ASYMPTOTIC HALF-GRID AND FULL-GRID MINORS

SANDRA ALBRECHTSEN AND MATTHIAS HAMANN*

ABSTRACT. We prove that every locally finite, quasi-transitive graph with a thick end whose cycle space is generated by cycles of bounded length contains the full-grid as an asymptotic minor and as a diverging minor. This in particular includes all locally finite Cayley graphs of finitely presented groups, and partially solves problems of Georgakopoulos and Papasoglu and of Georgakopoulos and Hamann.

Additionally, we show that every (not necessarily quasi-transitive) graph of finite maximum degree which has a thick end and whose cycle space is generated by cycles of bounded length contains the half-grid as an asymptotic minor and as a diverging minor.

1. INTRODUCTION

Fat minors are a coarse or metric variant of graph minors. They first appeared in works of Chepoi, Dragan, Newman, Rabinovich and Vaxes [7] and of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot and Scott [5]. They play an important role in many (open) problems at the intersection of structural graph theory and coarse geometry – an area which can be described as 'coarse graph theory'.

A *model* of a graph X in a graph G is a collection of connected *branch sets* and *branch paths* in G such that after contracting each branch set to a vertex, and each branch path to an edge, we obtain a copy of X. A model of X is *K-fat* (in G), for some $K \in \mathbb{N}$, if its branch sets and paths are pairwise at least K apart, except that we do not require this for incident branch set-path pairs (see also Section 2.6 for the definition). We say that X is a *(K-fat) minor* of G if G contains a (K-fat) model of X. The graph X is an *asymptotic minor* of G if X is a K-fat minor of G for every $K \in \mathbb{N}$. An important advantage of asymptotic minors over the usual minors is that they are preserved under quasi-isometries, and in particular, it does not depend on the choice of a finite generating set whether a Cayley graph of a finitely generated group contains a fixed graph as an asymptotic minor [21].

Recently, Georgakopoulos and Papasoglu [21] gave an overview of the area of 'coarse graph theory', where they presented results and open problems regarding the interplay of geometry and graphs, many of which concern fat minors. These problems have already attracted quite some attention; some (partial) solutions can be found in [1, 2, 7, 8, 16, 17, 27]. Our main contribution is a partial resolution of a problem of Georgakopoulos and Papasoglu about asymptotic grid minors in quasi-transitive graphs [21, Problem 7.3]. To state this problem, we first need some definitions.

An *end* of a graph G is an equivalence class of rays where two rays in G are equivalent if there are infinitely many pairwise disjoint paths between them in G. An end is *thick* if it has infinitely many

pairwise disjoint rays. The *full-grid* is the graph on $\mathbb{Z} \times \mathbb{Z}$ in which two vertices (m, n) and (m', n') are adjacent if and only if $|m - m'| + |n - n'| = 1$, and the *half-grid*[1] is its induced subgraph on $\mathbb{N} \times \mathbb{Z}$.

One of the cornerstones of infinite graph theory is *Halin's Grid Theorem* [22, Satz 4′], which asserts that every graph with a thick end contains the half-grid as a minor. Following this approach, Heuer [25] characterised the graphs containing the full-grid as a minor. These graphs form a proper subclass of the graphs with a thick end: while it is clearly true that every graph with a full-grid minor has a thick end, the converse is false in general, as the half-grid itself already witnesses. However, as it turned out, if we only consider graphs which are *quasi-transitive*, i.e. graphs whose vertex set has only finitely many orbits under its automorphism group, then these two graph classes coincide. Indeed, Georgakopoulos and the second author [20] showed that every locally finite, quasi-transitive graph with a thick end contains the full-grid as a minor.

Georgakopoulos and Papasoglu [21] asked whether this result can be generalised to the coarse setting in the following sense.

Problem 1.1. [21, Problem 7.3] *Let G be a locally finite Cayley graph of a one-ended finitely generated group. Must the half-grid be an asymptotic minor of G? Must the full-grid be an asymptotic minor of G?*

Note that every Cayley graph of a group is (quasi-)transitive. Moreover, the unique end of a one-ended, quasi-transitive graph is always thick [6, 28].

Our main theorem partially answers both questions in the affirmative, under the additional assumption that G is a locally finite Cayley graph of a finitely presented group. In fact, we show the following result.

Theorem 1. *Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. If G has a thick end, then the full-grid is an asymptotic minor of G.*

(We refer the reader to Section 2.3 for the definitions concerning the cycle space.)

Note that Theorem 1 includes all locally finite Cayley graphs of finitely presented groups. Examples such as inaccessible graphs and groups [13, 14] or Diestel-Leader graphs [12, 15] indicate that the geometry of arbitrary locally finite, quasi-transitive (or Cayley) graphs may be far more involved. This is why generalising Theorem 1 to locally finite Cayley graphs of arbitrary finitely generated groups or even to all locally finite, quasi-transitive graphs may be much harder, and will require a different approach to that presented in this paper (see the sketch of the proof in Section 3 for details).

For the proof of Theorem 1 we construct, for every such graph G, a single model of the full-grid (see Theorem 3.4), which can be turned into a K-fat model of the full-grid, for every $K \in \mathbb{N}$, by deleting some of its branch sets and paths. Moreover, it can be turned into a model of the full-grid that *diverges*: for any two diverging sequences of vertices and/or edges of the full-grid also their branch sets/paths diverge in G (see Section 2.7 for the definition).

Theorem 2. *Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. If G has a thick end, then the full-grid is a diverging minor of G.*

This partially solves a question of Georgakopoulos and the second author [20, Problem 4.1].

---

[1]Note that usually the grid on $\mathbb{N}^2$ is referred to as the half-grid. However, for us it will be more convenient to work with the grid on $\mathbb{N} \times \mathbb{Z}$. It is easy to see that our results about the half-grid also hold for the grid on $\mathbb{N}^2$.

Krön and Möller [26, Theorem 5.5] proved that a locally finite, quasi-transitive connected graph has no thick end if and only if it is quasi-isometric to a tree. Thus, instead of assuming that the graph G in Theorems 1 and 2 has a thick end, we may assume that G is not quasi-isometric to a tree (see Section 7.2 for details).

As a first step in the proof of Theorem 1, we find the half-grid as an asymptotic minor. For this, we do not need the transitivity assumption on G. Indeed, we prove the following theorem.

Theorem 3. *Let G be a graph of finite maximum degree whose cycle space is generated by cycles of bounded length. If G has a thick end, then the half-grid is an asymptotic minor of G.*

Note that every graph satisfying the premise of Theorems 1 and 2 has finite maximum degree as it is locally finite and quasi-transitive.

Similar as in the proof of Theorem 1, we again construct a single model of the half-grid (see Theorem 3.3), which can be turned into a K-fat model of the half-grid, for every $K \in \mathbb{N}$, and into a diverging model of the half-grid.

Theorem 4. *Let G be a graph of finite maximum degree whose cycle space is generated by cycles of bounded length. If G has a thick end, then the half-grid is a diverging minor of G.*

This partially solves a question of Georgakopoulos and the second author [20, Problem 4.2].

This paper is structured as follows. In Section 2 we recall some important definitions. Section 3 consists of three parts. We first introduce some new definitions in Sections 3.1 and 3.2. We then give a sketch of the proofs of Theorems 1 to 4 in Sections 3.3 and 3.4, where we also state Theorems 3.3 and 3.4, our stronger results on half-grid and full-grid minors, which we already briefly mentioned above. In Sections 3.5 and 3.6 we derive Theorems 1 to 4 from Theorems 3.3 and 3.4. Section 4 contains some preparatory work about diverging and quasi-geodesic rays. We then prove Theorems 3.3 and 3.4 in Sections 5 and 6, respectively. We finish in Section 7 by discussing some related problems.

2. Preliminaries

Our notions mainly follow [9]. In what follows, we recap some important definitions which we need later.

Given sets $U' \subseteq U$ of vertices of a graph G, a component C of $G - U$ *attaches* to U' if C has a neighbour in U'. The *boundary $\partial_G X$* of a subgraph X of G is the set $N_G(V(G - X))$ of vertices of X that send in G an edge outside of X. For example, the boundary $\partial_G C$ of a component C of $G - U$ is $N_G(U) \cap V(C)$.

A graph G is *quasi-transitive* if the automorphism group of G acts on $V(G)$ with only finitely many orbits, that is if $V(G)$ can be partitioned into finitely many sets $U_0, \ldots, U_n$ such that for all $i \in \{0, \ldots, n\}$ and $u, v \in U_i$ there exists an automorphism φ of G such that $\varphi(u) = v$. The *stabilizer* of a subgraph X of G consists of precisely those automorphisms of G that map X to itself.

2.1. **Paths, rays and combs.** For two sets A, B of vertices of G, an *A–B path* meets A precisely in its first vertex and B precisely in its last vertex. For a subgraph H of G, an *H-path* is a non-trivial path which meets H precisely in its endvertices.

A *ray* is a one-way infinite path, and a *double ray* is a two-way infinite path. A *tail* of a (double) ray R is any subray $S \subseteq R$. If $R = r_0 r_1 \ldots$ is a ray, then we denote by $r_i R r_j$ for $i, j \in \mathbb{N}$ the subpath $r_i \ldots r_j$

of R, and by $r_i R$ or $R_{\geq i}$ the tail $r_i r_{i+1} \ldots$ of R. Further, we denote by Rr_i or $R_{\leq i}$ the subpath $r_0 \ldots r_i$ of R. We use these notions analogously for double rays; in particular, if $R = \ldots r_{-1} r_0 r_1 \ldots$ is a double ray, then Rr_i and $R_{\leq i}$ denote the tail $r_i r_{i-1} \ldots$ of R.

A *comb* is a union of a ray R with infinitely many pairwise disjoint finite paths which have precisely their first vertex on R; we call the last vertices of these paths the *teeth* of the comb and refer to R as its *spine*. The following observation about combs in infinite graphs is well-known and follows immediately from the *Star-Comb Lemma*; see e.g. [9, Lemma 8.2.2] for a proof.

Lemma 2.1. *Let U be an infinite set of vertices in a locally finite, connected graph G. Then G contains a comb with all teeth in U.*

Moreover, every infinite, connected graph has a vertex of infinite degree or contains a ray.

2.2. **(Hexagonal) grids.** The *full-grid*, denoted by FG, is the graph on $\mathbb{Z}^2$ in which two vertices (m, n) and (m', n') are adjacent if and only if $|m - m'| + |n - n'| = 1$. The *hexagonal full-grid* is obtained from FG by deleting every other rung, as shown in Figure 2.1. The *(hexagonal) half-grid*, denoted by HG, is the induced subgraph of the (hexagonal) full-grid on vertex set $\mathbb{N} \times \mathbb{Z}$.

We call the double rays R^i of the (hexagonal) full- and half-grid its *vertical double rays* and the edges e_{ij} its *horizontal edges* (see Figure 2.1).
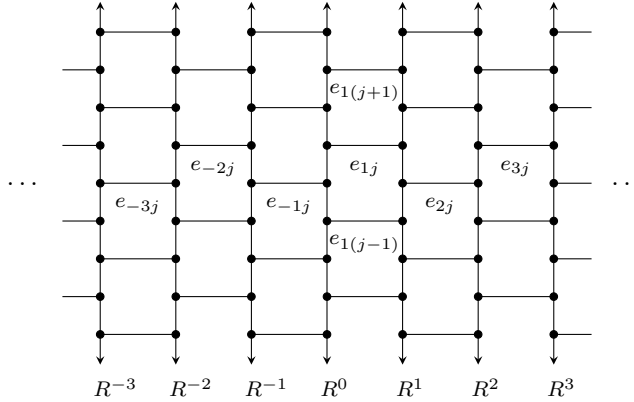

FIGURE 2.1. The hexagonal full-grid with vertical double rays R^i and horizontal edges e_{ij}.

2.3. **Cycle space.** Let G be a graph. The *edge space* of G is the vector space over the 2-element field $\mathbb{F}_2$ of all functions $E(G) \to \mathbb{F}_2$: its elements correspond to the subsets of $E(G)$ and vector addition corresponds to symmetric difference. The *cycle space* of G is the subspace of the edge space of G spanned by all the cycles in G – more precisely, by their edge sets; for simplicity, we will not distinguish between the edge sets in the cycle space and the subgraphs they induce in G.

We say that the cycle space of G is *generated by cycles of bounded length* if there is some $n \in \mathbb{N}$ such that the cycles in G of length at most n generate the cycle space of G.

2.4. **Ends.** An *end* ε of a graph G is an equivalence class of rays in G where two rays are equivalent if they are joined by infinitely many disjoint paths in G or, equivalently, if for every finite set $U \subseteq V(G)$ both rays have tails in the same component of $G - U$. A *(double) ε-ray* is a (double) ray whose tails are all

contained in ε. An end is *thick* if, for every $n \in \mathbb{N}$, there are n pairwise disjoint ε-rays. Halin [22, Satz 1] showed that this is the case if and only if there are infinitely many pairwise disjoint ε-rays.

A finite set $U \subseteq V(G)$ *distinguishes* two ends $\varepsilon, \varepsilon'$ of G if no component of $G - U$ contains rays from both ε and ε'. A graph G is *accessible* if there exists some $n \in \mathbb{N}$ such that every two distinct ends of G can be distinguished by a set of at most n vertices of G.

Theorem 2.2. [23, Corollary 3.2] *Every locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length is accessible.*

2.5. **Distance and balls.** Let G be a graph. We write $d_G(v, u)$ for the distance of the two vertices v and u in G. For two sets U and U' of vertices of G, we write $d_G(U, U')$ for the minimum distance of two elements of U and U', respectively. If one of U or U' is just a singleton, then we omit the braces, writing $d_G(v, U') := d_G(\{v\}, U')$ for $v \in V(G)$. If X is a subgraph of G, then we abbreviate $d_G(U, V(X))$ as $d_G(U, X)$.

Given a set U of vertices of G, the *ball (in G) around U of radius $r \in \mathbb{N}$*, denoted by $B_G(U, r)$, is the set of all vertices in G of distance at most r from U in G. If $U = \{v\}$ for some $v \in V(G)$, then we omit the braces, writing $B_G(v, r)$ for the ball (in G) around v of radius r. Additionally, we abbreviate the induced subgraph on $B_G(U, r)$ of G with $G[U, r] := G[B_G(U, r)]$. If X is a subgraph of G, then we abbreviate $B_G(V(X), r)$ and $G[V(X), r]$ as $B_G(X, r)$ and $G[X, r]$, respectively.

A subgraph X of G is *c-quasi-geodesic[2] (in G)* for some $c \in \mathbb{N}$ if for every two vertices $u, v \in V(X)$ we have $d_X(u, v) \leq c \cdot d_G(u, v)$. We call X *quasi-geodesic* if it is c-quasi-geodesic for some $c \in \mathbb{N}$ and *geodesic* if it is 1-quasi-geodesic.

Two rays R, S in G *diverge* if for every $n \in \mathbb{N}$ they have tails $R' \subseteq R$, $S' \subseteq S$ satisfying $d_G(R', S') > n$. A double ray R in G *diverges* if every two disjoint tails of R diverge.

2.6. **Fat and diverging minors.** Let G, X be graphs. A *model* $(\mathcal{V}, \mathcal{E})$ of X in G is a collection $\mathcal{V}$ of disjoint sets $V_x \subseteq V(G)$ for vertices x of X such that each $G[V_x]$ is connected, and a collection $\mathcal{E}$ of internally disjoint V_{x_0}–V_{x_1} paths E_e for edges $e = x_0 x_1$ of X which are disjoint from every V_x with $x \neq x_0, x_1$. The V_x are its *branch sets* and the E_e are its *branch paths*. A model $(\mathcal{V}, \mathcal{E})$ of X in G is *K-fat* for $K \in \mathbb{N}$ if $d_G(Y, Z) \geq K$ for every two distinct $Y, Z \in \mathcal{V} \cup \mathcal{E}$ unless $Y = E_e$ and $Z = V_x$ for some vertex $x \in V(X)$ incident to $e \in E(X)$, or vice versa. The graph X is a *(K-fat) minor* of G, denoted by $X \prec G$ $(X \prec_K G)$, if G contains a (K-fat) model of X. Moreover, X is an *asymptotic minor* of G, denoted by $X \prec_\infty G$, if X is a K-fat minor of G for all $K \in \mathbb{N}$. Let ε be an end of G. If X is a one-ended graph, then we write $X \prec_K^\varepsilon G$ if G contains a K-fat model $(\mathcal{V}, \mathcal{E})$ of X such that every ray in $\bigcup_{x \in V(X)} G[V_x] \cup \bigcup_{e \in E(X)} E_e$ is an ε-ray. Similarly, we write $X \prec_\infty^\varepsilon G$ if $X \prec_K^\varepsilon G$ for all $K \in \mathbb{N}$.

A model $(\mathcal{V}, \mathcal{E})$ of X in G *diverges* if for every two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ of vertices and/or edges of X such that $d_X(x_n, y_n) \to \infty$, we have $d_G(U_n, W_n) \to \infty$ where $U_n := V_{x_n}$ if $x_n \in V(X)$ and $U_n := V(E_{x_n})$ if $x_n \in E(X)$ and analogously $W_n := V_{y_n}$ or $W_n := V(E_{y_n})$.

2.7. **Fat and diverging subdivisions.** A *subdivision* of a graph X is a graph which arises from X by replacing every edge in X by a new path between its endvertices such that no new path has an inner

---

[2]Note that in general metric spaces one also allows for an additive error; this property is there called '(c, b)-quasi-geodesic'. So here a subgraph is c-quasi-geodesic if and only if it is $(c, 0)$-quasi-geodesic. It is easily verifiable that these two notions of quasi-geodesic, with or without an additive error, are equivalent for graphs.

vertex in $V(X)$ or on any other new path. The original vertices of X are the *branch vertices* of the subdivision and the new paths are its *branch paths*. Let G be a graph and let $H \subseteq G$ be a subdivision of X with branch vertices v_x for $x \in V(X)$ and branch paths E_e for $e \in E(X)$. Then H is *K-fat* (in G) if there are sets $V_x \subseteq V(H)$ with $v_x \in V_x$ for $x \in V(X)$ and paths $E'_e \subseteq E_e$ for $e \in E(X)$ such that $((V_x)_{x \in V(X)}, (E'_e)_{e \in E(X)})$ is a K-fat model of X. The subdivision H of X *diverges* (in G) if the model $(((\{v_x\})_{x \in V(X)}, (E_e)_{e \in E(X)})$ of X in G diverges.

3. FURTHER DEFINITIONS AND A SKETCH OF THE PROOF

In this section we first introduce ultra fat minors and escaping subdivisions of certain graphs (see Sections 3.1 and 3.2). We then give in Sections 3.3 and 3.4 a sketch of the proofs of Theorems 1 to 4. There, we also state two stronger theorems, Theorems 3.3 and 3.4, from which we then derive Theorems 1 to 4 in Sections 3.5 and 3.6.

3.1. **Ultra fat minors.** We say that a model $((V_i)_{i \in \mathbb{N}}, (E_{ij})_{i \neq j \in \mathbb{N}})$ of $K_{\aleph_0}$ in a graph G is *ultra fat* if

- $d_G(V_i, V_j) \geq \min\{i, j\}$ for all $i \neq j \in \mathbb{N}$,
- $d_G(E_{ij}, E_{k\ell}) \geq \min\{i, j, k, \ell\}$ for all $i, j, k, \ell \in \mathbb{N}$ with $\{i, j\} \neq \{k, \ell\}$, and
- $d_G(V_i, E_{k\ell}) \geq \min\{i, k, \ell\}$ for all $i, k, \ell \in \mathbb{N}$ with $i \notin \{k, \ell\}$.

Further, we say that $K_{\aleph_0}$ is an *ultra fat minor* of G, and write $K_{\aleph_0} \prec_{UF} G$, if G contains an ultra fat model of $K_{\aleph_0}$. The idea is that an ultra fat model of $K_{\aleph_0}$ in a graph G witnesses that G contains $K_{\aleph_0}$ as an asymptotic minor. Indeed, if $((V_i)_{i \in \mathbb{N}}, (E_{ij})_{i \neq j \in \mathbb{N}})$ is an ultra fat model of $K_{\aleph_0}$ in G, then $((V_i)_{i \in \mathbb{N}_{\geq K}}, (E_{ij})_{i \neq j \in \mathbb{N}_{\geq K}})$ is a K-fat model of $K_{\aleph_0}$ in G. In particular, we have the following observation.

Observation 3.1. *If a graph G contains $K_{\aleph_0}$ as an ultra fat minor, then it contains every countable graph as an asymptotic minor. Moreover, if $K_{\aleph_0} \prec^\varepsilon_{UF} G$ for some end ε of G, then also $X \prec^\varepsilon_\infty G$ for every one-ended, countable graph X.* $\square$

Moreover, the following observation is immediate from the definitions.

Observation 3.2. *If a graph G contains $K_{\aleph_0}$ as an ultra fat minor, then it contains every countable graph as a diverging minor, and in particular, it contains every countable graph of maximum degree at most 3 as a diverging subdivision. Moreover, if $K_{\aleph_0} \prec^\varepsilon_{UF} G$ for some end ε of G, then we may choose the diverging minor / subdivision so that all its rays lie in ε.* $\square$

3.2. **Escaping subdivisions.** We call the double rays in a subdivision of the hexagonal half- or full-grid corresponding to the vertical double rays R^i of the hexagonal half- or full-grid its *vertical (double) rays*, and the branch paths corresponding to the horizontal edges e_{ij} its *horizontal paths*, and we usually denote the former by S^i and the latter by P_{ij}. Whenever we introduce a subdivision of the hexagonal half- or full-grid with vertical double rays S^i without specifying the vertex sets of the S^i, we tacitly assume that $S^i = \ldots s^i_{-1} s^i_0 s^i_1 \ldots$ and that their tails $S^i_{\geq 0}$ are the image of the 'upper' half of the vertical double ray R^i of the hexagonal half- or full-grid.

Let G be a graph and let $H \subseteq G$ be a subdivision of the hexagonal half-grid with vertical double rays S^i and horizontal paths P_{ij}. We say that H is *escaping* if there are $0 := M_0 < M_1 < \ldots \in \mathbb{N}$ such that $M_i > M_{i-1} + 2i$ for all $i \geq 1$ and

(i) $S^i \subseteq G[S^0, M_i] - B_G(S^0, M_{i-1} + 2i)$ for all $i \in \mathbb{N}_{\geq 1}$, and

(ii) $P_{1j} \subseteq G[S^0, M_1]$ and $P_{ij} \subseteq G[S^0, M_i] - B_G(S^0, M_{i-2} + i)$ for all $i \in \mathbb{N}_{\geq 2}$ and $j \in \mathbb{Z}$.

A subdivision $H \subseteq G$ of the hexagonal full-grid with vertical double rays S^i and horizontal paths P_{ij} is *escaping* if the S^i and P_{ij} with $i \geq 0$ form an escaping subdivision of the hexagonal half-grid as well as the S^i and P_{ij} with $i \leq 0$, and if there is some $M \in \mathbb{N}$ such that the S^i with $i > 0$ are contained in a different component of $G - B_G(S^0, M)$ than the S^i with $i < 0$.

3.3. **Sketch of the proofs of Theorems 3 and 4.** We will prove Theorems 3 and 4 simultaneously by showing the following stronger result.

Theorem 3.3. *Let ε be a thick end of a graph G with finite maximum degree whose cycle space is generated by cycles of bounded length. Then either $K_{\aleph_0} \prec^\varepsilon_{UF} G$ or G contains an escaping subdivision H of the hexagonal half-grid whose rays all lie in ε.*

By Observations 3.1 and 3.2, an ultra fat model of $K_{\aleph_0}$ contains a diverging and a K-fat model of the half-grid for every $K \in \mathbb{N}$. So to derive Theorems 3 and 4 from Theorem 3.3 it suffices to show that an escaping subdivision of the hexagonal half-grid also contains a diverging and a K-fat subdivision of the hexagonal half-grid (see Sections 3.5 and 3.6).

For the proof of Theorem 3.3, we first show that G contains for every thick end ε a diverging double ε-ray R (see Theorem 4.1), and we then set $S^0 := R$. Second, we show that G contains double rays $S^1, S^2, \dots$ such that the S^i are contained in increasingly distant 'thickened cylinders' around R of the form $G[R, M_i] - B_G(R, M_{i-1} + 2i)$ for some $M_0 < M_1 < \dots \in \mathbb{N}$, as required by (i) for the vertical double rays of an escaping subdivision of the hexagonal half-grid. Finally, we connect the S^i by infinitely many paths so that infinitely many of them either form the vertical double rays of an escaping subdivision of the hexagonal half-grid or they form the branch sets of an ultra fat model of $K_{\aleph_0}$ (see Lemma 5.4).

Let us describe the second step in more detail. We will choose the S^i recursively, starting from the diverging double ε-ray $R = \dots r_{-1} r_0 r_1 \dots (=: S^0)$. For this, we first show that $C[\partial_G C, \lfloor \frac{\kappa - 2}{2} \rfloor] = C \cap G[R, L + \lfloor \frac{\kappa}{2} \rfloor]$ is connected for every $L \in \mathbb{N}$ and every component C of $G - B_G(R, L)$, where $\kappa \in \mathbb{N}$ is such that the cycle space of G is generated by cycles of length $\leq \kappa$ (see Lemma 5.1). Note that this is the only part in in the proofs of Theorems 1 to 4 where we use the assumption on the cycle space; nevertheless, the assumption is crucial here, and the rest of the proof relies on this lemma.[3]

We then show that for every $L \in \mathbb{N}$ some component C of $G - B_G(R, L)$ is 'long', i.e. it has a neighbour in $B_G(R_{\geq j}, L)$ and in $B_G(R_{\leq -j}, L)$ for all $j \in \mathbb{N}$ (see Lemma 5.3). Combining that C is long and $C[\partial_G C, \lfloor \frac{\kappa - 2}{2} \rfloor]$ is connected then allows us to find a double ray in $C[\partial_G C, \lfloor \frac{\kappa - 2}{2} \rfloor]$, which thus also lies in $G[R, L + \lfloor \frac{\kappa}{2} \rfloor] - B_G(R, L)$. Hence, we may proceed recursively by increasing the radius L of the ball around R by a summand of $\lfloor \frac{\kappa}{2} \rfloor + 2i$ in each step.

3.4. **Sketch of the proof of Theorems 1 and 2.** Similar as before, we prove Theorems 1 and 2 simultaneously, by showing the following stronger result.

Theorem 3.4. *Let ε be a thick end of a locally finite, quasi-transitive graph G whose cycle space is generated by cycles of bounded length. Then either $K_{\aleph_0} \prec^\varepsilon_{UF} G$ or G contains an escaping subdivision of the hexagonal full-grid whose rays all lie in ε.*

---

[3]This is not entirely true. We in fact prove stronger versions of Theorems 1 to 4 (see Section 3.6 below), which find the desired minors in a prescribed end. For this, in the case of Theorems 1 and 2, we need the assumption on the cycle space once more, to ensure that the graph is accessible.

Similar as above, Observations 3.1 and 3.2 together with results from Section 3.5 below will show that it suffices to prove Theorem 3.4 in order to obtain Theorems 1 and 2 (see Section 3.6).

The proof of Theorem 3.4 builds on Theorem 3.3. From the proof of Theorem 3.3 it follows that we have more control over where the escaping subdivision of the hexagonal half-grid lies (see Theorem 3.3', the detailed version of Theorem 3.3, in Section 5). For this, let ε be a thick end of G, and let R be a diverging double ε-ray. Given a 'thick' component C of $G - B_G(R, K)$ for some $K \in \mathbb{N}$, that is one which includes a long component of $G - B_G(R, L)$ for every $L \geq K$, we in fact obtain an escaping subdivision H of the hexagonal half-grid whose first vertical double ray is R and which is 'mostly' contained in C (unless Theorem 3.3' yields an ultra fat model of $K_{\aleph_0}$, in which case we are immediately done). Now suppose that for some large enough $L \in \mathbb{N}$ there is another thick component D of $G - B_G(R, L)$. Then Theorem 3.3 yields another escaping subdivision H' of the hexagonal half-grid whose first vertical double ray is R and which is 'mostly' contained in D (or an ultra fat model of $K_{\aleph_0}$). Gluing H and H' together along their common first vertical double ray R then yields the desired subdivision of the hexagonal full-grid (see Lemma 6.2).

It thus suffices to prove that G contains a diverging double ε-ray R such that, for some large enough $K \in \mathbb{N}$, there are two distinct thick components of $G - B_G(R, K)$. This step is mainly divided into two lemmas (see Lemmas 6.8 and 6.9). We first show that if R' is a double ε-ray which is not only diverging but even quasi-geodesic, then it is enough that for some large enough $K \in \mathbb{N}$ there are distinct components $C \neq D$ of $G - B_G(R', K)$ such that C is thick but D is only 'half-thick' (see Lemma 6.8) because then we can use the quasi-transitivity of G to find another quasi-geodesic double ε-ray R such that $G - B_G(R, K)$ has two distinct thick components (see Lemma 6.8). Here, a component of $G - B_G(R, K)$ is half-thick if it includes for every $L \geq K$ a component of $G - B_G(R, L)$ which is 'half-long', i.e. which has neighbours in $B_G(R_{\geq n}, L)$ *or* in $B_G(R_{\leq -n}, L)$ for all $n \in \mathbb{N}$.

Next, we show that such a double ray R' exists. For this, we first prove that G contains three ε-rays R_1, R_2, R_3 that intersect pairwise in a single common vertex such that $R_1 \cup R_2 \cup R_3$ is quasi-geodesic (see Theorem 4.5). Applying (the detailed version Theorem 3.3' of) Theorem 3.3 to the quasi-geodesic, and hence diverging, double ray $R_1 \cup R_2$ then yields an escaping subdivision H of the hexagonal half-grid whose first vertical double ray is $R_1 \cup R_2$. Now for every $K \in \mathbb{N}$, by the definition of escaping, H will lie 'mostly' in one component C_K of $G - B_G(R, K)$, which then needs to be thick. We then analyse where R_3 lies in relation to H. If, for some large enough $L \in \mathbb{N}$, R_3 has a tail in a component $D_L \neq C_L$ of $G - B_G(R_1 \cup R_2, L)$, then we are done since D_L needs to be half-thick as R_3 diverges from $R_1 \cup R_2$ but lies in the same end as R_1 and R_2.

Otherwise, again since R_3 diverges from $R_1 \cup R_2$, it has a tail in C_K for all $K \in \mathbb{N}$. We then distinguish two cases. First assume that R_3 is far away from H. Then, since R_3 has a tail in each C_K, we can connect R_3 and H by infinitely many paths. These paths together with R_3 then yield infinitely many H-paths that 'jump over' H. We then use these paths together with H to find an ultra fat model of $K_{\aleph_0}$. Otherwise, R_3 lies close to H. Then either R_3 separates H into an 'upper half' containing (a tail of) R_1 and a 'lower half' containing (a tail of) R_2, and then $R_1 \cup R_3$ (or symmetrically $R_2 \cup R_3$) is the desired double ray R', or there are infinitely many H-paths that 'jump over' R_3, which then again yield an ultra fat model of $K_{\aleph_0}$ (see Lemma 6.9).

3.5. **Obtaining fat and diverging minors from escaping subdivisions.** In this section we describe how one can turn an escaping subdivision H of the hexagonal half-grid (full-grid) into a K-fat or diverging minor of the hexagonal half-grid (full-grid).

Lemma 3.5. *Let G be a locally finite graph, and let $H \subseteq G$ be an escaping subdivision of the hexagonal half-grid (full-grid) whose vertical double ray S^0 diverges. Then the following assertions hold for all $K \in \mathbb{N}$:*

(i) *H contains a subdivision of the hexagonal half-grid (full-grid) which is K-fat in G, and*

(ii) *H contains a subdivision of the hexagonal half-grid (full-grid) which diverges in G.*

In fact, the subdivisions which we obtain from Lemma 3.5 (i) and (ii) will have the property that their sets of vertical double rays are a subset of the vertical double rays of H.

Since we will have to delete some of the branch paths from H in the proof of Lemma 3.5, we need the following auxiliary result.

Proposition 3.6. *Let H be a subdivision of the hexagonal half-grid (full-grid) with vertical double rays S^i and horizontal paths P_{ij}. Let H' be obtained from H by deleting some of the P_{ij}. If H' still contains, for every $i \in \mathbb{N} \setminus \{0\}$ ($i \in \mathbb{Z} \setminus \{0\}$), infinitely many P_{ij} with $j \in \mathbb{N}$ and infinitely many P_{ij} with $j \in \mathbb{Z}_{\leq 0}$, then H' contains a subdivision H'' of the hexagonal half-grid (full-grid) whose vertical double rays are the S^i and whose set of horizontal paths is a subset of the P_{ij}.*

Proof. To obtain the desired graph H'', one may recursively select paths $P_{ij} \subseteq H'$ with sufficiently large $|j|$ to represent the edges f_k in the order indicated in Figure 3.1 (and similarly for the hexagonal full-grid in the order $f_1, f_2, f'_2, f_3, f'_3, f_4, f'_4, f_5, f_6, f_7, f'_7, \ldots$). □
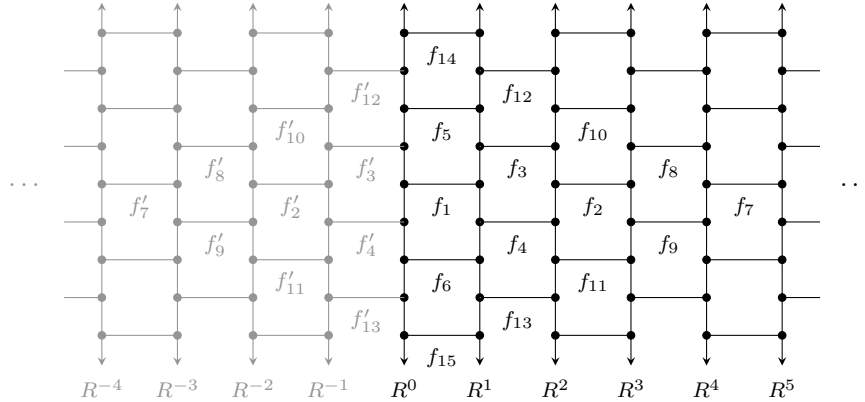

FIGURE 3.1. The hexagonal half-grid (full-grid) with an enumeration of its horizontal edges as needed for the proof of Proposition 3.6.

For the proof of Lemma 3.5 we need the following two auxiliary results, which assert that escaping subdivisions have some additional properties.

Lemma 3.7. *Let $R = \ldots r_{-1} r_0 r_1 \ldots$ be a diverging double ray in a locally finite graph G, let $L \in \mathbb{N}$, and let S be a ray in $G[R, L]$. Suppose there are infinitely many pairwise disjoint $R_{\geq 0}$–S paths P_i in $G[R, L]$. Then S has a tail T such that $T \subseteq G[R_{\geq 0}, L]$, and all but finitely many P_i are contained in $G[R_{\geq 0}, L]$.*

Proof. Since R diverges, there is some $n \in \mathbb{N}$ such that $R_{\geq n}$ and $R_{\leq -n}$ have distance at least $2L + 2$ from each other; in particular, $G[R_{\geq n}, L]$, and $G[R_{\leq -n}, L]$ are disjoint and not joined by an edge. Hence, they are separated in $G[R, L]$ by $B_G(r_{-n+1} R r_{n-1}, L)$. Since $B_G(r_{-n+1} R r_{n-1}, L)$ is finite as G is locally finite, it follows that S is eventually contained in either $G[R_{\geq n}, L]$ or $G[R_{\leq -n}, L]$ and that at most finitely many P_i meet $B_G(r_{-n+1} R r_{n-1}, L)$. As the P_i are disjoint and start in $R_{\geq 0}$, and hence infinitely many P_i start in $R_{\geq n}$, it follows that S has a tail T such that $T \subseteq G[R_{\geq n}, L] \subseteq G[R_{\geq 0}, L]$ and that all but finitely many P_i lie in $G[R_{\geq 0}, L]$. $\qquad\square$

Corollary 3.8. *Let G be a locally finite graph and let H be an escaping subdivision of the hexagonal half-grid with vertical double rays S^i and horizonal paths P_{ij} such that S^0 diverges. Then*

    (i) *for all $i, k \in \mathbb{N}$ there is $\ell \in \mathbb{N}$ such that $S^i_{\geq \ell} \subseteq G[S^0_{\geq k}, M_i]$ and $S^i_{\leq -\ell} \subseteq G[S^0_{\leq -k}, M_i]$, and*

    (ii) *for all $i, k \in \mathbb{N}$ there is $\ell \in \mathbb{N}$ such that $P_{ij} \subseteq G[S^0_{\geq k}, M_i]$ and $P_{i(-j)} \subseteq G[S^0_{\leq -k}, M_i]$ for all $j \geq \ell$.*

Proof. Let $i, k \in \mathbb{N}$ be given. Set $R := S^0$, where we enumerate $R = \ldots r_{-1} r_0 r_1 \ldots$ so that $r_0 = s^0_k$. Applying Lemma 3.7 to R, $L := M_i$ and $S := S^i_{\geq 0}$ and the paths $P_j := P_{ij}$ for $j \geq 0$ yields some $m \in \mathbb{N}$ such that $S^i_{\geq m}$ and all P_{ij} with $j \geq m$ are contained in $G[R_{\geq 0}, M_i] = G[S^0_{\geq k}, M_i]$. Similarly, we find some $n \in \mathbb{N}$ such that $S^i_{\leq -n}$ and all P_{ij} with $j \leq -n$ are contained in $G[S^0_{\leq -k}, M_i]$. Then $\ell := \max\{m, n\}$ is as desired. $\qquad\square$

The next lemma finds a diverging subdivision in an escaping subdivision of the hexagonal half- or full-grid.

Lemma 3.9. *Let H be an escaping subdivision of the hexagonal half-grid (full-grid) in a locally finite graph G with vertical double rays S^i and horizontal paths P_{ij} such that S^0 diverges. Then there exists an escaping subdivision $H' \subseteq H$ of the hexagonal half-grid (full-grid) whose vertical double rays are the S^i and whose horizontal paths P'_{ij} are a subcollection of the P_{ij} such that H' diverges and such that for every two non-incident edges of the hexagonal half-grid (full-grid) their images in H' are at least K apart in G if they are contained in $H'_K := \bigcup_{i \geq K} S^i \cup \bigcup_{i > K, j \in \mathbb{Z}} P'_{ij}$.*

Proof. We only give the proof for the hexagonal full-grid; the construction for the hexagonal half-grid is analogous. For the sake of this proof, we denote the horizontal edges e_{ij} of the hexagonal full-grid by $f_{i(2j)}$ if $i \in 2\mathbb{Z} + 1$, and by $f_{i(2j-1)}$ if $i \in 2\mathbb{Z}$, and we enumerate the P_{ij} accordingly. Let x^{i-1}_j, x^i_j denote the endvertices of P_{ij} on S^{i-1} and S^i, respectively. We will recursively select the branch paths Q_{ij} of the edges f_{ij} amongst the $P_{k\ell}$ such that

    (1) $d_G(Q_{ij}, Q_{k\ell}) \geq \max\{|i|, |j|, |k|, |\ell|\}$ for all $i, j, k, \ell \in \mathbb{Z}$ such that $\{i, j\} \neq \{k, \ell\}$,

    (2) $d_G(Q_{ij}, S^k y^k_{j-1} \cup y^k_{j+1} S^k) \geq \max\{|i|, |j|, |k|\}$ for all $i, j, k \in \mathbb{Z}$, and

    (3) $d_G(S^i y^i_{j-1}, y^k_j S^k) \geq \max\{|i|, |k|, |j|\}$ for all $i, j, k \in \mathbb{Z}$,

where y^{i-1}_j and y^i_j denote the endvertices of Q_{ij} on S^{i-1} and S^i, respectively. Let H' denote the graph obtained from the union of the S^i and the Q_{ij}. Clearly, H' is still an escaping subdivision of the hexagonal full-grid (whose horizontal paths P'_{ij} are essentially the Q_{ij}, except that they are again enumerated as usual). It follows from (1)–(3) and (i) of escaping subdivisions that H'_K has the desired property. Moreover, H' diverges. Indeed, the distances in G between the images U_n, W_n in H' of vertices and/or edges a_n, b_n of the hexagonal full-grid which form diverging sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ in H grow by (1)–(3) unless the U_n, W_n are of the form $Q_{ij}, y^k_{j-1} S^k y^k_j$ or $Q_{ij}, y^k_j S^k y^k_{j+1}$ or $y^i_{j-1} S^i y^i_j, y^k_{j-1} S^k y^k_j$. But their distances grow because of (i) and (ii) of escaping subdivisions; we omit the details.

We now describe how we choose the paths Q_{ij}. First, we set $Q_{i0} := P_{i0}$ for all $i \in 2\mathbb{N} + 1$ and $Q_{i0} := P_{(i-1)0}$ for $i \in 2\mathbb{Z}_{<0} + 1$. Now let $n \in \mathbb{N}$ be given, and assume that we have already chosen paths Q_{ij} for all $|j| < n$. Assume further, without loss of generality, that n is even. We now describe how we choose the paths Q_{in}, where $i \in 2\mathbb{Z} + 1$ since n is even. The choice of the paths $Q_{i(-n)}$ can be done analogously after the choice of the Q_{in}.
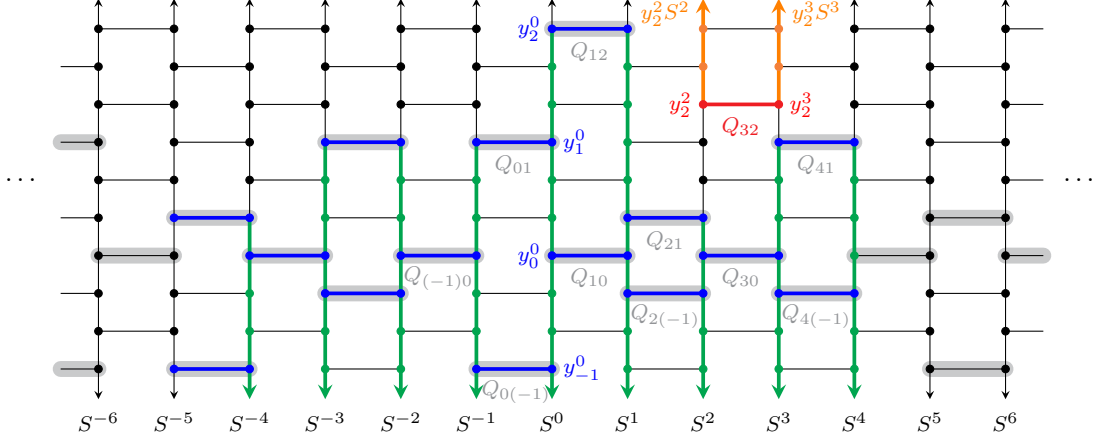

FIGURE 3.2. Depicted in blue and green are the subgraphs X_{32} and Y_{32} that are used to choose Q_{32}. The paths Q_{ij} that are chosen before Q_{32} are shown in grey.

So let $i \in 2\mathbb{N} + 1$ be given, and assume that we have already chosen paths Q_{kn} for all $k \in 2\mathbb{N} + 1$ with $k < i$. We will now select a path P_{ij} to be Q_{in}; again, the choice of the $Q_{(-i)n}$ can be done analogously after the choice of the Q_{in}.

By (i) and (ii), we have $d_G(P_{ij}, P_{k\ell}), d_G(S^i, P_{k\ell}) \geq \max\{i, |k|\}$ for all $k \notin \{i-1, i, i+1\}$ and $j, \ell \in \mathbb{Z}$. Hence, no matter which P_{ij} we choose to be Q_{in}, we will have that $d_G(Q_{in}, Q_{k\ell}) \geq \max\{i, n, |k|, |\ell|\}$ and $d_G(y_n^{i-1}S^{i-1}, Q_{k\ell}), d_G(y_n^i S^i, Q_{k\ell}) \geq \max\{i, n, |k|\}$ for all k with $|k| \geq k_{in} := \max\{i+2, n\}$ and $|\ell| \leq n$. So when choosing Q_{in}, we only need to consider those finitely many $Q_{k\ell}$ with $|k| < k_{in}$ and $|\ell| \leq n$. Since $X_{in} := \bigcup_{|k|<k_{in}, |\ell|<n} Q_{k\ell} \cup \bigcup_{0<k<i} Q_{kn}$ is finite and G is locally finite, all but finitely many P_{ij} have the property that $x_j^{i-1}S^{i-1} \cup P_{ij} \cup x_j^i S^i$ has distance at least $\max\{i, n\}$ from X_{in} (see Figure 3.2).

Moreover, by (i) and (ii), we have $d_G(P_{ij}, S^k), d_G(S^i, S^k) \geq \max\{i, |k|\}$ for all $k \neq i$. Hence, no matter which P_{ij} we choose to be Q_{in}, we will have that $d_G(Q_{in}, S^k), d_G(y_n^i S^i, S^k), d_G(y_n^{i-1}S^{i-1}, S^k) \geq \max\{i, n, |k|\}$ for all k with $|k| \geq \max\{i+1, n\}$ and $|\ell| \leq n$. So when choosing Q_{in}, we only need to consider those finitely many S^k with $|k| < \max\{i+1, n\}$. By Corollary 3.8 (i) and (ii) and because S^0 diverges, all but finitely many of the P_{ij} have the property that $x_n^{i-1}S^{i-1} \cup P_{ij} \cup x_j^i S^i$ has distance at least $\max\{i, n\}$ from $Y_{in} := \bigcup_{|k|<k_{in}} S^k y_{(n-1)}^k$. Hence, we can pick a path P_{ij} whose endvertices on S^i and S^{i-1} appear on S^{i-1} and S^i after the endvertices of $Q_{(i-1)(n-1)}$ and $Q_{(i+1)(n-1)}$, respectively, such that $x_j^{i-1}S^{i-1} \cup P_{ij} \cup x_j^i S^i$ has distance at least $\max\{i, n\}$ from $X_{in} \cup Y_{in}$, and we set $Q_{in} := P_{ij}$. Clearly, Q_{in} is as desired. □

We are now ready to prove Lemma 3.5.

Proof of Lemma 3.5. (ii): By Lemma 3.9 every escaping subdivision H of the hexagonal half- or full-grid whose vertical double ray S^0 diverges contains a diverging subdivision H' of the hexagonal half- or full-grid, respectively, as a subgraph.

(i): Assume first that H is an escaping subdivision of the hexagonal half-grid, and let $H' \subseteq H$ be obtained from H by applying Lemma 3.9. Further, let $\tilde{H}$ be obtained from H' by deleting all horizontal paths that are not of the form P'_{ij} for $i \in 2\mathbb{Z}$ and $j \in 3\mathbb{Z}$ or $i \in 2\mathbb{Z}+1$ and $j \in 3\mathbb{Z}+1$. Then also the subgraph $\tilde{H}_K$ of $\tilde{H}$ consisting of all S^i with $i \geq K$ and all $P'_{ij} \subseteq \tilde{H}$ with $i > K$ is a subdivision of the hexagonal half-grid, and we claim that it is K-fat. Indeed, to turn $\tilde{H}_K$ into a K-fat model of the hexagonal half-grid we may choose the following sets V_x as branch sets, for every $x \in V(\tilde{H}_K)$ of degree 3. Let $x \in V(S^i)$ and let E_e and E_f be the branch paths of $H'_K := \bigcup_{i \geq K} S^i \cup \bigcup_{i > K, j \in \mathbb{Z}} P'_{ij} \subseteq H'$ starting at x that are contained in S^i. We then choose as V_x the union over E_e and E_f. By construction and since the images in H'_K of any two non-incident edges of the hexagonal half-grid have distance at least K in G, it follows that the model is K-fat.

Second, assume that H is an escaping subdivision of the hexagonal full-grid, let $H' \subseteq H$ be obtained from H by applying Lemma 3.9, and let $\tilde{H} \subseteq H'$ be defined as above. Then the graph $\tilde{H}_{K+1}$ defined as above is K-fat for every $K \in \mathbb{N}$ by the argument above. Similarly, it follows by (the symmetry of) the construction of H' in the proof of Lemma 3.9 that also the subgraph $\tilde{H}_{-K-1}$ consisting of all S^i with $i \leq -K-1$ and all $P'_{ij} \subseteq \tilde{H}$ with $i \leq -K-1$ is K-fat for all $K \in \mathbb{N}$. Since G is locally finite, we then find infinitely many S^{-K-1}–S^{K+1} paths W_j in $\tilde{H}$ which are pairwise at least K apart. By the assumptions on $\tilde{H}$ and since $\tilde{H}$ is escaping, gluing the W_i with $\tilde{H}_{K+1}$ and $\tilde{H}_{-K-1}$ together yields (after possibly applying Proposition 3.6) a subdivision H'' of the hexagonal full-grid. By construction, H'' is K-fat. $\square$

3.6. **Proof of the main results given Theorems 3.3 and 3.4.** In this section we derive Theorems 1 to 4 from Theorems 3.3 and 3.4; in fact, we show the following more detailed versions.

Theorem 1'. *Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. Then $FG \prec_\infty^\varepsilon G$ for every thick end ε of G.*

Proof of Theorem 1 and Theorem 1' given Theorem 3.4. By Observation 3.1 and Lemma 3.5 (i), Theorem 3.4 yields Theorem 1' and hence also Theorem 1, where we note that the subdivision obtained from Lemma 3.5 (i) has all its rays in the same end as the full-grid obtained from Theorem 3.4. $\square$

Theorem 2'. *Let ε be a thick end of a locally finite, quasi-transitive graph G whose cycle space is generated by cycles of bounded length. Then G contains a diverging subdivision of the hexagonal full-grid whose rays all lie in ε.*

Proof of Theorem 2 and Theorem 2' given Theorem 3.4. By Observation 3.2 and Lemma 3.5 (ii), Theorem 3.4 yields Theorem 2', where we note that the subdivision obtained from Lemma 3.5 (ii) has all its rays in the same end as the full-grid obtained from Theorem 3.4. For Theorem 2, note that every diverging subdivision of the hexagonal full-grid can be contracted into a diverging minor of the full-grid. $\square$

Theorem 3'. *Let G be graph of finite maximum degree whose cycle space is generated by cycles of bounded length. Then $HG \prec_\infty^\varepsilon G$ for every thick end ε of G.*

Proof of Theorem 3 and Theorem 3' given Theorem 3.3. By Observation 3.1 and Lemma 3.5 (i), Theorem 3.3 yields Theorem 3', and hence also Theorem 3, where we note that the subdivision obtained from Lemma 3.5 (i) has all its rays in the same end as the half-grid obtained from Theorem 3.3. □

Theorem 4'. *Let ε be a thick end of a graph G of finite maximum degree whose cycle space is generated by cycles of bounded length. Then G contains a diverging subdivision of the hexagonal half-grid whose rays all lie in ε.*

Proof of Theorem 4 and Theorem 4' given Theorem 3.3. By Observation 3.2 and Lemma 3.5 (ii), Theorem 3.3 yields Theorem 4', where we note that the subdivision obtained from Lemma 3.5 (ii) has all its rays in the same end as the half-grid obtained from Theorem 3.3. For Theorem 4', note that a diverging subdivision of the hexagonal half-grid can easily be contracted into a diverging minor of the half-grid. □

4. Diverging double rays and quasi-geodesic 3-stars of rays in thick ends

In this section we prove two theorems about double rays and 3-stars of rays in thick ends, which we need for the proofs of Theorems 1 to 4.

4.1. **Diverging double rays.** Georgakopoulos and Papasoglu [21, Theorem 8.16] showed that every connected graph of finite maximum degree which has an infinite set of pairwise disjoint rays has a diverging double ray (whose tails may lie in two distinct ends). For the proofs of Theorems 3 and 4 we will need the following variant of that theorem, which lets us find the diverging double ray in any thick end we like.

Theorem 4.1. *Let G be a graph of finite maximum degree, and let ε be a thick end of G. Then G has a diverging double ε-ray.*

The proof of Theorem 4.1 uses the same idea as the one of [21, Theorem 8.16] by Georgakopoulos and Papasoglu, in that Lemma 4.3 and Lemma 4.4 below are variants of [21, Corollary 8.15 and Lemma 8.17]. However, our proof is more involved, as we need to take care that the tails of the double ray lie in the prescribed end.

Essentially, we will deduce Theorem 4.1 from the following coarse Menger's theorem for two paths, which was proven independently by the first author, Huynh, Jacobs, Knappe and Wollan [3, Theorem 1] and by Georgakopoulos and Papasoglu [21, Theorem 8.1]; the version we state here is from the latter authors. A *metric graph* is a pair (G, ℓ) of a graph G and an assignment of edge-lengths $\ell \colon E(G) \to \mathbb{R}_{>0}$.

Theorem 4.2. *Let G be a metric graph, and let $X, Y \subseteq V(G)$. For every $K > 0$, there is either*

    (i) *a set $B \subseteq V(G)$ of diameter $\leq K$ such that $G - B$ contains no path joining X to Y, or*

    (ii) *two X–Y paths at distance at least $d := K/272$ from each other.*

It follows by a compactness argument that we may replace in Theorem 4.2 the set Y by an end ε and the two paths in (ii) by ε-rays.

Lemma 4.3. *Let (G, ℓ) be a metric graph such that $B_{(G,\ell)}(v, n)$ is finite for all $v \in V(G)$ and $n \in \mathbb{N}$. Let A be a finite set of vertices in G, and let ε be an end of G. For every $K > 0$, there is either*

    (i) *a set $B \subseteq V(G)$ of diameter $\leq K$ such that $G - B$ contains no ε-ray starting at A, or*

    (ii) *two ε-rays starting at A at distance at least $d := K/272$ from each other.*

Proof. In the following, we abbreviate $d_{(G,\ell)}$ and $B_{(G,\ell)}$ with d_G and B_G, respectively.

By the assumption on ℓ and since A is finite, the balls $B_G(A, n)$, for $n \in \mathbb{N}$, are finite. Hence, there exists, for every $n \in \mathbb{N}$ a unique component C_n of $G - B_G(A, n)$ such that every ε-ray has a tail in C_n. Their neighbourhoods $N_G(C_n) \subseteq B_G(A, n)$ are finite; so we may set $k_n := |N_G(C_n)|$ and enumerate $N_G(C_n) =: \{v_1^n, \ldots, v_{k_n}^n\}$.

We apply Theorem 4.2 to the sets $X := A$ and $Y := N_G(C_n)$ in G. If, for some $n \in \mathbb{N}$, Theorem 4.2 yields a set $B \subseteq V(G)$ of diameter $\leq K$ that separates A and $N_G(C_n)$, then by the definition of C_n this B is as in (i). Hence, we may assume that for every $n \in \mathbb{N}$, we find two A–$N_G(C_n)$ paths P_n, Q_n that are at least $d := K/272$ apart in G.

For all $m \leq n \in \mathbb{N}$, we define a k_m-tuple

$$t_n^m = ((t_1^{nm}, \tilde{t}_1^{nm}, s_1^{nm}), \ldots, (t_{k_m}^{nm}, \tilde{t}_{k_m}^{nm}, s_{k_m}^{nm})) \in T_m := (\{0, \ldots, d\}^2 \times \{-4, \ldots, 4\})^{k_m}$$

of triples as follows. We let t_i^{nm} be the distance $d_G(P_n, v_i^m)$ between P_n and v_i^m if it is less than d; otherwise we set $t_i^{nm} := d$. Analogously, we let $\tilde{t}_i^{nm}$ be the distance $d_G(Q_n, v_i^m)$ between Q_n and v_i^m if it is less than d; otherwise we set $\tilde{t}_i^{nm} := d$. Further, if $t_i^{nm} \neq 0$ and $\tilde{t}_i^{nm} \neq 0$, then we set $s_i^{nm} := 0$. Otherwise, it follows that precisely one of P_n and Q_n meets v_i^m, and we then let s_i^{nm} encode whether v_i^m meets P_n ($s_i^{nm} \in \{1, 2, 3, 4\}$) or Q_n ($s_i^{nm} \in \{-4, -3, -2, -1\}$) and whether its predecessor and successor on P_n or Q_n both lie in C_m ($|s_i^{nm}| = 1$), both lie in $G - C_m$ ($|s_i^{nm}| = 2$), or its predecessor lies in C_m and its successor lies in $G - C_m$ ($|s_i^{nm}| = 3$) or vice versa ($|s_i^{nm}| = 4$).

Since all T_m are finite, there exist infinite index sets $\mathbb{N} \supseteq I_0 \supseteq I_1 \supseteq \ldots$ such that, for all $m \in \mathbb{N}$, all t_n^m with $n \in I_m$ are equal. We pick, for every $m \in \mathbb{N}$, some $i_m \in I_m$. Now set

$$\tilde{P}_{i_m} := P_{i_m} \cap (G[C_{m-1}, 1] - C_m) \quad \text{and} \quad \tilde{Q}_{i_m} := Q_{i_m} \cap (G[C_{m-1}, 1] - C_m),$$

and let $P := \bigcup_{m \in \mathbb{N}} \tilde{P}_{i_m}$ and $Q := \bigcup_{m \in \mathbb{N}} \tilde{Q}_{i_m}$. We claim that P and Q are at least d apart in G and that they both contain an ε-ray that starts in A. It then follows that these rays are as in (ii).

First, we show that P and Q are at least d apart in G. For this, recall that $d_G(P_n, Q_n) \geq d$ for all $n \in \mathbb{N}$ by the choice of P_n, Q_n. Now let $m \leq n \in \mathbb{N}$ be given. We show that $d_G(\tilde{P}_{i_m}, \tilde{Q}_{i_n}) \geq d$; the other case is symmetric. Clearly, if $m = n$, then $d_G(\tilde{P}_{i_m}, \tilde{Q}_{i_n}) \geq d$ holds by the choice of P_{i_m}, Q_{i_n}, so we may assume that $m < n$. Set $\ell := d_G(\tilde{P}_{i_m}, \tilde{Q}_{i_n})$, and let $W = w_0 \ldots w_\ell$ be a shortest $\tilde{P}_{i_m}$–$\tilde{Q}_{i_n}$ path. Then W meets $N_G(C_m)$ in a vertex v_j^m because $\tilde{Q}_{i_n} \subseteq G[C_{n-1}, 1] \subseteq G[C_m, 1]$ and $\tilde{P}_{i_m} \subseteq G - C_m$ as $m < n$. It follows that

$$\ell = d_G(w_0, v_j^m) + d_G(v_j^m, w_\ell) \geq d_G(\tilde{P}_{i_m}, v_j^m) + d_G(v_j^m, \tilde{Q}_{i_n}) \geq d_G(P_{i_m}, v_j^m) + d_G(v_j^m, Q_{i_n}) \geq t_j^{i_m m} + \tilde{t}_j^{i_n m}$$

$$= t_j^{i_n m} + \tilde{t}_j^{i_n m} = \min\{d_G(P_{i_n}, v_j^m), d\} + \min\{d_G(v_j^m, Q_{i_n}), d\} \geq \min\{d_G(P_{i_n}, Q_{i_n}), d\} = d.$$

where we used $t_j^{i_m m} = t_j^{i_n m}$ since $i_m, i_n \in I_m$. Hence, $d_G(P, Q) \geq d$ as desired.

So to conclude the proof, it remains to show that P and Q both contain an ε-ray that starts at A. We show the claim for P; the other case is symmetric. By definition, it is clear that P meets A in a unique vertex a, which is the endvertex of P_{i_1} in A; in particular, a has degree 1 in P. Hence, it suffices to show that all other vertices in P have degree 2 in P, as then P contains a ray that starts in a, and which then has to lie in ε since $P - C_m = \bigcup_{n \leq m} \tilde{P}_{i_n}$ is finite for all $m \in \mathbb{N}$. By definition of P, every vertex of P that is not contained in some $N_G(C_m)$ is contained in precisely one $\tilde{P}_{i_m}$, and has thus degree 2 in P. So

let some $v_j^m \in V(P) \cap N_G(C_m)$ be given. Then $s_j^{i_m m} \in \{1, 2, 3, 4\}$, and it follows that v_j^m has degree 2 in G because $s_j^{i_m m} = s_j^{i_{m+1} m}$ by the choice of i_m, i_{m+1}. $\qquad \square$

The remainder of the proof of Theorem 4.1 is now analogous to the one of [21, Theorem 8.16]. More precisely, we have the following auxiliary lemma.

Lemma 4.4. *Let G be a graph of finite maximum degree, and let ε be a thick end of G. Then there is a finite set A of vertices in G and an assignment of edge-lengths $\ell \colon E(G) \to \mathbb{R}_{>0}$ with the following properties:*

    (i) *no ball of radius 1 in the corresponding metric d_ℓ separates A from ε,*

    (ii) $\lim_{e \in E(G)} \ell(e) = 0$, *and*

    (iii) *every ball of finite radius in d_ℓ is finite.*

Proof. The proof is analogous to [21, Lemma 8.17] with just one exception: we choose the sequence $(R_n)_{n \in \mathbb{N}}$ of pairwise disjoint rays so that every R_n is an ε-ray, which is possible because ε is thick.

Note that (iii) follows easily from the proof, since the S^n are 'thickened rings' $B_G(o, r_n) \setminus B_G(o, r_{n-1})$ around a vertex $o \in V(G)$ and because $\sum_{i \in \mathbb{N}} 1/n$ is infinite. $\qquad \square$

Proof of Theorem 4.1. The proof is analogous to [21, Theorem 8.16] with just one exception: instead of [21, Corollary 8.15 & Lemma 8.17] we apply Lemmas 4.3 and 4.4. $\qquad \square$

4.2. **Quasi-geodesic 3-stars of rays.** By Theorem 4.1 every graph G of finite maximum degree contains for every thick end ε a diverging double ε-ray. For the proofs of Theorems 1 and 2, we need the following result, which strengthens Theorem 4.1 in the special case where G is quasi-transitive and accessible.

Theorem 4.5. *Let ε be a thick end of a locally finite, accessible, quasi-transitive graph G. Then there exists $c \in \mathbb{N}_{\geq 1}$ and ε-rays R_1, R_2, R_3 in G such that $R_1 \cap R_2 = R_1 \cap R_3 = R_2 \cap R_3 = \{v\}$ for some $v \in V(G)$ and such that $R_1 \cup R_2 \cup R_3$ is c-quasi-geodesic in G.*

For the proof of Theorem 4.5, we first need the following auxiliary lemma.

Lemma 4.6. *Let G be a locally finite, accessible, quasi-transitive graph that contains a thick end ε. Then there exists a connected, one-ended, quasi-geodesic subgraph H of G such that every ray in H is an ε-ray in G and such that the stabilizer of H acts quasi-transitively on H.*

Proof. By a result of Diestel, Jacobs, Knappe and Kurkofka [10, Lemma 7.12] and in particular its proof [11, Appendix A], there exists a connected, induced, one-ended subgraph H of G whose rays all lie in ε such that every component of $G - H$ has finite neighbourhood in H, such that there are only finitely many orbits of such components under the stabilizer Γ of H in the automorphism group of G and such that Γ acts quasi-transitively on H. It remains to prove that H is quasi-geodesic. Since there are only finitely many orbits of components of $G - H$ under Γ, each such component has finite neighbourhood in H and because H is connected, there exists $c \in \mathbb{N}_{\geq 1}$ such that for every component C of $G - H$ every two vertices in $N_G(C)$ have distance at most c in H.

We claim that H is c-quasi-geodesic. Indeed, let $x, y \in V(H)$ be given, and let P be a shortest x–y path in G. Further, let $Q_0, \ldots, Q_m$ be the maximal non-trivial subpaths of P that are internally disjoint from H. Then every Q_i is internally contained in some component C of $G - H$ and starts and ends in $N_G(C)$. By the choice of c, there exists a path Q_i' in H of length at most c which has the same

endvertices as Q_i. It follows that the union W over $P \cap H$ and the Q'_m is connected, contained in H, contains x, y, and hence contains an x–y path. Since all Q_i are non-trivial and have thus length at least 1, it follows that $d_H(x, y) \leq |E(W)| \leq |E(P \cap H)| + c(m + 1) \leq c \cdot |E(P)| = c \cdot d_G(x, y)$ as desired. $\qquad \square$

We also need the following result of the first author, Diestel, Elm, Fluck, Jacobs, Knappe and Wollan [2].

Lemma 4.7 ([2, Lemma 4.3]). *Let X be a c-quasi-geodesic subgraph of some graph G for some $c \in \mathbb{N}_{\geq 1}$. If P is a shortest v–X path in G for some vertex $v \in V(G)$, then $X \cup P$ is $(2c + 1)$-quasi-geodesic in G.*

We can now prove Theorem 4.5.

Proof of Theorem 4.5. Let us first assume that G is one-ended. In this case, we apply two compactness arguments. First, a standard compactness argument (see e.g. [29, Proposition 5.2]) implies the existence of a geodesic double ray R in the locally finite and quasi-transitive graph G.

To make the second compactness argument, we first show that $G[R, K] \neq G$ for all $K \in \mathbb{N}$. Since G is locally finite, the set $B_G(r_1 R r_{2K+1}, K)$ is finite. As R is geodesic, the sets $B_G(R_{\leq 0}, K)$ and $B_G(R_{\geq 2K+2}, K)$ are disjoint and not joint by an edge. Hence, every $B_G(R_{\leq 0}, K)$–$B_G(R_{\geq 2K+2}, K)$ path meets either $B_G(r_1 R r_{2K+1}, K)$ or $G - G[R, K]$. But since both $R_{\leq 0}$ and $R_{\geq 2K+1}$ lie in the unique end of G, there are infinitely many disjoint such paths, of which at most finitely many can meet the finite set $B_G(r_1 R r_{2K+1}, K)$. Hence, $G - G[R, K]$ is non-empty.

Thus, there exists vertices in G of arbitrary distance from R. Let x_i be a vertex at distance i from R, let r_{j_i} be a vertex of R with $d_G(x_i, r_{j_i}) = d_G(x_i, R)$, and let $P_i = p_0^i \ldots p_i^i$ be a shortest x_i–r_{j_i} path. Then $R \cup P_i$ is 3-quasi-geodesic by Lemma 4.7. Since G is quasi-transitive, there is an infinite index set $I \subseteq \mathbb{N}$ such that all r_{j_i} lie in the same orbit. Let $s \in V(G)$ be another vertex in that orbit. For all $i \in I$, let φ_i be an automorphism of G that maps r_{j_i} to s. Then, since G is locally finite, there exists an infinite index set $I_1 \subseteq I$ such that $\varphi_i(r_{j_i-1} R r_{j_i+1} \cup p_0^i P p_1^i)$ coincides for all $i \in I_1$, amongst which we again find an infinite index set $I_2 \subseteq I_1$ such that $\varphi_i(r_{j_i-2} R r_{j_i+2} \cup p_0^i P p_2^i)$ coincides for all $i \in I_2$ and so on. This results in three internally disjoint, geodesic rays starting in s whose union is 3-quasi-geodesic. Obviously, all three rays must lie in the unique end ε of G, so they are as desired.

Let us now assume that G has more than one end. Since G is accessible, there exists by Lemma 4.6 a connected, one-ended, c-quasi-geodesic, quasi-transitive subgraph H of G for some $c \in \mathbb{N}_{\geq 1}$ such that every ray in H is an ε-ray in G. By the first case, we find the desired three rays R_1, R_2, R_3 in H whose union is 3-quasi-geodesic. Since H is a c-quasi-geodesic subgraph of G, the rays R_1, R_2, R_3 form a $3c$-quasi-geodesic subgraph of G. $\qquad \square$

5. HALF-GRID MINORS

In this section we prove Theorem 3.3; in fact, we show a more detailed version, which we need in the next section for the proof of Theorem 3.4.

Let $R = \ldots r_{-1} r_0 r_1 \ldots$ be a double ray in a graph G, and let $K \in \mathbb{N}$. A component C of $G - B_G(R, K)$ is *long* if C has a neighbour in $B_G(R_{\geq i}, K)$ and in $B_G(R_{\leq -i}, K)$ for all $i \in \mathbb{N}$. Further, C is *thick* if, for every $L \geq K$, some long component of $G - B_G(R, L)$ is contained in C.

Theorem 3.3'. *Let R be a diverging double ray in a thick end ε of a locally finite graph G whose cycle space is generated by cycles of bounded length. Then either $K_{\aleph_0} \prec_{UF}^{\varepsilon} G$ or G contains an escaping subdivision H of the hexagonal half-grid whose first vertical ray is R.*

In particular, if $K_{\aleph_0} \not\preceq_{UF}^{\varepsilon} G$ and C is a thick component of $G - B_G(R, L)$ for some $L \in \mathbb{N}$, then we may choose the vertical double rays S^i of H so that $S^i \subseteq C$ for all $i \geq 1$.

Proof of Theorem 3.3 given Theorem 3.3'. By Theorem 4.1, there exists a diverging double ε-ray R in G. Apply Theorem 3.3 to R. □

In the remainder of this section we prove Theorem 3.3'; see Section 3.3 for a sketch of the proof.

Lemma 5.1. *Let G be a graph whose cycle space is generated by cycles of length at most $\kappa \in \mathbb{N}$, and let Y be a connected subgraph of G. Then for every component C of $G - Y$ that attaches to Y, the graph $C[\partial_G C, \lfloor \frac{\kappa-2}{2} \rfloor]$ is connected.*

Note that if G is connected, then every component of $G - Y$ attaches to Y.

Proof. Clearly, it suffices to show for every two vertices $v_0, v_1 \in \partial_G C$ that there exists a v_0–v_1 path in $C[\partial_G C, \lfloor \frac{\kappa-2}{2} \rfloor]$. So let $v_0, v_1 \in \partial_G C$ be given, and let u_0 and u_1 be vertices of Y which are adjacent to v_0 and v_1, respectively. Since Y is connected, there exists a u_1–u_0 path Q in Y. Let P be a v_0–v_1 path in C. Then $D := v_0 P v_1 u_1 Q u_0 v_0$ is a cycle in G. By assumption on the cycle space of G, we can write D as a finite sum of cycles $D_1, \ldots, D_n$ in G of length at most κ, i.e.

$$D = \sum_{D_i \in \mathcal{D}} D_i$$

where $\mathcal{D} := \{D_1, \ldots, D_n\}$. Let $\mathcal{D}' \subseteq \{D_1, \ldots, D_n\}$ consist of those D_i that do not lie completely in C, i.e. that contain a vertex of $G - C$. Note that $D_i \cap C \subseteq C[\partial_G C, \lfloor \frac{\kappa-2}{2} \rfloor]$ for all $D_i \in \mathcal{D}'$ since D_i has length at most κ and meets $G - C$. Let

$$H := \bigcup_{D_i \in \mathcal{D}'} D_i \cap C \subseteq C \left[\partial_G C, \left\lfloor \frac{\kappa - 2}{2} \right\rfloor \right] \subseteq C$$

be the subgraph of C consisting of all vertices and edges in C that lie on cycles from $\mathcal{D}'$. Note that $v_0, v_1 \in V(H)$ since $v_0 u_0, v_1 u_1 \in E(D)$. We claim that v_0 and v_1 lie in the same component of H, which clearly yields the claim. So suppose for a contradiction that v_0 and v_1 lie in distinct components H_0, H_1 of H. Then the set F of edges in G between H_0 and $G - H_0$ is a cut in G that separates H_0 and H_1; in particular, F is finite since G is locally finite and because $H_0 \subseteq H$ is finite as $\bigcup_{D_i \in \mathcal{D}'} D_i \supseteq H_0$ is a finite union of finite cycles. Obviously, F must contain an edge f from $P \subseteq C$. Then f cannot lie in $\sum_{D_i \in \mathcal{D}'} D_i \subseteq \bigcup_{D_i \in \mathcal{D}'} D_i$, since $f \in E(C)$ but $f \notin E(H)$. Hence, as $f \in E(P) \subseteq E(D)$, it lies in

$$H' := D + \sum_{D_i \in \mathcal{D}'} D_i = \sum_{D_i \in \mathcal{D}} D_i + \sum_{D_i \in \mathcal{D}'} D_i = \sum_{D_i \in \mathcal{D} \setminus \mathcal{D}'} D_i \subseteq C,$$

where for the last inclusion we used that $D_i \subseteq C$ for all $D_i \in \mathcal{D} \setminus \mathcal{D}'$ by the choice of $\mathcal{D}'$. In particular, the same argument also yields that $E(P) \cap F \subseteq E(H')$.

As H' is a finite sum of cycles in G, it is an element of the cycle space of G. Thus, H' meets the finite cut F in an even number of edges. As P is a finite path from $v_0 \in V(H_0)$ to $v_1 \in V(H_1) \subseteq V(G - H_0)$, it meets the finite cut F in an odd number of edges. Combining these two facts with $E(P) \cap F \subseteq E(H')$ yields that H' contains an edge $f' \neq f$ from F which does not lie on P. Since $H' \subseteq C$, the edge f' must lie in C. But since f' is not an edge of $P = D \cap C$, it is not an edge of D either. Hence, f' is an edge of $\sum_{D_i \in \mathcal{D}'} D_i$, and thus an edge of $\bigcup_{D_i \in \mathcal{D}'} D_i$. Since f' is also an edge of C, it lies in H, which is a contradiction to the choice of F. □

Lemma 5.2. *Let $R = \ldots r_{-1} r_0 r_1 \ldots$ be a diverging double ray in an end ε of a locally finite graph G. Then for every $K, n \in \mathbb{N}$ some component of $G - B_G(R, K)$ attaches to $B_G(R_{\leq -n}, K)$ and $B_G(R_{\geq n}, K)$.*

Proof. Since R diverges, there exists some $m \in \mathbb{N}$ such that $R_{\leq -m}$ and $R_{\geq m}$ are at least $2K + 2$ apart in G. Set $N := \max\{n, m\}$. As G is locally finite, the set $B_G(r_{-N} R r_N, K)$ is finite. Hence, as $R_{\leq -N}$ and $R_{\geq N}$ are both ε-rays and thus equivalent, there exists an $R_{\leq -N}$–$R_{\geq N}$ path $P = p_0 \ldots p_\ell$ in G that avoids $B_G(r_{-N} R r_N, K)$. Since $p_0 \in V(R_{\leq -N})$ and $p_\ell \in V(R_{\geq N})$, there is a first vertex p_i of P that is contained in $B_G(R_{\geq N}, K)$, and a last vertex p_j with $j \leq i$ that is still contained in $B_G(R_{\leq -N}, K)$.

We claim that $i \geq j + 2$, which then implies that $P' := p_{j+1} P p_{i-1}$ is non-empty. As P avoids $B_G(r_{-N} R r_N, K)$ and by the choice of p_i and p_j, it then follows that P' is contained in a component of $G - B_G(R, K)$, which then attaches to $B_G(R_{\leq -N}, K)$ and $B_G(R_{\geq N}, K)$ via $p_j p_{j+1}$ and $p_{i-1} p_i$, respectively, and which is thus as desired.

So suppose for a contradiction that $i - j \leq 1$. Then $d_G(p_j, p_i) \leq i - j \leq 1$, and thus

$$d_G(R_{\leq -N}, R_{\geq N}) \leq d_G(R_{\leq -N}, p_j) + d_G(p_j, p_i) + d_G(p_i, R_{\geq N}) \leq K + 1 + K = 2K + 1,$$

which is a contradiction since $d_G(R_{\leq -N}, R_{\geq N}) \geq 2K + 2$ by the choice of N. $\qquad\square$

Lemma 5.3. *Let R be a diverging double ray in a thick end of a locally finite graph G whose cycle space is generated by cycles of bounded length. Then for every $K \in \mathbb{N}$ some component of $G - B_G(R, K)$ is long.*

Proof. Let $\kappa \in \mathbb{N}$ such that the cycle space of G is generated by cycles of length at most κ. Suppose for a contradiction that no component of $G - B_G(R, K)$ is long. Since R diverges, there exists some $N \in \mathbb{N}$ such that $d_G(R_{\leq -N}, R_{\geq N}) \geq 2K + \kappa + 2$. As G is locally finite, $B_G(r_{-N} R r_N, K)$ is finite. Hence, the set $\mathcal{C}$ of components of $G - B_G(R, K)$ which attach to $B_G(r_{-N} R r_N, K)$ is finite. Since no $C \in \mathcal{C}$ is long by assumption and because $\mathcal{C}$ is finite, there exists some $m \in \mathbb{N}$ such that $N_G(C) \subseteq B_G(R_{\geq -m}, K)$ or $N_G(C) \subseteq B_G(R_{\leq m}, K)$ for all $C \in \mathcal{C}$.

By Lemma 5.2, some component C of $G - B_G(R, K)$ attaches to $B_G(R_{\leq -m-1}, K)$ and $B_G(R_{\geq m+1}, K)$; in particular, $C \notin \mathcal{C}$ by the choice of m. Let $U^-, U^+ \subseteq \partial_G C$ be the set of vertices in C that send an edge to $B_G(R_{\leq -N}, K)$ or to $B_G(R_{\geq N}, K)$, respectively. Then $U^- \cup U^+ = \partial_G C$ because $C \notin \mathcal{C}$. Since $C[\partial_G C, \lfloor \frac{\kappa-2}{2} \rfloor]$ is connected by Lemma 5.1, this implies that $B_C(U^-, \lfloor \frac{\kappa-2}{2} \rfloor)$ and $B_C(U^+, \lfloor \frac{\kappa-2}{2} \rfloor)$ either intersect non-emptily or there is an edge between them. Hence, there are vertices $u^- \in U^-$ and $u^+ \in U^+$ of distance at most $\lfloor \frac{\kappa-2}{2} \rfloor + 1 + \lfloor \frac{\kappa-2}{2} \rfloor$ from each other. Thus,

$$d_G(R_{-N}, R_N) \leq d_G(R_{-N}, u^-) + d_G(u^-, u^+) + d_G(u^+, R_N)$$

$$\leq (K+1) + \left(\left\lfloor \frac{\kappa-2}{2} \right\rfloor + 1 + \left\lfloor \frac{\kappa-2}{2} \right\rfloor \right) + (K+1)$$

$$\leq 2K + \kappa + 1$$

which is a contradiction since $d_G(R_{-N}, R_N) \geq 2K + \kappa + 2$ by the choice of N. $\qquad\square$

Lemma 5.4. *Let ε be an end of a locally finite graph G. Suppose there are $M_0 < M_1 < \ldots \in \mathbb{N}$ and double ε-rays $S^0, S^1, \ldots$ such that S^0 diverges, such that $S^i \subseteq G[S^0, M_i] - B_G(S^0, M_{i-1})$ for all $i \in \mathbb{N}_{\geq 1}$ and such that there are infinitely many disjoint $S^0_{\geq 0}$–$S^i_{\geq 0}$ paths and infinitely many disjoint $S^0_{\leq 0}$–$S^i_{\leq 0}$ paths in $G[S^0, M_i]$. Then either $K_{\aleph_0} \prec^\varepsilon_{UF} G$, or there are $0 = i_0 < i_1 < \ldots \in \mathbb{N}$ and an escaping subdivision H of the hexagonal half-grid whose vertical double rays are the S^{i_j}.*

Proof. By passing to a subsequence of the S^i if necessary, we may assume that $M_i > M_{i-1} + 2i$ and that

(a) $S^i \subseteq G[S^0, M_i] - B_G(S^0, M_{i-1} + 2i)$ for all $i \in \mathbb{N}$.

Set $T'_0 := S^0_{\geq 0}$ and $T''_0 := S^0_{\leq 0}$. By assumption and Lemma 3.7, every S^i has disjoint tails T'_i and T''_i that are contained in $G[T'_0, M_i]$ and in $G[T''_0, M_i]$, respectively. For each vertex t in T'_i we choose a shortest t–T'_0 path in G, which then has length $\leq M_i$ and lies in $G[T'_0, M_i]$. Then infinitely many of these paths are T'_i–T'_0 paths (i.e. they only have their first vertex on T'_i), of which infinitely many are pairwise disjoint since they have length $\leq M_i$ and because G is locally finite; let us denote these paths by Q_{ij}.

For every Q_{ij} let $k_{ij} \neq i$ be maximal such that $d_G(Q_{ij}, T'_{k_{ij}}) < k_{ij}$; if no such k_{ij} exists, we set $k_{ij} := 0$. Note that $k_{ij} < i$ since $d_G(Q_{ij}, T'_k) \geq d_G(Q_{ij}, S^k) \geq k$ for all $k > i$ by (a) and because $Q_{ij} \subseteq G[T'_0, M_i]$ for all $i, j \in \mathbb{N}$. We now obtain T'_i–$T'_{k_{ij}}$ paths Q'_{ij} by concatenating a suitable (initial) subpath of Q_{ij} with a shortest Q_{ij}–$T'_{k_{ij}}$ path. In particular, since G is locally finite, we may assume that the Q'_{ij} for every (arbitrary but fixed) $i \in \mathbb{N}_{\geq 1}$ are pairwise disjoint. By the choice of the k_{ij} it follows that

(b) $d_G(Q'_{ij}, T'_k) \geq k$ for all $k, j \in \mathbb{N}$ and $i \in \mathbb{N}_{\geq 1}$ with $k \notin \{i, k_{ij}\}$.

Moreover, since Q_{ij} is a shortest path between its first vertex and T'_0, it follows that once Q_{ij} meets $G[T'_0, M_{k_{ij}-1} + i]$ it will stay in there. By the definition of Q'_{ij} and k_{ij} and by (a), this implies that

(c) $Q'_{ij} \subseteq G[T'_0, M_i] - B_G(T'_0, M_{k_{ij}-1} + i)$ for all $i \in \mathbb{N}_{\geq 1}$ and $j \in \mathbb{N}$.

Let X be the auxiliary graph on the vertex set $\{T'_i \mid i \in \mathbb{N}\}$ where T'_i and $T'_{i'}$ are connected by an edge in X for $i' < i$ if and only if infinitely many of the Q'_{ij} have one endvertex on $T'_{i'}$. Clearly, every T'_i is adjacent to at least one $T'_{i'}$ with $i' < i$, and hence X is connected. Thus, since X is infinite, it either has a vertex of infinite degree or it contains a ray by Lemma 2.1.

Let us first assume that there is some $\ell \in \mathbb{N}$ and an infinite subset $I = \{i_0, i_1, \dots\} \subseteq \mathbb{N}$ with $i_0 < i_1 < \dots$ such that T'_ℓ is adjacent in X to all T'_i with $i \in I$. Then the T'_i for $i \in I$ form the branch sets $V_n := V(T'_{i_n})$ of an ultra fat model of $K_{\aleph_0}$ in G. Indeed, we have $d_G(V_n, V_m) \geq \min\{n, m\}$ by (a), so it remains to find suitable branch paths. Given any enumeration of $\mathbb{N}^2$, we may choose the branch paths P_{nm} between V_n and V_m recursively. Since G is locally finite, and because at step (n, m) we have only chosen finitely many branch paths $P_{n'm'}$, there exist paths $Q'_{i_n j}$, $Q'_{i_m j'}$ that both end in T'_ℓ such that the path P_{nm} consisting of $Q'_{i_n j}, Q'_{i_m j'}$ and a suitable subpath of T'_ℓ is at least $\min\{n, m\}$ apart from all earlier chosen branch paths $P_{n'm'}$. Then by construction and (a) and (b) it follows that also $d_G(P_{nm}, V_k) \geq k$ for all $k \notin \{n, m\}$, and hence the model of $K_{\aleph_0}$ is ultra fat. Moreover, since its branch sets are the vertex sets of the ε-rays T'_{i_n}, we find $K_{\aleph_0} \prec^\varepsilon_{UF} G$.

Hence, we may assume that there are $0 = i_0 < i_1 < \dots \in \mathbb{N}$ such that every T'_{i_n} is adjacent in X to $T'_{i_{n-1}}$. Then there are, for every $n \in \mathbb{N}$, infinitely many $Q'_{i_n j}$ that end in $T'_{i_{n-1}}$. We reindex these $Q'_{i_n j}$ by $\mathbb{N}_{\geq 1} \times \mathbb{N}$, and the S^{i_n} by $\mathbb{N}$.

We now apply the same argument to the tails T''_i of the (reindexed) S^i. This either yields $K_{\aleph_0} \prec^\varepsilon_{UF} G$, or we find indices $0 = i_0 < i_1 < \dots \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, there are pairwise disjoint $T''_{i_{n-1}}$–T''_{i_n} paths $Q''_{i_n j}$ that satisfy (b) and (c) with T''_k instead of T'_k. Now the S^{i_n} form the vertical double rays of an escaping subdivision of the hexagonal half-grid. Indeed, we can choose for every $n \in \mathbb{N}$ infinitely many $T'_{i_{n-1}}$–T'_{i_n} paths $P'_{i_n j}$ in $\bigcup_{i_{n-1} < k \leq i_n}(T'_k \cup \bigcup_{j \in \mathbb{N}} Q'_{kj})$. We also set $P'_{i_n(-j)} := Q''_{i_n j}$ for all $j \in \mathbb{N}$. Then combining (c) of the $Q'_{i_n j}$ and the $Q''_{i_n j}$ with Lemma 3.7 yields that

(c') $P'_{i_n j} \subseteq G[S^0, M_i] - B_G(S^0, M_{i_n-1} + i)$

for every $i \in \mathbb{N}_{\geq 1}$ and all but finitely $j \in \mathbb{N}$. Since G is locally finite, we now obtain a subdivision of the hexagonal half-grid with vertical double rays the S^{i_n} by recursively selecting paths P_{nj} amongst the $P'_{i_n k}$ to represent the horizontal edges e_{nj} (compare Proposition 3.6). $\qquad \square$

We are now ready to prove Theorem 3.3'.

Proof of Theorem 3.3'. Let $\kappa \in \mathbb{N}$ such that the cycle space of G is generated by cycles of length at most κ. Let N_n, for $n \in \mathbb{N}$, be such that $d_G(R_{\geq N_n}, R_{\leq -N_n}) > n$, which exists since R diverges.

By Lemma 5.4, it suffices to show that there are $M_0 < M_1 < \ldots \in \mathbb{N}$ and diverging double ε-rays $R := S^0, S^1, \ldots$ such that $S^i \subseteq G[S^0, M_i] - B_G(S^0, M_{i-1})$ for all $i \in \mathbb{N}_{\geq 1}$ and such that there are infinitely many disjoint $S^0_{\geq 0}$–$S^i_{\geq 0}$ paths and infinitely many disjoint $S^0_{\leq 0}$–$S^i_{\leq 0}$ paths in $G[S^0, M_i]$. We will prove the assertion with $M_0 := 0$ and $M_i := M_0 + \lfloor \kappa/2 \rfloor$ for all $i > 0$.

Let $i > 0$ be given. By Lemma 5.3, there exists a long component C_i of $G - B_G(R, M_{i-1})$. For the 'in particular' part we note that if we are given some $L \in \mathbb{N}$ and a thick component C of $G - B_G(R, L)$, then we may set $M_0 := L$ instead of $M_0 := 0$ and choose as C_i always a long component of $G - B_G(R, M_{i-1})$ which is contained in C.

Set $U^+ := \partial_G C_i \cap B_G(R_{\geq 0}, M_{i-1} + 1)$ and $U^- := \partial_G C_i \cap B_G(R_{\leq 0}, M_{i-1} + 1)$. Since G is locally finite and C_i is long, U^+ and U^- are infinite. As $C_i[\partial_G C_i, \lfloor \frac{\kappa-2}{2} \rfloor]$ is connected by Lemma 5.1, applying the Star-Comb Lemma (cf. Lemma 2.1) in $C_i[\partial_G C_i, \lfloor \frac{\kappa-2}{2} \rfloor]$ to U^+ and U^-, respectively, yields two combs D^+ and D^-. By Lemma 3.7, their spines S^+ and S^- are eventually contained in $G[R_{\geq N_{M_i}}, M_i]$ and $G[R_{\leq -N_{M_i}}, M_i]$, respectively, i.e. they have tails T^+, T^- such that $T^+ \subseteq G[R_{\geq N_{M_i}}, M_i]$ and $T^- \subseteq G[R_{\leq -N_{M_i}}, M_i]$. In particular, T^+, T^- are disjoint by the choice of N_{M_i}, so we can link them by a path in the connected $C_i[\partial_G C_i, \lfloor \frac{\kappa-2}{2} \rfloor]$ to obtain a double ray $S^i \subseteq C_i[\partial_G C_i, \lfloor \frac{\kappa-2}{2} \rfloor]$. Clearly, S^i is as desired. Indeed, the infinitely many $S^0_{\geq 0}$–$S^i_{\geq 0}$ paths can be obtained by extending the paths in D^+ from T^+ to its teeth by shortest paths to S^0, and analogously for T^-. $\qquad \square$

6. FULL-GRID MINORS

In this section we prove Theorem 3.4, which we restate here for convenience.

Theorem 3.4. *Let ε be a thick end of a locally finite, quasi-transitive graph G whose cycle space is generated by cycles of bounded length. Then either $K_{\aleph_0} \prec^\varepsilon_{UF} G$ or G contains an escaping subdivision of the hexagonal full-grid whose rays all lie in ε.*

In fact, we will prove the following variant of Theorem 3.4, which implies Theorem 3.4:

Theorem 6.1. *Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. If G has a thick end, then either $K_{\aleph_0} \prec_{UF} G$ or G contains an escaping subdivision of the hexagonal full-grid.*

We remark that instead of adding Theorem 6.1 as an intermediate step in the proof of Theorem 3.4 we could have also formulated the three lemmas below which we use to construct either an ultra fat model of $K_{\aleph_0}$ or an escaping subdivision of the hexagonal full-grid (Lemmas 6.2, 6.8 and 6.9) so that we may choose a thick end ε, and the lemma then returns the desired structure 'in' ε. However, while this would have been possible in Lemmas 6.2 and 6.9 without changing their proofs, this is not true for Lemma 6.8. There, we would then have to use the fact that G is accessible, to reduce the problem to one-ended graphs.

Proof of Theorem 3.4 given Theorem 6.1. By Theorem 2.2, G is accessible. Hence, by Lemma 4.6, there exists a connected, one-ended, c-quasi-geodesic, quasi-transitive subgraph X of G for some $c \in \mathbb{N}$ such that every ray in X is an ε-ray in G. Applying Theorem 6.1 to X yields either an ultra fat model $((V_i)_{i \in \mathbb{N}}, (E_{ij})_{i \neq j \in \mathbb{N}})$ of $K_{\aleph_0}$ in X or an escaping subdivision H of the hexagonal full-grid in X. Since X is a c-quasi-geodesic subgraph of G, we find in the former case that $((V_i)_{i \in c\mathbb{N}}, (E_{ij})_{i \neq j \in c\mathbb{N}})$ is an ultra fat model of $K_{\aleph_0}$ in G. In fact, since every ray in X is an ε-ray in G, we have $K_{\aleph_0} \prec_{UF}^{\varepsilon} G$. Similarly, in the latter case, H contains a subdivision of the hexagonal full-grid which is escaping in G. Indeed, we may choose vertical double rays $\ldots, S^{i_{-1}}, S^{i_0}, S^{i_1}, \ldots$ of H such that $i_0 = 0$ and $M_{i_{j-1}} + 2i_j \geq c(M_{i_{j-1}} + 2j)$ for $j > 0$, and similarly $M_{i_{j+1}} + 2|i_j| \geq c(M_{i_{j+1}} + 2|j|)$ for $j < 0$. Then adding suitable S^{i_j}–$S^{i_{j+1}}$ paths in H yields a subdivision $H' \subseteq H$ of the hexagonal full-grid which is escaping in G since X is a c-quasi-geodesic subgraph of G. Moreover, all rays in H' are ε-rays. $\qquad\square$

In the remainder of this section we prove Theorem 6.1. The formal proof of Theorem 6.1, which collects the tools from this whole section, can be found at the end of the last subsection, Section 6.2.

We first give a brief overview of this section; a more detailed sketch of the proof of Theorem 6.1 can be found in Section 3.4. Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length and which has a thick end. Further, let $R = \ldots r_{-1}r_0r_1 \ldots$ be a double ray, and let $K \in \mathbb{N}$. Recall that a component C of $G - B_G(R, K)$ is *long* if C has a neighbour in $B_G(R_{\geq i}, K)$ and in $B_G(R_{\leq -i}, K)$ for all $i \in \mathbb{N}$. Further, C is *thick* if, for every $L \geq K$, some long component of $G - B_G(R, L)$ is contained in C.

In Lemma 6.2 below, we show that if G contains a diverging double ray R such that, for some $L \in \mathbb{N}$, $G - B_G(R, L)$ has at least two thick components, then either $K_{\aleph_0} \prec_{UF} G$ or G contains an escaping subdivision of the hexagonal full-grid. Our remaining task then is to prove that G indeed contains such a double ray R. Showing this will be the main effort of this proof, and it will be done in Section 6.2 (see Lemmas 6.8 and 6.9). For this, in Section 6.1, we provide with Lemma 6.5 a sufficient condition for G to contain $K_{\aleph_0}$ as an ultra fat minor, which enables us to find an ultra fat $K_{\aleph_0}$ minor in G if we cannot find such a double ray R.

Lemma 6.2. *Let R be a diverging double ray in a locally finite graph G whose cycle space is generated by cycles of bounded length. Suppose that for some $L \in \mathbb{N}$, there are at least two thick components of $G - B_G(R, L)$. Then either $K_{\aleph_0} \prec_{UF} G$, or G contains an escaping subdivision of the hexagonal full-grid.*

Proof. Let $C \neq D$ be two distinct thick components of $G - B_G(R, L)$. Since we are done if $K_{\aleph_0} \prec_{UF} G$, we may assume that applying Theorem 3.3' to R, K and C or D, respectively, yields escaping subdivisions H_C and H_D, respectively, of the hexagonal half-grid. Let $M_0^C < M_1^C < \ldots$ and $M_0^D < M_1^D < \ldots$ witness that H^C and H^D are escaping. Further, let S_i^C, S_i^D and P_{ij}^C, P_{ij}^D be the vertical double rays and horizontal paths of H_C and H_D, respectively. By the 'in particular' part of Theorem 3.3', it follows that the S_i^C and the S_i^D are contained in C and D, respectively; in particular, they are disjoint. Moreover, by property (ii) of escaping subdivisions, we have that for $M := \max\{M_1^C + 2, M_1^D + 2\}$ the paths P_{ij}^C and P_{ij}^D with $i \geq M$ are contained in C and D, respectively, and are hence disjoint from each other. Let $H_C' \subseteq H_C$ be a subdivision of the hexagonal half-grid with vertical double rays S_0^C and S_i^C for $i \geq M$, which we may obtain by choosing as the new branch paths for the horizontal edges e_{1j} infinitely many disjoint S_0^C–S_M^C paths Q_j^C in H_C'. Let H_D' be chosen analogously. Clearly, H_C' and H_D' are still escaping.

Now since $S_0^C = R = S_0^D$, gluing H_C' and H_D' together along R yields a graph H' which is nearly as desired except that the paths Q_j^C and Q_ℓ^D may intersect. But since G is locally finite, we can delete some of the Q_j^C and Q_ℓ^D and apply Proposition 3.6 to obtain a subdivision $H \subseteq H'$ of the hexagonal full-grid. By construction, H is escaping. $\square$

6.1. **Half-grids with crosses.** In this section we establish a sufficient condition which ensures that a graph G contains $K_{\aleph_0}$ as an ultra fat minor. This condition essentially requires an escaping subdivision $H \subseteq G$ of the hexagonal half-grid, and infinitely many H-paths in G that 'jump over' the vertical double rays in H. For this, we first need the following two auxiliary statements about $K_{\aleph_0}$ minors in half-grids with certain additional edges.

Lemma 6.3. *Let G be obtained from the half-grid by adding all edges of the form $(i,0)(i+1,1)$ and $(i,1)(i+1,0)$ for $i \in \mathbb{N}$. Then G contains a model $(\mathcal{V}, \mathcal{E})$ of $K_{\aleph_0}$ such that $V_i, V(E_{ij}) \subseteq \mathbb{N}_{\geq i} \times \mathbb{Z}$ for all $i < j \in \mathbb{N}$ and such that the E_{ij} are pairwise disjoint.*

Proof. One can easily construct the $K_{\aleph_0}$ minor recursively starting from K_1 with branch set $V_1 := \{(0,0)\}$. For this, assume that we have already defined a model of K_n with branch sets $V_1, \ldots, V_n$, branch paths E_{ij} and integers $j_n, \ell_n \in \mathbb{N}$ such that $V_i, E_{ij} \subseteq \{i, \ldots, \ell_n\} \times \{-j_n, \ldots, 0, \ldots, j_n\}$ and such that either all V_i meet $\{i, \ell_n\} \times \{-j_n\}$ or all V_i meet $\{0, \ell_n\} \times \{j_n\}$ and such that the E_{ij} are pairwise disjoint. We can then extend the branch sets $V_1, \ldots, V_n$ and add a new branch set V_{n+1} as well as new branch paths $E_{(n+1)j}$ as depicted in Figure 6.1 such that $V_1, \ldots, V_{n+1}$ are the branch sets and the E_{ij} are the branch paths of a K_{n+1} minor. $\square$
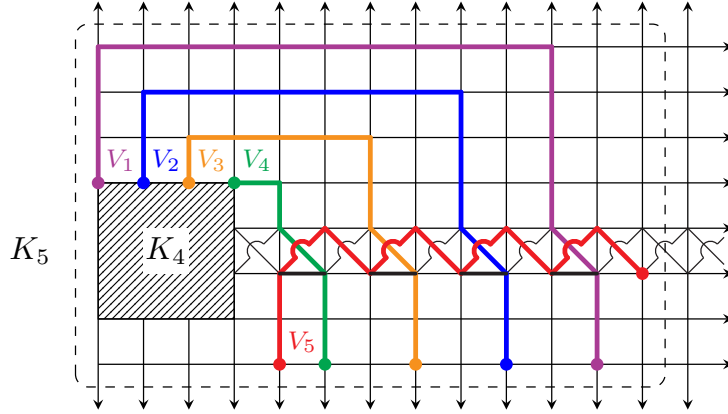

FIGURE 6.1. Sketch of a K_5 minor in a half-grid with all $(i,0)(i+1,1)$ and $(i,1)(i+1,0)$ edges. The branch paths E_{ij} between V_5 and the V_i are the thickened $(i,0)(i+1,0)$ edges.

In particular, we have the following corollary:

Corollary 6.4. *Let X be obtained from the half-grid by adding disjoint edges f_n between (i_n, k_n) and (j_n, ℓ_n), for every $n \in \mathbb{N}$, such that $i_n, j_n < i_{n'}, j_{n'}$ and $|i_n - j_n| \geq 2$ for all $n < n'$. Then X contains a model $(V, \mathcal{E})$ of $K_{\aleph_0}$ such that $V_i, V(E_{ij}) \subseteq \mathbb{N}_{\geq i} \times \mathbb{Z}$ for all $i < j \in \mathbb{N}$ and such that the E_{ij} are pairwise disjoint.*

Proof. It is straight forward to check that the lines $\{i_n\} \times \mathbb{Z}$ in X form the vertical double rays of a subdivision X' of the graph G in the premise of Lemma 6.3 such that every branch path in X' corresponding to an edge of G between $\{n\} \times \mathbb{Z}$ and $\{n+1\} \times \mathbb{Z}$ is contained in $X[\{i_n, \ldots, i_{n+1}\} \times \mathbb{Z}]$. Hence, Lemma 6.3 immediately yields the assertion. $\square$

Before we can state the main lemma of this subsection, we first need the following definition. Let H be an escaping subdivision of the hexagonal half-grid in a graph G with vertical double rays S^i and horizontal paths P_{ij}. An H-path Q in G with endvertices on S^i and S^j for some $i, j \in \mathbb{N}$ is *K-fat* for some $K \in \mathbb{N}$ if

- $d_G(Q, S^k) \geq \min\{k, K\}$ for all $k \neq i, j \in \mathbb{N}$, and
- $d_G(Q, P_{k\ell}) \geq \min\{k, K\}$ for all $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$.

Lemma 6.5. *Let H be an escaping subdivision of the hexagonal half-grid in a locally finite graph G with vertical double rays S^i. Suppose there are infinitely many pairwise disjoint H-paths Q_m with endvertices on S^{i_m} and S^{j_m} for some $i_m, j_m \in \mathbb{N}$ such that Q_m is m-fat and such that $i_m, j_m < i_{m'}, j_{m'}$ and $|i_m - j_m| \geq 2$ for all $m < m'$. Then H contains $K_{\aleph_0}$ as an ultra fat minor.*

Proof. Since G is locally finite and the Q_m are finite, we may assume that $d_G(Q_m, Q_{m'}) \geq m$ for all $m < m' \in \mathbb{N}$, by possibly deleting some of the Q_m. Further, we may assume, for every $Q_m =: q_0^m \ldots q_{n_m}^m$, that $Q_m \cap G[S^{i_m}, i_m'] = q_0^m \ldots q_{i_m'}^m$ and $Q_m \cap G[S^{j_m}, j_m'] = q_{n_m - j_m'}^m \ldots q_{n_m}^m$ where $i_m' := \min\{\lfloor i_m/3 \rfloor, m\}$ and $j_m' := \min\{\lfloor j_m/3 \rfloor, m\}$. In particular, if Q_m has distance less than $\lfloor j_m'/3 \rfloor$ to vertices $u, v \in V(S^{i_m})$, then $d_G(u, v) < i_m'$, and similarly for j_m. Indeed, let Q_m' be a subpath of Q_m which is a $B_G(S^{i_m}, i_m')$–$B_G(S^{j_m}, j_m')$ path. Then we can replace Q_m by a path that consists of Q_m' and a shortest S^{i_m}–Q_m' path and a shortest Q_m'–S^{j_m} path. Note that to regain that $d_G(Q_m, P_{k\ell}) \geq \min\{k, m\}$, we might have to delete some of the $P_{k\ell}$. But since the new Q_m can only be too close to paths $P_{k\ell}$ with $k \in \{i_m, i_m+1, j_m, j_m+1\}$ and because G is locally finite, we only need to delete at most finitely many $P_{k\ell}$ for every k, so by Proposition 3.6 this yields a subdivision $H'' \subseteq H'$ of the hexagonal half-grid with the same vertical double rays as H'.

Then the graph $\tilde{H}$ obtained from the union of H'' and the Q_m is a subdivision of a graph X as in the premise of Corollary 6.4. Hence, G contains a model $(\mathcal{V}, \mathcal{E})$ of $K_{\aleph_0}$ such that $V_i, V(E_{ij}) \subseteq \tilde{H}_i := H''_{\geq i} \cup \bigcup_{m \geq i} Q_m$ and such that the E_{ij} are pairwise disjoint. By Lemma 3.9 and the assumptions on the Q_m, we may assume for every $K \in \mathbb{N}$ that the images of every two non-incident edges of X have distance at least $\lfloor K/3 \rfloor$ in G if their images in $\tilde{H}$ are contained in $\tilde{H}_K$. Since every vertex of $K_{\aleph_0}$ has degree at least 2, we may assume that if some $V_i \in \mathcal{V}$ meets some branch paths of $\tilde{H}$ in an inner vertex, then it in fact contains it. Similarly, every $E_{ij} \in \mathcal{E}$ contains any branch path of $\tilde{H}$ as soon as it meets an inner vertex of it. Hence, since the V_i, E_{ij} are pairwise disjoint (except for incident branch set - path pairs) and contained in $\tilde{H}_i$, they have distance at least $\lfloor i/3 \rfloor$ to all $V_k, E_{k\ell}$ with $k \geq i$. It follows that $((V_i)_{i \in 3\mathbb{N}}, (E_{ij})_{i \neq j \in 3\mathbb{N}})$ is ultra fat. $\square$

6.2. **Finding two thick components.** Let G be a locally finite, quasi-transitive graph whose cycle space is generated by short cycles. In this subsection we show that if G has a thick end, then either $K_{\aleph_0} \prec_{UF} G$, or G contains a quasi-geodesic double ray such that, for some $L \in \mathbb{N}$, there are at least two thick components of $G - B_G(R, L)$. Together with Lemma 6.2, this then concludes the proof of Theorem 6.1. The proof of this assertion is mainly divided into two lemmas, Lemmas 6.8 and 6.9 below; see Section 3.4 for a sketch of the proof.

To carry out the proofs of Lemmas 6.8 and 6.9, we need the following two auxiliary statements.

Lemma 6.6. *Let $R = \ldots r_{-1}r_0r_1 \ldots$ be a quasi-geodesic double ray in a graph G of finite maximum degree whose cycle space is generated by cycles of bounded length. For every $K, N \in \mathbb{N}$ there exist $d = d(K, N), \ell = \ell(K) \in \mathbb{N}$ such that the following holds: If a component C of $G - B_G(R, K)$ has neighbours in $B_G(R_{\leq i}, K)$ and $B_G(R_{\geq i+N}, K)$ for some $i \in \mathbb{Z}$, then there exist vertices $x \in \partial_G C \cap B_G(r_{i-\ell}Rr_i, K+1)$ and $y \in \partial_G C \cap B_G(r_{i+N}Rr_{i+N+\ell}, K+1)$ such that $d_C(x, y) \leq d$.*

Proof. Let $c, \kappa \in \mathbb{N}$ such that R is c-quasi-geodesic and such that the cycle space of G is generated by cycles of length at most κ. By assumption, the maximum degree $\Delta(G)$ of G is finite. We prove the assertion with $d := N \cdot \Delta(G)^{K + \lfloor \frac{\kappa}{2} \rfloor + 1} + \kappa$ and $\ell := c(2K + \kappa + 1)$.

Let $U^+, U^*, U^- \subseteq B_C(\partial_G C, \lfloor \frac{\kappa-2}{2} \rfloor)$ be the set of vertices in C that have distance at most $K + \lfloor \frac{\kappa}{2} \rfloor$ in G from $R_{\geq i+N}, r_i R r_{i+N}$ and $R_{\leq i}$, respectively; and note that by the assumptions on C, the sets U^+ and U^- are non-empty. By Lemma 5.1, $C[\partial_G C, \lfloor \frac{\kappa-2}{2} \rfloor]$ is connected. Hence, since $U^+, U^- \subseteq B_C(\partial_G C, \lfloor \frac{\kappa-2}{2} \rfloor)$, there is a U^+–U^- path $P = p_0 \ldots p_n$ in $C[\partial_G C, \lfloor \frac{\kappa-2}{2} \rfloor]$. Clearly, since $B_C(\partial_G C, \lfloor \frac{\kappa-2}{2} \rfloor) = U^+ \cup U^* \cup U^-$, the choice of P guarantees that $\mathring{P} = p_1 \ldots p_{n-1}$ is contained in U^*, and hence P has length at most $|U^*| + 1 \leq N \cdot \Delta(G)^{K + \lfloor \frac{\kappa}{2} \rfloor + 1}$. Let $j, j' \in \mathbb{Z}$ such that P starts in $V(C) \cap B_G(r_j, K + \lfloor \frac{\kappa}{2} \rfloor)$ and ends in $V(C) \cap B_G(r_{j'}, K + \lfloor \frac{\kappa}{2} \rfloor)$, and let Q be a shortest $B_G(r_j, K+1)$–P path and Q' a shortest $B_G(r_{j'}, K+1)$–P path. Then the concatenation of Q, P and Q' yields a $B_G(r_j, K+1)$–$B_G(r_{j'}, K+1)$ path P' in G which starts in a vertex $x \in \partial_G C \cap B_G(r_j, K+1)$ and ends in some $y \in \partial_G C \cap B_G(r_{j'}, K+1)$. In particular, Q and Q' have length at most $\lfloor \frac{\kappa-2}{2} \rfloor$, and hence P' has length at most d, which implies $d_C(x, y) \leq d$. Moreover, $j \geq i + N$ and $j' \leq i$. It remains to show that $j \leq i + N + \ell$ and $j' \geq i - \ell$. Since $\mathring{P}$ lies in U^*, there exists a p_1–$r_i R r_{i+N}$ path Q'' of length at most $K + \lfloor \frac{\kappa}{2} \rfloor$. Then $Q p_0 p_1 Q''$ is an r_j–r_k path for some $k \leq i + N$. As $Q p_0 p_1 Q''$ has length at most $(K + \lfloor \frac{\kappa}{2} \rfloor) + 1 + (K + \lfloor \frac{\kappa}{2} \rfloor) \leq 2K + \kappa + 1$ and R is c-quasi-geodesic, it follows that $|j - k| = d_R(r_j, r_k) \leq c(2K + \kappa + 1) = \ell$, and hence $j \leq k + \ell \leq i + N + \ell$. The case $j' \geq i - \ell$ is analogous.   □

Corollary 6.7. *Let $L \in \mathbb{N}$, and let R be a quasi-geodesic double ray in some locally finite graph G whose cycle space is generated by cycles of bounded length. Then $G - B_G(R, L)$ has at most finitely many long components.*

In particular, if, for every $K \geq L$, $G - B_G(R, K)$ has a long component C_K, then $G - B_G(R, L)$ has a thick component which contains infinitely many C_K.

Proof. Applying Lemma 6.6 to R and $K := L$ and $N := 0$ yields some $\ell \in \mathbb{N}$ such that every long component of $G - B_G(R, L)$ has a neighbour in $B_G(r_0 R r_\ell, L)$. Hence, since $B_G(r_0 R r_\ell, L)$ is finite as $r_0 R r_\ell$ is finite and G is locally finite, there are at most finitely many long components.

For the 'in particular'-part note that since G is locally finite, every component of $G - B_G(R, L)$ which contains a long component of $G - B_G(R, K)$ for some $K \geq L$ is long. Hence, the assertion follows from the first part by the pigeonhole principle.   □

We can now prove the two main lemmas of this section. Given a double ray $R = \ldots r_{-1}r_0r_1 \ldots$ in a graph G, a component C of $G - B_G(R, L)$ is *half-long* if C has neighbours in $B_G(R_{\geq i}, L)$ or in $B_G(R_{\leq -i}, K)$ for all $i \in \mathbb{N}$, and C is *half-thick* if, for every $M \geq L$, some half-long component of $G - B_G(R, M)$ is contained in C.

Lemma 6.8. *Let $c, L \in \mathbb{N}$ and let R be a c-quasi-geodesic double ray in a locally finite, quasi-transitive graph G whose cycle space is generated by cycles of bounded length. Assume that $G - B_G(R, L)$ has distinct components $C \neq D$ such that C is thick and D is half-thick. Then G contains a c-quasi-geodesic double ray S such that $G - B_G(S, L)$ has two thick components.*

Proof. Let $\kappa \in \mathbb{N}$ such that the cycle space of G is generated by cycles of length at most κ. As we are done if D is thick, we may assume that we can enumerate $R =: \ldots r_{-1}r_0r_1 \ldots$ so that $N_G(D) \subseteq B_G(R_{\geq 0}, L)$. Since G is quasi-transitive, there is an infinite index set $I_0 \subseteq \mathbb{N}$ such that all r_i with $i \in I_0$ lie in the same $\mathrm{Aut}(G)$-orbit. Let v be another vertex in that orbit. Then there exists a sequence $(\varphi_i)_{i \in I_0}$ of automorphisms of G such that $\varphi_i(r_i) = v$. Since G is locally finite, there is an infinite index set $I_1 \subseteq I_0$ such that $\varphi_i(r_{i-1}r_ir_{i+1})$ coincides for all $i \in I_1$ amongst which we again find some infinite set $I_2 \subseteq I_1$ such that $\varphi_i(r_{i-2} \ldots r_{i+2})$ coincides for all $i \in I_2$ and so on. Now pick for every $n \in \mathbb{N}$ some $i_n \in I_n$, and let I consist of these i_n. This leads to a c-quasi-geodesic double ray S that contains v and such that every subpath of S of length $2\ell \in \mathbb{N}$ that contains v as central vertex is the image of $r_{i_n-\ell}Rr_{i_n+\ell}$ under φ_{i_n} for all $i_n \in I$ with $n \geq \ell$. We enumerate $S =: \ldots s_{-1}s_0s_1 \ldots$ where $s_0 := v$ and $s_1 = \varphi_{i_1}(r_{i_1+1})$.

We claim that S is as desired. For this, we show that, for every $K \geq L$, $G - B_G(S, K)$ has long components C'_K, D'_K such that every C'_K–D'_K path in G meets $B_G(S, L)$. Then the assertion follows. Indeed, by Corollary 6.7, $G - B_G(S, L)$ has a thick component E which contains infinitely many of the C'_K, D'_K. If infinitely many of the C'_K, D'_K do not lie in E, then applying Corollary 6.7 again yields a second thick component $E' \neq E$. Otherwise, at most finitely many of the C'_K, D'_K are not contained in E, which implies that E contains both C'_K and D'_K for some $K \geq L$. But since E is connected and avoids $B_G(S, L)$, this contradicts that every C'_K–D'_K path meets $B_G(S, L)$.

So let $K \geq L$ be given. It remains to show that $G - B_G(S, K)$ has long components C'_K, D'_K such that every C'_K–D'_K path in G meets $B_G(S, L)$. Since C is thick and D is half-thick there exist components $C_K \subseteq C$ and $D_K \subseteq D$ of $G - B_G(R, K)$ such that C_K is long and D_K is half-long.

Claim 1. There are long components C'_K, D'_K of $G - B_G(S, K)$ and vertices $x \in \partial_G C'_K$ and $y \in \partial_G D'_K$ such that $\varphi_i^{-1}(x) \in \partial_G C_K$ and $\varphi_i^{-1}(y) \in \partial_G D_K$ for infinitely many $i \in I$.

Proof. Let us first find a component D'_K. Since $N_G(D) \subseteq B_G(R_{\geq 0}, L)$ and $K \geq L$, we also have $N_G(D_K) \subseteq B_G(R_{\geq 0}, K)$. Let $m \in \mathbb{N}$ such that $N_G(D_K) \cap B_G(r_m, K) \neq \emptyset$. For every $n \in \mathbb{N}$, let ℓ, d_n be given by applying Lemma 6.6 to R, K and $N := 2n$ (note that ℓ does not depend on N).

Then there exists, for every $n \in \mathbb{N}$ and $i \geq m + n$, a $B_G(r_{i-n-\ell}Rr_{i-n}, K+1)$–$B_G(r_{i+n}Rr_{i+n+\ell}, K+1)$ path P_{in} in D_K of length at most d_n with endvertices $p_{in} \in \partial_G D_K \cap B_G(r_{i-n-\ell}Rr_{i-n}, K+1)$ and $q_{in} \in \partial_G D_K \cap B_G(r_{i+n}Rr_{i+n+\ell}, K+1)$.

By the choice of I, we have $\varphi_i(r_{i-\ell}Rr_{i+\ell}) = s_{-\ell}Ss_\ell$ for all large enough $i \in I$, and hence these automorphisms φ_i map $B_G(r_{i-\ell}Rr_{i+\ell}, K+1)$ to $B_G(s_{-\ell}Ss_\ell, K+1)$. As G is locally finite, the set $B_G(s_{-\ell}Ss_\ell, K+1)$ is finite. Combining these facts with $p_{i0}, q_{i0} \in B_G(r_{i-\ell}Rr_{i+\ell}, K+1)$ yields that there are $p_0, q_0 \in V(G)$ and an infinite index set $J_0 \subseteq I$ such that $\varphi_i(p_{i0}) = p_0$ and $\varphi_i(q_{i0}) = q_0$ for all $i \in J_0$. Using again that G is locally finite and P_{i0} has length at most d_0 for all $i \in J_0$, we find a p_0–q_0 path $P_0 \subseteq G$ and an infinite index set $J'_0 \subseteq J_0$ such that $\varphi_i(P_{i0}) = P_0$ for all $i \in J'_0$. By the same argument, we find a subsets $J'_0 \supseteq J'_1 \supseteq \ldots$ such that, for all $n \in \mathbb{N}$ and $i \in J'_n$, $\varphi_i(P_{in}) = P_n$ for some path $P_n \subseteq G$ with endvertices $p_n, q_n \in V(G)$. Pick for every $n \in \mathbb{N}$ some $i \in J'_n$ and let J consist of these i.

We claim that there is a component D'_K of $G - B_G(S, K)$ which contains infinitely many of the P_n, and which is hence long. For this, we first show that no P_n meets $B_G(S, K)$. So let $n \in \mathbb{N}$ be given, and set $d'_n := n + \ell + c(d_n + 2K + 2)$. By the choice of P_n, we have $P_n = \varphi_i(P_{in})$ for all large enough $i \in J$. Since also $\varphi_i(B_G(r_{i-d'_n} R r_{i+d'_n}, K)) = B_G(s_{-d'_n} S s_{d'_n}, K)$ for all large enough $i \in J$ and since P_{in} avoids $B_G(R, K)$, it follows that P_n avoids $B_G(s_{-d'_n} S s_{d'_n})$. Moreover, P_n is also disjoint from $B_G(S_{<-d'_n}, K)$ and $B_G(S_{>d'_n}, K)$ since $d'_n = n + \ell + c(d_n + 2K + 2)$ and because S is c-quasi-geodesic and P_n has length at most d_n and starts in $B_G(s_{-n-\ell} S s_{-n}, K+1)$ and ends in $B_G(s_n S s_{n+\ell}, K+1)$.

So every P_n avoids $B_G(S, K)$ and is hence contained in a component of $G - B_G(S, K)$. By the choice of ℓ via Lemma 6.6 and because P_n starts in $B_G(S_{\geq n}, K+1)$ and ends in $B_G(S_{\leq -n}, K+1)$, every component that contains some P_n with $n \geq \ell$ attaches to $B_G(s_0 S s_\ell, K)$. Since this set is finite as G is locally finite, infinitely many P_n lie in the same component D'_K of $G - B_G(S, K)$, which then needs to be long. Now since $\varphi_i^{-1}(P_n) = P_{in} \subseteq D_K$ for all $i \in I'$, where I' is an infinite subset of I, we may choose $y \in \partial_G D'_K$ as one endvertex of some P_n which is contained in D'_K.

The proof for C'_K is now completely analogous except that we have to use I' instead of I in order to ensure that $\varphi_i^{-1}(x) \in \partial_G C_K$ and $\varphi_i^{-1}(y) \in \partial_G D_K$ for (the same) infinitely many $i \in I$. $\diamond$

Let C'_K, D'_K and $x \in \partial_G C'_K$, $y \in \partial_G D'_K$ be given by Claim 1. To finish the proof, we are left to show that every C'_K–D'_K path in G meets $B_G(S, L)$. For this, suppose for a contradiction that there is a C'_K–D'_K path that avoids $B_G(S, L)$. Then, since $C'_K \ni x$ and $D'_K \ni y$ are connected, there also exists an x–y path Q that avoids $B_G(S, L)$. Denote by m the length of Q, let $\ell \in \mathbb{N}$ such that $x, y \in B_G(s_{-\ell} S s_\ell, K+1)$, and set $m' := c(K + m + L + 2) + \ell$. By the choice of S there is some i among the infinitely many $i \in I$ that satisfy $\varphi_i^{-1}(x) \in \partial_G C_K$ and $\varphi_i^{-1}(y) \in \partial_G D_K$ such that $\varphi_i(r_{i-m'} R r_{i+m'}) = s_{-m'} S s_{m'}$. Since Q avoids $B_G(S, L)$, it follows that $\varphi_i^{-1}(Q)$ avoids $B_G(r_{i-m'} R r_{i+m'}, L)$. But $\varphi_i^{-1}(Q)$ also avoids $B_G(R_{<i-m'}, L) \cup B_G(R_{>i+m'}, L)$: Otherwise, since $\varphi_i^{-1}(Q)$ has length m and starts in $B_G(r_{i-\ell} R r_{i+\ell}, K+1)$, there would be a path of length at most $L + m + (K+1)$ that joins a vertex from $R_{<i-m'} \cup R_{>i+m'}$ to a vertex from $r_{i-\ell} R r_{i+\ell}$. Since $m' - \ell = c(L + m + K + 2)$, this would contradict that R is c-quasi-geodesic. Hence, $\varphi_i^{-1}(Q)$ is an $\varphi_i^{-1}(x)$–$\varphi_i^{-1}(y)$ path which avoids $B_G(R, L)$. But since $\varphi_i^{-1}(x) \in \partial_G C_K \subseteq C_L$ and $\varphi_i^{-1}(y) \in \partial_G D_K \subseteq D_L$, this contradicts that C_L and D_L are distinct components of $G - B_G(R, L)$. $\square$

Lemma 6.9. *Let G be a locally finite, quasi-transitive graph with a thick end whose cycle space is generated by cycles of bounded length. If G does not contain $K_{\aleph_0}$ as an ultra fat minor, then there exists $L \in \mathbb{N}$ and a quasi-geodesic double ray R in G such that $G - B_G(R, L)$ has distinct components $C \neq D$ such that C is thick and D is half-thick.*

Proof. Let R_1, R_2, R_3 be given by applying Theorem 4.5 to some thick end of the locally finite, quasi-transitive graph G, which is accessible by Theorem 2.2. Then applying Theorem 3.3 to the quasi-geodesic, and hence diverging, double ray $R_1 \cup R_2$ yields an escaping subdivision H of the hexagonal half-grid with vertical double rays S^i and horizontal paths P_{ij} such that $S^0 = R_1 \cup R_2$. Since H is escaping, there exist $M_0 < M_1 < \ldots \in \mathbb{N}$ such that $M_i > M_{i-1} + 2i$ for all $i \geq 1$ and

(i) $S^i \subseteq G[S^0, M_i] - B_G(S^0, M_{i-1} + 2i)$ for all $i \in \mathbb{N}_{\geq 1}$, and

(ii) $P_{1j} \subseteq G[S^0, M_1]$ and $P_{ij} \subseteq G[S^0, M_i] - B_G(S^0, M_{i-2} + i)$ for all $i \in \mathbb{N}_{\geq 2}$ and $j \in \mathbb{Z}$.

Let us also note that, since $R_1 \cup R_2 \cup R_3$ is quasi-geodesic, we also have that

(iii) the set $V(R_3) \cap B_G(S^0, L)$ is finite for all $L \in \mathbb{N}$.

Hence, by (ii) and (iii) and because G is locally finite, we may assume, by deleting at most finitely many P_{ij} for every $i \in \mathbb{N}$ and applying Proposition 3.6, that

(iv) $d_G(R_3, P_{ij}) > 2i$ for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$.

Let $H_{\geq n} \subseteq H$, for $n \in \mathbb{N}$, be the subgraph consisting of all S^i, P_{kj} with $i \geq n$ and $k > n$. By (i) and (ii), we have, for every $L \in \mathbb{N}_{\geq 1}$, that $d_G(H_{\geq L}, S^0) > L$; let C_L be the component of $G - B_G(S^0, L)$ containing $H_{\geq L}$. Clearly, the C_L are long and hence, since $C_1 \supseteq C_2 \supseteq \ldots$, the C_L are thick.

If there is some $L \in \mathbb{N}_{\geq 1}$ such that $R_3 \cap C_L = \emptyset$, then we are done. Indeed, by (iii), there is, for every $L' \geq L$, a component $D_{L'}$ of $G - B_G(S^0, L')$ that contains a tail of R_3. Since R_3 and R_1 belong to the same end, the components $D_{L'}$ are half-long. As clearly $D_{L'} \subseteq D_L$ for all $L' \geq L$, we find that D_L is half-thick. Since also $D_L \neq C_L$ by assumption, $R := R_1 \cup R_2$, L, $C := C_L$ and $D := D_L$ are as desired.

Thus, we may assume that $R_3 \cap C_L \neq \emptyset$ for all $L \in \mathbb{N}_{\geq 1}$. We distinguish two cases.

Case 1: *For all $K \in \mathbb{N}$ there is some $N_K \in \mathbb{N}$ such that $d_G(R_3, S^i) > K$ for all $i \geq N_K$.* (See Figure 6.2.)

We will use R_3 to find fat H-paths as in Lemma 6.5, which then implies that G contains $K_{\aleph_0}$ as an ultra fat minor, concluding the first case of the proof.

Without loss of generality let $N_K \geq K$ for all $K \in \mathbb{N}$. By (i) and (iii), for every $K \in \mathbb{N}$, the ray R_3 has a tail T_K which avoids $B_G(S^0, M_{N_K-1} + K)$, and which thus satisfies $d_G(T_K, S^i) > K$ for all $i \in \mathbb{N}$ by (i). By (ii) and (iv), we also find $d_G(P_{ij}, T_K) > K$ for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, and hence $d_G(H, T_K) > K$.
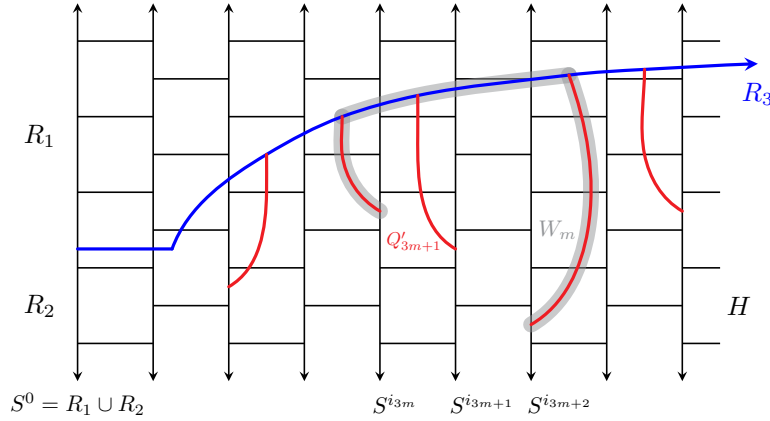

FIGURE 6.2. Sketch of Case 1 in the proof of Lemma 6.9: the hexagonal half-grid H and the ray R_3, which has large distance from H but is connected to H by paths Q'_m. The paths W_m consisting of Q'_{3m}, Q'_{3m+2} and a suitable subpath of R_3 are m-fat H-paths.

By (i), and because $R_3 \cap C_L \neq \emptyset$ as well as $H_{\geq L} \subseteq C_L$ for all $L \in \mathbb{N}_{\geq 1}$, there exists, for every $m \in \mathbb{N}$, an T_m–$\bigcup_{i \geq N_m} B_G(S^i, m)$ path Q_m that ends in some $B_G(S^{i_m}, m)$ for $i_m \geq m$ and avoids $B_G(S^0, M_{N_m} + m)$. We extend each Q_m to an T_m–S^{i_m} path Q'_m by adding a shortest S^{i_m}–Q_m path. Since $d_G(Q_m, S^i) > m$ for all $i \neq i_m$ by the choice of Q_m and again by (i), we find $d_G(Q'_m, S^i) > m$ for all $i \neq i_m$. Moreover, by (i), the paths Q'_m still avoid $B_G(S^0, M_{N_m} + m)$. Hence, by (ii), we have that, for every $i \in \mathbb{N}$, the paths Q'_m with $m \geq i$ have distance at least m from the paths P_{ij}. Since G is locally finite, this implies that we may assume, by deleting for every $i \in \mathbb{N}$ at most finitely many P_{ij} and applying Proposition 3.6, that $d_G(P_{ij}, Q'_m) > m$ for all $m, i \in \mathbb{N}$ and $j \in \mathbb{Z}$. All in all, we find $d_G(H - S^{i_m}, Q'_m) > m$ for all $m \in \mathbb{N}$.

Since Q'_m avoids $B_G(S^0, M_{N_m} + m)$ but is itself eventually contained in some $B_G(S^0, m')$ for $m' > m$ as Q'_m is finite, we may assume, by passing to a subsequence of the Q'_m if necessary, that the Q'_m are pairwise disjoint. Moreover, by (i) and again since Q'_m avoids $B_G(S^0, M_{N_m})$, we may assume that $i_1 < i_2 < \ldots$, by once again passing to a subsequence of the Q'_m if necessary.

Now since $d_G(H - S^{i_m}, Q'_m) > m$ and $d_G(H, T_m) > m$, the paths W_m that consist of Q'_{3m}, Q'_{3m+2} and a suitable subpath of T_{3m} are m-fat H-paths with endvertices on $S^{i_{3m}}$ and $S^{i_{3m+2}}$ (see Figure 6.2). In particular, since the Q'_m are pairwise disjoint, we may assume, by passing to a subsequence of the W_m, that also the W_m are pairwise disjoint. Hence, the W_m are m-fat H-paths as in Lemma 6.5, which implies that G contains $K_{\aleph_0}$ as an ultra fat minor. This concludes the first case of the proof.

Case 2: *There exists some K such that $d_G(R_3, S^i) < K$ for infinitely many $i \in \mathbb{N}$.* (See Figure 6.3.)

We first show that we may assume, by passing to a subgraph of H if necessary, that

(a) $d_G(R_3, S^i) < i$ for all $i \geq K$, and

(b) for all $K \leq j < i$, if $d_G(r, S^i) < i$ for some $r \in V(R_3)$, then $d_G(rR_3, S^j) \geq j$.

Indeed, set $i_j := j$ for all $j < K$, and let $i_K \geq K$ be minimal such that $d_G(R_3, S^{i_K}) < K$. By (i) and (iii), R_3 has a tail T that is disjoint from $B_G(S^0, M_{i_K} + i_K)$. Since $R_3 - T$ is finite and because of (i), we have $d_G(R_3 - T, S^i) > i$ for all large enough $i \in \mathbb{N}$, and hence, as $d_G(R_3, S^i) < K$ for infinitely many $i \in \mathbb{N}$, there is some $i_{K+1} > i_K$ such that $d_G(T, S^{i_{K+1}}) < i_{K+1}$ and $d_G(R_3 - T, S^i) \geq i$ for all $i \geq i_{K+1}$. By continuing in this way, we obtain a sequence $0 := i_0 < i_1 < \cdots \in \mathbb{N}$ such that the rays S^{i_j} satisfy (a) and (b). Now pick a subdivision $H' \subseteq H$ of the hexagonal half-grid whose vertical double rays are precisely the S^{i_j}. It is easy to see that H' still satisfies (i) to (iii). By deleting from H' at most finitely many P'_{ij} for every $i \in \mathbb{N}$ and applying Proposition 3.6, we also regain property (iv) for H'. Hence, H' is the desired subgraph of H, and we denote H' again by H.
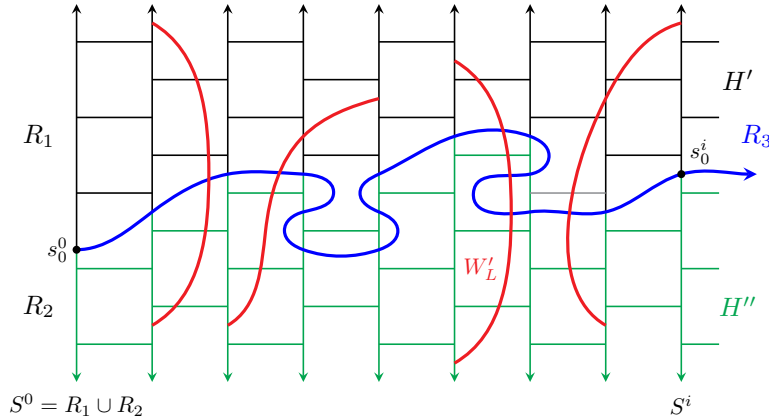

Figure 6.3. Sketch of Case 2 in the proof of Lemma 6.9: the hexagonal half-grid H and the ray R_3, which has distance $< i$ to every vertical double ray S^i of H by (a). By (b), once R_3 comes close to some S^i, it will never come close to some S^j with $j < i$ again. The paths W'_L have distance at least L from R_3 and 'jump over' R_3.

Let $S^i =: \ldots s^i_{-1} s^i_0 s^i_1 \ldots$ such that $d_G(R_3, s^i_0) < i$ and such that $S^i_{\geq 0}$ is the 'upper half' and $S^i_{\leq 0}$ is the 'lower half' of S^i (see Figure 6.3). Further, let H' be the 'upper half' of H with respect to this

enumeration, that is, let H' be the subgraph of H that consists of the $S^i_{\geq 0}$ and all paths P_{ij} whose endvertices lie on the $S^i_{\geq 0}$. Let H'' be the 'lower half' of H defined analogously.

Now choose for all $i, L \in \mathbb{N}$ maximal tails T'_{iL}, T''_{iL} of $S^i_{\geq 0}$ and $S^i_{\leq 0}$, respectively, such that

(c) $d_G(R_3, T'_{iL}) \geq 2L$ and $d_G(R_3, T''_{iL}) \geq 2L$.

Note that by (i) and (iii), the T'_{iL}, T''_{iL} are non-empty. Note further that, by (ii) and (iii), for all $i, L \in \mathbb{N}$ all but finitely many of the P_{ij} avoid $B_G(R_3, L)$. Hence, there are, for every $L \in \mathbb{N}$, escaping subdivisions $H'_L, H''_L \subseteq H_{\geq L}$ of the hexagonal 'quarter-grid' whose vertical rays are the T'_{iL} or T''_{iL}, respectively, for $i \geq L$ such that $H'_1 \supseteq H'_2 \supseteq \ldots$ as well as $H''_1 \supseteq H''_2 \supseteq \ldots$, and such that H'_L, H''_L avoid $B_G(R_1 \cup R_3, L)$.

Since H'_L, H''_L avoid $B_G(R_1 \cup R_3, L)$, they are contained in components C_L, D_L of $G - B_G(R_1 \cup R_3, L)$, respectively. Clearly, C_L is long and D_L is half-long. As $C_1 \supseteq C_2 \supseteq \ldots$ and $D_1 \supseteq D_2 \supseteq \ldots$, this implies that C_L is thick and D_L is half-thick. If there is some $L \in \mathbb{N}$ such that $C_L \neq D_L$, then we are done as then L, $R := R_1 \cup R_3$, $C := C_L$ and $D := D_L$ are as desired.

Hence, we may assume that $C_L = D_L$ for all $L \in \mathbb{N}$. We will now once again construct fat H-paths as in the premise of Lemma 6.5, which then yields that G contains $K_{\aleph_0}$ as an ultra fat minor, and which thus concludes the proof. Since $C_L = D_L$ and $H'_L \subseteq C_L$, $H''_L \subseteq D_L$ for all $L \geq K$, there are $(\bigcup_{i \in \mathbb{N}} B_G(T'_{iL}, \min\{i, L\}))$–$(\bigcup_{i \in \mathbb{N}} B_G(T''_{iL}, \min\{i, L\}))$ paths W_L that avoid $B_G(R_1 \cup R_3, 3L)$. We now modify W_L as follows. Let i'_L be such that W_L starts in $B_G(T'_{i'_L L}, \min\{i'_L, L\})$. If W_L meets $B_G(P_{ij}, \min\{i, L\})$ for some $P_{ij} \subseteq H'_{4L}$ with $i \notin \{i'_L, i'_L + 1\}$, then we let P_{ij} be the last such path, and we shorten W_L so that it meets $B_G(P_{ij}, \min\{i, L\})$ precisely in its first vertex. Then we extend W_L by a shortest W_L–P_{ij} path and a suitable subpath of P_{ij} so that W_L ends in T'_{iL} (or in $T'_{(i-1)L}$ if the shortest W_L–P_{ij} path has distance $< \min\{i, L\}$ from $T'_{(i-1)L}$). Otherwise, we extend W_L by a shortest W_L–$T'_{i'_L L}$ path. Analogously, we modify the end of W_L. Let W'_L be the path which we obtain in this way from W_L and let $i_L, j_L \in \mathbb{N}$ be such that W'_L starts in $T'_{i_L L}$ and ends in $T''_{j_L L}$. Further, let w^0_L, w^1_L be the endvertices of W'_L on $T'_{i_L L}$ and $T''_{j_L L}$, respectively. Then, for all $L \in \mathbb{N}$,

(α) W'_L starts at $w^0_L \in T'_{i_L L}$ and ends at $w^1_L \in T''_{j_L L}$,

(β) W'_L avoids $B_G(R_1 \cup R_3, 2L)$, and

(γ) $d_G(W'_L, T'_{kL} \cup T''_{kL}) \geq \min\{k, L\}$ for all $k \neq i_L, j_L \in \mathbb{N}$.

Since W'_L avoids $B_G(R_1 \cup R_3, 2L)$ but is eventually contained in $B_G(R_1 \cup R_3, L')$ for some $L' > 2L$ because W'_L is finite, we may assume, by passing to a subsequence if necessary, that the W'_L are pairwise disjoint. Moreover, by Corollary 3.8 (i), every $S^i_{\geq 0}$ is contained in $B_G(R_1, L)$ for some large enough $L \in \mathbb{N}$ and hence, by (β) and by once again passing to a subsequence, we may assume that

(δ) $i_1 < i_2 < \ldots$.

Finally, by construction, W'_L can have distance $< \min\{i, L\}$ to some P_{ij} only if $i \in \{i_L, i_L + 1, j_L, j_L + 1\}$. So since G is locally finite and because of (α) and (δ), we may assume, by deleting at most finitely many P_{ij} for every $i \in \mathbb{N}$ and applying Proposition 3.6, that

(ε) $d_G(W'_L, P_{ij}) > \min\{i, L\}$ for all P_{ij} with $i \notin \{j_L, j_L + 1\}$.

We now distinguish two cases.

Case 2a: *There is a sequence $L_1 < L_2 < \ldots \in \mathbb{N}$ such that $j_{L_1} < j_{L_2} < \ldots$.*

Since also $i_1 < i_2 < \ldots$ by (δ), we may assume, by passing to a subsequence of the W'_L, that $i_L, j_L < i_{L'}, j_{L'}$ for all $L < L' \in \mathbb{N}$. Then we obtain, similar as to (ε), that

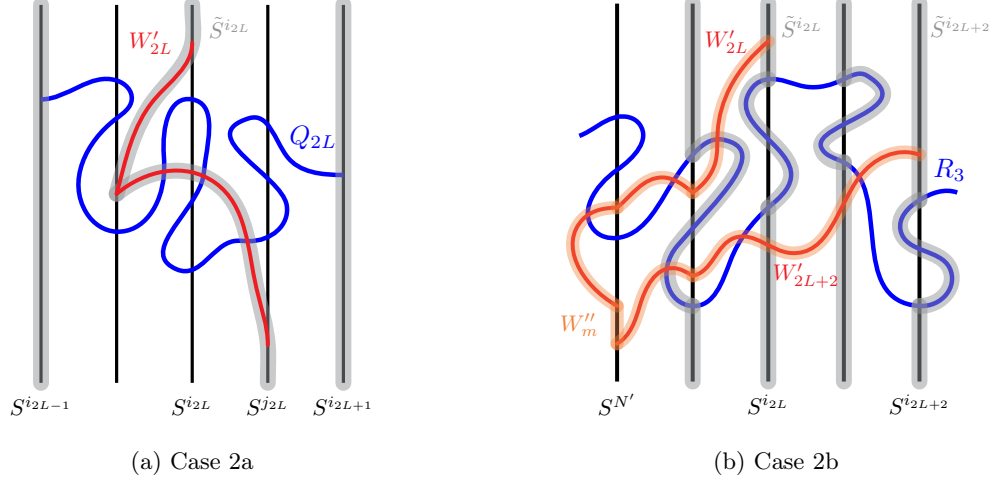(a) Case 2a   (b) Case 2b

FIGURE 6.4. The new double rays $\tilde{S}^{i_L}$ in Case 2a and 2b (indicated in grey), which again form the vertical double rays of an escaping subdivision $\tilde{H}$ of the hexagonal half-grid. In Case 2a, subpaths Q_{2L} of R_3 are fat $\tilde{H}$-paths, while in Case 2b, the paths W_m'' are fat $\tilde{H}$-paths.

(ζ) $d_G(W_L', P_{ij}) > \min\{i, L\}$ also for all P_{ij} with $i \in \{j_L, j_L + 1\}$.

Since every $\tilde{T}_i := S^i \setminus (T_{ii}' \cup T_{ii}'')$ is finite, it follows that $\tilde{T}_i \subseteq G[R_3, L']$ for some $L' \in \mathbb{N}$. So for every $i \in \mathbb{N}$ we have by (β) and (γ) that $d_G(S^i, W_L') \geq i$ for all large enough L. Conversely, since every W_L' is finite and by (i), there exist for every W_L' at most finitely many S^i such that $d_G(W_L', S^i) < i$. Hence, we may assume, by passing to a subsequence of the W_L', that every S^i has distance $< i$ to at most one W_L'. Indeed, we may pick some W_{L_1}' that has distance < 1 to S^1 (if such a W_{L_1}' exists; otherwise we choose W_{L_1}' arbitrarily) and delete all other W_L' that have distance < 1 to S^1. Then there are still infinitely many W_L' left, and also there are infinitely many S^i left that do not have distance $< i$ to W_{L_1}'; so we may pick a new path W_L' for the next such S^i, and so on. We enumerate these W_{L_i}' again by W_i'. In particular, it follows that $d_G(W_L', S^{i_{L'}}) \geq i_{L'}$ and $d_G(W_L', S^{j_{L'}}) \geq j_{L'}$ for all $L \neq L' \in \mathbb{N}$.

For every $2L \in \mathbb{N}$, we set $\tilde{S}^{i_{2L}} := S^{j_{2L}} w_{2L}^1 W_{2L}' w_{2L}^0 S^{i_{2L}}$ (see Figure 6.4 (a)). Since every $\tilde{S}^{i_{2L}}$ has a tail in $S_{\geq 0}^{i_{2L}}$ and in $S_{\leq 0}^{j_{2L}}$ and because of (ε) and (ζ), we can find in H infinitely many $S^{i_{2L}-1}$–$\tilde{S}^{i_{2L}}$ paths and infinitely many $\tilde{S}^{i_{2L}}$–$S^{i_{2L}+1}$ paths that make the $\tilde{S}^{i_{2L}}$ and the $S^{i_{2L}+1}$ into an escaping subdivision $\tilde{H}$ of the hexagonal half-grid. By (a) and (b), R_3 contains for every $2L$ a subpath Q_{2L} that starts in $B_G(S^{i_{2L}-1}, i_{2L}-1)$, ends in $B_G(S^{i_{2L}+1}, i_{2L}+1)$ and is otherwise disjoint from all $B_G(S^{i_{L'}}, i_{L'})$ and $B_G(S^{j_{L'}}, j_{L'})$ with $L' \neq 2L \in \mathbb{N}$. Since $d_G(R_3, W_{2L'}') \geq 2L'$ by (β), it follows that also $d_G(Q_{2L}, \tilde{S}^{i_{2L'}}) \geq 2L'$ for all $L' \neq L \in \mathbb{N}$. Moreover, by the definition of $\tilde{S}^{i_{2L}}$ and because of (β) and (c), we also have $d_G(Q_{2L}, \tilde{S}^{i_{2L}}) \geq 2L$. Hence, extending the Q_{2L} by shortest paths to $S^{i_{2L}-1}$–$S^{i_{2L}+1}$ paths yields fat $\tilde{H}$-paths as in Lemma 6.5. This implies that G contains $K_{\aleph_0}$ as an ultra fat minor, and hence concludes this case of the proof.

Case 2b: *There is some $N \in \mathbb{N}$ such that $j_L \leq N$ for infinitely many $L \in \mathbb{N}$.*

By passing to a subsequence of the W_L', we may assume that $j_L = N'$ for some $N' \in \mathbb{N}$ and all $L \in \mathbb{N}$. We again modify H as follows. For every $i \in \mathbb{N}$, let $\tilde{S}^i$ be obtained from S^i by replacing the subpath $\tilde{T}_i := S^i \setminus (T_{ii}' \cup T_{ii}'')$ of S^i by a path Q_i that consists of a suitable subpath of R_3 together with shortest

T'_{ii}–R_3 and R_3–T''_{ii} paths (see Figure 6.4 (b)). Then the $\tilde{S}^i$ are again double rays. By (i) and (b) and because G is locally finite and the Q_i are finite and pairwise disjoint, there are $N' < i_{L_1} < i_{L_2} < \ldots \in \mathbb{N}$ such that the $\tilde{S}^{i_{L_j}}$ again satisfy (i) with $M_{i_{L_j}}$ updated to $M_{i_{L(j+1)}-1}$. Since every $\tilde{S}^i$ still has a tail in $S^i_{\geq 0}$ and in $S^i_{\leq 0}$ and because of (iv), we can find in H, for every $j \in \mathbb{N}$, infinitely many $\tilde{S}^{i_{L_j}}$–$\tilde{S}^{i_{L_j+1}}$ paths that make the $\tilde{S}^{i_{L_j}}$ into an escaping subdivision $\tilde{H}$ of the hexagonal half-grid. Since all W'_{L_j} end in $S^{N'}$, we can connect the $W'_{L_{2j}}$ pairwise by suitable subpaths of $S^{N'}$, to obtain infinitely many pairwise disjoint $\tilde{H}$-paths W''_m. Note that a path W''_m obtained from W'_{L_i} and W'_{L_j} for $i, j \in 2\mathbb{N}$ is an $\tilde{H}$-path that starts in $\tilde{S}^{i_{L_i}}$ and ends in $\tilde{S}^{i_{L_j}}$. Moreover, by (β), (γ) and (ε) and because $\tilde{H}$ is escaping, W''_m is $\min\{L_i, L_j\}$-fat. Hence, by (δ) and because we only used the paths W'_{L_i} with $i \in 2\mathbb{N}$ to construct the W''_m, infinitely many of the W''_m yield fat $\tilde{H}$-paths as in Lemma 6.5, which implies that G contains $K_{\aleph_0}$ as an ultra fat minor, and which thus concludes the proof. $\square$

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1. The assertion follows immediately by first applying Lemma 6.9 and Lemma 6.8 and then Lemma 6.2. $\square$

7. FURTHER COMMENTS

Before we discuss other related topics to this paper, let us note that our main results give only partial answers to the problems that we mentioned in the introduction, that is to [21, Problem 7.3] and [20, Problems 4.1 and 4.2]. Hence, these problems are still open for arbitrary finitely generated groups that need not be finitely presented and, more generally, for locally finite, quasi-transitive graphs without any restrictions on their cycle spaces.

7.1. **Coarse embeddings.** Theorem 2' asserts that we can find a diverging subdivision of the hexagonal full-grid in every locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length and that has a thick end. The advantage of diverging subdivisions over arbitrary subdivisions is that they preserve some of the geometry of the original graph. One might wish to strengthen Theorem 2', by asking for a subdivision of the hexagonal full-grid whose geometry is even closer related to the geometry of G.

For two graphs G and H, a map $f: V(H) \to V(G)$ is a *coarse embedding* if there exist functions $\rho^-: [0, \infty) \to [0, \infty)$ and $\rho^+: [0, \infty) \to [0, \infty)$ such that $\rho^-(a) \to \infty$ for $a \to \infty$ and

$$\rho^-(d_H(u, v)) \leq d_G(f(u), f(v)) \leq \rho^+(d_H(u, v))$$

for all $u, v \in V(H)$. It is easy to check that a coarse embedding of the hexagonal full-grid always yields a diverging subdivision; however, conversely, a diverging subdivision is in general much weaker than a coarse embedding. One may thus ask whether we can always find a coarse embedding of the hexagonal full-grid in a locally finite, quasi-transitive graph with a thick end.

However, it was already discussed in [20] that for arbitrary locally finite, quasi-transitive graphs (without any condition on their cycle spaces) we cannot ask for coarse embeddings of the hexagonal full-grid. Indeed, coarse embeddings preserve the asymptotic dimension[4], that is, if H has asymptotic dimension at least n and H is coarsely embeddable into G, then the asymptotic dimension of G is at least n, too. Since every locally finite Cayley graph of the lamplighter group has asymptotic dimension 1, see Gentimis [19],

---

[4]See e.g. [5] for a definition of the asymptotic dimension.

and has a thick end, but the full-grid has asymptotic dimension 2, we cannot ask for coarse embeddings of the hexagonal full-grid into all locally finite, quasi-transitive graphs with thick ends.

However, a special case of a theorem by Fujiwara and Whyte [18] states that every locally finite, quasi-transitive graph with a thick end whose cycle space is generated by cycles of bounded length has asymptotic dimension at least 2. Thus, the asymptotic dimension of the full-grid does not prevent it from being coarsely embeddable into such graphs. This motivates the following problem.

Problem 7.1. *Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length and that has a thick end. Is the hexagonal full-grid coarsely embeddable into G?*

Note that a positive answer to this question would also yield a positive answer to [20, Problem 4.5].

7.2. **Quasi-isometries to trees.** For two graphs G and H, a map $f : V(H) \to V(G)$ is a *quasi-isometry* if there exist $c \geq 1$ and $d \geq 0$ such that

$$\frac{1}{c}(d_H(u, v)) - d \leq d_G(f(u), f(v)) \leq c(d_H(u, v)) + d$$

for all $u, v \in V(H)$ and

$$d_H(f(V(G)), w) \leq d$$

for all $w \in V(H)$. Two graphs are *quasi-isometric* if there exists a quasi-isometry between them.

As we have discussed in the introduction, a result by Krön and Möller [26, Theorem 5.5] asserts that a locally finite, quasi-transitive, connected graph is quasi-isometric to a tree if and only if it has no thick end. Hence, we obtain the following corollary from Theorems 1 and 2, which yields two new characterisations of quasi-transitive, locally finite, connected graphs that are quasi-isometric to trees for the special case that the cycle space is generated by cycles of bounded length.

Corollary 7.2. *Let G be a locally finite, quasi-transitive, connected graph whose cycle space is generated by cycles of bounded length. Then the following are equivalent:*

  (i) *G has a thick end.*
 (ii) *G contains the full-grid as an asymptotic minor.*
(iii) *G contains the full-grid as a diverging minor.*
(iv) *G is not quasi-isometric to a tree.*   □

For further characterisations of quasi-transitive, locally finite, connected graphs that are quasi-isometric to trees, we refer the reader to [4, 24, 26, 30].

7.3. **Quasi-isometries to planar graphs.** Finally, we would like to draw the reader's attention to another related problem, which is still open. As we discussed in the previous subsection, the (global) geometry of locally finite, quasi-transitive graphs without a thick end is well understood as they are quasi-isometric to forests. In that sense, our results, Theorems 1 and 2, can be seen as a step towards understanding the (global) geometry of the remaining locally finite, quasi-transitive graphs – those with a thick end. In the case where the cycle space of a locally finite, quasi-transitive graph G is generated by cycles of bounded length, we showed that G contains the full-grid as an asymptotic minor. Since asymptotic minors cannot hide in a ball of small radius, they will appear in the global structure of G. However, even if G is one-ended, this does not mean that the geometry of G resembles that of the full-grid or, more generally, of a one-ended, planar graph. The reason for this is simple: the global structure of G

may be far more involved, and may contain the full-grid only as a substructure. Indeed, even in our proof, we might have found the asymptotic full-grid inside an asymptotic minor of the infinite complete graph. However, Georgakopoulos and Papasoglu conjectured that this is in fact the only thing that can happen.

Conjecture 7.3. [21, Conjecture 9.3] *Let G be a locally finite, transitive graph. Then G either is quasi-isometric to a planar graph or contains every finite graph as an asymptotic minor.*

Note that this can be seen as a coarse version of Thomassen's [28] result that every locally finite, one-ended, transitive graph is either planar or can be contracted into the infinite complete graph.

MacManus [27] proved Conjecture 7.3 in the special case where G is a locally finite Cayley graph of a finitely presented group. Recall that every Cayley graph of a finitely presented group is transitive and has a cycle space which is generated by cycles of bounded length. So the assumption on the cycle space, which was already crucial for our proofs of Theorems 1 to 4, reappears here.

We remark that MacManus's proof uses deep group-theoretic results that have no counterpart for quasi-transitive graphs. Hence, Conjecture 7.3 is still open for arbitrary locally finite, quasi-transitive graphs whose cycle space is generated by cycles of bounded length.

REFERENCES

[1] S. Albrechtsen, R. W. Jacobs, P. Knappe, and P. Wollan, *A characterisation of graphs quasi-isometric to K_4-minor-free graphs*, arXiv preprint (2024), available at arXiv:2408.15335.

[2] S. Albrechtsen, R. Diestel, A.-K. Elm, E. Fluck, R. W. Jacobs, P. Knappe, and P. Wollan, *A structural duality for path-decompositions into parts of small radius*, arXiv preprint (2023), available at arXiv:2307.08497.

[3] S. Albrechtsen, T. Huynh, R. W. Jacobs, P. Knappe, and P. Wollan, *A Menger-type theorem for two induced paths*, SIAM Journal on Discrete Mathematics **38** (2024), no. 2, 1438–1450, available at arXiv:2305.04721.

[4] Y. Antolín, *On Cayley graphs of virtually free groups*, Groups. Complexity. Cryptology **3** (2011), no. 2, 301–327, available at arXiv:0911.2177.

[5] M. Bonamy, N. Bousquet, L. Esperet, C. Groenland, C.-H. Liu, F. Pirot, and A. Scott, *Asymptotic dimension of minor-closed families and Assouad–Nagata dimension of surfaces*, Journal of the European Mathematical Society **26** (2023), no. 10, 3739–3791, available at arXiv:2012.02435.

[6] J. Carmesin, F. Lehner, and R. G. Möller, *On tree-decompositions of one-ended graphs*, Mathematische Nachrichten **292** (2019), no. 3, 524–539, available at arXiv:1706.08330.

[7] V. Chepoi, F. F. Dragan, I. Newman, Y. Rabinovich, and Y. Vaxes, *Constant approximation algorithms for embedding graph metrics into trees and outerplanar graphs*, Discrete & Computational Geometry **47** (2012), 187–214, available at arXiv:1007.0489.

[8] J. Davies, R. Hickingbotham, F. Illingworth, and R. McCarty, *Fat minors cannot be thinned (by quasi-isometries)*, arXiv preprint (2024), available at arXiv:2405.09383.

[9] R. Diestel, *Graph theory* (6th edition), Springer-Verlag, 2024.

[10] R. Diestel, R. W. Jacobs, P. Knappe, and J. Kurkofka, *Canonical graph decompositions via coverings*, arXiv preprint (2024), available at arXiv:2207.04855.

[11] ———, *Canonical graph decompositions via coverings; extend version*, arXiv preprint (2024), available at arXiv:2207.04855v5.

[12] R. Diestel and I. Leader, *A conjecture concerning a limit of non-Cayley graphs*, Journal of Algebraic Combinatorics **14** (2001), no. 1, 17–25.

[13] M. J. Dunwoody, *An inaccessible graph*, Random walks, boundaries and spectra, 2011, pp. 1–14.

[14] M. J. Dunwoody, *An inaccessible group*, Geometric group theory, Vol. 1 (Sussex, 1991), 1993, pp. 75–78.

[15] A. Eskin, D. Fisher, and K. Whyte, *Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs*, Annals of Mathematics. Second Series **176** (2012), no. 1, 221–260, available at arXiv:math/0607207.

[16] L. Esperet and U. Giocanti, *Coarse geometry of quasi-transitive graphs beyond planarity*, Electronic Journal of Combinatorics **31(2)** (2024), available at arXiv:2312.08902.

[17] K. Fujiwara and P. Papasoglu, *A coarse-geometry characterization of cacti*, arXiv preprint (2023), available at arXiv:2305.08512.

[18] K. Fujiwara and K. Whyte, *A note on spaces of asymptotic dimension one*, Algebraic & Geometric Topology **7** (2007), 1063–1070, available at arXiv:math/0610391.

[19] T. Gentimis, *Asymptotic dimension of finitely presented groups*, Proceedings of the American Mathematical Society **136** (2008), no. 12, 4103–4110, available at arXiv:math/0610541.

[20] A. Georgakopoulos and M. Hamann, *A full Halin grid theorem*, Discrete & Computational Geometry (2024), available at arXiv:2403.05862.

[21] A. Georgakopoulos and P. Papasoglu, *Graph minors and metric spaces*, arXiv preprint (2023), available at arXiv:2305.07456.

[22] R. Halin, *Über die Maximalzahl fremder unendlicher Wege in Graphen*, Mathematische Nachrichten **30** (1965), 63–85.

[23] M. Hamann, *Accessibility in transitive graphs*, Combinatorica **38** (2018), 847–859, available at arXiv:1404.7677.

[24] M. Hamann, F. Lehner, B. Miraftab, and T. Rühmann, *A Stallings type theorem for quasi-transitive graphs*, Journal of Combinatorial Theory, Series B **157** (2022), 40–69, available at arXiv:1812.06312.

[25] K. Heuer, *Excluding a full grid minor*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **87** (2017), no. 2, 265–274, available at arXiv:1510.03599.

[26] B. Krön and R. G. Möller, *Quasi-isometries between graphs and trees*, Journal of Combinatorial Theory, Series B **98** (2008), no. 5, 994–1013.

[27] J. P. MacManus, *Fat minors in finitely presented groups*, arXiv preprint (2024), available at arXiv:2408.10748.

[28] C. Thomassen, *The Hadwiger number of infinite vertex-transitive graphs*, Combinatorica **12** (1992), no. 4, 481–491.

[29] C. Thomassen and W. Woess, *Vertex-transitive graphs and accessibility*, Journal of Combinatorial Theory, Series B **58** (1993), no. 2, 248–268.

[30] W. Woess, *Graphs and groups with tree-like properties*, Journal of Combinatorial Theory, Series B **47** (1989), no. 3, 361–371.

University of Hamburg, Department of Mathematics, Bundesstrasse 55 (Geomatikum), 20146 Hamburg, Germany

Email address: {sandra.albrechtsen,matthias.hamann}@uni-hamburg.de