Advanced Set Theory: Infinite Games and Determinacy

Assignment 2

Exercise 1. (Winning strategies and trees)

Let $T \subseteq \omega^{<\omega}$ be a tree. A node $t \in T$ is called *non-splitting* if it has only one successor in T, i.e., $\exists! n \, (t^\frown\langle n\rangle \in T)$. A node t is called *totally splitting* if it has every possible successor in T, i.e., $\forall n \, (t^\frown\langle n\rangle \in T)$.

(a) Let T be a non-empty tree with the property that for every $t \in T$:

- t is of even length $\implies$ t is non-splitting, and
- t is of odd length $\implies$ t is totally splitting.

Who has a winning strategy in $G([T])$? Describe (informally) that winning strategy.

(b) Show that Player I has a winning strategy in $G(A)$ if and only if there is a tree T of the type described above, such that $[T] \subseteq A$.

(c) Now describe a special type of tree with the property that Player II has a winning strategy in $G(A)$ if and only if there is a tree S of this type, such that $A \cap [S] = \varnothing$. Conclude by formulating AD in terms of the existence of special trees.

Exercise 2. (Topology and the Baire Space)

Recall that a set K in a topological space is called *compact* if every open cover of K has a finite subcover, i.e., if for every J with $K \subseteq \{O_j : j \in J\}$ with each O_j open, there exists a finite $I \subseteq J$ such that $K \subseteq \{O_i : i \in I\}$.

Let K be a closed subset of the Baire Space and T be the "tree of K", i.e., $T = \{x{\restriction}n : x \in K, n < \omega\}$. Show that K is compact if and only if every node of T is finitely-splitting.

Exercise 3. (Determinacy and complements)

Show that, in ZFC, the property of "being determined" is not closed under complements, i.e., show that there exists a set A such that A is determined but A^c is not determined.

(Hint: adapt the diagonalization proof, and it might help to think of Exercise 1 above).

Note: contrast this to the exercise in the previous assignment showing that the determinacy for a pointclass $\mathbf{\Gamma}$ is equivalent to the determinacy for the class $\breve{\mathbf{\Gamma}}$ of complements of sets in $\mathbf{\Gamma}$.