HOMOGENEOUS 2-PARTITE DIGRAPHS

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Abstract. We call a 2-partite digraph D *homogeneous* if every isomorphism between finite induced subdigraphs that respects the 2-partition of D extends to an automorphism of D that does the same. In this note, we classify the homogeneous 2-partite digraphs.

1. Introduction

A structure is *homogeneous* if every isomorphism between finite induced substructures extends to an automorphism of the whole structure. This notion is due to Fraïssé [4], see also [5]. Since his work appeared, several classification results of countable homogeneous structures have been proved. These include results on partial orders by Schmerl [13], graphs by Gardiner [6] and by Lachlan and Woodrow [11], tournaments by Lachlan [10], directed graphs by Lachlan [9] and Cherlin [2, 3], bipartite graphs by Goldstern, Grossberg, and Kojman [7], and, recently, ordered graphs by Cherlin [1]. For more details on homogeneous structures, we refer to Macpherson's survey [12].

In this note, we classify the homogeneous 2-partite digraphs (Theorem 3.1). This classification problem occured during the classification of the countable connected-homogeneous digraphs [8], where a digraph is *connected-homogeneous* if every isomorphism between finite induced connected subdigraphs extends to an automorphism of the whole digraph.

2. Preliminaries

In this note, a *bipartite graph* is a triple $G = (X, Y, E)$ of pairwise disjoint sets such that every $e \in E$ is a set consisting of one element of X and one element of Y. We call $V(G) = X \cup Y$ the *vertex set* of G and E the *edge set* of G. A *2-partite digraph* is a triple $D = (X, Y, E)$ of pairwise disjoint sets with $E \subseteq (X \times Y) \cup (Y \times X)$ and such that $(u, v) \in E$ implies $(v, u) \notin E$. Again, $V(D) = X \cup Y$ are the *vertices* of D and E are the *edges* of D. We write uv instead of (u, v) for edges of D. A 2-partite digraph (X, Y, E) is *bipartite* if either $E \subseteq X \times Y$ or $E \subseteq Y \times X$. The *underlying undirected bipartite graph* of a 2-partite digraph (X, Y, E) is defined by

$$(X, Y, \{\{u, v\} \mid uv \in E\}).$$

Two vertices u, v of a 2-partite digraph $D = (X, Y, E)$ are *adjacent* if either $uv \in E$ or $vu \in E$. The *successors* of $u \in V(D)$ are the elements of the *out-neighbourhood* $N^+(u) := \{w \in V(D) \mid uw \in E\}$ and its *predecessors* are the elements of its *in-neighbourhood* $N^-(u) := \{w \in V(D) \mid wu \in E\}$. For $x \in X$, we define

$$x^\perp = \{y \in Y \mid y \text{ not adjacent to } x\}$$

and, for $y \in Y$, we define

$$y^{\perp} = \{x \in X \mid x \text{ not adjacent to } y\}.$$

A bipartite graph $G = (X, Y, E)$ is *homogeneous* if every isomorphism φ between finite induced subgraphs A and B with $(V(A) \cap X)\varphi \subseteq X$ and $(V(A) \cap Y)\varphi \subseteq Y$ extends to an automorphism α of G with $X\alpha = X$ and $Y\alpha = Y$. Similarly, a 2-partite digraph $D = (X, Y, E)$ is *homogeneous* if every isomorphism φ between finite induced subdigraphs A and B with $(V(A) \cap X)\varphi \subseteq X$ and $(V(A) \cap Y)\varphi \subseteq Y$ extends to an automorphism α of D with $X\alpha = X$ and $Y\alpha = Y$.

A first step towards the classification of the homogeneous 2-partite digraphs was already done when Goldstern et al. [7] classified the homogeneous bipartite graphs. Thus, before moving on, we cite their result and discuss its effects towards the classification of the homogeneous 2-partite digraphs.

Theorem 2.1. [7, Remark 1.3] *A bipartite graph is homogeneous if and only if it is isomorphic to one of the following bipartite graphs:*

(i) *a complete bipartite graph;*
(ii) *an empty bipartite graph;*
(iii) *a perfect matching;*
(iv) *the bipartite complement of a perfect matching;*
(v) *a generic bipartite graph.* □

The *bipartite complement of a perfect matching* is a complete bipartite graph with sides of equal cardinality where a perfect matching is removed from the edge set. A bipartite graph $G = (X, Y, E)$ is *generic* if for any two disjoint finite subsets U_X, W_X of X and any two disjoint finite subsets U_Y, V_Y of Y there exist $y \in Y$ and $x \in X$ with $U_X \subseteq N(y)$ and $V_X \cap N(y) = \emptyset$ as well as with $U_Y \subseteq N(x)$ and $V_Y \cap N(x) = \emptyset$.

For bipartite digraphs (X, Y, E), Theorem 2.1 applies analogously in the following sense: as we have either $E \subseteq X \times Y$ or $E \subseteq Y \times X$, the underlying undirected bipartite graph is homogeneous, so belongs to some class of the list in Theorem 2.1. Conversely, every orientation of a homogeneous bipartite graph that results in a bipartite digraph gives a homogeneous bipartite digraph. Note that homogeneous bipartite digraphs are in particular homogeneous 2-partite digraphs. Hence, the above classification gives us a partial classification in the case of the homogeneous 2-partite digraphs in that it gives a full classification of the homogeneous bipartite digraphs. In the remainder of this note we extend this partial classification by classifying those homogeneous 2-partite digraphs that are not bipartite.

3. The main result

In this section, we shall prove our main theorem, the classification of the homogeneous 2-partite digraphs (Theorem 3.1).

Theorem 3.1. *A 2-partite digraph is homogeneous if and only if it is isomorphic to one of the following 2-partite digraphs:*

(i) *a homogeneous bipartite digraph;*
(ii) *an M_κ for some cardinal $\kappa \geq 2$;*
(iii) *a generic 2-partite digraph;*
(iv) *a generic orientation of a generic bipartite graph.*

For a cardinal $\kappa \geq 2$, let M_κ be a bipartite digraph (X, Y, E) with $|X| = \kappa = |Y|$ such that either $(X, Y, E \cap (X \times Y))$ or $(X, Y, E \cap (Y \times X))$ is a perfect matching and the other is the bipartite complement of the same perfect matching. In particular, the underlying undirected bipartite graph is a complete bipartite graph.

We call a 2-partite digraph (X, Y, E) *generic* if its underlying undirected bipartite graph is a complete bipartite graph and if for all pairwise disjoint finite subsets $A_X, B_X \subseteq X$ and $A_Y, B_Y \subseteq Y$ there are vertices $y \in Y$ and $x \in X$ with $A_X \subseteq N^+(y)$ and $B_X \subseteq N^-(y)$ as well as with $A_Y \subseteq N^+(x)$ and $B_Y \subseteq N^-(x)$. Similarly, we call a 2-partite digraph (X, Y, E) a *generic orientation of a generic bipartite graph* if for all pairwise disjoint finite subsets $A_X, B_X, C_X \subseteq X$ and $A_Y, B_Y, C_Y \subseteq Y$ there are vertices $y \in Y$ and $x \in X$ with $A_X \subseteq N^+(y)$, $B_X \subseteq N^-(y)$ and $C_X \subseteq y^\perp$ as well as with $A_Y \subseteq N^+(x)$, $B_Y \subseteq N^-(x)$ and $C_Y \subseteq x^\perp$. It is easy to verify that its underlying undirected graph is a generic bipartite graph.

Note that standard back-and-forth arguments show that, up to isomorphism, there are a unique countable generic 2-partite digraph and a unique countable generic orientation of the (unique) countable generic bipartite graph.

It is worthwhile noting that by Theorem 3.1 the underlying undirected bipartite graph of a homogeneous 2-partite digraph is always homogeneous, which is false for arbitrary homogeneous digraphs and their underlying undirected graphs.

The fact that the listed 2-partite digraphs in Theorem 3.1 are homogeneous is already discussed in the previous section for case (i), while in case (ii) it is a consequence of the fact that the bipartite complement of a perfect matching is homogeneous. The cases (iii) and (iv) can be easily verified by the above mentioned back-and-forth argument. (This can also be applied if they are not countable to show that they are homogeneous.) Before we start with the remaining direction of the proof of Theorem 3.1, we show some lemmas.

Lemma 3.2. *Let $D = (X, Y, E)$ be a homogeneous 2-partite digraph. If $N^+(v)$ and $N^-(v)$ are infinite and $v^\perp$ is finite for some $v \in V(D)$, then $v^\perp = \emptyset$.*

Proof. Let $x \in X$ and $m = |x^\perp|$. First, let us suppose $m = 1$. We note that any automorphism of D that fixes x must also fix the unique element $x_Y \in x^\perp$. Let y be a successor of x. As $N^+(x)$ is infinite, we find two vertices y_1, y_2 in Y that have a common predecessor. Homogeneity then implies that the two vertices y and x_Y in Y have a common predecessor z. Let z' be a successor of x_Y. By homogeneity, we find an automorphism β of D that fixes x and maps z to z'. As mentioned above, β must fix x_Y as it fixes x. But we have $zx_Y \in E$ and $(zx_Y)\alpha = z'x_Y \notin E$ because of $x_Y z' \in E$, which is impossible.

Now let us suppose $m \geq 2$. By homogeneity and as m is finite, we find for any subset A of Y of cardinality m a vertex $a \in X$ with $a^\perp = A$. As Y is infinite, there are two subsets A_1, A_2 of Y of cardinality m with $|A_1 \cap A_2| = m - 1$ and two such subsets B_1, B_2 with $|B_1 \cap B_2| = m - 2$. Let $a_i, b_i \in X$ with $a_i^\perp = A_i$ and $b_i^\perp = B_i$, respectively, for $i = 1, 2$. Then there is no automorphism of D that maps a_1 to b_1 and a_2 to b_2 even though D is homogeneous as the number of vertices that are not adjacent to a_1 and a_2 is larger than the corresponding number for b_1 and b_2. Analogous contradictions for any vertex in Y instead of $x \in X$ show the assertion. $\square$

Lemma 3.3. *Let $D = (X, Y, E)$ be a homogeneous 2-partite digraph. If $N^+(v)$ and $N^-(v)$ are infinite and $v^\perp = \emptyset$ for all $v \in V(D)$, then D is a generic 2-partite digraph.*

Proof. It suffices to show that for any two disjoint finite subsets A and B of X we find a vertex $v \in Y$ with $A \subseteq N^+(v)$ and $B \subseteq N^-(v)$. Indeed, the corresponding property for subsets of Y then follows analogously. Note that we find for every $y \in Y$ two sets $A_y \subseteq N^+(y)$ and $B_y \subseteq N^-(y)$ with $|A| = |A_y|$ and $|B| = |B_y|$. As D is homogeneous and as $A \cup B$ and $A_y \cup B_y$ induce (empty) isomorphic finite subdigraphs of D, there exists an automorphism α of D that maps A_y to A and B_y to B. So $y\alpha$ is a vertex we are searching for. □

Lemma 3.4. *Let $D = (X, Y, E)$ be a homogeneous 2-partite digraph. If $N^+(v)$, $N^-(v)$, and $v^\perp$ are infinite for all $v \in V(D)$, then D is a generic orientation of a generic bipartite graph.*

Proof. Similarly to the proof of Lemma 3.3, it suffices to show that for any three pairwise disjoint finite subsets A, B, C of X we find a vertex $v \in Y$ with $A \subseteq N^-(v)$ and $B \subseteq N^+(v)$ and $C \subseteq v^\perp$. For every $y \in Y$, we find (pairwise disjoint) subsets $A_y \subseteq N^+(y)$ and $B_y \subseteq N^-(y)$ and $C_y \subseteq y^\perp$ with $|A| = |A_y|$ and $|B| = |B_y|$ and $|C| = |C_y|$. Note that each of the two sets $A \cup B \cup C$ and $A_y \cup B_y \cup C_y$ has no edge. Applying homogeneity, we find an automorphism α of D that maps A_y to A and B_y to B and C_y to C. So $y\alpha$ is a vertex that has the desired properties. □

Now we are able to prove our main theorem.

Proof of Theorem 3.1. Let $D = (X, Y, E)$ be a homogeneous 2-partite digraph that is not bipartite. Then we find in X some vertex with a predecessor in Y and some vertex with a successor in Y. By homogeneity, we can map the first onto the second and conclude the existence of a vertex in X that has a predecessor and a successor in Y. Analogously, we obtain the same for some vertex of Y. By homogeneity, every vertex of D has predecessors and successors. In particular, we have $|X| \geq 2$ and $|Y| \geq 2$.

Let us suppose that two vertices $u, v \in X$ have the same successors, that is, $N^+(u) = N^+(v)$. By homogeneity, we can fix u and map v onto any vertex w of $X \smallsetminus \{u\}$ by some automorphism of D and thus obtain $N^+(w) = N^+(u)$ for every $w \in X$. So no vertex in $N^+(u)$ has successors in X, which is impossible as we saw earlier. Hence, we have $N^+(u) \neq N^+(v)$ for any two distinct vertices $u, v \in X$. Analogously, the same holds for any two distinct vertices in Y and also for the set of predecessors of every two vertices either in X or in Y. Thus, we have shown

$$N^+(u) \neq N^+(v) \quad \text{and} \quad N^-(u) \neq N^-(v) \quad \text{for all } u \neq v \in X \tag{1}$$

and

$$N^+(u) \neq N^+(v) \quad \text{and} \quad N^-(u) \neq N^-(v) \quad \text{for all } u \neq v \in Y. \tag{2}$$

Let us assume that $n := |N^+(u)|$ is finite for some $u \in X$. Note that, for any subset A of Y of cardinality n, we find a vertex $a \in X$ with $N^+(a) = A$ by homogeneity. If $|Y| > n + 1$ and $n \geq 2$, then we find two subsets of Y of cardinality n whose intersection has $n - 1$ elements and two such sets whose intersection has $n - 2$ elements. So we find two vertices in X with $n - 1$ common successors and we also find two vertices in X with $n - 2$ common successors. This is a contradiction to homogeneity, because we cannot map the first pair of vertices onto the second pair. Thus, we have either $n = 1$ or $|Y| = n + 1$. If $|Y| = n + 1$, then we directly obtain $D \cong M_{n+1}$ since every vertex in X also has some predecessor in Y. So let us assume $n = 1$. If we have $1 < k \in \mathbb{N}$ for $k := |N^-(u)|$, then we obtain $D \cong M_{k+1}$, analogously. So let us assume that either $|N^-(u)| = 1$ or $N^-(u)$ is infinite. First, we consider the case that $N^-(u)$ is infinite. An empty set $u^\perp$ directly implies $D \cong M_{|Y|}$. So let us suppose $u^\perp \neq \emptyset$. Let u^+ be the unique vertex in $N^+(u)$. Since $u^\perp \neq \emptyset$, we find for some and hence by homogeneity for every vertex in Y some vertex in X it is not adjacent to. Let $w \in (u^+)^\perp$ and let $v \in N^+(u^+)$. By homogeneity, we find an automorphism α of D that fixes u and maps v to w. Since α fixes u, it must also fix u^+. But since $u^+v \in E$ and $(u^+v)\alpha = u^+w \notin E$, this is not possible. Hence, if $N^+(u)$ is finite, it remains to consider the case $n = 1 = k$. Due to (1), no two vertices of X have a common predecessor or a common successor. Thus, also every vertex in Y has precisely one predecessor and one successor. Let $v \in Y$ and $w \in X$ with $uv, vw \in E$. Then we can map the pair (u, w) onto any pair of distinct vertices of X, as D is homogeneous. Thus, for all $x \neq z \in X$, there exists $y \in Y$ with $xy, yz \in E$. This shows $|X| = 2$ as every vertex of D has precisely one successor. Hence, D is a directed cycle of length 4, which is isomorphic to M_2.

Analogous argumentations in the cases of finite $N^-(u)$, $N^+(v)$ or $N^-(v)$ with $u \in X$ and $v \in Y$ show that the only remaining case is that every vertex in D has infinite in- and infinite out-neighbourhood. Due to Lemma 3.2, we know that $|u^\perp|$ is either 0 or infinite and that $|v^\perp|$ is either 0 or infinite. Since $x^\perp \neq \emptyset$ if and only if $y^\perp \neq \emptyset$ for all $x \in X$ and $y \in Y$, the assertion follows from Lemmas 3.3 and 3.4. $\square$

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