1982] "Proper Forcing." *Lecture Notes in Mathematics* **940**, Springer-Verlag, Berlin-New York, 1982. MR 84h:03002.

[1983] The singular cardinals problem: Independence results. *In* "Surveys in Set Theory" (A. R. D. Mathias, ed.). *London Math. Soc. Lecture Note Ser.* **87**, Cambridge University Press, Cambridge-New York, 1983, 116–134. MR 87b:03114; the entire collection 86m:03005.

[1984] Can you take Solovay's inaccessible away? *Israel J. Math.* **48** (1984), no. 1, 1–47. MR 86g:03082a.

[1987] Semiproper forcing axiom implies Martin's maximum but not PFA^+. *J. Symbolic Logic* **52** (1987), no. 2, 360–367. MR 89g:03072.

[1994] "Cardinal Arithmetic." *Oxford Logic Guides* **29**, The Clarendon Press, Oxford University Press, New York, 1994. MR 96e:03001.

[1998] "Proper and Improper Forcing." *Perspectives in Mathematical Logic* **29**, Springer-Verlag, Berlin, second edition, 1998. MR 98m:03002.

S. Shelah and W. H. Woodin

[1990] Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable. *Israel J. Math.* **70** (1990), no. 3, 381–394. MR 92m:03087.

J. C. Shepherdson

[1951] Inner models for set theory. I. *J. Symbolic Logic* **16** (1951), 161–190. MR 13,522a.

[1952] Inner models for set theory. II. *J. Symbolic Logic* **17** (1952), 225–237. MR 14,834h.

[1953] Inner models for set theory. III. *J. Symbolic Logic* **18** (1953), 145–167. MR 15,278f.

J. R. Shoenfield

[1954] A relative consistency proof. *J. Symbolic Logic* **19** (1954), 21–28. MR 15,668f.

[1959] On the independence of the axiom of constructibility. *Amer. J. Math.* **81** (1959), 537–540. MR 21 #5563.

[1961] The problem of predicativity. *In* "Essays on the Foundations of Mathematics" (Y. Bar-Hillel et al., eds.). Magnes Press, Hebrew Univ., Jerusalem, 1961, 132–139. MR 29 #2177; the entire collection MR 28 #3918.

W. Sierpiński

[1925] Sur une classe d'ensembles. *Fund. Math.* **7** (1925), 237–243.

[1927] Sur quelques propriétés des ensembles projectifs. *C. R. Acad. Sci. Paris* **185** (1927), 833–835.

[1928] Sur une décomposition d'ensembles. *Monatsh. Math.* **35** (1928), 239–242.

[1933] Sur un problème de la théorie des relations. *Ann. Scuola Norm. Sup. Pisa, II* **2** (1933), 285–287. Zbl. 007.09702.

W. Sierpiński and A. Tarski

[1930] Sur une propriété caracteristique des nombres inaccessibles. *Fund. Math.* **15** (1930), 292–300.

R. Sikorski

[1964] "Boolean Algebras." Springer-Verlag, Berlin-New York, second edition, 1964. MR 31 #2178.

J. H. Silver

[1970a] A large cardinal in the constructible universe. *Fund. Math.* **69** (1970), 93–100. MR 43 #43.

[1970b] Every analytic set is Ramsey. *J. Symbolic Logic* **35** (1970), 60–64. MR 48 #10807.

[1971a] Measurable cardinals and Δ^1_3 well-orderings. *Ann. of Math. (2)* **94** (1971), 414–446. MR 45 #8517.

[1971b] Some applications of model theory in set theory. *Ann. Math. Logic* **3** (1971), no. 1, 45–110. MR 53 #12950.

[1971c] The independence of Kurepa's conjecture and two-cardinal conjectures in model theory. *In* "Axiomatic Set Theory" (D. S. Scott, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1971, pp. 383–390. MR 43 #3112; the entire collection MR 43 #38.

[1971d] The consistency of the GCH with the existence of a measurable cardinal. *In* "Axiomatic Set Theory" (D. S. Scott, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1971, pp. 391–395. MR 43 #4663; the entire collection MR 43 #38.

[1975] On the singular cardinals problem. *In* "Proceedings of the International Congress of Mathematicians," Vancouer, B. C., 1974, Vol. 1. Canad. Math. Congress, Montreal, Que., 1975, pp. 265–268. MR 55 #2576; the entire collection MR 54 #7.

[1980] Counting the number of equivalence classes of Borel and coanalytic equivalence relations. *Ann. Math. Logic* **18** (1980), no. 1, 1–28. MR 81d:03051.

T. Skolem

[1970] "Selected Works in Logic." Edited by J. E. Fenstad. Universitetsforlaget, Oslo, 1970. MR 44 #2562.

R. M. Solovay

[1963] Independence results in the theory of cardinals. *Notices Amer. Math. Soc.* **10** (1963), 595. Abstract.

[1967] A nonconstructible Δ^1_3 set of integers. *Trans. Amer. Math. Soc.* **127** (1967), 50–75. MR 35 #2748.

[1969] On the cardinality of Σ^1_2 sets of reals. *In* "Foundations of Mathematics" (J. J. Buloff et al., eds.), Symposium Commemorating Kurt Gödel, Columbus, Ohio, 1966. Springer, New York, 1969, pp. 58–73. MR 43 #3115; the entire collection MR 39 #2594.

[1970] A model of set theory in which every set of reals is Lebesgue measurable. *Ann. of Math. (2)* **92** (1970), 1–56. MR 42 #64.

[1971] Real-valued measurable cardinals. *In* "Axiomatic Set Theory" (D. S. Scott, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967. Amer. Math. Soc., Providence, R.I., 1971, pp. 397–428. MR 45 #55; the entire collection MR 43 #38.

[1974] Strongly compact cardinals and the GCH. *In* "Proceedings of the Tarski Symposium" (L. Henkin et al., eds.), Proc. Sympos. Pure Math., Vol. XXV, Univ. of California, Berkeley, Calif., 1971. Amer. Math. Soc., Providence, R.I., 1974, pp. 365–372. MR 52 #106; the entire collection MR 50 #1829.

R. M. Solovay, W. N. Reinhardt, and A. Kanamori

[1978] Strong axioms of infinity and elementary embeddings. *Ann. Math. Logic* **13** (1978), no. 1, 73–116. MR 80h:03072.

R. M. Solovay and S. Tennenbaum

[1971] Iterated Cohen extensions and Souslin's problem. *Ann. of Math. (2)* **94** (1971), 201–245. MR 45 #3212.

E. Specker

[1949] Sur un problème de Sikorski. *Colloq. Math.* **2** (1949), 9–12. MR 12,597b.

[1957] Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom). *Z. Math. Logik Grundlagen Math.* **3** (1957), 173–210. MR 20 #5738.

J. R. Steel

[1996] "The Core Model Iterability Problem." *Lecture Notes in Logic* **8**, Springer-Verlag, Berlin, 1996. MR 99k:03043.

J. R. Steel and R. van Wesep

[1982] Two consequences of determinacy consistent with choice. *Trans. Amer. Math. Soc.* **272** (1982), no. 1, 67–85. MR 83i:03081.

M. H. Stone

[1936] The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.* **40** (1936), 37–111. Zbl. 014.34002.

M. Suslin

[1917] Sur une définition des ensembles mesurables B sans nombres transfinis. *C. R. Acad. Sci. Paris* **164** (1917), 88–91.

[1920] Problème 3. *Fund. Math.* **1** (1920), 223.

G. Takeuti

[1970] A relativization of axioms of strong infinity to ω_1. *Ann. Japan Assoc. Philos. Sci.* **3** (1970), 191–204. MR 43 #6083.

A. Tarski

[1925a] Sur les ensembles finis. *Fund. Math.* **6** (1925), 45–95.

[1925b] Quelques théorèmes sur les alephs. *Fund. Math.* **7** (1925), 1–14.

[1930] Une contribution à la théorie de la mesure. *Fund. Math.* **15** (1930), 42–50.

[1938] Über unerreichbare Kardinalzahlen. *Fund. Math.* **30** (1938), 68–89. Zbl. 018.34702.

[1939] On undecidable statements in enlarged systems of logic and the concept of truth. *J. Symbolic Logic* **4** (1939), 105–112. MR 1,34c.

[1945] Ideale in vollständingen Mengenkörpern. II. *Fund. Math.* **33** (1945), 51–65. MR 8,193b.

[1962] Some problems and results relevant to the foundations of set theory. *In* "Logic, Methodology and Philosophy of Science" (E. Nagel et al., eds.), Proc. 1960 Internat. Congr. Stanford Univ. Press, Stanford, Calif., 1962, pp. 125–135. MR 27 #1382; the entire collection MR 29 #3347.

S. Tennenbaum

[1968] Souslin's problem. *Proc. Natl. Acad. Sci. U.S.A.* **59** (1968), 60–63. MR 37 #55.

S. Todorčević

[1984] Trees and linearly ordered sets. *In* "Handbook of Set-Theoretic Topology" (K. Kunen and J. E. Vaughan, eds.). North-Holland Publishing Co., Amsterdam-New York, 1984, 235–293. MR 86h:54040; the entire collection MR 85k:54001.

[1987] Partitioning pairs of countable ordinals. *Acta Math.* **159** (1987), no. 3-4, 261–294. MR 88i:04002.

[1989] "Partition Problems in Topology." *Contemp. Math.* **84**, Amer. Math. Soc., Providence, R.I., 1989. MR 90d:04001.

[1997] "Topics in Topology." *Lecture Notes in Mathematics* **1652**, Springer-Verlag, Berlin, 1997. MR 98g:54002.

J. Truss

[1977] Sets having calibre $\aleph_1$. *In* "Logic Colloquium '76" (R. O. Gandy and J. M. E. Hyland, eds.), Oxford, 1976. *Stud. Logic Foundations Math.* **87**, North-Holland Publishing Co., Amsterdam, 1977, pp. 595–612. MR 57 #16073; the entire collection MR 57 #53.

S. Ulam

[1930] Zur Masstheorie in der allgemeinen Mengenlehre. *Fund. Math.* **16** (1930), 140–150.

B. Veličković

[1992] Forcing axioms and stationary sets. *Adv. Math.* **94** (1992), no. 2, 256–284. MR 93k:03045.

B. Veličković and W. H. Woodin

[1998] Complexity of reals in inner models of set theory. *Ann. Pure Appl. Logic* **92** (1998), no. 3, 283–295. MR 99f:03067.

G. Vitali

[1905] "Sul Problema della Mesura dei Gruppi di Punti di una Retta." Bologna, 1905.

D. A. Vladimirov

[1969] "Boolean Algebras." Izdat. Nauka, Moscow, 1969. (Russian). MR 41 #8314.

P. Vopěnka

[1964] The independence of the continuum hypothesis. *Comment. Math. Univ. Carolinae* **5** (1964), suppl. I, 48 pp. (Russian). MR 29 #5742.

[1965a] The limits of sheaves and applications on constructions of models. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13** (1965), 189–192. MR 32 #53.

[1965b] On ∇-model of set theory. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13** (1965), 267–272. MR 32 #54.

[1965c] Properties of ∇-model. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13** (1965), 441–444. MR 32 #7400.

[1966] ∇-models in which the generalized continuum hypothesis does not hold. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **14** (1966), 95–99. MR 34 #41.

[1967a] General theory of ∇-models. *Comment. Math. Univ. Carolinae* **8** (1967), 145–170. MR 35 #5310.

[1967b] The limits of sheaves over extremally disconnected compact Hausdorff spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **15** (1967), 1–4. MR 37 #912.

P. Vopěnka and B. Balcar

[1967] On complete models of the set theory. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **15** (1967), 839–841. MR 39 #3989.

P. Vopěnka, B. Balcar, and P. Hájek

[1968] The notion of effective sets and a new proof of the consistency of the axiom of choice. *J. Symbolic Logic* **33** (1968), 495–496. Abstract.

P. Vopěnka and P. Hájek

[1965] Permutation submodels of the model ∇. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13** (1965), 611–614. MR 33 #2530.

[1967] Concerning the ∇-models of set theory. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **15** (1967), 113–117. MR 37 #5094.

[1972] "The Theory of Semisets." Academia (Publishing House of the Czechoslovak Academy of Sciences), Prague, 1972. MR 56 #2824.

P. Vopěnka and K. Hrbáček

[1966] On strongly measurable cardinals. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **14** (1966), 587–591. MR 35 #2747.

[1967] The consistency of some theorems on real numbers. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **15** (1967), 107–111. MR 37 #1255.

W. H. Woodin

[1988] Supecompact cardinals, sets of reals, and weakly homogeneous trees. *Proc. Natl. Acad. Sci. U.S.A.* **85** (1988), no. 18, 6587–6591. MR 89m:03040.

[1999] "The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal." *de Gruyter Series in Logic and its Applications* **1**, Walter de Gruyter & Co., Berlin, 1999. MR 2001e:03001.

W. H. Woodin, A. R. D. Mathias, and K. Hauser

[∞] "The Axiom of Determinacy." *de Gruyter Series in Logic and its Applications*, Walter de Gruyter & Co.

W. H. Young

[1906] "The Theory of Sets of Points." Cambridge Univ. Press, London and New York, 1906.

J. Zapletal

[1996] A new proof of Kunen's inconsistency. *Proc. Amer. Math. Soc.* **124** (1996), no. 7, 2203–2204. MR 96i:03051.

E. Zermelo

[1904] Beweis, dass jede Menge wohlgeordnet werden kann. *Math. Ann.* **59** (1904), 514–516.

[1908] Untersuchungen über die Grundlagen der Mengenlehre. *Math. Ann.* **65** (1908), 261–281.

M. Zorn

[1935] A remark on method in transfinite algebra. *Bull. Amer. Math. Soc.* **41** (1935), 667–670. Zbl. 012.33702.

Notation

ZF	Zermelo-Fraenkel axiomatic theory	3
ZFC	the theory ZF with the Axiom of Choice	3
$x \in y$, $x = y$	atomic formulas: x is a member of y, x is equal to y	5
$\wedge, \vee, \neg, \rightarrow, \leftrightarrow$	logical connectives: conjunction, disjunction, negation, implication, equivalence	5
$\forall x$, $\exists x$	quantifiers: for all x, there exists x	5
$\{x : \varphi(x, p_1, \ldots, p_n)\}$	the class of all x satisfying $\varphi(x, p_1, \ldots, p_n)$	5
$C = D$	a class C is equal to a class D	5
V	universal class (universe, $\{x : x = x\}$)	6
$C \subset D$	a class C is included in a class D	6
$C \cap D$	the intersection of classes C and D	6
$C \cup D$	the union of classes C and D	6
$C - D$	the difference of classes C and D	6
$\bigcup C$	the union of sets from a class C	6
$\{a, b\}$	the pair	7
$\{a\}$	the singleton	7
(a, b)	the ordered pair	7
$(a_1, \ldots, a_{n+1})$	the ordered $n+1$-tuple	7
$\emptyset$	the empty set	8
$\bigcap C$	the intersection of sets from a class C	8
$\bigcup X$	the union	9
$\{a_1, \ldots, a_n\}$	the set with elements a_1, …, a_n	9
$X \triangle Y$	symmetric difference of X and Y	9
$P(X)$	the power set of X	9
$X \times Y$	the product of X and Y	10
$X_1 \times \ldots \times X_{n+1}$	the product of $n+1$ sets	10
X^n	the power of a set X	10
$\mathrm{dom}(R)$	the domain of a relation R	10
$\mathrm{ran}(R)$	the range of a relation R	10
$\mathrm{field}(R)$	the field of a relation R	10
$y = f(x)$	y is the value of f at x	11
$f : X \rightarrow Y$	f is a function from X to Y	11
Y^X	the set of functions from X to Y	11
$f \restriction X$	the restriction of a function f to a set X	11
$f \circ g$	composition of f and g	11
$f``X$, $f(X)$	the image of a set X by a function f	11
$f_{-1}(X)$	the inverse image of a set X by a function f	11
f^{-1}	the inverse of a function f	11
$F``C$, $F(C)$	the image of a class C by a class function F	12
$[x]$	the equivalence class of x	12
$X/\equiv$	the quotient of X by an equivalence relation $\equiv$	12

$\boldsymbol{N}$	the smallest inductive set	13
0, 1, 2, 3, ...	the natural numbers	13
$(P, <)$	a partially ordered set	17
$\sup X$	the supremum of X	17
$\inf X$	the infimum of X	17
Ord	the class of ordinals	19
$\alpha + 1$	the successor of an ordinal α	20
ω, $\boldsymbol{N}$	the set of finite ordinals (natural numbers)	20
$\langle a_\xi : \xi < \alpha \rangle$	an α-sequence; a (transfinite) sequence of length α	21
$s^\frown x$, sx	the extension of a sequence s by an element x	21
$\langle a_\alpha : \alpha \in Ord \rangle$	a sequence	21
$\lim_{\xi \to \alpha} \gamma_\xi$	the limit of a sequence $\langle \gamma_\xi : \xi < \alpha \rangle$	22
$\alpha + \beta$	the sum of ordinals α and β	23
$\alpha \cdot \beta$	the product of ordinals α and β	23
α^β	the power of an ordinal α by an ordinal β	23
ε_0	the least ordinal α such that $\alpha = \omega^\alpha$	24
$\rho(x)$	the rank of an element x in a well-founded relation E	25
$\lvert X \rvert = \lvert Y \rvert$	sets X, Y have the same cardinality	27
$\lvert X \rvert \leq \lvert Y \rvert$	the cardinality of a set X is less or equal to the cardinality of a set Y (there exists a one-to-one mapping of X into Y)	27
$\kappa + \lambda$	the sum of cardinals κ and λ	28
$\kappa \cdot \lambda$	the product of cardinals κ and λ	28
κ^λ	the power of a cardinal κ by a cardinal λ	28
χ_X	the characteristic function of a subset X of a given set	28
$\lvert W \rvert$	the cardinal of a well-ordered set W	29
α^+	the cardinal successor of an ordinal α	29
$h(X)$	Hartogs function	29
$\aleph_\alpha$	the αth infinite cardinal	30
ω_α	the αth infinite order-type of a well-ordered set	30
Γ	the canonical well-ordering of Ord^2	30
$\operatorname{cf} \alpha$	the cofinality of an ordinal α	31
$\boldsymbol{R}$	the set of real numbers	37
$\mathfrak{c}$	the cardinality of the continuum (continuum)	37
$\boldsymbol{Q}$	the set of rational numbers	37
$\boldsymbol{C}$	the Cantor set	37
CH	the Continuum Hypothesis	37
G_δ, F_σ	G_δ sets, F_σ sets	42
$\mathcal{N}$	the Baire space (ω^ω)	42
$O(s)$	a basic clopen set in the Baire space	42
Seq	the set of finite sequences of natural numbers	43
$[T]$	the set of infinite paths through a tree T	43
AC	the Axiom of Choice	47
DC	the Principle of Dependent Choices	50
$[A]^\lambda$	the set of subsets of A of cardinality λ	51
$[A]^{<\kappa}$, $P_\kappa(A)$	the set of subsets of A of cardinality less than κ	52
$\sum_{i \in I} \kappa_i$	the sum of cardinal numbers κ_i, $i \in I$	52
$\prod_{i \in I} X_i$	the product of sets X_i, $i \in I$	53
$\prod_{i \in I} \kappa_i$	the product of cardinal numbers κ_i, $i \in I$	53
GCH	the Generalized Continuum Hypothesis	55
$\beth_\alpha$	the beth function	55
$\gimel(\kappa)$	the gimel function ($\kappa^{\operatorname{cf} \kappa}$)	56
SCH	the Singular Cardinal Hypothesis	58
TC(S)	the transitive closure of a set S	64

V_α	the αth set of the cumulative hierarchy of sets	64
$\mathrm{rank}(x)$	the rank of a set x (in the cumulative hierarchy of sets)	64
$\hat{C}$	the set of elements of a class C with minimal rank	65
$[x]$	the type of an equivalence class of an equivalence relation on a proper class	65
$C/{\equiv}$	the quotient of a (proper) class C by an equivalence relation $\equiv$	65
$\mathrm{ext}_E(x)$	the extension of x by a binary relation E ($\{z : z\ E\ x\}$)	67
BG	Bernays-Gödel axiomatic theory	70
BGC	the theory BG with the Axiom of Choice	70
$\hat{P}$	$\{Q \in [A]^{<\omega} : P \subset Q\}$ for $P \in [A]^{<\omega}$	73
$u+u$, $u\cdot v$, $-u$	the Boolean operations: the sum, the product, and the complement	78
$[\varphi]$	the class of equivalent sentences of a first order language (member of the Lindenbaum algebra)	79
B^+	the set of all nonzero elements of a Boolean algebra B	79
$B{\restriction}a$	the Boolean algebra $\{u \in B : u \le a\}$ with the partial order inherited from B	79
$u \triangle v$	$(u-v)+(v-u)$	80
B/I, $B/{\sim}$	the quotient of a Boolean algebra B mod I	80
$\sum\{u : u \in X\}$	the supremum (sum) of a set X in a Boolean algebra	82
$\prod\{u : u \in X\}$	the infimum (product) of a set X in a Boolean algebra	82
$\mathrm{sat}(B)$	the least κ that B is κ-saturated	84
$f_*(U)$	the ultrafilter $\{X \subset T : f_{-1}(X) \in U\}$	86
$a = \lim_U a_n$	a is the U-limit of a_n, $n \in \omega$	86
$u \oplus v$	$(u-v)+(v-u)$	87
$\triangle_{\alpha<\kappa} X_\alpha$	the diagonal intersection of X_α, $\alpha < \kappa$	92
I_{NS}	the nonstationary ideal	93
$\sum_{\alpha<\kappa} X_\alpha$	the diagonal union of X_α, $\alpha < \kappa$	93
E^κ_λ	$\{\alpha < \kappa : \mathrm{cf}\,\alpha = \lambda\}$	94
$\mathrm{Tr}(S)$	the trace of a stationary set S	99
$\mathrm{Lim}(C)$	the set of all limit points of a set C	100
$o(A)$	the order of a stationary set A	100
$\triangle_{a\in A} X_a$	the diagonal intersection in $P_\kappa(A)$	101
$X{\restriction}A$	the projection of $X \in P_\kappa(B)$ to a set $A \subset B$	102
Y^B	the lifting of $Y \in P_\kappa(A)$ to $B \supset A$	102
$\kappa \to (\lambda)^n_m$	κ arrows λ	109
$\kappa \to (\alpha)^n_m$	κ arrows α	112
$\kappa \to (\alpha,\beta)^n$	κ arrows (α,β)	112
$o(x)$	the order-type of $\{y : y < x\}$ in a tree T	114
$\mathrm{height}(T)$	the height of a tree T, $\sup\{o(x)+1 : x \in T\}$	114
$\kappa \to (\alpha)^{<\omega}_m$	κ arrows α	121
$\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$	the hierarchy of Borel sets ($\mathbf{\Sigma}^0_\alpha$ sets, $\mathbf{\Pi}^0_\alpha$ sets)	140
$\mathcal{A}\{A_s : s \in Seq\}$	Suslin operation $(\bigcup_{a\in\omega^\omega} \bigcap_{n=0}^\infty A_{a\restriction n})$	143
$\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$	the hierarchy of projective sets ($\mathbf{\Sigma}^1_n$ sets, $\mathbf{\Pi}^1_n$ sets, $\mathbf{\Delta}^1_n$ sets)	144
$\mu^*(X)$	the outer measure of a set X	146
$v(I)$	the volume of an interval I	146
$\mu(A)$	the Lebesgue measure of a set A	147
$t^{\mathfrak{A}}[a_1,\dots,a_n]$	the value of a term t in a model $\mathfrak{A}$	155
$\mathfrak{A} \vDash \varphi[a_1,\dots,a_n]$	a formula φ holds in a model $\mathfrak{A}$	155
$\mathfrak{B} \prec \mathfrak{A}$	a model $\mathfrak{B}$ is an elementary submodel of a model $\mathfrak{A}$	156
$f =_F g$	the functions f and g are equal modulo a filter F	158
$\varphi^{M,E}$, φ^M, $(M,E) \vDash \varphi$	the relativization of a formula φ	161

$Form$	the set of all formulas of the language $\{\in\}$	162
$\ulcorner\varphi\urcorner$	the set coding a formula φ ($\ulcorner\varphi\urcorner \in Form$)	162
$\#\sigma$	the Gödel number of a sentence σ	162
$T(x)$	the truth definition	162
H_κ	the set of all x with $\lvert \mathrm{TC}(X)\rvert < \kappa$	171
$\mathrm{def}(M)$	the set of subsets of M definable over $(M, \in)$	175
L_α, L	the hierarchy of constructible sets	175
$G_1, \ldots, G_{10}$	Gödel operations	178
$\mathrm{cl}(M)$	the closure of a set M under Gödel operations $G_1, \ldots, G_{10}$	181
C^M	the class $\{x : \varphi^M(x)\}$ where $C = \{x : \varphi(x)\}$	182
F^M	the operation F defined in a class M	182
c^M	the constant c defined in a class M	182
Σ_n, Π_n, Δ_n	the hierarchy of properties, classes, relations, and functions	183
$\vDash_n$, $\vDash_n^M$	the satisfaction relation restricted to Σ_n formulas	186
$M \prec_{\Sigma_n} N$	M is a Σ_n-elementary submodel of N	187
$<^n_{\alpha+1}$	the end-extensions of canonical well-orderings of the subsets W_n^α of $L_{\alpha+1}$	189
$<_{\alpha+1}$	the canonical well-ordering of $L_{\alpha+1}$	189
$<_L$	the canonical well-ordering of L	190
$\diamondsuit$	the Diamond Principle	191
$\mathrm{def}_A(M)$	the set of subsets of M definable over $(M, \in, A \cap M)$	192
$L_\alpha[A]$, $L[A]$	the hierarchy of sets constructible from a set A	192
$L(\boldsymbol{R})$	the smallest inner model that contains all reals	193
$L_\alpha(A)$, $L(A)$	the hierarchy of sets constructible from elements of the transitive closure of a set A	193
OD	the class of ordinal-definable sets	194
HOD	the class of hereditarily ordinal-definable sets	194
$OD[A]$	the class of ordinal-definable sets from A	195
$HOD[A]$	the class of hereditarily ordinal-definable sets from A	195
$OD(A)$	the class of ordinal-definable sets over A	195
$HOD(A)$	the class of hereditarily ordinal-definable sets over A	196
ZF^-	Zermelo-Fraenkel set theory without the Power Set Axiom	198
$L[A]$	the class of sets constructible from a class A	199
$M[X]$	the least model of ZF such that $M \subset M[X]$ and $X \in M[X]$	199
$\dot{a}$	a name of a set from $V[G]$	203
$x \sim y$	the set of conditions compatible with x in a forcing notion is the same as that for y	205
$Q = P/\!\sim$	Q is the separative quotient of P	205
$e : P \to B(P)$	the Boolean completion of a partially ordered set P	206
$\lVert x = y\rVert$, $\lVert x \in y\rVert$	Boolean functions in a Boolean universe, the Boolean values of $x = y$ and $x \in y$	206
$\lVert\varphi\rVert$	the Boolean value of a formula in a Boolean-valued model	207
V^B	the Boolean-valued model	209
$\rho(x)$	the rank function in V^B	209
$u \Rightarrow v$	$-u + v$	209
$\lVert x \in y\rVert$, $\lVert x \subset y\rVert$, $\lVert x = y\rVert$	the Boolean values of atomic formulas in V^B	209
$\check{x}$	the canonical name for a set in the ground model	211
$\check{x}$	the canonical name for a set in the ground model	212
$\dot{G}$	the canonical name for generic ultrafilter	214
M^B	the Boolean-valued model inside a transitive model M	214
M^P	the class of P-names, $M^P = M^{B(P)}$	215
$\Vdash$, $\Vdash_P$	the forcing relation	215
$p \Vdash \varphi$	p forces φ	215

$\check{M}$	the canonical name for the ground model	215
x^G	the interpretation of a name by a generic ultrafilter	216
$M[G]$	the generic extension of a transitive model M	216
x^G	the interpretation of a P-name by a generic filter	218
$P \times Q$	the product forcing	229
$G = G_1 \times G_2$	a generic set G is the product of projections G_1 and G_2 in a product forcing	229
$\prod_{i\in I} P_i$	the product of forcing notions P_i, $i \in I$	230
$s(p)$	the support of a condition in an infinite product forcing, $s(p) = \{i \in I : p(i) \neq 1\}$	230
G_i, $i \in I$	the projections of a generic filter G on the coordinates of the product forcing $\prod_{i\in I} P_i$	230
$P^{\leq\lambda} \times P^{>\lambda}$	the decomposition of Easton product into two parts, one satisfying the λ^+-chain condition and the other being λ-closed	233
$\mathrm{Col}(\kappa, <\lambda)$	the Lévy collapsing algebra (λ is an inaccessible cardinal)	238
$(P_T, <)$	the forcing associated with a tree T	242
$p \leq_n q$	$p \leq q$ and every nth splitting node of q is an nth splitting node of p	244
$p\restriction s$	the tree $\{t \in p : t \subset s \text{ or } t \supset s\}$	245
$B_1\restriction a$	the algebra $\{x \cdot a : x \in B_1\}$ for an $a \in B_2 \supset B_1$, $a \neq 0$	248
ZFA	set theory with atoms	250
$P^\alpha(S)$, $P^\infty(S)$	the cumulative hierarchy in ZFA	250
$P^\infty(\emptyset)$	the kernel in ZFA	250
$\mathrm{sym}(x)$	the symmetry group of a set in ZFA, the group of permutations $\{\pi \in \mathcal{G} : \pi(x) = x\}$	251
$\mathrm{fix}(E)$	the subgroup of permutations fixed on a set E of a given group	252
$\mathrm{sym}(\dot{x})$	the symmetry group of a name $\dot{x} \in V^B$, the group of automorphisms of B, $\{\pi \in \mathcal{G} : \pi(\dot{x}) = \dot{x}\}$	253
HS	the class of hereditarily symmetric names	253
$x \mapsto \tilde{x}$	an embedding of a permutation model U with the set of atoms A into a symmetric model N of ZF so that $(P_\alpha(A))^U$ and $(P_\alpha(\tilde{A}))^N$ are $\in$-isomorphic	256
$\diamondsuit'$	a principle equivalent to the Diamond Principle $\diamondsuit$	263
$P * \dot{Q}$	two-step iteration of forcing notions	267
$\Vdash_P \varphi$	$\|\varphi\|_{B(P)} = 1$	267
$G * H$	two-step iteration of generic filters	267
$B * \dot{C}$	the iteration of two complete Boolean algebras	269
$D : B$	the quotient of a complete Boolean algebra D by a filter generated by the generic ultrafilter on a complete subalgebra B	269
P_α	the iteration of a sequence $\langle \dot{Q}_\beta : \beta < \alpha\rangle$ of names of forcing notions	270
MA, MA_κ	Martin's Axiom	272
SH	Suslin's Hypothesis	274
$s(p)$	the support of p, $s(p) = \{\beta : \text{not } \Vdash_\beta p(\beta) = 1\}$	280
$f =^* g$	f equals g modulo an ultrafilter U, $\{x \in S : f(x) = g(x)\} \in U$	285
$[f]$	the class of f in $=^*$	285
$f \in^* g$, $[f] \in^* [g]$	a function f is a member of a function g modulo an ultrafilter U, $\{x \in S : f(x) \in g(x)\} \in U$	285
Ult, $\mathrm{Ult}_U(V)$, $(\mathrm{Ult}, \in^*)$	the ultraproduct of the universe	285

j, j_U	an elementary embedding of V in Ult	285
$\mathrm{ext}(f)$	the extension of the equivalence class $[f]$, $\{[g] : g \in^* f\}$	286
$M(X)$	the Mahlo operation for a class X, $M(X) = \{\alpha : X \cap \alpha$ is stationary in $\alpha\}$	290
$\mathcal{L}_{\kappa,\omega}$	a language with κ variables, and infinitary connectives $\bigvee_{\xi<\alpha}$, $\bigwedge_{\xi<\alpha}$, for $\alpha < \kappa$	293
$\mathcal{L}_{\kappa,\kappa}$	a language with κ variables, infinitary connectives $\bigvee_{\xi<\alpha}$, $\bigwedge_{\xi<\alpha}$, and infinitary quantifiers $\exists_{\xi<\alpha} v_\xi$, $\forall_{\xi<\alpha} v_\xi$ for $\alpha < \kappa$	293
c^φ_ξ, $\xi < \alpha$	Skolem constants	293
$\exists_{\xi<\alpha} v_\xi\, \varphi(v_\xi, \ldots)_{\xi<\alpha} \to \varphi(c^\varphi_\xi, \ldots)_{\xi<\alpha}$	a Skolem sentence for a formula φ in an $\mathcal{L}_{\kappa,\kappa}$ language	294
Π^n_m, Σ^n_m	the hierarchy of higher order formulas	295
$h_\varphi(x_1, \ldots, x_n)$	a definable Skolem function for a formula $\varphi(u, v_1, \ldots, v_n)$	300
$0^\sharp$	zero-sharp, $0^\sharp = \{\varphi : L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n]\}$	312
$h_\varphi(v_1, \ldots, v_n)$	the canonical Skolem function for $\varphi(u, v_1, \ldots, v_n)$	313
$H^{\mathfrak{A}}(X)$	the Skolem hull of X	314
$\Sigma(\mathfrak{A}, I)$	the set of all formulas $\varphi(v_1, \ldots, v_n)$ true in $\mathfrak{A}$ for increasing sequences of elements of a set of indiscernibles I	314
$\mathrm{Ult} = \mathrm{Ult}_D(M)$	the ultraproduct of M by an M-ultrafilter D	324
$x^\sharp$	x-sharp, $x^\sharp = \{\varphi : (L_{\aleph_\omega}[x], \in, x) \vDash \varphi[\aleph_1, \ldots, \aleph_n]\}$, for $x \subset \omega$	328
$H^\delta(\alpha \cup p)$	the Skolem hull of $\alpha \cup p$ in $(L_\delta, \in)$	331
$H^\rho_n(Z)$	the Σ_n Skolem hull of a set Z in $(L_\rho, \in)$	333
$\{(\mathrm{Ult}^{(\alpha)}, E^{(\alpha)}), i_{\alpha,\beta} : \alpha, \beta \in Ord\}$, $\mathrm{Ult}^{(\alpha)}_U(V)$	the iterated ultrapowers	342
$U^{(\alpha)}$	the $\kappa^{(\alpha)}$-complete ultrafilter on $\kappa^{(\alpha)}$, $U^{(\alpha)} = i_{0,\alpha}(U)$	342
$\kappa^{(\alpha)}$	the measurable cardinal in $\mathrm{Ult}^{(\alpha)}$, $\kappa^{(\alpha)} = i_{0,\alpha}(\kappa)$	344
U_n	the product ultrafilters, $U_1 = U$, $U_{n+1} = \{X \subset \kappa^{n+1} : \{\alpha : \{\langle\alpha_1, \ldots, \alpha_n\rangle : \langle\alpha, \alpha_1, \ldots, \alpha_n\rangle \in X\} \in U_n\} \in U\}$	345
U_E	the ultrafilter induced by U_n via the order isomorphism between $n = \lvert E\rvert$ and E	346
$\mathrm{in}_{E,S}(X)$	the inclusion map, $\mathrm{in}_{E,S}(X) = \{t \in \kappa^S : t{\restriction}E \in X\}$ for $X \subset \kappa^E$	346
$(B_\alpha, \subset)$	the Boolean algebra of sets $Z \subset \kappa^\alpha$ having a finite support, i.e., $Z = \mathrm{in}_{E,\alpha}(X)$ for some $X \subset \kappa^E$ with finite $E \subset \alpha$	347
$0^\dagger$	zero-dagger	353
$\langle M_\gamma : \gamma \le \lambda\rangle$	the iterated ultrapower of an inner model M	356
$o(U)$	the order of a normal measure U (the rank of U in the Mitchell order)	358
$o(\kappa)$	the order of a cardinal κ (height of the Mitchell order)	358
$U <_{\mathcal{U}} W$	$\mathcal{U}$ is a closed set of normal measures, $U, W \in \mathcal{U}$, and $U \in j_W(\mathcal{U})$	358
$o^{\mathcal{U}}(U)$	the order of $U \in \mathcal{U}$ in $<^{\mathcal{U}}$	358
$o^{\mathcal{U}}(\kappa)$	the order of a cardinal κ in $<^{\mathcal{U}}$	358
$l(\mathcal{U})$	the length of a set of normal measures $\mathcal{U}$	359
$L\langle A_\alpha : \alpha < \theta\rangle$	the model $L[A]$ where $A = \{(\alpha, X) : X \in A_\alpha\}$	360
$L[\mathcal{U}]$	the model $L\langle U_{\alpha,\beta} : \alpha, \beta\rangle$ where $\mathcal{U} = \{U_{\alpha,\beta} : \alpha, \beta\}$	360
$\hat{x}$	$\{y \in P_\kappa(A) : x \subset y\}$	365
Ult^-, $\mathrm{Ult}^-_U(V)$	the version of ultrapower considering only functions on λ^+ that assume at most λ values; U is an ultrafilter on λ^+ for a cardinal λ	367
$[f]^-$	the element of the transitive collapse of $\mathrm{Ult}^-(V)$ represented by the function f	367
κ_x, λ_x	$\kappa_x = x \cap \kappa$ and $\lambda_x =$ the order-type of x, for $x \in P_\kappa(\lambda)$	374

α_x	the order type of $x \cap \alpha$	377
VP	Vopěnka's Principle	380
$E = \{E_a : a \in [\lambda]^{<\omega}\}$	the (κ, λ)-extender derived from an elementary embedding j with critical point κ	382
Ult_E	the direct limit of the directed system $\{\mathrm{Ult}_{E_a}, i_{a,b} : a \subset b \in [\lambda]^{<\omega}\}$ associated with an extender E	382
$j_E : V \to \mathrm{Ult}_E$	the elementary embedding associated with an extender E	382
$\dot{P}^{(\alpha)}_\beta$	the forcing iteration of $\langle \dot{Q}_{\alpha+\xi} : \xi < \beta \rangle$ inside V^{P_α} so that $P_{\alpha+\beta}$ is isomorphic to $P_\alpha * \dot{P}^{(\alpha)}_\beta$	396
$\triangle_s A_s$	$\{\alpha < \kappa : \alpha \in \bigcap\{A_s : \max(s) < \alpha\}\}$	402
$A \setminus s$	$A - (\max(s) + 1)$ for $A \subset \kappa$ and $s \in [\kappa]^{<\omega}$	403
$\mathrm{sat}(I)$	$\mathrm{sat}(P(\kappa)/I)$ where I is an ideal on κ	409
$f_*(I)$	the ideal $\{X \subset \kappa : f_{-1}(X) \in I\}$ where I is an ideal and f is an ideal on κ	410
$f_*(\mu)$	the measure ν defined by $\nu(X) = \mu(f_{-1}(X))$ where $f : \kappa \to \kappa$ and μ is a (real-valued) measure on κ	410
$g < h$	$\mathrm{dom}(g) \subset \mathrm{dom}(h)$ and $g(\alpha) \leq h(\alpha)$ for $\alpha \in \mathrm{dom}(g)$ where g and h are functions into κ defined on a set of positive measure	411
$\mathrm{Ult}_G(M)$	the generic ultrapower where G is a generic ultrafilter on $P(\kappa)/I$	421
j_G	the canonical embedding from M into $\mathrm{Ult}_G(M)$	421
$W_1 \leq W_2$	the I-partition W_1 is a refinement of the I-partition W_2	424
W_F	the I-partition $\{\mathrm{dom}(f) : f \in F\}$ associated with a functional F	424
$\mathcal{G}_I$	the infinite game on sets of positive I-measure played by the players Empty and Nonempty	426
$\diamondsuit(E)$	the Diamond Principle restricted to a stationary set E	442
$\diamondsuit_\kappa$	the Diamond Principle $\diamondsuit(\kappa)$	442
$\square_\kappa$	Jensen's Square Principle	443
P_S	the forcing shooting a closed unbounded set (conditions are bounded closed subsets of a stationary set S; p is stronger than q if $q = p \cap \alpha$ for some α)	445
I^+	$\{S \subset \kappa : S \notin I\}$	450
$I \restriction S$	$\{X \subset \kappa : X \cap S \in I\}$, the ideal concentrating on a set S	450
Reg	$\{\alpha < \kappa : \alpha$ is a regular cardinal$\}$	452
$\|\varphi\|$	the (rank) norm of a function $\varphi : \omega_1 \to \omega_1$	458
$f =_I g$, $f \leq_I g$, $f <_I g$	the relations between functions modulo an ideal on an infinite set	460
$f =_F g$, $f \leq_F g$, $f <_F g$	the relations between functions modulo the dual ideal to a filter F	460
$\|f\|$	Galvin-Hajnal norm of an ordinal function f	460
f_η, $\eta < \kappa^+$	the canonical ordinal functions	461
$\mathrm{cof}\, D$, $\mathrm{cof} \prod A/D$	the cofinality of the ultraproduct $\prod A/D$ in the ordering $<_D$	466
$\mathrm{pcf}\, A$	the set of all cofinalities of ultraproducts $\prod A/U$	466
M^a_α, $\alpha < \omega_k$	an elementary chain of submodels of some $(H_\vartheta, \in, \prec)$ where $\prec$ is a well-ordering of H_ϑ with $M^a_\alpha \supset a \cup \omega_k$ for a countable set $a \subset \omega_k$	468
χ^a_α	the characteristic function of M^a_α for a countable set $a \subset \omega_k$ and $\alpha < \omega_k$, $\chi^a_\alpha(n) = \sup(M^a_\alpha \cap \omega_n)$	468
M^a	$M^a = \bigcup_{\alpha < \omega_k} M^a_\alpha$ for a countable set $a \subset \omega_k$	468
χ^a	the characteristic function of M^a for a countable set $a \subset \omega_k$	468

$B_\lambda \subset A$, $\lambda \in \operatorname{pcf} A$	the generators of pcf A	470
J_λ	the ideal generated by the sets B_ν, $\nu < \lambda$	470
$J_\kappa[B_\kappa]$	the ideal generated by $J_\kappa \cup \{B_\kappa\}$	471
$\overline{B_\lambda}$, $\lambda \in A$	the transitive generators of pcf A	474
Σ^1_n, Π^1_n, Δ^1_n	the lightface hierarchy of projective sets	479
$\Sigma^1_n(a)$, $\Pi^1_n(a)$, $\Delta^1_n(a)$	the relativization of the hierarchy of projective sets	479
Σ^0_n, Π^0_n	the lightface Borel hierarchy (hierarchy of arithmetical sets)	480
u_m, $m \in \boldsymbol{N}$ (or z_m, $m \in \boldsymbol{N}$)	the canonical homeomorphism between $\mathcal{N}$ and $\mathcal{N}^\omega$; $u_m(n) = u(\Gamma(m,n))$	482
Seq_r	the set of r-tuples of sequences of natural numbers of the same length	483
$T(x)$	the tree $\{(s_1,\dots,s_r) \in Seq_r : (x\restriction n, s_1,\dots,s_r) \in T$ where $n = \text{length } s_i\}$	483
T/s	the tree $\{t : s^\frown t \in T\}$	484
$\|T\|$	the height of a well-founded tree T	484
$\rho_T(t)$	the rank of an element t of a well-founded tree T	484
$[T]$	$\{f \in X^\omega : \forall n\, f\restriction n \in T\}$	484
$Seq(K)$	the set of all finite sequences in K	485
$p[T]$	$\{x \in \mathcal{N} : T(x)$ is ill-founded$\}$	485
E_x	the relation $\{(m,n) : x(\Gamma(m,n)) = 0\}$ coded by $x \in \mathcal{N}$	485
WF	$\{x \in \mathcal{N} : x$ codes a well-founded relation$\}$	485
WO	$\{x \in \mathcal{N} : x$ codes a well-ordering on $\boldsymbol{N}\}$	485
$\preccurlyeq_\varphi$	the prewellordering induced by a norm φ; $a \preccurlyeq_\varphi b \leftrightarrow \varphi(a) \le \varphi(b)$	496
$\boldsymbol{\delta}^1_2$	$\sup\{\alpha : \alpha$ is the length of a $\boldsymbol{\Sigma}^1_2$ prewellordering$\}$	502
I_1, I_2, …, I_k, …	a recursive enumeration of open intervals with rational endpoints	504
$u(c)$, $v_i(c)$	the elements of $\mathcal{N}$ defined, for $c \in \mathcal{N}$ and $i \in \boldsymbol{N}$, by $u(c)(n) = c(n+1)$, $v_i(c)(n) = c(\Gamma(i,n)+1)$	504
Σ_α, Π_α	the set of $\boldsymbol{\Sigma}^0_\alpha$-codes and the set of $\boldsymbol{\Pi}^0_\alpha$-codes, respectively, $0 < \alpha < \omega_1$	504
BC	the set of all Borel codes $\bigcup_{\alpha<\omega_1} \Sigma_\alpha = \bigcup_{\alpha<\omega_1} \Pi_\alpha$	504
A_c	the Borel set coded by a $c \in$ BC	504
I_m, I_c	the ideals $\{B \in \mathcal{B} : \mu(B) = 0\}$ and $\{B \in \mathcal{B} : B$ is meager$\}$, respectively	511
$\mathcal{B}_m$, $\mathcal{B}_c$	the quotient algebras $\mathcal{B}/I_m$ and $\mathcal{B}/I_c$, respectively	511
B^*	the Borel set A_c if $B = A^M_c$ for some $c \in M$	511
$R(M)$, $C(M)$	the sets of all random and all Cohen reals over M, respectively	514
$\operatorname{Col}(\aleph_0, \lambda)$	the collapsing algebra	516
$A \smallsetminus s$	$A - (\max(s)+1)$ for $A \subset \omega$ and $s \in [\omega]^{<\omega}$	524
$[s, A]$	$\{X \in [\omega]^\omega : s \subset X$ and $X \smallsetminus s \subset A\}$	524
add(LM), cov(LM), unif(LM), cof(LM)	the cardinal invariants of Lebesgue measure	532
add(BP), cov(BP), unif(BP), cof(BP)	the cardinal invariants of the Baire property	532
$\mathfrak{d}$, $\mathfrak{b}$	the dominating number and the bounding number, respectively	533
$\mathfrak{t}$	the least cardinality of a tower	540
$\mathfrak{u}$	the least cardinality of a family of subsets of ω that generates an ultrafilter	540
$\operatorname{rud}(M)$	the rudimentary closure of $M \cup \{M\}$	548
J_α, $\alpha \in Ord$	the Jensen hierarchy of constructible sets	548

ρ_α^n	the Σ_n-projectum of α, i.e., the least $\rho \leq \alpha$ such that there is a $\Sigma_n(J_\alpha)$ function such that $f\text{“}J_\rho = J_\alpha$	549
$p\restriction s$	the tree $\{t \in p : t \subset s \text{ or } t \supset s\}$ for a tree p and $s \in p$	557
$\mathcal{F}(T)$	the fusion $\bigcap_{n=0}^\infty \bigcup_{s\in\{0,1\}^n} T(s)$ for a fusionable collection of perfect trees $T = \{T(s) : s \in Seq(\{0,1\})\}$	558
T'	the tree $\{t \in T : t$ has $\aleph_2$ extensions in $T\}$ where $T \subset \omega_2^{<\omega}$ is a tree	561
$h_T(t)$	the least α such that $t \notin T_{\alpha+1}$ where T_α is defined by induction: $T_0 = T$, $T_{\alpha+1} = T'_\alpha$, and $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$ if α is limit	561
s_p	the stem of a Laver tree p	565
$S^p(t)$	the set $\{a \in \omega : t^\frown a \in p\}$ where p is a Laver tree and $t \in p$	565
s_i^p, $i = 0, 1, \ldots$	a canonical enumeration of nodes in a Laver tree p	565
$q \leq_n p$	$q \leq p$ and $s_i^p \in q$ for all $i = 0, \ldots, n$ where p, q are Laver trees	565
$U + V$	the ultrafilter $\{X \subset \boldsymbol{N} : \{m \in \boldsymbol{N} : X - m \in V\} \in U\}$ where $X - m = \{n : m + n \in X\}$ and U, V are ultrafilters on $\boldsymbol{N}$	573
$\beta\boldsymbol{N}$	the Stone-Čech compactification of $\boldsymbol{N}$	573
A^*	the clopen set $\{V \in \beta\boldsymbol{N} : A \in V\}$ in $\beta\boldsymbol{N}$ for $A \subset \boldsymbol{N}$	574
OCA	the Open Coloring Axiom	576
$I \times J$	the ideal of sets $X \subset S \times T$ such that $\{x \in S : \{y \in T : (x, y) \in X\} \notin J\} \in I$ where I and J are ideals on S and T, respectively	580
$\boldsymbol{C}_\kappa$	the complete Boolean algebra of the forcing for adding κ Cohen reals	588
$\overline{B}$	the completion of a Boolean algebra B	588
$A \leq_{\text{reg}} B$	A is a regular subalgebra of a Boolean algebra B	588
$\text{pr}^A(b)$, $\text{pr}_A(b)$	the projections of b to a subalgebra A	589
$\langle X\rangle$	the subalgebra generated by a set X	589
$A(b_1, \ldots, b_n)$	the subalgebra generated by the set $A \cup \{b_1, \ldots, b_n\}$ where A is a subalgebra	589
P_S, $\boldsymbol{C}_S$	$\boldsymbol{C}_S = B(P_S)$ and P_S is the forcing consisting of finite 0–1 functions with domain $\subset S$	589
Fr_G	the free Boolean algebra with a set G of free generators	590
$\limsup_n a_n$	$\prod_{n=0}^\infty \sum_{k\geq n} a_n$ (a Boolean operation)	598
$\liminf_n a_n$	$\sum_{n=0}^\infty \prod_{k\geq n} a_n$ (a Boolean operation)	598
$\lim_n a_n$	the common value of $\limsup_n a_n$ and $\liminf_n a_n$ provided that they are equal	598
$M[G]$	$\{\dot{x}^G : \dot{x} \in M\}$ where $M \prec H_\lambda$ and G is V-generic	605
PFA	the Proper Forcing Axiom	607
$T\restriction C$	the tree $\{t \in T : o(t) \in C\}$ where T is an ω_1-tree and $C \subset \omega_1$ is a closed unbounded set	612
PFA^+	if $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ are dense subsets of a proper forcing P and if $\Vdash \dot{S} \subset \omega_1$ is stationary, then there exists a $\mathcal{D}$-generic filter G such that $\dot{S}^G$ is stationary	613
PFA^-	if P is proper such that $\lvert P\rvert \leq \aleph_1$ and if $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ are dense then there exists a $\mathcal{D}$-generic filter	614
G_A	the game of players I and II in which the players choose the consecutive members of a sequence of natural numbers $\langle a_0, b_0, a_1, b_1, \ldots\rangle$; I wins if the sequence is in the set $A \subset \omega^\omega$ and otherwise II wins	627
AD	the Axiom of Determinacy	627

$\sigma * b$	a play played by player I by a strategy σ in the game G_A	627
$a * \tau$	a play played by player II by a strategy τ in the game G_A	627
PD	the Projective Determinacy	628
$\mathrm{AD}^{L(\boldsymbol{R})}$	the Axiom of Determinacy in $L(\boldsymbol{R})$	628
$\mathrm{cone}(x_0)$	the cone $\{x \in \mathcal{N} : x_0 \in L[x]\}$	633
$\boldsymbol{\delta}^1_n$	$\sup\{\xi : \xi$ is the length of a $\boldsymbol{\Delta}^1_n$ prewellordering of $\mathcal{N}\}$ (the projective ordinal)	636
Θ	$\sup\{\xi : \xi$ is the length of a prewellordering of $\mathcal{N}\}$	636
$G_A^{\langle a_0,b_0,\ldots,a_n,b_n\rangle}$	the game in which player I plays $\langle a_{n+1}, a_{n+2}\ldots\rangle$, player II plays $\langle b_{n+1}, b_{n+2}\ldots\rangle$, and in which II wins when $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in A$	637
$G_A^{a_0}$	the game in which II makes a first move b_0, then I plays a_1, etc., and II wins if $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in \mathcal{N} - A$	637
$\preccurlyeq$	the linear ordering of *Seq* that extends the partial ordering $\supset$	638
T_s	$\{t \in Seq : (u,t) \in T$ for some $u \subset s\}$ where $T \subset Seq_2$ is a tree and $s \in Seq$	638
K_s	the set $\{t_0, \ldots, t_{n-1}\} \cap T_s$ where $\lvert s\rvert = 2n$, $\{t_n : n \in \omega\}$ is an enumeration of *Seq* and $T \subset Seq_2$ is a tree	638
k_s	the size of the finite set K_s	638
T_s	the set $\{t \in Seq : (s,t) \in T$ for some $s \in Seq\}$ and a tree T on $\omega^{\tau} \times K$	642
μ_s, $s \in Seq$	the measures on T_s's ensuring that the tree T on $\omega \times K$ is homogeneous	642
$\mu_{s,t}$	the natural projection map from T_t to T_s for $s \subset t$ in *Seq*	642
Q, $Q_{<\kappa}$	the stationary tower forcing	653
$f =_G g$, $f \in_G g$	the predicates in the generic ultrapower by the stationary tower forcing	653
K	the core model up to a measurable cardinal	660
$\mathrm{rud}_A(M)$	the closure of $M \cup \{M\}$ under functions rudimentary in A	660
J^A_α, $\alpha \in Ord$	the relativized Jensen hierarchy of sets	660
$\mathcal{C}_\lambda$	the closed unbounded filter on λ	661
$M < M'$	the well-ordering of mice	662
K^m	the core model up to $o(\kappa) = \kappa^{++}$	664
K^{strong}	the core model up to a strong cardinal	666
ρ^1_M	the Σ_1-projectum of M	667
$\mathrm{MS} = \bigcup_{n=0}^{\infty} \mathrm{MS}_n$	the class of all measure sequences	676
R_U	the Radin forcing for a measure sequence U	677
MM	Martin's Maximum	681
SPFA	Semiproper Forcing Axiom	681
RCS	revised countable support iteration	682
$X^{\perp}$	$X^{\perp} = \{M \in [H_\lambda]^\omega : M \prec H_\lambda$ and $N \notin X$ for every countable N that satisfies $M \prec N \prec H_\lambda$ and $N \cap \omega_1 = M \cap \omega_1\}$	684
RP, $\mathrm{RP}(\lambda)$	the Reflection Principle	688
SRP, $\mathrm{SRP}(\lambda)$	the Strong Reflection Principle	688
IA	the set of all internally approachable models	699
$\square_{\kappa,\nu}$, $\square_{\kappa,<\nu}$	weaker square principles	702
$\square_{\kappa,\kappa}$, $\square^*_\kappa$	the Weak Square	702
F^1_n	the Π^1_n filter	703
$\mathrm{AD}_{\boldsymbol{R}}$	the determinacy of games where moves are real numbers	705

Name Index

Abraham, Uri, 583, 607, 610, 614
Addison, John W. Jr., 153, 509, 645
Aleksandrov, Pavel Sergeevich, (1896–1982), 105, 153
Aronszajn, Nachman, (1907–1980), 116, 117, 120, 122, 123, 274–276, 283, 563, 568, 569, 571, 611, 612, 614, 702

Bagaria, Joan, 571, 691
Baire, René Louis, (1874–1932), 41–45, 87, 139, 140, 143, 148–153, 277, 479, 495, 515, 519–521, 523, 525–527, 529, 531, 533–537, 561, 615–618, 620, 621, 623–626, 628–633, 640, 644, 647, 650
Balcar, Bohuslav, 200, 599
Baldwin, Stewart, 387, 658
Banach, Stefan, (1892–1945), 50, 133, 137, 138, 256, 630, 631, 644
Bartoszyński, Tomek, 532–534, 540
Baumgartner, James E., 105, 112, 274, 282, 283, 439, 456, 568, 583, 603, 604, 609, 612, 614, 688, 694, 704, 705
Bekkali, Mohamed, 610, 614, 693
Bendixson, Ivar Otto, (1861–1935), 40, 42, 43, 45, 139, 151, 153, 492, 561
Bernays, Isaac Paul, (1888–1977), 61, 70, 71, 197, 200, 237
Bernstein, Felix, (1878–1956), 28, 35, 37, 61, 88
Blackwell, David Harold, 645
Blass, Andreas R., 265
Bloch, Gérard, 105
Bolzano, Bernard *in full* Bernardus Placidus Johann Nepomuk, (1781–1848), 15
Boolos, George S., (1940–1996), 200, 337
Booth, David, 89, 122, 278, 283
Borel, Félix Edouard Justin Émile, (1871–1956), 35, 40–42, 44, 45, 50, 82, 131, 139, 140, 142, 144–147, 149, 150, 152, 153, 243, 480, 486–488, 490, 497, 504–507, 509, 511–515, 520, 521, 523, 537, 540, 564, 568, 571, 615–618, 621, 624, 637, 645
Bukovský, Lev, 61, 407, 571
Burgess, John P., 617, 626
Burke, Maxim R., 465, 466, 469, 470, 477, 705

Cantor, Georg Ferdinand Ludwig Philipp, (1845–1918), 15, 24, 26–29, 33, 35, 37, 38, 40, 42–45, 55, 88, 139, 148, 151, 153, 244, 492, 500, 522, 524, 539, 561, 619, 623, 632
Carr, Donna M., 105
Cartan, Henri Paul, 89
Cauchy, Augustin-Louis, (1789–1857), 40, 42
Čech, Eduard, (1893–1960), 573
Chang, Chen Chung, 172, 328, 337, 582, 704
Cohen, Paul Joseph, 201, 202, 219, 221, 224–226, 236, 241, 243, 255, 256, 262, 264, 265, 273, 281, 282, 419, 437, 456, 511, 513–515, 520, 523, 536, 537, 539, 563, 564, 569, 571, 582, 588, 589, 591, 592, 599, 667, 691
Comfort, William Wistar, 89
Cummings, James, 673, 680, 702, 705

Davis, Morton, 644
Dedekind, Julius Wilhelm Richard, (1831–1916), 34, 35, 39, 45, 48, 83
Dehornoy, Patrick, 407
De Morgan, Augustus, (1806–1871), 78, 88
Devlin, Keith J., 330, 337, 545, 547, 550, 551
Dodd, Anthony J., 387, 658–661, 663, 664, 666, 668
Donder, Hans-Dieter, 682, 705

Douwen, Eric Karel van, (1946–1987), 540
Dushnik, Ben, 112, 123

Easton, William Bigelow, 232–235, 237, 262, 264, 393, 395, 396, 398, 399, 670, 672, 674
Ehrenfeucht, Andrzej, 313, 314, 337
Ellentuck, Erik, 524, 525, 527, 528, 540
Ellis, R., 583
Engelking, Ryszard, 264
Erdős, Paul, (1913–1996), 89, 105, 111–113, 122, 123, 138, 302–304, 309, 336, 456, 462, 540, 541, 582

Feferman, Solomon, 259, 265
Feng, Qi, 623, 626, 692–694
Fichtenholz, Grigorii Mikhajlovich, (1888–1959), 89
Fodor, Géza, 93–95, 98, 101, 104, 105, 118, 121, 122, 271, 281, 336, 453
Foreman, Matthew, 597–599, 658, 680, 684, 686–688, 692–694, 701, 702, 705
Fraenkel, Abraham Adolf, (1891–1965), 3, 15, 71, 198, 265
Fréchet, Maurice-René, (1878–1973), 73, 75
Fremlin, David Heaver, 533, 586, 599
Friedman, Harvey M., 105
Friedman, Sy D., 545, 547
Frolík, Zdeněk, (1933–1989), 89
Fubini, Guido, (1879–1943), 148, 523, 536
Fuchino, Sakae, 599
Fuchs, Ulrich, (1962–1998), 694
Fukson, V. I., 265

Gaifman, Haim, 337, 343, 364
Gale, David, 644
Galvin, Fred, 89, 123, 172, 457, 460, 477, 524, 525, 540, 554, 583, 672
Gitik, Moti, 387, 395, 443, 450, 456, 665, 669, 672–675, 680, 696, 698, 699, 703–705
Glazer, S., 573, 583
Gödel, Kurt Friedrich, (1906–1978), 70, 157, 161–163, 167, 168, 172, 175–178, 180–182, 187, 188, 190, 191, 194, 195, 197, 199–201, 237, 251, 332, 494, 496, 508, 509, 583
Goldstern, Martin, 691
Graham, Ronald L., 573, 583
Gray, C., 614
Gregory, John, 442, 456, 550
Groszek, Marcia J., 626, 691

Hager, Anthony W., 89
Hahn, Hans, (1879–1934), 50, 256
Hájek, Peter, 200, 224, 265
Hajnal, András, 105, 112, 122, 123, 138, 200, 309, 439, 457, 460, 477, 554, 556, 672
Hales, Alfred W., 574, 583
Halmos, Paul Richard, 153
Halpern, James D., 256, 265
Hanf, William P., (1934–1989), 138, 296, 297, 309
Harrington, Leo A., 453, 615, 626, 637, 644, 645
Hausdorff, Felix, (1869–1942), 15, 35, 57, 59, 61, 89, 153, 256, 575, 583, 623
Hechler, Stephen H., 61, 278, 283
Hessenberg, Gerhard, (1874–1925), 35
Hilbert, David, (1862–1943), 44, 45
Hindman, Neil, 573, 583
Hrbáček, Karel, 366, 386, 540

Ihoda, Jaime I., *see* Judah, Haim

Jackson, Steve, 645
Jech, Thomas J., 61, 100, 101, 105, 256, 264, 265, 364, 407, 421, 438, 452, 456, 477, 554–556, 583, 594, 595, 598, 599, 691–694, 696, 698, 703, 705
Jensen, Ronald Björn, 191, 200, 241, 264, 265, 309, 329, 330, 337, 387, 401, 407, 441, 443, 453, 456, 540, 545–548, 550, 551, 556, 557, 561, 571, 595, 658–661, 663, 664, 666, 668, 672, 675, 697, 702, 705
Jewett, Robert I., 574, 583
Jónsson, Bjarni, 305, 308, 328, 337
Judah, Haim, 537, 540

Kanamori, Akihiro, 330, 332, 335, 337, 387, 439, 537, 621, 645, 668
Kantorovich, Leonid Vitalyevich, (1912–1986), 89
Karłowicz, Monika, 264
Karp, Carol Ruth, (1926–1972), 200
Kechris, Alexander S., 508, 626, 636, 637, 645
Keisler, H. Jerome, 86, 89, 138, 172, 308, 309, 386
Ketonen, Jussi A., 283, 309, 387
Kleene, Stephen Cole, (1909–1994), 153, 508
Knaster, Bronisław, (1893–1980), 230

Koepke, Peter G., 547, 668
Kondô, Motokiti, 496, 498, 499, 508, 625, 640
König, Dénes, (1884–1944), 122, 123
König, Julius, (1849–1913), 33, 54, 61, 232, 467
Koppelberg, Sabine, 592, 599
Kripke, Saul Aaron, 516, 540
Kueker, David W., 102, 105, 705
Kunen, Keneth, 282, 290, 309, 323, 324, 326, 327, 337, 339, 348, 349, 364, 367, 368, 381, 386, 387, 392, 407, 438, 456, 463, 503, 509, 540, 563
Kuratowski, Kazimierz (*also* Casimir), (1896–1980), 61, 89, 133, 137, 138, 153, 508, 509
Kurepa, Đuro, (1907–1993), 119, 123, 551, 553–556

Laver, Richard, 378, 387, 564–568, 570, 571, 594, 604, 607, 608, 684, 691
Lebesgue, Henri-Léon, (1875–1941), 42, 44, 50, 82, 87, 125, 131, 137, 146–148, 150, 153, 243, 479, 495, 508, 511, 512, 515, 519–523, 529, 531, 534, 535, 537, 540, 618, 620, 621, 623, 624, 628, 629, 640, 647, 650
Lévy, Azriel, 172, 183, 200, 237, 238, 255, 256, 259, 264, 265, 389, 392, 407, 434, 435, 439, 445, 446, 454, 493, 508, 516, 518–522, 529, 553, 555, 570, 626, 650, 651, 656, 674, 675
Lindenbaum, Adolf, (1904–1941), 79
Łoś, Jerzy, (1920–1998), 159, 161, 172, 285, 363, 366, 367, 384, 421, 661, 667
Louveau, Alain, 626
Löwe, Benedikt, 668
Löwenheim, Leopold, (1878–1957), 157, 168, 169, 171
Luzin, Nikolai Nikolaevich, (1883–1950), 153, 508, 539, 540

Magidor, Menachem, 330, 332, 335, 337, 387, 446, 453, 455, 456, 465, 466, 469, 470, 477, 545, 611, 621, 623, 626, 658, 663, 664, 668, 672–678, 680, 684, 686–688, 692–694, 697, 698, 701, 702, 705
Maharam, Dorothy, 585, 599
Mahlo, Paul, 95, 96, 99, 103–105, 135, 289, 290, 297, 298, 309, 311, 313, 384, 390, 391, 398, 406, 412, 428, 429, 452, 453, 547, 555, 623, 624, 648, 691, 696–698, 703
Makinson, David C., 89
Malitz, Jerome I., 274, 283
Mansfield, Richard B., 492, 500, 508, 509, 620, 626
Marczewski, Edward, (1907–1976), E. Szpilrajn *before the World War II*, 153, 540
Martin, Donald A., 267, 272–274, 276–278, 282, 283, 387, 502, 503, 509, 529–531, 533, 539, 540, 607, 612, 620, 621, 626, 637, 642, 643, 645, 658, 666, 668, 681, 688, 690, 693
Máté, Attila, 439
Matet, Pierre, 705
Mathias, Adrian R. D., 402, 407, 524, 527–529, 540, 564, 566, 568, 604, 626, 691
Matsubara, Yo, 705
Mazur, Stanisław, (1905–1981), 630, 631, 644
McAloon, Kenneth, 595, 599
Mekler, Alan H., 697, 705
Menas, Telis K., 102, 105, 387, 407
Miller, Arnold W., 533, 534, 540
Miller, Edwin W., 112, 123
Milner, Eric Charles, (1928–1997), 60, 61, 105
Mirimanov, Dmitri, 71
Mitchell, William John, 283, 356–358, 361, 362, 364, 368, 378, 387, 395, 438, 554, 569–571, 658, 664, 665, 667–669, 673, 675, 676, 678, 680, 696, 702, 703
Monk, J. Donald, 309
Montague, Richard, 71, 172
Morgenstern, Carl F., 387
Moschovakis, Yiannis N., 496, 508, 509, 636, 640, 645
Mostowski, Andrzej, (1913–1975), 69, 71, 265, 286, 313, 314, 337, 484, 494, 508
Mycielski, Jan, 644
Myhill, John R., 200

Namba, Kanji, 561, 571, 675
Neumann, John von, (1903–1957), 26, 71
Novikov, Peter Sergeevich, (1901–1975), 509

Paris, Jeffrey B., 364, 392, 407
Patai, L., 61
Pawlikowski, Janusz, 534, 540

Peano, Giuseppe, (1858–1932), 157
Pierce, Richard S., (1927–1992), 89
Pincus, David F., 265
Pospíšil, Bedřich, (1912–1944), 75, 89
Powell, William C., 387
Prikry, Karel L., 105, 400–403, 406, 407, 410, 421, 438, 524, 525, 540, 566, 583, 664, 669, 674–677, 680, 682, 692
Prisco, Carlos Augusto Di, 705

Radin, Lon Berk, 675–678, 680
Rado, Richard, (1906–1989), 60, 61, 111–113, 122, 123, 456, 462, 540, 582
Raisonnier, Jean, 533, 540
Ramsey, Frank Plumpton, (1903–1930), 76, 77, 86, 87, 89, 107–110, 121–123, 135, 136, 138, 283, 299–302, 305, 308, 311, 315, 318, 320, 328, 390, 391, 524–527, 529, 537, 538, 540, 573, 574, 626
Reinhardt, William Nelson, (1939–1998), 138, 274, 283, 309, 386, 387
Rothberger, Fritz, (1902–2000), 540, 541, 583
Rothschild, Bruce L., 573
Rowbottom, Frederick, 138, 282, 283, 300, 305, 308, 309, 406, 412
Rubin, Matatyahu, 583, 610, 614
Rudin, Mary Ellen, 86, 89
Rudin, Walter, 89
Russell, Bertrand Arthur William, (1872–1970), 4

Sacks, Gerald E., 244, 264, 595, 604, 691
Scheepers, Marion, 575, 583
Schimmerling, Ernest, 611, 697, 702, 704, 705
Schlindwein, Chaz, 682, 694
Scott, Dana Stewart, 200, 224, 265, 285, 296, 297, 300, 309, 323
Shanin, Nikolai Aleksandrovich, 118, 123
Shelah, Saharon, 76, 89, 387, 442, 443, 450, 451, 453, 456, 457, 460, 461, 465, 466, 469, 470, 473, 476, 477, 537, 540, 550, 555, 556, 563, 570, 571, 583, 592, 595, 598, 599, 601, 604, 607, 610, 614, 620, 626, 658, 667, 672, 673, 680, 682, 684, 686–688, 690–695, 697, 698, 702, 703, 705
Shepherdson, John Cedric, 200
Shoenfield, Joseph R., (1927–2000), 70, 71, 200, 490, 491, 495, 507, 508, 557, 618
Sierpiński, Wacław, (1882–1969), 61, 123, 153, 490, 508, 540
Sikorski, Roman, (1920–1983), 88, 89, 265
Silver, Jack H., 96, 105, 232, 283, 309, 311–313, 318–320, 322, 323, 328, 330, 336, 337, 339, 353, 364, 394, 395, 407, 421, 422, 438, 439, 457, 524, 540, 553, 556, 569, 571, 583, 615, 617, 618, 623, 626, 647, 661, 663, 669, 670, 672, 686
Skolem, Thoralf Albert, (1887–1963), 15, 71, 155–157, 168, 169, 171, 172, 198, 293, 294, 300, 301, 304, 305, 308, 313–318, 320, 321, 323, 326, 331–334, 336, 348–352, 370, 371, 412, 545–547, 549, 550, 660, 662, 685, 689
Slaman, Theodore A., 626
Smith, Edgar, 438
Sochor, Antonín, 256, 265
Solovay, Robert M., 95, 105, 138, 224, 264, 267, 272, 274, 276, 282, 337, 364, 372, 386, 387, 389, 392, 407, 410, 424, 428, 429, 438, 439, 450, 492, 508, 509, 511, 515, 519–523, 529, 540, 547, 551, 556, 599, 620, 626, 633, 634, 636, 645, 650, 699, 705
Specker, Ernst P., 265, 571
Spencer, Joel H., 573
Steel, John R., 387, 642, 643, 645, 666–668, 695, 702, 705
Steinhaus, Hugo, (1887–1972), 644
Stern, Jacques, 533, 540
Stewart, Frank M., 644
Stone, Marshall Harvey, (1903–1989), 78, 81, 89, 573
Suslin, Mikhail Yakovlevich, (1894–1919), 39, 45, 113–116, 123, 143–146, 153, 239, 241, 242, 262–264, 267, 274, 282, 484, 485, 488, 497, 503, 527, 551, 556, 559, 563, 571, 593–595, 599, 612, 615, 621, 642–644, 656
Świerczkowski, Stanisław, 644
Szpilrajn, Edward, *see* Marczewski, Edward

Takeuti, Gaisi, 407

Tarski, Alfred, (1902–1983), 14, 15, 61, 75, 81, 89, 123, 137, 138, 162, 172, 308, 309, 386, 438
Taylor, Alan D., 688, 705
Tennenbaum, Stanley, 264, 267, 272, 274, 282
Tikhonov, Andrei Nikolaevich, (1906–1993), 50, 256
Todorčević, Stevo, 563, 571, 574, 577, 583, 609, 610, 612, 614, 688, 689, 693, 694
Truss, John K., 533, 540

Ulam, Stanisław Marcin, (1909–1984), 126, 131–133, 137, 153, 409
Urysohn, Pavel Samuilovich, (1898–1924), 105

van der Waerden, B. L., *see* Waerden, Bartel Leendert van der
van Douwen, E. K., *see* Douwen, Eric Karel van
van Wesep, R., *see* Wesep, Robert van
Vaught, Robert L., (1926–2002), 172
Veličković, Boban, 614, 626
Vitali, Giuseppe, (1875–1932), 48, 137, 138, 147, 149, 152
Vladimirov, Denis Artemievich, (1929–1994), 598, 599
von Neumann, J., *see* Neumann, John von
Vopěnka, Petr, 200, 224, 249, 265, 366, 380, 386, 540

Waerden, Bartel Leendert van der, (1903–1996), 574, 583
Welch, Philip, 668
Wesep, Robert van, 705
Witzany, Jiří, 698
Woodin, W. Hugh, 365, 384, 387, 390, 395, 430, 450, 452, 456, 607, 611, 623, 626, 636, 639, 640, 642, 643, 645, 647–650, 653–659, 666–669, 673, 674, 678–680, 689, 695–697, 702, 705

Young, William H., (1863–1942), 153

Zapletal, Jindřich, 463, 593, 599
Zermelo, Ernst Fridrich Ferdinand, (1871–1953), 3, 15, 26, 48, 50, 61, 198
Zorn, Max A., (1906–1993), 49, 50, 59, 61, 75, 88, 575, 581

Index

Absolute formula (property)
– downward absolute formula, 185
– formula absolute for a transitive model, 163
– upward absolute formula, 185
$\mathbf{\Sigma}^1_3$-Absolute universe for generic extensions, 624
Addition of cardinals, $\kappa + \lambda$, 28
Addition of ordinals, $\alpha + \beta$, 23
κ-Additive measure, 129
Adequate model, 184
Adequate set, 188
Aleph, 29
Algebra, *see* algebra of sets, Boolean algebra, subalgebra
Algebra of sets, 41
– κ-complete algebra of sets, 82
– σ-algebra, 41
– σ-algebra of Borel sets, 41
– σ-algebra of Lebesgue measurable sets, 42
– σ-algebra of sets having the Baire property, 149
Algebraic real number, 44
Almost all, 99, 159, 409
Almost disjoint family
– almost disjoint family of functions, 97
– almost disjoint family of sets, 118
– strongly almost disjoint family of sets, 451
Almost disjoint forcing, 277
Almost disjoint functions, 118, 419
Almost disjoint sets, 118, 578, 651
Almost everywhere, 159
Almost universal class, 182
Alphabet, 574
Amalgamation
– amalgamation of Laver trees, 565
– amalgamation of perfect trees, 245
Amenable model, 549
Analytic set, 142
Antichain
– antichain below a condition, 202
– antichain in a Boolean algebra, 84
– antichain in a tree, 114
– antichain of conditions, 201
– maximal antichain, 84
Arithmetic progression, 574
Arithmetical set, Σ^0_n, Π^0_n, 480
Aronszajn tree, 116
Arrow notation
– $\kappa \to (\alpha)^n_m$, 112
– $\kappa \to (\alpha, \beta)^n$, 112
– $\kappa \to (\lambda)^n_m$, 109
Atom
– atom in the theory ZFA, 250
– atom of an ideal, 410
– atom of a Boolean algebra, 79
– atom of a measure, 126
Atomic formulas, $x \in y$, $x = y$, 5
Atomless ideal, 410
Atomless measure, 126
Automorphism
– automorphism of a Boolean algebra, 79
– automorphism of a Boolean-valued universe V^B, 221
– automorphism of a partially ordered set, 18
– automorphism of a tree, 262
– $\in$-automorphism of the universe, 66
– $\in$-automorphism of the universe of ZFA, 251
Axiom
– a weak form of the Axiom of Choice, 628
– $\mathrm{AD}^{L(\boldsymbol{R})}$, 628
– Analytic Determinacy, 637
– axiom (20.30), 381

– axiom consistent relative to a theory, 163
– axiom consistent with a theory, 163
– axiom independent of a theory, 163
– Axiom of Choice, AC, 3, 47, 70
– Axiom of Constructibility, $V = L$, 175
– Axiom of Determinacy, AD, 627
– Axiom of Determinateness, AD, 644
– Axiom of Extensionality, 3, 6, 70, 250
– Axiom of Infinity, 3, 13, 70
– Axiom of Pairing, 3, 6, 70
– Axiom of Power Set, 3, 9, 70
– axiom of predicate calculus, 6
– Axiom of Regularity, 3, 63, 70
– Axiom of Replacement, 70
– Axiom of Union, 3, 9, 70
– Continuum Hypothesis, CH, 37
– Countable Axiom of Choice, 50
– $\mathbf{\Delta}^1_n$ Determinacy, 639
– Diamond Principle, $\diamondsuit$, 191
– Generalized Continuum Hypothesis, GCH, 55
– Martin's Axiom, MA, 272
– Martin's Axiom, MA_κ, 272
– Open Coloring Axiom, OCA, 576
– *see* principle
– Projective Determinacy, PD, 628
– Proper Forcing Axiom, PFA, 607, PFA^+, 613, PFA^-, 614
– Reflection Principle, RP, 688
– restricted Diamond Principle, $\diamondsuit(E)$, $\diamondsuit_\kappa$, 442
– Singular Cardinal Hypothesis, SCH, 58
– Strong Reflection Principle, SRP, 688
– Suslin's Hypothesis, SH, 274
Axiom schema
– Axiom Schema of Comprehension, 4, 70
– Axiom Schema of Replacement, 3, 13
– Axiom Schema of Separation, 3, 4, 7
– Collection Principle, 65
– Vopěnka's Principle, VP, 380
Axioms of Bernays-Gödel, BG, BGC, 70
Axioms of Zermelo-Fraenkel, ZF, ZFC, 3
Axiom A, 614

Baire Category Theorem, (T4.8) 41
Baire property, 148
Baire space, $\mathcal{N}$, 42
Bernays-Gödel axioms, BG, BGC, 70
Beth function, $\beth_\alpha$, 55
Binary relation, *see* relation
Boolean algebra, 78
– (κ, λ)-distributive Boolean algebra, 246
– atomic Boolean algebra, 79
– atomless Boolean algebra, 79
– Boolean algebra completely generated by a set, 83
– Boolean algebra generated by a set, 79
– class Boolean algebra, 235
– Cohen algebra, 588
– collapsing algebra $\mathrm{Col}(\aleph_0, \lambda)$, 516
– complete Boolean algebra, 82
– countably complete Boolean algebra, 82
– free Boolean algebra, Fr_G, 590
– κ-complete Boolean algebra, 82
– κ-distributive Boolean algebra, 85
– κ-generated Boolean algebra, 247
– κ-saturated Boolean algebra, 84
– Lévy collapsing algebra $\mathrm{Col}(\kappa, <\lambda)$, (T15.22) 238
– Lindenbaum algebra, 79
– measure algebra, 415, 585
– minimal algebra, 595
– quotient algebra, 80
– rigid algebra, 595
– σ-complete Boolean algebra, 82
– semi-Cohen algebra, 591
– simple algebra, 595
– strategically ω-closed Boolean algebra, 596
– *see* subalgebra
– Suslin algebra, 593
– trivial Boolean algebra, $\{0, 1\}$, 81
– weakly (κ, λ)-distributive Boolean algebra, 246
Boolean operation
– $+$, $\cdot$, $-$, 78
– $\Rightarrow$, 209
– $\oplus$, 87
– infinite Boolean operations, $\sum$, $\prod$, 82, $\limsup_n a_n$, $\liminf_n a_n$, 598
Boolean universe, 206
Boolean value of a formula, $\|\varphi\|$, 207
Boolean-valued model, 206
– Boolean valued model V^B, 209
– Boolean-valued model with a class Boolean algebra, 235

– extensional Boolean-valued model, 209
– full Boolean-valued model, 208
Borel code, 504
– $\mathbf{\Pi}^0_\alpha$-code, 504
– $\mathbf{\Sigma}^0_\alpha$-code, 504
Borel set, 41, 140
Borel's Conjecture, 564
Bounded MA($\mathcal{P}$), 691
Bounded set in a tree, 240
Bounded subset of an ordinal, 32
Boundedness Lemma, (C25.14) 487
Bounding number, $\mathfrak{b}$, 533
Branch in a tree, 114
α-Branch in a tree, 114
Branching family, 579

Canonical embedding, *see* embedding
Canonical inner model, *see* inner model
Canonical name, *see* name
Canonical stationary set, *see* stationary set
Canonical well-ordering, *see* well-ordering
Cantor space
– Cantor set, $\boldsymbol{C}$, 37
– generalized Cantor space, λ^ω, 623
Cantor's Normal Form Theorem, (T2.26) 24
Capture (model N captures dense set A), 655
Cardinal invariant, 529
– additivity, add, 532
– bounding number, $\mathfrak{b}$, 533
– cofinality, cof, 532
– covering, cov, 532
– dominating number, $\mathfrak{d}$, 533
– uniformity, unif, 532
Cardinal number, 29
– α-Mahlo cardinal, 696
– *see* cardinal invariant
– cardinal satisfying full reflection, 698
– Erdős cardinal, η_α, 302
– extendible cardinal, 379
– finite cardinal, 27
– greatly Mahlo cardinal, 697
– huge cardinal, 380
– inaccessible cardinal, 58
– ineffable cardinal, 308
– Jónsson cardinal, 305
– λ-strong cardinal, 382
– λ-strong cardinal for a set A, 647
– λ-supercompact cardinal, 375
– limit cardinal, 30
– Mahlo cardinal, 95
– measurable cardinal, 127
– n-huge cardinal, 386
– Π^n_m-indescribable cardinal, 295
– Ramsey cardinal, 121
– real-valued measurable cardinal, 130
– reflecting cardinal, 697
– regular cardinal, 32
– Rowbottom cardinal, 305
– singular cardinal, 32
– strong cardinal, 381
– strong limit cardinal, 58
– strongly compact cardinal, 136
– sufficiently large cardinal, 602
– supercompact cardinal, 136, 374
– superstrong cardinal, 657
– weakly compact cardinal, 113
– weakly inaccessible cardinal, 33
– weakly Mahlo cardinal, 96
– Woodin cardinal, 384, 647
Cardinal successor of an ordinal, α^+, 29
Cardinality, $|X|$, 27
Chain
– chain in a partially ordered set, 49
– E-chain, 611
– elementary chain of models, 611
Chain condition
– countable chain condition, c.c.c., 84, 220
– κ-chain condition, κ-c.c., 84, 227
Chang's Conjecture, 328
Characteristic function
– characteristic function of a model, $\chi^a_\alpha(n)$, 468
Choice function, 47
Class, 5, 70
– almost universal class, 182
– class closed under an operation, 177
– class definable from parameters, 5
– class of forcing conditions, 235
– class of hereditarily ordinal-definable sets from a set, $HOD[A]$, 195
– class of hereditarily ordinal-definable sets over a set, $HOD(A)$, 196
– class of hereditarily ordinal-definable sets, HOD, 194
– class of hereditarily symmetric names, HS, 253
– class of ordinal-definable sets from a set, $OD[A]$, 195
– class of ordinal-definable sets over a set, $OD(A)$, 195

– class of ordinal-definable sets, OD, 194
– class of sets constructible from a set, $L[A]$, 192
– class of Silver indiscernibles, 311
– δ-closed class, 325
– definable class, 5
– Δ_n class, 184
– extensional class, 68
– generic class over a model, 235
– Π_n class, 183
– proper class, 6
– Σ_n class, 183
– the smallest inner model containing a set, $L(A)$, 193
– universal class, V, 6
Class Boolean algebra, 235
Closed forcing
– essentially $<\kappa$-closed forcing, 455
– κ-closed forcing, 228
– $<\kappa$-closed forcing, 228
Closed set
– closed set of normal measures, 358
– closed subset of $P_\kappa(A)$, 100
– $\equiv$-closed set, 633
– set closed under an operation, 102
– set of indiscernibles closed in $Ord^{\mathfrak{A}}$, 318
Closed unbounded filter
– closed unbounded filter on $P_\kappa(A)$, 100
– closed unbounded filter on κ, 91
– closed unbounded filter $\mathcal{C}_\kappa$ survives at β, 663
Closed unbounded set
– closed unbounded set in κ, 91
– closed unbounded subset of $P_\kappa(A)$, 100
– locally closed unbounded set, 691
– strongly closed unbounded set, 704
Closure
– closure point of a function, 102, 678
– closure under functions rudimentary in A, $\mathrm{rud}_A(M)$, 660
– rudimentary closure, $\mathrm{rud}(M)$, 548
– transitive closure, $\mathrm{TC}(S)$, 64
Club-guessing property, 442
Club-guessing sequence, 442
Club-isomorphic trees, 612
Co-dense subalgebras, 591
Code
– Borel code, 504
– code of an uncountable set in almost disjoint forcing, 277
– code of a binary relation on $\boldsymbol{N}$, 485
– code of a formula by a set, $\ulcorner\varphi\urcorner$, 162
– code of a set by a set of ordinals, 196
– $\mathbf{\Pi}^0_\alpha$-code, 504
– $\mathbf{\Sigma}^0_\alpha$-code, 504
– standard code, 549
Cofinal sequence, 31
Cofinal set
– cofinal subset of an ordinal, 31
– cofinal subset of a partially ordered set, 461
Cofinality
– *see* cardinal invariant
– cofinality of an ordinal, $\mathrm{cf}\,\alpha$, 31
– cofinality of a partially ordered set, 461
– possible cofinalities of ultraproducts, pcf, 466
– true cofinality of a partially ordered set, 461
Cohen algebra, 588
Cohen forcing, 202
Cohen real, 202, 225, 513
Coherent system of measures, 382
Coiteration, 358
Collapse forcing, 237
Collapsing a cardinal, 203
Collapsing cardinals, 237
Collection, *see* family
Collection Principle, 65
Compactness of pcf, 472
Compactness Theorem, 293
Comparison of mice, 661
Compatible conditions, 201
Complete Boolean algebra, 82
Complete embedding, 83
Complete homomorphism, 83
Complete linearly ordered set, 38
Complete metric space, 40
Complete subalgebra, 83
Completely generated algebra (subalgebra), 83
Completely Ramsey set, 525
Completion of a Boolean algebra, 82
n-Component of a Laver tree, 565
Composition of functions, 11
Condensation Lemma, (L13.17) 188, (L18.38) 332, (L27.5) 550
Condensation point, 45
Condition, *see* forcing condition, forcing conditions

Cone, 633
Conjecture
– Borel's Conjecture, 564
– Chang's Conjecture, 328
Connectives, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, 5
Consistent set of sentences, 157
Constant function, c_a, 161
Constructible set, 175
Constructible universe, L, 175
Continued fraction, 42
Continuous sequence, 22
Continuum (real line), $\boldsymbol{R}$, 37
Continuum (the cardinality of $\boldsymbol{R}$), $\mathfrak{c}$, 37
Continuum function, 2^κ, 55
Continuum Hypothesis, CH, 37
Core model
– core model up to a measurable cardinal, K, 660
– core model up to a strong cardinal, K^{strong}, 666
– core model up to a Woodin cardinal, K^{steel}, 667
– core model up to $o(\kappa) = \kappa^{++}$, K^m, 664
Correct position in a strategy, 631, 632
Correspondence, 12
Countable Axiom of Choice, 50
Countable chain condition, c.c.c., 39, 84, 220
Countable set, 30
Countably complete Boolean algebra, 82
Countably complete filter, 77
Countably complete ideal, 77
Critical point
– critical point of an elementary embedding, 323
– critical point of an extender, 382
Cumulative hierarchy, 171
– constructible hierarchy of sets, L_α, 175
– hierarchy $L_\alpha(A)$, 193
– hierarchy of sets constructible from a set, $L_\alpha[A]$, 192
– hierarchy of sets with atoms, $P^\alpha(S)$, 250
– hierarchy of sets, V_α, 64
– Jensen hierarchy, J_α, 548
– relativized Jensen hierarchy, J^A_α, 660
Cut
– cut in a Boolean algebra, 83
– Dedekind cut in a linearly ordered set, 39
– regular cut in a Boolean algebra, 83
– regular cut in a separative partially ordered set, 205

D-finite set, 34
D-infinite set, 34
Δ-Lemma, 118
Δ-system, 117
Decide a formula, 204
Dedekind cut, 39
Definable class, 5
Definable element, 157
Definable set
– definable class, 5
– definable set in a model, 157
– definable set over a model, 157
ω_1-Dense ideal, 695
Dense linearly ordered set, 38
Dense open set, 41
Dense set, 201, 205
Dense subalgebra, 82
Dense subset, 38
Density
– density of a Boolean algebra, 589
– subset of $\boldsymbol{N}$ of density 0, 74
– uniform density of a Boolean algebra, 589
Determinacy, 627
– $\mathrm{AD}_{\boldsymbol{R}}$, 705
– $\mathrm{AD}^{L(\boldsymbol{R})}$, 628
– Analytic Determinacy, 637
– Borel Determinacy, 637
– $\boldsymbol{\Delta}^1_n$ Determinacy, 639
– Determinacy of open games, 636
– the Axiom of Determinacy, AD, 627
– the Projective Determinacy, PD, 628
Diagonal function, d, 287, 374
Diagonal intersection
– $\triangle_{\alpha<\kappa} X_\alpha$, 92
– $\triangle_{s\in[\kappa]^{<\omega}} A_s$, 402
– diagonal intersection in $P_\kappa(A)$, $\triangle_{a\in A} X_a$, 101
Diagonal limit of ordinal functions, 461
Diagonal union, $\sum_\alpha X_\alpha$, 93
Diamond Principle
– $\diamondsuit$, 191
– $\diamondsuit(E)$, 442
– $\diamondsuit_\kappa$, 442
– $\diamondsuit_\kappa(E)$, 550
Difference of sets, $X - Y$, 8
Direct limit, 158
λ-Directed closed forcing notion, 395
Directed set, 157

Directed subset of $P_\kappa(A)$, 101
Directed system of models, 157
Disjoint elements, 78
Disjoint family of sets, 12
Disjoint sets, 8
I-Disjoint sets, 580
(κ, λ)-Distributive Boolean algebra, 246
κ-Distributive Boolean algebra, 85
κ-Distributive forcing, 228
$<\kappa$-Distributive forcing, 228
Distributivity, 85, 228, 246
Domain of a relation, 10
Dominating family, 532
Dominating number, $\mathfrak{d}$, 533
Domination
– a function dominates a function, 243
– eventual domination, 580
– function eventually dominates a function, 278
Dual filter, 73
Dual ideal, 73

E.M. set (Ehrenfeucht-Mostowski), 314
Easton product, 233
Easton support, 395
Element
– definable element, 157
– E-minimal element, 25
– element hereditarily in a set, 66
– greatest element, 17
– independent element over a subalgebra, 590
– least element, 17
– maximal element, 17, 49
– minimal element, 17
– stable element, 84, 88
Elementarily equivalent models, 156
Elementary chain of models, 611
Elementary embedding, 157
Σ_n-Elementary submodel, $M \prec_{\Sigma_n} N$, 187
Elementary submodel, $\mathfrak{B} \prec \mathfrak{A}$, 156
Ellentuck topology, 524
Embedding, 79, 156
– canonical embedding, j, 161, j_G, 421
– complete embedding, 83
– elementary embedding, 157
– elementary embedding of the universe, 285
– nontrivial elementary embedding (of the universe), 287
Empty set, $\emptyset$, 8
End-extension
– end-extension of a model, 654
– end-extension of a set, 406
– end-extension of $<_\alpha$ (on L_α), 188
Enumeration, 21
Enumeration of alephs, $\aleph_\alpha$, 30
Equivalence class, $[x]$, 12
Equivalence relation, 12
Erdős cardinal η_α, 302
Erdős-Rado Partition Theorem, (T9.6) 111
Essentially $<\kappa$-closed forcing, 455
Eventual domination, 580
Exact upper bound of a set of ordinal functions, 461
Exponentiation of cardinals, κ^λ, 28
Exponentiation of ordinals, α^β, 23
Extender
– (κ, λ)-extender derived from an elementary embedding, 382
– (κ, λ)-extender derived from the direct limit embedding, 384
– two extenders overlap, 666
Extendible cardinal, 379
Extension
– *see* end-extension
– extension of a function, 11
– extension of a set in a binary relation, 67
– generic extension, 216
Extensional Boolean-valued model, 209
Extensional class, 68
Extensional relation, 68

F_σ set, 42
Factor Lemma, (L19.5) 342, (L21.8) 396, (C26.11) 518
Family, 12
– almost disjoint family, 118
– branching family, 579
– disjoint family, 12
– dominating family, 532
– functional, 424
– fusionable collection of perfect trees, 558
– independent family, 75
– *see* set
– strongly almost disjoint family of sets, 451
– unbounded family, 532
Field of a relation, 10
Filter, 73, 80, 202
– *see* closed unbounded filter
– countably complete filter, 77

– dual filter, 73
– filter generated by a set, 74
– filter on a group, 251
– Fréchet filter, 73
– *see* generic filter
– κ-complete filter, 77, 82
– maximal filter, 74
– normal filter, 96
– Π^1_n filter on κ, 703
– principal filter, 73, 80
– σ-complete filter, 77
– trivial filter, $\{S\}$, 73, $\{1\}$, 80
– *see* ultrafilter
Fine measure, *see* measure
Finite cardinal, 27
Finite intersection property, 74, 80
Finite ordinal, 20
Finite sequence, 21
Finite set, 14, 20
– D-finite set, 34
– T-finite set, 14
First category, 148
Fixed point, 26
Forces, p forces σ, $p \Vdash \sigma$, 204
Forcing, 201
– almost disjoint forcing, 277
– Cohen forcing, 202
– collapse forcing, 237
– λ-directed closed forcing, 395
– forcing adding $\aleph_2$ eventually different reals, 593
– forcing adding a closed unbounded set with finite conditions, 613
– forcing collapsing a cardinal, 203
– forcing collapsing cardinals, 237
– forcing preserving cardinals, 219
– forcing satisfying Axiom A, 604
– forcing shooting a closed unbounded set, 445
– forcing with a class of conditions, 235
– forcing with countable normal trees, 239
– forcing with perfect trees (Sacks forcing), 244
– Hechler forcing, 278
– *see* iterated forcing
– κ-closed forcing, 228
– $<\kappa$-closed forcing, 228
– κ-distributive forcing, 228
– $<\kappa$-distributive forcing, 228
– Laver forcing, 565
– Lévy collapse, 238
– Magidor's forcing, 674
– Mathias forcing, 524
– Prikry forcing, 401
– product forcing, 229
– proper forcing, 601
– Radin forcing, R_U, 677
– random forcing, 243
– semiproper forcing, 649
– stationary set preserving forcing, 681
– stationary tower forcing, 653, 678
– strategically ω-closed forcing, 596
Forcing axiom
– Bounded MA($\mathcal{P}$), 691
– MA($\mathcal{P}$), 690
– MA$^+$($\mathcal{P}$), 690
– Martin's Maximum, MM, 681
– Semiproper Forcing Axiom, 681
Forcing condition
– empty condition, $\emptyset$, 230
– master condition, 400
– (M, P)-generic condition, 602
– (M, P)-semigeneric condition, 649
– stronger forcing condition, 201
Forcing conditions, 201
– compatible forcing conditions, 201
– incompatible forcing conditions, 201
– semiproper set of conditions, 654
Forcing language, 215
Forcing notion, *see* forcing
Forcing relation, $\Vdash_P$, $\Vdash$, 215
Forcing Theorem, (T14.6) 204
Formula, 5, 155
– atomic formulas, $x \in y$, $x = y$, 5
– formula of nth order, 295
– formula valid in a Boolean-valued model, $\|\varphi\| = 1$, 207
– Π_n formula, 183
– Π^n_m formula, 295
– Σ_n formula, 183
– Σ^n_m formula, 295
Fréchet filter, 73
Free Boolean algebra, Fr_G, 590
Free variable, 5
Full Boolean-valued model, 208
Full reflection, 698
Function, 11
– beth function, $\beth_\alpha$, 55
– canonical ordinal function on κ, f_η, 460
– *see* characteristic function
– choice function, 47
– constant function, c_a, 161
– continuum function, 2^κ, 55
– diagonal function, d, 287, 374

– Δ_n function, 184
– exact upper bound of a set of ordinal functions, 461
– function eventually dominates a function, 278
– function from a set to a set, 11
– function on a set, 11
– function on κ^α with finite support, 347
– function onto a set, 11
– gimel function, $\beth(\kappa)$, 56
– Hartogs function, $h(X)$, 29
– inclusion map, 346
– increasing function, 17
– inverse of a function, 11
– Laver function, 378
– monotone function, 306
– n-ary function on a set, 11
– normal (ordinal) function, 92
– one-to-one function, 11
– order-preserving function, 17
– Π_n function, 183
– rank function, $\rho(x)$, 68
– regressive (ordinal) function, 93
– rudimentary function, 548
– rudimentary function in A, 660
– Skolem function, 156
– Σ_n function, 183
– Σ_n Skolem function, 333
– unbounded function on a set of positive measure, 410
Functional, 424
Fusion, 244
– fusion of a fusion sequence, 565
– fusion of a fusionable collection, 558
Fusion sequence
– fusion sequence of Laver trees, 565
– fusion sequence of perfect trees, 244
Fusionable collection of perfect trees, 558

G_δ set, 42
Game
– analytic game, 637
– Banach-Mazur game, 630
– Borel game, 637
– closed game, 637
– covering game, 629
– determined game G_A, 627
– G_A, 627
– game $\mathcal{G}$ on a Boolean algebra, 596
– $\mathcal{G}_I$, 426
– open game, 636
– perfect set game, 632
– projective game, 628
– proper game for P below p, 603
– Solovay game, 634
Gap
– Hausdorff gap, 575
– (κ, λ)-gap, 575
Generalized Continuum Hypothesis, GCH, 55
σ-Generates (a set σ-generates a Boolean algebra), 585
Generators of pcf, 470
– transitive generators of pcf, 473
Generic class, 235
(M, P)-Generic condition, 602
Generic extension, 216
Generic filter
– $\mathcal{D}$-generic filter, 202
– generic filter minimal over the ground model, 244
– generic ultrafilter, 206
Generic Model Theorem, (T14.5) 203
Generic real
– Cohen real, 202, 225, 513
– Laver real, 565
– Mathias real, 527
– random real, 243, 513
– Sacks real, 244
Generic set, 202
Generic ultrafilter, 206
Generic ultrapower, $\mathrm{Ult}_G(M)$, 421, 653
Gimel function, $\beth(\kappa)$, 56
Gödel number of a sentence, $\#\sigma$, 162
Gödel numbering, 162
Gödel operations, $G_1, \ldots, G_{10}$, 177
Gödel's Completeness Theorem, 157
Gödel's First Incompleteness Theorem, 157
Gödel's Normal Form Theorem, (T13.4) 177
Greatest element, 17
Greatly Mahlo cardinal, 697
Ground model, 201

Hausdorff formula, 57
Hausdorff gap, 575
Hechler forcing, 278
Height
– height of a tree, 114
– height of a well-founded relation, 25
Hereditarily countable set, 171
Hereditarily finite set, 171
Hereditarily in a set, 66

Hereditarily ordinal-definable set, 194
Hereditary set (in ZFA), 250
Hierarchy
– *see* cumulative hierarchy
– hierarchy of Borel sets, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Delta}^0_\alpha$, 140
– hierarchy of large cardinals, 302, 358, 389, 620
– hierarchy of projective sets, $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$, 144
– hierarchy of stationary sets, 99
– Kleene's hierarchy of analytic predicates, 153
– lightface Borel hierarchy, Σ^0_n, Π^0_n, 480
– lightface projective hierarchy, Σ^1_n, Π^1_n, Δ^1_n, 479
– relativization of the projective hierarchy, $\Sigma^1_n(a)$, $\Pi^1_n(a)$, $\Delta^1_n(a)$, 479
– Σ_n hierarchy, 184
Homogeneous measure algebra, 585
Homogeneous set, 107, 524
– 0-homogeneous set, 609
– 1-homogeneous set, 609
Homogeneous tree, 262, 642
κ-Homogeneous tree, 642
Homomorphism, 79
– complete homomorphism, 83
Huge cardinal, 380
n-Huge cardinal, 386

Ideal, 73, 80
– atomless ideal, 410
– countably complete ideal, 77
– dual ideal, 73
– ideal of nonstationary sets, 93
– κ-complete ideal, 77, 82
– λ-saturated ideal, 409
– normal ideal, 96
– ω_1-dense ideal, 695
– Π^1_n ideal on κ, 703
– precipitous ideal, 424
– prime ideal, 74, 81
– principal ideal, 73, 80
– σ-ideal, 77, 82
– σ-saturated ideal, 128
– trivial ideal, $\{\emptyset\}$, 73, $\{0\}$, 80
Idempotent element of a semigroup, 573
Idempotent ultrafilter, 573
Ill-founded sequential tree, 483
Image
– image of a set by a function, 11
– inverse image of a set by a function, 11
Inaccessible cardinal, 58
Inclusion map, 346
Inclusion of classes, $C \subset D$, 6
Incomparable nodes of a tree, 43
Incompatible conditions, 201
Increasing function, 17
Indecomposable ordinal, 26
Independence results, 219
Independent element over a subalgebra, 590
Independent family, 75
Independent sentence of the axioms, 219
Independent set in a Boolean algebra, 590
Indiscernibles, 299
– indiscernibles for $\mathfrak{A}$, 314
– indiscernibles for L, 312
– Silver indiscernibles, 311
Induction
– $\in$-Induction Theorem, (T6.4) 66
– Induction Theorem, (E1.9) 14
– Transfinite Induction, (T2.14) 21
– Well-Founded Induction, (T6.10) 68
Inductive set, 12
Ineffable cardinal, 308
Infimum, $\inf X$, 17
Infinite game, *see* game
Infinite product, 53
Infinite product, $\prod_i \kappa_i$, 53
Infinite sequence, 21
Infinite set, 14, 20
– D-infinite set, 34
– T-infinite set, 14
Infinite sum, $\sum_i \kappa_i$, 52
Initial segment
– initial segment of a closed set of normal measures, 359
– initial segment of a well-ordered set, 18
Inner model
– canonical inner model of a measurable cardinal of order 2, 360
– *see* core model
– inner model of ZF, 182
– the class of constructible sets, L, 175
– the class of sets constructible from a set, $L[A]$, 192
– the smallest inner model containing a set, $L(A)$, 193
Instance of Δ_0 separation, 177
Instance of a variable word, 574
Internally approachable model, 699

Interpretation, 155
– interpretation by a generic ultrafilter, 216
– G-interpretation of P-names, 218
Intersection of sets, $X \cap Y$, 8
Interval of regular cardinals, 466
Inverse image of a set by a function, 11
Inverse limit, 280
Inverse of a function, 11
Isomorphic models, 156
Isomorphism, 79
– isomorphism between models, 156
– isomorphism of partially ordered sets, 18
– isomorphism of trees, 262
– measure-preserving isomorphism of Boolean algebras, 587
Iterable M-ultrafilter, 354
Iterated forcing, 280
– iteration with I-support, 280
– iteration with countable support, 280
– iteration with finite support, 270
– revised countable support iteration, RCS, 682
– two-step iteration, $P * \dot{Q}$, 267, $B * \dot{C}$, 269
Iterated ultrapower of an inner model, 356
Iterated ultrapower, $\mathrm{Ult}_U^{(\alpha)}(V)$, 342

Jensen hierarchy, J_α, 548
Jensen's Covering Theorem, (T18.30) 329
Jónsson cardinal, 305

Kernel of an homomorphism, 80
Kernel of the universe in ZFA, $P^\infty(\emptyset)$, 250
Kondô's Uniformization Theorem, (T25.36) 498
Kurepa tree, 119

Language
– first order language, $\mathcal{L}$, 155
– forcing language, 215
– infinitary language, $\mathcal{L}_{\kappa,\omega}$, $\mathcal{L}_{\kappa,\kappa}$, 293
– language of BG, BGC, 70
– language of ZF, ZFC, 4
– language of ZFA, 250
– nth order language, 295
Laver forcing, 565
Laver function, 378
Laver real, 565
Laver tree, 565
Least element, 17
Lebesgue measurable set, 147
Lebesgue measure, 44, 147
Left-topological semigroup, 573
Lemma on Transitive Generators, (L24.31) 473
Length
– length of an extender, 382
– length of a branch, 114
– length of a set of normal measures, 359
Level of a tree, 114
– αth level of a tree, 114
Lévy collapse, 238
Lifting, 102
Limit
– diagonal limit of ordinal functions, 461
– direct limit, 158
– inverse limit, 280
– limit of a sequence, 22
– of a convergent sequence in a Boolean algebra, 598
– U-limit of a sequence, 86
Limit cardinal, 30
Limit ordinal, 20
Limit point of a set of ordinals, 91
Lindenbaum algebra, 79
Linear ordering, 17
Linearly ordered set, 17
– dense linearly ordered set, 38
– unbounded linearly ordered set, 38
Localization Lemma, (L24.32) 475
Locally closed unbounded set, 691
Logical symbols, 155, 293, 295
Lower bound, 17
Luzin set, 539

Magidor's forcing, 674
Mahlo cardinal, 95
α-Mahlo cardinal, 696
Mahlo operation, $\mathrm{Tr}(S)$, 99, $M(X)$, 290
Mapping, *see* function
Martin's Axiom
– MA, 272
– MA_κ, 272
Martin's Maximum, MM, 681
Master condition, 400
Mathias forcing, 524
Mathias real, 527
Maximal antichain, 84
Maximal element, 17, 49
Maximal filter, 74

Meager set, 87, 148
Measurable cardinal, 127
Measure
– atomless measure, 126
– fine measure, 136, 365
– κ-additive measure, 129
– Lebesgue measure, 147
– measure of a set, 125
– measure on a Boolean algebra, 585
– nontrivial σ-additive probabilistic, 125
– normal fine measure, 136
– normal measure on $P_\kappa(A)$, 374
– normal real-valued measure, 410
– probabilistic measure on a Boolean algebra, 586
– product measure, 415
– signed measure on a Boolean algebra, 586
– strictly positive measure on a Boolean algebra, 586
– strongly normal measure, 385
– translation invariant measure, 125
– two-valued measure, 126
Measure algebra, 415, 585
– homogeneous measure algebra, 585
Measure Problem, 125
Measure sequence, 676
Measure space, $(S, \mathcal{F}, \mu)$, 415
Measure-preserving isomorphism of Boolean algebras, 587
Membership relation, 4
Menge, 15
Milner-Rado paradox, 60
Minimal Boolean algebra, 595
Minimal degree of constructibility, 244
Minimal element, 17
– E-minimal element, 25
– $\in$-minimal element, 14
Minimal generic filter, 244
Minimal ultrafilter, 87
Mitchell order, 358
Model, 162
– adequate model, 184
– amenable model, 549
– *see* Boolean-valued model
– end-extension of a model, 654
– ground model, 201
– *see* inner model
– internally approachable model, 699
– model for a given language, 155
– model of type (κ, λ), 304
– N captures A, 655
– permutation model, 251
– remarkable (Σ, α)-model, 317
– (Σ, α)-model, 315
– transitive model, 163
– unbounded (Σ, α)-model, 316
Monotone function, 306
Mostowski's Absoluteness Theorem, (T25.4) 484
Mostowski's Collapsing Theorem, (T6.15) 69
Mouse, 660
Move
– move by a strategy, 627
– move in the game G_A, 627
Multiplication of cardinals, $\kappa \cdot \lambda$, 28
Multiplication of ordinals, $\alpha \cdot \beta$, 23
Mutually stationary sets, 701

Name, 215
– canonical name for a generic ultrafilter, $\dot{G}$, 214
– canonical name for a set from the ground model, $\check{x}$, 211–212
– canonical name for the ground model, $\check{M}$, 215
– hereditary symmetric name, 253
– P-name
–– P-name, 215
– symmetric name, 253
Natural number, 20
Near, 576
Node, *see* splitting node
Nondecreasing sequence, 22
Nonstationary ideal, 93
Nontrivial elementary embedding of the universe, 287
Norm, 496
– norm of a function (rank), $\|\varphi\|$, 458
– Π^1_1-norm, 496
– Π^1_n-norm, 640
– Σ^1_2-norm, 496
– Σ^1_n-norm, 640
Normal filter, 96
– normal filter on $P_\kappa(A)$, 104
Normal fine measure, 136
Normal Form Theorem for Π^1_1, (T25.3) 483
Normal function, 92
Normal ideal, 96
Normal measure, 134
– real-valued normal measure, 410
Normal measure on $P_\kappa(A)$, 374
Normal sequence, 22

Normal tree, 114
Notion of forcing, *see* forcing
Nowhere dense set, 41, 148
Null set, 146
Number, *see* cardinal number, natural number, ordinal number, real number

One-to-one function, 11
Open Coloring Axiom, OCA, 576
Open dense set, 85, 202
– open dense set below a condition, 202
Open set
– dense open set, 41
– open subset of $[X]^2$, 576
Operation
– absolute operation on Borel sets, 507
– arithmetic operation on cardinals, $\kappa + \lambda$, $\kappa \cdot \lambda$, κ^λ, 28
– arithmetic operations on ordinal numbers, $\alpha + \beta$, $\alpha \cdot \beta$, α^β, 23
– *see* Boolean operation
– *see* diagonal intersection, diagonal union
– Gödel operations, $G_1, \ldots, G_{10}$, 177
– Mahlo operation, $\mathrm{Tr}(S)$, 99, $M(X)$, 290
– n-ary operation on a set, 11
– operation + on $\beta\boldsymbol{N}$, 573
– operation on a set, 102
– operations on classes, $\cap$, $\cup$, $-$, $\bigcup$, 6
– Suslin operation $\mathcal{A}$, 143
Order
– Mitchell order of normal measures, $U_1 < U_2$, 358
– order of a cardinal number (height of the Mitchell order), $o(\kappa)$, 358
– order of a normal measure, $o(U)$, 358
– order of a stationary set, $o(A)$, $o(\kappa)$, 100
Order-preserving function, 17
Order-type, 19
Order-type of an ordinal in the generic ultrapower, 421
Ordered n-tuple, 7
Ordered pair, 7
Ordering
– linear ordering, 17
– partial ordering, 17
– prewellordering, 496
– Rudin-Keisler ordering, 86
– strict ordering, 17
– well-ordering, 18
Ordinal function, *see* function
Ordinal number, 19
– finite ordinal, 20
– indecomposable ordinal, 26
– projective ordinal, $\boldsymbol{\delta}^1_n$, 636
Ordinal-definable set, 194
– hereditarily ordinal-definable set, 194
– ordinal-definable set from a set, 195
– ordinal-definable set over a set, 195
Outer measure, 146
Overlapping extenders, 666

Pair, 7
Paradox
– Milner-Rado Paradox, 60
– Russell's Paradox, 4
Parameter
– parameter in a formula (property), 5
– standard parameter, 549
Partial ordering, 17
Partial play, 597
Partially ordered set, 17
– separative partially ordered set, 204
Partition, 12, 84
I-Partition, 424
Partition property, 109
Partition relation
– $\kappa \to (\alpha)^n_m$, 112
– $\kappa \to (\alpha, \beta)^n$, 112
– $\kappa \to (\lambda)^n_m$, 109
Path, 43
pcf (possible cofinalities), 466
– compactness of pcf, 472
Perfect set, 40
Perfect tree, 43, 244, 561
– superperfect tree, 618
Permutation, 251
Permutation model, 251
Pigeonhole principle, 107
Play
– in the game G_A, 627
– partial play, 597
– play by a strategy, 627
Player
– Empty and Nonempty, 426
– I and II, 596, 627
– in the infinite game $\mathcal{G}$ on a Boolean algebra, 596
p-Point, 76
Polish space, 44
Possible cofinalities, pcf(A), 466
Power of a set, X^n, 10
Power set, $P(X)$, 9
Precipitous ideal, 424

Predense set, 202
Predense set below a condition, 202
Prewellordering, 496
Prikry forcing, 401
Prime ideal, 74, 81
Prime Ideal Theorem, (T7.10) 81
Principal filter, 73, 80
Principal ideal, 73, 80
Principle
– Collection Principle, 65
– *see* Diamond Principle
– pigeonhole principle, 107
– Principle of Dependent Choices, DC, 50
– reduction principle, 507
– Reflection Principle, (T12.14) 168
– Reflection Principle, RP, 688
– $\mathbf{\Sigma}^1_1$-Separation Principle, (L11.11) 146
– Δ_0-Separation, 177
– separation principle, 507
– *see* Square
– Strong Reflection Principle, SRP, 688
– Vopěnka's Principle, VP, 380
Probabilistic measure on a Boolean algebra, 586
Problem
– Measure Problem, 125
– Problem 36.7, 673
– Singular Cardinal Problem, 672
– Suslin's Problem, 39
Product
– infinite product, 53
– infinite product, $\prod_i \kappa_i$, 53
– product of cardinals, $\kappa \cdot \lambda$, 28
– product of linear orders, 24
– product of measure spaces, 415
– reduced product, 159
Product forcing, 229
– Easton product, 233
– κ-product, 230
– product of a collection of forcing notions, 230
– product of two forcing notions, 229
Product Lemma, (L15.9) 229
Product measure, 415
Product of two sets, 10
Projection, 102, 142
– projection of a generic filter on a coordinate of a product forcing, 230
– projection to a subalgebra, 589
Projection of a set, 34
Projective Determinacy, PD, 628
Projective ordinal, $\boldsymbol{\delta}^1_n$, 636
Projective set, 145
Projective stationary set, 688
Σ_n-Projectum, 549
n-Projectum, 547
Proper class, 6
Proper forcing, 601
Proper Forcing Axiom, PFA, 607, PFA$^+$, 613, PFA$^-$, 614
Proper game for P below p, 603
Proper subset, 9
Property, 4
– absolute property, 185
– Axiom A, 614
– Baire property, 148
– club-guessing property, 442
– Δ_n property, 184
– Knaster property (K), 231
– partition property, 109
– perfect set property, 150
– Π_n property, 183
– Σ_n property, 183
– the finite intersection property, 74, 80
– the prewellordering property, 496, 640
– the scale property, 498, 640
– the tree property, 120
– the uniformization property, 498

Quantifiers, $\forall x$, $\exists x$, 5
Quotient
– of a Boolean algebra, B/I, 80
– separative quotient, 206
– $X/\equiv$, 12

Radin forcing, R_U, 677
Ramsey cardinal, 121
Ramsey null set, 525
Ramsey set, 524
– completely Ramsey set, 525
– Ramsey null set, 525
Ramsey ultrafilter, 76
Random forcing, 243
Random real, 243, 513
Range of a relation, 10
Rank
– Galvin-Hajnal norm of an ordinal function, $\|f\|$, 460
– rank function, $\rho(x)$, 68
– rank of an element in a relation, $\rho(x)$, 25

– rank of a set, rank(x), 64
Rapid sequence of ordinal functions, 464
Real line, $\boldsymbol{R}$, 37
Real number
– algebraic real number, 44
– *see* generic real
– real of minimal degree of constructibility, 244
– transcendental real number, 44
Real-valued measurable cardinal, 130
Real-valued measure, 125
Real-valued normal measure, 410
Recursion
– $\in$-Recursion, (T6.5) 66
– Transfinite Recursion, (T2.15) 22
– Well-Founded Recursion, (E2.15) 26, (T6.11) 68
Reduced product, 159
Reduction principle, 507
Refinement
– refinement of an I-partition, 424
– refinement of a partition, 85
Reflecting cardinal, 697
Reflection Principle, (T12.14) 168
Reflection Principle, RP, 688
Reflexive relation, 12
Regressive ordinal function, 93
Regular cardinal, 32
Regular cut, 83
Regular open set, 152
Regular subalgebra, 588
Relation
– binary relation, 10
– Δ_n relation, 184
– equality predicate, $=$, 4
– equivalence relation, 12
– extensional relation, 68
– forcing relation, $\Vdash_P$, $\Vdash$, 215
– membership relation, $\in$, 4
– n-ary relation, 10
– *see* partition relation
– Π_n relation, 183
– reflexive relation, 12
– relation between Boolean algebras and logic, 79
– relation between Determinacy and large cardinals, 633
– relation between ideals and homomorphisms, 80
– relation between pcf and cardinal arithmetic, 466
– relation on X, 10
– Σ_n relation, 183
– symmetric relation, 12
– transitive relation, 12
– well-founded relation, 25
Relative constructibility, 192
Relativization of a formula, $\varphi^{M,E}$, φ^M, 161
Remarkable (Σ, α)-model, 317
Representation Lemma, (L19.13) 347
Restriction of a function, 11
Rigid Boolean algebra, 595
Rigid tree, 262
Rowbottom cardinal, 305
Rudimentary closure, rud(M), 548
Rudimentary function, 548
Rudimentary function in A, 660
Rudin-Keisler ordering, 86
Russell's Paradox, 4

Sacks real, 244
Satisfaction, 155
Satisfaction for an nth order formula, 295
σ-Saturated ideal, 128
κ-Saturated Boolean algebra, 84
λ-Saturated ideal, 409
Saturation, 84
Scale, 498
– κ-scale, 133
– Π^1_1-scale, 498
– Π^1_n-scale, 640
– Σ^1_n-scale, 640
– scale in $\prod_{a\in A}\gamma_a$, 461
Second category, 148
Semi-Cohen Boolean algebra, 591
(M, P)-Semigeneric condition, 649
Semiproper forcing, 649
Semiproper Forcing Axiom, 681
Semiproper set of conditions, 654
Sentence, 5
Sentence independent of the axioms, 219
Separable metric space, 40
Separated sets, 145
Separation principle, *see* principle
Separative partially ordered set, 204
Separative quotient, 206
Sequence, 21
– α-sequence, 21
– club-guessing sequence, 442
– cofinal sequence, 31
– continuous sequence, 22
– $\diamondsuit$-sequence, 191

– finite sequence, 21
– fusion sequence of Laver trees, 565
– fusion sequence of perfect trees, 244
– infinite sequence, 21
– measure sequence, 676
– nondecreasing sequence, 22
– normal sequence, 22
– Prikry generic sequence, 402
– rapid sequence of ordinal functions, 464
– sequence in a set, 21
– sequence of length α, 21
– sequence of length n, 21
– square-sequence, 443
– transfinite sequence, 21
Sequential tree, 43
– ill-founded sequential tree, 483
– r-dimensional sequential tree, 483
– well-founded sequential tree, 483
Set, 70
– adequate set, 188
– analytic set, 142
– arithmetical set, Σ^0_n, Π^0_n, 480
– Borel set, 41, 140
– bounded set in a tree, 240
– bounded set of ordinal functions, 464
– closed subset of $P_\kappa(A)$, 100
– *see* closed unbounded set
– cofinal set, 31
– cofinal set of ordinal functions, 464
– constructible set, 175
– countable set, 30
– D-finite set, 34
– D-infinite set, 34
– *see* definable set
– dense set, 201, 205
– directed set, 157
– directed subset of $P_\kappa(A)$, 101
– $\equiv$-closed set, 633
– E.M. set of formulas, 314
– empty set, $\emptyset$, 8
– F_σ, G_δ set, 42
– *see* family
– finite set, 14, 20
– generates a model (Skolem hull), 300
– has n-elements, 14
– has strong measure zero, 538
– has the Baire property, 148
– hereditarily countable set, 171
– hereditarily finite set, 171
– hereditary set (in ZFA), 250
– homogeneous set, 107, 524
– homogeneously Suslin set, 642
– independent subset of a Boolean algebra, 590
– inductive set, 12
– infinite set, 14, 20
– κ-homogeneously Suslin set, 642
– κ-Suslin set, 485
– κ-weakly homogeneously Suslin set, 643
– Lebesgue measurable set, 147
– Luzin set, 539
– meager set, 87, 148
– nowhere dense set, 41, 148
– null set, 146
– open dense set, 85, 202
– open dense set below a condition, 202
– open subset of $[X]^2$, 576
– *see* ordinal-definable set
– partially ordered set, 17
– perfect set, 40
– predense set, 202
– predense set below a condition, 202
– projective set, 145
– *see* Ramsey set
– recursive closed set, Π^0_1 set, 480
– recursive open set, recursively enumerable set, Σ^0_1 set, 480
– reflects a formula, 168
– regular open set, 152
– relative constructible set, 192
– remarkable set of formulas, 318
– $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Delta}^0_\alpha$ set, 140
– $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$ set, 144
– semiproper set of conditions, 654
– set accepts a set, 525
– set at most countable, 30
– set captures a Mathias forcing condition, 527
– set closed under an operation, 102
– set constructible from a set, 192
– set definable from a sequence of ordinals, 519
– set generic over M, 202
– set of all Borel codes, BC, 504
– set of all infinite paths, 43
– set of first category, 148
– set of indiscernibles, 299
– set of indiscernibles closed in $Ord^{\mathfrak{A}}$, 318
– set of indiscernibles for $\mathfrak{A}$, 314
– set of measure one, 409
– set of measure zero, 409
– set of positive measure, 409

– set of possible cofinalities, pcf(A), 466
– set of second category, 148
– set of the first category, 41
– set rejects a set, 525
– set σ-generates a Boolean algebra, 585
– Solovay set of reals, 515
– *see* stationary set
– superperfect set in $\mathcal{N}$, 618
– symmetric set, 251
– T-finite set, 14
– T-infinite set, 14
– transitive set, 13, 19
– unbounded E.M. set of formulas, 317
– unbounded linearly ordered set, 38
– unbounded set in $P_\kappa(A)$, 100
– uncountable set, 30
– universal $\mathbf{\Sigma}^0_n$ set, 141
– universal $\mathbf{\Sigma}^1_n$ set, 145
– universally Baire set, 623
– weakly homogeneously Suslin set, 643
– well-founded E.M. set, 318
Sets
– mutually stationary sets, 701
Sets of ZFA, 250
Sharp
– sharp for a strong cardinal, 666
– x-sharp, $x^\sharp$, 328
– zero-sharp, $0^\sharp$, 312
Shoenfield's Absoluteness Theorem, (T25.20) 490
Signed measure on a Boolean algebra, 586
Sikorski's Extension Theorem, (E7.30) 88
Silver indiscernibles, 311
Simple Boolean algebra, 595
Singleton, $\{a\}$, 7
Singular cardinal, 32
Singular Cardinal Hypothesis, SCH, 58
Singular Cardinal Problem, 672
Skolem constants, c^φ_ξ, 293
Skolem function, 156
– canonical Skolem function, 313
– definable Skolem function, 300
– Σ_n Skolem function, 333
Skolem hull, 156
Skolem sentence, 294
Skolem term, 300
Σ_n Skolem hull, 333
Solovay set of reals, 515
Special $\aleph_2$-Aronszajn tree, 571
Special Aronszajn tree, 117
Splitting node, 244
– nth splitting node, 244
– ω-splitting node, 618
Square
– Square Principle, $\square_\kappa$, 443
– $\square_{\kappa,\nu}$, 702
– Weak Square, $\square_{\kappa,\kappa}$, $\square^*_\kappa$, 702
Square-sequence, 443
Stable element, 84, 88
Standard code, 549
Standard parameter, 549
Stationary set
– canonical stationary set, 477
– projective stationary set, 688
– stationary set reflects at α, 444
– stationary subset of $P_\kappa(A)$, 100
– stationary subset of a cardinal number, 91
– stationary subset of $P(A)$, 678
Stationary set preserving forcing, 681
Stationary tower forcing, 653, 678
Stem, 557
– stem of a Laver tree, 565
Stone's Representation Theorem, (T7.11) 81
Strategically ω-closed Boolean algebra, 596
Strategically ω-closed notion of forcing, 596
Strategy
– strategy in the game $\mathcal{G}$, 596
– strategy in the game $\mathcal{G}_I$, 426
– strategy in the game G_A, 627
– winning strategy, 426, 596, 627
Strength
– consistency strength, 358, 367, 382, 389, 394, 404, 426, 429, 450, 456, 547, 610, 623, 624, 636, 647, 659, 665, 669, 672, 679, 680, 691, 695–698, 702
– relative strength, 249, 387
– strength of an embedding, 676
Strict ordering, 17
Strictly positive measure on a Boolean algebra, 586
Strong cardinal, 381
λ-Strong cardinal, 382
λ-Strong cardinal for a set A, 647
Strong limit cardinal, 58
Strong measure zero, 538
Strong Reflection Principle, SRP, 688
Stronger forcing condition, 201

Strongly closed unbounded set, 704
Strongly compact cardinal, 136
Strongly normal measure, 385
Subalgebra, 79
– co-dense subalgebra, 591
– complete subalgebra, 83
– dense subalgebra, 82
– regular subalgebra, 84, 588
– subalgebra completely generated by a set, 83
– subalgebra generated by a set, 79
Subclass of a class, $C \subset D$, 6
Submodel, 156
– elementary submodel, $\mathfrak{B} \prec \mathfrak{A}$, 156
– Σ_n-elementary submodel, $M \prec_{\Sigma_n} N$, 187
– symmetric submodel of a generic extension, 253
Subset, 9
– dense subset, 38
– proper subset, 9
– symmetric subset, 404
Successor cardinal, $\aleph_{\alpha+1}$, 30
Successor of an ordinal, 20
Successor ordinal, $\beta + 1$, 20
Sufficiently large cardinal number, 602
Sum
– infinite sum of cardinals, $\sum_i \kappa_i$, 52
– sum of cardinals, $\kappa + \lambda$, 28
– sum of linear orders, 23
Supercompact cardinal, 136, 374
λ-Supercompact cardinal, 375
Superperfect set in $\mathcal{N}$, 618
Superperfect tree, 618
Superstrong cardinal, 657
Support
– countable support, 230, 280
– Easton support, 395
– finite support, 230, 270
– I-support, 280
– κ-support, 230
– $<\kappa$-support, 230
– support for a set in a permutation model, 252
– support of a condition, 230
– support of a condition in the stationary tower forcing, 653
– support of a subset of κ^α, 347
Supremum, $\sup X$, 17
Suslin algebra, 593
Suslin line, 114
Suslin operation $\mathcal{A}$, 143
κ-Suslin set, 485
Suslin tree, 114
κ^+-Suslin tree, 551
Suslin's Hypothesis, SH, 274
Suslin's Problem, 39
Symmetric difference, 9
Symmetric name, 253
– hereditarily symmetric name, 253
Symmetric relation, 12
Symmetric set, 251
Symmetric submodel of a generic extension, 253
Symmetric subset, 404
Symmetry, 253
Symmetry Lemma, (L14.37) 221

T-finite set, 14
T-infinite set, 14
Tarski's Ultrafilter Theorem, (T7.5) 75
Term, 155
– Skolem term, 300
– Σ_n Skolem term, 333
Theorem
– Aronszajn's Theorem, (T9.16) 116
– Bagaria's Theorem, (T37.28) 691
– Bartoszyński-Raisonnier-Stern's Theorem, (T26.46) 533
– Baumgartner-Malitz-Reinhardt's Theorem, (T16.17) 274
– Booth's Theorem, (T16.27) 278
– Boundedness Lemma, (C25.14) 487
– Cantor's Normal Form Theorem, (T2.26) 24
– Cantor's Theorem, (T3.1) 27, (T4.1) 37, (T4.3) 38
– Cohen's Theorem, (T14.36) 221
– Condensation Lemma, (L27.5) 550
– Dodd-Jensen's Covering Theorem for K, (T35.14) 663
– Dodd-Jensen's Covering Theorem for $L[U]$, (T35.16) 664
– Dodd-Jensen's Theorem, (T35.6) 661, (T35.13) 663
– Dushnik-Miller's Theorem, (T9.7) 112
– Easton's Theorem, (T15.18) 232
– Erdős-Rado Partition Theorem, (T9.6) 111
– Fodor's Theorem, (T8.7) 93
– Foreman's Theorem, (T30.25) 597
– Foreman-Magidor's Theorem, (T38.16) 701
– Foreman-Magidor-Shelah's Theorem, (T37.9) 684, (T37.13) 686, (T37.16) 687

– Fremlin's Lemma, (L30.5) 586
– Fremlin's Theorem, (T26.45(ii)) 533
– Gaifman's Theorem, (T19.7) 343
– Galvin-Hajnal Theorem, (T24.1) 457
– Galvin-Prikry-Silver's Theorem, (T26.22) 524
– Gitik's Theorem, 669
– Gitik-Shelah Theorem, (T23.17) 450
– Glazer's Lemma, (L29.2) 573
– Gödel's Condensation Lemma, (L13.17) 188, (L18.38) 332
– Gödel's Normal Form Theorem, (T13.4) 177
– Gödel's Theorem, (T13.18) 188, (T13.20) 190, (T25.26) 494
– Gödel's Completeness Theorem, 157
– Gödel's First Incompleteness Theorem, 157
– Gödel's Second Incompleteness Theorem, 157
– Gödel's Second Incompleteness Theorem, 157
– Gödel's Theorem, (T13.16) 187
– Gregory's Theorem, (T23.2) 442
– Hajnal's Theorem, (T27.11) 554
– Hanf-Scott's Theorem, (T17.18) 297
– Hausdorff's Theorem, (T29.7) 575
– Hindman's Theorem, (T29.1) 573
– Jech's Theorem, (T8.20) 100, (T8.22) 101, (T8.24) 101
– Jech-Shelah's Theorem, (T27.12) 555
– Jech-Sochor's Theorem, (T15.53) 256
– Jensen's Covering Theorem, (T18.30) 329
– Jensen's Theorem, (T13.21) 191, (T15.26) 241, (T27.1) 546, (T27.7) 551, (T28.1) 557
– Kondô's Uniformization Theorem, (T25.36) 498
– König's Theorem, (T5.10) 54
– König's Lemma, (E9.5) 122
– Kripke's Theorem, (C26.8) 516
– Kueker's Theorem, (T8.28) 102
– Kunen's Theorem, (T17.7) 290, (T18.20) 323, (T19.14) 348, (T20.4) 367
– Kunen-Martin Theorem, (T25.43) 503
– Lévy's Theorem, (T15.22) 238
– Laver's Theorem, (T20.21) 378, (T28.14) 564
– Laver-Shelah's Theorem, (T28.25) 570
– Lemma on Transitive Generators, (L24.31) 473
– Localization Lemma, (L24.32) 475
– Łoś's Theorem, (T12.3) 159
– Magidor's Theorem, (T23.10) 446, (T23.23) 453, (T32.16) 621, (T36.3) 672
– Magidor-Shelah-Gitik's Theorem, (T36.5) 673
– Maharam's Theorem, (T30.1) 585
– Mansfield's Theorem, (T25.39) 500
– Mansfield-Solovay's Theorem, (T25.23) 492
– Martin's Theorem, (T25.42) 502, (T32.15) 621, (T33.18) 637
– Martin-Solovay's Theorem, (T16.20) 276, (T26.39) 529
– Martin-Solovay-Mansfield's Theorem, (T32.14) 620
– Martin-Steel's Theorem, (T33.28) 642, (T33.34) 643, 666
– Martin-Steel-Woodin's Theorem, (T33.26) 642
– Mathias' Theorem, (T21.14) 402, (T26.23) 524, (T26.35) 527
– Menas' Theorem, (T8.27) 102
– Mitchell's Theorem, (T19.30) 356, (T19.37) 361, (T19.38) 361, (T19.39) 362, (T28.24) 569, (C35.18) 665, (T35.17) 665, (T36.16) 678
– Moschovakis' Theorem, (T33.23) 640
– Mostowski's Absoluteness Theorem, (T25.4) 484
– Mostowski's Collapsing Theorem, (T6.15) 69
– Namba's Theorem, (T28.10) 561
– Normal Form Theorem for Π^1_1, (T25.3) 483
– Pawlikowski's Theorem, (T26.47) 534
– Pospíšil's Theorem, (T7.6) 75
– Prikry's Theorem, (T21.10) 401, (T22.2) 410
– Properties of Forcing, (T14.7) 204
– Ramsey's Theorem, (T9.1) 107
– Rowbottom's Theorem, (T17.27) 300
– Sacks' Theorem, (T15.34) 244
– Schimmerling's Theorem, (T31.30) 611
– Scott's Theorem, (T17.1) 285
– Shanin's Theorem (Δ-Lemma), (T9.18) 118
– Shelah's Theorem, (T23.3) 442, (T24.8) 461, (T24.16) 465, (T24.18)

466, (T24.25) 470, (T24.33) 476, (T28.12) 563, (T31.15) 604, (T32.13) 620, (T37.4) 682, (T37.10) 684, (T37.26) 690, (T38.1) 695
– Shoenfield's Absoluteness Theorem, (T25.20) 490
– Sierpiński's Theorem, 490
– Sikorski's Extension Theorem, (E7.30) 88
– Silver's Theorem, (T8.12) 96, (T8.13) 96, (T18.1) 311, (T18.1 relativized) 328, (T19.3) 339, (T21.4) 395, (T27.9) 553, (T28.23) 569, (T32.1) 615, (T32.20) 623
– Solovay's Theorem, (T8.10) 95, (T20.8) 372, (T22.1) 410, (T22.26) 428, (T26.14) 519, (T26.20) 522, (T27.8) 551, (T33.12) 633
– Solovay-Tennenbaum's Theorem, (T16.13) 272
– Steel's Theorem, (T35.22) 667
– Stone's Representation Theorem, (T7.11) 81
– Suslin's Theorem, (T11.10) 145
– Symmetry Lemma, (L14.37) 221
– Tarski's Theorem, (T12.7) 162
– Tarski's Ultrafilter Theorem, (T7.5) 75
– the Baire Category Theorem, (T4.8) 41
– the Cantor-Bendixson Theorem, (T4.6) 40
– the Cantor-Bernstein Theorem, (T3.2) 28
– the Compactness Theorem, 293
– the Ehrenfeucht-Mostowski Theorem, 313
– the Factor Lemma, (L19.5) 342, (L21.8) 396, (C26.11) 518
– the Forcing Theorem, (T14.6) 204, 236
– the Fubini Theorem, (L11.12) 148
– the Generic Model Theorem, (T14.5) 203
– the Hales-Jewett Theorem, (T29.4) 574
– the Homogeneity of the Lévy Collapse, (T26.12) 518
– the Induction Theorem, (E1.9) 14
– the $\in$-Induction Theorem, (T6.4) 66
– the Kunen-Paris Theorem, (T21.3) 392
– the Lévy-Solovay Theorem, (T21.1) 389
– the Löwenheim-Skolem Theorem, (T12.1) 157
– the Prime Ideal Theorem, (T7.10) 81
– the Product Lemma, (L15.9) 229
– the $\in$-Recursion Theorem, (T6.5) 66
– the Reflection Principle, (T12.14) 168
– the Representation Lemma, (L19.13) 347
– the $\mathbf{\Sigma}^1_1$-Separation Principle, (L11.11) 146
– the Transfinite Induction, (T2.14) 21
– the Transfinite Recursion, (T2.15) 22
– the Vopěnka-Hrbáček Theorem, (T20.3) 366
– the Weak Compactness Theorem, 293
– the Well-Founded Induction, (T6.10) 68
– the Well-Founded Recursion, (E2.15) 26, (T6.11) 68
– Todorčević's Lemma, (L31.24) 609
– Todorčević's Theorem, (T29.8) 577, (T31.23) 609, (T31.28) 610, (T37.18) 688
– Truss-Miller Theorem, (T26.45(i)) 533
– Ulam's Theorem, (T10.1) 126
– van der Waerden's Theorem, (T29.3) 574
– Vitali's Theorem, (E10.1) 137
– Vladimirov's Lemma, (E30.11) 598
– Vopěnka's Theorem, (T15.46) 249
– Woodin's Theorem, (T33.16) 636, (T33.27) 642, (T33.33) 643, (T34.6) 650, (T34.14) 655, (T34.16) 656, (T36.2) 669, (T36.18) 678, (T38.2) 695, (T38.3) 696
– Woodin-Cummings' Theorem, (T36.6) 673
– Woodin-Gitik's Theorem, (T36.4) 673
– Zermelo's Well-Ordering Theorem, (T5.1) 48
– Zorn's Lemma, (T5.4) 49
Theory
– axiomatic set theory with atoms, ZFA, 250
– axiomatic set theory with classes, BG, BGC, 70
– axiomatic set theory, ZF, ZFC, 3

– combinatorial set theory, 107–123, 267, 573–583
– descriptive set theory, 42, 44, 50, 126, 139, 148, 479–509, 529, 595, 615–626, 637, 645
– fine structure theory, 330, 337, 443, 545–550, 556
– measure theory, 42, 73, 125–138, 146–148, 409–420, 522–523, 585–588
– model theory, 155–161, 172, 283, 285, 292, 313, 381
– pcf theory, 457, 460, 461, 463, 466–477, 669, 672, 702
– Peano Arithmetic, 157
– theory of cardinal and ordinal numbers, 15
– theory of core models, 659–661, 663–666, 672
– theory of inner models, 659, 660, 668, 675, 702
– theory of iteration trees, 666
– theory of large cardinals, 58, 113–117, 120–138, 157, 285–439, 441, 442, 444, 620–623, 633–636, 649
– theory of mice, 661
– theory of ordinal and cardinal numbers, 17–35
– theory of partition relations, 107–123
– theory of stationary sets, 91–105, 441–456
– theory of transitive models, 63–71, 163–200
– theory of well-ordered sets, 17–26
Topology
– Baire topology, 42
– Ellentuck topology, 524
– Σ^1_1-topology, 615
– topology based on the Mathias forcing, 524
– topology of the real line, 40
Tower, 540
Trace of a stationary set, $\mathrm{Tr}(S)$, 99
Transcendental real number, 44
Transfinite Induction, (T2.14) 21
Transfinite Recursion, (T2.15) 22
Transfinite sequence, 21
Transitive closure, $\mathrm{TC}(S)$, 64
Transitive collapse, 68
Transitive generators of pcf, 473
Transitive model, 163
Transitive relation, 12
Transitive set, 13, 19
Translation invariant measure, 125
Tree, 114
– Aronszajn tree, 116
– club-isomorphic trees, 612
– homogeneous tree, 262, 642
– ill-founded sequential tree, 483
– κ-homogeneous tree, 642
– κ^+-Suslin tree, 551
– κ-weakly homogeneous tree, 643
– Kurepa tree, 119
– Laver tree, 565
– normal α-tree, 114
– perfect tree, 43, 244, 561
– r-dimensional sequential tree, 483
– rigid tree, 262
– sequential tree, 43
– special $\aleph_2$-Aronszajn tree, 571
– special Aronszajn tree, 117
– superperfect tree, 618
– Suslin tree, 114
– tree of 0–1 sequences, 244
– tree on $\omega \times K$, 485
– tree on ω_2, 561
– weakly homogeneous tree, 643
– well-founded sequential tree, 483
Tree property, 120
Trivial Boolean algebra, $\{0, 1\}$, 81
Trivial filter, $\{S\}$, 73, $\{1\}$, 80
Trivial ideal, $\{\emptyset\}$, 73, $\{0\}$, 80
True cofinality, 461
Truth definition, 162
Two-step iteration
– two-step iteration of complete Boolean algebras, $B * \dot{C}$, 269
– two-step iteration of forcing notions, $P * \dot{Q}$, 267
Two-valued measure, 126

Ulam matrix, 131
– $(\aleph_1, \aleph_0)$-matrix, 131
– (λ^+, λ)-matrix, 132
Ultrafilter, 74, 81
– fine measure, 136
– generic ultrafilter, 206
– idempotent ultrafilter, 573
– (κ, λ)-regular ultrafilter, 373
– *see* M-ultrafilter
– minimal ultrafilter, 87
– normal fine measure, 136
– normal measure, 134
– p-point, 76
– Ramsey ultrafilter, 76
– uniform ultrafilter, 75
M-Ultrafilter, 323

– iterable M-ultrafilter, 354
– κ-complete M-ultrafilter, 323
– normal iterable M-ultrafilter, 354
– normal M-ultrafilter, 323
Ultrapower
– $\mathrm{Ult}_U \mathfrak{A}$, 161
– generic ultrapower, $\mathrm{Ult}_G(M)$, 421, 653
– iterated ultrapower, $\mathrm{Ult}_U^{(\alpha)}(V)$, 342
Ultraproduct, $\mathrm{Ult}_U\{\mathfrak{A}_x : x \in S\}$, 159
Unbounded family, 532
Unbounded function on a set of positive measure, 410
Unbounded linearly ordered set, 38
Unbounded subset of $P_\kappa(A)$, 100
Unbounded subset of an ordinal, 32
Uncountable set, 30
Uniform density of a Boolean algebra, 589
Uniform ultrafilter, 75
Uniformization, 498
Uniformization Theorem (Kondô), (T25.36) 498
Union
– $\bigcup X$, 9
– diagonal union, $\sum_\alpha X_\alpha$, 93
Universal class, V, 6
Universal set
– universal $\mathbf{\Sigma}^0_n$ set, 141
– universal $\mathbf{\Sigma}^1_n$ set, 145
Universally Baire set, 623
Universe
– Boolean universe, 206
– constructible universe, L, 175
– $\mathbf{\Sigma}^1_3$-absolute universe for generic extensions, 624
– universal class, V, 6
Upper bound, 17, 49
– exact upper bound of a set of ordinal functions, 461
– least upper bound of ordinal functions, 460
– upper bound of ordinal functions, 460

Valid formula, $\|\varphi\| = 1$, 207
Value of a function at a set, 11
Value of a term, 155
Variable, 5, 574
– free variable, 5
– variable of order k, 295
Variable word, 574
Vopěnka's Principle, VP, 380

Weak Covering Theorem, (T35.17(v)) 665
Weak Square, $\square^*_\kappa$, 702
Weakly compact cardinal, 113
Weakly homogeneous tree, 643
κ-Weakly homogeneous tree, 643
Weakly inaccessible cardinal, 33
Weakly (κ, λ)-distributive Boolean algebra, 246
Weakly Mahlo cardinal, 96
Weight of measure algebra, 585
Well-Founded Induction, (T6.10) 68
Well-Founded Recursion, (E2.15) 26, (T6.11) 68
Well-founded relation, 25
Well-founded sequential tree, 483
Well-ordering, 18
– canonical well-ordering of $\alpha \times \alpha$, 30
– canonical well-ordering of $Ord \times Ord$, 30
– canonical well-ordering of L, 190
Winning strategy in an infinite game, *see* strategy, game
Woodin cardinal, 384, 647
Word, 574
– variable word, 574

Zermelo's Well-Ordering Theorem, (T5.1) 48
Zermelo-Fraenkel axioms, ZF, ZFC, 3
Zero-dagger, $0^\dagger$, 353
Zero-sharp, $0^\sharp$, 312
ZFA, 250

*S*pringer *M*onographs in *M*athematics

This series publishes advanced monographs giving well-written presentations of the "state-of-the-art" in fields of mathematical research that have acquired the maturity needed for such a treatment. They are sufficiently self-contained to be accessible to more than just the intimate specialists of the subject, and sufficiently comprehensive to remain valuable references for many years. Besides the current state of knowledge in its field, an SMM volume should also describe its relevance to and interaction with neighbouring fields of mathematics, and give pointers to future directions of research.

Abhyankar, S.S. **Resolution of Singularities of Embedded Algebraic Surfaces** 2nd enlarged ed. 1998
Alexandrov, A.D. **Convex Polyhedra** 2005
Andrievskii, V.V.; Blatt, H.-P. **Discrepancy of Signed Measures and Polynomial Approximation** 2002
Angell, T. S.; Kirsch, A. **Optimization Methods in Electromagnetic Radiation** 2004
Ara, P.; Mathieu, M. **Local Multipliers of C*-Algebras** 2003
Armitage, D.H.; Gardiner, S.J. **Classical Potential Theory** 2001
Arnold, L. **Random Dynamical Systems** corr. 2nd printing 2003 (1st ed. 1998)
Arveson, W. **Noncommutative Dynamics and E-Semigroups** 2003
Aubin, T. **Some Nonlinear Problems in Riemannian Geometry** 1998
Auslender, A.; Teboulle M. **Asymptotic Cones and Functions in Optimization and Variational Inequalities** 2003
Bang-Jensen, J.; Gutin, G. **Digraphs** 2001
Baues, H.-J. **Combinatorial Foundation of Homology and Homotopy** 1999
Brown, K.S. **Buildings** 3rd printing 2000 (1st ed. 1998)
Cherry, W.; Ye, Z. **Nevanlinna's Theory of Value Distribution** 2001
Ching, W.K. **Iterative Methods for Queuing and Manufacturing Systems** 2001
Crabb, M.C.; James, I.M. **Fibrewise Homotopy Theory** 1998
Chudinovich, I. **Variational and Potential Methods for a Class of Linear Hyperbolic Evolutionary Processes** 2005
Dineen, S. **Complex Analysis on Infinite Dimensional Spaces** 1999
Dugundji, J.; Granas, A. **Fixed Point Theory** 2003
Ebbinghaus, H.-D.; Flum J. **Finite Model Theory** 2006
Elstrodt, J.; Grunewald, F. Mennicke, J. **Groups Acting on Hyperbolic Space** 1998
Edmunds, D.E.; Evans, W.D. **Hardy Operators, Function Spaces and Embeddings** 2004
Engler, A.; Prestel, A. **Valued Fields** 2005
Fadell, E.R.; Husseini, S.Y. **Geometry and Topology of Configuration Spaces** 2001
Fedorov, Y.N.; Kozlov, V.V. **A Memoir on Integrable Systems** 2001
Flenner, H.; O'Carroll, L. Vogel, W. **Joins and Intersections** 1999
Gelfand, S.I.; Manin, Y.I. **Methods of Homological Algebra** 2nd ed. 2003
Griess, R.L. Jr. **Twelve Sporadic Groups** 1998
Gras, G. **Class Field Theory** corr. 2nd printing 2005
Hida, H. ***p*-Adic Automorphic Forms on Shimura Varieties** 2004
Ischebeck, F.; Rao, R.A. **Ideals and Reality** 2005
Ivrii, V. **Microlocal Analysis and Precise Spectral Asymptotics** 1998
Jech, T. **Set Theory** (3rd revised edition 2002) corr. 4th printing 2006
Jorgenson, J.; Lang, S. **Spherical Inversion on SLn (R)** 2001
Kanamori, A. **The Higher Infinite** corr. 2nd printing 2005 (2nd ed. 2003)
Kanovei, V. **Nonstandard Analysis, Axiomatically** 2005
Khoshnevisan, D. **Multiparameter Processes** 2002
Koch, H. **Galois Theory of *p*-Extensions** 2002
Komornik, V. **Fourier Series in Control Theory** 2005
Kozlov, V.; Maz'ya, V. **Differential Equations with Operator Coefficients** 1999
Landsman, N.P. **Mathematical Topics between Classical & Quantum Mechanics** 1998
Leach, J.A.; Needham, D.J. **Matched Asymptotic Expansions in Reaction-Diffusion Theory** 2004
Lebedev, L.P.; Vorovich, I.I. **Functional Analysis in Mechanics** 2002

Lemmermeyer, F. **Reciprocity Laws: From Euler to Eisenstein** 2000
Malle, G.; Matzat, B.H. **Inverse Galois Theory** 1999
Mardesic, S. **Strong Shape and Homology** 2000
Margulis, G.A. **On Some Aspects of the Theory of Anosov Systems** 2004
Miyake, T. **Modular Forms** 2006
Murdock, J. **Normal Forms and Unfoldings for Local Dynamical Systems** 2002
Narkiewicz, W. **Elementary and Analytic Theory of Algebraic Numbers** 3rd ed. 2004
Narkiewicz, W. **The Development of Prime Number Theory** 2000
Parker, C.; Rowley, P. **Symplectic Amalgams** 2002
Peller, V. (Ed.) **Hankel Operators and Their Applications** 2003
Prestel, A.; Delzell, C.N. **Positive Polynomials** 2001
Puig, L. **Blocks of Finite Groups** 2002
Ranicki, A. **High-dimensional Knot Theory** 1998
Ribenboim, P. **The Theory of Classical Valuations** 1999
Rowe, E.G.P. **Geometrical Physics in Minkowski Spacetime** 2001
Rudyak, Y.B. **On Thom Spectra, Orientability and Cobordism** 1998
Ryan, R.A. **Introduction to Tensor Products of Banach Spaces** 2002
Saranen, J.; Vainikko, G. **Periodic Integral and Pseudodifferential Equations with Numerical Approximation** 2002
Schneider, P. **Nonarchimedean Functional Analysis** 2002
Serre, J-P. **Complex Semisimple Lie Algebras** 2001 (reprint of first ed. 1987)
Serre, J-P. **Galois Cohomology** corr. 2nd printing 2002 (1st ed. 1997)
Serre, J-P. **Local Algebra** 2000
Serre, J-P. **Trees** corr. 2nd printing 2003 (1st ed. 1980)
Smirnov, E. **Hausdorff Spectra in Functional Analysis** 2002
Springer, T.A. Veldkamp, F.D. **Octonions, Jordan Algebras, and Exceptional Groups** 2000
Sznitman, A.-S. **Brownian Motion, Obstacles and Random Media** 1998
Taira, K. **Semigroups, Boundary Value Problems and Markov Processes** 2003
Talagrand, M. **The Generic Chaining** 2005
Tauvel, P.; Yu, R.W.T. **Lie Algebras and Algebraic Groups** 2005
Tits, J.; Weiss, R.M. **Moufang Polygons** 2002
Uchiyama, A. **Hardy Spaces on the Euclidean Space** 2001
Üstünel, A.-S.; Zakai, M. **Transformation of Measure on Wiener Space** 2000
Vasconcelos, W. **Integral Closure. Rees Algebras, Multiplicities, Algorithms** 2005
Yang, Y. **Solitons in Field Theory and Nonlinear Analysis** 2001
Zieschang, P.-H. **Theory of Association Schemes** 2005