Report on Kripke Platek Set Theory

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1 Why another set theory ?

To begin with, one may ask : why do we need to be concerned with alternative set theories ? Indeed, ZFC does most of the work that mathematicians ask of "a foundation of mathematics", namely, it serves as a universal language for mathematics, since most of the objects of mathematics can be seen as some kind of sets, and most of the theorems of mathematics can be formalized in ZFC. Well, if someone has indeed such demands and only such, that it is possible that there is incdd not much in alternative set theories for him, except maybe for the case of a system stronger than ZFC. But we can also ask more, or ask other things of a fondation for mathematics. For instance, we can ask, as Hilbert did we finitism, that our fondation "embodies" in a a proper way, a certain type of reasoning and that most (since all is impossible) of mathematics could be done in that formal system. Nevertheless we now know that if we demand finitism, then we will only get what is doable in PRA, and so we will be lacking a great deal of fundamental theorems. So maybe we should be searching in between the full power of ZFC and the very weak power of PRA. But what such a theory could be/look like? Well, to answer that question we should think of the aspect that we can see as problematic in ZFC. One of the first things that we could drop is the axiom of choice. Indeed, it is quite common to reject this axiom since it allows us to introduce an object, a choice function, but without any idea of what such an object may look like. For this reason this axiom is often seen as highly inconstructive and this can be a reason to chose to drop it. One other reason is that, as it's well known, the Axiom of Choice implies the LEM, which is often seen as the one of the most non constructive principles. From this, it also follows that we have to drop the axiom of foundation. An other principle of ZF that is usually sees as inconstructive is the power set axiom. Let's begin by noticing that if this axiom is inconstructive, it is on another sense than the previous one. Indeed, with the axiom of Choice we had absolutely no idea of what the choice function will look like. On the contrary, with the power set axiom we have at least an idea of what sets will be in the power set of a set X. Nevertheless, it is true that this idea is not really precise. Indeed, it is well-known that we have no way, given an infinite set X to enumerate all its subsets, and that there are some subsets of X that are not definable by a formula whose parameters are all elements of X. It has been argued that the inconstructivity of this axiom might be one of the reasons why we are unable to decide the Continuum hypothesis in ZF (or ZFC): there is no precise notion of what counts as a subset of X (See Bairwise, 1975) (remark that the actual proof is quite far from this "common sense" argument). We could continue in this direction for a very long time. But lets introduce the axioms of KP and let's see how it is indeed a theory that somehow match with some philosophical conceptions.

2 The axioms of KP

Axiom of the empty set : $\exists x \forall y (y \notin x)$

Extensionality: $\forall x \forall y \forall z (z \in x \leftrightarrow z \in y) \rightarrow (x = y)$.

Pairing: $\forall x \forall y \exists z \forall w (w \in z) \rightarrow (w = x \vee w = y)$

Union: $\forall x \exists y \forall z (z \in y \rightarrow \exists w (w \in x \wedge z \in w))$

Inducion schema: $\forall \vec{a} (\forall x (\forall y (y \in x) \rightarrow \phi(y, \vec{a})) \rightarrow \phi(x, \vec{a})) \rightarrow \forall x \forall \vec{a} \phi(x, \vec{a})$

Σ_0-seperation: $\forall \vec{a} \forall x \exists y \forall z ((z \in y) \leftrightarrow (z \in x \wedge \phi(z, \vec{a}))$ where ϕ is a Σ_0 formula.

Σ_0-collection: $\forall \vec{a} (\forall x \exists y (\phi(x, y, \vec{a}) \rightarrow \forall u \exists v (\forall x \in u)(\exists y \in v) \phi(x, y, \vec{a}))$, where ϕ is a Σ_0 formula.

2.1 Discussion of the axioms

As the reader can check, accordingly to what we have said during the introduction, the Axiom of Choice and the Power set have been deleted. But this is not the only way that ZFC have been weakened. Indeed, the separation axiom has been restricted to the Σ_0 formula and the remplacement scheme has been deleted and remplaced by the Collection-Schema for Σ_0 formulas. It is straightforward that KP is a sub-theory of ZF. Indeed, the only axiom of KP that is not explicitly a axiom of ZF is the Collection-Schema for Σ_0 formula. But the full Collection schema is actually a theorem of ZF (see for instance Jech, 2003). It is also clear that the Collection-Schema implies the Remplacement-Schema (in fact the Remplacement-Schema is a just a particular case of the Collection-Schema, namely, the case where we demand a functional formula insted of just a serial one), and therefore we have also the Σ_0 Remplacement-Schema available in KP. One could ask: why this restriction to Σ-0 formulas? There is two main answers that can be given. The first one is mainly philosophical: the Σ_0 formulas are somehow connected to the idea of computation. In the case of arithmetical hierarchy (where the concept of Σ_0 appeared for the first time) we know that a Σ_0 can always be decided in a finite time by a automatic computation. In the case of set theory (Levy hierarchy) the situation is a little bit more abstract, since the bound of a formula does not need to be a natural number but can be any set. However, the idea is somehow the same : to verify if a Σ_0 formula holds, we will never have to run through the all universe of sets : in the worst case scenario we will have to reach the bound. The other reason is more technical : the Σ_0 formulas are "absolute" for transitive models. By "absolute" we mean the following : We say that a formula ϕ of LST is downard absolute for some transitive class M iff $\forall \vec{x} \in M(\phi(\vec{x}) \rightarrow \phi^M(\vec{x})$.

Similarly, we say that a a formula ϕ of LST is downard absolute for some transitive class M iff $\forall \vec{x} \in M(\phi^M(\vec{x}) \rightarrow \phi(\vec{x})$. Finally, we say that formula ϕ is absolute for M iff it is both D-absolute and U-absolute. Among the usual formulas of set theory, the ones that express that α is an ordinal, that x is transitive, that x is a subset of y and many others are Σ_0 and therefore absolute. In the other hand there is also very common formulas of set theory that are not absolute. For instance the formula expressing that x is countable is not D-absolute: the fact that x is countable does not imply that the bijection between it and $\mathbb{N}$ exists in our model. As an example of formula that is not U-absolute we have the formula expressing tha κ is a cardinal. Indeed, it is not enough to know that κ is an ordinal and that there is no bijection between κ and any of its elements in our model, since it is possible that our model is "too small" and that the bijcetion between κ and one of its elements exist in a "bigger" model.

The formulas that are Σ_0^{KP} i.e the formulas ϕ such that KP $\vdash \phi \leftrightarrow \psi$ where ψ is Σ_0 are absolute.

2.2 Additional axioms

Infinity: $\exists \alpha Lim(\alpha)$ where "$Lim(x)$"is a predicate satisfied iff x is a limit ordinal.

Urelements: $\exists x \forall y(y \in x \leftrightarrow \exists p(y = p)$

The first axiom is well-known. The second just asserts that there is set of urelements. As Bairwise argues (Bairwise, 1975), it is often necessary to work with urelements in some sub-theories of ZF as KP since all the objects mathematics can not necessarily be constructed inside this sub-theory.

3 Some theorems of KP

3.1 Existence of the cartesian product

Theorem 1. Given two sets, x and y, their cartesian product exists.

Proof. Given $a \in x$ $\forall b \in y$ the ordered pair (a,b) exists.

By Σ_0 collection the set $w_a = \{(a, b) : b \in y\}$ exists for every every $a \in x$.

By Σ_0 collection the set $c' = \{w_a : a \in x\}$ exists.

Finally $x \times y = c = \bigcup c'$ □

We have proved the existence of the cartesian product without the power set axiom.

3.2 Some stronger versions of collection and separation provable in KP

We will show that in fact KP allows us to use stronger versions oF collection and separation than the one appearing in the axioms.

Theorem . Σ_1 Collection Let $\phi(x, y)$ be a Σ_1 formula such that $\forall a \in x \exists y \phi(x, y)$. Then $KP \vdash \exists y \forall a \in x \exists b \in y(\phi(a, b))$.

Proof. Claim: If ϕ is Σ_1 then we can always find a set y such that $\phi \leftrightarrow \exists y \phi^{(y)}$. (See the Σ Reflection Principle in (Bairwise, 1975)).

Now by the claim we have : $\exists y \forall a \in x \exists b \in y \phi^{(y)}$.

Now let $w = \{b \in y | \exists a \in x \phi^{(y)}(a, b)\}$($\Delta_0$ separation).

Clearly $\phi^{(y)}(a, b) \to \phi(a, b)$.

Therefore we $\forall a \in x \exists b \in w \phi(a, b)$. □

Theorem . Δ_1-Separation: Let $\phi(z, \vec{a})$ be a Δ_1^{KP} formula of LST. Then:

KP$\vdash \forall \vec{a} \forall x \exists y \forall z(z \in y \leftrightarrow z \in x \wedge \phi(z, \vec{a})$.

Proof. By the claim of the previous proof we have :

KP$\vdash \phi(z, \vec{a}) \leftrightarrow \forall v \theta(v, z, \vec{a})$ KP$\vdash \phi(z, \vec{a}) \leftrightarrow \exists v \psi(v, z, \vec{a})$ where θ and ψ are Δ_0

$\forall z(\phi(z, \vec{a}) \vee \neg\phi(z, \vec{a})$ and therefore $\forall z \exists v(\psi(v, z, \vec{a}) \vee \neg\theta(v, z, \vec{a})$.

Σ_0 collection there is a u such that: $\forall z \in x \exists v \in u(\psi(v, z, , \vec{a}) \vee \neg\theta(v, z, \vec{a}))$.

Σ_0 comprehension: $y = \{z \in x | \exists v \in u(\psi(v, z, \vec{a}))\}$.

It follows easily that y is the desire set. □

3.3 The theorem of recursion.

One of the main theroem that will be used for our main result is the recursion theorem. We don't reproduce the proof here but it can be found in (Devlin, 1984).

4 The construction of L in KP

Now we arrive to the main point of the report : the construction of L (the constructible) inside KP. L is usually defined like this :

$L_0 = \emptyset$ $L_{\alpha+1} = Def(L_\alpha)$

$L_\lambda = \bigcup_{\alpha\in\lambda} L_\alpha$ $L = \bigcup_{\alpha\in Ord} L_\alpha$.

The problem is now to define "$Def(X)$" that intuitively is the set of all definable subset of X, that is, there is a formula of LST whose parameters are elements of X that definies each elemetns of $Def(X)$. The issue is that, in a theory expressed inside first order logic, we cannot quantify over formulas. The way the operation Def is defined is using a predicate $Def(u, v)$ saying that $u = def(v)$. This can be done by way of coding. We arenot going to go in the details of this coding, which can be fined in (Devlin, 1984). Nevertheless, we will provide an idea of how this is done.

4.1 A language of sets

The idea is first to develop a "language of sets", which is a language so that it's "components" (variable, constants, formulas, etc...) will be sets. This language will be $\mathcal{L}_V$ and here are its main components.

Variables: $v_n = (2, n)$

Constants: $x = (3, x) = x^*$

Atomic formulas 1: $(x \in y)$ =<0 4 x y 1>

Atomic formulas 2: $(x = y)$ =<0 5 x y 1>

We can then reflect some facts about $\mathcal{L}_\mathcal{V}$ in LST. For instance we can define $Var(x) \leftrightarrow$ (x is an ordered pair) $\wedge (Pr_1^2[x] = 2) \wedge (Pr_2^2[x] = n)$ for some $n \in \omega$.

Following that strategy, we can define $Const(x)$ (saying that x is a constant,$FinSeq(x)$ (saying that x is a finite sequence), $FrAt(x)$ (saying that x is an atomic formula), $F_\in(\theta, x, y)$, saying that (θ is an atomic formula expressing the fact that $x \in y$), $F_=(\theta, x, y)$ (the analogue for the equality), $F_\square(\theta, \phi, \psi)$ (for any connector $\square$), $Build(\phi, \psi)$ (saying that ψ is a sequence of formula of length n such that $\psi_n = \phi$ and every formula in the list is either atomic or the composition of two other formulas in the list) with Σ_0 formulas.

Once we have that result, we can show that the predicate $Fml(x)$ saying tha x is a formula of our new language is Δ_1^{KP}. It is clearly Σ_1 provided the relation $Build$ (ϕ is a formula iff there is a ψ such that $Build(\phi, \psi)$) and in fact, we can find a Π_1 formula equivalent to it ("equivalent" in the sense that KP proves their equivalence), so it is Δ_1^{KP}. Likewise, the formulas $Fr(\phi, x)$ (saying that x is the set of variables occurring free in ϕ), $Sub(\phi, \psi, v_0, z^*)$ (saying that ϕ is a formula with v_0 among its free variable, and that ψ is the formula that we obtain when we substitute the constant z^* to every free occurrence of v_0 in ϕ), and $Sat(u, \phi)$ (which is basically a truth predicate for the structure $< u, \in >$) are all Δ_1^{KP}.

4.2 L can be constructed inside KP

Once we have this result, we can show that the predicate $D(v, u)$ and the function $Def(u)$ are Δ_1^{KP}. The fact that we can defined those two using Δ_1^{KP} formula is crucial. On the one hand it ensures us that the predicate $Def(v, u)$ is absolute (sine the Δ_1 formulas are the only one that are absolute except for the Σ_0 formulas), which allows us to conclude that the definable set of definable subsets of u will be the same in any transitive model. On the other hand, it ensure that L can actually be constructed, since we only have remplacement (collection in fact) for Σ_0 and Σ_1 formulas.

5 Further topics

It would have been intersting to study Krike Platek more as an actual foundations for mathematics. For instance, it would have been intersting to see what part of mathematics could actually be carry on inside of KP. In this stand point, it will also have been enlightening to compare its "foundational power" to other weak theories (or at least weaker than ZFC), such as CZF and some subsystems of second order arithmetic.