MINOR EXCLUSION IN QUASI-TRANSITIVE GRAPHS

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Abstract. In this note, we show that locally finite quasi-transitive graphs are quasi-isometric to trees if and only if every other locally finite quasi-transitive graph quasi-isometric to them is minor excluded. This generalizes results by Ostrovskii and Rosenthal and by Khukhro on minor exclusion for groups.

1. Introduction

For two graphs G, H, we call H a *minor* of G if H can be obtained from G by contracting edges and deleting edges and vertices. A graph is *minor excluded* if there exists some finite graph that is not a minor of it.

In graphs, minor exclusion has played an important role for a long time, e. g. via Kuratowski's planarity criterion. Considering groups, minor exclusion was mostly considered in the case of planar groups, where a finitely generated group is *planar* if it has a planar locally finite Cayley graph. Ostrovskii and Rosenthal [7] looked at minor exclusion for groups from a broader viewpoint: do there exist locally finite groups all of whose locally finite Cayley graphs are minor excluded (not minor excluded)? They answered both questions positively: they proved for an infinite class of finitely generated groups that all of their locally finite Cayley graphs are not minor excluded and they proved that every locally finite Cayley graph of any finitely generated virtually free group is minor excluded. This latter result was extended by Khukhro [5]. She showed the reverse direction, i. e. characterised the finitely generated groups all of whose locally finite Cayley graphs are minor excluded as the finitely generated virtually free groups.

We generalise this characterisation to quasi-transitive graphs, where a graph is *quasi-transitive* if its automorphism group acts on it with only finitely many orbits. The analogue of looking at Cayley graphs in this situation is that we ask for minor exclusion for all locally finite quasi-transitive graphs that are quasi-isometric to the original one. For two graph G and H a map $\varphi\colon V(G) \to V(H)$ is *quasi-isometric* if there exist $\gamma \geq 1$ and $c \geq 0$ such that the following holds for all $x, y \in (G)$:

$$\frac{1}{\gamma} d_H(\varphi(x), \varphi(y)) - c \leq d_G(x, y) \leq \gamma d_H(\varphi(x), \varphi(y)) + c,$$

where d_G and d_H denote the distance functions in G and H, respectively. We will prove the following theorem.

Theorem 1.1. *Let G be a locally finite quasi-transitive graph. Then G is quasi-isometric to a tree, if and only if every locally finite quasi-transitive graph quasi-isometric to G is minor excluded.*

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Bonamy et al. [1] proved that all locally finite Cayley graphs of finitely generated groups of asymptotic dimension at least 3 are not minor excluded, moreover from their discussion follows that all quasi-transitive locally finite graphs of asymptotic dimension at least 3 are not minor excluded. On the other side, quasi-transitive locally finite graph that are quasi-isometric to trees have asymptotic dimension 1. However there are finitely generated groups of asymptotic dimension 1 that are not virtually free, see Gentimis [2]. The following question remains open: do there exist locally finite quasi-transitive graphs of asymptotic dimension at most 2 such that all locally finite quasi-transitive graphs quasi-isometric to it are not minor excluded?

2. Proof

Before we start the proofs, we need some definitions. Let G and H be graphs. We call H a *minor* of G, if there exists a set $\{G_x \mid x \in V(H)\}$ of disjoint subsets of $V(G)$ such that if $xy \in E(H)$, then there exists $uv \in E(G)$ with $u \in G_x$ and $v \in G_y$. The vertex sets G_x are the *branch sets*.

By G^2 we denote the graph with vertex set $V(G)$ such that two vertices are adjacent if and only if their distance in G is either 1 or 2.

A *ray* is a one-way infinite path. Two rays in G are *equivalent* if for every finite $S \subseteq V(G)$ there exists a component of $G - S$ that contains all but finitely many vertices of both rays. This is an equivalence relation whose classes are the *ends* of G. Let $m \in \mathbb{N}$. An end has *degree at least* m if it contains m pairwise disjoint rays. An end is *thin* if there exists an $n \in \mathbb{N}$ such that the end does not have degree at least n. It is *thick* if it is not thin.

Proposition 2.1. *Let G be an infinite graph with an end of degree at least m. Then G^2 contains a K_m-minor.*

Proof. Let ω be an end of G of degree at least m and let $R_1, \dots, R_m$ be m disjoint rays in ω. Let $\mathcal{P}_1, \dots, \mathcal{P}_n$ with $n = \frac{m(m-1)}{2}$ be an enumeration of the two-element subsets of $\{R_1, \dots, R_m\}$.

For all $i \in \{0, \dots, n\}$ we construct a finite vertex set S_i of G, m disjoint rays $R_1^i, \dots, R_m^i$ in G^2 and a set $\mathcal{Q}_i$ of i disjoint paths in G^2 such that the following holds:

(i) R_m^i and R_m coincide outside of S_i;
(ii) the last vertex of R_m^i in S_i lies on R_m;
(iii) the vertices of R_m^i outside of S_i form a tail of R_m^i;
(iv) every $Q \in \mathcal{Q}_i$ is internally disjoint from all R_j^i;
(v) every $Q \in \mathcal{Q}_i$ has all its vertices in S_i;
(vi) for every $1 \leq j \leq i$, there is a path $Q_j \in \mathcal{Q}_i$ joining the rays in $\mathcal{P}_j$.

Note that (i) implies that R_m^i has a tail in G.

For $i = 0$, set $S_i = \emptyset$ and $\mathcal{Q}_i = \emptyset$ and $R_j^i = R_j$ for all $1 \leq j \leq m$. This satisfies (i)–(vi) trivially.

Now let us assume that we have constructed S_{i-1}, $R_1^{i-1}, \dots, R_m^{i-1}$ and $\mathcal{Q}_{i-1}$. Let $\mathcal{P}_i = \{R_k, R_\ell\}$ and let Q be a path in $G - S_{i-1}$ joining R_k and R_ℓ. Note that Q also joins R_k^{i-1} and R_ℓ^{i-1}. We may assume that Q meets those two rays only in its end vertices. Let $R_{i_1}^{i-1}$ be the ray that Q meets first after its starting vertex. Let x_1 be the first common vertex on Q of Q and $R_{i_1}^{i-1}$ and let x_2 be the last such vertex. Now we modify Q and $R_{i_1}^{i-1}$ as shown in Figure 1: we remove every second vertex of

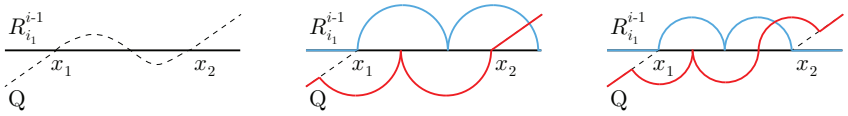

FIGURE 1. Modifying x_1Qx_2 and $x_1R_{i_1}^{i-1}x_2$ depending on the length of $x_1R_{i_1}^{i-1}x_2$: the middle figure shows an example for odd distance and the rightmost figure for even distance.

$x_1R_{i_1}^{i-1}x_2$ from that ray, so that the remaining vertices still form a ray in G^2. Now we replace x_1Qx_2 by the vertices that we just removed from $x_1R_{i_1}^{i-1}x_2$. Together with Qx_1 without x_1 and x_2Q without x_2 it forms a path in G^2. We continue doing these modifications for all remaining intersections of the new path with other rays $R_{i_j}^{i-1}$, where we put the vertices with even distance to the first vertex of Q and $R_{i_j}^{i-1}$ into the modification of Q if the last common vertex of $R_{i_{j-1}}^{i-1}$ and Q was not put into that path and otherwise the vertices with odd distance. If we have not modified the ray R_j^{i-1}, then we set $R_j^i := R_j^{i-1}$. By the choice of when to put the first common vertex of Q and the rays into the modification of Q or the new rays, the resulting path Q' and the rays $R_{i_j}^i$ are indeed a path and rays. Then Q' will be added to the set $\mathcal{Q}_{i-1}$ to obtain the set $\mathcal{Q}_i$. Let S_i be S_{i-1} together with all vertices of Q' and, for each $1 \leq j \leq m$, a finite starting path of R_j^i such that R_j^i coincides with R_j after this path and such that the last vertex of that path also lies on R_j. By construction, (i)–(vi) hold.

For $1 \leq i \leq m$, let P_i be a subpath of R_i^n that contains all vertices of R_i^n that lie on paths of $\mathcal{Q}_n$. Then it is easy to see that G^2 contains a K_m-minor, where each P_i lies in a different branch set and each $Q \in \mathcal{Q}_n$ that connects P_i and P_j is split among those branch sets. □

Using a result of Thomassen [8], we obtain the following corollary of Proposition 2.1.

Corollary 2.2. *Let G be a one-ended quasi-transitive locally finite graph. Then G^2 is not minor excluded.*

Proof. By Thomassen [8, Proposition 5.6], the unique end of a one-ended quasi-transitive locally finite graph is thick. Thus, Proposition 2.1 implies the assertion. □

Proposition 2.3. *Let G be a graph of bounded degree that is quasi-isometric to a tree of bounded degree. Then G is minor excluded.*

Proof. Let T be a tree of bounded degree and let $\varphi\colon V(G) \to V(T)$ be a (γ, c)-quasi-isometry for some $\gamma \geq 1$ and $c \geq 0$. Let D_G, D_T be the maximum degrees of G and T, respectively. Then

$$M_T := \sum_{i=0}^{\gamma+c-2} (D_T - 1)^i, \qquad M_G := \sum_{i=0}^{c-1} (D_G - 1)^i$$

are the maximum sizes of balls around vertices of T, of G of radius $\gamma + c - 1$, of radius c, respectively. Let H be a minor of G that is isomorphic to K_n for some

$n \in \mathbb{N}$. We will show that $n \leq \max\{2M_T^2M_G^2, D_TM_TM_G\}$, which implies that G is minor excluded. For $x \in V(H)$, we denote by G_x the branch set of x in G. For an edge $e = st \in E(T)$, let T_s, T_t be the component of $T - e$ that contains s, t, respectively. Note that since φ is a (γ, c)-quasi-isometry, we have $|\varphi^{-1}(t)| \leq M_G$.

Let us assume that there is an edge $e = st \in E(T)$ and $x, y \in V(H)$ with $\varphi(G_x) \subseteq V(T_s)$ and $\varphi(G_y) \subseteq V(T_t)$. Since H is a complete graph, there is a G_x-G_y edge uv in G. By the assumption on φ, we have $d(\varphi(u), \varphi(v)) \leq \gamma + c$. Let us assume that $\varphi(u) \in V(T_s)$. Then $\varphi(u)$ lies in the ball of radius at most $\gamma + c - 1$ around s and $\varphi(v)$ lies in the ball of radius at most $\gamma + c - 1$ around t. There are at most M_T^2 many such pairs (u, v). Since $|\varphi^{-1}(a)| \leq M_G$ for every $a \in V(T)$, there are at most $M_T^2M_G^2$ many such pairs (u, v). Now let $z \in V(H)$ such that $\varphi(G_z)$ meets T_s and T_t. Then there are adjacent $u, v \in G_z$ with $\varphi(u) \in V(T_s)$ and $\varphi(v) \in V(T_t)$. So as above, we find at most $M_T^2M_G^2$ such edges uv and hence such vertices z. This implies $n \leq 2M_T^2M_G^2$.

Now let us assume that for every edge $e = st \in E(T)$ there is at most one component of $T - e$ that contains some $\varphi(G_x)$. If T_s, T_t contains some $\varphi(G_x)$, then we orient the edge e towards s, t, respectively. If neither T_s nor T_t contains any $\varphi(G_x)$, we do not orient e at all. Note that this orientation is *consistent*: if $s_1t_1, s_2t_2 \in E(T)$ and they are oriented towards s_1, s_2, respectively, then we must have either $T_{s_2} \subseteq T_{s_1}$ or $T_{s_1} \subseteq T_{s_2}$. Thus and since H is finite, there is a unique non-empty subtree T' of T whose inner edges are not directed at all but such that every edge outside of T' is directed towards T'. If T' has an edge $e = st$, then for every $x \in V(H)$, the set $\varphi(G_x)$ meets T_s and T_t. By the same argument as in the previous case, there are at most $M_T^2M_G^2$ such branch sets. Thus, we have $n \leq M_T^2M_G^2$ in this case. So let us assume that T' consists of a unique vertex t. Then every branch set G_x contains a vertex u with $d(\varphi(u), t) \leq \gamma + c$. There are at most D_TM_T many possibilities for $\varphi(u)$ and thus at most $D_TM_TM_G$ many possibilities for u. This implies $n \leq D_TM_TM_G$ in this situation.

So we have $n \leq \max\{2M_T^2M_G^2, D_TM_TM_G\}$, which proves the assertion as discussed above. □

Now we are able to prove our main result.

Proof of Theorem 1.1. Assume that G is not quasi-isometric to any tree. Since locally finite quasi-transitive graphs without thick ends are quasi-isometric to trees by Krön and Möller [6, Theorem 2.8], G has a thick end. Thus, G^2 is not minor excluded by Proposition 2.1. Since G^2 is quasi-isometric to G, it is not minor excluded, either.

Let us now assume that G is quasi-isometric to a tree. Note that G has bounded degree by assumption. According to Krön and Möller [6, Theorem 2.8], every end of G is thin. Thus, [4, Theorem 7.5] and [3, Lemma 2.9] imply that G is quasi-isometric to a 3-regular tree. By Proposition 2.3, G is minor excluded. Since being quasi-isometric is an equivalence relation, every graph that is quasi-isometric to G is also quasi-isometric to a tree. Hence, they are minor excluded as well. □

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