Set Theory Project: Introduction to Forcing

Assignment 4

Part A: Nowhere dense and meager sets

Recall the following definition from classical topology.

Definition. *A set $X \subseteq \mathbb{R}$ is* ***dense*** *if for every rational open interval (p, q), $X \cap (p, q) \neq \varnothing$. A set $X \subseteq \mathbb{R}$ is* ***somewhere dense*** *if for some (p, q), the set $X \cap (p, q)$ is dense in (p, q). A set $X \subseteq \mathbb{R}$ is* ***nowhere dense*** *if it is not somewhere dense, i.e., if*

$$\forall p < q \in \mathbb{Q}\ \exists p', q' \in \mathbb{Q} \text{ s.t. } p \leq p' < q' \leq q \text{ and } (p', q') \cap X = \varnothing.$$

(alternatively, a set X is nowhere dense if its closure has empty interior, but this is harder to use).

By the Baire Category Theorem, the whole space $\mathbb{R}$ cannot be covered by countably many nowhere dense sets.[1] Can $\mathbb{R}$ be covered by $\aleph_1$-many nowhere dense sets? Clearly, if CH holds then $\mathbb{R} = \bigcup_{r \in \mathbb{R}} \{r\}$ is an $\aleph_1$-union of singletons, each singleton being nowhere dense. But is it true in general?

Consider the forcing $\mathbb{P}$ such that:

- The conditions are non-empty closed intervals $[p, q] \subseteq \mathbb{R}$, with $p, q \in \mathbb{Q} \cup \{-\infty, +\infty\}$ and $p < q$ (and we assume $-\infty < q < +\infty$ for all $q \in \mathbb{Q}$).
- $[p', q'] \leq [p, q]$ iff $[p', q'] \subseteq [p, q]$ (iff $p \leq p' < q' \leq q$).
- $\mathbf{1}_{\mathbb{P}} = (-\infty, +\infty) = \mathbb{R}$.

1. (a) Show that $\mathbb{P}$ has the ccc.
   (b) Write down what it means for $[p, q]$ and $[p', q']$ to be compatible.
   (c) Suppose $G \subseteq \mathbb{P}$ is a filter. Show that, in general, we could have $\bigcap G = \varnothing$.
   (d) Let $D := \{[p, q] \ :\ p, q \notin \{-\infty, +\infty\}\}$. Show that this set is dense and, if $G \cap D \neq \varnothing$, then $\bigcap G \neq \varnothing$.

   [Hint: use the compactness of $[p, q]$]

2. Prove the following: if $\mathsf{MA} + \neg\mathsf{CH}$ holds, then $\mathbb{R}$ cannot be covered by $\aleph_1$-many nowhere dense sets (this can be viewed as a generalization of the Baire Category Theorem).

   [Hint: start with a collection of $\aleph_1$-many nowhere dense sets. Define suitable dense sets and find a subset of $\mathbb{R}$ which is disjoint from all of them.]

[1] There are several formulations of BCT. An equivalent formulation is "non-empty open sets are not countable unions of dense sets", and another one is "countable intersections of open dense sets are dense". Countable unions of nowhere dense sets are also called **meager** or **of first category**.

Part B: Dominating Reals and the Bounding Number

Definition: For two functions $f, g : \omega \to \omega$, we say that g *dominates* f iff $\exists N \forall n \geq N \ (f(n) < g(n))$. One such function $d \in \omega^\omega$ is called a *dominating real over* M if for all $f \in \omega^\omega \cap M$ ($f \leq^* d$).

1. We define the so-called "Hechler Forcing" partial order $\mathbb{D}$ as follows:

   - The conditions are pairs (s, f) such that $s \in \omega^{<\omega}$, $f \in \omega^\omega$ and $s \subseteq f$ (i.e., s is an initial segment of f).
   - The relation is given by: $(t, g) \leq (s, f)$ iff $s \subseteq t$ and $\forall n \ (f(n) \leq g(n))$.
   - $\mathbf{1} = (\varnothing, \vec{0})$ (where $\vec{0}$ denotes the constant-0 function).

   (a) Show that Hechler forcing satisfies the ccc.

   (b) Let G be $\mathbb{D}$-generic over a model M. Let $f_G := \bigcup\{s \ : \ \exists f((s, f) \in G)\}$. Show that f_G is a function from ω to ω.

   (c) Show that for every $(s, f) \in \mathbb{D}$ we have $(s, f) \Vdash \check{s} \subseteq \dot{f}_G \ \wedge \ \forall n > |\check{s}| \ (\check{f}(n) \leq \dot{f}_G(n))$. Note that $\dot{f}_G$ refers to the canonical name for the function defined from the generic filter as in (b). So, what you really have to prove here is the following:

   For every $\mathbb{D}$-generic filter G over M such that $(s, f) \in G$, we have
   $$M[G] \models s \subseteq f_G \ \wedge \ \forall n > |s| \ (f(n) \leq f_G(n))$$

   (d) Show that f_G is a dominating real over M.

2. Let $\mathfrak{b}$ (the "bounding number") be the least cardinality of a set of functions from ω to ω such there is **no** d which dominates all of them. In other words:

   $$\mathfrak{b} = \min\{\kappa \ : \ \exists\{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega \ \nexists d \in \omega^\omega \ \forall \alpha (f_\alpha \leq^* d)\}$$

   In yet other words: $\kappa < \mathfrak{b}$ if and only if for any κ-collection of functions from ω to ω, **there is** a $d \in \omega^\omega$ which dominates them all.

   (a) Show (in ZFC) that $\omega_1 \leq \mathfrak{b} \leq 2^{\aleph_0}$.

   (b) Show that $\mathsf{MA} \to \mathfrak{b} = 2^{\aleph_0}$ (thus, if $\mathsf{MA} + \neg\mathsf{CH}$, then $\omega_1 < \mathfrak{b} = 2^{\aleph_0}$ holds, i.e., for any collection of ω_1-many functions, there is one which dominates all of them).