GROUP ACTIONS ON METRIC SPACES: FIXED POINTS AND FREE SUBGROUPS

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Abstract. We look at group actions on graphs and other metric spaces, e. g., at group actions on geodesic hyperbolic spaces. We classify the types of automorphisms on these spaces and prove several results about the density of the hyperbolic limit set of the group in the whole limit set of the group.

1. Introduction

In many situations, groups acting on some topological space offer the alternative between the existence of a free subgroup $\mathbb{Z} * \mathbb{Z}$ and the existence of a fixed point in the space under the action of the group. For example, if the space is a connected locally finite graph, then such results can be found in [10, 14, 22, 28, 29]. For the case of proper geodesic hyperbolic spaces we refer to [1, 4, 8, 9, 29] for these results.

The investigation for locally finite graphs was started by Halin [10]. He distinguished automorphisms into two type: *Type I* are those that fix some finite set of vertices and *Type II* are all other automorphisms. He proved that the latter are the translations of the graph. The proofs in [10] do not need the assumption of local finiteness and in further investigations of group actions on graphs Jung [14] noticed that. However, just omitting it does not reflect what is really happening there as we shall see in the following example.

Consider the complete graph on the vertex set $\mathbb{Z}$. We look at two different automorphisms. The first just maps each vertex i to $i + 1$. The second automorphism does the same except for $i = -1, 0$: it fixes 0 and maps -1 to 1. In the sense of Halin's types, the first is of Type II, while the second is of Type I. But the automorphisms do not differ much. Indeed, both automorphisms leave a *bounded* vertex set invariant, not just a finite one. With this in mind, we are able to prove the following results, where X is any graph, G a group of automorphisms of X, and $\hat{X}$ the completion of X with all ends all of whose rays eventually leave every bounded ball (see Theorem 3.1). (We refer to Section 2 for definitions.)

- Every automorphism of X is either elliptic, hyperbolic, or parabolic (Theorem 2.3);
- a group G of automorphisms fixes either a bounded subset of G or a unique limit point of G in its boundary ∂X, or X has precisely two limit points of G, or G contains two hyperbolic elements that freely generate a free subgroup (Theorem 2.8);
- the hyperbolic limit set of G is dense in the limit set of G (Theorem 2.7);
- the hyperbolic limit set of G is bilaterally dense in the limit set of G if and only if either X has precisely two limit points of G or G contains two hyperbolic elements without a common fixed point (Theorem 2.9);
- if the limit set of G is infinite, then it is a perfect set (Theorem 2.10).

Unfortunately, the results do not hold if we take all vertex ends: we shall discuss a graph that violates the third and fourth result if ∂X are all vertex ends (see Example 2). Another possibility (Theorem 3.3) is to take metric ends instead of vertex ends. We are able to prove this almost simultaneously, as we are building up a general topological setting (*contractive G-completions*), in which we prove our results. This topological setting will extend Woess's contractive G-compactifications [29] to spaces that need not be proper. And both mentioned completions of infinite graphs will be examples of these contractive G-completions.

A further class of metric spaces that are contractive G-completions are geodesic hyperbolic spaces, see Section 4. So the above mentioned results also hold for them with $\hat{X}$ being the geodesic hyperbolic space with its hyperbolic boundary. We note that the first two facts are already known by the experts, i. e. the corresponding proofs for proper hyperbolic spaces in [4] carry over to geodesic hyperbolic spaces that need not be proper. Note that all mentioned results are known to be true for proper geodesic hyperbolic spaces, cp. [24, 29].

We note that, for certain classes of groups, Karlsson and Noskov [16] considered group actions on generalisations of contractive G-compactifications. We also note that our notion of contractive G-completions has similarities with *convergence groups* as defined by Gehring and Martin [6] and that were also investigated by Tukia [26] but differ from them just as Woess's contractive G-compactifications do.

2. Contractive G-completions

Let X be a metric space, let $\hat{X} \supseteq X$ be a regular Hausdorff space, and let G be a group of automorphisms (i.e. self-isometries) on X. If X is proper and $\hat{X}$ compact, Woess [29] called $\hat{X}$ a *G-compactification* if the following axioms (C1) and (C2) hold.

(C1) The identity $X \to \hat{X}$ is a homeomorphism and X is open and dense in $\hat{X}$.

(C2) Every element of G extends to a homeomorphism of $\hat{X}$.

Unfortunately, if we do not have the additional assumptions that X is proper and $\hat{X}$ compact and also no further axioms, then we run into some problems as we shall illustrate in the following.

One fact in the case of G-compactifications is that for any sequence $(g_i)_{i\in\mathbb{N}}$ in G and any $x \in X$ such that the set $\{xg_i \mid i \in \mathbb{N}\}$ is unbounded it has an accumulation point in ∂X. This is false in general as the following example shows.

Let $\mathcal{G}$ be a free group with free generating set $\mathcal{S} := \{s_i \mid i \in \mathbb{N}\}$ and let $\mathcal{X}$ be the Cayley graph of $\mathcal{G}$ with respect to $\mathcal{S}$. So $\mathcal{X}$ is an $\aleph_0$-regular tree. We consider the natural action of $\mathcal{G}$ on $\mathcal{X}$. Let $\hat{\mathcal{X}}$ be the completion of $\mathcal{X}$ with its ends. (We refer to Section 3 for the definition of an end.) Let o be the vertex corresponding to $1_{\mathcal{G}}$. Then the sequence $(os_i^i)_{i\in\mathbb{N}}$ is unbounded but has no accumulation point in $\partial\mathcal{X}$, since o separates any two of its vertices. We do not want to forbid such situations but we have to deal with them. So we have to require that, if this happens, we still have some structure in G and $\hat{X}$. In our example, we may take the automorphism s_1 which gives us all we need in these situations: a sequence $(s_1^i)_{i\in\mathbb{N}}$ of automorphisms such that each of the two sequences $(os_1^i)_{i\in\mathbb{N}}$ and $(os_1^{-i})_{i\in\mathbb{N}}$ converges. This is why we introduce the following new axiom.

(C3) Let $(g_i)_{i\in\mathbb{N}}$ be a sequence in G and $x \in X$ with $d(x, xg_i) \to \infty$ for $i \to \infty$. Then either the set $\{xg_i \mid i \in \mathbb{N}\}$ has an accumulation point in ∂X or

$\langle g_i \mid i \in \mathbb{N}\rangle$ contains an automorphism γ such that each of the two sequences $(x\gamma^i)_{i\in\mathbb{N}}$ and $(x\gamma^{-i})_{i\in\mathbb{N}}$ converges to some point in ∂X.

Note that (C3) immediately implies that, if $\{xg^i \mid i \in \mathbb{N}\}$ converges for some $x \in X$ and some $g \in G$, then $\{xg^{-i} \mid i \in \mathbb{N}\}$ has an accumulation point in ∂X.

If we want to show that the hyperbolic limit set of G is dense in the limit set of G, this new axiom is still not enough:[1] if we have a sequence $(g_i)_{i\in\mathbb{N}}$ such that for some $x \in X$ the sequence $(xg_i)_{i\in\mathbb{N}}$ converges, we would like the set $\{xg_i^{-1} \mid i \in \mathbb{N}\}$ to have an accumulation point – just as it is true in the case $g_i = g^i$ for all $i \in \mathbb{N}$ and some $g \in G$ as seen above. Once more, this need not be true as we shall demonstrate on our earlier example $\mathcal{X}$. Consider the sequence $(s_1^i s_i^{-1})_{i\in\mathbb{N}}$ in $\mathcal{G}$. Then the vertex set $\{os_1^i s_i^{-1} \mid i \in \mathbb{N}\} \subseteq V(\mathcal{X})$ converges the end containing the ray $o, os_1, os_1^2, \ldots$, the direction of s_1. But the set $\{os_i s_1^{-1} \mid i \in \mathbb{N}\}$ has no accumulation point, since o separates any two of its vertices. Of course, we can use (C3) to obtain some $g \in \langle s_1^i s_i^{-1} \mid i \in \mathbb{N}\rangle$ such that $(og^i)_{i\in\mathbb{N}}$ and $(og^{-i})_{i\in\mathbb{N}}$ converges. But this is not much help, if we want to find a hyperbolic limit point close to some previously chosen limit point η, as both new limit points can lie abritrarily far away from η. So in our example, (C3) might give us $s_1^2 s_2^{-1} \in \mathcal{G}$, but neither of its directions is (close to) the direction of s_1. Therefore, we introduce the following axiom.

(C4) Let $(g_i)_{i\in\mathbb{N}}$ be a sequence in G and $x \in X$ such that the set $\{xg_i \mid i \in \mathbb{N}\}$ converges to some boundary point η but the set $\{xg_i^{-1} \mid i \in \mathbb{N}\}$ has no accumulation point in ∂X. Then there is a sequence $(h_j)_{j\in\mathbb{N}}$ in $\langle g_i \mid i \in \mathbb{N}\rangle$ such that each of the sets $\{xh_j^i \mid i \in \mathbb{N}\}$ and $\{xh_j^{-i} \mid i \in \mathbb{N}\}$ converges to distinct boundary points η_j, $\mu_j \in \partial X$, respectively. In addition, the sequence $(\eta_j)_{j\in\mathbb{N}}$ converges to η.

We call $\hat{X}$ a *G-completion* if the axioms (C1)–(C4) hold. A completion $\hat{X}$ of X is *projective* if for all sequences $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}$ in X such that $(x_i)_{i\in\mathbb{N}}$ converges to $\eta \in \partial X$ and such that $d(x_i, y_i) \leq M$ for some $M < \infty$ also the sequence $(y_i)_{i\in\mathbb{N}}$ converges to η. A G-completion $\hat{X}$ of X is *contractive* if it is projective and if for all sequences $(g_i)_{i\in\mathbb{N}}$ in G with

$$xg_n \to \eta \in \partial X \quad \text{and} \quad xg_n^{-1} \to \mu \in \partial X$$

for some $x \in X$ the sequence $(yg_n)_{n\in\mathbb{N}}$ *converges uniformly* to η outside every neighbourhood of μ in $\hat{X}$, that is, that for any open neighbourhoods U of η and V of μ, there is an $n_0 \in \mathbb{N}$ such that $yg_n \in U$ for all $y \in \hat{X} \smallsetminus V$ and all $n \geq n_0$.

Lemma 2.1. *Let $\hat{X}$ be a projective G-completion. No bounded sequence in X converges to any $\eta \in \partial X$.*

Proof. Let us suppose that we find an $\eta \in \partial X$ and a bounded sequence $(x_i)_{i\in\mathbb{N}}$ that converges to η. Then any constant sequence $(x)_{i\in\mathbb{N}}$ in X converges to η due to projectivity. But this contradicts the fact that $\hat{X}$ is Hausdorff. □

In a slight abuse of notation, we write for $U, V \subseteq \hat{X}$:

$$d(U, V) := \inf\{d(u, v) \mid u \in U \cap X, v \in V \cap X\}$$

[1] Readers not familiar with the definitions of (hyperbolic) limit points and directions of group elements may skip the following motivation till the next axiom (C4) without losing much; they might return here later, after they read the necessary definitions just before Lemma 2.6.

Lemma 2.2. *Let $\hat{X}$ be a projective G-completion and let η and μ be distinct elements of ∂X. For every open neighbourhood U of η with $\mu \notin \overline{U}$, there exists an open neighbourhood V of μ with $d(U,V) > 0$ and $\overline{U} \cap \overline{V} = \emptyset$.*

Furthermore, for any $x \in X \smallsetminus U$ we may choose V so that $x \notin V$.

Proof. As $\hat{X}$ is regular, we find an open neighbourhood $V' \subseteq X \smallsetminus (U \cup \{x\})$ of μ and an open neighbourhood U' of $\overline{U} \cup \{x\}$ that are disjoint. Projectivity gives us that any sequence within a fixed distance $M > 0$ to $U \cap X$ converges to a boundary point in $\overline{U}$ and hence not to μ. So $V = V' \smallsetminus \overline{B}_M(U)$ is open, still has μ as an accumulation point, and satisfies the other assertions. □

We call an automorphism $g \in G$ on X

- *elliptic* if it fixes a bounded non-empty subset of X;
- *hyperbolic* if it is not elliptic and if it fixes precisely two boundary points $\eta, \mu \in \partial X$;
- *parabolic* if it is not elliptic and if it fixes precisely one boundary point $\eta \in \partial X$.

Theorem 2.3. *Let $\hat{X}$ be a contractive G-completion of a metric space X. Then each $g \in G$ is either elliptic, hyperbolic, or parabolic.*

Furthermore, if g is hyperbolic and fixes the two boundary points η and μ, then $xg^n \to \eta$ and $xg^{-n} \to \mu$ for all $x \in X$ or vice versa, and if g is parabolic, then for every $x \in X$ the set $\{xg^n \mid n \in \mathbb{Z}\}$ has precisely one accumulation point, the boundary point fixed by g.

Remark 2.4. *Note that in general for a parabolic element g the analogous convergence property as for hyperbolic elements need not be true, that is, at the end of Section 3 we shall give an example of a contractive G-completion X that has a boundary point η such that $xg^n \not\to \eta$ for all $x \in X$. Due to projectivity, this implies $yg^n \not\to \eta$ for every $y \in X$.*

Proof of Theorem 2.3. Let $g \in G$ and $x \in X$. Let us assume that g is not elliptic. Then the set $\{d(xg^n, xg^m) \mid m,n \in \mathbb{Z}\}$ is unbounded and hence, the same is true for $\{d(x, xg^n) \mid n \in \mathbb{N}\}$. So we conclude by (C3) that $A := \{xg^n \mid n \in \mathbb{N}\}$ has an accumulation point $\eta \in \partial X$ and $B := \{xg^{-n} \mid n \in \mathbb{N}\}$ has an accumulation point $\mu \in \partial X$.

Let $(g^{n_i})_{i\in\mathbb{N}}$ be a subsequence of $(g^i)_{i\in\mathbb{N}}$ such that $xg^{n_i} \to \eta$ for $i \to \infty$.Since the elements of G are homeomorphisms on $\hat{X}$, we know by projectivity of $\hat{X}$ that

$$\eta g = (\lim xg^{n_i})g = \lim(xg)g^{n_i} = \eta.$$

So we have $\eta g = \eta$ and, analogously, we also have $\mu g = \mu$.

Let ∂A, ∂B be the sets of accumulation points of A, B in ∂X, respectively. Then the sets ∂A and ∂B are non-empty closed subsets of ∂X. First, we show

$$|\partial A| = 1 = |\partial B|. \tag{1}$$

Let us suppose that there is a second accumulation point η' of A. We have $\eta' g = \eta'$, too. The sequence $(xg^{-n_i})_{i\in\mathbb{N}}$ is unbounded because of $d(x, xg^n) = d(xg^{-n}, x)$. If $(xg^{-n_i})_{i\in\mathbb{N}}$ has no accumulation point in ∂X, then there is a $z \in \mathbb{Z}$ such that $(xg^{kz})_{k\in\mathbb{N}}$ and $(xg^{-kz})_{k\in\mathbb{N}}$ converge in $\hat{X}$ by (C3). But then we have $|\partial A| = 1 = |\partial B|$, as $\hat{X}$ is projective and as $\{d(xg^z, xg^{z+i}) \mid 0 \leq i \leq z\}$ is bounded. So $(xg^{-n_i})_{i\in\mathbb{N}}$ converges to μ, a contradiction. Hence, $(xg^{-n_i})_{i\in\mathbb{N}}$ has an accumulation point in ∂X, say μ. Let us take an infinite subsequence of $(n_i)_{i\in\mathbb{N}}$

such that $(xg^{-n_i})_{i\in\mathbb{N}}$ converges for this subsequence to μ. We may assume that $(n_i)_{i\in\mathbb{N}}$ itself is this subsequence. If $\eta \neq \mu$, let U and V be open neighbourhoods of η and μ, respectively, with $x \notin U \cup V$ and $\overline{U} \cap \overline{V} = \emptyset$. Let Z be an open neighbourhood of η' with $\eta \notin \overline{Z}$. Set $W := U \setminus Z$. Due to contractivity, there exists $m \in \mathbb{N}$ with $(X \setminus V)g^{n_m} \subseteq W$ and we conclude $xg^{\ell n_m} \in W$ for all $\ell \in \mathbb{N}$ inductively. Due to projectivity, every accumulation point of A lies in $\overline{W}$ as $\{d(xg^{n_m}, xg^{i+n_m}) \mid 0 \leq i \leq n_m\}$ is bounded. This contradicts the choice of W. If $\eta = \mu$, let $U = V$ be an open neighbourhood of η with $\eta' \notin \overline{U}$ and let $(y_i)_{i\in\mathbb{N}}$ be a sequence in $X \setminus U$ that converges to η'. As $\hat{X}$ is contractive, there is an $m \in \mathbb{N}$ with $y_i g^{n_m} \in U$ for all $i \in \mathbb{N}$. But then we have $\eta' = \eta' g^{n_m} \in \overline{U}$, a contradiction. This shows that η is the unique element of ∂A. Analogously, we obtain that μ is the unique element of ∂B, which shows (1).

Next, we show

(2) *if $\nu g = \nu$ for some $\nu \in \partial X$, then either $\nu = \eta$ or $\nu = \mu$.*

Let us suppose that there is $\nu \in \partial X \setminus \{\eta, \mu\}$ with $\nu g = \nu$. As $\hat{X}$ is regular, we may take open neighbourhoods U and V of η and μ, respectively, with $\overline{U} \cap \overline{V} = \emptyset$ such that $\nu \notin \overline{U} \cup \overline{V}$. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in X converging to ν. As $\nu \notin \overline{U} \cup \overline{V}$, only finitely many x_i lie in $U \cup V$. In particular, we may have chosen $(x_i)_{i\in\mathbb{N}}$ in $X \setminus (U \cup V)$. By contractivity, we find an $n \in \mathbb{N}$ with $x_i g^n \in U$ for all $i \in \mathbb{N}$. Hence, we have $\nu = \nu g^n \in \overline{U}$, a contradiction that shows (2).

Thus, there are at most two boundary points, η and μ, of X fixed by g and g is either parabolic or hyperbolic. If g is parabolic, then we just showed that the set $\{xg^n \mid n \in \mathbb{Z}\}$ has precisely one accumulation point, as we showed earlier $|\partial A| = 1$, and the same is true for $\{yg^n \mid n \in \mathbb{Z}\}$ for any $y \in X$ by projectivity.

So let us assume that η and μ are distinct, that is, that g is hyperbolic. We have to show the convergence property of hyperbolic automorphisms. Let us first show that xg^n and xg^{-n} for $n \in \mathbb{N}$ converge to η and μ, respectively. Therefore, we show that we can find a sequence $(n_i)_{i\in\mathbb{N}}$ such that xg^{n_i} converges to η and xg^{-n_i} converges to μ. Let us take an arbitrary sequence $(n_i)_{i\in\mathbb{N}}$ such that xg^{n_i} converges to η. Let us suppose that μ is no accumulation point of xg^{-n_i}. As $d(x, xg^{-n_i})$ is unbounded, we know by (C3) that there is an $n \in \mathbb{N}$ such that $(xg^{nk})_{k\in\mathbb{N}}$ and $(xg^{-nk})_{k\in\mathbb{N}}$ converge. So their limit points must be η and μ, respectively. By projectivity, this holds also for g instead of g^n. Now, let $y \in X$. As $\hat{X}$ is projective and $d(x, y) = d(xg^n, yg^n)$ for all $n \in \mathbb{N}$, also the sequence $(yg^i)_{i\in\mathbb{N}}$ converges to η and the sequence $(yg^{-i})_{i\in\mathbb{N}}$ converges to μ. This shows the additional statement on hyperbolic automorphisms. $\square$

Remark 2.5. *Note that we have not used axiom (C4) in the proof of Theorem 2.3.*

For a hyperbolic element g, let the boundary point to which the sequence $(xg^n)_{n\in\mathbb{N}}$ for $x \in X$ converges be the *direction* of g. Note that this definition does not depend on the point x by projectivity. By g^+ we denote the direction of g and by g^- the direction of g^{-1}. For parabolic elements, we denote by g^+ and g^- the unique fixed boundary point. For a contractive G-completion $\hat{X}$ of X, let the *limit set* $\mathcal{L}(G)$ of G be the set of accumulation points in ∂X of xG for any $x \in X$ and let the *hyperbolic limit set* $\mathcal{H}(G)$ of G be the set of directions of hyperbolic elements. Again, these sets do not depend on the choice of x due to projectivity.

Notice that due to Theorem 2.3, the automorphism γ mentioned in (C3) is either hyperbolic or parabolic and in (C4) we find infinitely many hyperbolic automorphisms whose directions converge to η.

Lemma 2.6. *Let $\hat{X}$ be a contractive G-completion of a metric space X, let U and V be non-empty open subsets of $\hat{X}$ with $d(U,V) > 0$, $\overline{U} \cap \overline{V} = \emptyset$, and $\overline{U} \cup \overline{V} \neq \hat{X}$, and let $g \in G$. If $(\hat{X} \setminus V)g \subseteq U$, then g is hyperbolic with $g^+ \in \overline{U}$ and $g^- \in \overline{V}$.*

Proof. First, we notice that $\hat{X} \setminus U \subseteq Vg$ and hence $(\hat{X} \setminus U)g^{-1} \subseteq V$. As U and V are disjoint, we obtain inductively that $(\hat{X} \setminus V)g^n \subseteq U$ and $(\hat{X} \setminus U)g^{-n} \subseteq V$ for all $n \geq 1$. Since X is dense in $\hat{X}$ and $\overline{U} \cap \overline{V} \neq \hat{X}$, we find an $x \in X \setminus (U \cup V)$. Let us show that the orbit of x under g is not bounded. Indeed, as $xg^{-1} \in V$ and $xg \in U$, we have $d(x, xg^2) \geq d(U,V)$ and thus

$$d(x, xg^n) \geq \frac{(n-1)}{2} d(U,V)$$

holds and shows that g is not elliptic. Hence, g is either parabolic or hyperbolic according to Theorem 2.3. Due to (C3), the set $\{xg^n \mid n \in \mathbb{N}\}$ has an accumulation point, which lies in $\overline{U}$, and $\{xg^{-n} \mid n \in \mathbb{N}\}$ has an accumulation point, which lies in $\overline{V}$. According to Theorem 2.3, the automorphism g cannot be parabolic, so it must be hyperbolic and we have $g^+ \in \overline{U}$ and $g^- \in \overline{V}$. □

Theorem 2.7. *Let $\hat{X}$ be a contractive G-completion of a metric space X.*

(i) *If $\mathcal{L}(G)$ has at least two elements, then $\mathcal{H}(G)$ is dense in $\mathcal{L}(G)$.*
(ii) *The set $\mathcal{L}(G)$ has either none, one, two, or infinitely many elements.*
(iii) *The set $\mathcal{H}(G)$ has either none, two, or infinitely many elements.*

Proof. To prove (i), let $\eta, \mu \in \mathcal{L}(G)$ be distinct and let $x \in X$. Then there are sequences $(g_i)_{i\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ in G with $xg_i \to \eta$ and $xh_i \to \mu$. We show that in any neighbourhood of η we find a direction of a hyperbolic element.

We may assume that $(xg_i^{-1})_{i\in\mathbb{N}}$ has at most one accumulation point: if it has more than one, then we take a subsequence of $(g_i)_{i\in\mathbb{N}}$ such that $(xg_i^{-1})_{i\in\mathbb{N}}$ converges in $\hat{X}$. If $(xg_i^{-1})_{i\in\mathbb{N}}$ has no accumulation point, then we find with condition (C4) a sequence $(f_j)_{j\in\mathbb{N}}$ of hyperbolic automorphisms such that $f_j^+ \to \eta$ for $j \to \infty$. Thus, we may assume that $(xg_i^{-1})_{i\in\mathbb{N}}$ converges to $\nu \in \partial X$.

We distinguish several cases. First, let us assume that $\nu \neq \eta$. Due to Lemma 2.2, we find open neighbourhoods U, V of η, ν, respectively, with $\overline{U} \cap \overline{V} = \emptyset$, with $d(U,V) > 0$, and with $x \notin V$. As $\hat{X}$ is contractive, there is an $n \in \mathbb{N}$ with $(\hat{X} \setminus V)g_i \subseteq U$ for all $i \geq n$. According to Lemma 2.6, for all $i \geq n$, the automorphism g_i is hyperbolic with $g_i^+ \in \overline{U}$ and $g_i^- \in V$. So we have found directions of hyperbolic automorphism arbitrarily close to η.

In the situation that $(xh_i^{-1})_{i\in\mathbb{N}}$ does not have μ as an accumulation point, an analogous proof as above gives us a direction of a hyperbolic automorphism $f \in \langle h_i \mid i \in \mathbb{N}\rangle$ in every neighbourhood of μ. If either $f^+ = \eta$ or $f^- = \eta$, then η itself is a direction of a hyperbolic element. Hence, we may assume that $f^+ \neq \eta$ and we may also assume that $f^+ \neq \nu$ by taking f^{-1} instead of f. Applying contractivity, we obtain that $f^+ g_n \in U$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. As $f^+ g_n$ is the direction of the hyperbolic automorphism $g_n f g_n^{-1}$, we obtain the direction of a hyperbolic automorphism in U, too.

Let us now assume that $(xg_i^{-1})_{i\in\mathbb{N}}$ converges to η and that $(xh_i^{-1})_{i\in\mathbb{N}}$ converges to μ. As $\eta \neq \mu$, there are again open neighbourhoods U and V of η and μ, respectively, with $x \notin \overline{U} \cup \overline{V}$, with $d(U,V) > 0$, and with $\overline{U} \cap \overline{V} = \emptyset$ due to Lemma 2.2. By contractivity, we find an $n \in \mathbb{N}$ such that

$$(\hat{X} \setminus U)g_i \subseteq U \quad \text{ and } \quad (\hat{X} \setminus V)h_i \subseteq V$$

for all $i \geq n$. For $f := h_n g_n$ this implies

$$(\hat{X} \setminus V)f \subseteq V g_n \subseteq U.$$

By Lemma 2.6, the automorphism f is hyperbolic with $f^+ \in \overline{U}$ and $f^- \in \overline{V}$. As we may haven chosen U so that $\overline{U}$ lies in some previously chosen open neighbourhood of η, we have shown (i).

For the proof of (ii) and (iii), let us assume that $\mathcal{L}(G)$ contains at least three elements. As $\mathcal{H}(G)$ is dense in $\mathcal{L}(G)$ according to (i) and as $\hat{X}$ is Hausdorff, there are two hyperbolic automorphisms g and h that do not fix the same two boundary points of X. Let $\eta \in \partial X$ with $\eta g = \eta$ and $\eta h \neq \eta$. Then due to contractivity, the sequence $(\eta h^n)_{n\in\mathbb{N}}$ converges to h^+ but does not contain h^+. Hence, the set $\{\eta h^n \mid n \in \mathbb{N}\}$ is infinite. On the other hand, the boundary point ηh^n is fixed by $h^{-n} g h^n$ which is, as it is conjugated to a hyperbolic automorphism, also hyperbolic. Hence $\mathcal{H}(G)$ and $\mathcal{L}(G)$ are infinite. Since every hyperbolic automorphism fixes two boundary points, we also have $|\mathcal{H}(G)| \neq 1$. □

A group G acts *discontinuously* on a metric space X, if there is a non-empty open subset $O \subseteq X$ with $Og \cap O = \emptyset$ for all non-trivial elements g of G.

Theorem 2.8. *Let $\hat{X}$ be a contractive G-completion of a metric space X. Then one of the following cases holds:*

(i) *G fixes a bounded subset of X;*
(ii) *G fixes a unique element of $\mathcal{L}(G)$;*
(iii) *$\mathcal{L}(G)$ consists of precisely two elements;*
(iv) *G contains two hyperbolic elements that have no common fixed point and that freely generate a free subgroup of G that contains aside from the identity only hyperbolic elements and that acts discontinuously on X.*

Proof. First, let us assume that G does not contain any hyperbolic automorphism. Then Theorem 2.7 (i) implies that $|\mathcal{L}(G)| \leq 1$. If $|\mathcal{L}(G)| = 1$, then the unique element of $\mathcal{L}(G)$ has to be fixed by G which shows that (ii) is true in this situation. Thus, we may assume that $\mathcal{L}(G)$ is empty. By (C3), the set xG must be bounded for any $x \in X$. So (i) holds.

Let us now assume that G contains a hyperbolic automorphism. Then we have $|\mathcal{L}(G)| \geq 2$. If $|\mathcal{L}(G)| = 2$, then (ii) holds. So we may assume that $|\mathcal{L}(G)| \neq 2$. Thus, we have $|\mathcal{H}(G)| > 2$, since $\mathcal{H}(G)$ is dense in $\mathcal{L}(G)$ due to Theorem 2.7 (i). So G contains more than one hyperbolic element. We shall show that either (ii) or (iv) holds.

Let us first consider the case that every two hyperbolic automorphisms have a common fixed point. Then we shall show the existence of a boundary point in $\mathcal{L}(G)$ that is fixed by all elements of G. Suppose that no such fixed point exists. Let $g \in G$ be hyperbolic. As G contains more than one hyperbolic element that have in total more than two distinct directions, we know that $\{g^+, g^-\}$ is not G-invariant. For every $h \in G$, the automorphism $h^{-1}gh$ is hyperbolic. As every two hyperbolic automorphisms have a common fixed point, either $g^+h = (h^{-1}gh)^+$ or $g^-h = (h^{-1}gh)^-$ lies in $\{g^+, g^-\}$, in particular, we have $\{g^+, g^-\}h \cap \{g^+, g^-\} \neq \emptyset$.

Let us suppose that there are $h_1, h_2 \in G$ with

$$\{g^+, g^-\}h_1 \cap \{g^+, g^-\} = \{g^+\} \quad \text{and} \quad \{g^+, g^-\}h_2 \cap \{g^+, g^-\} = \{g^-\}.$$

As $g_i := h_i^{-1}gh_i$ for $i = 1, 2$ are hyperbolic, they have a common fixed point η. But this fixed point is neither g^+ nor g^- by the choices of h_1 and h_2. Let U and V be disjoint open neighbourhoods of g^+ and g^-, respectively, such that none of them contains η. As $\hat{X}$ is contractive, there is an $n \in \mathbb{N}$ such that $(\hat{X} \smallsetminus V)g^n \subseteq U$ and $(\hat{X} \smallsetminus U)g^{-n} \subseteq V$. Again, the automorphisms $f_1 := g^{-n}g_1g^n$ and $f_2 := g^n g_2 g^{-n}$ are both hyperbolic and we have

$$\{f_1^+, f_1^-\} = \{\eta g^n, g^+g^n\} \subseteq U \quad \text{and} \quad \{f_2^+, f_2^-\} = \{\eta g^{-n}, g^-g^{-n}\} \subseteq V$$

which implies that f_1 and f_2 have no common fixed point even though they are hyperbolic. This contradiction shows that there is a $\mu \in \partial X$ that lies in $\{g^+, g^-\}f$ for all $f \in G$. Let ν be the other element of $\{g^+, g^-\}$.

Since G fixes no element of $\mathcal{L}(G)$, there is an $f \in G$ with $\mu f \neq \mu$. Then we have $\nu f = \mu$ and $\nu f^2 = \mu f \neq \mu$. As $\mu \in \{\mu, \nu\}f^2$, we conclude that $\mu f^2 = \mu$. Since f is a homeomorphism on $\hat{X}$ and $\nu f = \mu f^2$, we have $\mu f = \nu$. Because of $|\mathcal{L}(G)| \neq 2$, there is a hyperbolic automorphism f' in G with precisely one fixed point in $\{g^+, g^-\}$, as any two hyperbolic automorphisms have a common fixed point. If this fixed point is ν, then we conclude $\mu f' = \mu$ as $\mu \in \{\mu, \nu\}f'$. By the choice of f', this is not possible. So f' fixes μ. Hence, the automorphism $f'f$ maps μ to ν and ν to $\nu f'f \neq \nu f = \mu$. So μ does not lie in $\{g^+, g^-\}f'f$. This contradiction shows that some element of $\mathcal{L}(G)$ is fixed by G in the situation that every two hyperbolic automorphisms have a common fixed point. Since $|\mathcal{H}(G)| \geq 2$, this fixed boundary point must be unique.

Let us consider the remaining case, that is, that there are two hyperbolic elements g and h in G without common fixed point. We shall show that there is some $k \geq 1$ such that g^k and h^k satisfy the condition (iv). Let U_1, V_1, U_2, and V_2 be open neighbourhoods in $\hat{X}$ of g^-, g^+, h^-, and h^+, respectively, that have pairwise positive distance from each other, such that their closures are disjoint and such that

$$\overline{U_1} \cap \overline{U_2} \cap \overline{V_1} \cap \overline{V_2} \neq \hat{X}.$$

We can find these neighbourhoods similarly as in the proof of Lemma 2.2. Let O be a non-empty open subset of X that is disjoint from all four just defined subsets of $\hat{X}$. As $\hat{X}$ is contractive, there is an $n_0 \geq 1$ with

$$(\hat{X} \smallsetminus U_1)g^n \subseteq V_1 \quad \text{and} \quad (\hat{X} \smallsetminus V_1)g^{-n} \subseteq U_1$$

as well as

$$(\hat{X} \smallsetminus U_2)h^n \subseteq V_2 \quad \text{and} \quad (\hat{X} \smallsetminus V_2)h^{-n} \subseteq U_2$$

for all $n \geq n_0$. Set $f_1 := g^{n_0}$ and $f_2 := h^{n_0}$. We shall show that f_1 and f_2 freely generate $F := \langle f_1, f_2 \rangle$ and that this group acts discontinuously on X. But as this proof is basically the well-known ping-pong argument, we omit it here and refer to the corresponding proof by Woess [29, Proposition 1] for G-compactifications of proper metric spaces. $\square$

The hyperbolic limit set is *bilaterally dense* in $\mathcal{L}(G)$ if $\mathcal{H}(G)$ is not empty and if for any two disjoint non-empty open sets $A, B \subseteq \mathcal{L}(G)$ there is a hyperbolic element $g \in G$ with $g^+ \in A$ and $g^- \in B$. Our next theorem says that $\mathcal{H}(G)$ is bilaterally dense in $\mathcal{L}(G)$ if and only if either (iii) or (iv) of Theorem 2.8 hold.

Theorem 2.9. *Let $\hat{X}$ be a contractive G-completion of a metric space X. The following statements are equivalent.*

(i) *The hyperbolic limit set of G is bilaterally dense in $\mathcal{L}(G)$.*
(ii) *Either $|\mathcal{L}(G)| = 2$ or G contains two hyperbolic elements that have no common fixed point.*

Proof. Let us assume that (i) holds and that $|\mathcal{L}(G)| \neq 2$. As $\mathcal{H}(G) \neq \emptyset$ by the definition of bilateral denseness, we know that $\mathcal{L}(G)$ and $\mathcal{H}(G)$ are infinite according to Theorem 2.7 (ii) and (iii). As $\hat{X}$ is Hausdorff, we may take four pairwise disjoint open subsets $V_1, \ldots, V_4$ of ∂X and conclude that there are two hyperbolic elements g, h in G with $g^+ \in V_1$, $g^- \in V_2$, $h^+ \in V_3$, and $h^- \in V_4$. Obviously, these two hyperbolic automorphisms have no common fixed point.

To show the converse, let us assume that (ii) holds. For every open neighbourhood Y in $\mathcal{L}(G)$ of any element $\eta \in \mathcal{L}(G)$, there is a neighbourhood Y' in $\hat{X}$ with $Y' \cap \mathcal{L}(G) \subseteq Y$ as $\hat{X}$ is regular. Thus, we may take disjoint non-empty open subsets A and B of $\hat{X}$ with $A' := A \cap \mathcal{L}(G) \neq \emptyset$ and $B' := B \cap \mathcal{L}(G) \neq \emptyset$ and just have to show that there is a hyperbolic element f in G with $f^+ \in A'$ and $f^- \in B'$.

If $|\mathcal{L}(G)| = 2$, then each of the two sets A' and B' consists of precisely one point and according to Theorem 2.7 (i) there is a hyperbolic element f in G with $f^+ \in A'$. This implies $f^- \in B'$. Hence, we may assume that G contains two hyperbolic elements without common fixed point.

Let $\eta \in A'$ and $\mu \in B'$, let U be an open neighbourhood of η with $\overline{U} \subseteq A$, and let V be an open neighbourhood of μ with $\overline{V} \subseteq B$ such that $d(U, V) > 0$, $\overline{U} \cap \overline{V} = \emptyset$, and $\overline{U} \cup \overline{V} \neq \hat{X}$. For the existence of U and V, we refer again to the proof of Lemma 2.2. Let us show:

(3) *there are hyperbolic elements $g, h \in G$ with $g^+, g^- \in U$ and $h^+, h^- \in V$.*

As $\mathcal{H}(G)$ is dense in $\mathcal{L}(G)$, we find a hyperbolic automorphism a in G with $a^+ \in U$. Since there are two hyperbolic elements in G without common fixed point, we find a hyperbolic automorphism b that fixes neither a^+ nor a^-. Applying contractivity to open neighbourhoods U' and V' of a^+ and a^-, respectively, with $U' \subseteq U$ we obtain an $n \in \mathbb{N}$ with $b^+a^n \in U$ and $b^-a^n \in U$. Let $g = a^{-n}ba^n$. Then g is hyperbolic as it is conjugated to a hyperbolic automorphism and for every $x \in \hat{X} \setminus U$ we have

$$xg^m = xa^{-n}b^m a^n \to b^+a^n = g^+ \text{ for } m \to \infty$$

and

$$xg^{-m} = xa^{-n}b^{-m}a^n \to b^-a^n = g^- \text{ for } m \to \infty.$$

Thus, g^+ and g^- lie in U. Analogously, we find a hyperbolic element h of G with $h^+, h^- \in V$, which shows (3).

By contractivity, there is some $m \in \mathbb{N}$ with $xg^m \in U$ and $xg^{-m} \in U$ for all $x \in \hat{X} \setminus U$ as well as $xh^m \in V$ and $xh^{-m} \in V$ for all $x \in \hat{X} \setminus V$. Let $f = h^m g^m$. Then we conclude

$$xf = xh^m g^m \in Vg^m \subseteq U$$

for all $x \in \hat{X} \smallsetminus V$ and

$$xf^{-1} = xg^{-m}h^{-m} \in Uh^{-m} \subseteq V$$

for all $x \in \hat{X} \smallsetminus U$. As $d(U,V) > 0$, $\overline{U} \cap \overline{V} = \emptyset$, and $\overline{U} \cup \overline{V} \neq \hat{X}$, Lemma 2.6 implies that f is hyperbolic with $f^+ \in \overline{U}$ and $f^- \in \overline{V}$ as desired. □

Theorem 2.10. *Let $\hat{X}$ be a contractive G-completion of a metric space X and such that $\mathcal{L}(G)$ is infinite. Then $\mathcal{L}(G)$ is a perfect set.*

Furthermore, the following statements are equivalent.

(i) *The set $\{(g^+, g^-) \mid g \in G,\ g \text{ is hyperbolic}\}$ is dense in $\mathcal{L}(G) \times \mathcal{L}(G)$.*
(ii) *The hyperbolic limit set of G is bilaterally dense in $\mathcal{L}(G)$.*
(iii) *There are two hyperbolic elements in G that have no common fixed point.*

Proof. To show that $\mathcal{L}(G)$ is perfect, we have to show that $\mathcal{L}(G)$ contains no isolated point. Let us suppose that $\eta \in \mathcal{L}(G)$ is isolated. As $\mathcal{H}(G)$ is dense in $\mathcal{L}(G)$ according to Theorem 2.7 (i), we find a hyperbolic element $g \in G$ with $g^+ = \eta$. Let $\mu \in \mathcal{L}(G)$ with $\mu g \neq \mu$. This limit point exists as $\mathcal{L}(G)$ is infinite. Since g is hyperbolic and $\hat{X}$ is contractive, the sequence $(\mu g^i)_{i \in \mathbb{N}}$ converges to g^+ but none of its elements is g^+. Hence, g^+ cannot be isolated in $\mathcal{L}(G)$.

For the additional statement, we note that (ii) is oviously a direct consequence of (i). The fact that $\mathcal{L}(G)$ is perfect implies the inverse direction and the equivalence of (ii) and (iii) follows from Theorem 2.9 (ii) as $|\mathcal{L}(G)| \neq 2$. □

3. Graphs with their ends

Contractive G-completions are natural generalizations of the contractive G-compactifications defined by Woess [29]. Besides proper geodesic hyperbolic spaces[2], examples for those contractive G-compactifications are locally finite connected graphs X with *vertex ends* as boundary (see [29]) that are the equivalence classes of *rays* (i.e. one-way infinite paths) where two rays are *equivalent* if and only if they lie eventually in the same component of $X \smallsetminus S$ for any finite vertex set S. A base for the topology on a graph with its vertex ends is given by sets that are open in the distance metric of the graph and by vertex sets C that have a finite neighbourhood (vertices in $V(G) \smallsetminus C$ that are adjacent to some vertex of C) and such that some ray lies in C. In this latter situation, the set C is a neighbourhood of all vertex ends that have a ray in C.

For our theorems, we dropped the hypothesis on X being a proper metric space, that is, we do not require the graphs to have finite degrees. Thus, the canonical guess would be to ask if arbitrary connected graphs X with their vertex ends are examples of G-completions. Unfortunately, this is not the case: the first obstacle is that such a space is not projective and the second is that the uniform convergence property of the contractivity does not hold for the space. We give an example for these two obstacles.

Example 1. Let X be a graph such that every vertex is a cut vertex and lies in λ blocks each of which is a copy of the complete graph on κ vertices, where κ and λ are infinite cardinals. These graphs have a large symmetry group: its automorphisms

[2]We will look at geodesic hyperbolic spaces in Section 4.

do not only act transitively on the graph. Indeed, the graphs are *distance-transitive* graphs[3], cp. [13, 21].

Considering the completion $\hat{X}$ of X with its vertex ends, any two rays in distinct blocks have bounded distance to each other but they lie in distinct vertex ends. Thus, $\hat{X}$ is not projective.

To see that also the second part of the definition of contractivity – the uniform convergence property – does not hold, let Y be a block in X and C be a component of $X - y$ for a vertex $y \in Y$ with $C \cap Y = \emptyset$. Let $(y_i)_{i\in\mathbb{N}}$ be a sequence in Y such that its elements are pairwise distinct and also all distinct from y and such that $Y \smallsetminus \{y_i \mid i \in \mathbb{N}\}$ is infinite. Let $(C_i)_{i\in\mathbb{N}}$ be a sequence of components of $X \smallsetminus \{y_i\}$ with $C_i \cap Y = \emptyset$. Then there is an automorphism g of X with $C_i g = C_{i+1}$ that fixes C pointwise. Thus, we have $xg^i \to \eta$ for $i \to \infty$ and for every $x \in C_1$, where η is the end that contains all rays in Y, and also $xg^{-i} \to \eta$ for $i \to \infty$. There is a neighbourhood U of η that intersects with C trivially. Hence, xg^n has to converge to η for every $x \in C$ if the uniform convergence property holds, but g fixes C pointwise, so we have $xg^n = x$. This shows that also uniform convergence fails for X and it finishes Example 1.

Let us modify Example 1 a bit so that we obtain a graph which also shows that Theorems 2.7 and 2.9 do not hold for graphs with all their vertex ends as completion.

Example 2. Let X be the graph from Example 1. For every $x \in V(X)$, let Y_x be a complete graph on $\aleph_0$ vertices, and let $y_x \in V(Y_x)$. Let Z be obtained from the disjoint union of X and all Y_x by identifying each x with y_x. Unfortunately, the limit set $\mathcal{L}(G)$ with $G := \mathrm{Aut}(Z)$ depends on the choice of the vertex used for its definition: taking a vertex from X leads to a limit set consisting of all vertex ends that belong to X and taking any other vertex implies that $\mathcal{L}(G)$ is the set of all vertex ends of Z. Since for contractive G-completions, the independence of $\mathcal{L}(G)$ from the chosen vertex x was implied by projectivity, which is not given in our situation, it would be natural to define $\mathcal{L}(G)$ to be the union of all accumulations points of zG for all $z \in V(Z)$.

To show that the concludions of Theorems 2.7 and 2.9 do not hold for X, it obviously suffices to show it for the conclusion of Theorem 2.7. But this is easy to see: every hyperbolic limit point can be separated by $x \in V(X)$ from the end in Y_x, so $\mathcal{H}(G)$ cannot be dense in $\mathcal{L}(G)$.

But nevertheless, Theorem 2.3 and Theorem 2.8 are true for connected graphs with their vertex ends as boundary considering finite vertex sets instead of bounded ones for the definition of elliptic elements, see Halin [10] and Jung [14]. Although the vertex ends fail to make $\hat{X}$ a G-completion in general, there is on one side a natural subclass of the ends and on the other side another notion of ends, the metric ends as defined by Krön in [17] (see also Krön and Möller [19, 20]), which our situation fits to.

We call a ray a *local ray* if there is a vertex set of finite diameter that contains infinitely many vertices of the ray. As we have seen in Example 1, existence of two distinct ends each of which contains a local ray is an obstruction for any completion

[3]A graph is called *distance-transitive* if, for each $k \in \mathbb{N}$, its automorphisms act transitively on those pairs of vertices that have distance k to each other.

of a graph to be projective and any end that contains a local ray might be an obstacle for the uniform convergence property in the definition of contractivity. This motivates us to consider only those ends for the contractive G-completion that do not contain any local ray. And indeed, we obtain the following result:

Theorem 3.1. *Let X be a connected graph and $\hat{X}$ the completion of X with all those vertex ends of X that do not contain any local ray. Then $\hat{X}$ is a contractive $\mathrm{Aut}(X)$-completion and the theorems of Section 2 hold for $\hat{X}$.*

The proof of Theorem 3.1 is similar to the one of Theorem 3.3 but uses finite vertex sets instead of vertex sets of finite diameter for the definition of the ends. Notice that Sprüssel [25, Theorem 2.2] showed that graphs with their ends form a normal topological space. We omit the proof of Theorem 3.1 and prove the results for connected graphs with their metric ends instead.

A ray in a graph X is a *metric ray* if it eventually lies outside every ball of finite diameter. So a ray is a metric ray if and only if it is not a local ray. Two metric rays are *equivalent* if they eventually lie in the same component of $X \setminus S$ for any vertex set S of finite diameter. This is an equivalence relation and its equivalence classes are the *metric ends* of X. A *metric double ray* is a *double ray* (i.e. a two-way infinite path) such that no ball of finite diameter contains infinitely many of its vertices. So any subray of a metric double ray is a metric ray. Let us define a base for the topology on a graph with its metric ends: it consists of all those sets that are open in the distance metric of the graph and of all those sets C of vertices that have a neighbourhood of finite diameter and such that some metric ray lies in C – in this situation the set C is a neighbourhood of all metric ends that have a metric ray which lies in C. For more details on metric ends, we refer to [17, 19, 20].

To prove that a connected graph X with its metric ends is an $\mathrm{Aut}(X)$-completion, we need a result due to Krön and Möller [19], which is (for a connected graph) a stronger version of Lemma 2.6.

Theorem 3.2. [19, Theorem 2.12] *Let X be a connected graph and $g \in \mathrm{Aut}(X)$. If there is a non-empty vertex set S of finite diameter, a component C of $X \setminus S$ and an $n \in \mathbb{N}$ with $(S \cup C)g^n \subseteq C$, then there is a metric double ray L and an $m \in \mathbb{N}$ such that g^m acts as a non-trivial translation on L.* □

Theorem 3.3. *Let X be a connected graph and $\hat{X}$ be X with its metric ends. Then $\hat{X}$ is a contractive $\mathrm{Aut}(X)$-completion of X and the theorems of Section 2 hold for $\hat{X}$.*

Proof. First, we mention that $\hat{X}$ is Hausdorff and regular, cp. [17, Theorem 4] and that the canonical extensions of automorphisms of X are homeomorphisms of $\hat{X}$, cp. [17, Theorem 6]. Furthermore, X is open and dense in $\hat{X}$. Thus, it remains to prove (C3) and (C4) for $G = \mathrm{Aut}(X)$ and then that the G-completion is contractive.

We note that the condition for $\hat{X}$ being projective is – as a direct consequence of the definition of metric ends – valid even though we have not proved yet that $\hat{X}$ is a G-completion. But we may use the property during the remainder of the proof.

To prove (C3), let $(g_j)_{j\in\mathbb{N}}$ be a sequence in G with $d(x, xg_j) \to \infty$ for $j \to \infty$. Let B_i be the ball with centre x and radius i. Either there is for each i precisely one component of $X \setminus B_i$ that contains all but finitely many vertices of $\{xg_j \mid j \in \mathbb{N}\}$ or there are two components C_1, C_2 of $X \setminus B_i$ and $k, \ell \in \mathbb{N}$ with $B_i g_k \subseteq C_1$

and $B_ig_\ell \subseteq C_2$ as well as with $d(B_ig_k, B_i) \geq 2$ and $d(B_ig_\ell, B_i) \geq 2$. In the first case, those components D_i that contain all but finitely many of the vertices of $\{xg_j \mid j \in \mathbb{N}\}$ define a unique metric end η as the radii of the balls B_i increase strictly: take the unique element in $\bigcap_{i\in\mathbb{N}} \overline{D}_i$. As the sequence $(xg_j)_{j\in\mathbb{N}}$ eventually lies in each of these components, the sequence must have η as an accumulation point.

Thus, we may assume that there are two distinct components C_1, C_2 of $X \smallsetminus B_i$ and $k, \ell \in \mathbb{N}$ with $B_ig_k \subseteq C_1$ and $B_ig_\ell \subseteq C_2$ and with $d(B_ig_k, B_i) \geq 2$ and $d(B_ig_\ell, B_i) \geq 2$. If either g_k or g_ℓ satisfies the assumptions of Theorem 3.2, then there is a vertex z on the metric double ray L of the conclusion of Theorem 3.2 such that the set $\{zg_j^n \mid n \in \mathbb{Z}\}$, for either $j = k$ or $j = \ell$, has the metric ends to which every subray of L converges as accumulation points. By projectivity, we conclude that each of the two sets $\{xg_j^n \mid n \in \mathbb{N}\}$ and $\{xg_j^{-n} \mid n \in \mathbb{N}\}$ has an accumulation point in ∂X. So we assume that neither g_k nor g_ℓ satisfies the assumptions of Theorem 3.2. This implies that $B_ig_k^2$ must lie in the same component of $X \smallsetminus B_ig_k$ in which B_i lies. Analogously, $B_ig_\ell^2$ lies together with B_i in a component of $X \smallsetminus B_ig_\ell$. Let us consider the automorphism $g := g_k^{-1}g_\ell$. Let $y \in C_1$ with $d(y, B_i) < d(B_i, B_ig_k)$. We have $yg_k^{-1} \in C_1g_k^{-1} \cap C_1$ and the vertices yg_k^{-1} and x must lie in the same component of $X \smallsetminus B_ig_k$. Hence, x and yg do not lie in the same component of $X \smallsetminus B_ig_\ell$ and the same is true for x and xg. This implies for the component C of $X \smallsetminus B_ig_k$ that contains x, that we have $(B_ig_k \cup C)g \subseteq C$. According to Theorem 3.2, there is a metric end that is a limit point of $\{xg^j \mid j \in \mathbb{N}\}$ and the same holds for $\{xg^{-j} \mid j \in \mathbb{N}\}$. This finishes the proof of (C3).

For the proof of (C4), let $x \in X$ and let $(g_k)_{k\in\mathbb{N}}$ be a sequence in G with $xg_k \to \eta$ for $k \to \infty$ for some $\eta \in \partial X$ such that $\{xg_k^{-1} \mid k \in \mathbb{N}\}$ has no accumulation point. So there is an i_0 such that for all $i \geq i_0$ and for the ball B_i of radius i and centre x all but finitely many of the balls B_ig_k lie in the same component C_i of $X \smallsetminus B_i$ and all but finitely many of the balls $B_ig_k^{-1}$ lie outside C_i. If we find infinitely many g_k that satisfy the assumptions of Theorem 3.2, then the sets $\{xg_k^n \mid n \in \mathbb{N}\}$ and $\{xg_k^{-n} \mid n \in \mathbb{N}\}$ have distinct limit points η_1 and η_2 in the set of metric ends and we find a sequence in the limit points of the sets $\{xg_j^n \mid n \in \mathbb{N}\}$ that converges to η since for all k with $B_ig_k \subseteq C_i$ one of the two limit points η_1 and η_2 lies in C_i, that is, contains a metric ray inside C_i. If we do not find these infinitely many g_k, then let k be such that B_ig_k lies in C_i and let ℓ be such that $B_ig_\ell^{-1}$ lies in a component of $X \smallsetminus B_i$ distinct from C_i and such that $d(B_i, B_ig_k) > 2$ and $d(B_i, B_ig_\ell^{-1}) > 2$. As in the proof of (C3), the automorphism $g_\ell g_k$ satisfies the assumptions of Theorem 3.2 and, as we can choose k among infinitely many natural numbers, we obtain our sequence of limit points of the sets $\{x(g_\ell g_k)^n \mid n \in \mathbb{N}\}$ that converges to η similarly to the previous case. This shows (C4).

Let us now prove that $\hat{X}$ is contractive. We have already seen that $\hat{X}$ is projective. So let $(g_i)_{i\in\mathbb{N}}$ be a sequence in G with $xg_i \to \eta$ and $xg_i^{-1} \to \mu$ for $i \to \infty$, some $x \in X$ and metric ends η and μ. Let U be a neighbourhood of η and V be a neighbourhood of μ. We may assume that there are vertex sets S_U and S_V of finite diameter such that U is a component of $X \smallsetminus S_U$ and V is a component of $X \smallsetminus S_V$. As $xg_n \to \eta$, there is by projectivity an $n_1 \in \mathbb{N}$ such that S_Vg_n lies in the same component of $X \smallsetminus S_U$ as η and such that $d(xg_n, S_U) > d(x, S_V) + \operatorname{diam}(S_V)$ for all $n \geq n_1$. Then we have $S_Vg_n \subseteq U$ and in particular $S_Vg_n \cap S_U = \emptyset$ for all $n \geq n_1$.

Similarly, we find $n_2 \in \mathbb{N}$ such that $S_U g_n^{-1}$ lies in the same component of $X \smallsetminus S_V$ as η and such that $d(xg_n^{-1}, S_V) > d(x, S_U) + \operatorname{diam}(S_U)$ for all $n \geq n_2$. Again, we have $S_U g_n^{-1} \subseteq V$ and $S_U g_n^{-1} \cap S_V = \emptyset$ for all $n \geq n_2$ and hence also $S_U \subseteq V g_n$. Let $n_0 := \max\{n_1, n_2\}$, let $n \geq n_0$, and let $y \in \hat{X} \smallsetminus V$. As $S_U \subseteq V g_n$ and S_V separates y and $S_U g_n^{-1}$, the vertex yg_n must lie outside the component of $X \smallsetminus (S_V g_n)$ that contains S_U. Since it is cannot be separated from η by S_U, we have $yg_n \in U$. This shows that $(yg_n)_{n\in\mathbb{N}}$ converges uniformly to η outside every neighbourhood of all accumulation points of $\{xg_i^{-1} \mid i \in \mathbb{N}\}$ in $\hat{X}$. □

In the case of locally finite graphs with their vertex ends as boundary, a parabolic automorphism g has the additional property that the sequence $(xg^i)_{i\in\mathbb{N}}$ converges to the unique fixed end for any vertex x. This is not true in the case of arbitrary graphs with their metric ends as boundary: Krön and Möller [19, Example 3.16] constructed a graph with precisely one metric end and an automorphism that fixes no bounded vertex set but leaves a double ray invariant that is neither bounded nor a metric double ray. This implies that for any vertex x on that double ray, its orbit is unbounded but there is a vertex set of finite diameter that contains infinitely many of the vertices in its orbit. This shows that, for contractive G-compactifications, an analogous convergence property as for hyperbolic automorphisms does not hold in the case of parabolic automorphisms.

4. Hyperbolic spaces

In this section, we consider hyperbolic spaces that are not necessarily proper[4] but *geodesic*, that is for every two points x, y there is an isometric image of $[0, d(x, y)]$ joining x and y. We shall show that the geodesic hyperbolic spaces with their hyperbolic boundary are contractive G-completions and hence, that the theorems of Section 2 are true for them. To obtain an overview which basic properties of geodesic hyperbolic spaces are known, we refer to [2, 27] and for an introduction to proper geodesic hyperbolic spaces, we refer to [1, 4, 5, 8, 9, 15]. Since we deal with spaces that are not necessarily proper, we will cite from the first list and mainly from [2]. Let us briefly recall the main definitions for hyperbolic spaces.

Let X be a metric space. The *Gromov-product* $(x, y)_o$ of $x, y \in X$ with respect to the *base-point* $o \in X$ is defined as follows:

$$(x, y)_o := \tfrac{1}{2}\left(d(o, x) + d(o, y) - d(x, y)\right).$$

For $\delta \geq 0$, the space X is *δ-hyperbolic* if for given base-point $o \in X$ we have

$$(x, y)_o \geq \min\{(x, z)_o, (y, z)_o\} - \delta$$

for all $x, y, z \in X$. A space is *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

It is easy to show that the definition of being hyperbolic does not depend on o, that is, if the space is δ-hyperbolic with respect to $o \in X$, then for $o' \in X$ there exists $\delta' \geq 0$ such that X is δ'-hyperbolic with respect to o'.

To define the completion $\hat{X}$ of a geodesic hyperbolic space X, we define a further metric on X. For this, let $\varepsilon > 0$ with $\varepsilon' = \exp(\varepsilon\delta) - 1 < \sqrt{2} - 1$. For $x, y \in X$, let

$$\varrho_\varepsilon(x, y) = \begin{cases} \exp(-\varepsilon(x, y)_o) & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

[4]A metric space is *proper* if all closed balls of finite diameter are compact.

Then

$$d_\varepsilon(x,y) = \inf\{\sum_{i=1}^{n-1} \varrho_\varepsilon(x_i,x_{i+1}) \mid x_i \in X,\ x_1 = x,\ x_n = y\}$$

for all $x, y \in X$ defines a metric on X with

$$(1-2\varepsilon')\varrho_\varepsilon(x,y) \leq d_\varepsilon(x,y) \leq \varrho_\varepsilon(x,y) \tag{4}$$

for all $x, y \in X$, see e.g. [2, Theorem 2.2.7]. Let $\hat{X}$ be the completion of the metric space (X, d_ε) and let $\partial X = \hat{X} \setminus X$ be the *hyperbolic boundary* of X. A subset S of X *separates* two sets $U, V \subseteq \hat{X}$ *geodesically* if every geodesic between a point of U and a point of V intersects non-trivially with S.

Let A and B be two subsets of a metric space Y. We say that A lies δ-*close* to B for some $\delta \geq 0$ if $d(a, B) \leq \delta$ for all $a \in A$. A *triangle* xyz in a geodesic metric space Y is a union of three geodesics – called *sides* of the triangle –, one between every two of the *vertices* x, y, and z of the triangle. A triangle is δ-*thin* if any of its sides lies δ-close to the union of its other two sides. Due to [2, Proposition 2.1.3], every triangle in a geodesic δ-hyperbolic space is 4δ-thin.

A useful property of the Gromov-product in geodesic hyperbolic spaces is the following:

Lemma 4.1. *Let X be a geodesic δ-hyperbolic space and let $x, y, z \in X$. Then we have for all geodesics π between y and z:*

$$d(x,\pi) - 8\delta \leq (y,z)_x \leq d(x,\pi).$$

For a proof of Lemma 4.1 we refer to any introductory text on hyperbolic spaces, e. g. [8].

We call a map $\varphi : Y \to Z$ between metric spaces *quasi-isometric* if there are $\gamma \geq 1$ and $c \geq 0$ such that

$$\frac{1}{\gamma} d_Y(y,y') - c \leq d_Z(\varphi(y),\varphi(y')) \leq \gamma d_Y(y,y') + c$$

for all $y, y' \in Y$. A *quasi-geodesic* is the image of a quasi-isometric map $\varphi : [0, r] \to Z$ with $r \in \mathbb{R}_{\geq 0}$ and an *infinite quasi-geodesic* is the image of a quasi-isometric map $\varphi : \mathbb{R}_{\geq 0} \to Z$.

Equipped with these definitions we are able to prove that the hyperbolic completions of geodesic hyperbolic spaces are contractive G-completions. The following lemma is similar to [4, Lemme 2.2].

Lemma 4.2. *Let X be a geodesic δ-hyperbolic space and let $g \in \mathrm{Aut}(X)$ with $d(x, xg^2) \geq d(x, xg) + 8\delta + \gamma$ for some $\gamma > 0$ and $x \in X$. Then there are two distinct boundary points η, μ of X with $(xg^n)_{n\in\mathbb{N}} \to \eta$ and $(xg^{-n})_{n\in\mathbb{N}} \to \mu$.*

Furthermore, the map $\mathbb{Z} \to \{xg^z \mid z \in \mathbb{Z}\}$, $z \mapsto xg^z$ is quasi-isometric.

Proof. Let us first show that the inequalities

$$m\,\gamma - \gamma \leq d(x, xg^m) \leq m\, d(x, xg) \tag{5}$$

hold for all $m \in \mathbb{N}$. The second inequality is obvious by triangle-inequality, so we just have to prove the first one. Let $m \in \mathbb{N}$. Using the quadruple conditions for hyperbolic spaces (cp. Section 2.4.1 and Proposition 2.1.3 in [2]) for the points x, xg, xg^2, and xg^m, we obtain

$$d(x,xg^2) + d(xg,xg^m) \leq \max\{d(x,xg) + d(xg^2,xg^m), d(x,xg^m) + d(xg,xg^2)\} + 8\delta.$$

Hence, we have

$$\begin{aligned}(6)\qquad \max\{d(x,xg^{m-2}),d(x,xg^{m})\} &\geq d(x,xg^{2})+d(x,xg^{m-1})-d(x,xg)-8\delta\\ &\geq d(x,xg^{m-1})+\gamma.\end{aligned}$$

An easy induction using $d(x,xg^2) \geq d(x,xg)+8\delta+\gamma$ and (6) shows

$$d(x,xg^{n+1}) \geq d(x,xg^{n})+\gamma$$

for all $n \in \mathbb{N}$ and hence, we have $d(x,xg^m) \geq (m-1)\gamma$.

Due to (5), the map $\mathbb{Z} \to X$, $z \mapsto xg^z$ is quasi-isometric. So we conclude with Theorem 4.4.1 and Proposition 5.2.10 of [2] that $\{xg^n \mid n \in \mathbb{N}\}$ and $\{xg^{-n} \mid n \in \mathbb{N}\}$ converge to distinct boundary points. □

Lemma 4.3. *Let X be a geodesic δ-hyperbolic space and let $x \in X$. Let $g,h \in \mathrm{Aut}(X)$ such that $d(x,xg^2) \leq d(x,xg)+8\delta$ and $d(x,xh^2) \leq d(x,xh)+8\delta$ and such that neither g nor h satisfies the conclusions of Lemma 4.2. If there is a ball B with centre x and radius R such that any geodesic between x and xgh intersects non-trivially with Bg and if we have $d(B',B'g) > 8\delta$ and $d(B'g,B'gh) > 8\delta$ for the ball B' with centre x and radius $R+16\delta$, then*

$$d(x,x(gh)^2) > d(x,xgh)+8\delta.$$

Proof. We consider the following points in X: x, xg, xh, xgh^2, xg^2h, and $x(gh)^2$. If we can show that xgh lies 16δ-close to any geodesic between x and $x(gh)^2$, then this geodesic must intersect non-trivially with $B'gh$ and we obtain

$$d(x,x(gh)^2) \geq 2d(x,xgh)-2(R+16\delta) > d(x,xgh)+8\delta.$$

Let us consider a geodesic between xg and xgh^2. If it intersects non-trivially with $B'gh$, then we conclude

$$d(xg,xgh^2) \geq d(xg,xgh)+d(xgh,xgh^2)-2(R+16\delta)+8\delta > d(xg,xgh)+8\delta$$

and we apply Lemma 4.2 to obtain a contradiction to our assumptions. Hence, no such geodesic intersects non-trivially with $B'gh$. Similarly, if we consider any geodesic between x and xg^2, then we obtain that it does not intersect non-trivially with $B'g$. So the same holds for any geodesic between xh and xg^2h with the ball $B'gh$.

Since triangles are 4δ-thin, we obtain that $[xh,xgh^2]$ lies 16δ-close to

$$[xgh^2,xg] \cup [xg,x] \cup [x,x(gh)^2] \cup [x(gh)^2,xg^2h] \cup [xg^2h,xh]$$

where the brackets denote any geodesic between the two points. As $[xh,xgh^2]$ intersects non-trivially with Bgh by assumption, one of the other five geodesics intersects non-trivially with $B'gh$. We have already shown that this is neither $[xgh^2,xg]$ nor $[xg^2h,xh]$. The geodesics $[xg,x]$ and $[x(gh)^2,xg^2h]$ do not intersect non-trivially with Bgh, too, since Bg separates x and xgh geodesically and since $d(B'g,B'gh) > 8\delta$, and the same is true for Bg^2h with $x(gh)^2$ and xgh. So $[x,x(gh)^2]$ intersects non-trivially with $B'gh$ and the assertion follows as described above. □

Now we are able to deduce the following.

Proposition 4.4. *Let X be a geodesic hyperbolic space and $\hat{X}$ the completion of X with the hyperbolic boundary. Then $\hat{X}$ is a contractive $\mathrm{Aut}(X)$-completion of X.*

Proof. Let X be δ-hyperbolic. By its definition, $\hat{X}$ is a completion of X and from [2, Section 2.2.3] we deduce that automorphisms of X extend to homeomorphisms of $\hat{X}$. As $(\hat{X}, d_\varepsilon)$ is a metric space, it is regular. Let $(g_i)_{i\in\mathbb{N}}$ be a sequence in $\mathrm{Aut}(X)$ such that $d(x, xg_i)$ is unbounded for some $x \in X$. We will show (C3). Let us consider closed balls B_i with centre x and radius i. Either, for all i, all but finitely many xg_j are not geodesically separated by B_i or there are a ball B_i and $k, \ell \in \mathbb{N}$ such that B_i separates xg_k and xg_ℓ geodesically and $d(B_i g_k, B_i) > 8\delta$ and $d(B_i g_\ell, B_i) > 8\delta$. In the first case, we obtain $(xg_k, xg_\ell) \to \infty$ for $k, \ell \to \infty$ because of Lemma 4.1, so the sequence converges to some boundary point. In the second case, either one of g_k^{-1} and g_ℓ or due to Lemma 4.3 the automorphism $g_k^{-1} g_\ell$ has the desired limit points by Lemma 4.2. This shows (C3).

For the proof of (C4) let $(g_i)_{i\in\mathbb{N}}$ be a sequence in G such that for some $x \in X$ and $\eta \in \partial X$ we have $xg_i \to \eta$ for $i \to \infty$ and such that $\{xg_i^{-1} \mid i \in \mathbb{N}\}$ has no accumulation point in ∂X. Notice that the convergence of the sequence $(xg_i)_{i\in\mathbb{N}}$ implies $d(x, xg_i) \to \infty$ for $i \to \infty$. Analogously as in the proof of (C3), we find $k, \ell \in \mathbb{N}$ such that one of the automorphisms g_k, g_ℓ, and $g_k g_\ell$ satisfies the assumption of Lemma 4.2 and hence fulfills the conclusions of that lemma. Furthermore, we find for each $k, \ell \in \mathbb{N}$ further integers $k', \ell' \in \mathbb{N}$ both larger than k and ℓ such that among $g_{k'}$, $g_{\ell'}$ and $g_{k'} g_{\ell'}$ we find another automorphism that satisfies the conclusions of Lemma 4.2. So we find an infinite sequence $(f_i)_{i\in\mathbb{N}}$ of such automorphisms: f_i is either some g_m or some $g_m g_n$ and for $i \to \infty$ also the indices m and n grow. Using this sequence, we shall construct another sequence $(h_i)_{i\in\mathbb{N}}$ of automorphisms such that each of the two sets $\{xh_i^n \mid n \in \mathbb{N}\}$ and $\{xh_i^{-n} \mid n \in \mathbb{N}\}$ has a limit point η_i and μ_i, respectively, such that these two limit points are distinct and such that $\eta_i \to \eta$ for $i \to \infty$. The first two properties are also true for each f_i and we will use that for the proof of the properties for the h_i.

Let us consider the open balls $B_{1/n}(\eta)$. For f_n, there is a constant Δ_n due to [2, Theorem 1.3.2] such that any geodesic between xf_n^{-m} and xf_n^m lies Δ_n-close to $\{xf_n^j \mid |j| \leq m\}$. As $xg_i \to \eta$ for $i \to \infty$, we find i_n such that the Δ_n-ball B with centre xg_{i_n} lies completely in $B_{1/n}(\eta)$. Let $h_n = g_{i_n}^{-1} f_n g_{i_n}$. Since h_n is conjugated to f_n, the conclusions of Lemma 4.2 also hold for h_n. Let us consider the two sets $Q_1 := \{xf_n^j \mid j \in \mathbb{N}\}$ and $Q_2 := \{xf_n^{-j} \mid j \in \mathbb{N}\}$ and, for $i = 1, 2$, quasi-geodesics R_i that contain all elements of Q_i and a geodesic between any xf_n^j and xf_n^{j+1}. The ball B separates any $q_1 \in R_1$ from any $q_2 \in R_2$ geodesically by its choice. Thus, one of the two quasi-geodesics, say R_i, has distance at least $d(x, B)/2$ to x. As R_i is quasi-geodesic, it has a limit point $\eta_n \in \partial X$. Using 4δ-thin triangles $z_\ell(xg_{i_n})q_\ell$ for sequences $(z_\ell)_{\ell\in\mathbb{N}}$ in $B_{1/n}(\eta)$ and $(q_\ell)_{\ell\in\mathbb{N}}$ in R_i converging to η and η_n, respectively, we obtain by Lemma 4.1

$$(z_\ell, q_\ell) \geq d(x, \pi_\ell) - 8\delta = d(x, B)/2 - 12\delta,$$

where π_ℓ is a geodesic between z_ℓ and q_ℓ. So due to (4), we know that the sequence $(\eta_k)_{k\in\mathbb{N}}$ converges to η. Notice that we might have to change some h_i in the sequence $(h_i)_{i\in\mathbb{N}}$ to h_i^{-1} to obtain precisely the statement of (C4).

For the projectivity property, let $(x_i)_{i\in\mathbb{N}}$ be a sequence in X that converges to some $\eta \in \partial X$ and let $(y_i)_{i\in\mathbb{N}}$ be another sequence in X such that there is an $M \geq 0$ with $d(x_i, y_i) \leq M$ for all $i \in \mathbb{N}$. As $(x_i)_{i\in\mathbb{N}}$ converges to a boundary point, we have $d(o, x_i) \to \infty$ and thus also $d(o, y_i) \to \infty$. This implies $(x_i, y_i) \to \infty$, so $d_\varepsilon(x_i, y_i) \to 0$. Hence, $\hat{X}$ is a projective G-completion.

To show contractivity, let $(g_i)_{i\in\mathbb{N}}$ be a sequence in $\mathrm{Aut}(X)$ such that for the base point $x \in X$ of the Gromov-product, the sequence $(xg_i)_{i\in\mathbb{N}}$ converges to $\eta \in \partial X$ and $(xg_i^{-1})_{i\in\mathbb{N}}$ converges to $\mu \in \partial X$. Let U and V be open neighbourhoods of η and μ, respectively. Then there are $\theta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\{xg^m \mid m \geq n_0\} \cup \{\eta\} \subseteq B_{\theta/3}(xg_n) \quad \text{and} \quad B_\theta(xg_n) \subseteq U$$

as well as

$$\{xg^{-m} \mid m \geq n_0\} \cup \{\mu\} \subseteq B_{\theta/3}(xg_{-n}) \quad \text{and} \quad B_\theta(xg_{-n}) \subseteq V$$

for all $n \geq n_0$. Let $y \in X \smallsetminus B_{2\theta/3}(xg_{n_0}^{-1})$. Then we have $d_\varepsilon(y,\mu) \geq \theta/3$ and $\exp(-\varepsilon(xg_n^{-1}, y)) \geq d_\varepsilon(xg_n^{-1}, y) \geq \theta/3$. We conclude

$$\begin{aligned}(xg_n, yg_n) &= \tfrac{1}{2}(d(x,xg_n) + d(x,yg_n) - d(xg_n,yg_n))\\ &= \tfrac{1}{2}(d(x,xg_n) + d(x,yg_n) - d(x,y))\\ &= d(x,xg_n) - (xg_n^{-1}, y).\end{aligned}$$

As $d(x,xg_n) \to \infty$ for $n \to \infty$, we find $n_1 \in \mathbb{N}$ such that we have

$$\begin{aligned}d_\varepsilon(xg_n, yg_n) &\leq \varrho_\varepsilon(xg_n, yg_n)\\ &= \exp(-\varepsilon d(x,xg_n) + \varepsilon(xg_n^{-1}, y))\\ &\leq \exp(-\varepsilon d(x,xg_n) - \log(\theta/3))\\ &< \theta/3.\end{aligned}$$

for all $n \geq n_1$. So yg_n lies in $B_{\theta/3}(xg_n) \subseteq U$. Let $\nu \in \partial X \smallsetminus V$. Then we can find a sequence $(y_i)_{i\in\mathbb{N}}$ in $X \smallsetminus B_{2\theta/3}(xg_{-n})$ that converges to ν. Since $y_i g_n \in B_{\theta/3}(xg_n)$, we conclude that νg_n lies in $B_\theta(xg_n)$. This shows contractivity and hence, we have shown that $\hat{X}$ is a contractive $\mathrm{Aut}(X)$-completion. □

We directly obtain:

Corollary 4.5. *Let X be a geodesic hyperbolic space and $\hat{X}$ the completion of X with its hyperbolic boundary. Then the theorems of Section 2 hold for $\hat{X}$.* □

5. Concluding remarks

Apart from the general investigation of groups acting on locally finite graphs or on proper geodesic hyperbolic spaces, there are several more detailed investigations most of which take either Theorem 2.8 (ii) or Theorem 2.8 (iv) as starting point and investigate these situations in more detail: Möller [23] showed that locally finite graphs with infinitely many ends for which a group of automorphisms acts transitively on the graph but fixes an end are quasi-isometric to trees. The same result was obtained in [11] for arbitrary graphs with infinitely many ends. Caprace et al. [3] showed an analogous result for locally finite hyperbolic graphs where the fixed end is replaced by a fixed hyperbolic boundary point (the planar situation was settled earlier in [7]).

In [18], Krön and Möller started with the situation of Theorem 2.8 (iv) and showed that if a group acts on a connected graph such that no vertex end is fixed by the group, then the group has a free subgroup containing (except for the trivial element) only hyperbolic automorphisms and the directions of these hyperbolic automorphisms are dense in the set of all limit points of the group. In the same paper, they also mentioned that an analogous proof holds for metric ends instead of vertex ends. The analogous statement also holds for proper geodesic hyperbolic spaces, see [12].

References

1. J.M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short, *Notes on word hyperbolic groups*, Group Theory from a Geometrical Viewpoint (Trieste, 1990) (E. Ghys, A. Haefliger, and A. Verjovsky, eds.), World Scientific, 1991, pp. 3–63.
2. S. Buyalo and V. Schroeder, *Elements of Asymptotic Geometry*, EMS Monographs in Mathematics, EMS, Zürich, 2007.
3. P.-E. Caprace, Y. de Cornulier, N. Monod, and R. Tessera, *Amenable hyperbolic groups*, arXiv:1202.3585, 2012.
4. M. Coornaert, T. Delzant, and A. Papadopoulos, *Notes sur les groupes hyperboliques de Gromov*, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, 1990.
5. M. Coornaert and A. Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Springer Lecture Notes, vol. 1539, Springer-Verlag, 1993.
6. F.W. Gehring and G.J. Martin, *Discrete quasiconformal groups. I*, Proc. London Math. Soc. **55** (1987), 331–358.
7. A. Georgakopoulos and M. Hamann, *On fixing boundary points of transitive hyperbolic graphs*, Arch. Math. (Basel) **99** (2012), no. 1, 91–99.
8. E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques, d'après M. Gromov*, Progress in Math., vol. 83, Birkhäuser, Boston, 1990.
9. M. Gromov, *Hyperbolic Groups*, Essays in group theory (S.M. Gersten, ed.), MSRI, vol. 8, Springer, New York, 1987, pp. 75–263.
10. R. Halin, *Automorphisms and Endomorphisms of Infinite Locally Finite Graphs*, Abh. Math. Sem. Univ. Hamburg **39** (1973), no. 1, 251–283.
11. M. Hamann, *End-transitive graphs*, Israel J. Math. **189** (2012), no. 1, 437–459.
12. M. Hamann and R.G. Möller, *Free groups acting on hyperbolic spaces*, in preparation.
13. M. Hamann and J. Pott, *Transitivity conditions in infinite graphs*, Combinatorica **32** (2012), no. 6, 649–688.
14. H.A. Jung, *On finite fixed sets in infinite graphs*, Discrete Math. **131** (1991), no. 1-3, 115–125.
15. I. Kapovich and N. Benakli, *Boundaries of hyperbolic groups*, Combinatorial and Geometric Group Theory (R. Gilman et al., ed.), Contemporary Mathematics, vol. 296, 2002, pp. 39–94.
16. A. Karlsson and G. Noskov, *Some groups having only elementary actions on metric spaces with hyperbolic boundaries*, Geom. Dedicata **104** (2004), 119–137.
17. B. Krön, *End compactifications in non-locally-finite graphs*, Math. Proc. Cambridge Phil. Soc. **131** (2001), no. 3, 427–443.
18. B. Krön and R.G. Möller, *Free groups and ends of graphs*, Abh. Math. Sem. Univ. Hamburg **78** (2008), no. 1, 1–15.
19. ———, *Metric ends, fibers and automorphisms of graphs*, Math. Nachr. **281** (2008), no. 1, 62–74.
20. ———, *Quasi-isometries between graphs and trees*, J. Combin. Theory (Series B) **98** (2008), no. 5, 994–1013.
21. H.D. Macpherson, *Infinite distance transitive graphs of finite valency*, Combinatorica **2** (1982), no. 1, 63–69.
22. R.G. Möller, *Ends of graphs*, Math. Proc. Cambridge Phil. Soc. **111** (1992), no. 2, 255–266.
23. ———, *Ends of graphs II*, Math. Proc. Cambridge Phil. Soc. **111** (1992), no. 3, 455–460.
24. M. Pavone, *Bilateral denseness of the hyperbolic limit points of groups acting on metric spaces*, Abh. Math. Sem. Univ. Hamburg **67** (1997), no. 1, 123–135.
25. P. Sprüssel, *End spaces of graphs are normal*, J. Combin. Theory (Series B) **98** (2008), no. 4, 798–804.
26. P. Tukia, *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand J. Math. **23** (1994), 157–187.
27. J. Väisälä, *Gromov hyperbolic spaces*, Expo. Math. **23** (2005), no. 3, 187–232.
28. W. Woess, *Amenable group actions on infinite graphs*, Math. Ann. **284** (1989), no. 2, 251–265.
29. ———, *Fixed sets and free subgroups of groups acting on metric spaces*, Math. Z. **214** (1993), no. 3, 425–440.

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