Asymptotic half-grid and full-grid minors

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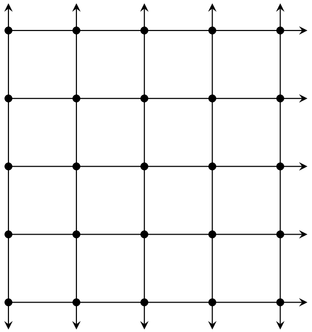
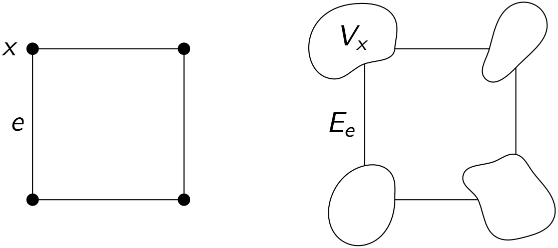


Minors



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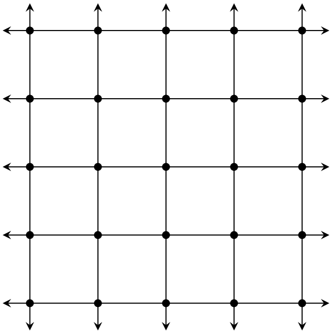

Quasi-isometries

Let G, H be graphs. A map $\varphi\colon V(G) \to V(H)$ is a quasi-isometry (and we call G and H quasi-isometric) if there exist $\gamma \geq 1$, $c \geq 0$ such that

1. $\frac{1}{\gamma} d_G(u,v) - c \leq d_H(\varphi(u), \varphi(v)) \leq \gamma d_G(u,v) + c$ for all $u, v \in V(G)$ and
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Quasi-isometries play an important role in geometric group theory since Cayley graphs of the same finitely generated group but for distinct finite generating sets are quasi-isometric.

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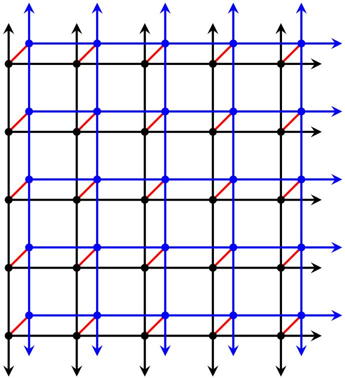

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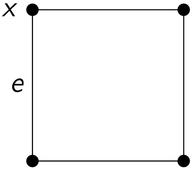
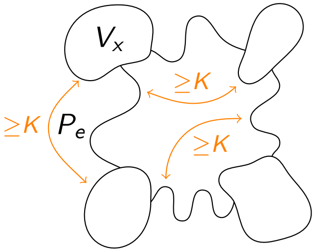
$\Rightarrow$ We look for minor-notions that appear in the coarse structure of the graphs.

Asymptotic minors

For $K \in \mathbb{N}$, a graph H is a K-fat minor if:





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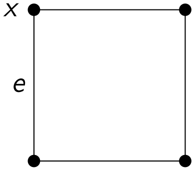

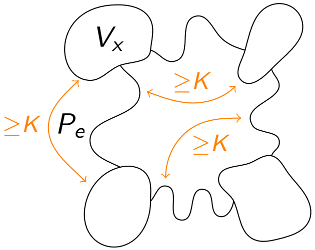

A graph G contains a graph H as an asymptotic minor if G contains H as a K-fat minor for every $K \in \mathbb{N}$.

A graph G contains a graph H as a diverging minor if G contains a model $(\mathcal{V}, \mathcal{E})$ of H with the following property:
for every two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of vertices and/or edges of H such that $d_H(x_n, y_n) \to \infty$, we have $d_G(X_n, Y_n) \to \infty$ where $X_n := V_{x_n}$ if $x_n \in V(H)$ and $X_n := V(P_{x_n})$ if $x_n \in E(H)$ and analogously $Y_n := V_{y_n}$ or $Y_n := V(P_{y_n})$.

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An end is thick if it contains infinitely many pairwise disjoint rays.

Coarse minor questions

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Theorem (Thomassen 1992)

Let G be a one-ended, locally finite, connected, quasi-transitive graph. Then its unique end is thick.

Main theorem

The cycle space of a graph G is generated by cycles of bounded length if there is some $n \in \mathbb{N}$ such that for each cycle C there exist finitely many cycles $C_1, \ldots, C_k$ of length at most n such that the edges of C are exactly those that lie in an odd number of C_i.

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Theorem

Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. Then G is not quasi-isometric to a tree if and only if G contains the full-grid as an asymptotic minor and as a diverging minor.

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Theorem (Krön & Möller 2008)

Let G be a locally finite, quasi-transitive graph. Then G contains a thick end if and only if it is not quasi-isometric to a tree.

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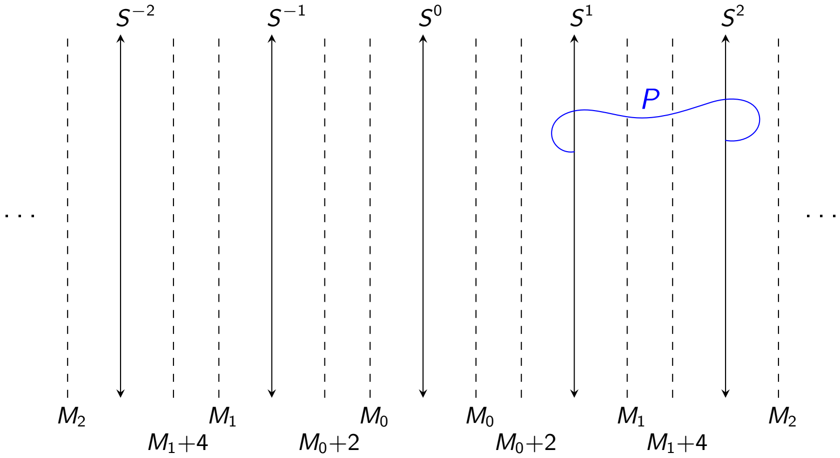
Corollary

Let Γ be a finitely presented group. Then Γ is not virtually free if and only if none of its locally finite Cayley graphs contain the full-grid as an asymptotic minor.

Escaping subdivisions of the full-grid



- $S^i \subseteq G[S^0, M_i] - B_G(S^0, M_{i-1} + 2i)$ for all $i \geq 1$ and
- $P \subseteq G[B_G(S^0, M_i)] - B_G(S^0, M_{i-2} + i)$

A model $((V_i)_{i \in \mathbb{N}}, (E_{ij})_{i \neq j \in \mathbb{N}})$ of $K_{\aleph_0}$ is ultra fat if

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is n-fat for every $n \in \mathbb{N}$.

Ultra fat $K_{\aleph_0}$-minors

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Proposition

Escaping subdivision of the full-grid and ultra fat $K_{\aleph_0}$-minors contain diverging and K-fat minors of the full-grid for all $K \in \mathbb{N}$.

Unifying result

Theorem

Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. Then G is not quasi-isometric to a tree if and only if G contains either an ultra fat $K_{\aleph_0}$-minor or the full-grid as an escaping subdivision.

Dropping the symmetry condition

Theorem

Let G be a graph of finite maximum degree whose cycle space is generated by cycles of bounded length. If G has a thick end, then G contains the half-grid as an asymptotic minor and as a diverging minor.

Final remarks

Question (Georgakopoulos & Papasoglu 2025, Georgakopoulos & H. 2024)

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Observation (Georgakopoulos)

Every locally finite Cayley graph of the lamplighter group has an ultra fat $K_{\aleph_0}$-minor.

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These theorems solve problems by Georgakopoulos and Papasoglu.