The planar Cayley graphs are effectively enumerable II

Agelos Georgakopoulos*[1] and Matthias Hamann†[2]

[1]Mathematics Institute, University of Warwick, CV4 7AL, UK
[2]Department of Mathematics, University of Hamburg, Germany

December 15, 2022

Abstract

We show that a group admits a planar, finitely generated, Cayley graph if and only if it admits a special kind of group presentation we introduce, called a planar presentation. Planar presentations can be recognised algorithmically. As a consequence, we obtain an effective enumeration of the planar Cayley graphs, yielding in particular an affirmative answer to a question of Droms et al. asking whether the planar groups can be effectively enumerated.

1 Introduction

In this paper we complete an effort, started in [13], and building upon [10, 11], the aim of which is to understand the planar Cayley graphs. In [13] we handled the special case of 3-connected Cayley graphs, and more generally, Cayley graphs G that admit a *consistent* embedding in $\mathbb{R}^2$, that is, an embedding the facial paths of which are preserved by the action on G by its group (see Section 2.3 for a more detailed definition). The groups having such Cayley graphs are exactly the Kleinian function groups, or equivalently, those groups admitting a faithful, properly discontinuous, action by homeomorphisms on a 2-manifold contained in $\mathbb{S}^2$ [12, 17]. For more information about this classical class of groups see e.g. [19, 20, 21, 22]. However, there are planar Cayley graphs

*Supported by EPSRC grant EP/L002787/1, and by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 639046). The first author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme 'Random Geometry' where work on this paper was undertaken.

†Supported by the Heisenberg-Programme of the Deutsche Forschungsgemeinschaft (DFG Grant HA 8257/1-1).
Both authors have been supported by FWF grant P-19115-N18.

the groups of which cannot act faithfully and properly discontinuously on $\mathbb{S}^2$ [12]. Therefore, the aforementioned groups form a proper subclass of the *planar groups*, i.e. the groups admitting a planar Cayley graph. In this paper we broaden the group presentations introduced in [13] so that we capture exactly the planar, locally finite, Cayley graphs. In particular, we capture the planar finitely generated groups, and we show that they can be effectively enumerated, answering a question of Droms et al. [5, 7].

The *Cayley complex* X corresponding to a group presentation $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is the 2-complex obtained from the Cayley graph G of $\mathcal{P}$ by glueing a 2-cell along each closed walk of G induced by a relator $R \in \mathcal{R}$. We say that X is *almost planar*, if it admits a map $\rho : X \to \mathbb{R}^2$ such that the 2-simplices of X are *nested* in the following sense. We say that two 2-simplices of X are nested, if the images of their interiors are either disjoint, or one is contained in the other, or their intersection is the image of a 2-cell bounded by two parallel edges corresponding to an involution $s \in \mathcal{S}$.[1] We call the presentation $\mathcal{P}$ a *planar presentation* if its Cayley complex is almost planar. We will show that every planar, finitely generated, Cayley graph admits a planar presentation. However, we prove something much stronger than that. We are going to introduce a specific type of planar presentation, called a *general planar presentation*, and show that every planar, finitely generated, Cayley graph admits such a presentation and, conversely, every general planar presentation has a planar Cayley graph (Theorem 5.10). This converse is the hardest result of this paper. Our main result is:

Theorem 1.1. *A finitely generated group admits a planar Cayley graph if and only if it admits a general planar presentation.*

The main idea of its proof is that if two relators in a presentation induce cycles whose interiors overlap but are not nested (in a sense similar to the nestedness of 2-simplices), then we replace a subword of one relator by a subword of the other to produce an equivalent presentation with less overlapping; our proof that a presentation with no such overlaps exists is based on a dual version of the machinery of Dunwoody cuts [3], but for cycles instead of cuts.

As a corollary of Theorem 1.1 we obtain that the planar, locally finite, Cayley graphs, and hence their groups, can be effectively enumerated (Theorem 6.3). This answers a question of Droms et al. [5, 7] and generalises a result of Renault [24] that verifies the question of Droms et al. in the case of planar Cayley graphs that can be embedded into the plane without accumulation points of the vertices. M. Dunwoody (private communication) informs us that the fact that the planar groups can be effectively enumerated should also follow from his result [8, Theorem 3.8] with a little bit of additional work (the main issue here is whether the 'or a subgroup of index two' proviso can be dropped).

For more on the motivation of this work and the general background we refer the reader to [13]. We think of this paper as a continuation of the latter, and although we have made an effort to keep it as self-contained as possible, some

[1] The third option can be dropped by considering the *modified* Cayley complex in the sense of [18], i.e. by representing involutions in $\mathcal{S}$ by single, undirected edges.

familiarity with [13] may be necessary. The reader willing to check all details of the backward direction of Theorem 1.1 is expected to have read the proof of [13, Theorem 3.3] as we explain in Section 5.1.

1.1 Planar presentations

The formal definition of a general planar presentation is given in Section 6 and it is a slight generalisation of the notion of a *generic* planar presentation defined in Section 3. Here, we are going to sketch the most interesting special case of this concept, called a special planar presentation. Such presentations always exist for a 3-connected planar Cayley graph, or more generally, for a Cayley graph that can be embedded in the plane in such a way that its label-preserving automorphisms carry facial paths to facial paths.

We say that a (finite) group presentation $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is a *special planar presentation*, if it can be endowed with a cyclic ordering σ —from now on called a *spin*— of the symmetrization $\mathcal{S}' = \{s, s^{-1} \mid s \in \mathcal{S}\}$ of its generating set, with the following property. Suppose $W_1 = sUt$ and $W_2 = s'Ut'$, where $s, s', t, t' \in \mathcal{S}'$, are two words, each contained in some rotation of a relator in $\mathcal{R}$ (possibly the same relator), where U is any (possibly trivial) word with letters in $\mathcal{S}'$. Observe that σ allows us to answer the following question: if we could embed the Cayley graph of $\mathcal{P}$ in the plane in such a way that for every vertex the cyclic ordering of the labels of its incident edges we observe coincides with σ, do the paths induced by these words W_1, W_2 cross each other or not? To make this more precise, we embed a tree consisting of a 'middle' path P with edges labelled by the letters in U, and two leaves attached at each endvertex of P labelled with s, s', t, t' as in Figure 1, where the spin we use at each endvertex of P is the one induced by σ on the corresponding 3-element subset of $\mathcal{S}'$. There are essentially two situations that can arise, both shown in that figure. Naturally, we say that W_1, W_2 *cross* each other in the right-hand situation, and they do not in the left-hand one.

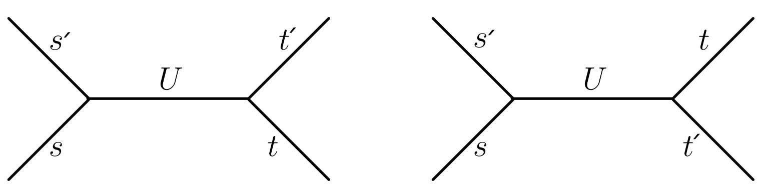

Figure 1: The definition of *crossing*; $W_1 = sUt$ crosses $W_2 = s'Ut'$ in the right, but not in the left.

We then say that $\mathcal{P}$ is a special planar presentation, if there is a spin σ on $\mathcal{S}'$ such that no two words as above cross each other. Note that this is an abstract property of sets of words, and it is defined without reference to the Cayley graph of $\mathcal{P}$; in fact, it can be checked algorithmically. The essence of this paper is that this is enough to guarantee the planarity of the Cayley graph, and that a converse statement holds.

This generalises an idea from [9], where it was shown that every planar discontinuous group admits a special planar presentation where every relator is *facial*, i.e. it crosses no other word (where we consider words that are not necessarily among our relators).

Our actual definition of a special planar presentation, given in Section 3.1, is in fact a bit more general than the above sketch. Consider for example the Cayley graph of the presentation $\langle a, b \mid a^n, b^2, aba^{-1}b \rangle$. Its Cayley graph is a prism graph with an essentially unique embedding in $\mathbb{R}^2$. Note that the spin of half of its vertices is the reverse of the spin of other half. This is a general phenomenon: every 3-connected Cayley graph has an essentially unique embedding, and in that embedding all vertices have the same spin *up to reflection*. However, for every generator s, either the two end-vertices of all edges labelled s have the same spin, or they always have reverse spins. This yields a classification of generators into *spin-preserving* and *spin-reversing* ones, and our definition of a special planar presentation takes this into account; still, everything can be checked algorithmically.

The situation becomes much more complex however if one wants to consider planar Cayley graphs that are not 3-connected. Such graphs do not always have an embedding with all vertices having the same spin up to reflection; perhaps the simplest such example is the one of Figure 2.

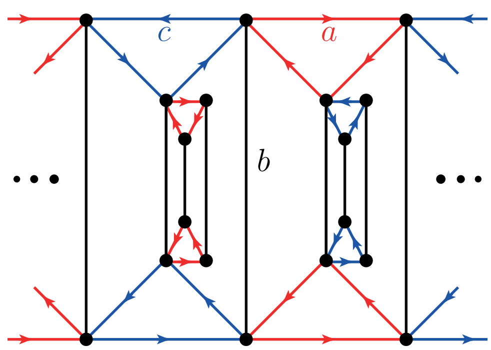

Figure 2: A 2-connected planar Cayley graph of Droms et al. [7], obtained by amalgamating two 6-element groups along an involution, which does not admit a consistent embedding.

In order to capture such Cayley graphs we had to come up with the notion we call a *general planar presentation* (defined in Section 6), which in particular translates, into abstract, algorithmically checkable, properties of words as above situations as in Figure 2, where a certain generator s with $s^2 = 1$ separates the graph into two parts, and behaves in a spin-preserving way in one part and in a spin-reversing way in the other part. That such general planar presentations always give rise to planar Cayley graphs is the hardest result of this paper, many of its complications arising from the fact that given a general planar presentation with such a 'separating' generator s, it is impossible to predict whether $s = 1$, which would imply that our Cayley graph does not quite have the structure anticipated by the presentation. The situation is complicated further by the fact that separating generators need not be involutions; an example is given in Figure 3.

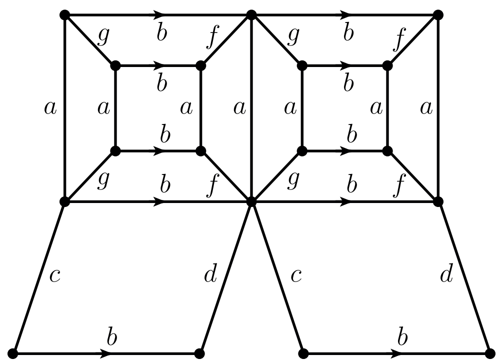

Figure 3: An (infinite) planar Cayley graph, corresponding to the presentation $\langle a, b, c, d, f, g \mid a^2, c^2, d^2, f^2, g^2, (af)^2, (ag)^2, abab^{-1}gbfb^{-1}, cbdb^{-1} \rangle$, with a separating edge b which is not an involution.

This paper is structured as follows. After some general definitions in Section 2, we introduce generic planar presentations in Section 3, and show that every Cayley graph of every generic planar presentation is planar in Section 5. In Section 4 we prove the reverse direction, i. e. that every planar Cayley graph admits a generic planar presentation. In Section 6 we slightly generalise from *generic* to *general* planar presentations, and put those facts together to obtain the results stated above. We finish with some open problems in Sections 6 and 7.

2 Definitions

2.1 Graph-theoretical concepts

Let $G = (V, E)$ be a connected graph fixed throughout this section. The reader may assume that all graphs in this paper are locally finite, although parts of our proofs extend almost verbatim to graphs with countably infinite vertex degrees. We chose to focus on locally finite Cayley graphs because an effective enumeration as in Theorem 6.3 can only be carried out for those and because working with spins becomes technical in the presence of infinite degrees.

A *walk* is a sequence of vertices $W = x_1x_2 \dots x_n$ of G such that $x_ix_{i+1} \in E$ for every $i < n$. A *path* can be thought of as a walk all vertices of which are distinct, although we will not distinguish between its two possible directions. Two paths in G are *independent*, if they do not meet at any vertex except perhaps at common endpoints. A *cycle* is a walk $x_1x_2 \dots x_n$ such that $x_1 = x_n$ and $x_i \neq x_j$ for all other pairs of values i, j.

If P is a path or cycle we will use $|P|$ to denote the number of vertices in P and $||P||$ to denote the number of edges of P. Let xPy denote the subpath of P between its vertices x and y.

A *hinge* of G is an edge $e = xy$ such that the removal of the pair of vertices x, y disconnects G. A hinge should not be confused with a *bridge*, which is an edge whose removal separates G although its endvertices are not removed.

The set of neighbours of a vertex x is denoted by $N(x)$.

G is called *k-connected* if $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. Note that if G is k-connected then it is also $(k-1)$-connected. The *connectivity* $\kappa(G)$ of G is the greatest integer k such that G is k-connected.

A 1-way infinite path is called a *ray* and a *double ray* is a directed 2-way infinite path.

The set of all finite sums of edge-sets of (finite) cycles forms a vector space over $\mathbb{F}_2$, the *cycle space* of G.

2.2 Cayley graphs and group presentations

We will follow the terminology of [4] for graph-theoretical terms and that of [2] and [23] for group-theoretical ones. Let us recall the definitions most relevant for this paper.

A *group presentation* $\langle \mathcal{S} \mid \mathcal{R} \rangle$ consists of a set $\mathcal{S}$ of distinct symbols, called the *generators* and a set $\mathcal{R}$ of words with letters in $\mathcal{S} \cup \mathcal{S}^{-1}$, where $\mathcal{S}^{-1}$ is the set of symbols $\{s^{-1} \mid s \in \mathcal{S}\}$, called *relators*. Each such group presentation uniquely determines a group, namely the quotient group $F_\mathcal{S}/N$ of the (free) group $F_\mathcal{S}$ of words with letters in $\mathcal{S} \cup \mathcal{S}^{-1}$ over the (normal) subgroup $N = N(\mathcal{R})$ generated by all conjugates of elements of $\mathcal{R}$.

The Cayley graph $Cay(\mathcal{P}) = Cay \langle \mathcal{S} \mid \mathcal{R} \rangle$ of a group presentation $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is an edge-coloured directed graph $G = (V, E)$ constructed as follows. The vertex set of G is the group $\Gamma = F_\mathcal{S}/N$ corresponding to $\mathcal{P}$. The set of *colours* we will use is $\mathcal{S}$. For every $g \in \Gamma, s \in \mathcal{S}$ join g to gs by an edge coloured s directed from g to gs. Note that Γ acts on G by multiplication on the left; more precisely, for every $g \in \Gamma$ the mapping from $V(G)$ to $V(G)$ defined by $x \mapsto gx$ is an *automorphism* of G, that is, an automorphism of G that preserves the colours and directions of the edges. In fact, Γ is precisely the group of such automorphisms of G. Any presentation of Γ in which $\mathcal{S}$ is the set of generators will also be called a presentation of $Cay(\mathcal{P})$.

Note that some elements of $\mathcal{S}$ may represent the identity element of Γ, and distinct elements of $\mathcal{S}$ may represent the same element of Γ; therefore, $Cay(\mathcal{P})$ may contain loops and parallel edges of the same colour.

If $s \in \mathcal{S}$ is an *involution*, i.e. $s^2 = 1$, then every vertex of G is incident with a pair of parallel edges coloured s (one in each direction). If s^2 is a relator in $\mathcal{R}$, we will follow the convention of replacing this pair of parallel edges by a single, undirected edge. This convention is common in the literature [18], and is convenient when studying planar Cayley graphs.

If G is a Cayley graph, then we use $\Gamma(G)$ to denote its group.

If R is any (finite or infinite) word with letters in $\mathcal{S} \cup \mathcal{S}^{-1}$, and g is a vertex of $G = Cay\langle \mathcal{S} \mid \mathcal{R} \rangle$, then starting from g and following the edges corresponding to the letters in R in order we obtain a walk W in G. We then say that W is *induced* by R at g, and we will sometimes denote W by gR; note that for a given R there are several walks in G induced by R, one for each starting vertex $g \in V(G)$.

Let $H_1(G)$ denote the first simplicial homology group of G over $\mathbb{Z}$. We will use the following well-known fact which is easy to prove.

Lemma 2.1. *Let $G = Cay\langle \mathcal{S} \mid \mathcal{R} \rangle$ be a Cayley graph. Then the (closed) walks in G induced by relators in $\mathcal{R}$ generate $H_1(G)$.*

2.3 Embeddings in the plane

An *embedding* of a graph G will always mean a topological embedding of the corresponding 1-complex in the euclidean plane $\mathbb{R}^2$; in simpler words, an embedding is a drawing in the plane with no two edges crossing.

A *face* of an embedding $\rho : G \to \mathbb{R}^2$ is a component of $\mathbb{R}^2 \setminus \rho(G)$. The *boundary* of a face F is the set of vertices and edges of G that are mapped by ρ to the closure of F. The *size* of F is the number of edges in its boundary. Note that if F has finite size then its boundary is a cycle of G.

A walk in G is called *facial* with respect to ρ if it is contained in the boundary of some face of ρ.

An embedding of a Cayley graph is called *consistent* if, intuitively, it embeds every vertex in a similar way in the sense that the group action carries faces to faces. Let us make this more precise. Given an embedding ρ of a Cayley graph G with generating set S, we consider for every vertex x of G the embedding of the edges incident with x, and define the *spin* of x to be the cyclic ordering of the set $L := \{xy^{-1} \mid y \in N(x)\}$ in which xy_1^{-1} is a successor of xy_2^{-1} whenever the edge xy_2 comes immediately after the edge xy_1 as we move clockwise around x. Note that L coincides with $S \cup S^{-1}$ and hence depends only on S and on our convention on whether to draw one or two edges per vertex for involutions. This allows us to compare spins of different vertices. Call an edge of G *spin-preserving* if its two endvertices have the same spin in ρ, and call it *spin-reversing* if the spin of one of its endvertices is the reverse of the spin of its other endvertex. Call a colour in S *consistent* if all edges bearing that colour are spin-preserving or all edges bearing that colour are spin-reversing in ρ. Finally, call the embedding ρ *consistent* if every colour is consistent in ρ. Note that if ρ is consistent, then there are only two types of spin in ρ, and they are the reverse of each other.

The following classical result was proved by Whitney [28, Theorem 11] for finite graphs and by Imrich [16] for infinite ones.

Theorem 2.2. *Let G be a 3-connected graph embedded in the sphere. Then every automorphism of G maps each facial path to a facial path.*

This implies in particular that if ρ is an embedding of the 3-connected Cayley graph G, then the cyclic ordering of the colours of the edges around any vertex of G is the same up to orientation. In other words, at most two spins are allowed in ρ. Moreover, if two vertices x, y of G that are adjacent by an edge, bearing a colour b say, have distinct spins, then any two vertices x', y' adjacent by a b-edge also have distinct spins. We just proved

Lemma 2.3. *Let G be a 3-connected planar Cayley graph. Then every embedding of G is consistent.*

Cayley graphs of connectivity 2 do not always admit a consistent embedding [7]. However, in the cubic case they do; see [11].

An embedding is *Vertex-Accumulation-Point-free*, or *accumulation-free* for short, if the images of the vertices have no accumulation point in $\mathbb{R}^2$.

A *crossing* of a path X by a path or walk Y in a plane graph is a subwalk $Q = e\mathring{Q}f$ of Y where the end-edges e, f of Q are incident with X on opposite sides of X (but not contained in X) and (the image of) $\mathring{Q}$ is contained in X (Figure 4). Note that $\mathring{Q}$ may be a trivial path and note that if Q is a crossing of X by Y, then X contains a crossing $Q' = g\mathring{Q}h$ of Y by X, which we will call the *dual crossing* of Q.

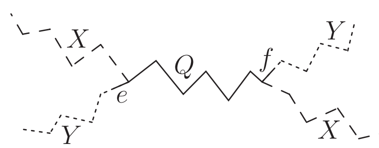

Figure 4: A crossing of X by Y.

For a closed walk W and $n \in \mathbb{N}$, let W^n be the n-times concatenation of W with itself. Two closed walks R and W *cross* if there are $i, j \in \mathbb{N}$ such that R^i contains a crossing of a subwalk of W^j. They are *nested* if they do not cross.

2.4 Fundamental groups of planar graphs

Let G be a graph. Let $W = x_1 x_2 \ldots x_n$ be a walk. The *inverse* of W is $x_n \ldots x_1$. If $x_{i-1} = x_{i+1}$ for some i, we call the walk $W' := x_1 \ldots x_{i-1} x_{i+2} \ldots x_n$ a *reduction* of W. Conversely, we *add* the *spike* $x_{i-1} x_i x_{i+1}$ to W' to obtain W. If W is a closed walk, i.e. if $x_1 = x_n$, we call $x_i \ldots x_n x_1 \ldots x_{i-1}$ a *rotation* of W. We emphasize that the x_i are vertices of G rather than elements of a group; we do not assume G to be a Cayley graph in this section.

Let $\mathcal{V}$ be a set of closed walks. The set of closed walks *generated by* $\mathcal{V}$ is defined to be the smallest set $\overline{\mathcal{V}} \supseteq \mathcal{V}$ of closed walks that is closed under taking concatenations, reductions, rotations and the inverse and under adding spikes. We also say that any $V \in \overline{\mathcal{V}}$ is *generated by* $\mathcal{V}$. A closed walk is *indecomposable* if it is not generated by closed walks of strictly smaller length. Note that no indecomposable closed walk W has a *shortcut*, i. e. a (possibly trivial) path between any two of its vertices that has smaller length than any subwalk of any rotation of W between them. In particular, indecomposable closed walks induce cycles.

For any $\eta \in \pi_1(G)$, let $W_\eta \in \eta$ be the unique reduced closed walk in η, and let W_η° be its (unique) cyclical reduction. For $\mathcal{V} \subseteq \pi_1(G)$, set

$$\mathcal{V}^\circ := \{W_\eta^\circ \mid \eta \in \mathcal{V}\}.$$

By $\mathcal{W}(G)$ we denote the set of all closed walks in G.

The following theorem is an immediate consequence of [15, Theorem 6.2], which is a generalisation of the main theorem of [14].

Theorem 2.4. *Let G be a planar locally finite 3-connected graph and Γ a group acting on G. Then $\pi_1(G)$ has a generating set $\mathcal{V}$ such that $\mathcal{V}^\circ$ is a Γ-invariant nested generating set for $\mathcal{W}(G)$ that consists of indecomposable closed walks.*

3 Planar presentations

In this section we introduce our notion of planar presentation, which is the central definition of this paper. For the convenience of the reader, we start by recalling the definition of a special planar presentation from [13]. We then define the more involved generic planar presentations in Section 3.2.

3.1 Special planar presentations

The intuition behind special planar presentations comes from the notion of a consistent embedding given above: a planar presentation is a group presentation endowed with some additional data (forming what we will call an embedded presentation) which describe the local structure of a consistent embedding of the corresponding Cayley graph, that is, the spin and the information of which generators preserve or reflect it.

Given a group presentation $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$, where $\mathcal{S}$ is finite, we will distinguish between two types of generators s: those for which we have s^2 as a relator in $\mathcal{R}$ and the rest. The reasons for this distinction will become clear later. Generators t for which the relation t^2 is provable but not explicitly part of the presentation might exist, but do not cause us any concerns. Given a group presentation $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$, we thus let $\mathcal{I} = \mathcal{I}(\mathcal{P})$ denote the set of elements $s \in \mathcal{S}$ such that $\mathcal{R}$ contains the relator s^2 or s^{-2}.

Let $\mathcal{S}' = \mathcal{S} \cup (\mathcal{S} \setminus \mathcal{I})^{-1}$. For example, if $\mathcal{P} = \left\langle a, b, c \mid a^2, b^2 \right\rangle$, then $\mathcal{S}' = \{a, b, c, c^{-1}\}$.

A *spin* on $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is a cyclic ordering of $\mathcal{S}'$ (to be thought of as the cycling ordering of the edges that we expect to see around each vertex of our Cayley graph once we have proved that it is planar)

An *embedded presentation* is a triple $\mathcal{P}, \sigma, \tau$ where $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is a group presentation, σ is a spin on $\mathcal{P}$, and τ is a function from $\mathcal{S}$ to $\{0, 1\}$ (encoding the information of whether each generator is spin-preserving or spin-reversing).

To every embedded presentation $\mathcal{P}, \sigma, \tau$ we can associate a tree $\mathbb{T}$ with an accumulation-free embedding in $\mathbb{R}^2$ that is a universal cover of $Cay \langle \mathcal{S} \mid \mathcal{R} \rangle$. As a graph, we let $\mathbb{T}$ be $Cay \left\langle \mathcal{S} \mid s^2, s \in \mathcal{I} \right\rangle$. Easily, we can embed $\mathbb{T}$ in $\mathbb{R}^2$ in such a way that for every vertex v of $\mathbb{T}$, one of the two cyclic orderings of the colours of the edges of v inherited by the embedding coincides with σ and moreover, for every two adjacent vertices v, w of $\mathbb{T}$, the clockwise cyclic ordering of the colours of the edges of v coincides with that of w if and only if $\tau(a) = 0$ where a is the colour of the v–w edge. (If $\tau(a) = 1$, then the clockwise ordering of v coincides with the anti-clockwise ordering of w.)

Given a word W, we let W^∞ be the 2-way infinite word obtained by concatenating infinitely many copies of W. We say that two words $W, Z \in \mathcal{R}$ *cross*, if there is a 2-way infinite path R of $\mathbb{T}$ induced by W^∞ and a 2-way infinite path L induced by Z^∞ such that L meets both components of $\mathbb{R}^2 \setminus R$. Note that, if two non-trivial words form closed walks in the Cayley graph, then the words cross if and only if the closed walks cross.

For example, consider the presentation $\mathcal{P} = \langle n, e, s, w \mid n^2, e^2, s^2, w^2 \rangle$, the spin n, e, s, w, n (read 'north, east, south, west'), and τ identically 0. Thus $\mathbb{T}$ is embedded in such a way that at every vertex v, the cyclic sequence of labels we read on the edges emanating from v as we move clockwise around v is n, e, s, w, n. Therefore, any word W containing ns as a subword crosses any word Z containing ew, because we can find a double ray induced by W^∞ containing the edges labelled n, s incident with the identity, and a double ray induced by Z^∞ containing the edges labelled e, w incident with the identity. The word $nesw$ however crosses no other word, and indeed adding that word to the above presentation yields a planar Cayley graph: the square grid.

Definition 3.1. A *special planar presentation* is an embedded presentation $(\mathcal{P}, \sigma, \tau)$ such that

(sP1) no two relators $W, Z \in \mathcal{R}$ cross, and

(sP2) for every relator R, the number of occurrences of letters s in R with $\tau(s) = 1$ (i.e. spin-reversing letters) is even; here, the symbol s^n counts as $|n|$ occurrences of s.

Requirement (sP2) is necessary, as the spin of the starting vertex of a cycle must coincide with that of the last vertex.

In [13] we proved the following results about special planar presentations.

Theorem 3.2 ([13, Theorem 3.3])**.** *Every planar, locally finite,* 3*-connected Cayley graph admits a special planar presentation.*

Theorem 3.3 ([13, Theorem 4.2])**.** *If* $(\mathcal{P}, \sigma, \tau)$ *is a special planar presentation, then its Cayley graph* $Cay(\mathcal{P})$ *is planar. Moreover,* $Cay(\mathcal{P})$ *admits a consistent embedding, with spin* σ *and spin-behaviour of generators given by* τ*.*

3.2 General planar presentations

We now extend the above definition of a planar presentation, to a more general one, the advantage of which is that it can capture Cayley graphs with

2-separators that do not admit consistent embeddings, which will allow us to extend Theorem 3.2 and Theorem 3.3 to all planar Cayley graphs.

Let again $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a group presentation, and define $\mathcal{S}'$ as above.

A *spin structure* $\mathcal{C}$ on $\mathcal{P}$ consists of a cover $B_1, \ldots, B_k$ of $\mathcal{S}'$ (i.e. $\bigcup_i B_i = \mathcal{S}'$) with the following properties

(S1) for every generator b, the number of B_i's containing b equals the number of B_i's containing b^{-1}, and

(S2) the auxiliary graph X on $\mathcal{C} \cup \mathcal{S}'$ with $s \sim B_i$ whenever $s \in B_i$, is a tree.

(It will become clear later that a special planar presentation is a special case of a general one in which $\mathcal{C}$ consists of a single set coinciding with $\mathcal{S}'$.)

The *hinges* of this spin structure are the elements of $\mathcal{S}'$ that have degree at least 2 in X; in other words, $h \in \mathcal{S}'$ is a hinge if $h \in B_i \cap B_j$ for some $i \neq j$. Hinges of a spin structure correspond to edges of our Cayley graph G whose two endvertices separate G.

Intuitively, the sets B_i correspond to the maximal 2-connected subgraphs of the Cayley graph and the hinges of the spin structure correspond to the hinges of the Cayley graph. We will prove this in Theorem 4.4.

For example, a, b are the hinges of the presentation

$$\left\langle a, b, c, d, f, g \mid a^2, c^2, d^2, f^2, g^2, (af)^2, (ag)^2, abab^{-1}gbfb^{-1}, cbdb^{-1} \right\rangle$$

given in Figure 3, and b is the only hinge in Figure 2. The tree X of condition (S2) corresponding to the presentation of Figure 3 is shown in Figure 5. Figure 6 shows the corresponding tree X that would result if we amalgamated the above group with two more groups each of which being isomorphic to the subgroup generated by b, c, d along the subgroup spanned by b.

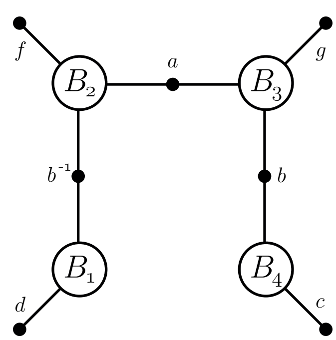

Figure 5: An example: the tree X of condition (S2) corresponding to Figure 3.

Condition (S2) has the following important consequences:

$$B_i \cap B_j \text{ is either empty or a singleton for every } i \neq j, \tag{1}$$

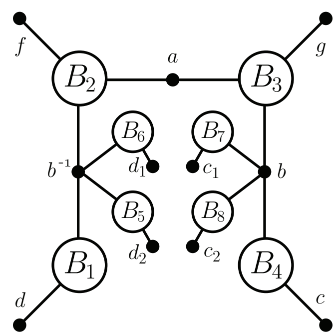

Figure 6: The tree X of condition (S2) corresponding to a variant of Figure 3.

because if $h, g \in B_i \cap B_j$ then h, g, B_i, B_j span a 4-cycle in X, which cannot happen when X is a tree, and

$$\text{every } B_i \text{ contains at least one hinge unless } k = 1, \text{ i.e. } \mathcal{C} \text{ is the singleton } \{\mathcal{S}'\}, \tag{2}$$

because if each neighbour of B_i in X has degree 1, then B_i and its neighbours form a component of X.

A *generic embedded presentation* is a quintuple $\mathcal{P}, \mathcal{C}, \sigma, \tau, \mu$ as follows; $\mathcal{P}$ is a group presentation and $\mathcal{C}$ a spin structure on $\mathcal{P}$ as above; σ is a function of $i \in \{1, \ldots, k\}$ assigning a spin (i.e. a cyclic ordering) to each $B_i \in \mathcal{C}$;
$\tau \colon \mathcal{S} \times \{1, \ldots, k\} \to \{0, 1\}$ encodes the information of whether each generator is spin-preserving or spin-reversing in each B_i it participates in (if $s \in \mathcal{S} \setminus B_i$, then the value of $\tau(s, i)$ will be irrelevant in the sequel); and for every $b \in \mathcal{S}$, and every i for which $b \in B_i$, $\mu(b, i)$ is a B_j such that $b^{-1} \in B_j$, and $\mu(b, i) \neq \mu(b, m)$ for $m \neq i$. This μ encodes the information of which pairs of B_i incident with the two endvertices of a given hinge belong to the same block of G. The use of $\mathcal{S}$ rather than $\mathcal{S}'$ in the definition of μ and τ is intended: the values we assign to each $b \in S$ give us enough information about how to treat b^{-1}.

For the time being, the data σ, τ, μ are abstract objects describing the intended structure and embedding of our Cayley graph given by $\mathcal{P}$. But we will indeed prove that if these data satisfy certain conditions, then the Cayley graph is indeed planar and can be embedded in the intended way.

As an example, the presentation $\langle \mathcal{S} \mid b^2, a^3, c^3, aba^{-1}b, cbcb \rangle$ of the graph of Figure 2 can be endowed with the following data. The spin structure $\mathcal{C}$ consists of two sets $B_1 = \{b, c, c^{-1}\}, B_2 = \{b, a, a^{-1}\}$. We can then let $\sigma(1) = (b, c, c^{-1})$, $\sigma(2) = (b, a^{-1}, a)$ —but any other σ would do in this case as there are only two cyclic orderings of a set of three elements, and they are the reflection of each other— $\tau(b, 1) = 0$, $\tau(b, 2) = 1$ —this is the most interesting aspect of this graph: any b edge is spin-preserving in one of its incident blocks and spin-reversing in the other— and $\mu(b, 1) = B_1$, $\mu(b, 2) = B_2$ —because b stabilises the two components into which it splits the graph.

Our general definition of a planar presentation will be very similar to that of Section 3.1, and still based on the idea of non-crossing relators. One difference is that we have to embed the tree $\mathbb{T} = Cay \left\langle \mathcal{S} \mid s^2, s \in \mathcal{I} \right\rangle$ in $\mathbb{R}^2$ more carefully. We will give the formal embedding in Section 5.1 but the idea is that rather than demanding every vertex to have the same cyclic ordering of its incident colours in the embedding, which would in general make it impossible to adhere to the spin-behaviour encoded by τ, we only demand that the cyclic orderings of the edges with colours in each B_i are preserved. To make this precise, we use an (accumulation-free) embedding $\rho : \mathbb{T} \to \mathbb{R}^2$ with the following properties. Given a vertex $x \in V(\mathbb{T})$ and $B_i \in \mathcal{C}$, we write $B_i(x)$ for the edges of x with labels in B_i.

(B1) σ is respected, i.e. for every vertex x of $\mathbb{T}$, and every $B_i \in \mathcal{C}$, the cyclic ordering induced on $B_i(x)$ by the ρ-image of the neighbourhood of x coincides with $\sigma(i)$ up to reflection.

(B2) τ is respected, i.e. for every edge $e = vw$ of $\mathbb{T}$, and every i such that the label s of e is in $B_i \in \mathcal{C}$, we have $1_{\sigma(i)}(B_i(v)) = 1_{\sigma(j)}(B_j(w))$ if and only if $\tau(s, i) = 0$, where $B_j = \mu(s, i)$ and $1_{\sigma(i)}(B_i(v))$ is 1 if the clockwise cyclic ordering of the colours of the edges of $B_i(v)$ coincides with $\sigma(i)$ and 0 otherwise.

We repeat the definition of *crossing* from Section 3.1 verbatim: given a word W, we let W^∞ be the 2-way infinite word obtained by concatenating infinitely many copies of W. We say that two words $W, Z \in \mathcal{R}$ *cross*, if there is a 2-way infinite path R of $\mathbb{T}$ induced by W^∞ and a 2-way infinite path L induced by Z^∞ such that L meets both components of $\mathbb{R}^2 \setminus R$.

The second and final difference of our generalised definition of a planar presentation compared to that of Section 3.1 will be an additional condition reflecting the idea that in a planar Cayley graph of connectivity 2, we can choose the relators in such a way that each cycle they induce is contained in a block. Recalling that our spin structure $\mathcal{C}$ is intended to capture the decomposition into blocks, the following definition should not be too surprising.

We say that a relator R is *blocked* with respect to $\mathcal{C}$, if it satisfies the following two properties. Firstly, for every two (possibly equal) consecutive letters st appearing in R^∞ or $(R^{-1})^\infty$, there is some $B_i \in \mathcal{C}$ containing both s^{-1}, t. Secondly, for every three consecutive letters sbt, where b is a hinge, appearing in R^∞ or $(R^{-1})^\infty$, if B_i is the unique element of $\mathcal{C}$ containing s^{-1}, b, then $\mu(b, i)$ contains both b^{-1}, t, unless $s = b = t$ and $b^2 \in \mathcal{R}$; here, the existence of such a B_i is guaranteed by the previous requirement, and its uniqueness is a consequence of (1) in the definition of a spin structure.

Definition 3.4. A *generic planar presentation* is a generic embedded presentation such that

(P1) every relator in $\mathcal{R}$ is blocked with respect to $\mathcal{C}$;

(P2) no two relators $W, Z \in \mathcal{R}$ cross;

(P3) for every relator R, the number of occurrences of letters t in R with $\tau(t,i) = 1$ (i.e. spin-reversing letters), where i is the unique value for which $s^{-1}, t \in B_i$ for the letter s preceding t in R, is even[2]; here, the symbol s^n counts as $|n|$ occurrences of s;

(P4) no relator is a sub-word of a rotation of another relator.

Note that a planar presentation as defined in Section 3.1 is a special case of a generic one when $\mathcal{C}$ consists of a single set coinciding with $\mathcal{S}'$.

In Section 6 we will slightly generalise the notion of a generic planar presentation further, by allowing the removal of certain redundancies, to obtain the notion of general planar presentation discussed in the introduction.

4 Every planar Cayley graph admits a generic planar presentation

In this section we prove that every planar Cayley graph admits a generic planar presentation.

We start by showing that every planar Cayley graph of connectivity 1 can be extended into a 2-connected one using redundant generators; see Lemma 4.1 below. We then show that every 2-connected planar Cayley graph admits a generic planar presentation in Section 4.2.

4.1 Planar Cayley graphs of connectivity 1

Lemma 4.1. *Every planar, locally finite, Cayley graph of connectivity 1 can be extended into a planar 2-connected, locally finite, Cayley graph by adding redundant generators.*

Proof. We proceed by induction on the number of blocks incident with the vertex o, where a *block* means a maximal 2-connected subgraph in this subsection. Pick two such blocks B, C, an edge from B corresponding to some generator b, and an edge from C corresponding to some generator c. Introduce a new redundant generator x and the relation $x = b^{-1}c$. Clearly, the resulting Cayley graph G' obtained from the original Cayley graph G by adding the generator x has less blocks incident with o than G.

We claim that G' is still planar. If none of b^2 or c^2 is a relator, then this is an easy exercise, based on the observation that G can be embedded in such a way that for every vertex v, the edges labelled b and c emanating from v lie in a common face boundary.

If however b^2, say, is a relator, then it is a bit harder to avoid that the two x edges emanating out of o and ob cross in our embedding. Still, the following observation will help us embed G' in this case (and it is also applicable to

[2] The existence and uniqueness of this B_i is a consequence of (P1); see the definition of 'blocked'.

the case where none of b^2 or c^2 is a relator). A good example to bear in mind throughout the rest of the proof is where G is the Cayley graph $Cay\left\langle b, c \mid b^2, c^2 \right\rangle$ of the free product of two copies of $\mathbb{Z}/2\mathbb{Z}$, and $x = bc$.

Let H_0 be the graph consisting of a single vertex, and suppose that for every $i \in \mathbb{N}$, the graph H_i is obtained from H_{i-1} by attaching a planar graph P_i to H_{i-1} by identifying some vertex $p_i \in V(P_i)$ with some vertex $h_i \in V(H_{i-1})$, and possibly joining a neighbour p'_i of p_i to a neighbour h'_i of h_i with an edge. Then $\bigcup_{i \geq 0} H_i$ is planar. (3)

To prove this, we first use induction to show that H_i is planar: given an embedding of H_{i-1}, observe that h'_i, h_i lie in a common face F_i since they are neighbours. Likewise, p'_i, p_i lie in a common face F'_i of P_i, and we may assume that that face is the outer face by embedding P_i appropriately. Indeed, given an embedding $\phi : P_i \to \mathbb{S}^2$, we can compose ϕ with a stereographic projection from $\mathbb{S}^2$ to $\mathbb{R}^2$ using a point inside F'_i as the projection point. We now embed H_i by drawing P_i inside F_i and, if there is a $h'_i - p'_i$ edge in H_i, joining h'_i to p'_i with an arc in F_i that avoids the rest of the graph.

The fact that $\bigcup_{i \geq 0} H_i$ is planar now follows from a standard compactness argument.

To complete our proof, we will show that our G' can be constructed as described in (3).

Indeed, let $\mathcal{H}$ be the set of blocks (i.e. maximal 2-connected subgraphs) of G, and let $H_1, H_2, \dots$ be an enumeration of $\mathcal{H}$ such that for $i > 1$, H_i is incident with some H_j for $j < i$. Then G' has the claimed structure, with the x-edges playing the role of the $h'_i - p'_i$ edges. □

4.2 Cayley graphs of connectivity 2

In this section, we will complete the proof of our main theorem by showing that every locally finite 2-connected planar Cayley graphs admits a generic planar presentation.

A *cut* in a graph G is a set of vertices C spanning a connected subgraph of G, such that the *boundary*

$$\partial C := \{x \in V(G) \setminus C \mid x \text{ has a neighbour in } C\}$$

of C is finite and $C \cup \partial C \neq V(G)$. The *order* of C is the cardinality of ∂C.

We call two cuts C, D *nested* if, setting $C^* := V(G) \setminus C$ and $D^* := V(G) \setminus D$, one of the four relations holds:

$$C \subseteq D, \quad C \subseteq D^*, \quad C^* \subseteq D, \quad C^* \subseteq D^*.$$

We call a set of cuts *nested*, if every two of its elements are nested.

Definition 4.2. Given a nested set $\mathcal{C}$ of cuts, a *block* is a maximal subgraph H such that for every cut C, we have either $V(H) \subseteq C \cup \partial C$ or $V(H) \subseteq C^*$ but not both.

To obtain a *torso* of a block H from H we add all edges xy such that $\{x, y\} \subseteq V(H)$ is a boundary of a cut in $\mathcal{C}$.

Tutte [27] showed that every finite 2-connected graph G has an $\mathrm{Aut}(G)$-invariant nested set $\mathcal{C}$ of cuts of order 2 whose torsos are either 3-connected or cycles. This fact also holds for locally finite graphs as proved by Droms et al. [6], and we will still refer to it as *Tutte's theorem*. To each such nested set of cuts, there is an associated tree T that admits a bijection from $V(T)$ to the blocks and boundaries of cuts in $\mathcal{C}$ such that, for any $t_1, t_2 \in V(T)$ and any t on the unique t_1–t_2 path in T, the image of t separates the images of t_1 and t_2.[3] We call this tree T the *decomposition tree* of the set of cuts.

A 2-*separator* is the boundary of a cut of order 2. Lemma 4.3 allows us to assume that all 2-separators of G are joined by an edge, i.e. they are hinges in the sense of Section 3.2. Given two Cayley graphs G, H, we call G a *Tietze-supergraph* of H if there are presentations $\langle \mathcal{S}_G \mid \mathcal{R}_G \rangle$ of $\Gamma(G)$ and $\langle \mathcal{S}_H \mid \mathcal{R}_H \rangle$ of $\Gamma(H)$ with $G = Cay \langle \mathcal{S}_G \mid \mathcal{R}_G \rangle$ and $H = Cay \langle \mathcal{S}_H \mid \mathcal{R}_H \rangle$ and with $\mathcal{S}_G \supseteq \mathcal{S}_H$ and $\mathcal{R}_G \supseteq \mathcal{R}_H$. Note that if the presentations $\langle \mathcal{S}_G \mid \mathcal{R}_G \rangle$ and $\langle \mathcal{S}_H \mid \mathcal{R}_H \rangle$ belong to the same group, they can be transferred to each other via *Tietze-transformations*, that is, we can add redundant relations or remove them or we can add a new generator s together with a relation $s = w$, where w is a word over the old generating set, or remove such a generator with the corresponding relation.

Lemma 4.3. *Every planar 2-connected Cayley graph G has a planar Tietze-supergraph H in which every pair of vertices that separates H is connected by an edge. In addition, the new edges are labelled by a new redundant generator. (Moreover, if G is locally finite, then so is H.)*

Proof. To begin with, pick a $\Gamma(G)$-invariant nested set $\mathcal{C}$ of cuts of order 2. This set exists due to Tutte's theorem mentioned above. For every pair of non-adjacent vertices x, y such that one component of $G - \{x, y\}$ lies in $\mathcal{C}$, we add a new redundant generator a and relation $a = x^{-1}y$. Let us show that the nestedness of $\mathcal{C}$ implies that we do not lose planarity.

Note that every 2-separator lies on the boundary of some face. So if we join x_1 and y_1 by a new edge and also want to join x_2 and y_2, then the only reason why we cannot do this is because the edge x_1y_1 separates the face on whose boundary the vertices x_2 and y_2 lie. So, originally, all four vertices x_1, x_2, y_1, y_2 are distinct and lie on a boundary of some face F in this order (either clockwise or anticlockwise). For $i = 1, 2$, let P_i be an x_i–y_i path whose inner vertices lie in a component of $G - \{x_i, y_i\}$ that avoids x_j and y_j for $j \neq i$. We find these paths since the cuts from Tutte's theorem are nested and since G is 2-connected. As the two paths P_i lie outside of F, the path P_2 connects a vertex in the inner face of $P_1 + y_1x_1$ to one in its outer face, which is impossible due to the Jordan curve theorem. This proves that we can indeed add the aforementioned redundant generators and relations without losing planarity.

[3] Readers that are familiar with tree-decompositions of graphs might notice that this just says that for every nested set of cuts, we find a tree-decomposition of the graph whose parts are the blocks and boundaries of cuts.

Since every vertex has only finitely many neighbours and every two of them can be separated by only finitely many 2-separators (see e.g. [26, Proposition 4.2]), the resulting Cayley graph G' is still locally finite. $\square$

Call a graph *well-separated* if it is 2-connected and every 2-separator is joined by an edge.

Theorem 4.4. *Every planar, locally finite, well-separated Cayley graph G with $\kappa(G) = 2$ admits a generic planar presentation.*

Proof. Let $\mathcal{C}$ be a $\Gamma(G)$-invariant nested set of cuts of order 2 as in Tutte's Theorem. Let $\mathcal{B}_o$ be the set of blocks (in the sense of Definition 4.2) that contain the vertex o. For $B \in \mathcal{B}_o$, let S_B be the set of those generators $s \in \mathcal{S} \cup \mathcal{S}^{-1}$ such that the edge with label s starting at o lies in B. Then $\mathcal{S} \cup \mathcal{S}^{-1}$ is covered by the sets S_B. We fix an embedding ρ of G in $\mathbb{R}^2$, and endow every S_B with the cyclic order induced by ρ at o. Let $\mathcal{B}'_o \subseteq \mathcal{B}_o$ be maximal such that no two distinct $B, B' \in \mathcal{B}_o$ are of the form $B = g(B')$ for any $g \in \Gamma(G)$. We can apply Theorem 2.4 to each $B \in \mathcal{B}'_o$ to obtain a set $\mathcal{D}_B \subseteq \pi_1(B)$ that generates $\pi_1(B)$, and such that $\mathcal{D}_B^\circ$ is a nested set of indecomposable closed walks that is invariant under the stabiliser of B in $\Gamma(G)$. Then it is easy to see that

$$\mathcal{D} := \bigcup_{\substack{B \in \mathcal{B}'_o \\ g \in \Gamma(G)}} g(\mathcal{D}_B)$$

generates $\pi_1(G)$. Let $\mathcal{R}_\mathcal{D}$ be the set of words corresponding to closed walks in $\mathcal{D}^\circ$. Easily, $\langle \mathcal{S} \mid \mathcal{R}_\mathcal{D} \rangle$ is a presentation of $\Gamma(G)$. Once more, we use Tietze-transformations to obtain a finite subset $\mathcal{R} \subseteq \mathcal{R}_\mathcal{D}$ with $\langle \mathcal{S} \mid \mathcal{R}_\mathcal{D} \rangle = \langle \mathcal{S} \mid \mathcal{R} \rangle$, which is possible as $\Gamma(G)$ is finitely presented (Droms [5, Theorem 5.1]). To see that the set $\mathcal{C} := \{B_1, \ldots, B_n\} := \{S_B \mid B \in \mathcal{B}_o\}$ is a spin structure of $\mathcal{P} := \langle \mathcal{S} \mid \mathcal{R} \rangle$, it remains to show that the graph $\mathcal{T} := (\mathcal{C} \cup \mathcal{S}', \mathcal{E})$, where $xy \in \mathcal{E}$ if and only if $x \in y$ or $y \in x$, is a tree.

Let us suppose that $\mathcal{T}$ is not a tree. Obviously, $\mathcal{T}$ is connected. So it contains some cycle $S_1 s_1 \ldots S_m s_m S_1$ with $S_i \in \mathcal{C}$ and $s_i \in \mathcal{S}'$. For each $i \leq m$, let $B(S_i) \in \mathcal{B}_o$ be such that $S_i = S_{B(S_i)}$. As each element of $\mathcal{B}_o$ is a block, there is some path P_i in $B(S_i)$ connecting the end vertices of s_{i-1} and s_i distinct from o (with $s_0 = s_m$). The concatenation of all these paths P_i is a cycle C in G that crosses all hinges s_i precisely once as $S_i \neq S_{i+1}$ (with $S_{m+1} = S_1$). But this is not possible as each cycle, and hence also C, must lie in a unique block of G.

For $i \leq n$, let $B(i)$ be that element of $\mathcal{B}_o$ with $S_{B(i)} = B_i$. For every hinge $b \in \mathcal{S}$ incident with o and every $i \leq n$ with $b \in B_i$, let $\mu(b, i)$ be that B_j with $b(B(i)) = B(j)$. So we have $b^{-1} \in B_j$. Let $\sigma(i)$ be the spin of B_i at o. To define whether every generator is spin-preserving or spin-reversing in each element of the spin-structure (it participates in), we remember that the blocks —being either 3-connected or cycles— have a unique embedding in the plane. So for $s \in \mathcal{S}$ and $i \leq n$, we define $\tau(s, i)$ to be 0 if s is spin-preserving in $B(i)$ and 1 otherwise. (Note that τ is also defined if $s \notin B_i$.) Clearly, $(\mathcal{P}, \mathcal{C}, \sigma, \mu, \tau)$ is a generic embedded presentation.

As every element of $\mathcal{D}^\circ$ lies in a unique block, every $R \in \mathcal{R}$ is blocked with respect to $\mathcal{C}$ by definition, and the number of spin-reversing generators in R is even. As $\mathcal{D}$ is nested, it is easy to check that no two relators cross. The fact that no cycle is a subgraph of any other cycle implies that no relator is a sub-word of a rotation of another relator, and hence our generic embedded presentation is a generic planar presentation. □

With an argument similar to the proof of [13, Corollary 3.4], we obtain:

Corollary 4.5. *Every planar well-separated Cayley graph G with $\kappa(G) = 2$ is the 1-skeleton of an almost planar Cayley complex of $\Gamma(G)$.*

Proof. Since G is planar, there is an embedding $\rho' \colon G \to \mathbb{R}^2$ by definition. We will extend ρ' to the desired map ρ from the Cayley complex X of $\Gamma(G)$ with respect to the presentation $\langle \mathcal{S} \mid \mathcal{R} \rangle$ from above. For this, given any 2-cell Y of X with boundary cycle C, we embed Y in the finite component of $\mathbb{R}^2 \setminus C$. It is a straightforward consequence of the nestedness of $\mathcal{D}$ that the resulting map ρ has the desired property. □

4.3 Consistent embeddings lead to special planar presentations

In the previous section, we have seen that 2-connected planar Cayley graphs admit generic planar presentations. However, if the Cayley graph has a consistent embedding, we obtain a bit more even for 1-connected graphs:

Theorem 4.6. *Every planar Cayley graph with a consistent embedding admits a special planar presentation.*

Proof. Let G be such a graph. First note that, by repeating the arguments of the proof of Lemma 4.3, we can join the two vertices of any 2-separator $\{x, y\}$ by a new edge whenever $xy \notin E(G)$ and $G - \{x, y\}$ has two components C with $\partial C = \{x, y\}$, while keeping the embedding consistent. So we may assume that every maximal 2-connected subgraph of G is well-separated.

Let $\mathcal{B}$ be a set of blocks of the maximal 2-connected subgraphs of G consisting of one block from each $\Gamma(G)$-orbit. As before, Theorem 2.4 gives us for each $B \in \mathcal{B}$ a set $\mathcal{D}_B$ that generates $\pi_1(B)$ such that $\mathcal{D}_B^\circ$ is a nested set of indecomposable closed walks that is invariant under the stabiliser in $\Gamma(G)$ of B. Let $\mathcal{R}_B$ be the set of words corresponding to the elements of $\mathcal{D}_B^\circ$. As above, Tietze-transformations give us a finite $\mathcal{R} \subseteq \bigcup_{B \in \mathcal{B}} \mathcal{R}_B$ such that $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is a finite presentation of $\Gamma(G)$, where $\mathcal{S}$ is the generating set of G.

If we let σ be the spin of one fixed vertex x and $\tau(s) = 0$ if the edge from x labelled s is spin-preserving and $\tau(s) = 1$ otherwise, then $(\mathcal{P}, \sigma, \tau)$ is a special planar presentation of $\Gamma(G)$. Indeed, nestedness of the closed walks in $\mathcal{D}_B^\circ$ implies that the corresponding words are non-crossing, the fact that they are indecomposable implies that no relator is a subword of any other relator, and the embedding implies that every relator contains an even number of spin-reversing letters. □

5 Proof of planarity of the Cayley graph of a generic planar presentation

In this section we prove that the Cayley graph defined by any generic planar presentation is planar (Theorem 5.10).

Let $(\mathcal{P}, \mathcal{C}, \sigma, \tau, \mu)$ be a generic planar presentation. For a hinge $h \in \mathcal{S}$, we let $\mathcal{C}(h) := \{B_i \in \mathcal{C} \mid h \in B_i\}$ and let $n(h)$ be the cardinality $|\mathcal{C}(h)|$. Note that $|\mathcal{C}(h)| = deg_X(h)$, where the tree X is as in (S2) of the definition of a spin structure.

Every hinge $b = xy \in E(\mathbb{T})$ of $\mathbb{T}$ labelled h naturally splits $\mathbb{T}$ into $n(h)$ subtrees: each of these subtrees contains b, it contains all edges of x with labels in some $B_i \in \mathcal{C}(h)$ and no other edges of x, and it contains those edges of y with labels in $\mu(h, i)$ and no other edges of y; moreover, each such subtree is maximal with these properties. Let $Sep_b = \{T_1, T_2, \ldots, T_{n(h)}\}$ denote the set of those subtrees, and note that $\bigcap Sep_b = \{b\}$.

Definition 5.1. A *pre-block* of $\mathbb{T}$ is a maximal subtree $A \subseteq \mathbb{T}$ not separated by any Sep_b; that is, for every hinge b of $\mathbb{T}$, A is contained in some element of Sep_b.

Alternatively, we can define a pre-block as a maximal subtree A of $\mathbb{T}$ such that for every $x, y \in V(A)$, if we let $s_1 s_2 \ldots s_k$ denote the word (with letters in $\mathcal{S}$) read along the x–y path, then s_{j-1}^{-1}, s_j lie in a common element of $\mathcal{C}$ for every $j > 1$, and whenever s_j is a hinge, and $s_{j-1}^{-1}, s_j \in B_i \in \mathcal{C}$, then $s_j^{-1}, s_{j+1} \in \mu(s_j, i)$.

5.1 The embedding ρ of $\mathbb{T}$

Our proof of Theorem 3.3, which we are generalising here as Theorem 5.10, started with an embedding of the corresponding tree $\mathbb{T}$ respecting the spin data. Using the fact that $Cay(\mathcal{P})$ is a quotient of $\mathbb{T}$ with respect to the normal closure of $\mathcal{R}$, we reduced the planarity of $Cay(\mathcal{P})$ to a property of a fundamental domain D of $\mathbb{T}$ with respect to $\mathcal{R}$ as follows. We obtain $Cay(\mathcal{P})$ by identifying pairs of points of D that are midpoints of edges of $\mathbb{T}$. It is then not hard to see that $Cay(\mathcal{P})$ is planar if any two pairs of such points are *nested*, i.e. they do not alternate as we move around the boundary of D.

In our new setup of a generic embedded presentation however, we have to work harder to make this idea work: our spin data give us some restrictions but do not uniquely determine an embedding of $\mathbb{T}$, and in fact we have to be careful with our choices in order for the proof in subsection 5.2 to work.

We will assume below that the reader is familiar with our proof of Theorem 3.3 in [13], since all its arguments will be needed here as well. We chose not to repeat those arguments here, not so much to save space, but because reading that proof first offers a good warm up before the much more involved proof that follows.

Recall that our generic embedded presentation consists of the data $\mathcal{P}, \mathcal{C}, \sigma, \tau, \mu$. For $B \in \mathcal{C}$ and a vertex $x \in V(\mathbb{T})$, recall that $B_i(x)$ denotes the edges going out of x whose labels are in B. We will show that there is an embedding $\rho : \mathbb{T} \to \mathbb{R}^2$ satisfying all of the following (the first two were also used in the definition of crossing relators in Section 3.2).

(ρ1) σ is respected, i.e. for every vertex $x \in V(\mathbb{T})$, and every $B_i \in \mathcal{C}$, the cyclic ordering induced on $B_i(x)$ by ρ coincides with $\sigma(i)$ up to reflection.

(ρ2) τ is respected, i.e. for every edge $e = vw$ of $\mathbb{T}$, and every i such that the label s of e is in $B_i \in \mathcal{C}$, we have $1_{\sigma(i)}(B_i(v)) = 1_{\sigma(j)}(B_j(w))$ if and only if $\tau(s,i) = 0$, where $B_j = \mu(s,i)$ and $1_{\sigma(i)}(B_i(v))$ is 1 if the clockwise cyclic ordering of the colours of the edges of $B_i(v)$ coincides with $\sigma(i)$ and 0 otherwise.

(ρ3) μ is respected: let $b \in E(\mathbb{T})$ be a hinge, and U, W two paths containing b contained in distinct pre-blocks containing b. Then U, W do not cross each other (at b).

(ρ4) If x, y belong to the same $N(\mathcal{R})$-orbit (where $N(\mathcal{R})$ is the normal subgroup generated by $\mathcal{R}$ as in Section 2.2), and b is a hinge at x with label in $h \in \mathcal{I}$, and $h \neq 1$, then the *local spin* at x with respect to b coincides up to reflection with the local spin at y with respect to the corresponding hinge labelled h.

Here, the *local spin* with respect to a generator $h \in \mathcal{S}'$ at a vertex x is the cyclic ordering on $N_X(h)$ induced by the embedding, where X denotes the tree from Section 3.2.

If G is a planar Cayley graph, then the results of Section 4.2 imply that if we embed the universal cover $\mathbb{T}$ of G into $\mathbb{R}^2$ in a way that locally imitates an embedding of G, then all above properties are satisfied.

An *open star* is a subspace of a graph consisting of a single vertex and all open half-edges incident with it. A *star* is the union of an open star with some of the midpoints in its closure.

Properties (ρ1) to (ρ3) are not hard to satisfy: we can embed $\mathbb{T}$ by starting with the star $E(o)$ and then recursively attaching the star $E(v)$ of a new vertex to the subtree embedded so far, and it is always possible to embed $E(v)$ without violating any of (ρ1)–(ρ3). In fact we could have several ways to extend the current embedding to $E(v)$, arising by 'permuting' those $B_i(v), 1 \leq i \leq k$ that do not contain the edge of v embedded before, and by 'reflecting' any such $B_i(v)$. These choices are in direct analogy to the flexibility we have in the embedding of any planar Cayley graph of connectivity 2: permuting the $B_i(v)$ corresponds to 'activating' a hinge b incident with v to exchange the order in which blocks separated by b are embedded. Reflecting a $B_i(v)$ corresponds to flipping such a block around.

These choices mean that (ρ4) will be violated unless we make them carefully. To achieve this, recall from (S2) of Section 3.2 that the auxiliary graph X on $\mathcal{C} \cup \mathcal{S}'$ with $s \sim B_i$ whenever $s \in B_i$, is a tree. Let X^ℓ denote the tree obtained from X by attaching to each vertex v in $\mathcal{S}' \subset V(X)$ a new leaf, which leaf we denote by $\ell(v)$.

Fix an embedding $\chi : X^\ell \to \mathbb{R}^2$ of that tree with the following two properties. Firstly, the spin of any vertex $B_i \in \mathcal{C}$ of X^ℓ coincides with $\sigma(i)$ up to reflection.

Recall that $N(v) = N_G(v)$ denotes the neighbourhood of v in a graph G. For every hinge $h \in \mathcal{S} \setminus \mathcal{I}$, note that $\mu(h, \cdot)$ defines a bijection between $N_X(h)$ and $N_X(h^{-1})$ by the definition of μ. We extend that bijection to $N_{X^\ell}(h)$ and $N_{X^\ell}(h^{-1})$ by mapping $\ell(h)$ to $\ell(h^{-1})$. The second property we impose on χ is that the spin it induces on $N_{X^\ell}(h)$ coincides up to reflection with the μ-image of that spin induced by χ on $N_{X^\ell}(h^{-1})$, and this holds for every such h.

For an involution hinge $h \in \mathcal{I}$, $\mu(h, \cdot)$ still defines a bijection between $N_X(h)$ and $N_X(h^{-1}) = N_X(h)$, and we do not impose any requirement on χ as we did for $h \in \mathcal{S} \setminus \mathcal{I}$. Instead, we let χ embed $N_{X^\ell}(h)$ with an arbitrary spin $\phi = \phi(h)$, and define the dual spin of ϕ as follows:

Definition 5.2. The *dual spin* of ϕ is the cyclic ordering on $N_{X^\ell}(h)$ obtained by composing ϕ with $\mu(h, \cdot)$.

To satisfy (ρ4), we will construct ρ in such a way that the local spin with respect to h at every vertex in a given $N(\mathcal{R})$-orbit either always coincides with ϕ or it always coincides with the dual of ϕ. We remark that we cannot construct ρ algorithmically since we cannot predict which vertices of $\mathbb{T}$ are in the same $N(\mathcal{R})$-orbit; we can only prove the existence of such a ρ abstractly.

We think of this χ as providing instructions about how to construct ρ. As an example, if the set $\mathcal{I}$ of involutions in $\mathcal{S}$ is empty, then every vertex of $\mathbb{T}$ will have the same spin up to reflection in ρ, and that spin can be read from χ by contracting all non-leaves of X^ℓ into a single vertex; that vertex has the right spin in the resulting star.

Let $o = x_1, x_2, \ldots$ be an enumeration of $V(\mathbb{T})$ such that $\{x_1, \ldots, x_k\}$ spans a connected subgraph for all k. We will construct ρ by embedding the x_i one at a time as indicated above. To begin with, we embed one edge e_0 incident with $x_1 = o$ in the 0th step. From now on, each step i begins with some vertices being embedded fully, i.e. with all incident edges, and some vertices having exactly one of their edges embedded in the current embedding ρ_{i-1} of some subtree of $\mathbb{T}$. Let j be the smallest index such that x_j has exactly one of its edges e_i embedded in ρ_{i-1}. We may assume without loss of generality that $j = i$ by changing our enumeration.

We extend ρ_{i-1} to ρ_i by embedding the remaining edges incident with x_i. This will be done by performing the following recursive procedure on X^ℓ to obtain an embedded star S_i with its edges labelled by $\mathcal{S}'$, and then embedding $N_{\mathbb{T}}(x_i)$ with the same spin as S_i.

To begin with, let ℓ be the unique leaf of X^ℓ such that $\ell = \ell(s)$ for the label $s \in \mathcal{S}$ of the edge e_i considered as outgoing from x_i. In order to embed $N(s)$, we distinguish the following cases.

Case 1: If $s \notin \mathcal{I}$, and s is a hinge, then we embed the star $N(s)$ of s in X^ℓ into $\mathbb{R}^2$ so that the spin of s in this embedding coincides with the spin of s in χ up to reflection; due to the second assumption on the embedding of X^ℓ, there are exactly two possibilities for this —up to reflection— and we choose the unique one guaranteeing ($\rho 3$): unless we are in step $i = 1$, in which case we just embed $N(s)$ with the spin of s in χ without reflection, the other endvertex x of e_i has already been fully embedded, and the local spin with respect to e_i (which now label s^{-1} as seen from x) at x coincides up to reflection with that induced on $N(s^{-1})$ by χ by induction hypothesis. We use the possibility to reflect or not in order to guarantee that the clockwise ordering of the B_i in $N(s)$ coincides with the counterclockwise ordering of the $\mu(s,i)$ induced by the spin of x in the embedding ρ_{i-1}.

Case 2: If $s \notin \mathcal{I}$, and s is not a hinge, then it has exactly two neighbours in $N(s)$ ($\ell(s)$ and the unique $B \in \mathcal{C}$ containing s), and so reflection does not change the spin; we just embed $N(s)$ in the unique possible way.

Case 3: If $s \in \mathcal{I}$, and s is not a hinge, then again we just embed $N(s)$ in the unique possible way.

Case 4: Finally, if $s \in \mathcal{I}$, and s is a hinge, then we follow a similar approach to the $s \notin \mathcal{I}$ case, except that we now do not insist that the spin of s in the embedding of $N(s)$ we produce coincides with the spin of s in χ up to reflection; we just make sure that ($\rho 3$) is satisfied, by embedding $N(s)$ so that the clockwise ordering of the B_i in $N(s)$ coincides with the counterclockwise ordering of the $\mu(s,i)$ induced by the spin of x in the embedding ρ_{i-1}; again this is well-defined by the second assumption on the embedding of X^ℓ unless we are in step $i = 1$, in which case we just embed $N(s)$ with the spin of s in χ.

Once $N(s)$ is embedded as above, we set $X^\ell_0 := N(s)$ and proceed by the following recursive procedure, which produces embeddings of an increasing sequence $X^\ell_1, \ldots, X^\ell_k (= X^\ell)$ of subtrees of X^ℓ to embed the rest of X^ℓ.

For $j = 1, 2, \ldots$, pick a leaf v_j of X^ℓ_{j-1} which is not a leaf of X^ℓ; if no such leaf exists then $X^\ell_{j-1} = X^\ell$ and we stop. Then we extend the current embedding of X^ℓ_{j-1} by embedding $N(v_j)$ in such a way that the spin of v_j coincides up to reflection with that induced by χ, unless $v_j \in \mathcal{I} \subseteq \mathcal{S}$ and $v_j \neq 1$, in which case we do the following. Let $y_i = x_i v_j$ be the vertex of $\mathbb{T}$ joined to x_i by the edge labelled v_j. If no vertex of $\mathbb{T}$ from the $N(\mathcal{R})$-orbit of x_i or y_i has been embedded yet by ρ_i, then we embed $N(v_j)$ with local spin given by χ. If some vertex of $\mathbb{T}$ from the $N(\mathcal{R})$-orbit of x_i has already been embedded by ρ_i, we embed $N(v_j)$ with same spin up to reflection as we used so far for all $x_j, j < i$, that are $N(\mathcal{R})$-equivalent to x_i; (we make this choice in order to satisfy ($\rho 4$)). Otherwise, we embed $N(v_j)$ with the dual spin —recall Definition 5.2— up to reflection of the spin we used so far for all $x_j, j < i$ that are $N(\mathcal{R})$-equivalent to y_i. Note that these choices ensure that $N(v_j)$ is embedded with the same spin up to reflection —namely, either that induced by χ or its dual— for all vertices in an $N(\mathcal{R})$-orbit, where we use the fact that, as $v_j \neq 1$, x_i and y_i are never in the same orbit.

In all cases, we still have the option of reflecting. If $v_j \in N(s)$, which means that $v_j \in \mathcal{C}$ and v_j contains the label s of e_i, then we have to worry about satisfying (ρ2); but one of the two choices we have due to the option of reflecting will satisfy (ρ2) for $e = e_i$ and $B_i = v_j$ and we make that choice. (If $v_j \notin N(s)$ then we do not worry about μ and τ; the other endvertices of the edges incident with x_i will make sure that this data is respected, just as we were careful above when embedding $N(s)$ for the label s of e_i.)

Let $X^\ell_j := X^\ell_{j-1} \cup N(v_j)$.

The procedure finishes when all of X^ℓ has been embedded. Then, we contract all non-leafs of X^ℓ to obtain the desired embedded star S_i out of that embedding. Finally, we embed $N_{\mathbb{T}}(x_i)$ with the same spin as S_i to extend ρ_{i-1} to ρ_i.

Let $\rho = \bigcup \rho_i$ be the limit of the ρ_i. We claim that ρ satisfies conditions (ρ1)–(ρ4). Indeed, if any of them is violated, then there is a first step in the above procedures violating it. But we designed all steps so that none of those conditions are violated: condition (ρ1) is never violated because we chose χ so that the spin of every $B_i \in \mathcal{C}$ coincides with $\sigma(i)$ up to reflection, which implies that the corresponding edges of x_i appear in that cyclic order up to reflection in S_i, and therefore in ρ, by the construction of the embedded star S_i. Condition (ρ2) is never violated because of the way we embedded $N(v_j)$ for $v_j \in N(s)$ in the construction of S_i. Condition (ρ3) is never violated because of the way we embedded $N(s)$ in the first step of the construction of S_i. Finally, condition (ρ4) is never violated because of the way we embedded $N(v_j)$ for $v_j \in \mathcal{I}$ in the construction of S_i.

In fact, we obtain a slightly stronger property than (ρ4), and this will be useful later:

Condition (ρ4) remains true if we define *local spin* using X^ℓ instead of X. (4)

5.2 Planarity of blocks

A *block* of G is an image $\pi([A])$ under the covering map $\pi\colon \mathbb{T} \to G$, where A denotes a pre-block of $\mathbb{T}$ and $[A] := \{x \in V(\mathbb{T}) \mid x \simeq_N y \text{ for some } y \in A\}$ denotes its $N(\mathcal{R})$-equivalence class.

Note that every block of G is connected: given vertices x, z in a block $K = \pi([A])$, we can find $x', z' \in A$ (and not just in the $N(\mathcal{R})$-orbit of A) with $\pi(x') = x, \pi(z') = z$, and so the x'–z' path P in A yields the x–z path $\pi(P)$ in K.

The main result of this section is

Lemma 5.3. *Every block of G is planar.*

In fact, we will prove a stronger statement similar to Theorem 3.3 ([13, Theorem 4.2]), namely, that every block admits an embedding into $\mathbb{R}^2$ respecting σ and τ.

The proof of this follows the lines of our proof of the planarity of G in the consistent case ([13, Theorem 4.2]), and we assume that the reader is familiar with that proof. Here we will point out the differences.

The proof of Lemma 5.3 begins here, and occupies the rest of Section 5.2, which includes further lemmas needed for its proof.

Proof of Lemma 5.3: Let K be a block of G. Let D be a fundamental domain of K in $\mathbb{T}$; that is, D is a subset of $\mathbb{T}$ containing exactly one point from each $N(\mathcal{R})$-orbit O such that $\pi(O) \in K$. This exists assuming Zorn's Lemma. We may assume that D is connected since K is, see [13, Lemma 4.1] for details. Moreover, we may assume without loss of generality that D is a union of stars. Thus the closure $\overline{D}$ of D in $\mathbb{T}$ is still the union of D with all midpoints of edges that have exactly one half-edge in D, and K can be obtained from $\overline{D}$ by identifying pairs of $N(\mathcal{R})$-equivalent midpoints. As in the proof of [13, Theorem 4.2], we will prove that any two pairs of such $N(\mathcal{R})$-equivalent midpoints are *nested*, where we say that two pairs of midpoints x, x' and y, y' in $\overline{D} \setminus D$ are nested, if the x-x' path in D does not cross the y-y' path, where we define crossing similarly to Section 2.3.

In order to guarantee this nestedness, we will have to embed $\mathbb{T}$ appropriately; in our general setup, $\mathbb{T}$ cannot be embedded consistently as in the case of special planar presentations, and this is why we are now only trying to prove the planarity of a block, and not of all of G at once.

For a relator W, we use W_o to denote the closed walk $o_G W$ in G induced by W at o_G, and let $\mathbb{T}_W := \pi^{-1}(W_o)$, which is a union of a set of double-rays of $\mathbb{T}$, which set we denote by $\mathbb{T}[W_o]$.

Recall we have chosen an embedding ρ of $\mathbb{T}$ in Section 5.1. For a pre-block C of $\mathbb{T}$, we define a *super-face* of C to be a face of the embedding $\rho(C)$ of C inherited by ρ. The super-faces of $\mathbb{T}$ are the super-faces of all of the pre-blocks of $\mathbb{T}$. Note that a super-face can contain several faces of $\mathbb{T}$.

The *dual* graph $\mathbb{T}^*$ of $\mathbb{T}$ is the graph whose vertex set is the set of faces of $\mathbb{T}$, and two faces of $\mathbb{T}$ are joined with an edge e^* of $\mathbb{T}^*$ whenever their boundaries share an edge e of $\mathbb{T}$. For two faces F, H of $\mathbb{T}$ and an F–H path P_{FH} in $\mathbb{T}^*$, let $Cr(\mathbb{T}[W_o], P_{FH})$ denote the *number of crossings* of $\mathbb{T}[W_o]$ by P_{FH}; to make this more precise, for a double-ray T in $\mathbb{T}[W_o]$, we write $cr(T, P_{FH})$ for the number of edges e in T such that P_{FH} contains e^*, and we let $Cr(\mathbb{T}[W_o], P_{FH}) := \sum_{T \in \mathbb{T}[W_o]} cr(T, P_{FH})$. We claim that

$$\text{for every two faces } F, H \text{ of } \mathbb{T}, \text{ the parity of the number of crossings } Cr(\mathbb{T}[W_o], P_{FH}) \text{ is independent of the choice of the path } P_{FH}. \tag{5}$$

To see this, note that if C is a cycle in $\mathbb{T}^*$, then $Cr(\mathbb{T}[W_o], C)$ —defined similarly to $Cr(\mathbb{T}[W_o], P_{FH})$— is even because the embedding of $\mathbb{T}$ is accumulation-free and so any ray entering the bounded side of C has to exit it again. This immediately implies (5).

We will define our relation $\sim_K$, or just $\sim$ if K is fixed, on the set of super-faces of pre-clusters contained in $\pi^{-1}(K)$. Given two super-faces F, H lying in pre-clusters contained in $\pi^{-1}(K)$, let $\mathbb{T}[W_o]_K$ denote the subset of $\mathbb{T}[W_o]$

contained in $\pi^{-1}(K)$. Now pick two faces $F' \subseteq F, H' \subseteq H$ contained in the super-faces F, H, and write $F \sim H$ if for each F'–H' path $P_{F'H'}$ in $\mathbb{T}^*$, the number of crossings $Cr(\mathbb{T}[W_o]_K, P_{F'H'})$ of $\mathbb{T}[W_o]_K$ by $P_{F'H'}$ is even. Since $Cr(\mathbb{T}[W_o]_K, P_{F'H'})$ is independent of the choice of $P_{F'H'}$ by (5), it is also independent of the choice of F', H', because if F'' is another face contained in F, then the F'–F'' path of $\mathbb{T}^*$ contained inside F crosses no element of $\mathbb{T}[W_o]_K$, because a super-face of any pre-cluster C in $\pi^{-1}(K)$ meets no element of $\mathbb{T}[W_o]_K$ by the definitions.

5.2.1 The bipartitions $\{I, O\}$

An important part of our planarity proof in the consistent case was that $\sim$ was invariant under the action of $N(\mathcal{R})$, see [13, Lemma 4.4]. Below (Lemma 5.8) we prove an analogous statement for the general case, namely that the restriction of $\sim$ to the super-faces of the pre-blocks in $\pi^{-1}(K)$ is $N(\mathcal{R})$-invariant.

The rest of our proof of Lemma 5.3 is almost identical to that of [13, Theorem 4.2], except that we are now working with the block K of G rather than the whole graph.

The equivalence relation $\sim$, now restricted on the set of super-faces $\mathcal{F}$ of $\pi^{-1}(K)$, uniquely determines a bipartition $\{I, O\}$ on $\mathcal{F}$ by choosing one super-face $F \in \mathcal{F}$ and letting $I := \{H \in \mathcal{F} \mid H \sim F\}$ and $O := \mathcal{F} \setminus I$.

Next, we adapt the material of [13, Section 4.3.1] to our new setup. For every super-face F in $\pi^{-1}(K)$, glue a copy of the domain $\overline{F} \subset \mathbb{R}^2$ to K by identifying each point of ∂F with $\pi(\partial F)$. If F, F' are equivalent face boundaries, in other words, if $\pi(\partial F) = \pi(\partial F')$, then we identify the corresponding 2-cells glued onto K. Let K^2 denote the set of these 2-cells, and let $\overline{K} = K \cup K^2$ denote the 2-complex consisting of K and these 2-cells. Notice that every edge $e \in E(K)$ has only 1 or 2 incident 2-cells in K^2.

Lemma 5.8 now means that if Z is a closed walk of G (here we really mean G and not just K) induced by a relator, then $\{I, O\}$ induces a bipartition $\pi[I], \pi[O]$ of K^2. Let us still denote this bipartition of K^2 by B_Z.

We extend that bipartition to an arbitrary cycle in K: given a cycle C of K, we choose a 'proof' P of C; that is, a sequence of closed walks $W_i, 1 \leq i \leq k$ of G induced by rotations of relators such that $C = \sum_{1 \leq i \leq k} W_i$. The existence of such a sequence (W_i) is not affected by the fact that we are focusing on a subgraph K; the W_i are allowed to be arbitrary relators. For every W_i, let I_{W_i}, O_{W_i} denote the two sides of the bipartition B_{W_i} of K^2 from above, and define the bipartition $B_C := \{I_C, O_C\}$ of K^2 by $I_C := \triangle_i I_{W_i}$ and $O_C := G^2 \triangle I_C$.

While in the definition of B_C it appears that it depends on the proof P, it actually does not as we shall see later. Until then, we denote it by $B_C(P)$. Our next aim is to show that, in a certain way, $B_C(P)$ behaves like the bipartition of the faces of a plane graph induced by a cycle C: to move between the two sides, one has to cross an edge of C. This is achieved by Lemma 5.5 below, for the proof of which we need the following.

Lemma 5.4. *Let e be a directed edge of K, let $W \in \mathcal{R}$ be a relator which is not of the form $b^2 = 1$ for $b \in \mathcal{S}$, and let $o_K W$ be the closed walk of K rooted at some vertex o_K of K induced by W. Then the number of double-rays in $\mathbb{T}[W_o]$ containing e equals the number of times that $o_K W$ traverses $\pi(e)$.*

Proof. If $o_K W$ does not traverse $\pi(e)$ then $\mathbb{T}[W_o]$ avoids e and we are done. So suppose that $o_K W$ does traverse $\pi(e)$. Let $o_K W^\infty$ denote the two-way infinite walk on K obtained by repeating $o_K W$ indefinitely. Let $T \in \mathbb{T}[W_o]$ be the lift of $o_K W^\infty$ to $\mathbb{T}$ (via π^{-1}) sending $\pi(e)$ to e, and note that T is a double-ray containing e. Let Q be the subpath of T that starts with e and finishes when a rotation of the word W is completed. By the definition of $\mathbb{T}[W_o]$, there is a 1–1 correspondence between the elements of $\mathbb{T}[W_o]$ containing e and the directed edges e' in Q that are $N(\mathcal{R})$-equivalent to e: each such element of $\mathbb{T}[W_o]$ can be obtained by translating T by the automorphism of $\mathbb{T}$ sending e' to e.

Now note that $o_K W$ traverses $\pi(e)$ whenever its lift T traverses one of those e'. Combined with the above observations this proves our assertion. □

Lemma 5.5. *For every $e \in E(K)$, the bipartition $B_C(P)$ separates 2-cells of e if and only if $e \in C$.*

Proof. Let I, O be the two elements of $B_C(P)$ as defined above. Then, letting $1_{F \in I}$ denote the indicator function of $F \in I$, we have

$$1_{F \in I} = N_F := |\{W_i \mid F \in I_{W_i}\}| \pmod 2,$$

and similarly

$$1_{H \in I} = N_H := |\{W_i \mid H \in I_{W_i}\}| \pmod 2.$$

But

$$N_F + N_H = |\{W_i \mid W_i \text{ separates } F \text{ from } H\}| \pmod 2$$

by the construction of I, O. We claim that $|\{W_i \mid W_i \text{ separates } F \text{ from } H\}|$ is odd if and only if $e \in E(C)$. Indeed, B_{W_i} separates F from H exactly when W_i traverses e an odd number of times by

for every edge e of $\mathbb{T}$, the two faces F, H of e lie in distinct elements of $\{I, O\}$ if and only if $e \in \mathbb{T}_W$ and e lies in an odd number of elements of $\mathbb{T}[W_o]$ (6)

and Lemma 5.4, and e is in C exactly when there is an odd number of W_i that traverse e an odd number of times.

Since that number is even if $e \notin E(C)$ and odd otherwise, our last congruence yields $N_F + N_H = 1 \pmod 2$ if and only if $e \in E(C)$. Therefore, the previous congruences imply that $1_{F \in I} = 1_{H \in I}$ if $e \notin E(C)$ and $1_{F \in I} \neq 1_{H \in I}$ if $e \in E(C)$, which is our claim. □

Lemma 5.5 implies in particular that $B_C(P)$ is characterised by C alone and is therefore independent of P, since $\overline{K}$ was defined without reference to P. Thus we can denote it by just B_C from now on.

In the following, we use again the definition of a crossing from Section 2.3.

Lemma 5.6. *Let C' be a finite path of $\mathbb{T}$ such that $C := \pi(C')$ is a cycle of K, and let $Q = e\mathring{Q}f$ be a crossing of C' in $\mathbb{T}$. Then B_C separates the 2-cells incident with $\pi(e)$ from the 2-cells incident with $\pi(f)$. Moreover, if Q_2 is a path of $\mathbb{T}$ such that $\pi(Q_2)$ is a cycle of K, then Q_2 crosses C' an even number of times.*

Proof. Let F be a face incident with the first edge e of Q, and let H be a face incident with the last edge f of Q. By the definition of a crossing, we can find a finite sequence $(F =)F_1, \dots, F_k(= H)$ of faces of $\mathbb{T}$ such that each F_i shares an edge e_i with F_{i+1} and exactly one of the e_i lies in C': we can visit all faces incident with Q until we reach H. By Lemma 5.5 and Lemma 5.8, B_C separates $\pi(F_1)$ from $\pi(F_k)$. This proves our first assertion.

For the second assertion, note that $\pi(Q_2)$ can be written as a concatenation of subarcs $C_1D_1C_2D_2 \dots C_k = C_1$ where each C_i lifts to a crossing of C' by Q_2 and each D_i avoids C and shares exactly one end-edge with each of C_i and C_{i+1}. We proved above that the 2-cells incident with end-edges of each C_i are separated by B_C. The same arguments imply that the 2-cells incident with end-edges of each D_i are *not* separated by B_C. Since $\pi(Q_2)$ is a cycle, this implies that Q_2 crosses C' an even number of times. □

As in the end of the proof of Theorem 3.3, the last lemma says that any two cycles of K cross each other an even number of times, and therefore any two pairs of identified points of $\overline{D}$ are nested.

This completes the proof of Lemma 5.3, except that we still have to prove Lemma 5.8. For this, we will need the following lemma.

Lemma 5.7. *For $b \in \mathcal{I}$ with $b = 1$, and any relator W in $\mathcal{R}$, the number of elements of $\mathbb{T}[W_o]$ containing any edge e labelled by b is even.*

Proof. Let T be an element of $\mathbb{T}[W_o]$ containing e. The automorphism β of $\mathbb{T}$ exchanging the two endvertices of e maps T to an element T' of $\mathbb{T}[W_o]$ because $b = 1$ and so the two end-vertices of e are $N(\mathcal{R})$-equivalent. Note that $T \neq T'$ even if T, T' contain the same vertices, because they have opposite directions (remember that double-rays are directed by definition). Note that $\beta(T') = T$. Therefore, β establishes a bijection without fixed points on the elements of $\mathbb{T}[W_o]$ containing e, which means that the number of those elements is even. □

Lemma 5.8. *For every block K of G, the restriction of $\sim_K$ to the super-faces of $\pi^{-1}(K)$ is invariant under the action of $N(\mathcal{R})$ on $\mathbb{T}$.*

Proof. We will adapt the proof of [13, Lemma 4.4]. Since K is fixed, let us just write $\sim$ instead of $\sim_K$.

We need to prove that if F, H are super-faces of $\pi^{-1}(K)$ in the same orbit of $N(\mathcal{R})$, then $F \sim H$. Again, we may assume that there are vertices x, y in the boundaries of F, H respectively, such that $y = xwRw^{-1}$ for some word w and some relator $R \in \mathcal{R}$: by the definition of the normal closure $N(\mathcal{R})$, if we can prove $F \sim H$ in this case, we can prove $F \sim H$ for every two F, H in the same orbit of $N(\mathcal{R})$.

Let α_{FH} be the automorphism of $\mathbb{T}$ mapping x to y.

Decompose the path $Q := xwRw^{-1}$ into (inclusion-)maximal subpaths contained in a pre-block. Then we can write

$$Q = P_1 \cup P_2 \cup \ldots \cup P_k (= P'_k) \cup P'_{k-1} \cup \ldots \cup P'_1,$$

where the P_i, P'_i are those maximal subpaths, P'_i is $N(\mathcal{R})$-equivalent to P_i for every $i < k$, and P_k contains the subpath of Q induced by R (such a P_k exists because every relator R is blocked). Note that the intersection of any two subsequent P_i or P'_i is either a hinge separating the corresponding pre-blocks, or a single vertex incident with such a hinge.

Since we are free to choose any F–H walk P_{FH} in $\mathbb{T}^*$ to decide whether $F \sim H$, we will choose a convenient one, which we construct now.

Recall that every $P_i, i > 1$ starts and ends at hinges, which we will call h_{i-1}, h_i, separating its pre-block from the pre-blocks containing P_{i-1}, P_{i+1} respectively; here h_{i-1}, h_i may or may not be contained in P_i as end-edges.

Let C_i be the pre-block containing P_i and let C'_i be the pre-block containing P'_i.

Let $\Pi_i, k > i > 1$, be an (inclusion-)minimal path in $\mathbb{T}^*$ joining a super-face incident with h_{i-1} to a super-face incident with h_i —where we say that a super-face F is *incident* with an edge if the boundary of F contains that edge— such that all vertices of Π_i are faces sharing a vertex with P_i, and Π_i does not intersect P_i (at a midpoint of any edge); see Figure 7. Define Π'_i similarly using P'_i instead of P_i. Note that there are exactly two such paths Π_i to choose from, one on either side of P_i; it doesn't matter much which of the two we will choose, but let us make 'the same' choice for both Π_i and Π'_i; more precisely, we ensure that

$$\begin{gathered}\Pi_i \text{ crosses an edge } e \text{ of } C_i \text{ (incident with } P_i\text{) if and only if } \Pi'_i \text{ crosses} \\ \text{the edge } \alpha_{FH}(e) \text{ of } C'_i.\end{gathered} \tag{7}$$

This is possible because ρ embeds C_i the same way as C'_i up to reflection, and Π_i is uniquely determined once we choose which of the two super-faces of C_i incident with h_i we want it to contain; by choosing Π'_i to contain the corresponding super-face incident with h'_i, our claim is satisfied. Note that Π_i does not cross h_i, because if it did we could shorten it.

For $i = 1$ we let Π_1 be a minimal path in $\mathbb{T}^*$ joining F to a super-face incident with h_1, and otherwise be defined similarly to $\Pi_i, k > i > 1$. Define Π'_1 similarly. Finally, let $\Pi_k = \Pi'_k$ be a minimal path in $\mathbb{T}^*$ joining a super-face incident with h_{k-1} to a super-face incident with $\alpha_{FH}(h_{k-1})$ without crossing P_k.

Let $\sqcup_i, k > i \geq 1$ be a path in $\mathbb{T}^*$ joining the last vertex of Π_i to the first vertex of Π_{i+1} such that all vertices of $\sqcup_i$ are faces sharing a vertex with $P_i \cap P_{i+1}$, and define $\sqcup'_i$ similarly for Π'_i, Π'_{i+1}; there are several choices for this $\sqcup_i$, so let us make it uniquely determined: if $P_i \cap P_{i+1}$ is a single vertex, then there are two candidates, and we always choose the one crossing h_i. If $P_i \cap P_{i+1}$ is the hinge h_i, then there are up to four choices, and we choose the one that crosses h_i and is contained in the two super-faces of C_i incident with h_i and in the two super-faces of C_{i+1} incident with h_i.

It follows from the choice of $\sqcup_i$ that it behaves well with respect to elements of $\mathcal{C}$:

$$\text{If } \sqcup_i \text{ meets an edge in } B_i(v)\setminus\{h_i\} \text{ (where } B_i\in\mathcal{C}) \text{ where the vertex } v \text{ is incident with } h_i, \text{ then } \sqcup_i \text{ meets every edge of } B_i(v). \tag{8}$$

A similar but slightly stronger is true for Π_i:

$$\text{If } \Pi_i \text{ meets an edge lying inside some super-face of } C_i, \text{ then } \Pi_i \text{ visits all faces incident with } P_i \text{ inside that super-face.} \tag{9}$$

Indeed, Π_i is by definition a minimal path joining certain super-faces of C_i; therefore, it crosses any super-face either completely or at a single boundary edge.

Finally, we obtain P_{FH} by concatenating all the $\Pi_i, \sqcup_i, \Pi'_i$ and $\sqcup'_i$:

$$P_{FH} := \Pi_1 \cup \sqcup_1 \cup \Pi_2 \ldots \cup \sqcup_{k-1} \cup \Pi_k (= \Pi'_k) \cup \sqcup'_{k-1} \ldots \cup \sqcup'_1 \cup \Pi'_1.$$

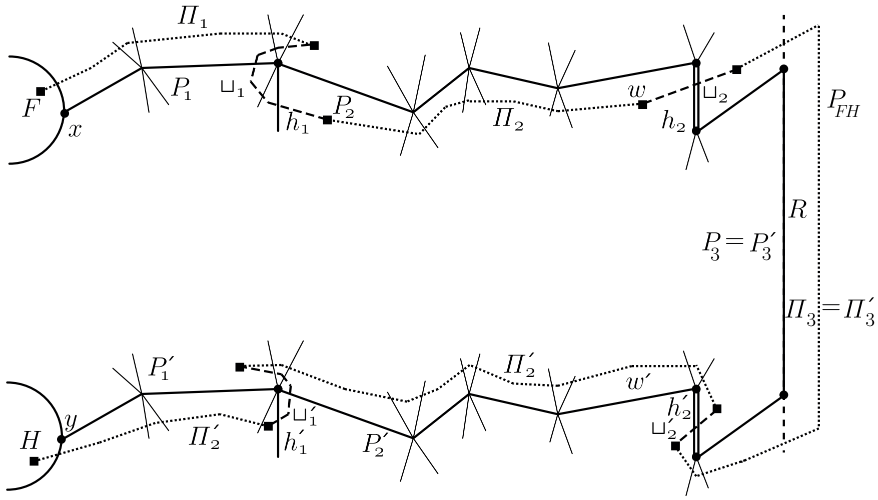

Figure 7: The path P_{FH} (dashed) in the proof of Lemma 5.8 with the paths Π_i, Π'_i, $\sqcup_i$, $\sqcup'_i$.

We need to check that $Cr(\mathbb{T}[W_o]_K, P_{FH})$ is even. We will do so by showing that the contributions of the Π_i to $Cr(\mathbb{T}[W_o]_K, P_{FH})$ cancel with those of the Π'_i, and the contributions of the $\sqcup_i$ cancel with those of the $\sqcup'_i$.

Let T be an element of $\mathbb{T}[W_o]_K$ with odd $cr(T, P_{FH})$, i.e. with an odd number of crossings of T by P_{FH}; only such T matter. Let $T' := \alpha_{FH}(T)$.

Let us first consider the total number of crossings of such T by the subpaths $\Pi_i, \Pi'_i, i < k$, of P_{FH}.

If T is contained in C_i, then $cr(T, \Pi_i) = cr(T', \Pi'_i)$ by (7).

If T is not contained in C_i, then Π_i crosses T an even number of times (0 or 2): this is easy to see when $T \cap P_i$ is a single vertex v by applying (9) to that vertex. The situation is slightly subtler when $T \cap P_i$ is a hinge g —no other option is possible as distinct pre-blocks intersect at an edge at most by construction. In this case, we remark that the pre-block D containing T lies in some super-face of C_i by the construction of ρ, and again Π_i must cross all faces incident with g inside that super-face by (9), therefore crossing both edges of T incident with g.

Finally, it is not hard to see that $\Pi_k = \Pi'_k$ has an even contribution to $Cr(\mathbb{T}[W_o]_K, P_{FH})$.

These facts combined show that $\sum_{T \in \mathbb{T}[W_o]_K} cr(T, \bigcup_i \Pi_i)$ is even.

Next, we consider the total number of crossings of such T by the subpaths $\sqcup_i, \sqcup'_i$. Suppose $cr(T, \sqcup_i)$ is odd. Then it must equal 1 as $\sqcup_i$ is too short to cross a double-ray three times, where we used property ($\rho 3$) of our embedding ρ that pre-blocks do not cross each other.

Let v_i be the last vertex of P_i and v'_i the last vertex of P'_i. If the local spin at v_i with respect to h_i coincides up to reflection with the local spin at v'_i with respect to h'_i, then $cr(T, \sqcup_i) = cr(T', \sqcup'_i)$ (here, local spin refers to X^ℓ rather than X; recall (4)). Therefore, the total contribution of the pair T, T' to $Cr(\mathbb{T}[W_o]_K, P_{FH})$ is even and can be ignored.

If those local spins do not coincide up to reflection, then by the choice of ρ ($\rho 4$), the label of h_i is an involution $b \in \mathcal{I}$ with $b = 1$. In this case however, Lemma 5.7 applies, yielding that the set H of elements of $\mathbb{T}[W_o]_K$ containing h_i is even. We claim that $T \in H$ (i.e. $h_i \subset T$): this follows from $cr(T, \sqcup_i) = 1$, the fact that $\sqcup_i$ only contains faces of $\mathbb{T}$ incident with h_i by its construction, and (8). Moreover, (8) also implies that $cr(R, \sqcup_i) = 1$ for every other $R \in H$. But as $|H|$ is even, the total contributions $\sum_{R \in H} cr(R, \sqcup_i)$ of its elements are even and can be ignored as well.

Summing up, we proved that both

$$\sum_{T \in \mathbb{T}[W_o]_K} cr(T, \bigcup_i \Pi_i) \quad \text{and} \quad \sum_{T \in \mathbb{T}[W_o]_K} cr(T, \bigcup_i \sqcup_i)$$

are even. Therefore $Cr(\mathbb{T}[W_o]_K, P_{FH})$ is even as well, since it is the sum of those two sums by definition. □

The proof of Lemma 5.3 is now complete. □

5.3 From the planarity of blocks to the planarity of G

The main aim of this section is to prove

Lemma 5.9. *Every hinge of G separates its incident blocks.*

Proof. The statement is equivalent to the statement that every cycle of G *crosses* each hinge b an even number of times, where the number of crosses of b by C is the maximum number of edge disjoint subpaths P_i of C such that b separates each P_i into two (possibly trivial, but non-empty) subpaths that lie in distinct blocks.

To prove the latter, let $C = c_0c_1 \dots c_k$ with $c_k = c_0$ be a cycle, and let $L = t_0t_1 \dots t_k$ be a lift of C to $\mathbb{T}$ via π^{-1}. Fix a hinge b. We may assume without loss of generality that c_0 is not a vertex of b. Let $P = w_1R_1w_1^{-1} \dots w_kR_kw_k^{-1}$ be a proof of C in our presentation.

Since $c_0 \notin b$ and since the end vertices of $w_iR_iw_i^{-1}$ are $N(\mathcal{R})$-equivalent to c_0, any crossings of b by P occur inside the subpaths $w_iR_iw_i^{-1}$ and not when switching from w_{i-1} to w_i. We have no crossings of b inside any R_i because our relators are blocked. Moreover, any crossings of b inside a w_i are paired up by crossings of b inside w_i^{-1}. Thus the number of crossings of b by P, and hence by C, is even. □

This, combined with the planarity of blocks we proved in the previous section, easily implies the planarity of G:

Theorem 5.10. *Let G be the Cayley graph of a generic planar presentation. Then G is planar.*

Proof. Combining Lemma 5.3 with Lemma 5.9 easily yields that G is planar. Indeed, we can embed G one block at a time: since incident blocks share a hinge only by Lemma 5.9, if we have already embedded a block A meeting a block B at a hinge b, then it is easy to embed B inside one of the two faces (we are free to choose) of the current embedding whose boundary contains b. □

6 Conclusions

We now put the above results together to prove the statements of the introduction. Because of the redundant generators used in Lemmas 4.1 and 4.3, we need to generalise our notion of planar presentation slightly. We say that $s \in \mathcal{S}$ is an *obviously redundant* generator of a presentation $\langle \mathcal{S} \mid \mathcal{R} \rangle$, if there is exactly one relator $W_s \in \mathcal{R}$ in which either s or s^{-1} appears exactly once and the other does not appear in it at all. A *general planar presentation* is a presentation obtained from a generic planar presentation by recursively removing zero or more obviously redundant generators s along with the corresponding relator W_s. The last two sections prove the two directions of Theorem 1.1:

Proof of Theorem 1.1. If G is a finitely generated planar Cayley graph, then by Lemmas 4.1 and 4.3 we may find a Tietze-supergraph that is is 2-connected and well-separated. Theorem 4.4 then yields a generic planar presentation, from which we can remove any generators that were not present in G to obtain a general planar presentation of G, which proves the forward direction.

For the backward direction, if G admits a general planar presentation, then some supergraph G' admits a generic planar presentation, and is thus planar by Theorem 5.10. Since planarity is preserved under deleting edges, so is G. □

A similar result holds when we insist that there is a consistent embedding, and we can even allow our Cayley graphs to have infinitely many generators:

Theorem 6.1. *A Cayley graph admits a consistent embedding in the plane if and only if it admits a special planar presentation.*

The two directions of Theorem 6.1 are given by Theorem 4.6 and Theorem 3.3.

Next, we use our presentations to obtain effective enumerations.

Theorem 6.2. *The Cayley graphs that admit a consistent embedding in the plane are effectively enumerable.*

Proof. By Theorem 6.1, it suffices to produce an effective enumeration of the special planar presentations. For this, it suffices to produce an enumeration of the embedded presentations, and output those embedded presentations that satisfy the three conditions in the definition of a special planar presentation (Definition 3.1); it is easy to see that these conditions can be checked algorithmically. □

Theorem 6.3. *The planar, locally finite, Cayley graphs are effectively enumerable.*

Proof. Similarly to the proof of Theorem 6.2, we remark that any effective enumeration of the general planar presentations gives rise to an effective enumeration of the planar Cayley graphs by Theorem 1.1.

To effectively enumerate the general planar presentations, we start with an enumeration of the generic embedded presentations, and output those that satisfy the four conditions of Definition 3.4, which can be checked algorithmically. Having thus effectively enumerated the generic planar presentations, we remove any obviously redundant generators to effectively enumerate the general planar presentations: for each output $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$, check for every $s \in \mathcal{S}$ whether s is an obviously redundant generator. For every such s found, output the presentation $G' := \langle \mathcal{S} \setminus \{s\} \mid \mathcal{R} \setminus \{W_s\} \rangle$. Then, recursively apply the same check to G', removing any obviously redundant generators of that presentation and so on. □

We conclude with some related questions concerning embeddings of Cayley complexes. Let $CC(\mathcal{P})$ denote the Cayley complex of a presentation $\mathcal{P}$. Call a map $\rho\colon CC(\mathcal{P}) \to \mathbb{R}^2$ *consistent* if its restriction to $Cay(\mathcal{P})$ is consistent. Call ρ *nested* if it witnesses the fact that $CC(\mathcal{P})$ is almost planar, i.e. if the images under ρ of the interiors of any two 2-cells are either disjoint, or one is contained in the other.

The following might be interesting as it exhibits a geometric property of Cayley complexes which can be decided by an algorithm.

Theorem 6.4. *There is an algorithm that given a presentation $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ decides whether $CC(\mathcal{P})$ admits a nested, consistent map into $\mathbb{R}^2$.*

Proof. We claim that $CC(\mathcal{P})$ admits a nested, consistent map into $\mathbb{R}^2$ if and only if there is a spin σ on $\mathcal{S}$ and a 'spin-behaviour' function τ from $\mathcal{S}$ to $\{0,1\}$ such that the triple $(\mathcal{P}, \sigma, \tau)$ is a special planar presentation.

To prove the backward direction, note that if $\mathcal{P}, \sigma, \tau$ is a special planar presentation, then $Cay(\mathcal{P})$ admits a consistent embedding ρ into $\mathbb{R}^2$ by Theorem 3.3. Extend this embedding into a map ρ' from $CC(\mathcal{P})$ to $\mathbb{R}^2$ by mapping each 2-cell inside the closed curve to which ρ maps its boundary. Then ρ' is nested because no two words in $\mathcal{R}$ cross each other by the definition of a special planar presentation.

For the forward direction, given such a map $\rho \colon CC(\mathcal{P}) \to \mathbb{R}^2$, we can read the spin data σ, τ from ρ since ρ is consistent. Then $\mathcal{P}, \sigma, \tau$ is an embedded presentation. To prove that it is a special planar presentation it remains to show that no two words in $\mathcal{R}$ cross each other, which follows immediately from the nestedness of ρ. □

By using general planar presentations instead of special ones, Theorem 6.4 can be generalised to yield a further decidable property of Cayley complexes, but instead of maps into $\mathbb{R}^2$ we have to consider maps into larger spaces obtained by glueing copies of $\mathbb{R}^2$ along (possibly closed) bounded simple curves —to which we map the hinges of our Cayley graphs— in a tree like fashion. We leave the details to the interested reader.

Our results do not yet answer the following

Problem 6.5. *Is there an algorithm that given a presentation $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ decides whether $CC'(\mathcal{P})$ is planar?*

In this problem $CC'(\mathcal{P})$ denotes the complex obtained from $CC(\mathcal{P})$ by removing redundant 2-cells, that is, if a set of 2-cells have the same boundary, we remove all but one of them. Some authors still call $CC'(\mathcal{P})$ the Cayley complex of $\mathcal{P}$. (In Theorem 6.4 it does not make a difference whether we consider $CC(\mathcal{P})$ or $CC'(\mathcal{P})$.)

We remark that it is not true that $CC(\mathcal{P})$ is planar if and only if $\mathcal{P}$ is a facial presentation in the sense of [9]; the presentation $\mathcal{P} = \left\langle a, b \mid a^2, b^3, ab^{-1} \right\rangle$ if facial, but its Cayley complex consists of a single vertex, two loops, a 2-cell winding twice around a loop, and a 2-cell winding three times around the other loop.

Having studied embeddings of Cayley complexes in $\mathbb{R}^2$, the following problem suggests itself

Problem 6.6. *Which groups admit a Cayley complex embeddable in $\mathbb{R}^3$?*

7 Further remarks

We proved that every planar Cayley graph G admits a planar presentation such that every relator induces a cycle of G (rather than an arbitrary closed walk with repetitions of vertices). It would be interesting if we could strengthen the definition of a planar presentation in such a way that this is always the case in the resulting planar Cayley graph. Some strengthening will be necessary as shown by the example $\mathcal{P} = \left\langle a, b \mid a^2, b^3, ab^{-1} \right\rangle$ from the previous section. This is a planar presentation —even stronger, every relator is facial— but it is easy to see that its group is the group of one element. Our optimism that this may be possible stems from the fact that it was possible in the cubic case [10].

A further interesting question, also asked in [10], is whether for every $n \in \mathbb{N}$ there is an upper bound $f(n)$, such that every n-regular planar Cayley graph admits a planar presentation with at most $f(n)$ relators. This would strengthen Droms' result [5, Theorem 5.1] that finitely generated planar groups are finitely presented.

It is known that the fundamental group of a finite graph of groups with residually finite vertex groups and finite edge groups is residually finite [25, II.2.6.12]. Dunwoody [8, Theorem 3.8] proved that planar groups have this structure, and so we obtain the following corollary, to which this paper has no contribution

Corollary 7.1. *Every planar group is residually finite.*

From this we deduce that the finitely generated planar groups have a uniformly solvable word problem, as this is the case more generally for finitely presented residually finite groups [1, Lemma 4.3]. The standard algorithm is however impractical. We would be interested to see bounds on the complexity of the word problem for our groups.

References

[1] M. Aschenbrenner, S. Friedl, and H. Wilton. Decision problems for 3?manifolds and their fundamental groups. *Geometry & Topology Monographs*, 19:201–236, 2015.

[2] O. Bogopolski. *Introduction to Group Theory.* EMS, Zuerich, Switzerland, 2008.

[3] W. Dicks and M. J. Dunwoody. *Groups acting on graphs.* Cambridge University Press, 1989.

[4] Reinhard Diestel. *Graph Theory* (3rd edition). Springer-Verlag, 2005. Electronic edition available at: `http://www.math.uni-hamburg.de/home/diestel/books/graph.theory`.

[5] C. Droms. Infinite-ended groups with planar Cayley graphs. *J. Group Theory*, 9(4):487–496, 2006.

[6] C. Droms, B. Servatius, and H. Servatius. The structure of locally finite two-connected graphs. *Electron. J. Combin.*, 2:#R17, 1995.

[7] C. Droms, B. Servatius, and H. Servatius. Connectivity and planarity of Cayley graphs. *Beiträge Algebra Geom.*, 39(2):269–282, 1998.

[8] M.J. Dunwoody. Planar graphs and covers. arXiv:0708.0920, 2007.

[9] A. Georgakopoulos. Characterising planar Cayley graphs and Cayley complexes in terms of group presentations. *European J. Combin.*, 36:282–293, 2014.

[10] A. Georgakopoulos. The planar cubic Cayley graphs. *Mem. Amer. Math. Soc.*, 250(1190), 2017.

[11] A. Georgakopoulos. The planar cubic Cayley graphs of connectivity 2. *European J. Combin.*, 64:152–169, 2017.

[12] A. Georgakopoulos. On planar Cayley graphs and Kleinian groups. *Trans. Amer. Math. Soc.*, 373:4649–4684, 2020.

[13] A. Georgakopoulos and M. Hamann. The planar Cayley graphs are effectively enumerable I: consistently planar graphs. *Combinatorica*, 39(5):993–1019, 2019.

[14] M. Hamann. Generating the cycle space of planar graphs. *Electron. J. Combin.*, 22:#P2.34, 2015.

[15] M. Hamann. Planar transitive graphs. *Electron. J. Combin.*, 25:#P4.8, 2018.

[16] W. Imrich. On Whitney's theorem on the unique embeddability of 3-connected planar graphs. In *Recent Adv. Graph Theory, Proc. Symp. Prague 1974*, pages 303–306. 1975.

[17] H. Levinson and B. Maskit. Special embeddings of Cayley diagrams. *J. Combin. Theory Ser. B*, 18(1):12–17, 1975.

[18] Roger C. Lyndon and Paul E. Schupp. *Combinatorial Group Theory*. Springer Science & Business Media, January 2001.

[19] A. Marden. Geometrically finite Kleinian groups and their deformation spaces. In *Discrete groups and automorphic functions, W. Harvey (ed.)*, pages 259–293. Academic Press, London, 1977.

[20] Albert Marden. *Outer Circles*. Cambridge University Press, 2007.

[21] B. Maskit. On the classification of Kleinian groups: I—Koebe groups. *Acta Math.*, 135:249–270, 1975.

[22] Bernard Maskit. *Kleinian Groups*, volume 287 of *Grundlehren der mathematischen Wissenschaften*. Springer, 1988.

[23] J. Meier. *Groups, Graphs and Trees*. Cambridge University Press, 2008.

[24] D. Renault. Enumerating Planar Locally Finite Cayley Graphs. *Geom. Dedicata*, 112:25–49, 2005.

[25] J.-P. Serre. Arbres, amalgames, SL_2. *Astérisque*, 2(46), 1977.

[26] C. Thomassen and W. Woess. Vertex-transitive graphs and accessibility. *J. Combin. Theory Ser. B*, 58:248–268, 1993.

[27] W.T. Tutte. *Connectivity in graphs.* Mathematical Expositions. 15. Toronto: University of Toronto Press; London: Oxford University Press. IX, 145 p. , 1966.

[28] H. Whitney. Congruent graphs and the connectivity of graphs. *Amer. J. Math.*, 54(1):150–168, 1932.