A Stallings type theorem for quasi-transitive graphs

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Abstract

We consider locally finite, connected, quasi-transitive graphs and show that every such graph with more than one end is a tree amalgamation of two other such graphs. This can be seen as a graph-theoretical version of Stallings' splitting theorem for multi-ended finitely generated groups and indeed it implies this theorem. Our result also leads to a characterisation of accessible graphs. We obtain applications of our results for planar graphs (answering a variant of a question by Mohar in the affirmative) and graphs without thick ends.

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1 Introduction

Stallings [25] proved that every finitely generated group with more than one end is either a free product with amalgamation over a finite subgroup or an HNN-extension over a finite subgroup. While amalgamated free products and HNN-extensions are group theoretical concepts, they can also be interpreted as operations on the Cayley graphs. For instance, if Γ_1 and Γ_2 are groups both of which contain a finite subgroup isomorphic to C, then Cayley graphs G_1 of Γ_1 and G_2 of Γ_2 can be glued together along copies of cosets of C in a treelike way in order to obtain a Cayley graph of the amalgamated free product $\Gamma_1 *_C \Gamma_2$.

Mohar [19] proposed a similar operation (called tree amalgamation) for arbitrary graphs. Roughly speaking, a tree amalgamation of two graphs G_1 and G_2 is obtained by gluing copies of G_1 onto G_2 and vice versa in a treelike way along finite subgraphs, called *adhesion sets*; we refer the reader to Section 5 for a precise definition. The main result of this paper can thus be seen as an analogue of Stallings' theorem for connected, quasi-transitive graphs.

Theorem 1.1. *Every connected, quasi-transitive, locally finite graph with more than one end is a non-trivial tree amalgamation of finite adhesion of two connected, quasi-transitive, locally finite graphs.*

The connection between Theorem 1.1 and Stallings' theorem goes even further: we show that we can restrict ourselves to two specific kinds of tree amalgamations which we refer to as 'Type 1' and 'Type 2', respectively. It turns out that the Cayley graph (with respect to the generators of the factors) of an amalgamated free product can always be seen as a Type 1 tree amalgamation, and similarly the Cayley graph of an HNN-extension can be seen as a Type 2 tree amalgamation, see Examples 5.5 and 5.6. Moreover, if Γ is a group acting quasi-transitively on a graph G, then the tree amalgamation of G obtained by Theorem 1.1 gives rise to an action of Γ on a semiregular tree. In the case that G is a Cayley graph of Γ, we can apply Bass-Serre theory to recover Stallings' theorem from Theorem 1.1, see Section 7.1.

We also consider several applications of our result. The first application concerns accessibility of graphs. Recall that a group is called accessible if it can be obtained from finite and one-ended groups by iterated amalgamated free products and HNN-extensions over finite subgroups. Similarly, one can ask what graphs can be obtained from finite and one-ended quasi-transitive graphs by iterated (Type 1 and 2) tree amalgamations over finite sets of vertices. Thomassen and Woess [27] defined accessibility for graphs as follows: a quasi-transitive, locally finite graph is *accessible in the sense of Thomassen of Woess*, or *TW-accessible* for short, if there is some $n \in \mathbb{N}$ such that every two ends can be separated by at most n edges.[1] They showed in [27] that a finitely generated group is accessible if and only if each of its locally finite Cayley graphs is accessible. We show that an analogous result holds for tree amalgamations of quasi-transitive graphs: the class of accessible, connected, quasi-transitive, locally finite graphs is precisely the class of graphs obtained by iterated tree amalgamations of finite adhesion starting with finite or one-ended connected, locally finite, quasi-transitive graphs.

[1] We will define accessibility differently and which is why we refer to their notion as TW-accessibility.

In Section 7.3 we obtain an answer to a question by Mohar from 1988 [19]. He asked, whether tree amalgamations are powerful enough to yield a classification of infinitely-ended transitive planar graphs in terms of finite and one-ended infinite planar transitive graphs. More precisely, he asked whether every 3-connected, planar, transitive graph can be obtained by iterated tree amalgamations of finite or one-ended, planar, transitive graphs. Georgakopoulos [9] gives examples where this is not possible, but suggests that Mohar intended to allow subdivisions of finite and one-ended, planar, transitive graphs as well. Our theorems provide an affirmative answer for quasi-transitive graphs because Dunwoody [8] proved that they are TW-accessible. It is worth noting that even if we start with a transitive graphs, the parts we end up with may still be quasi-transitive.

Additionally, as mentioned above, we obtain Stallings' theorem as a corollary to Theorem 1.1, see Section 7.1. We also obtain a new characterisation of quasi-transitive locally finite graphs that are quasi-isometric to trees, see Section 7.2. In Section 7.4 we discuss some further applications of our main result regarding hyperbolic graphs, quasi-isometries of graphs and asymptotic dimensions of graphs.

Our main tool to prove Theorem 1.1 are tree-decompositions that are invariant under the automorphisms of the graph. While some proofs of Stallings' theorem rely on edge separators and their structure trees, see for instance Dunwoody [5], it turns out that tree-decompositions and vertex separators work better in combination with tree amalgamations. However, due to the similar natures of structure trees and tree-decompositions, it is not surprising that some results that we prove here (in particular Propositions 4.7 and 4.8) have also been proved for structure trees, see e. g. Thomassen and Woess [27] and Möller [20, 21].

2 Preliminaries

We follow the general notations of [4] unless stated otherwise. In the following we will state the most important definitions for convenience.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a subset X of $V(G)$ we denote by $G[X]$ the subgraph of G *induced* by X, that is, $G[X] = (X, \{uv \in E(G) \mid u, v \in X\})$; $G - X$ denotes the subgraph $G[V(G) \setminus X]$ induced by the complement of X. A *geodesic* is a shortest path between two vertices. A *ray* is a one-way infinite path, the infinite subpaths of a ray are its *tails*. Two rays are *equivalent* if they have tails contained in the same component of $G - S$ for every finite set S of vertices. The equivalence classes of rays in a graph are its *ends*. The *degree* of an end is the maximum number of disjoint rays in that end, if this maximum exists. If the maximum does not exist, that is, if an end contains n disjoint rays for every $n \in \mathbb{N}$, then we say that this end has *infinite degree* and we call it *thick*. An end with finite degree is called *thin*. An end ω is *captured* by a set X of vertices if every ray of ω has infinite intersection with X and it *lives* in X if every ray of ω has a tail in X.

Let $X \subseteq V(G)$. We consider the graph with vertex set $(V(G) \smallsetminus X) \cup \{v_X\}$, where v_X is a new vertex, and and the following edge set:

$$\{uv \in E(G) \mid u, v \in V(G) \smallsetminus X\} \cup \{v_X u \mid \exists x \in X : xu \in E(G)\}.$$

We call this graph the *contraction* of X in G and we say that it is obtained from G by *contracting* X. Since edges are just vertex sets of size 2, the definition carries over to edges.

Let Γ be a group acting on G and let $X \subseteq V(G)$. The (*setwise*) *stabilizer* of X with respect to Γ is the set

$$\Gamma_X := \{g \in \Gamma \mid g(x) \in X \text{ for all } x \in X\}.$$

The Γ-*orbit* of a vertex $x \in V(G)$ is the set $\{g(x) \mid g \in \Gamma\}$; if Γ is clear from the context we omit it and speak of the *orbit* of x. We say that Γ acts *transitively* on G if $V(G)$ is one Γ-orbit and that Γ acts *quasi-transitively* on G if $V(G)$ consists of finitely many Γ-orbits.

3 Tree-decompositions

In this section we introduce the main tool for our proofs: tree-decompositions. A *tree-decomposition* of a graph G is a pair $(T, \mathcal{V})$ where T is a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ is a family of vertex sets of G such that the following three conditions are satisfied:

(T1) $V(G) = \bigcup_{t \in V(T)} V_t$.

(T2) For every edge $e \in E(G)$ there is a $t \in V(T)$ such that V_t contains both vertices that are incident with e.

(T3) $V_{t_1} \cap V_{t_2} \subseteq V_{t_3}$ whenever t_3 lies on the t_1-t_2 path in T.

The sets V_t are called the *parts* of $(T, \mathcal{V})$, the tree T is called the *decomposition tree*, and the vertices of T are called its *nodes* (to distinguish them from the vertices of G). The sets $V_{t_1} \cap V_{t_2}$ with $t_1 t_2 \in E(T)$ are called *adhesion sets*. We say that $(T, \mathcal{V})$ has *adhesion at most* k for $k \in \mathbb{N}$ if all adhesion sets have size at most k and it has *finite adhesion* if all adhesion sets are finite.

Remark 3.1. Let $(T, \mathcal{V})$ be a tree-decomposition and let $t_1 t_2$ be an edge of T. For $i = 1, 2$, let T_i be the component of $T - t_1 t_2$ that contains t_i. It follows from (T3) that $V_{t_1} \cap V_{t_2}$ separates the vertices in $\bigcup_{t \in T_1} V_t$ from those in $\bigcup_{t \in T_2} V_t$.

We say that $(T, \mathcal{V})$ *distinguishes* two ends ω_1 and ω_2 if there is a finite adhesion set $V_{t_1} \cap V_{t_2}$ such that one end lives in $\bigcup_{t \in T_1} V_t$ and the other one lives in $\bigcup_{t \in T_2} V_t$, where T_i is the maximal subtree of $T - t_1 t_2$ containing t_i. It distinguishes them *efficiently* if no vertex set in G of smaller size than $V_{t_1} \cap V_{t_2}$ separates them. For $k \in \mathbb{N}$, two ends of G are k-*distinguishable* if there is a set of k vertices of G that separates them.

Let Γ be a group acting on G. If every $\gamma \in \Gamma$ maps each part of $(T, \mathcal{V})$ to a part and thereby induces an automorphism of T we say that $(T, \mathcal{V})$ is Γ-*invariant*.

The following theorem by Carmesin et al. will be the main result we are building on.

Theorem 3.2. [2] *Let G be a locally finite graph, let Γ be a group acting on G and let $k \in \mathbb{N}$. Then there is a Γ-invariant tree-decomposition of G of adhesion at most k that efficiently distinguishes all k-distinguishable ends.* □

4 Splitting tree-decompositions

In this section, we first modify the tree-decomposition of Theorem 3.2, mainly to make its parts connected. Then we will prove some properties of the newly obtained tree-decomposition, in particular, where the tree-decomposition captures the ends of the graph. Our first step in modifying the tree-decomposition of Theorem 3.2 will be to make all adhesion sets connected while keeping the action of Γ on $(T, \mathcal{V})$.

Proposition 4.1. *Let Γ be a group acting on a locally finite graph G and let $(T, \mathcal{V}) = (T, (V_t)_{t \in V(T)})$ be a Γ-invariant tree-decomposition of G of finite adhesion. Then there is a Γ-invariant tree-decomposition $(T, \mathcal{V}') = (T, (V'_t)_{t \in V(T)})$ of G such that every adhesion set of $(T, \mathcal{V}')$ is finite and connected, and such that $V_t \subseteq V'_t$ for every $t \in V(T)$.*

Proof. Let u and v be two vertices of an adhesion set of $(T, \mathcal{V})$. Let $\mathcal{P}_{uv}$ be the set of all geodesics between u, and v and let V_{uv} be the set of all vertices of G that lie on the paths of $\mathcal{P}_{uv}$. For a part V_t, let V'_t be the union of V_t with all sets V_{uv} where u and v lie in an adhesion set contained in V_t. Let $\mathcal{V}' := \{V'_t \mid t \in V(T)\}$.

We claim that $(T, \mathcal{V}')$ is a tree-decomposition. As every element of $\mathcal{V}'$ is a superset of some element of $\mathcal{V}$, we only have to verify (T3). Let $x \in V'_{t_1} \cap V'_{t_2}$ for $t_1, t_2 \in V(T)$, and let t_3 be on the t_1-t_2 path $s_1, \ldots, s_n$ in T with $s_1 = t_1$ and $s_n = t_2$. If $x \in V_{t_1} \cap V_{t_2}$, then we have $x \in V_{t_3} \subseteq V'_{t_3}$ as $(T, \mathcal{V})$ is a tree-decomposition. If $x \in (V'_{t_1} \smallsetminus V_{t_1}) \cap V_{t_2}$, then it lies on a geodesic P between two vertices x_1, x_2 of an adhesion set of $(T, \mathcal{V})$ in V_{t_1}. Since every adhesion set $V_{s_i} \cap V_{s_{i+1}}$ separates V_{s_1} from V_{s_n} and since $x \in V_{t_2}$, the path P must pass through $V_{s_i} \cap V_{s_{i+1}}$. Thus, either P contains two vertices u, v of $V_{s_i} \cap V_{s_{i+1}}$ such that x lies on the u-v subpath P' of P, or x lies in $V_{s_i} \cap V_{s_{i+1}}$. In the first case, we added P' to the adhesion set $V_{s_i} \cap V_{s_{i+1}}$ because P' is a geodesic with its end vertices in $V_{s_i} \cap V_{s_{i+1}}$. Thus, in both cases x lies in $V_{s_i} \cap V_{s_{i+1}}$ and thus in V'_{t_3}. If $x \in (V'_{t_1} \smallsetminus V_{t_1}) \cap (V'_{t_2} \smallsetminus V_{t_2})$, let $t_4 \in V(T)$ with $x \in V_{t_4}$. By the previous case, x lies in V'_t for every t on the t_1-t_4 or t_2-t_4 paths in T. Since T is a tree, these cover the path $s_1, \ldots, s_n$ and hence $x \in V'_{t_3}$. This proves that $(T, \mathcal{V}')$ is a tree-decomposition.

It remains to show that $(T, \mathcal{V}')$ has the desired properties. By construction, every adhesion set is connected and $V_t \subseteq V'_t$. Since G is locally finite and the adhesion sets of $(T, \mathcal{V})$ are finite, every adhesion set of $(T, \mathcal{V}')$ is finite. Since we made no choices when adding all possible geodesics to the adhesion sets, Γ acts on $(T, \mathcal{V}')$ in the same way as on $(T, \mathcal{V})$. □

We call a tree-decomposition of a graph G *connected* if all parts induce connected subgraphs of G.

The step to make the adhesion sets connected is just an intermediate step for us: we aim for connected tree-decompositions. The connection between these two notions is given by our next lemma.

Lemma 4.2. *If all adhesion sets of a tree-decomposition $(T, \mathcal{V})$ of a connected graph G are connected, then $(T, \mathcal{V})$ is connected.*

Proof. Let u and w be two vertices of V_t for some $t \in V(T)$. Since G is connected, there is a path $P = p_1, \ldots, p_n$ with $p_1 = u$ and $p_n = w$. We choose P

with as few vertices outside of V_t as possible. Let us suppose that P leaves V_t. Let $p_i \in V_t$ such that $p_{i+1} \notin V_t$ and let p_j be the first vertex of P after p_i that lies in V_t. As $p_n = w \in V_t$ we know that such a vertex always exists. Let $t' \in V(T)$ be such that $p_{i+1} \in V_{t'}$. Then the adhesion set $V_t \cap V_s$, where s is the neighbour of t on the t-t' path in T, separates V_t from p_{i+1}. Hence, the definition of a tree-decomposition implies that p_j must lie in $V_t \cap V_s$, too. But then we can replace the subpath of P between p_i and p_j by a path in $V_t \cap V_s$. The resulting walk contains a path between u and w with fewer vertices outside of V_t than P. This contradiction shows that all vertices of P lie in V_t and hence $G[V_t]$ is connected. □

Most of the time we do not need the full strength of Theorem 3.2 in that it suffices to consider Γ-invariant tree-decompositions with few Γ-orbits that still distinguish some ends.

Let Γ be a group acting on a connected, locally finite graph G with at least two ends. A Γ-invariant tree-decomposition $(T, \mathcal{V})$ of G is a *splitting tree-decomposition* (*with respect to* Γ) if it has the following properties:

(i) $(T, \mathcal{V})$ distinguishes at least two ends.

(ii) Every adhesion set of $(T, \mathcal{V})$ is finite.

(iii) Γ acts on $(T, \mathcal{V})$ with precisely one orbit on $E(T)$.

It follows from Theorem 3.2 that splitting tree-decompositions always exist.

The term 'splitting' is used in analogy to group splittings, since the splitting tree-decompositions catch up the properties of group splittings for tree-decompositions.

Corollary 4.3. *Let Γ be a group acting on a locally finite graph G with at least two ends. Then there is a splitting tree-decomposition $(T, \mathcal{V})$ of G.*

Proof. By Theorem 3.2, we find a Γ-invariant tree-decomposition $(T, \mathcal{V})$ of bounded adhesion that separates some ends. Let tt' be an edge of T such that $V_t \cap V_{t'}$ separates some ends. Let $E_{tt'}$ be the orbit of tt', that is, the set $\{g(tt') \mid g \in \Gamma\}$, and let T' be obtained from T by contracting each component C of $T - E_{tt'}$ to a single vertex t_C. We set $V_{t_C} := \bigcup_{s \in C} V_s$ and set $\mathcal{V}'$ be the set of those sets V_{t_C}. It is easy to see that $(T', \mathcal{V}')$ is a splitting tree-decomposition with respect to Γ: the only non-trivial requirement is that $(T', \mathcal{V}')$ distinguishes at least two ends. But this follows from the fact that $V_t \cap V_{t'}$ separates two ends. □

Let us combine our results on connected, splitting tree-decompositions.

Corollary 4.4. *Let Γ be a group acting on a connected, locally finite graph G with at least two ends. Then the following hold.*

(i) There is a splitting tree-decomposition of G with respect to Γ whose adhesion sets are connected; in particular this tree-decomposition is connected.

(ii) If $(T, (V_t)_{t \in V(T)})$ is a splitting tree-decomposition of G with respect to Γ, then there is a connected, splitting tree-decomposition $(T, (V'_t)_{t \in V(T)})$ of G with respect to Γ such that $V_t \subseteq V'_t$ for every $t \in V(T)$.

Proof. By Corollary 4.3, there is a splitting tree-decomposition of G. Given a splitting tree-decomposition $(T, (V_t)_{t\in V(T)})$, Proposition 4.1 implies the existence of a splitting tree-decomposition $(T, (V'_t)_{t\in V(T)})$ with $V_t \subseteq V'_t$ for every $t \in V(T)$ and connected adhesion sets. Lemma 4.2 implies that such a tree decomposition is connected. □

Now we investigate connections between a graph and the parts of any connected, splitting tree-decomposition thereof. Some of these connections are similar to connections between graphs are their structure trees based on edge separators, see e. g. Thomassen and Woess [27] and Möller [20, 21]. We start by showing that connected, splitting tree-decompositions behave well with respect to the class of quasi-transitive graphs.

Proposition 4.5. *Let Γ be a group acting quasi-transitively on a connected, locally finite graph G with at least two ends and let $(T, \mathcal{V})$ be a connected, splitting tree-decomposition of G. Then for each part $V_t \in \mathcal{V}$ the stabilizer Γ_{V_t} acts quasi-transitively on $G[V_t]$.*

Proof. If $u \in V_t$ does not lie in any adhesion set, then none of its images $v \in V_t$ under elements of Γ lie in an adhesion set. Hence, if $\gamma \in \Gamma$ maps u to v, it must fix V_t setwise, as it acts on $(T, \mathcal{V})$, so it lies in the stabilizer of V_t. Thus, the intersection of V_t with the Γ-orbit of u is the Γ_{V_t}-orbit of u.

Now consider the vertices contained in adhesion sets. Fix an adhesion set $V_t \cap V_s$. As $(T, \mathcal{V})$ is splitting, for every adhesion set $V_t \cap V'_t$ there exists $\gamma \in \Gamma$ that maps $V_t \cap V_{t'}$ to $V_t \cap V_s$. This automorphism either stabilizes V_t, or it maps V_t to V_s. If there is an adhesion set which cannot be mapped to $V_t \cap V_s$ by an automorphism which stabilizes V_t, then fix one such adhesion set $V_t \cap V_{s'}$ and let $\gamma_0 \in \Gamma$ be an automorphism mapping $V_t \cap V_{s'}$ to $V_t \cap V_s$.

Now let $V_t \cap V_{t'}$ be an adhesion set, and let $\gamma \in \Gamma$ be an automorphism mapping $V_t \cap V_{t'}$ to $V_t \cap V_s$. If γ fixes V_t setwise, then every vertex of $V_t \cap V_{t'}$ lies in the Γ_{V_t}-orbit of some vertex of $V_t \cap V_s$. If γ does not stabilize V_t, then γ maps V_t to V_s, and consequently $\gamma_0^{-1}\gamma$ maps $V_t \cap V_{t'}$ to $V_t \cap V_{s'}$ and stabilizes V_t. It follows that every vertex of $V_t \cap V_s$ lies in the Γ_{V_t}-orbit of some vertex of $V_t \cap V_{s'}$. Since all adhesion sets are finite, this immediately implies that there are only finitely many Γ_{V_t}-orbits on vertices contained in adhesion sets. □

Subtrees of connected, splitting tree-decompositions that contain a common adhesion set cannot be too large as the following lemma shows.

Lemma 4.6. *Let Γ be a group acting quasi-transitively on a connected, locally finite graph G with at least two ends and let $(T, \mathcal{V})$ be a connected, splitting tree-decomposition of G with respect to Γ. For an adhesion set X let T_X be the maximal subtree of T such that $X \subseteq V_t$ for all $t \in V(T_X)$. Then the diameter of T_X is at most 2.*

Proof. The set X is contained in every V_t for $t \in V(T_X)$, and thus also in every adhesion set $V_t \cap V_{t'}$ for $tt' \in E(T_X)$. Since all adhesion sets have the same size, we have $V_t \cap V_{t'} = X$ for every $tt' \in E(T_X)$.

Suppose the diameter of T_X is at least 3, and let $R = \dots t_0 t_1 \dots$ be a maximal path in T_X. We shall show that R is a double ray.

Let us suppose that t_{i+3} is the last vertex on R. As $(T, \mathcal{V})$ is splitting, we find $\gamma \in \Gamma$ such that $\gamma(t_i t_{i+1}) = t_{i+2} t_{i+3}$. Note that γ fixes $X = V_{t_i} \cap V_{t_{i+1}} =$

$V_{t_{i+2}} \cap V_{t_{i+3}}$ setwise. If $\gamma(t_i) = t_{i+2}$, then $\gamma(t_{i+2})$ is a neighbour of t_{i+3} distinct from t_{i+2} that contains X, a contradiction to the choice of i. If $\gamma(t_i) = t_{i+3}$, then γ fixes the edge $t_{i+1}t_{i+2}$ but neither of its incident vertices. Let $\gamma' \in \Gamma$ map $t_{i+1}t_{i+2}$ to $t_{i+2}t_{i+3}$. Note that γ' fixes X setwise, too. Then either γ' or $\gamma'\gamma$ maps t_i to a neighbour of t_{i+3} distinct from t_{i+2}. This is again a contradiction, which shows that R has no last vertex. Analogously, R has no first vertex. So it is a double ray.

Note that the part of some node of T_X contains X properly as $G = X$ is finite otherwise. But as Γ acts transitively on $E(T)$, we have at most two Γ-orbits on $V(T)$. Hence infinitely many parts of R contain X properly. Since each V_{t_i} is connected, one vertex of X must have infinitely many neighbours. This contradiction to local finiteness shows the assertion. □

Our next result is a characterisation of the finite parts of a connected, splitting tree-decomposition.

Proposition 4.7. *Let Γ be a group acting quasi-transitively on a connected, locally finite graph G with at least two ends and let $(T, \mathcal{V})$ be a connected, splitting tree-decomposition of G. Then the degree of a node $t \in V(T)$ is finite if and only if V_t is finite.*

Proof. Let V_t be finite. Since $(T, \mathcal{V})$ is splitting, we have only one Γ-orbit on the adhesion sets. Local finiteness of G thus implies that each vertex of V_t lies in only finitely many distinct adhesion sets and that each of these adhesion sets separates the graph in only finitely many components. Therefore, the degree of t is finite.

Now let us assume that the degree of t is finite. Let U be a subset of V_t that consists of one vertex from each Γ_{V_t}-orbit that meets V_t. By Proposition 4.5 the set U is finite. The vertices in U have bounded distance to the union W of all adhesion sets in V_t. As they meet all Γ_{V_t}-orbits and Γ_{V_t} fixes W setwise, all vertices in V_t have bounded distance to W. Note that W is finite as t has finite degree. Since G is locally finite, V_t must be finite. □

Let $(T, \mathcal{V})$ be a tree-decomposition of a graph G. We say that an end η of T *captures* an end ω of G if for every ray $R = t_1, t_2, \ldots$ in η the union $\bigcup_{i \in \mathbb{N}} V_{t_i}$ captures ω. A node of T *captures* ω if its part does so.

Let us now investigate where the ends of G lie in $(T, \mathcal{V})$.

Proposition 4.8. *Let G be a graph and let $(T, \mathcal{V})$ be a connected tree-decomposition of G such that the maximum size of its adhesion sets is at most $k \in \mathbb{N}$. Then the following hold.*

(i) Each end of G is captured either by an end or by a node of T.

(ii) Every thick end of G is captured by a node of T.

(iii) Every end of T captures a unique thin end of G, which has degree at most k.

(iv) Assume that Γ acts quasi-transitively on G and that $(T, \mathcal{V})$ is Γ-invariant with finitely many Γ-orbits on $E(T)$. Every end of G that is captured by a node $t \in V(T)$ corresponds to a unique end of $G[V_t]$, that is, for every end ω of G that is captured by $t \in V(T)$ there is a unique end ω_t of $G[V_t]$ with $\omega_t \subseteq \omega$.

Proof. Let ω be an end of G and let Q, R be two rays in ω. For an edge $st \in E(T)$, let T_s and T_t be the subtrees of $T - st$ with $s \in V(T_s)$ and $t \in V(T_t)$. If the ray Q has all but finitely many vertices in $\bigcup_{x \in V(T_s)} V_x$ and R has all but finitely many vertices in $\bigcup_{x \in V(T_t)} V_x$ or vice versa, then we have a contradiction as Q and R cannot lie in the same end if they have tails that are separated by the finite vertex set $V_s \cap V_t$. We now orient the edge st from s to t if tails of Q and R lie in $\bigcup_{x \in V(T_t)} V_x$, and we orient it from t to s if tails of Q and R lie in $\bigcup_{x \in V(T_s)} V_x$. Obviously, every node of T has at most one outgoing edge. Let t_Q, t_R be nodes of T such that the first vertex of Q lies in V_{t_Q}, and the first vertex of R lies in V_{t_R}, and let P_Q and P_R be the maximal (perhaps infinite) directed paths in our orientation of T that start at t_Q and t_R, respectively. Note that if P_Q and P_R meet at a vertex, they continue in the same way. Thus, if they meet, they either end at a common vertex or have a common infinite subpath. We shall show that P_Q and P_R meet. Let P be the t_Q-t_R path in T. Then there is a unique sink x on it as every node of T has at most one outgoing edge. This sink is a common node of P_Q and P_R. If P_Q and P_R end at a node, this node captures ω and if they share a common infinite subpath, this is a ray whose end captures ω. We proved (i).

Now let us assume that ω has degree at least $k+1$. Then there are $k+1$ pairwise disjoint rays $R_1, \ldots, R_{k+1}$ in ω. Let t_i, P_i be a node and a path of T defined for R_i as we defined t_R and P_R for the ray R. By an easy induction, we can extend the above argument that P_Q and P_R meet to obtain that all $P_1, \ldots, P_{k+1}$ have a common node x. Let us suppose that ω is captured by an end η of T. Let y be the node of T that is adjacent to x and that separates x and η. Then all rays R_i must contain a vertex of $V_x \cap V_y$. This is not possible as $V_x \cap V_y$ contains at most k vertices and the rays R_i are disjoint. This contradiction shows (ii) and the second part of (iii).

Let R, Q be two rays that lie in ends of G that are captured by the same end η of T. With the notations P_Q, P_R as above, the intersection $P_Q \cap P_R$ is a ray in ω. As G is locally finite and $(T, \mathcal{V})$ is a connected tree-decomposition, there are infinitely many disjoint paths between Q and R and thus, they are equivalent and lie in the same end of G. This proves (iii).

To prove (iv), let us assume that Γ acts quasi-transitively on G and has finitely many orbits on the edges of the decomposition tree T. Let ω be an end of G that is captured by a node $t \in V(T)$ and let R be a ray in ω that starts at a vertex in V_t. Since V_t captures ω, there are infinitely many vertices of V_t on R. Whenever R leaves V_t through an adhesion set, it must reenter it through the same adhesion set by Remark 3.1. We replace every such subpath P, where the end vertices of P lie in a common adhesion set and the inner vertices of P lie outside of V_t, by a geodesic in $G[V_t]$ between the end vertices of P. We end up with a walk W with the same starting vertex as R. We shall see that W contains a one-way infinite path. First, we recursively delete closed subwalks of W to end up with a path R'. Since G is locally finite and R meets V_t infinitely often, R contains vertices of V_t that are arbitrarily far away from the starting vertex of R. As we only took geodesics to replace the subpaths of R that were outside of V_t and as Γ acts on $(T, \mathcal{V})$ with only finitely many orbits on the edges of T, these replacement paths have a bounded length. Hence, W eventually leaves every ball of finite diameter around its starting vertex. This implies that R' is a ray. Obviously, R and R' are equivalent. Thus $G[V_t]$ contains a ray in ω. Let

ω_t be the end of $G[V_t]$ that contains R' and let Q be a ray in ω_t. Since no finite separator can separate Q and R' in $G[V_t]$, the rays are also equivalent in G. Thus, we have shown $\omega_t \subseteq \omega$.

Let ω'_t be an end in $G[V_t]$ different from ω_t, let S be a finite subset of V_t that separates ω_t from ω'_t, and let P be a path in G connecting vertices in different components of $G[V_t] - S$. As before, whenever P leaves V_t through an adhesion set, it must reenter it through the same adhesion set by Remark 3.1. We again replace every such subpath, where the end vertices lie in a common adhesion set and the inner vertices lie outside of V_t, by a geodesic in $G[V_t]$ to obtain a walk P' in $G[V_t]$. Since P and P' have the same endpoints and P' must meet S, we know that P either contains a vertex in S, or it contains a vertex in an adhesion set which meets S. Let S' be the set containing all vertices of S and all vertices contained in adhesion sets that meet S. There are only finitely many orbits of vertices in adhesion sets, hence there is an upper bound on the diameter of the adhesion sets. Since S is finite and G is locally finite, this implies that S' is finite. By definition, there is no path in $G - S'$ connecting vertices in different components of $G[V_t] - S$. In particular, S' separates every ray in ω_t from every ray in ω'_t, and hence (iv) holds. □

5 Tree amalgamations

In this section, we prove our main result, Theorem 1.1. Before we move on to that proof, we need to state some definitions, in particular, the main definition: tree amalgamations, a notion introduced by Mohar [19].

For the definition of tree amalgamations, let G_1 and G_2 be graphs. Let $(S^i_k)_{k \in I_i}$ be a family of subsets of $V(G_i)$. Assume that all sets S^i_k have the same cardinality and that the index sets I_1 and I_2 are disjoint. For all $k \in I_1$ and $\ell \in I_2$, let $\phi_{k\ell} \colon S^1_k \to S^2_\ell$ be a bijection and let $\phi_{\ell k} = \phi^{-1}_{k\ell}$. We call the maps $\phi_{k\ell}$ and $\phi_{\ell k}$ *bonding maps.*

Let T be a $(|I_1|, |I_2|)$-*semiregular* tree, that is, a tree in which for the canonical bipartition $\{V_1, V_2\}$ of $V(T)$ the vertices in V_i all have degree $|I_i|$. Denote by $D(T)$ the set obtained from the edge set of T by replacing every edge xy by two directed edges $\vec{xy}$ and $\vec{yx}$. For a directed edge $\vec{e} = \vec{xy} \in D(T)$, we denote by $\overleftarrow{e} = \vec{yx}$ the edge with the reversed orientation. Let $f \colon D(T) \to I_1 \cup I_2$ be a labelling, such that for every $t \in V_i$, the labels of edges starting at t are in bijection to I_i.

For every $i \in \{1, 2\}$ and for every $t \in V_i$, take a copy G_t of the graph G_i. Denote by S^t_k the corresponding copies of S^i_k in $V(G_t)$. Let us take the disjoint union of the graphs G_t for all $t \in V(T)$. For every edge $\vec{e} = \vec{st}$ with $f(\vec{e}) = k$ and $f(\overleftarrow{e}) = \ell$ we identify each vertex x in the copy of S^s_k with the vertex $\phi_{k\ell}(x)$ in S^t_ℓ. Note that this does not depend on the orientation we pick for $\vec{e}$, since $\phi_{\ell k} = \phi^{-1}_{k\ell}$. The resulting graph is called the *tree amalgamation* of the graphs G_1 and G_2 over the *connecting tree* T and is denoted by $G_1 * G_2$ or by $G_1 *_T G_2$ if we want to specify the tree.

In the context of tree amalgamations the sets S^i_k and S^s_k are called the *adhesion sets* of the tree amalgamation. More specifically, the sets S^1_k are the adhesion sets of G_1 and the sets S^2_k are the adhesion sets of G_2. If the adhesion sets of a tree amalgamation are finite, then this tree amalgamation has *finite adhesion.* For every node $t \in V(T)$, there is a canonical map mapping each vertex $x \in V(G_t)$ to the vertex of $G_1 *_T G_2$ obtained from x by all identifications. We call a tree amalgamation $G_1 *_T G_2$ *trivial* if for some $t \in V(T)$ this canonical map is a bijection. Note that if the tree amalgamation has finite adhesion, it is trivial if $V(G_i)$ is the only adhesion set of G_i and $|I_i| = 1$ for some $i \in \{1, 2\}$.

The tree amalgamation $G = G_1 * G_2$ *distinguishes* ends if there is some adhesion set $S_k^s = S_\ell^t$ for adjacent $s, t \in V(T)$ such that for every component C of $T - st$ the subgraph of G induced by $\bigcup_{x \in C} G_i^x$ contains an end.

We remark that the map described in the definition of a trivial tree amalgamation does not necessarily induce a graph isomorphism $G_t \to G_1 *_T G_2$: it is a bijection $V(G_t) \to V(G_1 *_T G_2)$ but need not induce a bijection $E(G_t) \to E(G_1 *_T G_2)$.

The *identification size* of a vertex $x \in V(G_1 *_T G_2)$ is the size of the subtree T' of T induced by all nodes t for which a vertex of G_t is identified with x. The tree amalgamation has *finite identification* if all identification sizes are finite. The *identification length* of a vertex $x \in V(G_1 *_T G_2)$ is the diameter of the subtree T' of T induced by all nodes t for which a vertex of G_t is identified with x. The *identification length* of the tree amalgamation is the supremum of the identification lengths of its vertices. We note that if a quasi-transitive tree amalgamation has finite identification, then its identification length is finite.

We remark that in Mohar's definition of a tree amalgamation [19] the identification length is always at most 2. But apart from this, our definition is equivalent to his.

It is worth noting that every tree amalgamation gives rise to a tree decomposition in the following sense.

Remark 5.1. Let G be a graph. If G is a tree amalgamation $G_1 *_T G_2$ of finite adhesion, then there is a naturally defined tree-decomposition of G. For $t \in V(T)$ let V_t be the set obtained from $V(G_t)$ after all identifications in $G_1 * G_2$. Set $\mathcal{V} := \{V_t \mid t \in V(T)\}$. Obviously, all vertices of G lie in $\bigcup_{t \in V(T)} V_t$ and for each edge there is some $V_t \in \mathcal{V}$ containing it. Property (T3) of a tree-decomposition is satisfied as the copies G_i^v are arranged in a treelike way and identifications to obtain a vertex take place in subtrees of T. So $(T, \mathcal{V})$ is a tree-decomposition. If $G_1 *_T G_2$ has finite adhesion, so does $(T, \mathcal{V})$. If the tree amalgamation distinguishes ends, then so does the tree-decomposition.

So far, the tree amalgamations do not interact with any group actions on G_1 and G_2. In particular, it is easy to construct a tree amalgamation of two quasi-transitive graphs which is not quasi-transitive: for instance, let G_1 be a double ray and let G_2 be any finite non-trivial graph. Pick precisely two adhesion sets in G_1 sets and at least two adhesion sets in G_2, all of finite size. Then it is easy to see that the tree amalgamation $G_1 * G_2$ is not quasi-transitive by noting that vertices of G_1 at different distances from the adhesion sets cannot be mapped to one another.

In the following, we describe some conditions on tree amalgamations which will ensure that tree amalgamations of quasi-transitive graphs are again quasi-transitive; this will be proved in Lemma 5.8.

Let Γ_i be a group acting on G_i for $i = 1, 2$, let $t \in V_i$, let $\gamma \in \Gamma_i$ and let $j \in \{1, 2\} \setminus \{i\}$. We say that the tree amalgamation *respects* γ if there is a permutation π of I_i such that for every $k \in I_i$ there is $\ell \in I_j$ and τ in the setwise stabiliser of S_ℓ in Γ_j such that

$$\phi_{k\ell} = \tau \circ \phi_{\pi(k)\ell} \circ \gamma \mid_{S_k} .$$

Note that this in particular implies that $\gamma(S^i_k) = S^i_{\pi(k)}$. The tree amalgamation *respects* Γ_i if it respects every $\gamma \in \Gamma_i$.

Let $k \in I_i$ and let $\ell, \ell' \in I_j$. We call the bonding maps from k to ℓ and ℓ' *consistent* if there is $\gamma \in \Gamma_j$ such that

$$\phi_{k\ell} = \gamma \circ \phi_{k\ell'}.$$

We say that the bonding maps between two sets $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$ are *consistent* if they are consistent for any $i \in \{1, 2\}$, $k \in J_i$, and $\ell, \ell' \in J_j$.

We say that the tree amalgamation $G_1 * G_2$ is of *Type 1 respecting the actions of* Γ_1 *and* Γ_2 or $(G_1, \Gamma_1) * (G_2, \Gamma_2)$ *is a tree amalgamation of Type 1* for short if the following holds:

(i) The tree amalgamation respects Γ_1 and Γ_2.

(ii) The bonding maps between I_1 and I_2 are consistent.

We say that the tree amalgamation $G_1 * G_2$ is of *Type 2 respecting the actions of* Γ_1 *and* Γ_2 or $(G_1, \Gamma_1) * (G_2, \Gamma_2)$ *is a tree amalgamation of Type 2* for short if the following holds:

(o) $G_1 = G_2 =: G$, $\Gamma_1 = \Gamma_2 =: \Gamma$, and $I_1 = I_2 =: I$,[2] and there is $J \subseteq I$ such that $f(\vec{e}) \in J$, if and only if $f(\overleftarrow{e}) \notin J$.

(i) The tree amalgamation respects Γ.

(ii) The bonding maps between J and $I \setminus J$ are consistent.

In this second case we also say that $G_1 * G_2 = G * G$ is a *tree amalgamation of* G *with itself.*

We say that $G_1 * G_2$ is a tree amalgamation *respecting the actions of* Γ_1 *and* Γ_2 if it is of either Type 1 or Type 2 respecting the actions Γ_1 and Γ_2 and we speak about the *tree amalgamation* $(G_1, \Gamma_1) * (G_2, \Gamma_2)$. It is worth noting that the tree amalgamation also depends on the choices of adhesion sets and bonding maps, but we usually suppress this dependency.

We note that in the case of a tree amalgamation of Type 1, all adhesion sets S^i_k lie in the same Γ_i-orbit, while in the case of a tree amalgamation of Type 2 we potentially have two Γ_i-orbits of adhesion sets S^i_k.

We now give some examples illustrating the behaviour of tree amalgamations which respect group actions. The first set of examples shows that the groups acting on the factors have a substantial impact on the outcome of the tree amalgamation if we insist that it must respect the actions. They also show that sometimes (if the setwise stabilisers of the adhesion sets do not agree), we have to take multiple copies of the same adhesion set in order to be able to amalgamate consistently.

Example 5.2. Let G_1 consist of a single edge with end vertices u, v and let $\Gamma_1 = \mathbb{Z}_2$ acting in the obvious way. Let G_2 be a 4-cycle on vertices x, y, z, w, and let $\Gamma_2 = \mathbb{Z}_2$ where the non-trivial element swaps x with y, and z with w.

[2]Technically this is not allowed, in particular since for the definition of $\phi_{k\ell}$ we needed I_1 and I_2 to be disjoint. These technicalities can be easily dealt with by an appropriate notion of isomorphism the details of which we leave to the reader.

Let $S_1^1 = S_2^1 = \{u, v\}$, let $S_a^2 = \{x, w\}$, and let $S_b^2 = \{z, y\}$ and define bonding maps by

$$\phi_{1a}(u) = x, \phi_{1a}(v) = w, \qquad \phi_{1b}(u) = y, \phi_{1b}(v) = z,$$
$$\phi_{2a}(u) = w, \phi_{2a}(v) = x, \qquad \phi_{2b}(u) = z, \phi_{2b}(v) = y.$$

and $\phi_{ji} = \phi_{ij}^{-1}$. The resulting tree amalgamation is an infinite double ladder with the infinite dihedral group D_∞ acting on it, and a straightforward case check shows that this tree amalgamation respects the action of the groups.

It is worth pointing out that once we have picked S_1^1 and S_a^2, this is the smallest tree amalgamation which respects the groups. We must have the adhesion set S_b^2, otherwise the tree amalgamation cannot respect the action of Γ_2. Furthermore, the adhesion set $\{u, v\}$ must appear at least twice in order to respect the nontrivial element of Γ_1 (since the stabilisers of S_a^2 and S_b^2 are trivial). It is also worth noting that any other choice of bonding maps would either violate consistency or not respect one of the group actions. We would also like to point out that the action on the tree amalgamation is transitive although the action of Γ_2 on G_2 was not transitive.

For a different example of a tree amalgamation, choose (G_1, Γ_1) and (G_2, Γ_2) as above, pick $S_1^1 = \{u, v\}$, but choose $S_a^2 = \{x, y\}$. Then the bonding map $\phi_{1a}(u) = x, \phi_{1a}(v) = y$ results in a tree amalgamation respecting the actions. In fact, this is an example of a trivial tree amalgamation. Moreover, the tree amalgamation is isomorphic to (G_2, Γ_2): the vertices of G_1 are simply identified with a subset of G_2.

Let us give two examples for tree amalgamations of Type 2 that use the same graph as factors but with different adhesion sets.

Example 5.3. Let G_i be the graph obtained from a complete graph on three vertices $\{b_i, c_i, d_i\}$ with a new vertex a_i attached to b_i and let T_3 be a 3-regular tree, see Figure 1. The adhesion sets are $\{a_i\}$, $\{c_i\}$, and $\{d_i\}$. Let $\phi_{a_j c_i}^{-1} = \phi_{c_i a_j}\colon \{c_i\} \to \{a_j\}$ and $\phi_{a_j d_i}^{-1} = \phi_{d_i a_j}\colon \{d_i\} \to \{a_j\}$ be the bonding maps, where $i \neq j \in \{1, 2\}$. These bonding maps already define $f\colon D(T_3) \to I_1 \cup I_2$, where I_i is the index set of the adhesion sets in G_i such that I_1 and I_2 are disjoint. For $i = 1, 2$, the automorphism groups of G_i are C_2, cyclic groups of order 2. Let us show that $(G, C_2) *_{T_3} (G, C_2)$ is a tree amalgamation of Type 2. Let J_i be the set consisting of the index set of the adhesion sets $\{c_i\}$, and $\{d_i\}$. Then for $i \neq j$ and for every edge $x_i x_j \in E(T_3)$, we have $f(\overrightarrow{x_i x_j}) \in J_i$, if and only if $f(\overleftarrow{x_i x_j}) \notin J_j$. In addition, it is a straightforward case to check that the bonding maps between J_i and $I_j \setminus J_j$ are consistent.

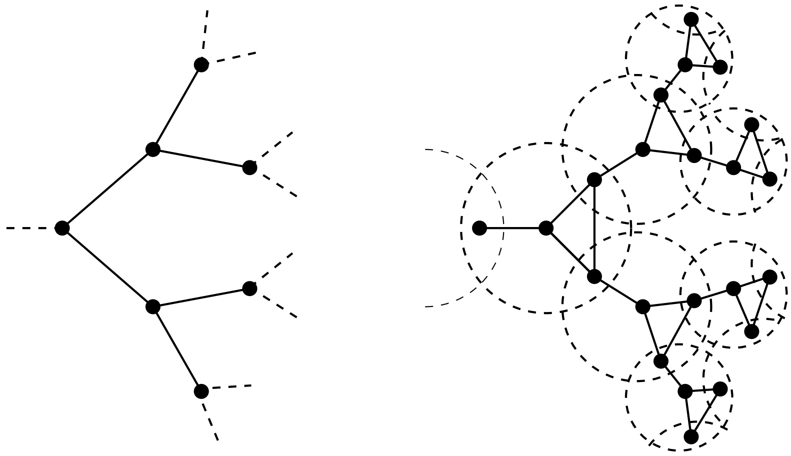

Figure 1: The connecting tree T_3 on the left and $G_1 *_{T_3} G_2$ on the right side.

Example 5.4. As in the previous example, let G_i be the graph obtained from a complete graph on three vertices $\{b_i, c_i, d_i\}$ with a new vertex a_i attached to b_i. Let T_4 be a 4-regular tree, see Figure 2. Let the adhesion sets be $\{a_i\}$, $\{a_i\}$, $\{c_i\}$, and $\{d_i\}$ for $i = 1, 2$. Note that the adhesion set $\{a_i\}$ occurs twice in this example. Let $\phi_{a_j c_i}^{-1} = \phi_{c_i a_j} : \{c_i\} \to \{a_j\}$, $\phi_{c_i d_j} : \{c_i\} \to \{d_j\}$, and $\phi_{d_i a_j} : \{d_i\} \to \{a_j\}$ be the bonding maps, where $i \neq j \in \{1, 2\}$. This defines a map $f : D(T_4) \to I_1 \cup I_2$, where I_i is the index set of the adhesion sets in G_i such that I_1 and I_2 are disjoint. Again, C_2 are the automorphism groups of G_1 and G_2. The proof that $(G_1, C_2) *_{T_4} (G_2, C_2)$ is a tree amalgamation of Type 2 follows analogously to the proof in Example 5.3.

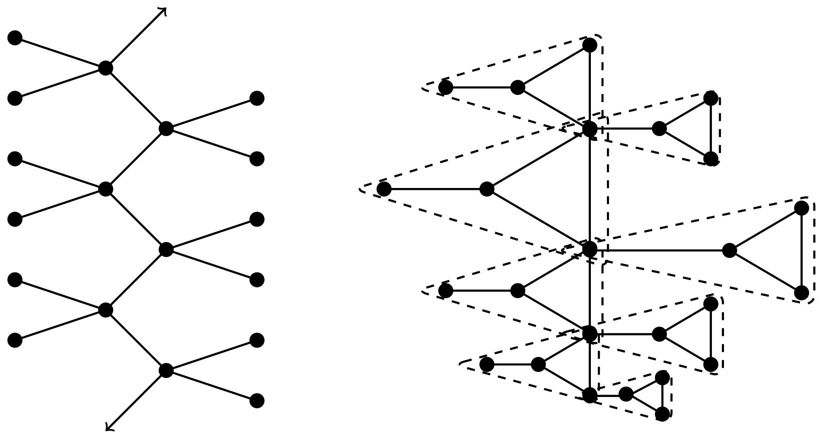

Figure 2: The connecting tree T_4 on the left and $G_1 *_{T_4} G_2$ on the right side.

The next two examples illustrate the relation of Type 1 and 2 tree amalgamations to amalgamated free products and HNN-extensions of groups.

We first recall the definition of the free product with amalgamation. Let $\Gamma_i = \langle S_i \mid R_i \rangle$ be a finitely generated group with a subgroup Δ_i for $i = 1, 2$ such that an isomorphism $\phi : \Delta_1 \to \Delta_2$ exists. Then the *free product* of Γ_1 and Γ_2

with amalgamation over Δ_1 and Δ_2 is the group defined by

$$\langle S_1 \cup S_2 \mid R_1, R_2, \{\delta^{-1}\phi(\delta) \mid \delta \in \Delta_1\}\rangle,$$

and we denote it by $\Gamma_1 *_{\Delta_1} \Gamma_2$,

Example 5.5. Let $\Gamma_i = \langle S_i \rangle$ be a finitely generated group with a finite subgroup Δ_i for $i = 1, 2$ such that an isomorphism $\phi\colon \Delta_1 \to \Delta_2$ exists. In addition let G_i be the Cayley graph of Γ_i with respect to S_i for $i = 1, 2$. We show that the Cayley graph of $\Gamma = \Gamma_1 *_{\Delta_1} \Gamma_2$ with respect to $S_1 \cup S_2$ is a tree amalgamation of G_1 and G_2.

In order to make sense of the above statement, we need to define adhesion sets and bonding maps for the tree amalgamation. The adhesion sets are the left cosets of Δ_i in Γ_i, formally we can pick a system $\{t^i_k\}_{k \in I_i}$ of coset representatives and let $S^i_k = t^i_k \Delta_i$. Note that $\{S^i_k\}_{k \in I_i}$ is a family of pairwise disjoint sets. Next we define the bonding maps. For $k \in I_1$ and $\ell \in I_2$, define $\phi_{k\ell}\colon S^1_k \to S^2_\ell$ by $\phi_{k\ell}(t^1_k g) = t^2_\ell \phi(g)$ and $\phi_{\ell k} = \phi^{-1}_{k\ell}$. Let T be a $(|I_1|, |I_2|)$-semiregular with the canonical partition (V_1, V_2) of $V(T)$. We can assume that the vertices of T are the left cosets of Γ_1 and Γ_2 and the edges correspond to the left cosets of Δ_1. Set the label under $f\colon D(T) \to I_1 \cup I_2$ of the edge corresponding to $g\Delta_1$ directed towards $h\Gamma_i$ as $k \in I_i$ such that $h^{-1}g\Delta_1 = S^i_k$. It is easy to see that the edges directed towards $h\Gamma_i$ have all of I_i as their labels. We have to show that $(G_1, \Gamma_1) * (G_2, \Gamma_2)$ is a tree amalgamation of Type 1. Let $\gamma \in \Gamma_i$ and let $j \in \{1, 2\} \setminus \{i\}$. For every k in I_i, there exists $k' \in I_i$ such that $\gamma t^i_k \Delta_i = t^i_{k'} \Delta_i$ and such that, for every $m \in I_i$ with $m \neq k$, we have $k' \neq m'$. This mapping of k to k' for every $k \in I_i$ defines a permutation π. Then for $\ell \in I_j$ the following holds

$$\phi_{k\ell} = \mathrm{id} \circ \phi_{\pi(k)\ell} \circ \gamma \mid_{S_k} .$$

In order to see that the bonding maps between I_1 and I_2 are consistent, let $k \in I_i$ and let $\ell, \ell' \in I_j$. Let $\gamma \in \Gamma_j$ such that $\gamma t^j_\ell \Delta_j = t^j_{\ell'} \Delta_j$. Then it is straightforward to check that

$$\phi_{k\ell'} = \gamma \circ \phi_{k\ell}.$$

Therefore we proved that $(G_1, \Gamma_1) * (G_2, \Gamma_2)$ is a tree amalgamation of Type 1. By the choice of f, we obtain an isomorphism Φ from G to $(G_1, \Gamma_1) * (G_2, \Gamma_2)$.

Let $\Gamma = \langle S \mid R \rangle$ be a finitely generated group. Let Δ_1 and Δ_2 be finite subgroups such that there is an isomorphism $\phi\colon \Delta_1 \to \Delta_2$. Moreover, let t be a symbol which is not an element of Γ. Then the *HNN-extension* of Γ over Δ_i with respect to ϕ is given by

$$\Gamma *_\phi = \langle S, t \mid R, \{\delta t = t\phi(\delta) \mid \delta \in \Delta\}\rangle.$$

Example 5.6. Let $\Gamma = \langle S \rangle$ be a finitely generated group with isomorphic finite subgroups Δ_1 and Δ_2 and let $\phi\colon \Delta_1 \to \Delta_2$ is an isomorphism. In addition let G be the Cayley graph of Γ with respect to S. Let G^* be the graph obtained from G by adding for each vertex v of G a new vertex vt and joining these by an edge. We note that the action of Γ on G extends to an action of Γ on G^*. Let $\{g^i_k \Delta_i\}_{k \in J_i}$ be the set of all left cosets of Δ_i in Γ for $i = 1, 2$. The adhesion sets are the elements of $\{S_k\}_{k \in J_1} = \{g^1_k \Delta_1 t\}_{k \in J_1}$ and of $\{T_\ell\}_{\ell \in J_2} = \{g^2_\ell \Delta_2\}_{\ell \in J_2}$. Without loss of generality we can assume $S_{k_0} = \Delta_1 t$ and $T_{\ell_0} = \Delta_2$. The bonding maps are $\phi_{k\ell_0}\colon S_k \to T_{\ell_0}$ and $\phi_{k\ell_0}\colon T_\ell \to S_{k_0}$ with $\phi_{k\ell_0}(g_k \delta t) = \phi(\delta)$

and $\phi_{\ell k_0}(g_k\delta) = \phi^{-1}(\delta)$. Let T be the tree whose vertices are the left cosets of Γ and whose edges correspond to the left cosets of Δ_1. Set the label under $f\colon D(T) \to J_1 \cup J_2$ of the edges corresponding to $g\Delta_1$ directed towards $h\Gamma$ with $g\Delta_1 \subseteq h\Gamma$ as $k \in J_2$ such that $h^{-1}g\Delta_1 = T_k$ and set the label of the same edge but with the reverse direction as $\ell \in J_1$ such that $S_\ell = T_k$. To show that $G^* * G^*$ respects Γ, let $\gamma \in \Gamma$. For every $k \in J_1$, there exists $k' \in J_1$ such that $\gamma g_k^1 \Delta_1 t = g_{k'}^1 \Delta_1 t$ and such that, for every $m \in J_1$ with $m \neq k$, we have $k' \neq m'$. This mapping of k to k' for every $k \in J_1$ defines a permutation π. Then for $\ell \in J_2$ the following holds

$$\phi_{k\ell} = \mathrm{id} \circ \phi_{\pi(k)\ell} \circ \gamma\mid_{S_k}.$$

Analogously, we obtain a permutation π' such that for every $k \in J_2$ there is $\ell \in J_1$ with

$$\phi_{k\ell} = \mathrm{id} \circ \phi_{\pi'(k)\ell} \circ \gamma_{T_k}.$$

Thus, the tree amalgamation respects γ. In order to prove that the bonding maps between J_i and J_j for $i \neq j$ are consistent, let $k \in J_i$ and $\ell, \ell' \in J_j$. Let $\gamma \in \Gamma$ such that $\gamma g_\ell^1 \Delta_1 t = g_{\ell'}^1 \Delta_1 t$ if $j = 1$ and such that $\gamma g_\ell^2 \Delta_2 = g_{\ell'}^2 \Delta_2$ if $j = 2$. Then it is easy to see that

$$\phi_{k\ell'} = \gamma \circ \phi_{k\ell}.$$

Thus, $G^* * G^*$ is a tree amalgamation of Type 2. By the choice of f, we obtain an isomorphism Φ from the Cayley graph of $\Gamma *_\phi$ with respect to $S \cup \{t\}$ to $G^* * G^*$.

Note that conditions (i) and (ii) in both cases of the definition of tree amalgamations respecting the actions do not depend on the specific labelling of the tree. This is no coincidence. In fact we will show that any two legal labellings of $D(T)$ give isomorphic tree amalgamations, see Lemma 5.8. Furthermore, any $\gamma \in \Gamma_i$ (interpreted as an isomorphism between parts of two such tree amalgamations) can be extended to an isomorphism of the tree amalgamations, which also implies that the tree amalgamations obtained this way are always quasi-transitive.

Before we turn to the proof of these facts, we need some notation. A *legally labelled star centred at* V_i is a function Λ from I_i to I_j. If the tree amalgamation is of Type 2, we further require that $\Lambda(k) \in J$ if and only if $k \notin J$. Informally, think of this as a star whose labels on directed edges could appear on a subtree of T induced by a vertex $t \in V_i$ and its neighbours: for $\vec{e}$ with label k, the value $\Lambda(k)$ tells us the label of $\overleftarrow{e}$.

An isomorphism of two legally labelled stars Λ, Λ' is a triple $(\gamma, \pi, (\gamma_k)_{k \in I_i})$ consisting of some $\gamma \in \Gamma_i$, a permutation π of I_i, and a family $(\gamma_k)_{k \in I_i}$ of elements of Γ_j such that for every $k \in I_i$

$$\phi_{k,\Lambda(k)} = \gamma_k \circ \phi_{\pi(k)\Lambda'(\pi(k))} \circ \gamma\mid_{S_k}.$$

In our interpretation of legally labelled stars as subtrees of T, this corresponds to an isomorphism of the corresponding subgraphs of the tree amalgamation.

Proposition 5.7. *Let Λ, Λ' be two legally labelled stars with respect to a tree amalgamation $(G_1, \Gamma_1) *_T (G_2, \Gamma_2)$ centred at V_i and let $\gamma \in \Gamma_i$. Then γ extends to an isomorphism $(\gamma, \pi, (\gamma_k)_{k \in I_i})$ of Λ and Λ'. Furthermore, if we are given $\tilde{k}, \tilde{k}' \in I_i$ and $\tilde{\gamma}_k \in \Gamma_j$ such that*

$$\phi_{\tilde{k},\Lambda(\tilde{k})} = \tilde{\gamma}_k \circ \phi_{\tilde{k}'\Lambda'(\tilde{k}')} \circ \gamma\mid_{S_k},$$

then we can choose $\pi(\tilde{k}) = \tilde{k}'$ *and* $\gamma_{\tilde{k}} = \tilde{\gamma}_k$.

Proof. Since the tree amalgamation respects γ, there are π and $\bar{\Lambda}\colon I_i \to I_j$ and τ_k in the stabiliser of $S_{\Lambda(k)}$ in Γ_j such that

$$\phi_{k\bar{\Lambda}(k)} = \tau_k \circ \phi_{\pi(k)\bar{\Lambda}(k)} \circ \gamma \restriction_{S_k} .$$

Let $\gamma'_k \in \Gamma_j$ be such that $\phi_{k\Lambda(k)} = \gamma'_k \circ \phi_{k\bar{\Lambda}(k)}$, and let $\gamma''_k \in \Gamma_j$ be such that $\phi_{\pi(k)\bar{\Lambda}(k)} = \gamma''_k \circ \phi_{\pi(k)\Lambda'(\pi(k))}$. These exist by (ii); for Type 2 recall that by the definition of legally labelled stars $k \in J$ if and only if $\Lambda(k) \notin J$. Now clearly

$$\phi_{k\Lambda(k)} = \gamma'_k \circ \tau_k \circ \gamma''_k \circ \phi_{\pi(k)\Lambda'(\pi(k))} \circ \gamma \restriction_{S_k},$$

thus showing that the two stars are isomorphic by means of the isomorphism $(\gamma, \pi, (\gamma_k)_{k\in I_j})$, where $\gamma_k = \gamma'_k \circ \tau_k \circ \gamma''_k$.

For the second part, let $(\gamma, \pi, (\gamma_k)_{k\in I_i})$ be an isomorphism between Λ and Λ'. Let $\tilde{k}'' = \pi^{-1}(\tilde{k}')$. Define $\rho(\tilde{k}) = \tilde{k}'$ and $\rho(\tilde{k}'') = \pi(\tilde{k})$. Let $\delta_{\tilde{k}} = \tilde{\gamma}_k$ and let

$$\delta_{\tilde{k}''} = \gamma_{\tilde{k}''} \circ \tilde{\gamma}_k^{-1} \circ \gamma_{\tilde{k}}.$$

For the remaining $k \in I_i$, let $\rho(k) = \pi(k)$ and $\delta_k = \gamma_k$. It is straightforward to check that γ, ρ, and $(\delta_k)_{k\in I_i}$ define an isomorphism between Λ and Λ' with the desired properties. □

Lemma 5.8. *Let G_1 and G_2 be connected, locally finite graphs and let Γ_i be a group acting quasi-transitively on G_i for $i = 1, 2$. Then any tree amalgamation $(G_1, \Gamma_1) *_T (G_2, \Gamma_2)$ is quasi-transitive and independent (up to isomorphism) of the particular labelling of T.*

Proof. Let T and T' be two labelled trees giving rise to tree amalgamations $G = (G_1, \Gamma_1) *_T (G_2, \Gamma_2)$ and $G' = (G_1, \Gamma_1) *_{T'} (G_2, \Gamma_2)$, respectively, such that the adhesion sets as well as the bonding maps for both tree amalgamations are the same. Let $t \in V(T)$ and let $t' \in V(T')$ be such that G_t and $G_{t'}$ are both isomorphic to G_i. Let $\gamma_t \in \Gamma_i$. We claim that there is an isomorphism $\bar{\gamma}\colon G \to G'$ such that

$$\bar{\gamma} \restriction_{G_t} = \mathrm{id}_{t'} \circ \gamma_t \circ \mathrm{id}_t^{-1},$$

where id_t and $\mathrm{id}_{t'}$ denote the canonical isomorphisms from G_i to G_t and $G_{t'}$ respectively. Clearly, the lemma follows from this claim.

For the proof of the claim define the star around $s \in V(T)$ by the map Λ_s mapping k to the label of $\overleftarrow{e_k}$, where $\overrightarrow{e_k}$ is the unique edge with label k starting at s. By Proposition 5.7, there are a bijection $\pi\colon N(t) \to N(t')$ and a family $(\gamma_s \in \Gamma_j)_{s\in N(t)}$ which extend γ_t to an isomorphism of the stars around t and t'. Iteratively apply Proposition 5.7 to vertices at distance $n = \{1, 2, 3, \dots\}$ from t. We obtain an isomorphism $\pi\colon T \to T'$ and maps $\gamma_s \in \Gamma_i$ for each $s \in V_i$ such that the restriction of π to s and its neighbours and the corresponding maps γ_x form an isomorphism between the stars at s and $\pi(s)$.

For $v \in V(G_s)$, define $\bar{\gamma}(v) = \mathrm{id}_{\pi(s)} \circ \gamma_s \circ \mathrm{id}_s^{-1}(v)$. If ss' is an edge, $v \in V(G_s)$ and $u \in V(G_{s'}$ such that u and v get identified in the construction of G, then $\bar{\gamma}(u) = \bar{\gamma}(v)$. Hence $\bar{\gamma}$ is well defined, and since it obviously maps edges to edges and non-edges to non-edges, it is the desired isomorphism. □

A closer inspection of the proof of Lemma 5.8 together with Remark 5.1 shows that tree amalgamations respecting the actions of quasi-transitive groups give rise to splitting tree-decompositions of $(G_1, \Gamma_1) * (G_2, \Gamma_2)$. The following lemma shows that the converse also holds, that is, splitting tree-decompositions of quasi-transitive graphs give rise to tree amalgamations respecting the actions of some quasi-transitive group on the parts.

Lemma 5.9. *Let Γ be a group acting quasi-transitively on a connected, locally finite graph G and let $(T, \mathcal{V})$ be a connected, splitting tree-decomposition of G with respect to Γ. Then one of the following holds.*

(1) There are $V_t, V_{t'} \in \mathcal{V}$ such that G is a non-trivial tree amalgamation

$$G[V_t] *_T G[V_{t'}]$$

of Type 1 respecting the actions of the stabilisers of $G[V_t]$ and $G[V_{t'}]$ in Γ.

(2) There is $V_t \in \mathcal{V}$ such that G is a non-trivial tree amalgamation

$$G[V_t] *_T G[V_t]$$

of Type 2 respecting the actions of the stabiliser of $G[V_t]$ in Γ.

Proof. Choose an oriented edge $\vec{e_0} \in D(T)$. We say that $\vec{e} \in D(T)$ is positively oriented, if there is $\gamma \in \Gamma$ mapping $\vec{e_0}$ to $\vec{e}$. Otherwise we say that $\vec{e}$ is negatively oriented. If Γ contains an element that reverses an edge of T, then let Γ' be the subgroup preserving the bipartition of T. This subgroup has index 2, and still acts quasi-transitively on G and transitively on the edges of T. Hence we can without loss of generality assume that no element of Γ swaps the endpoints of an edge, and thus every edge is either positively or negatively oriented, but not both.

Let s and t be the start and end point of $\vec{e_0}$ respectively. Let $(\vec{e_k})_{k\in K}$ be the positively oriented edges starting at s and let $(\vec{e_\ell})_{\ell\in L}$ be the negatively oriented edges starting at t. Without loss of generality, assume that K and L are disjoint, and that $\vec{e_0} = \vec{e_{k_0}} = \overleftarrow{e_{\ell_0}}$. For every $k \in K$ pick a $\gamma_k \in \Gamma$ which maps $\vec{e_0}$ to $\vec{e_k}$ (with $\gamma_{k_0} = \mathrm{id}$). For every $\ell \in L$ pick $\gamma_\ell \in \Gamma$ which maps $\overleftarrow{e_0}$ to $\vec{e_\ell}$ (with $\gamma_{\ell_0} = \mathrm{id}$). If there is an element γ_{st} of Γ that maps s to t, then for every $k \in K, \ell \in L$ let $\gamma'_k = \gamma_k \circ \gamma_{st}$ and $\gamma'_\ell = \gamma_\ell \circ \gamma_{st}^{-1}$.

Note that e_0 can be mapped to any edge incident to e_0 by a unique element of the form γ_k or γ'_k for some $k \in K \cup L$. For an arbitrary edge $e \neq e_0$, let e' be the first edge of the path connecting e to e_0. If $\gamma_{e'} \in \Gamma$ maps e_0 to e', then by the above remark there is a unique element δ_e of the form γ_k or γ'_k such that $\gamma_e \circ \delta_e$ maps e_0 to e. Use this to inductively construct (starting from $\delta_{e_0} = \mathrm{id}$) for each $e \in E(T)$ an automorphism $\gamma_e \in \Gamma$ such that $\gamma_e(e_0) = e$. Let $\vec{e}$ be the orientation of e pointing away from e_0 if $e \neq e_0$ and $\vec{e} = \vec{e_0}$ otherwise. Define the label $f(\vec{e})$ to be the unique $k \in K \cup L$ such that the δ_e from above equals γ_k or γ'_k. Note that $k \in K$ if and only if $\vec{e}$ is positively oriented. In this case define $f(\overleftarrow{e}) = \ell_0$, otherwise define $f(\overleftarrow{e}) = k_0$.

The following observation will be useful later. Let v be a vertex of T, and let $\vec{e}$ be the first edge of the path from v to e_0 (in case v is s or t this is an orientation of e_0). Let $\Delta_v = \{\delta_f \mid v \in f, f \neq e\}$.

- If all edges starting at v are positively (resp. negatively) oriented, then $\Delta_v = \{\gamma_k \mid k_0 \neq k \in K\}$ (resp. $\Delta_v = \{\gamma_\ell \mid \ell_0 \neq \ell \in L\}$).
- Otherwise, if $\vec{e}$ is positively (resp. negatively) oriented, then $\Delta_v = \{\gamma_k, \gamma'_\ell \mid k_0 \neq k \in K, \ell \in \ell\}$ (resp. $\Delta_v = \{\gamma'_k, \gamma_\ell \mid k \in K, \ell_0 \neq \ell \in L\}$).

In particular, taking into account the label of $\vec{e}$, in the first case the edges starting at v are labelled bijectively by K (resp. L), while in the second case they are labelled bijectively by $K \cup L$.

Next we show how this labelling defines a tree amalgamation. First assume there is no automorphism $\gamma \in \Gamma$ mapping s to t. Then all positively oriented edges must point from V_1 to V_2, where $V_1 \cup V_2$ is the bipartition of T with $s \in V_1$—this corresponds to the first case in the above observation. Let G_1 be isomorphic to $G[V_s]$, and let G_2 be isomorphic to $G[V_t]$. Let id_s and id_t be the respective isomorphisms.

For the definition of the adhesion sets let $I_1 = K$ and $I_2 = L$. For $k \in K$ let t_k be the endpoint of $\vec{e_k}$ and define $S_k = \mathrm{id}_s^{-1}(V_s \cap V_{t_k})$. Similarly, for $\ell \in L$, let s_ℓ be the endpoint of $\vec{e_\ell}$ and define $S_k = \mathrm{id}_t^{-1}(V_t \cap V_{s_\ell})$. Finally, define the adhesion maps by $\phi_{k\ell} = \mathrm{id}_t^{-1} \circ \gamma_\ell \circ \gamma_k^{-1} \circ \mathrm{id}_s |_{S_k}$.

The labels of directed edges starting at each vertex are in bijection to K or L depending on whether the vertex is in V_1 or V_2. Hence the above information together with the labelling defines a tree amalgamation. If Γ_i is a group acting on G_i in the same way as the setwise stabiliser (in Γ) of G_s, of G_t acts on G_s, on G_t respectively, then it is straightforward to verify that this tree amalgamation is of Type 1 respecting the actions. Note that the possible replacement of Γ by Γ' changes neither Γ_1 nor Γ_2.

It only remains to show that the tree amalgamation is isomorphic to G. Let e_v be the first edge on the path from $v \in V(T)$ to e_0. If $v \in V_1$, then set $\mathrm{id}_v = \gamma_{e_v} \circ \mathrm{id}_s$. Otherwise set $\mathrm{id}_v = \gamma_{e_v} \circ \mathrm{id}_t$. It is easy to verify that for an edge $e = uv$ with labels $f(\vec{e}) = k, f(\overleftarrow{e}) = \ell$ we have that $\mathrm{id}_v^{-1} \circ \mathrm{id}_u = \phi_{k\ell}$, and this clearly shows that the tree amalgamation is isomorphic to G.

The proof in the case where there is γ_{st} mapping s to t is very similar to the first case. Define G_1 and G_2 as before, but make sure that $\gamma_{st} \circ \mathrm{id}_s = \mathrm{id}_t$. This ensures that the actions Γ_1 on G_1 and Γ_2 on G_2 are the same, hence we can without loss of generality assume that $G_1 = G_2$ and $\Gamma_1 = \Gamma_2$.

Set $I_1 = I_2 = K \cup L$ and let $J = K$. Recall that $f(\vec{e}) \in K$ if and only if $f(\overleftarrow{e}) \in L$. Since we can map s to t, there are positively and negatively oriented edges starting at each vertex, hence the labels of edges starting at any vertex are in bijection with $K \cup L$. Hence (o) for tree amalgamations of Type 2 holds. Define the adhesion sets and adhesion maps exactly as above (but note that all adhesion sets end up in the same graph since $G_1 = G_2$). This gives a tree amalgamation of Type 2 by construction which is isomorphic to G by the same argument as above. □

Now we are ready to prove the main result of this section, the graph-theoretical analogue of Stallings' theorem, Theorem 1.1. We are proving a slightly stronger version than the one we stated in the introduction.

Theorem 5.10. *Let Γ be a group acting quasi-transitively on a connected, locally finite graph G with more than one end. Then there are connected subgraphs G_1, G_2 of G and groups Γ_1, Γ_2 acting quasi-transitively on G_1, G_2, respectively,* *such that G is a non-trivial tree amalgamation $(G_1, \Gamma_1) * (G_2, \Gamma_2)$ of finite adhesion and finite identification distinguishing ends.*

Furthermore, Γ_i can be chosen to be the setwise stabiliser of G_i in Γ.

Proof. By Corollary 4.4, G has a splitting tree-decomposition $(T, \mathcal{V})$ with connected adhesion sets. Using Lemma 5.9, G is a non-trivial tree amalgamation $(G_1, \Gamma_1) *_T (G_2, \Gamma_2)$, where Γ_i is the setwise stabiliser of G_i in Γ for $i = 1, 2$. Proposition 4.5 implies that Γ_i acts quasi-transitively on G_i for $i = 1, 2$. It remains to show that the tree amalgamationhas finite identification. Since Γ acts transitively on adhesion sets, each adhesion set induces a connected subgraph of the same size n. If some vertex x was contained in infinitely many distinct adhesion sets, then there would be infinitely many distinct paths of length at most n starting at x, contradicting local finiteness of G. In particular for a node $t \in V(T)$ only finitely many edges incident with t correspond to adhesion sets that contain any fixed vertex x. If the tree amalgamation does not have finite identification, then there must be a ray $t_1 t_2 \dots$ in T such that all edges $t_i t_{i+1}$ correspond to adhesion sets that contain x. Since there are only finitely many distinct adhesion sets that contain x, we may assume by (T3) that all $t_i t_{i+1}$ correspond to the same adhesion set S. As $(T, \mathcal{V})$ is connected, in every $V_{t_i} \neq S$ there is a neighbour of some vertex of S. Applying (T3) shows that these are all distinct neighbours of S. Thus, we have $V_{t_i} = S$ for all but finitely many $i \in \mathbb{N}$. In particular, there is an edge $t_i t_{i+1}$ with $V_{t_i} = V_{t_{i+1}}$. Since Γ acts transitively on $E(T)$, this implies $V(G) = V_{t_i}$. This is not possible by (i) from the definition of a splitting tree-decomposition. Thus, the tree amalgamation has finite identification. It distinguishes ends, since $(T, \mathcal{V})$ does that. □

6 Accessible graphs

Let G be a connected, quasi-transitive, locally finite graph with more than one end and let Γ act quasi-transitively on G. We say that G *splits (non-trivially)* into connected, quasi-transitive, locally finite graphs G_1, G_2 if it is a non-trivial tree amalgamation $G = G_1 * G_2$ of finite adhesion respecting the actions of groups Γ_i acting quasi-transitively on G_i and if the tree-decomposition defined by $G_1 * G_2$ (as in Remark 5.1) is splitting with respect to Γ. Note that the stabilizer in Γ of G_i acts quasi-transitively on G_i by Proposition 4.5. Now if one of the *factors* G_1 or G_2 also has more than one end, we can split it with respect to its stabilizer, too. We can continue this for every factor and call this a *process of splittings*. Note that it is important in a process of splittings to use the group action of the stabiliser of the factor in order to split the factor. If we eventually end up with factors that are either finite or have at most one end, that is, if the process of splittings terminates, we call the (multi-)set of these factors a *terminal factorisation* of G. (Also, if G is one-ended, we say it is a *terminal factorisation* of itself.) We call G *accessible* if it has a terminal factorisation.

Remark 6.1. Let G be an accessible connected, quasi-transitive, locally finite graph. Then there are connected, quasi-transitive, locally finite graphs $G_1, \dots,$ G_n, $H_1, \dots, H_{n-1}$ with $G = H_{n-1}$ and trees $T_1, \dots, T_{n-1}$ such that the following hold:

(i) every G_i has at most one end;

(ii) for every $i \leq n-1$, the graph H_i is a tree amalgamation $H *_{T_i} H'$ with respect to group actions of finite adhesion, where

$$H, H' \in \{G_j \mid 1 \leq j \leq n\} \cup \{H_j \mid 1 \leq j < i\}.$$

Remark 6.2. Let $\mathcal{G}_0$ be the class of all connected, quasi-transitive, locally finite graphs with at most one end. For $i > 0$, let $\mathcal{G}_i$ be the class obtained by tree amalgamations of finite adhesion of elements in $\bigcup_{j<i} \mathcal{G}_j$ respecting group actions. Set $\mathcal{G} := \bigcup_{i \in \mathbb{N}} \mathcal{G}_i$. Then $\mathcal{G}$ is the class of all accessible connected, quasi-transitive, locally finite graphs.

The following result generalizes a graph theoretical characterisation of accessibility of finitely generated groups: Thomassen and Woess [27, Theorem 1.1] proved that a finitely generated group is accessible if and only if it has a locally finite Cayley graph that is accessible in their sense.

Theorem 6.3. *Let G be a connected, locally finite, quasi-transitive graph. Then the following statements are equivalent.*

(1) *G is accessible.*

(2) *G is TW-accessible.*

Before we prove Theorem 6.3, we need another result. Recall that a tree-decomposition efficiently distinguishes two ends if there is an adhesion set $V_{t_1} \cap V_{t_2}$ separating them such that no set of smaller size than $V_{t_1} \cap V_{t_2}$ separates them.

Theorem 6.4. *Let G be a connected, locally finite graph such that there is an $n \in \mathbb{N}$ such that every two ends of G can be separated by at most n vertices. Let Γ be a group acting quasi-transitively on G. Then there exists a Γ-invariant tree-decomposition $(T, \mathcal{V})$ of G of finite adhesion such that $(T, \mathcal{V})$ distinguishes all ends of G efficiently and such that there are only finitely many Γ-orbits on $E(T)$.*

Proof. By Theorem 3.2 we find a Γ-invariant tree-decomposition $(T, \mathcal{V})$ of G of adhesion at most k that distinguishes all ends efficiently.

For every adhesion set $V_t \cap V_{t'}$ that does not separate any two ends efficiently, we contract the edge tt' in T and assign the vertex set $V_t \cup V_{t'}$ to the new node. It is easy to check that the resulting pair $(T', \mathcal{V}')$ is again a tree-decomposition. It only has adhesion sets that distinguish ends efficiently. Note that Γ still acts on $(T', \mathcal{V}')$ as the set of adhesion sets that do not separate ends efficiently is Γ-invariant. A result of Thomassen and Woess [27, Proposition 4.2] says that there are only finitely many vertex sets S of size at most n containing a fixed vertex such that for two components C_1, C_2 of $G - S$ every vertex of S has a neighbour in C_1 and in C_2. Since G is locally finite and quasi-transitive, it follows that there are only finitely many orbits of adhesion sets that separate ends efficiently. This proves the assertion. □

Proof of Theorem 6.3. To prove that (2) implies (1), let G be TW-accessible and let Γ be a group acting on G with only finitely many orbits. As G is quasi-transitive, there is an $n \in \mathbb{N}$ such that their ends of G can be separated by at most n vertices. By Theorem 6.4 we find a Γ-invariant tree-decomposition $(T, \mathcal{V})$ of G of finite adhesion such that $(T, \mathcal{V})$ distinguishes all ends of G

efficiently and such that there are only finitely many Γ-orbits on $E(T)$. We apply Proposition 4.1, and obtain a tree-decomposition $(T,\mathcal{U})$ with connected adhesion sets and $V_t \subseteq U_t$ for all $t \in V(T)$. We prove the assertion by induction on the number of Γ-orbits of adhesion sets of $(T,\mathcal{U})$. Let $tt' \in E(T)$. For every edge $t_1t_2 \in E(T)$ that does not lie in the same Γ-orbit as tt', we contract the edge t_1t_2 in T and assign the vertex set $U_{t_1} \cup U_{t_2}$ to the new node. Let T' be the resulting tree and $\mathcal{U}' = \{U_s \mid s \in V(T')\}$. It is easy to verify that $(T',\mathcal{U}')$ is a tree-decomposition. The only edges of T' are those that have their origin in the Γ-orbit of the edge $tt' \in E(T)$ and Γ still acts on $(T',\mathcal{U}')$ so that $(T',\mathcal{U}')$ is a connected, splitting tree-decomposition of G with only connected adhesion sets. Lemma 5.9 implies that G is a non-trivial tree amalgamation $G_1 *_{T'} G_2$ with respect to group actions, where the graphs G_1 and G_2 are induced by the parts of $(T',\mathcal{U}')$. The tree-decomposition $(T,\mathcal{U})$ induces a tree-decomposition $(T_W,\mathcal{W})$ on the parts W of $(T',\mathcal{U}')$ and there are fewer Γ_W-orbits on the adhesion sets of $(T_W,\mathcal{W})$ than Γ-orbits on the adhesion sets of $(T,\mathcal{U})$. Thus, we can apply induction on the number of orbits of adhesion sets. This shows (1).

To prove that (1) implies (2), we will use the graph classes $\mathcal{G}_i$ and $\mathcal{G}$ as defined in Remark 6.2 and show inductively that every $\mathcal{G}_i$ contains only connected, quasi-transitive, locally finite graphs for which there exists an $n \in \mathbb{N}$ such that any two of its ends can be separated by at most n vertices. This clearly implies that every $\mathcal{G}_i$ only contains graphs that are TW-accessible. This is obviously true for $\mathcal{G}_0$. Let $G \in \mathcal{G}_i$ for $i > 0$. Then there are $G_1, G_2 \in \bigcup_{j<i} \mathcal{G}_j$ such that G is a tree amalgamation $G_1 *_T G_2$ of finite adhesion respecting group actions. By induction, we may assume that G_1 and G_2 are TW-accessible and quasi-transitive. Note that quasi-transitivity of G follows from Lemma 5.8 since G_1 and G_2 are quasi-transitive. For $i = 1,2$, let k_i be a positive number such that any two ends of G_i can be separated by at most k_i many vertices. Let $(T,\mathcal{V})$ be the tree-decomposition we obtain from the tree amalgamation $G_1 *_T G_2$ according to Remark 5.1. Let k be the maximum of k_1, k_2 and the size of adhesion sets of $G_1 *_T G_2$.

Let Q, R be two rays in different ends ω_Q, ω_R of G, respectively. If there is some adhesion set $V_t \cap V_{t'}$ such that Q and R have tails that are separated by $V_t \cap V_{t'}$, then the ends they lie in must be separated by that adhesion set as well. Hence, they are separable by a separator of order at most k. So we may assume that, eventually, they lie on the same side of each adhesion set. By Proposition 4.8 (i) every end of G is captured either by an end or by a node of T. Since no separator separates any tails of Q and R, their ends are captured by the same node or end of T. By Proposition 4.8 (iii) an end of T captures a unique end of G. Thus, ω_Q and ω_R are captured by the same node of T. By Proposition 4.8 (iv) every end of G that is captured by a node $t \in V(T)$ corresponds to a uniquely determined end of $G[V_t]$. These ends can be separated by a separator S in $G[V_t]$ of order at most k by assumption. However, S need not be a separator of G that separates those ends. Still, it is possible to enlarge S to a separator of G that separates ω_Q and ω_R and still has bounded size: if K is the maximum diameter of the adhesion sets measured in G_1 and in G_2, then every vertex of S has distance at most K to only finitely many adhesion sets that are contained in V_t as G is locally finite; so we can add all these adhesion sets to S and obtain a set S'. As G is quasi-transitive, the size of S' only depends on k, the number of orbits of vertices of G, the maximum number of adhesion sets in V_t that have distance at most K to a common vertex and the size of any adhesion set of $(T, \mathcal{V})$, in particular, it is bounded by some $\ell \in \mathbb{N}$ and it is independent of the chosen ends. If we show that S' separates ω_Q and ω_R, then it follows immediately that there is an $n \in \mathbb{N}$ such that every two ends of G can be separated by at most n vertices.

Let $P = \dots, x_{-1}, x_0, x_1, \dots$ be a double ray with its tail $x_0, x_1, \dots$ in ω_Q and its tail $x_0, x_{-1}, \dots$ in ω_R. Since both ends ω_Q and ω_R are captured by V_t, there are infinitely many x_i with $i > 0$ that lie in V_t and infinitely many x_i with $i < 0$ that lies in V_t. Let us assume $x_0 \in V_t$. Whenever the ray $P^+ := x_0 x_1 \dots$ leaves V_t through an adhesion set $V_t \cap V_{t'}$, it must reenter V_t and this must happen through the same adhesion set. Since S is finite and separates ω_Q and ω_R, there are $i_1, i_2 \in \mathbb{Z}$ such that no $x_i \in V_t$ with $i \geq i_1$ is separated in $G[V_t]$ by S from ω_Q and no $x_i \in V_t$ with $i \leq i_2$ is separated in $G[V_t]$ by S from ω_R. Then there must be some path $x_i, \dots, x_j$ with $j \geq i+1$ and whose inner vertices lie outside of V_t such that x_j is not separated by S from ω_Q and x_i is not separated by S from ω_R. Thus, the shortest x_i-x_j path in $G[V_t]$ meets S. As x_i and x_j lie in a common adhesion set, we conclude that this lies in S'. Thus, S' separates ω_Q from ω_R in G. This shows (2). □

In the proof of the implication (2) to (1) of Theorem 6.3 we chose a specific way to split the factors. (It was based on a Γ-invariant tree-decomposition of G.) In an earlier version of this paper, we did not know if we can split arbitrarily in each step and still have to end in a terminal factorisation. But we conjectured that this is true.

Conjecture 6.5. *Let G be an accessible, connected, quasi-transitive, locally finite graph. Every process of splittings must end after finitely many steps.*

This conjecture has been verified in [15].

Accessibility of finitely generated groups received a lot of attention after Wall [28] conjectured that all finitely generated groups are accessible and among the main results in this area are Dunwoody's results that Wall's conjecture is false in general [7] but true for (almost) finitely presented groups [6]. In the case of quasi-transitive, locally finite graphs, the investigation focused on graphs that are TW-accessible, see [13, 22, 27]. However, Theorem 6.3 enables us to carry over these results to graphs that are accessible in our sense.

7 Applications

7.1 Stallings' theorem

There are several proofs of Stallings' theorem in the literature, see [5, 16, 23, 25]. In this section we will discuss how to obtain Stallings' theorem from our results.

Let Γ be a finitely generated group with infinitely many ends and let G be a locally finite Cayley graph of Γ. Then G has infinitely many ends, too. By Theorem 5.10, G is a non-trivial tree amalgamation $G_1 *_T G_2$ of finite adhesion respecting group actions. Since it has finite adhesion and Γ acts regularly[3] on G, the stabiliser in Γ of an edge of T, which is a subgroup of the stabiliser in Γ of the corresponding adhesion set, is finite. If the induced action of Γ on T is with inversion of edges, then we subdivide each edge of T once. On the edges

[3] that is, for every two $u, v \in V(G)$ there is a unique element of Γ mapping u to v

of the resulting tree, the group Γ acts transitively but without inversion. Now we apply Bass-Serre theory via the following theorem.

Theorem 7.1. [24] *Let T be a tree without leaves and let Γ act on T without inversion of edges but transitively on $E(T)$. If Γ acts transitively on T then Γ is an HNN-extension of the stabilizer of a vertex over the pointwise stabilizer of an edge. If there are two Γ-orbits on $V(T)$, then Γ is the free product of the stabilizers of two adjacent vertices with amalgamation over the pointwise stabilizer of the incident edge.* □

We thus obtain Stallings' theorem.

Theorem 7.2. [25] *A finitely generated group has more than one end, if and only if it is either a free product with amalgamation over a finite subgroup or an HNN-extension over a finite subgroup.* □

Note that the groups acting on tree amalgamations of Type 1 respecting the actions of groups acting on the factors act on the connecting trees with two orbits and thus lead to free products with amalgamation via Theorem 7.1. Similarly, groups acting on tree amalgamations of Type 2 respecting the action of groups act transitively on the connecting trees and thus lead to HNN-extension via Theorem 7.1.

7.2 Graphs without thick ends

In this section, we prove that connected, quasi-transitive, locally finite graphs with only thin ends are the connected, quasi-transitive, locally finite graphs that have terminal factorisations with only finite factors. But before we go into the proof, we need some definitions.

Let G and H be graphs. A map $\varphi\colon V(G) \to V(H)$ is a (γ, c)-*quasi-isometry* if there are constants $\gamma \geq 1, c \geq 0$ such that

$$\gamma^{-1} d_G(x, y) - c \leq d_H(\varphi(x), \varphi(y)) \leq \gamma d_G(x, y) + c$$

for all $x, y \in V(G)$ and such that $\sup\{d_H(x, \varphi(V(G))) \mid x \in V(H)\} \leq c$. We then say that G is *quasi-isometric* to H.

Krön and Möller [17, Theorem 5.5] showed that a connected, quasi-transitive, locally finite graph has only thin ends if and only if it is quasi-isometric to a tree. Trees are obviously TW-accessible and it follows from their definition of accessibility that the class of quasi-transitive, locally finite such graphs is invariant under quasi-isometries. Thus, we have verified the following.

Proposition 7.3. *Every connected, locally finite, quasi-transitive graph that has only thin ends is TW-accessible.* □

We mention that Thomassen and Woess [27, Theorem 5.3] showed Proposition 7.3 for transitive graphs directly with a nice graph theoretical argument. It is not too hard to modify their argument in such a way that the proof works for quasi-transitive graphs as well.

Another result we need for our investigation here is due to Thomassen.

Proposition 7.4. [26, Proposition 5.6.] *If G is an infinite, connected, quasi-transitive, locally finite graph with only one end, then the end is thick.* □

Recently, Carmesin et al. [3, Theorem 5.1] extended Proposition 7.4 to graphs that need not be locally finite.

Now we are able to give a new characterisation of connected, quasi-transitive, locally finite graphs with only thin ends.

Theorem 7.5. *A connected, quasi-transitive, locally finite graph has only thin ends if and only if it has a terminal factorisation of only finite graphs.*

Proof. Let G be a connected, quasi-transitive, locally finite graph. First, let us assume that every end of G is thin. By Proposition 7.3, G is TW-accessible. So Theorem 6.3 implies that G is accessible and hence has a terminal factorisation. All the factors of that terminal factorisation have at most one end. Since they are quasi-transitive by Proposition 4.5, they cannot have one end due to Proposition 7.4. So they are locally finite graphs without ends, which implies that they are finite graphs.

For the other direction, we follow the steps to factorise G, factorise each of its factors and so on until we end up with a terminal factorisation. Note that by Proposition 4.8 (ii) every thick end of G is captured by nodes of the involved splitting tree-decompositions. So if G had a thick end, then one of the factors of the terminal factorisation must have a thick end, which is impossible as these factors are finite by assumption. Thus, all ends of G are thin. □

Note that there are several characterisations of (quasi-transitive or Cayley) graphs that are quasi-isometric to trees, see e. g. Antolín [1], Krön and Möller [17], Manning [18] and Woess [29]. We enlarged their list of characterisations by our theorem.

A natural class of quasi-transitive graphs are Cayley graphs. So our theorems apply in particular for such graphs and we obtain as a corollary of Theorem 7.5 a result for virtually free groups. A group Γ is *virtually free* if it contains a free subgroup of finite index.

Woess [29] showed that a finitely generated group is virtually free if and only if every end of any of its locally finite Cayley graphs is thin. Thus we directly obtain the following corollary.

Corollary 7.6. *A finitely generated group is virtually free if and only if any of its locally finite Cayley graphs has a terminal factorisation of only finite graphs.* □

In [12] the interplay between tree amalgamations and quasi-isometries is investigated further and the results of this section are extended to graphs other than trees in two ways. First, it is shown that the quasi-isometry type of (iterated) tree amalgamations only depend on the quasi-isometry types of the infinite factors. Then, in the case of accessible infinitely-ended graphs, it is shown that the quasi-isometry types of the graphs determine the quasi-isometry types of the infinite factors in any of its terminal factorisations.

7.3 Planar graphs

Mohar [19] raised the question whether tree amalgamations are powerful enough to characterise (3-connected) planar, transitive, locally finite graphs in terms of finite or one-ended, locally finite, planar, transitive graphs. We are able to give an affirmative answer in case of planar, quasi-transitive graphs. This does not give a complete answer Mohar's question, since we cannot guarantee the factors in the case of transitive graphs to be transitive again: we only prove that they are quasi-transitive. Also, we remind the reader that our notion of tree amalgamations differs slightly from Mohar's notion, since his tree amalgamations have identifications length at most 2.

Dunwoody [8] proved that planar, quasi-transitive, locally finite graphs are TW-accessible, see also [14]. This allows us to apply Theorem 6.3 to these graphs. We directly obtain the following result.

Theorem 7.7. *For every planar, connected, quasi-transitive, locally finite graph G there are finitely many planar, connected, quasi-transitive, locally finite graphs $G_1, \dots, G_n$ with at most one end such that G can be obtained by finitely many (iterated) tree amalgamations of $G_1, \dots, G_n$.* □

We point out that examples given in [9] show that we cannot replace the term 'quasi-transitive' by 'transitive' or 'Cayley' in the above theorem; Georgakopoulos [9] suggests that Mohar's question is to be interpreted in terms of subdivisions, that is, we can replace 'quasi-transitive' by 'transitive' if in addition to tree amalgamations we allow subdivisions.

7.4 Further applications

In this section we briefly mention further applications of our main results. In [11] we prove that tree amalgamations and hyperbolic graphs fit well together in that we prove that a locally finite, quasi-transitive graph with more than one end is hyperbolic if and only if it is the non-trivial tree amalgamation of two locally finite, quasi-transitive, hyperbolic graphs. Additionally, the homeomorphism type of the hyperbolic boundary is uniquely determined by the homeomorphism types of the hyperbolic boundaries of their factors [11, Theorem 1.2]. Since hyperbolic, locally finite, quasi-transitive graphs are TW-accessible [13], this implies by Theorem 6.3 that the homeomorphism type of the hyperbolic boundary is uniquely determined by the homeomorphism types of the hyperbolic boundaries in any terminal factorisation.

Similarly, we consider in [12] quasi-isometry types of tree amalgamations and prove that they only depend on the quasi-isometry types of their factors. In the case of accessible graphs with the same number of ends, we obtain that two graphs are quasi-isometric if and only if all terminal factorisations have the same quasi-isometry types of infinite factors.

As a third application, we obtain in [10] a sharp upper bound of the asymptotic dimension of tree amalgamations depending on the asymptotic dimensions of their factors.

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