TREE AMALGAMATIONS AND QUASI-ISOMETRIES

MATTHIAS HAMANN

ABSTRACT. We investigate the connections between tree amalgamations and quasi-isometries. In particular, we prove that the quasi-isometry type of multi-ended accessible quasi-transitive connected locally finite graphs is determined by the quasi-isometry type of their one-ended factors in any of their terminal factorisations. Our results carry over theorems of Papasoglu and Whyte on quasi-isometries between multi-ended groups to those between multi-ended graphs. In the end, we discuss the impact of our results to a question of Woess.

1. INTRODUCTION

Tree amalgamations can be thought of as an analogue of free products with amalgamation or HNN-extensions for graphs. (We refer to Section 2 for the precise definitions.) The following theorem of [6] says that every multi-ended locally finite connected quasi-transitive graph splits over a finite subgraph as a tree amalgamation. It is a graph theoretic version of Stallings' splitting theorem of multi-ended groups [10]. (We refer to Section 2 for the precise definitions needed for Theorem 1.1.)

Theorem 1.1. [6, Theorem 5.3] *Every connected quasi-transitive locally finite graph with more than one end is a non-trivial tree amalgamation of finite adhesion and finite identification length of two connected quasi-transitive locally finite graphs that respects the group actions and distinguishes ends.*

Just like Stallings' theorem enables us to prove theorems about multi-ended groups having knowledge of their factors, we are aiming for similar results for graphs. Two such examples are already proved in [6] and [5]: tree amalgamations respect hyperbolicity, i. e. two locally finite quasi-transitive graphs are hyperbolic if and only if their tree amalgamation is hyperbolic, see [5, Theorem 1.1], and a connected locally finite quasi-transitive graph has only thin ends if and only if it has a terminal factorisation of finite graphs, see [6, Theorem 7.5]. Here, a terminal factorisation can be seen as analogue of a terminal graph of groups: whenever we split our multi-ended graphs, we may ask if their factors have more than one end and apply to those our splitting theorem, too. If we iterate this splitting and eventually have only factors with at most one end, a *terminal factorisation* of the original graph consists of these finite or one-ended factors. Krön and Möller [7, Theorem 5.5] proved that a connected locally finite quasi-transitive graph has only thin ends if and only if it is quasi-isometric to a tree. Before we state the result of [6, Section 7.2], we define quasi-isometries. A map $\varphi\colon V(G) \to V(H)$ is a *quasi-isometry* if

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there are constants $\gamma \geq 1$, $c \geq 0$ such that

$$\gamma^{-1} d_G(u, v) - c \leq d_H(\varphi(u), \varphi(v)) \leq \gamma \, d_G(u, v) + c$$

for all $u, v \in V(G)$ and such that $\sup\{d_H(v, \varphi(V(G))) \mid v \in V(H)\} \leq c$. We then say that G is *quasi-isometric* to H. Note the being quasi-isometric is an equivalence relation.

Theorem 1.2. [6, Section 7.2] *A connected quasi-transitive locally finite graph is quasi-isometric to a tree if and only if it has a terminal factorisation of only finite graphs.*

In this paper we are looking more into the connections of tree amalgamations with quasi-isometries. Since tree amalgamations of two graphs do not only depend on the two graphs themselves but also on how we glue their copies together, the first natural question is whether the quasi-isometry type of a tree amalgamation depends on these choices. We will prove that (under some mild assumptions) it does not depend on these choices (see Corollary 3.4).

Our main result consists of two parts. First, we will generalize the property that the quasi-isometry type of the tree amalgamation of two graphs G_1 and G_2 does not depend on the particular tree amalgamation: we will prove that a tree amalgamation of G_1 and G_2 is quasi-isometric to the tree amalgamation of two graphs, one of which is quasi-isometric to G_1 the other to G_2. More generally, our next result says that even iterated tree amalgamations only depend on the quasi-isometry type of their infinite factors if the tree amalgamations have infinitely many ends. (This is the first statement of Theorem 1.3.)

An obvious question is whether we can also obtain a reverse statement of that, i.e. if G and H are quasi-isometric graphs, do all factorisations of G and H have the same set of quasi-isometry types of infinite parts? In general this is false as an easy example shows: take a tree; this has a factorisation into finite graphs, but also the trivial factorisation into just itself.

In this example, we can still factorise the tree while this is not possible for the finite graphs. So we might just ask for a reverse of the above mentioned statement (the first statement in Theorem 1.3) for terminal factorisations. We call a connected quasi-transitive locally finite graph *accessible* if it has a terminal factorisation. Note that not all connected locally finite quasi-transitive graphs have a terminal factorisation, i.e. there are inaccessible connected quasi-transitive locally finite graphs, as examples by Dunwoody [1, 2] show. But the class of accessible connected quasi-transitive locally finite graphs is indeed a class where we obtain the reverse statement for terminal factorisations.

Theorem 1.3. *Let G and H be connected locally finite quasi-transitive graphs with infinitely many ends and let $(G_1, \ldots, G_n), (H_1, \ldots, H_m)$ be factorisations of G, H, respectively. If $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_m)$ have the same set of quasi-isometry types of infinite factors, then G and H are quasi-isometric.*

Furthermore, if G and H are quasi-isometric and accessible and the factorisations are terminal, then $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_n)$ have the same set of quasi-isometry types of infinite factors.

Let us point out that we just ask for the same sets and not the same multi-sets in Theorem 1.3.

Papasoglu and Whyte [9] proved a group-theoretic versions of Theorem 1.3. That is, they proved the following result.

Theorem 1.4. [9, Theorems 0.3 and 0.4] *Let G and H be finitely generated infinitely ended groups. If they are the fundamental groups of finite graphs of groups with the same sets of quasi-isometry types of vertex groups, then G and H are quasi-isometric.*

Furthermore, if G and H are quasi-isometric accessible groups, then any terminal graph of group decompositions of G and H have the same quasi-isometry types of one-ended vertex groups.

Our proofs are inspired by their proof ideas from free products with amalgamations and HNN-extensions of groups and indeed their results can be obtained from ours using Bass-Serre theory. (We refer to [6, Section 7.1] for a discussion of tree amalgamations and group products via Bass-Serre theory.) After some preliminary work in Section 2, we prove the first statement of Theorem 1.3 in Section 3 and the second part of Theorem 1.3 in Section 4.

In Section 5, we turn our attention towards a question of Woess that seems to be a bit unrelated at a first glance but which is also a reason why the results for quasi-transitive locally finite graphs cannot be obtained from the results for finitely generated groups. Woess [12, Problem 1] posed the problem whether there are locally finite transitive graphs that are not quasi-isometric to any locally finite Cayley graph. His problem was settled in the negative by Eskin et al. [3] who proved that the Diestel-Leader graphs are counterexamples. While we do not construct any new counterexamples, we will obtain as a corollary of Theorem 4.2 that in order to find further counterexamples, it suffices to look at either one-ended graphs or inaccessible ones.

2. PRELIMINARIES

In this section, we state our major definitions and cite several results that we are going to use in the proofs of our main theorems.

Let G and H be graphs. We call G *quasi-transitive* if there are only finitely many $\mathrm{Aut}(G)$-orbits on the vertex set of G. We call G and H *bilipschitz equivalent* if they are quasi-isometric with $c = 0$.

A tree is *semiregular* or (p_1, p_2)-*semiregular* if for the canonical bipartition V_1, V_2 of its vertex set into independent sets all vertices in V_1 have the same degree $p_1 \in \mathbb{N} \cup \{\infty\}$ and all vertices in V_2 have the same degree $p_2 \in \mathbb{N} \cup \{\infty\}$.

Let G_1 and G_2 be two graphs and let T be a (p_1, p_2)-semiregular tree with canonical bipartition V_1, V_2 of its vertex set into independent sets. Let

$$c \colon E(T) \to \{(k, \ell) \mid 1 \le k \le p_1, 1 \le \ell \le p_2\}$$

such that for all $v \in V_i$ the i-th coordinates of the elements of $\{c(e) \mid v \in e\}$ exhaust the set $\{k \mid 1 \le k \le p_i\}$. Note that in particular the i-th coordinates of the elements of $\{c(e) \mid v \in e\}$ are all distinct. Let $\{S_k^i \mid 1 \le k \le p_i\}$ be a set of subsets of $V(G_i)$ such that all S_k^i have the same cardinality. Let $\varphi_{k,\ell} \colon S_k^1 \to S_\ell^2$ be a bijection.

For each $v \in V_i$ with $i = 1, 2$, let G_i^v be a copy of G_i. We denote by S_k^v the copy of S_k^i in G_i^v. Let $H := G_1 + G_2$ be the graph obtained from the disjoint union of all G_i^v by adding an edge between all $x \in S_k^v$ and $\varphi_{k,\ell}(x) \in S_\ell^u$ for every edge $vu \in E(T)$ with $c(vu) = (k, \ell)$ and $v \in V_1$. Let G be the graph obtained from H by contracting all new edges, i. e. all edges outside of the G_i^v. We call G the *tree*

amalgamation of G_1 and G_2 over T (with respect to the sets S_k and the maps $\varphi_{k,\ell}$), denoted by $G_1 *_T G_2$. If T is clear from the context, we simply write $G_1 * G_2$. See Figure 1 for an image of $G_1 + G_2$ and of $G_1 * G_2$. The sets S_k^i and their copies in G are the *adhesion sets* of the tree amalgamation. If the adhesion sets of a tree amalgamation are finite, then this tree amalgamation has *finite adhesion*. Let $\psi \colon V(H) \to V(G)$ be the canonical map that maps every $x \in V(H)$ to the vertex of G it ends up after all contractions. A tree amalgamation $G_1 *_T G_2$ is *trivial* if there is some G_i^v such that the restriction of ψ to G_i^v is bijective. Note that a tree amalgamation of finite adhesion is trivial if $V(G_i)$ is the only adhesion set of G_i and $p_i = 1$ for some $i \in \{1, 2\}$.
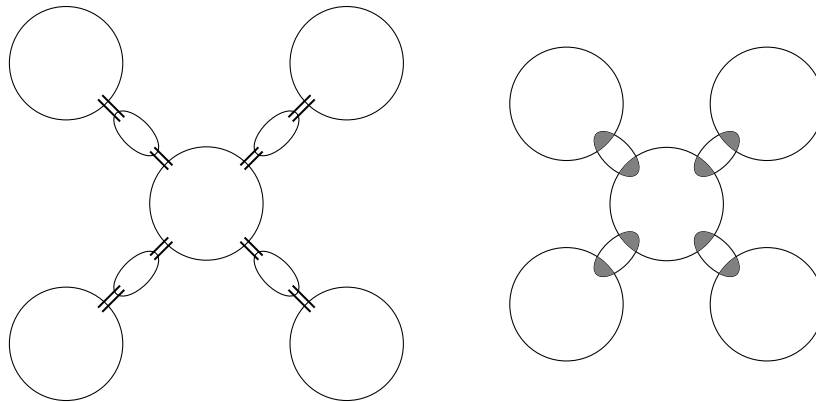

FIGURE 1. The left figure is part of the graph $G_1 + G_2$ with copies of G_1 being the smaller balls and those of G_2 the larger ones. On the right, the same copies are shown after identification. The grey parts are the adhesion sets.

Rays are one-way infinite paths and two rays in a graph G are equivalent if they lie eventually in the same component of $G - S$ for every finite set S. The equivalence classes of rays are the *ends* of G.

The tree amalgamation $G = G_1 * G_2$ *distinguishes* ends if there is some adhesion set $S_k^v = S_\ell^u$ for adjacent vertices u, v of T such that for every component C of $T - uv$ the graph induced by $\bigcup_{w \in C} G_i^w$ contains an end.

The *identification size*[1] of a vertex $x \in V(G)$ is the smallest size of subtrees of T over which the contractions to obtain x take place, i.e. the size n of the smallest subtree T' of T such that x is obtained by contracting only edges between vertices in $\bigcup_{u \in V(T')} V(G_j^u)$. The tree amalgamation has *finite identification* if every vertex has finite identification size. It has *bounded identification* if the supremum of the identification sizes is finite. We note that tree amalgamations may have finite but not bounded identification. E.g. this happens if there exists a sequence of vertices $(v_i)_{i \in \mathbb{N}}$ such that v_i lies in more than i but finitely many adhesion sets.

Remark 2.1. For a tree amalgamation $G_1 * G_2$ of bounded identification, the canonical map $\psi \colon V(G_1 + G_2) \to V(G_1 * G_2)$ is a quasi-isometry whose constants

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[1]Equivalently, as in [8], you can define G via identifications of vertices in the disjoint union of the G_i^v instead of contraction of the newly added edges. From this point of view, we get the justification of the term 'identification size'.

depend only on the identification sizes of the vertices. Thus, if the identification sizes and the diameters of adhesion sets are bounded, then G_1 and G_2 embed quasi-isometrically into $G_1 + G_2$ and thus into $G_1 * G_2$.

If G_1 is finite we define the *finite extension of G_2 by G_1 (with respect to $G_1 * G_2$)* to be isomorphic to a subgraph of $G_1 * G_2$ that is induced by a copy G_2^v of G_2 and all copies G_1^u of G_1 with $uv \in E(T)$. Figure 2 shows a finite extension. It is straightforward to see that the finite extension of G_2 is quasi-isometric to G_2 if all adhesion sets in G_2 have bounded diameter.
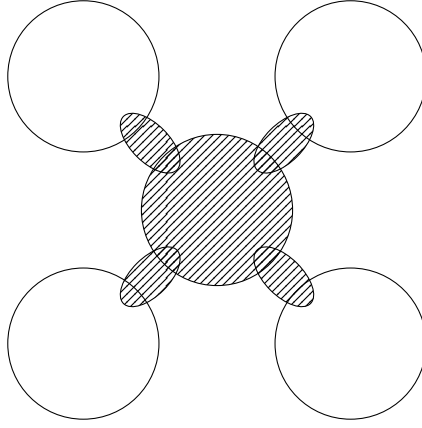

FIGURE 2. As in Figure 1, the copies of G_1 are the smaller balls and those of G_2 the larger ones. For finite G_1, the shaded graph shows the finite extension of G_2 by G_1.

Remark 2.2. Let T be a (p_1, p_2)-semiregular tree and let $G_1 *_T G_2$ be a tree amalgamation, where G_1 is finite. Let G_2' be the finite extension of G_2 by G_1 with respect to $G_1 *_T G_2$. We will define a tree amalgamation $G_2' *_{T'} G_2$ with $G_2' *_{T'} G_2 = G_1 *_T G_2$, where T' will be a $(p_2(p_1 - 1), p_2)$-semiregular tree. For that choose $u \in V_2$ and let U_1 be the set of vertices $v \in V_2$ with distance 0 mod 4 to u and $U_2 := V_2 \setminus U_1$. To obtain T' we start with T and contract all edges of T that are incident with a vertex of U_1. The resulting graph is a $(p_2(p_1 - 1), p_2)$-semiregular tree. We may assume that the vertex set of T' is $U_1 \cup U_2$. For each vertex in U_1 we take a copy of G_2' and for each vertex in U_2 we take a copy of G_2. There is a canonical way of assigning the labels to the edges of T' so that the tree amalgamation $G_2' *_{T'} G_2$ is the same as the tree amalgamation $G_1 *_T G_2$.

We will use Theorem 1.1 in a slightly different form to avoid the definition of a tree amalgamation with respect to the group actions. We note that the statement about the finite identification varies a bit from [6, Theorem 5.3], but the version we use here is mentioned in its proof in [6].

Theorem 2.3. [6, Theorem 5.3] *Every connected quasi-transitive locally finite graph with more than one end is a non-trivial tree amalgamation $G_1 * G_2$ that distinguishes ends and has finite adhesion and finite identification of two connected quasi-transitive locally finite graphs such that the set of adhesion sets in each factor has at most two orbits under some group acting quasi-transitively on that factor.*

Note that the properties of the tree amalgamation of Theorem 2.3 imply that it has bounded identification.

A *factorisation* of a connected locally finite quasi-transitive graph G is a tuple $(G_1, \ldots, G_n)$ of connected locally finite quasi-transitive graphs such that G is obtained from the elements of the tuple by iterated tree amalgamations of finite adhesion and finite identification such that for each step some group of automorphisms of each factor acts quasi-transitively on this factor and the set of its adhesion sets. Note that this implies that the tree amalgamation is of bounded adhesion. So from now on, we may assume that the conclusion of Theorem 2.3 the tree amalgamation is of bounded identification. A factorisation is *terminal* if every element of the tuple has at most one end.[2]

Remark 2.4. Let G be an accessible connected quasi-transitive locally finite graph. Then there are connected quasi-transitive locally finite graphs $G_1, \ldots, G_n$, $H_1, \ldots,$ H_{n-1} with $G = H_{n-1}$ and trees $T_1, \ldots, T_{n-1}$ such that the following hold:

(1) every G_i has at most one end;
(2) for every $i \leq n - 1$, the graph H_i is a tree amalgamation $H *_{T_i} H'$ of finite adhesion and bounded identification, where

$$H, H' \in \{G_j \mid 1 \leq j \leq n\} \cup \{H_j \mid 1 \leq j < i\}.$$

Papasoglu and Whyte used a construction in [9] that we will use for our proofs, too. However, we can express it in terms of tree amalgamations and thus stick to our notations. Let G_1 and G_2 be graphs and let $v_i \in V(G_i)$ for $i = 1, 2$ be their *base vertices*. Let $(S_k^i)_{k \leq |G_i|}$ be the adhesion sets such that $(S_k^i)_{k \leq |G_i|}$ forms a partition of $V(G_i)$ into sets of size 1. Assume that $S_1^1 = \{v_1\}$ and $S_1^2 = \{v_2\}$. Let T be a $(|G_1|, |G_2|)$-semiregular tree with canonical vertex partition $\{V_1, V_2\}$, and let $u \in V_1$. Let

$$c \colon E(T) \to \{(k, \ell) \mid 1 \leq k \leq |G_1|, 1 \leq \ell \leq |G_2|\}$$

be as required for a tree amalgamation with the following additional property for all edges $xy \in E(T)$, where x is closer to u than y:

- if $x \in V_1$, let the second coordinate of $c(xy)$ be 1;
- if $x \in V_2$, let the first coordinate of $c(xy)$ be 1.

We denote the graph $G_1 + G_2$ by $G_1 +_{v_1, v_2} G_2$. Note that there is a unique edge in T with label $(1,1)$. We call the corresponding edge in $G_1 +_{v_1, v_2} G_2$ the *base edge* of $G_1 +_{v_1, v_2} G_2$. It is the unique edge in $G_1 +_{v_1, v_2} G_2$ that connects two base vertices.

Papasoglu and Whyte proved several lemmas about (their version of) this special kind of tree amalgamations. Before we state them, we need a further definition. We call a graph G *unvarying* if there is some γ such that for every $u, v \in V(G)$ there is a γ-bilipschitz map $G \to G$ that maps u to v. Note that every quasi-transitive graph is unvarying.

Lemma 2.5. [9, Lemma 1.1] *Let G and H be unvarying graphs. Let F be a graph such that there is some $\gamma > 0$ such that the following hold.*

(1) *F contains families $(G_i)_{i \in I}$ and $(H_j)_{j \in J}$ of disjoint subgraphs so that $V(F)$ is covered by them.*

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[2]We note that the definitions for (terminal) factorisations are weaker than in [6] as we do not care about the specific group actions for our results here.

(2) *Every vertex of F is incident with a unique edge that lies outside all G_i, H_j.*
(3) *Every edge of F outside of all G_i, H_j is incident with a vertex in some G_i and with a vertex in some H_j.*
(4) *For every $i \in I$, $j \in J$ there is a γ-bilipschitz equivalence $G_i \to G$, $H_j \to H$, respectively.*
(5) *Contracting all edges inside the graphs G_i and H_j results in a tree.*

Then there is a constant δ (depending only on γ and the unvaryingness-constants of G and H) such that for any edge $e \in E(F)$ connecting some G_i to some H_j and any base vertices u in G and v in H there is a δ-bilipschitz equivalence $F \to G+_{u,v}H$ that maps e to the base edge of $G +_{u,v} H$.

Lemma 2.5 is our main tool to switch from the tree amalgamations we are starting with to tree amalgamations that are defined using base vertices, in order to apply the two following lemmas.

Let G_1 and G_2 be graphs with base vertices. The *wedge* of G_1 and G_2 is their disjoint union with an edge joining their base vertices.

Lemma 2.6. [9, Lemma 1.3] *Let G_1 and G_2 be infinite unvarying graphs with base vertices u, v, respectively. Then $G := G_1 +_{u,v} G_2$ is bilipschitz equivalent to the wedge of any finite number of copies of G.*

Lemma 2.7. [9, Lemma 1.4] *Let G_1 and G_2 be infinite unvarying graphs and u, v be their base vertices, respectively. Then $G_1 +_{u,v} G_2$ and $G_1 +_{u,v} (G_2 +_{v,v} G_2)$ are bilipschitz equivalent.*

Together with the quasi-isometry from Remark 2.1, Lemmas 2.5 and 2.7 imply the following for tree amalgamations.

Corollary 2.8. *Let G and H be connected locally finite graphs. Then $G * H$ and $G * (H * H)$ are quasi-isometric, where all adhesion sets of the tree amalgamations have size 1 and such that every vertex lies in an adhesion set.*

Proof. By Remark 2.1, $G * H$ is quasi-isometric to $G + H$, which is bilipschitz equivalent to $G +_{u,v} H$ for some base vertices u of G and v of H. This is bilipschitz equivalent to $G +_{u,v} (H +_{v,v} H)$ by Lemma 2.7, which in turn is bilipschitz equivalent to $G + (H + H)$ by Lemma 2.5, since $H + H$ is bilipschitz equivalent to $H +_{v,v} H$ by the same lemma. Remark 2.1 shows that $G + (H + H)$ is quasi-isometric to $G * (H * H)$ and thus finishes the proof. $\square$

We end this section by proving two quasi-isometry results regarding infinitely-ended trees.

Lemma 2.9. *Every two quasi-transitive locally finite trees with infinitely many ends are quasi-isometric.*

Proof. It suffices to prove that every locally finite tree T with infinitely many ends and finitely many orbits of vertices is quasi-isometric to the 3-regular tree. First, we note that there are only finitely many orbits of vertices of degree 2 in T, so replacing every maximal path with all internal vertices of degree 2 by an edge results in a tree T' that is quasi-isometric to T. Thus, we may assume that T has no vertices of degree 2. Now we note that, if P is a path with k edges in a tree such that all vertices of P have degree 3 in that tree, then contracting P to a single vertex leads to a vertex of degree $k + 3$. We fix a root u in the 3-regular tree T_3 and a root v

in T. Let P_v be a path in T_3 starting at u of length $d(v) - 3$, where $d(v)$ denotes the degree of v. Now let us assume that for a subtree T' of T that contains v we have the following:

(1) for each vertex w of T' there is a path P_w in T_3;
(2) distinct such paths are disjoint; and
(3) $x, y \in V(T')$ are adjacent if and only if T_3 has an edge with one end vertex in P_x and and the other in P_y.

We pick a vertex $w \in T - T'$ that is adjacent to a vertex x of T'. By (3) there is a vertex a in T_3 adjacent to P_x. Also (3) implies that the vertices in T_3 that lie on the paths P_y for $y \in V(T')$ form a subtree. Hence, we find a path P_w in T_3 that starts at a, is disjoint from any P_y with $y \in V(T')$ and has length $d(w) - 3$. We end up with a collection of paths in T_3, one for every vertex of T, that partitions $V(T_3)$. Contracting these paths defines a quasi-isometry, since the paths have bounded length, and results in a tree isomorphic to T due to the requirement on the lengths of the paths P_w. $\square$

The last result of this section is a sharpening of the easy direction of Theorem 1.2. Generally, this is follows from Theorem 1.2 and Lemma 2.9 combined, if the tree obtained from Theorem 1.2 was quasi-transitive. Since this is not stated in that theorem, we will have to prove it.

Lemma 2.10. *A connected locally finite quasi-transitive graph with infinitely many ends that has a terminal factorisation of only finite graphs is quasi-isometric to a 3-regular tree.*

Proof. Let $(G_1, \ldots, G_n)$ be a terminal factorisation of a connected locally finite quasi-transitive graph G, where all G_i are finite. First, we prove by induction on n that G is quasi-isometric to a tree with infinitely many ends with at most n orbits on its vertex set. This suffices to prove the assertion since such trees are quasi-isometric to a 3-regular tree by Lemma 2.9. If $n = 2$, then the map $G_1 +_T G_2 \to T$ that maps the vertices in G_i^u to u is a quasi-isometry. So by Remark 2.1 and as the tree amalgamation of a factorisation is of finite identification, and thus bounded identification, G is quasi-isometric to T. Now let $n \geq 3$ and let (H_1, H_2) be a factorisation of G such that H_1 and H_2 have terminal factorisations that are subsequences of $(G_1, \ldots, G_n)$. Note that these are proper subsequences. So by induction H_i is quasi-isometric to some tree T_i with finitely many orbits. Let $\alpha_i \colon H_i \to T_i$ be a quasi-isometry and let T be a tree such that $G = H_1 *_T H_2$. For each $\mathrm{Aut}(G)$-orbit of $E(T)$ we choose an edge $e = uv$ in it and an edge $x_u^e x_v^e \in E(H_1 + H_2)$ with $x_u^e \in H_i^u$ and $x_v^e \in H_j^v$. We extend the definition of x_u^e and x_v^e to all edges of T in such a way that it is compatible with the action of $\mathrm{Aut}(G)$. Now we construct a new tree T'. For that, we replace in T each vertex u with a copy T_i^u of T_i and replace the edge $uv \in E(T)$ by an edge $\alpha_i(x_u^{uv})\alpha_j(x_v^{uv})$. By its construction and as T, T_1 and T_2 are trees, T' is a tree, too. There are only finitely many orbits on $V(T')$ as this is true for the other three involved trees and as the construction of T' respects the action of $\mathrm{Aut}(G)$. Finally, we note that the quasi-isometries α_1 and α_2 extend to a quasi-isometry $\alpha \colon G \to T'$ as the adhesion sets of $H_1 *_T H_2$ have bounded diameter. $\square$

3. Tree amalgamations and quasi-isometries

In this section, we shall prove the first part of Theorem 1.3. In preparations for that, we prove that we may assume – up to quasi-isometry – that the tree amalgamations that we consider have adhesion 1, disjoint adhesion sets and no vertices outside adhesion sets.

Lemma 3.1. *Let G be a locally finite connected quasi-transitive graph and let (G_1, G_2) be a factorisation of G. Then there is a locally finite connected quasi-transitive graph H that has a factorisation (H_1, H_2) such that the following hold.*

(1) *G is quasi-isometric to H;*
(2) *G_i is quasi-isometric to H_i for $i = 1, 2$;*
(3) *$H_1 * H_2$ has adhesion 1;*
(4) *all adhesion sets of $H_1 * H_2$ are distinct;*
(5) *the adhesion sets of H_i cover H_i for $i = 1, 2$.*

Proof. We will modify the tree amalgamation $G_1 *_T G_2$ and the involved graphs G_1, G_2 so that the resulting graphs and tree amalgamation satisfy our assertions. We will do this step by step. I. e., first we modify them so that (1)–(3) are true, then modify the resulting so that (1)–(4) are true and finally modify a last time to satisfy all five statements.

First, choose for each edge $uv \in E(T)$ such that u gets replaced by a copy of G_1 when moving to $G_1 + G_2$ a vertex x_{uv} in the adhesion set in G_1^u belonging to this edge. We do this so that our choices are invariant under $\mathrm{Aut}(G_1 + G_2)$. Now we delete in $G_1 + G_2$ all edges between copies of G_1 and copies of G_2 except for those between x_{uv} and $\varphi_{k,\ell}(x_{uv})$ where $(k, \ell) = c(uv)$, where φ and c are given by the definition of $G_1 *_T G_2$. Since (G_1, G_2) is a factorisation of G, there is a unique orbit of adhesion sets. Thus, all adhesion sets have bounded diameter and it follows that the identity map on $V(G_1 + G_2)$ is a quasi-isometry between these two graphs. Contracting all edges between the copies of G_1 and G_2 leads to a tree amalgamation F_1 of G_1 and G_2 of adhesion 1 that is quasi-isometric to G. It follows from the choice of the vertices x_{uv} that F_1 is quasi-transitive and (G_1, G_2) is a factorisation of F_1. So F_1, G_1, G_2 satisfy (1)–(3).

Now we replace each vertex x in G_1 that lies in some adhesion set by copies $x_1, \ldots, x_{n_x}$, where n_x denotes the number of adhesion sets in G_1 that contain x. Note that n_x is finite as the tree amalgamation has finite identification. We add

- all edges $x_i x_j$ for $1 \leq i, j \leq n_x$ with $i \neq j$ and for every x in adhesion sets,
- all edges $v x_i$ for every $1 \leq i \leq n_x$ and for every edge vx if v lies in no adhesion set but x lies in an adhesion set, and
- all edges $x_i y_j$ for all $1 \leq i \leq n_x$ and $1 \leq j \leq n_y$ and edges xy if $x \neq y$ lie in adhesion sets.

Let G_1' be the new graph. Let $\alpha_1 \colon G_1' \to G_1$ be the map that fixes all vertices of G_1' outside of adhesion sets and maps a vertex x_i to its origin x otherwise. Analogously, we define G_2' for the graph G_2 and a map $\alpha_2 \colon G_2' \to G_2$. It is easy to see that α_1 and α_2 are quasi-isometries. Since G_1 and G_2 are quasi-transitive, so are G_1' and G_2'. Choose the adhesion sets of G_i' such that they are mapped by α_i to the adhesion sets of G_i, have size 1, are disjoint and cover all vertices that get mapped into adhesion sets of G_i by α_i. Set $F_2 := G_1' * G_2'$. By construction, F_2 is quasi-transitive. Obiously, we can extend the maps α_1 and α_2 to a map $\alpha \colon G_1' + G_2' \to G_1 + G_2$

that is a quasi-isometry. By Remark 2.1, we obtain a quasi-isometry $F_2 \to G_1 * G_2$. Thus, F_2, G_1', G_2' satisfy (1)–(4).

As G is quasi-transitive, every vertex has finite distance to each of the finitely many orbits. Since each orbit of vertices either is covered by all adhesion sets or intersects them trivially, there exists $c_i \geq 0$ such that every vertex of G_i' has distance at most c_i to some vertex of the adhesion sets in G_i'. We define a new graph H_i whose vertex set consists of the vertices of G_i' that lie in adhesion sets. It has edges between every two vertices that have distance at most $2c_i + 1$ in G_i'. Note that this condition ensures that H_i is connected. The identity maps $V(H_1) \to V(G_1')$ and $V(H_2) \to V(G_2')$ are quasi-isometries that extend to a quasi-isometry $H_1 + H_2 \to G_1' + G_2'$. Since $\mathrm{Aut}(G_1'), \mathrm{Aut}(G_2')$ acts quasi-transitively on the adhesion sets in G_1', in G_2', respectively, the analogue is true for H_1 and H_2 since their construction is compatible with the automorphisms. Also, $H := H_1 * H_2$ is quasi-transitive and quasi-isometric to $H_1 + H_2$ by Remark 2.1. Then H, H_1, H_2 satisfy (1)–(5). $\qquad\square$

Lemma 3.2. *Let G and H be connected graphs. If G is finite and H infinite, then $G * H$ is quasi-isometric to $H * \ldots * H$, where the number of factors equals the number of adhesion sets of G.*

Proof. Let m be the number of adhesion sets in G. Consider the map that fixes in $G + H$ all vertices in copies of H and that maps the vertices in copies of G to some neighbour of that copy in a copy of H. Since G is finite, this map induces a quasi-isometry $G + H \to H + \ldots + H$ with m summands and its image is quasi-isometric to $H * \ldots * H$ with m factors. $\qquad\square$

Theorem 3.3. *Let F, G and H be connected quasi-transitive locally finite graphs and let T and T' be semi-regular trees with infinitely many ends. If F and G are quasi-isometric, then any locally finite tree amalgamations $F *_T H$ and $G *_{T'} H$ of finite adhesion and bounded identification are quasi-isometric.*

Proof. By Lemma 3.1, we may assume that the adhesion sets have size 1, are disjoint and cover F, G and H.

First, we consider the case that F is finite. Then G is finite, too. If H is finite, then with the notation used in the definition of tree amalgamations we can map F^v to v and H^w to w to obtain a quasi-isometry $F + H \to T$. Similarly, $G + H$ is quasi-isometric to T'. Note that finite identification implies that T and T' are locally finite since F, G and H are finite. Since T and T' have infinitely many ends, they are quasi-isometric by Lemma 2.9 and hence $F *_T H$ and $G *_{T'} H$ are quasi-isometric.

Now assume that H is infinite, but F is still finite. By Lemma 3.2, $F *_T H$ is quasi-isometric to $H * \ldots * H$ with $|F|$ factors and $G *_{T'} H$ is quasi-isometric to $H * \ldots * H$ with $|G|$ factors. By Lemma 2.5, these are quasi-isometric to $H +_{v,v} \ldots +_{v,v} H$ and $H +_{v,v} \ldots +_{v,v} H$, respectively, for some $v \in V(H)$. By Lemma 2.7, we know that these are quasi-isometric, and hence $F *_T H$ is quasi-isometric to $G *_{T'} H$.

Let us now consider the case that F and thus also G are infinite. We first assume that H is infinite. We apply Corollary 2.8 to see that $F * H$ is quasi-isometric to $F * H * H$ and $G * H$ is quasi-isometric to $G * H * H$. By replacing H by $H * H$, we may assume that H is a non-trivial tree amalgamation.

Let $\varphi \colon F \to G$ be a quasi-isometry. Let B be the image of φ and let $A \subseteq V(F)$ such that the restriction of φ to A is a bijection $A \to B$. Note that the vertices of F, of G have bounded distance to A, to B, respectively, as φ is a quasi-isometry.

We consider the graph $F +_{u,v} H$, where $u \in A$ is the base vertex of F and $v \in B$ is the base vertex of H. Let α map vertices inside copies F^t of F to copies of A such that $\sup\{d(x, \alpha(x)) \mid x \in \bigcup F^t\}$ is finite and such that every base vertex is the image of itself but of no other vertex. Note that α is finite-to-one as the graphs are locally finite.

Let us modify $F +_{u,v} H$. We replace every edge that is incident with a vertex x in a copy of F and a vertex y in a copy of H by a new edge $\alpha(x)y$. Since $d(x, \alpha(x))$ is bounded, the new graph X is quasi-isometric to $F +_{u,v} H$ and thus it is quasi-isometric to $F * H$ by Lemma 2.5. We equip A with a graph structure by adding edges between any two vertices with distance at most $\sup\{2d(x, \alpha(x)) + 1 \mid x \in \bigcup F^t\}$. As noted earlier, this results in a connected graph. Then A with this new metric is bilipschitz equivalent to A with the metric induced by F. We change C accordingly, i. e. we replace every copy of F by a copy of A with the new edges, and obtain a graph X'. Extending the map α by the identity on the copies of H, we obtain a quasi-isometry $F + H \to X'$.

We change X' in the following way: for each a in a copy of A we choose a neighbour u_a outside of the copies of A and we replace all edges au with u outside of copies of A by uu_a. Let Y be the resulting graph. By the choice of α, the base vertex of every copy of F has a unique neighbour outside of its copy of F. It follows that every component of Y with all copies of A deleted is a wedge of finitely many copies of H. As H is a non-trivial tree amalgamation, Lemma 2.6 shows that each of the components of Y with the copies of A removed is bilipschitz equivalent to H. So Lemma 2.5 implies that $F * H$ is quasi-isometric to $A +_{u,v} H$. Analogously, $G * H$ is quasi-isometric to $B +_{w,v} H$, where $w \in B$ is the base vertex of G. Since A and B are bilipschitz equivalent, Lemma 2.5 implies that $A +_{u,v} H$ and $B +_{w,v} H$ are quasi-isometric and so are $F * H$ and $G * H$.

If H is finite, then Lemma 3.2 implies that $F * H$ is quasi-isometric to $F * \ldots * F$ with $|H|$ copies and $G * H$ is quasi-isometric to $G * \ldots * G$ with $|H|$ copies. Then Corollary 2.8 implies that $F * H$ is quasi-isometric to $F * F$ and $G * H$ is quasi-isometric to $G * G$. The previous case with both F and H infinite shows that $F * F$ is quasi-isometric to $F * G$ which in turn is quasi-isometric to $G * G$, which completes the proof. $\qquad\square$

Theorem 3.3 is the major step in showing that the quasi-isometry type of a tree amalgamation does not depend on the particular adhesion sets, provided they are finite. Let us now show this result.

Corollary 3.4. *Let G_1 and G_2 be infinite connected locally finite quasi-transitive graphs and G and H be two non-trivial tree amalgamations of G_1 and G_2 both having finite adhesion and bounded identification. Assume that for $i = 1, 2$ some group of automorphisms of G_i acts quasi-transitively on G_i and on its adhesion sets for both tree amalgamations. Then G is quasi-isometric to H.*

Proof. Let $G = G_1 *_T G_2$ and $H = G_1 *_{T'} G_2$. Since G_i is infinite and some group acts quasi-transitively on its vertices and its adhesion sets, it follows from [6, Proposition 4.7] and the connection between tree amalgamations and tree-decompositions as discussed in [6, Section 5] that all degrees of T are infinite. As the tree amalgamations are non-trivial, both T and T' have infinitely many ends. Thus, the assertion follows from Theorem 3.3. $\qquad\square$

Before we turn our attention to the first part of Theorem 1.3, we prove another small corollary.

Corollary 3.5. *Let G be an infinite locally finite quasi-transitive connected graph and let (G_1, G_2) be a factorisation of G such that G_1 is infinite, G_2 is finite and the amalgamating tree is not a star. Then G is quasi-isometric to $G_1 * G_1$.*

Proof. By Lemma 3.2, G is quasi-isometric to $G_1 * \ldots * G_1$ with as many factors as there are adhesion sets in G_2. As the amalgamating tree is not a star, this number is at least 2. Applying Theorem 3.3 repeatedly implies that G is quasi-isometric to $G_1 * G_1$. $\qquad\qquad\square$

We are going to prove a slightly more technical version of the first statement of Theorem 1.3 that together with Theorem 3.3 will imply it.

Theorem 3.6. *Let G be a locally finite quasi-transitive graph with infinitely many ends and let $(G_1, \ldots, G_m)$ be a factorisation of G. Let $H_1, \ldots, H_n$ be infinite factors in $(G_1, \ldots G_m)$ consisting of one representative per infinite quasi-isometry type. Then exactly one of the following is true.*
(1) *If $H := H_1 * \ldots * H_n$ has infinitely many ends, then G is quasi-isometric to H.*
(2) *If H does not have infinitely many ends, but there exists a finite graph H_{fin} such that some non-trivial tree amalgamation $H * H_{\mathrm{fin}}$ has infinitely many ends, then $H * H_{\mathrm{fin}}$ is quasi-isometric to G.*
(3) *G is quasi-isometric to a 3-regular tree.*

Proof. We prove the assertion by induction on the number n of infinite quasi-isometry types of $(G_1, \ldots, G_m)$. If $n = 0$, then G is quasi-isometric to a 3-regular tree by Lemma 2.10. Let $n = 1$. We distinguish the cases whether $(G_1, \ldots, G_m)$ has one or more than one infinite factor. Let us first assume that $(G_1, \ldots, G_m)$ has only one infinite factor, say G_1. If it has no finite factor, the assertion follows immediately. So we may assume that $m \geq 2$. We may assume $G = (\ldots (G_1 * G_2) * \ldots * G_m)$. Repeatedly applying Corollary 3.5 shows that G is quasi-isometric to $G_1 * \ldots * G_1$ with $m \geq 2$ factors. Applying Lemma 2.5 and then Lemma 2.7 repeatedly, implies that G is quasi-isometric to $G_1 +_{u,u} G_1$ for the base vertex u of G_1. Now if G_1 does not have infinitely many ends, we obtain analogously that for any finite graph H_{fin} and non-trivial tree amalgamation $G_1 * H_{\mathrm{fin}}$ we have that $G_1 * H_{\mathrm{fin}}$ is quasi-isometric to $G_1 * G_1$, which shows the assertion in this case. If G_1 has infinitely many ends, then it has a factorisation (G_1^1, G_1^2). If both factors are infinite, we can apply Lemmas 2.5 and 2.7 to see that $G_1 +_{u,u} G_1$ is quasi-isometric to G_1, since Lemma 2.7 shows that $(G_1^1 +_{u,v} G_1^2) +_{u,u} (G_1^1 +_{u,v} G_1^2)$ is quasi-isometric to $G_1^1 +_{u,v} G_1^2$. If G_1^1 is infinite but G_1^2 is finite, G_1 is quasi-isometricto $G_1^1 * G_1^1$ by Corollary 3.5. But then Lemmas 2.5 and 2.7 show that $G_1^1 * G_1^1$ is quasi-isometric to $(G_1^1 * G_1^1) +_{u,u} (G_1^1 * G_1^1)$, which is quasi-isometric to $G_1 * G_1$ by Lemma 2.5. If G_1^1 and G_1^2 are finite, then G_1 is quasi-isometric to a 3-regular tree by Lemma 2.10 and so is $G_1 * G_1$. So by Lemma 2.9, G_1 is quasi-isometric to $G_1 * G_1$. Thus, we have shown the assertion in the case that $(G_1, \ldots, G_m)$ has only one infinite factor.

So let us assume that $(G_1, \ldots, G_m)$ has more than one infinite factor. Since $n = 1$, all of them are in the same quasi-isometry class. By Theorem 3.3 and Corollary 3.5, we may assume that $G = H_1 * \ldots * H_1$. Corollary 2.8 implies that G is quasi-isometric to $H_1 * H_1$. If H_1 is one-ended, let H_{fin} be a finite graph and T be a semi-regular tree with infinitely many ends such that $H_{\mathrm{fin}} *_T H_1$ has infinitely

many ends. It follows that $H_{\mathrm{fin}} *_T H_1$ is non-trivial. By Corollary 3.5, $H_{\mathrm{fin}} *_T H_1$ is quasi-isometric to $H_1 * H_1$. So $H_{\mathrm{fin}} *_T H_1$ is quasi-isometric to G.

If H_1 has more than one end, then it splits as a tree amalgamation $H^1 * H^2$ by Theorem 1.1. If H^1 and H^2 are finite, then $H = H_1$ is quasi-isometric to a 3-regular tree and so is G by Lemma 2.10. Hence, G is quasi-isometric to H. If H^1 and H^2 are infinite, then $H_1 = H^1 * H^2$ is quasi-isometric to $(H^1 * H^1) * (H^2 * H^2) = H_1 * H_1$ by Corollary 2.8. Theorem 3.3 implies that G is quasi-isometric to $G_{\mathrm{fin}} * H_1$ for any finite G_{fin} such that $G_{\mathrm{fin}} * H_1$ is non-trivial. If H^1 is finite and H^2 is infinite, then Corollary 3.5 implies that H_1 is quasi-isometric to $H^2 * H^2$. So Corollary 2.8 implies that H_1 is quasi-isometric to $H_1 * H_1$. As G is quasi-isometric to $H_1 * H_1$, it is quasi-isometric to H_1.

Let us now assume that $n \geq 2$. For $i = 0, \ldots, n$, let $G_i^1, \ldots, G_i^{k_i}$ be the factors of $(G_1, \ldots, G_m)$ that are quasi-isometric to H_i, where H_0 is any connected finite graph. Then

$$G = G_0^1 * \ldots * G_0^{k_0} * \ldots * G_n^1 * \ldots * G_n^{k_n}.$$

Corollary 3.5 shows that we can replace each finite factor by G_1^1. So we may assume that $k_0 = 0$. If $n = 2$ and $k_1 = 1$, then we apply Corollary 2.8 and obtain that G is quasi-isometric to $G_1^1 * G_2^1$. Thus, Theorem 3.3 implies that $G_1^1 * G_2^1$, and hence G, is quasi-isometric to $H_1 * H_2 = H$. If $n \geq 3$ or $k_1 \geq 2$, then $G' := G_1^1 * \ldots * G_{n-1}^{k_{n-1}}$ has infinitely many ends and by induction it is quasi-isometric to either $H_1 * \ldots * H_{n-1}$ or $G_{\mathrm{fin}} * H_1$ for any finite G_{fin} such that $G_{\mathrm{fin}} * H_1$ is non-trivial. Once again, Corollary 3.5 shows that we can replace G_{fin} by H_1 and thus G' is quasi-isometric to either $H_1 * \ldots * H_{n-1}$ or $H_1 * H_1$. By Theorem 3.3 we may assume that $G_n^j = H_n$ for every $1 \leq j \leq k_n$. By applying Corollary 2.8, we reduce the multiple factors of H_1 and H_n to just one each and obtain that G is quasi-isometric to H. $\qquad\square$

4. Accessibility and quasi-isometries

In this section we shall prove Theorem 4.2. The first step is to see that if a one-ended connected quasi-transitive locally finite graph embeds quasi-isometrically into a tree amalgamation, then it embeds already quasi-isometrically into one of the factors.

Lemma 4.1. *Let G and H be connected quasi-transitive locally finite graphs and let (G_1, G_2) be a factorisation of G. If H has precisely one end and embeds quasi-isometrically into G, then it embeds quasi-isometrically into either G_1 or G_2.*

Proof. Let $\varphi\colon H \to G$ be a (γ, c)-quasi-isometric embedding. Let S be an adhesion set in G and let S' be the set of vertices of G of distance at most $\gamma + c$ from S. If there are vertices of $\varphi(H)$ in different components of $G - S'$, then their preimages are not connected in $H - \varphi^{-1}(S')$ by the choice of S', and as H is one-ended and S' finite, there is only one infinite component C_S^∞ of $H - \varphi^{-1}(S')$ and only finitely many finite components. So we find Δ_S such that the images of all vertices of finite components of $H - \varphi^{-1}(S')$ have distance at most Δ_S from S. Note that since we have only finitely many orbits of adhesion sets, the numbers Δ_S are globally bounded by some Δ.

Let uv be an edge of T. Then $G_i^u \cap G_j^v$ is an adhesion set. The infinite component of $C_{G_i^u \cap G_j^v}^\infty$ gets mapped into either

$$G_{T_u} := \bigcup_{a \in V(T_u)} G_i^a \qquad \text{or} \qquad G_{T_v} := \bigcup_{a \in V(T_v)} G_i^a$$

but not into both, where T_w is the component of $T - uv$ that contains w for $w \in \{u, v\}$. We orient the edge uv towards u if the infinite component gets mapped into G_{T_u} and we orient it towards v otherwise. It is easy to see that every vertex has at most one out-going edge and that there is at most one vertex without outgoing edges in this orientation of T. To see that there is at least one sink, let us suppose that this is not the case. Then there is a directed ray in the orientation of T. As $G_1 * G_2$ has finite identification, every $\varphi(a)$ for $a \in V(H)$ lies eventually outside the adhesion sets on that ray and for every adhesion set on that ray, we find a later one that is disjoint from it. Obviously, there is an adhesion set S on that ray separating $\varphi(a)$ from C_S^∞. But this is impossible as a must lie eventually within the Δ-neighbourhood of all later adhesion sets of that ray. So T has a unique sink G_i^x such that every vertex of H gets mapped by φ into the Δ-neighbourhood of G_i^x. We can easily modify φ and obtain a maps φ' that maps H quasi-isometrically into G_i^x with respect to the metric of G. But since G_i^x has only finitely many orbits of adhesion sets, φ' also maps H quasi-isometrically into G_i^x with the metric of G_i^x. $\square$

Theorem 4.2. *Let G be a connected accessible locally finite quasi-transitive graph. A connected locally finite quasi-transitive graph H is quasi-isometric to G if and only if it satisfies the following conditions:*

  (i) *H has the same number of ends as G;*
 (ii) *H is accessible; and*
(iii) *any terminal factorisation of H has the same set of quasi-isometry types of one-ended factors as any terminal factorisation of G.*

Proof. Let us first assume that H satisfies (i) to (iii). If G and H have infinitely many ends, then it follows directly from Theorem 3.6 with Theorem 3.3 that G is quasi-isometric to H. If each has exactly two ends, then no factor can be one-ended, so all are finite and hence G and H are quasi-isometric to the double ray and thus quasi-isometric to each other. If each has exactly one end, then each of the two terminal factorisations has at most one one-ended factor and all others are finite as any tree amalgamation of finite adhesion and finite identification of two or more (quasi-transitive) locally finite graphs has at least two ends that lie in distinct copies of the factors and are separated by a finite adhesion set. Moreover, a tree amalgamation of a one-ended graph and a finite graph is one-ended only if the non-trivial amalgamation tree is a star, i. e. a tree of diameter at most 2. For such tree amalgamations the finite extension of the one-ended factor by the finite factor is just the tree amalgamation itself and thus the tree amalgamation is quasi-isometric to the one-ended factor by Remark 2.2. Thus, G is also quasi-isometric to H in this case.

Now let us assume that G is quasi-isometric to H. In [6, Theorem 6.3], it was shown that accessibility is equivalent to the existence of some $n \in \mathbb{N}$ such that every two ends can be separated by at most n vertices, which is the accessibility as in the sense of Thomassen and Woess [11]. Since this latter property is invariant

under quasi-isometries, so is accessibility. Since the number of ends is preserved by quasi-isometries, too, it only remains to prove (iii). Let $(G_1, \ldots, G_n)$ be a terminal factorisation of G. Then there exist $F_1, \ldots, F_{n-1}$ such that $G = F_{n-1}$ and such that for every $i \leq n-1$ the graph F_i is a tree amalgamation of finite adhesion with factors in

$$\{G_j \mid 1 \leq j \leq n\} \cup \{F_j \mid 1 \leq j < i\}.$$

Let H' be a factor in a terminal factorisation $(H_1, \ldots, H_m)$ of H. By Remark 2.1, H' maps quasi-isometrically into H and thus into G. Applying Lemma 4.1 recursively, we conclude that for some G_i there is a quasi-isometric embedding $\varphi \colon H' \to G_i$. Similarly, G_i embeds quasi-isometrically into some factor F of the terminal factorisation $(H_1, \ldots, H_m)$ of H by a map ψ with $\psi \circ \varphi = id$. Since $\psi \circ \varphi = id$, we know that F must be mapped by $\psi \circ \varphi$ into H'. As both are one-ended, we conclude $F = H'$. Thus, H' is quasi-isometric to G_i. □

We are now ready to combine our results in order to obtain Theorem 1.3.

Proof of Theorem 1.3. Let G and H be connected locally finite quasi-transitive graphs with infinitely many ends and let $(G_1, \ldots, G_n), (H_1, \ldots, H_m)$ be factorisations of G, H, respectively. Let $F_1, \ldots, F_\ell$ be representatives of the quasi-isometry types of infinite factors in $(G_1, \ldots, G_n)$. First, we assume that $F_1, \ldots, F_\ell$ are also the representatives of the quasi-isometry types of infinite factors in $(H_1, \ldots, H_m)$. If there are no infinite factors, then both graphs G and H are quasi-isometric to the 3-regular tree by Theorem 3.6 (3), so G is quasi-isometric to H in that case. If $F := F_1 * \ldots * F_\ell$ has infinitely many ends, then Theorem 3.6 (1) implies that G and H are quasi-isometric to F, so again G is quasi-isometric to H. If F does not have infinitely many ends, then for every finite graph F_{fin} such that $F * F_{\text{fin}}$ has infinitely many ends, G is quasi-isometric to $F * F_{\text{fin}}$, which is quasi-isometric to H. So we also have that G and H are quasi-isometric in this last case, which finishes the proof of the first statement of Theorem 1.3.

For the second statement, assume that G and H are quasi-isometric, that each of them is accessible and that $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_n)$ are terminal factorisations. By Theorem 4.2 (iii), the set of quasi-isometry types of one-ended factors of both factorisations are the same. □

5. Quasi-isometries between transitive graphs and Cayley graphs

Woess [12, Problem 1] asked whether there are transitive locally finite graphs that are not quasi-isometric to some locally finite Cayley graph. Eskin et al. [3] showed that the Diestel-Leader graphs are examples of transitive graphs that are not quasi-isometric to some locally finite Cayley graph. Since the Diestel-Leader graphs are one-ended, the question arises what can be said about Woess' question for graphs that need not have one-ended graphs as building blocks in our tree amalgamation sense, i.e. inaccessible graphs. Dunwoody [2] constructed an inaccessible locally finite transitive graph that is another example for a negative answer to Woess' question. As an application of our previous results, we obtain that one-ended and inaccessible examples are basically the only ones you have to consider when understanding the quasi-isometry differences between locally finite transitive graphs and locally finite Cayley graphs.

Theorem 5.1. *Let G be a locally finite transitive accessible graph that is not quasi-isometric to any locally finite Cayley graph. Then there is a one-ended locally finite*

transitive graph that is quasi-isometric to some factor in a terminal factorisation of G and that is not quasi-isometric to any Cayley graph.

Proof. If G has precisely two ends, then it is quasi-isometric to the double ray by Theorem 1.2, which is a locally finite Cayley graph. So we may assume that G has infinitely many ends. Let $(G_1, \ldots, G_n)$ be a terminal factorisation of G. Note that every G_i is quasi-transitive and thus quasi-isometric to some transitive locally finite graph, see e. g. Krön and Möller [7, Theorem 5.2]. Suppose that every G_i is quasi-isometric to some locally finite Cayley graph H_i. Then $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_n)$ have the same quasi-isometry types of infinite factors. Let $\Gamma_i = \langle S_i \mid R_i \rangle$ be a group that has H_i as Cayley graph. Let Γ be the free product of all Γ_i with presentation $\langle S \mid R \rangle$, where $S = \bigcup_{1 \leq i \leq n} S_i$ and $R = \bigcup_{1 \leq i \leq n} R_i$. Then the Cayley graph H of Γ with respect to $\langle S \mid R \rangle$ is the tree amalgamation of all H_i, i. e. $H = (\ldots (H_1 * H_2) * \ldots) * H_n$. By Theorem 1.3, G is quasi-isometric to H, which is a contradiction. $\qquad\square$

We can also restrict Woess' question to certain classes of graphs and groups. If this class is invariant under quasi-isometries, factorisations and tree amalgamations, then the proof of Theorem 5.1 stays true for it.

One such class are the hyperbolic graphs and group: a graph is *hyperbolic* if there is some $\delta \geq 0$ such that for every three vertices x, y, z and every three paths, one between each pair of $\{x, y, z\}$, each of the paths lies in the δ-neighbourhood of the union of the other two paths; and a finitely generated group is *hyperbolic* if it has a locally finite hyperbolic Cayley graph.

It follows from the definition that hyperbolicity is preserved under quasi-isometries. By [5, Theorem 1.1], hyperbolicity is also preserved under factorisations and tree amalgamations. Additionally, hyperbolic quasi-transitive locally finite graphs are accessible by [4, Theorem 4.3]. Thus, if there is a hyperbolic locally finite quasi-transitive graph that shows that Woess' question is false, then there is already a one-ended such graph.

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Matthias Hamann, Mathematics Institute, University of Warwick, Coventry, UK
Current address: Matthias Hamann, Department of Mathematics, University of Hamburg, Hamburg, Germany