Hyperbolic digraphs

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Outline

1. motivation
2. hyperbolic digraphs
3. quasi-isometries
4. hyperbolic boundary ∂D
5. final remark

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2. As part of increasing interest in geometric semigroup theory Gray and Kambites (2014) came up with a geometric notion of hyperbolicity in the directed setting

   their main interest: decision problems (such as word problem, Green's relations) and finite presentability

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A geodesic triangle of D consists of three vertices and, for every two of them, x and y, a geodesic either from x to y or from y to x.

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and the k-out-ball of u is the set

$$B_k^+(u) := \{v \in V(D) \mid d(u, v) \leq k\}.$$

Let $\delta \geq 0$. A geodesic triangle is δ-thin if each of its geodesics P satisfies the following property: if Q and R are the other two geodesics such that Q has the first vertex of P as either its first or last vertex and R has the last vertex from P as either its first or last vertex, then P lies in the union of the δ-out-ball of Q and of the δ-in-ball of R.

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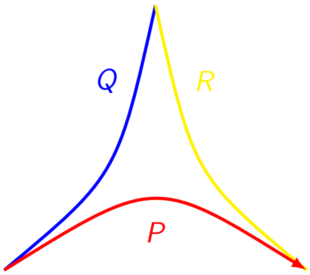

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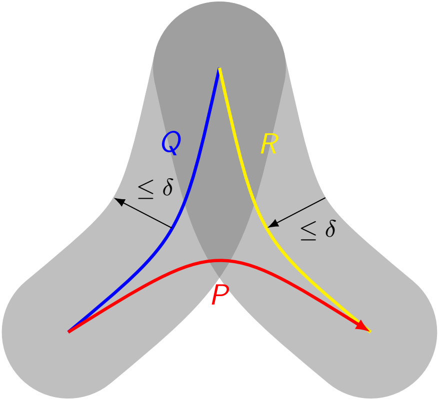

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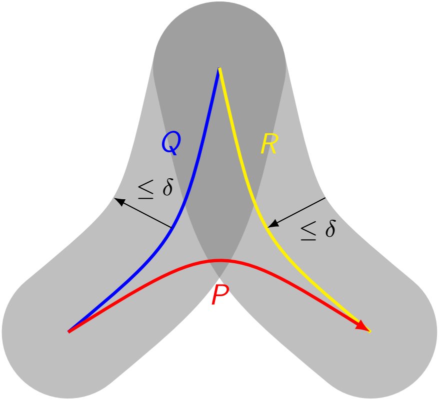
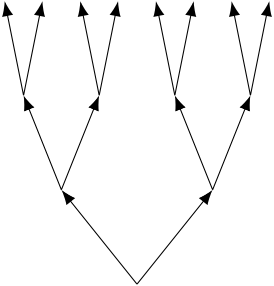


A digraph is hyperbolic if there exists $\delta \geq 0$ such that all geodesic triangles are δ-thin.

Examples

Example

1. Oriented trees are examples for hyperbolic digraphs.



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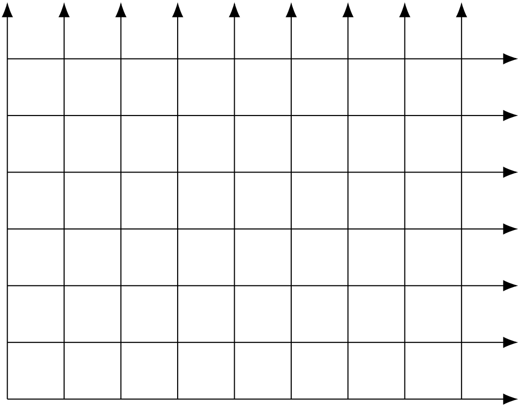
2. $\mathbb{N} \times \mathbb{N}$ is not a hyperbolic digraph.



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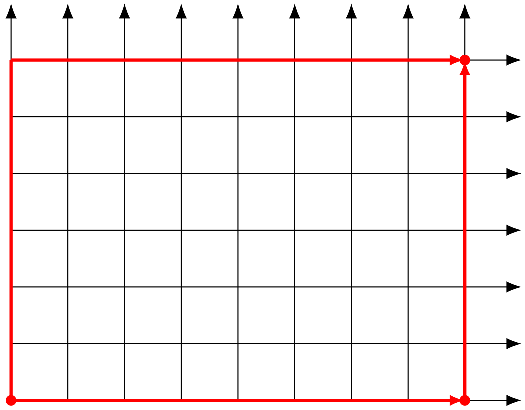

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Let D_1, D_2 be digraphs and let $\gamma \geq 1$ and $c \geq 0$. A map $f\colon V(D_1) \to V(D_2)$ is a (γ, c)-quasi-isometry if the following hold:

1. for all $x, y \in V(D_1)$ we have
$$\frac{1}{\gamma} d_{D_1}(x,y) - c \leq d_{D_2}(f(x), f(y)) \leq \gamma d_{D_1}(x,y) + c;$$
2. for every $x \in V(D_2)$ there exists $y \in V(D_1)$ with $d_{D_2}(f(x), y) \leq c$ and $d_{D_2}(y, f(x)) \leq c$.

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2. for every $x \in V(D_2)$ there exists $y \in V(D_1)$ with $d_{D_2}(f(x), y) \leq c$ and $d_{D_2}(y, f(x)) \leq c$.

Theorem (H.)

Quasi-isometries between digraphs of bounded degree preserve hyperbolicity.

Problem (Gray and Kambites)

If one Cayley digraph (wrt a finite generating set) of a finitely generated semigroup is hyperbolic, then is every such Cayley digraph hyperbolic?

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Our results on quasi-isometries leads to:

THEOREM (H.)

If one Cayley digraph (wrt a finite generating set) of a finitely generated right cancellative semigroup is hyperbolic, then every such Cayley digraph is hyperbolic.

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In a digraph D, a geodesic ray is a sequence $R = v_0 v_1 \dots$ such that $d(v_i, v_j) = j - i$ for all $i \leq j \in \mathbb{N}$ and a geodesic anti-ray is a sequence $Q = \dots v_{-1} v_0$ such that $d(v_i, v_j) = j - i$ for all $i \leq j \leq 0 \in \mathbb{Z}$.

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In hyperbolic digraphs of bounded degree, we can define an equivalence relation $\approx$ on the geodesic rays and anti-rays as follows:
$R_1 \approx R_2$ for geodesic rays or anti-rays R_1, R_2 if there exists $m \in \mathbb{N}$ and infinitely many pairwise disjoint R_1-R_2 and R_2-R_1 paths of length at most m.

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$R_1 \approx R_2$ for geodesic rays or anti-rays R_1, R_2 if there exists $m \in \mathbb{N}$ and infinitely many pairwise disjoint R_1-R_2 and R_2-R_1 paths of length at most m.
The equivalence classes of this relation $\approx$ are the hyperbolic boundary points of D. We denote by ∂D the hyperbolic boundary of D, i. e. the set of hyperbolic boundary points.

Pseudo-semimetrics

Theorem (H.)

Let D be a rooted hyperbolic digraph of bounded degree. Then there is a visual pseudo-semimetric d_h on $D \cup \partial D$.

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Let X be a set. A pseudo-semimetric is a function $d\colon X \times X \to [0, \infty]$ that satisfies the following properties

- $d(x, x) = 0$ for all $x \in X$ and
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

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Here, being a visual pseudo-semimetric means roughly that $d_h(x, y)$ is about $e^{-\varepsilon d^{\leftrightarrow}(o,P)}$, where P is any x-y geodesic, o is the root and

$$d^{\leftrightarrow}(o, P) = \min\{d(o, P), d(P, o)\}.$$

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Theorem (H.)

Quasi-isometries $D_1 \to D_2$ between hyperbolic digraphs of bounded degree extend to homeomorphisms $D_1 \cup \partial D_1 \to D_2 \cup \partial D_2$ (wrt to both topologies).

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Our results hold in a more general case than bounded degree.

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $x \in V(D)$, for every $n \in \mathbb{N}$ and for all $y, z \in B_n^+(x)$ the distance $d(y, z)$ is either ∞ or bounded by $f(n)$.

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Semimetric spaces (also known as quasi-metric or asymmetric spaces) are pseudo-semimetric spaces X with the property

$$d(x, y) = 0 \text{ if and only if } x = y \text{ for all } x, y \in X.$$

Remark

Most of our results hold for semimetric spaces satisfying the condition on end points of geodesics instead of bounded degree digraphs.

The only results that fail in this setting are those that used some compactness arguments: in the undirected setting, we usually apply the Arzelá-Ascoli theorem. In general, this is false in the case of semimetric spaces.