THE CLASSIFICATION OF CONNECTED-HOMOGENEOUS DIGRAPHS WITH MORE THAN ONE END

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Abstract. We classify the connected-homogeneous digraphs with more than one end. We further show that if their underlying undirected graph is not connected-homogeneous, they are highly-arc-transitive.

1. Introduction

A graph is called *homogeneous* if every isomorphism between two finite induced subgraphs extends to an automorphism of the graph. If only isomorphisms between finite *connected* induced subgraphs are required to extend to an automorphism, the graph is called *connected-homogeneous*, or simply *C-homogeneous*. In the context of digraphs, the same notion of homogeneity and C-homogeneity applies, connectedness being taken in the underlying undirected graph. There are classification results for

- the homogeneous graphs, [3, 9, 11, 22, 29],
- the C-homogeneous graphs, [9, 12, 13, 17, 18],
- the homogeneous digraphs, [3, 20, 21],

but not for the C-homogeneous digraphs. Our aim in this paper is to classify the C-homogeneous digraphs with more than one end. Partial results towards such a classification are known for the locally finite case; they are due to Gray and Möller [14].

We classify the connected C-homogeneous digraphs, of any cardinality, that have more than one end. The most important tool we use is the concept of structure trees based on vertex cut systems, introduced recently by Dunwoody and Krön [8] and used before in [16, 17, 19]. A crucial feature of this new technique is its applicability to arbitrary infinite graphs: the previously available theory of structure trees in terms of edge cuts, due to Dunwoody [7] (see also [4, 26, 28, 31]) and used by Gray and Möller [14], only allows for the treatment of locally finite graphs. Our proof is based on the classification of the countable homogeneous tournaments of Lachlan [21] and homogeneous bipartite graphs of Goldstern, Grossberg and Kojman [15] and is otherwise from first principles. We reobtain the results of Gray and Möller [14] but do not use them.

We further study the relationship between the C-homogeneous digraphs and the C-homogeneous graphs. A natural question arising here is whether or not the underlying undirected graph of a C-homogeneous digraph is C-homogeneous, and,

2010 *Mathematics Subject Classification.* Primary 05C20, 05C63.

vice versa, whether a C-homogeneous undirected graph admits a C-homogeneous orientation. Interestingly, neither need be the case.

We say that a C-homogeneous digraph is of *Type I* if its underlying undirected graph is C-homogeneous; otherwise it is of *Type II*. Combining the results of Gray and Möller [14] with those of Gray and Macpherson [13] we know that there exist digraphs of both types, and that there are C-homogeneous graphs that do not admit a C-homogeneous orientation. In Section 4 we show that connected C-homogeneous digraphs with more than one end are of Type I, if and only if they either are a tree or contain a triangle.

Another widely studied class of digraphs are the *highly-arc-transitive* digraphs, those that are k-arc-transitive[1] for all $k \in \mathbb{N}$. As a corollary of our methods, we find that the connected C-homogeneous digraphs of Type II with more than one end are highly-arc-transitive. This was previously known for locally finite such digraphs [14]. Unlike its undirected counterpart (cf. [17, 31]), the class of highly-arc-transitive digraphs is far from understood. See [1, 24, 25, 30] for papers related to highly-arc-transitive digraphs.

2. Basics

2.1. **Digraphs.** A digraph $D = (VD, ED)$ consists of a non-empty set VD, its set of *vertices*, and an asymmetric (i.e. irreflexive and anti-symmetric) binary relation ED over VD, the set of *edges* of D.

We write xy for an edge $(x, y) \in ED$ and say that xy is directed *from x to y*. For $x \in VD$ we define its *out-neighborhood* as $N^+(x) := \{y \in VD \mid xy \in ED\}$, its *in-neighborhood* as $N^-(x) := \{z \in VD \mid zx \in ED\}$ and finally its *neighborhood* as $N(x) := N^+(x) \cup N^-(x)$. Two vertices are called *adjacent* if one is in the other's neighborhood. For a vertex set $X \subseteq VD$ the neighborhood of X is defined as $N(X) := \left(\bigcup_{x\in X} N(x)\right) \setminus X$ and $N^+(X), N^-(X)$ are defined analogously. For all $x \in VD$ we denote with $d^+(x), d^-(x)$ the cardinality of $N^+(x), N^-(x)$, respectively.

A sequence $x_0x_1 \dots x_k$ of pairwise distinct vertices of D with $k \in \mathbb{N}$ and $x_i \in N^+(x_{i-1})$ for all $1 \leq i \leq k$ is called a *k-arc* from x_0 to x_k. Given two vertices x and y we say that y is a *descendant* of x if there is a k-arc from x to y for some $k \in \mathbb{N}$ and we define the *descendant-digraph* of x to be the subgraph $\operatorname{desc}(x) \subseteq D$ that is induced by the set of all its descendants.

If $x_0x_1 \dots x_n$ is a sequence of vertices such that any two subsequent vertices are adjacent then it is called a *walk* and a walk of pairwise distinct vertices is called a *path*. A path that is also an arc is called a *directed path*. A digraph is called *connected* if any two vertices are joined by a path.

A walk $x_0x_1 \dots x_n$ such that $x_i \in N^+(x_{i+1}) \Leftrightarrow x_{i+1} \in N^-(x_{i+2})$ is called *alternating*. If $e = xy$ and $e' = x'y'$ are contained in a common alternating walk then they are called *reachable* from each other. This clearly defines an equivalence relation, the *reachability relation*, on ED which we denote by $\mathcal{A}$, and for $e \in ED$ we refer to the equivalence class that contains e by $\mathcal{A}(e)$. See also [1].

One-way infinite paths are called *rays* and two rays R_1, R_2 are called *equivalent* if for every finite vertex set S both R_1 and R_2 lie eventually in the same component. This is indeed an equivalence relation on the rays of D the classes of which we call the *ends* of D. An end ω is thus a set of rays and we say that ω is *contained* in a

[1] A (di)graph is called *k-arc-transitive* if every (directed) path of length k can be mapped to any other by an automorphism.

subgraph $H \subset D$ if there is a ray R in H such that $R \in \omega$. The same notion of an end is used for (undirected) graphs (see [5, p. 202]).

2.2. **Structure trees.** In this section we introduce the terms of cuts and structure trees. Compared with Dunwoody and Krön [8] we use a different notation for the cut systems in order to indicate the relation of cut systems with the well-known graph theoretic concept of separations, see [5].

Let G be a connected graph and let $A, B \subseteq VG$ be two vertex sets. The pair (A, B) is a *separation* of G if $A \cup B = VG$ and $EG[A] \cup EG[B] = EG$.

The *order* of a separation (A, B) is the order of its *separator* $A \cap B$ and the subgraphs $G[A \setminus B]$ and $G[B \setminus A]$ are the *wings* of (A, B). A separation (A, B) *separates* two vertices if they lie in distinct wings of (A, B). With $(A, \sim)$ we refer to the separation $(A, (VG \setminus A) \cup N(VG \setminus A))$. A separation (A, B) of finite order with non-empty wings is called *essential* if the wing $G[A \setminus B]$ is connected and no proper subset of $A \cap B$ separates the wings of (A, B). A *cut system* of G is a non-empty set $\mathcal{C}$ of essential separations (A, B) of G satisfying the following three conditions:

(i) If $(A, B) \in \mathcal{C}$ then there is an $(X, Y) \in \mathcal{C}$ with $X \subseteq B$.
(ii) Let $(A, B) \in \mathcal{C}$ and C be a component of $G[B \setminus A]$. If there is a separation $(X, Y) \in \mathcal{C}$ with $X \setminus Y \subseteq C$, then the separation $(C \cup N(C), \sim)$ is also in $\mathcal{C}$.
(iii) If $(A, B) \in \mathcal{C}$ with wings X, Y and $(A', B') \in \mathcal{C}$ with wings X', Y' then there are components C_1 in $X \cap X'$ and C_2 in $Y \cap Y'$ or components C_1 in $Y \cap X'$ and C_2 in $X \cap Y'$ such that both C_1 and C_2 are wings of separations in $\mathcal{C}$.

A separation $(A, B) \in \mathcal{C}$ is called a *$\mathcal{C}$-cut*. Two $\mathcal{C}$-cuts $(A_0, A_1), (B_0, B_1)$ are *nested* if there are $i, j \in \{0, 1\}$ such that one wing of $(A_i \cap B_j, \sim)$ does not contain any component C with $(C \cup N(C), \sim) \in \mathcal{C}$ and $A_{1-i} \cap B_{1-j}$ contains $(A_0 \cap A_1) \cup (B_0 \cap B_1)$. A cut system is *nested* if each two of its cuts are nested.

Remark 2.1.

1. If, for two $\mathcal{C}$-cuts $(A_0, A_1), (B_0, B_1)$, the separator $A_0 \cap A_1$ contains vertices of both wings of (B_0, B_1), then the two cuts are not nested.
2. In any transitive graph G with an $\mathrm{Aut}(G)$-invariant cut system $\mathcal{C}$, any two nested cuts (A_0, A_1) and (B_0, B_1) with $(A_0 \cap A_1) \cup (B_0 \cap B_1) \subseteq A_{1-i} \cap B_{1-j}$ have the property that $A_i \cap B_j$ is empty by [8, Lemma 3.5].

A $\mathcal{C}$-cut is *minimal* if there is no $\mathcal{C}$-cut with smaller order. A *minimal cut system* is a cut system all whose cuts are minimal and thus have the same order.

A *$\mathcal{C}$-separator* is a vertex set S that is a separator of some separation in $\mathcal{C}$. Let $\mathcal{S}$ be the set of $\mathcal{C}$-separators. A *$\mathcal{C}$-block* is a maximal induced subgraph X of G such that

(i) for every $(A, B) \in \mathcal{C}$ either $VX \subseteq A$ or $VX \subseteq B$ but not both;
(ii) there is some $(A, B) \in \mathcal{C}$ with $VX \subseteq A$ and $A \cap B \subseteq VX$.

Let $\mathcal{B}$ be the set of $\mathcal{C}$-blocks. For a nested minimal cut system $\mathcal{C}$ let $\mathcal{T}(\mathcal{C})$ be the graph with vertex set $\mathcal{S} \cup \mathcal{B}$ and such that two vertices X, Y of $\mathcal{T}(\mathcal{C})$ are adjacent if and only if either $X \in \mathcal{S}$, $Y \in \mathcal{B}$, and $X \subseteq Y$ or $X \in \mathcal{B}$, $Y \in \mathcal{S}$, and $Y \subseteq X$. Then $\mathcal{T} = \mathcal{T}(\mathcal{C})$ is called the *structure tree* of G and $\mathcal{C}$ and by Lemma 6.2 of [8] it is indeed a tree.

For a component X of $\mathcal{T}$ let $\bigcup X := \{v \in VG \mid \exists x \in VX \text{ with } v \in x\}$.

A cut system $\mathcal{C}$ of G is called *basic* if the following conditions hold:

(i) $\mathcal{C}$ is non-empty, minimal, nested and $\mathrm{Aut}(G)$-invariant.
(ii) $\mathrm{Aut}(G)$ acts transitively on $\mathcal{S}$.
(iii) For each $\mathcal{C}$-cut (A, B) both A and B contain an end of G and there is no separation of smaller order that has this property.

With the results of Dunwoody's and Krön's work on (vertex) cut systems [8] we can deduce the following theorem.

Theorem 2.2. *For any graph with more than one end there is a basic cut system.* □

If we take, for a connected graph G, all those cuts whose separators consist of one vertex, each, then we obtain as the structure tree the well-known block-cutvertex tree. So in this case the two obtained trees coincide, but have different notations. That is, in the proof of Theorem 4.2 we could argue using the block-cutvertex tree instead of the structure tree. But for consistency reasons we have not done so.

Lemma 2.3. *Let G be a graph and let $\mathcal{C}$ be a nested cut system of G such that no $\mathcal{C}$-separator contains any edge. For any path P that has both its end vertices in the same $\mathcal{C}$-separator S, there is a $\mathcal{C}$-block with maximal distance to S in $\mathcal{T}(\mathcal{C})$ that contains edges of P. This $\mathcal{C}$-block contains at least two edges of P.*

Proof. Any two vertices that are not in a common $\mathcal{C}$-block, are separated by some $\mathcal{C}$-separator. So we conclude that for each edge of G there is a unique $\mathcal{C}$-block that contains this edge, as it is not contained in any $\mathcal{C}$-separator. The path P has only finitely many edges, so there are just finitely many $\mathcal{C}$-blocks that contain edges of P and we may pick one, X say, with maximal distance to S in $\mathcal{T}(\mathcal{C})$. Let xy be an edge on P that lies in X. Then either x or y does not lie in that $\mathcal{C}$-separator S' that separates X from S and lies in X. We assume that this is y. Let z be the other neighbor of x on P. The edge yz cannot lie further away from S than X in the structure tree, but since $y \notin S'$, we have $yz \in EX$. So X contains at least two edges of P. □

In the context of a digraph D all concepts introduced in this section are related to the underlying undirected graph G of D except for one definition: We call a cut system $\mathcal{C}$ for a digraph D *basic* if it has the following properties.

(i) $\mathcal{C}$ is non-empty, minimal, nested and $\mathrm{Aut}(D)$-invariant.
(ii) $\mathrm{Aut}(D)$ acts transitively on $\mathcal{S}$.
(iii) For each $\mathcal{C}$-cut (A, B) both A and B contain an end of D and there is no separation of smaller order that has this property.

So we just replaced the automorphism group of the underlying undirected graph by that of the digraph and the remainder of the definition stays the same.

Then Theorem 2.2 does not only hold for any graph but also for any digraph by the results in [8]. We have to define the property of being basic differently, because we know in general only that we may consider $\mathrm{Aut}(D)$ as a subgroup of $\mathrm{Aut}(G)$ but we do not know whether it is a proper subgroup or not. Thus, our cut system could have more than one $\mathrm{Aut}(D)$-orbit of separators which would be more difficult to deal with.

2.3. **Bipartite digraphs.** Let κ, λ be arbitrary cardinals, and $m \in \mathbb{N}$. We define the *directed semi-regular tree* $T_{\kappa,\lambda}$ to be the directed tree with bipartition $A \cup B$ such that $d(a) = d^+(a) = \kappa$ for all $a \in A$ and $d(b) = d^-(b) = \lambda$ for all $b \in B$, the *complete bipartite digraph* $K_{\kappa,\lambda}$ to be the digraph with bipartition $A \cup B$ such that $|A| = \kappa$, $|B| = \lambda$ and all edges point from A to B, the *directed complement of a perfect matching* CP_κ to be the digraph obtained from $K_{\kappa,\kappa}$ by removing a perfect matching, and the *cycle* C_{2m} to be the digraph obtained by orienting the undirected cycle on $2m$ vertices such that no 2-arc arises. In the context of graphs we use the same notation to refer to the underlying undirected graph.

We call a bipartite graph G with bipartition $X \cup Y$ *generic bipartite*, if it has the following property: For any finite disjoint subsets U and W of X (of Y) there is a vertex v in Y (in X) such that $U \subseteq N(v)$ and $W \cap N(v) = \emptyset$. Any generic bipartite graph contains any countable bipartite graph as an induced subgraph, and thus up to isomorphism there is a unique countable generic bipartite graph (cp. [5, p. 213] and [10, p. 98]). A *generic bipartite digraph* is a digraph D whose underlying undirected graph G is generic bipartite with bipartition $A \cup B$ and such that all edges of D are directed from A to B.

2.4. **C-homogeneous graphs.** In order to study the C-homogeneous digraphs of Type I we make use of the classification of connected C-homogeneous graphs with more than one end from [17], which we briefly summarize in Theorem 2.4.

With $X_{\kappa,\lambda}$ we denote a graph with connectivity 1 such that every *block*, that is a maximal 2-connected subgraph, is a complete graph on κ vertices and every vertex lies in λ distinct blocks.

Theorem 2.4. *A connected graph with more than one end is C-homogeneous if and only if it is isomorphic to an $X_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$.* □

3. Local structure

In this chapter we summarize some preliminary results of the connection between a connected C-homogeneous digraph with more than one end and a basic cut system $\mathcal{C}$ of this digraph which exists by Theorem 2.2. In particular we investigate the local structure around $\mathcal{C}$-separators.

Lemma 3.1. *Let D be a connected C-homogeneous digraph with more than one end. Let $\mathcal{C}$ be a basic cut system of D and let S be a $\mathcal{C}$-separator. Then there is no edge in D with both end vertices in S. In particular, no two $\mathcal{C}$-blocks can share an edge.*

Proof. Let $(A, B) \in \mathcal{C}$ with $A \cap B = S$ and let us suppose that there is $xy \in ED$ with $x, y \in S$. By the minimality of $\mathcal{C}$ each vertex in a $\mathcal{C}$-separator has neighbors in both wings of the corresponding separation. Let $a \in A \setminus B$ and $b \in B \setminus A$ be such neighbors of y. Then there are different possibilities for the direction of their connecting edges. Let us first consider the case that $ay, by \in ED$. Then there is an automorphism α that maps xy onto by. So S^α lies in B, since $\mathcal{C}$ is nested and $b \in S^\alpha$, and we have either $A \subseteq A^\alpha$ or $A \subseteq B^\alpha$ by the nestedness of $\mathcal{C}$. Hence there is a vertex b_0, which is either a^α or b^α, that lies either in $B \cap B^\alpha$ or in $B \cap A^\alpha$ such that $b_0 y \in ED$ and $S_0 := S^\alpha$ separates a and b_0. Let $\{A_0, B_0\} = \{A^\alpha, B^\alpha\}$ such that $a \in A_0$ and $b_0 \in B_0$.

Now let α_0 be an automorphism of D such that $a^{\alpha_0} = a$ and $(by)^{\alpha_0} = b_0 y$. Hence the vertex $b_1 := b_0^{\alpha_0}$ lies in $B_1 \setminus A_1$, where $A_1 := A_0^{\alpha_0}$ and $B_1 := B_0^{\alpha_0}$, and we have $b_1 y \in ED$. Since A_0 meets A_1, $S_1 := S_0^{\alpha_0} \neq S_0$ lies in B_0, and $\mathcal{C}$ is nested, we know that A_0 is a proper subset of A_1, B_1 is a proper subset of B_0 and S_0 lies in

A_1, which implies $b \in A_1$. Furthermore b_1 and a are separated from each other by both S_0 and S_1. By repeating this process recursively, we have an α_i that fixes a and y and maps b_{i-1} onto b_i and we get a further vertex $b_{i+1} = b_i^{\alpha_i} \in B_{i+1} \setminus A_{i+1}$ that is separated by $S_{i+1} := S_i^{\alpha_i}$ from $a \in A_{i+1} \setminus B_{i+1}$. By the same argument as before we have that A_i is a proper subset of $A_{i+1} := A_i^{\alpha_i}$, that $B_{i+1} := B_i^{\alpha_i}$ is a proper subset of B_i and that $b_i \in A_{i+1}$. Hence, $b_j \in A_{i+1}$ for all $j \leq i$, which implies $b_i \neq b_j$ for all $i \neq j$.

Thus, after the step $m := |S|+1$ there has to be some $k < |S|$ such that b_k is not contained in S_m and therefore lies in $A_m \setminus B_m$. That is, b_k is also separated from b_m by S_m and $I := \{a, b_k, b_m\}$ forms an independent set. Note that by construction all elements in I have y as a common out-neighbor. Hence, due to the C-homogeneity of D, there is an automorphism β of D which interchanges a and b_k and fixes y and b_m. Remark that $a \in A \setminus B \subset A_{k+1} \setminus B_{k+1}$ and $b_m \in B_m \setminus A_m \subset B_{k+1} \setminus A_{k+1}$. Thus, S_{k+1} is a separator containing b_k that separates a and b_m, which implies that S_{k+1}^{β} is a separator containing a that separates b_k and b_m. But due to the minimality of $\mathcal{C}$, there is a b_k-b_m-path in B_{k+1} that meets S_{k+1} only in b_k and therefore S_{k+1}^{β} meets both $A_{k+1} \setminus B_{k+1}$ and $B_{k+1} \setminus A_{k+1}$, contradicting the nestedness of $\mathcal{C}$ (cp. Remark 2.1).

Let us suppose $by, ya \in ED$. Let α be an automorphism of D with $(xy)^\alpha = ya$ and choose $\{X, Y\} = \{A^\alpha, B^\alpha\}$ such that $b \in X \setminus Y$. Then there is a neighbor c of y in $Y \setminus X$, which is separated from b by S^α. Note that by nestedness, and since both X and Y meet A, we have that either $X \cap B$ or $Y \cap B$ is empty. So $b \in X \cap B$ yields $Y \cap B =$ and thus $c \in A$. If $cy \in ED$ then we may take the vertices c, b instead of a, b and get a contradiction by the first case above. Thus we may assume that $yc \in ED$. But then we can map the digraph $D[b, y, a]$ onto $D[b, y, c]$ and hence the number of separators that separate b from a without containing one of them equals the number of separators that separate b and c without containing one of them. Due to [8, Lemma 4.1] this number is finite. Because of the nestedness of $\mathcal{C}$, every such separator that separates the vertices b and a lies entirely in X, and since a and c are joined by a path that lies except for a in $Y \setminus X$, it also separates b and c. But S^α contains a and separates b from c, a contradiction. The case $ay, yb \in ED$ is analogous.

Finally, we suppose that $ya, yb \in ED$. By considering instead of D the digraph obtained from D by reversing the direction of each edge we also may assume that there are $a' \in A \setminus B$ and $b' \in B \setminus A$ with $a'x, b'x \in ED$. Let α be an automorphism of D with $(xy)^\alpha = yb$. Then there is a vertex $b'' \in B \setminus A$ that is separated by S^α from a and such that $b''y \in ED$. But then we have the situation of the previous case and thus we know that no such edge xy exists.

For the remaining statement, let X and Y be distinct $\mathcal{C}$-blocks. Then there is $x \in X \setminus Y$. Since Y, being a $\mathcal{C}$-block, is maximally $\mathcal{C}$-inseparable, there is a $\mathcal{C}$-separator S that separates x from Y. As X is also $\mathcal{C}$-inseparable, we have $X \cap Y \subseteq S$. Therefore X and Y cannot share an edge. $\square$

Lemma 3.2. *Let D be a connected C-homogeneous digraph with more than one end and let $\mathcal{C}$ be a basic cut system of D. Then for each induced 2-arc xyz in D there is a $\mathcal{C}$-separation (A, B) separating x and z (and therefore $A \cap B$ contains y).*

Proof. Let xyz be an induced 2-arc in D. Since $\mathcal{C}$ is a non-empty set of cuts, there is a $\mathcal{C}$-separator S which is non-empty, as D is connected. Thus by C-homogeneity we may assume $y \in S$. Let $(A, B) \in \mathcal{C}$ with $A \cap B = S$. If x and z lie in distinct wings of (A, B) we are done. Otherwise, if x and z lie in the same wing, consider a neighbor v of y in the other wing. Then either xyv or vyz is an induced 2-arc P in D whose end vertices are separated by (A, B). By C-homogeneity there is an automorphism α of D with $P^\alpha = xyz$, such that (A^α, B^α) separates x and z. □

Lemma 3.3. *Let D be a connected C-homogeneous digraph with more than one end and let $\mathcal{C}$ be a basic cut system of D. Then for each 2-arc P in D we have $|P \cap S| \leq 1$ for all $\mathcal{C}$-separators S.*

Proof. Let $P = xyz$ be a 2-arc in D and S a $\mathcal{C}$-separator. By Lemma 3.1 we only have to show that S cannot contain both x and z. So assume $\{x, z\} \subseteq S$. Then again by Lemma 3.1 P is induced, such that Lemma 3.2 yields a $\mathcal{C}$-separation (A, B) that separates x from z. By Remark 2.1 this contradicts the nestedness of $\mathcal{C}$. □

Lemma 3.4. *Let D be a connected C-homogeneous triangle-free digraph with more than one end, let $\mathcal{C}$ be a basic cut system of D, and let S be a $\mathcal{C}$-separator. Then there is no directed path in D with both endvertices in S.*

Proof. Assume that there is a directed path $P = x_0 \dots x_n$ and a $\mathcal{C}$-separator S with $P \cap S = \{x_0, x_n\}$. We may choose P and S such that P has minimal length. By Lemma 3.1 and Lemma 3.3 P has length at least 3, so we may consider the directed subpath $P' = x_0 x_1 x_2 \subseteq P$ of length precisely 2. Since D is triangle-free, P' is induced and Lemma 3.2 yields a $\mathcal{C}$-separation (A, B) separating x_0 from x_2. As $\mathcal{C}$ is nested, Remark 2.1 implies that S cannot separate any two vertices of $S' := A \cap B$. Therefore x_0 and x_n, which are both contained in S, must lie on the same side of (A, B), either in A or in B. But this means that S' either contains x_n or that (A, B) separates x_2 from x_n and therfore S' has to contain an inner vertex of $x_2 P x_n$. In both cases S' contains x_1 and a vertex x_i with $3 \leq i \leq n$, contradicting the minimality of P. □

Lemma 3.5. *Let D be a connected C-homogeneous triangle-free digraph with more than one end, and let $\mathcal{C}$ be a basic cut system of D. Then for any cut $(A, B) \in \mathcal{C}$ there is no 2-arc xyz in $D[A]$ with $y \in A \cap B$.*

Proof. By Lemma 3.1 we only have to show that given a cut $(A, B) \in \mathcal{C}$ there is no 2-arc xyz in D such that $y \in S := A \cap B$ and $x, z \in A \setminus B$. So let us suppose there is such a path. Then y has a neighbor $b \in B \setminus A$. We may assume that their connecting edge is pointing towards y, since otherwise changing the direction of each edge gives a digraph D' which is C-homogeneous and has this property.

Suppose that there is a second neighbor $c \in B \setminus A$ of y. If $cy \in ED$, then the lack of triangles in D implies that there is an $\alpha \in \mathrm{Aut}(D)$ that fixes b, y, z and with $x^\alpha = c$, $c^\alpha = x$. But then the separations (A, B) and (A^α, B^α) are not nested. Thus we may assume that $yc \in ED$. In this situation let β be an automorphism of D that fixes x, y, b and maps z onto c and vice versa—a contradiction as before.

So b is the unique neighbor of y in B. We may assume that there is another vertex a, say, that lies in S, since otherwise we could map the 2-arc byz onto xyz, as D is C-homogeneous and triangle-free, and, thus, y would separate x from z, contradicting the fact that x and z lie in the same component of $D - S$. Now consider a path P in D connecting a and y and let $\mathcal{T}$ denote the structure tree of D and $\mathcal{C}$. Let $\mathcal{M}$ denote the set of $\mathcal{C}$-blocks containing edges of P. By Lemma 3.1

we can apply Lemma 2.3 to obtain a $\mathcal{C}$-block $M \in \mathcal{M}$ whose distance to S in $\mathcal{T}$ is maximal with respect to $\mathcal{M}$.

Now every nontrivial component of $P \cap M$ has to contain exactly two edges: An isolated edge would either be contained in a separator, in contradiction to Lemma 3.1, or it would connect M to two distinct neighbors in $\mathcal{T} \cap \mathcal{M}$, contradicting the choice of M. If there is a segment of P in M with a length of at least three, then it contains either a directed subsegment, isomorphic to byz, or a subsegment isomorphic to $by \cup xy$. In each case there exists an isomorphism φ such that S^φ separates the endvertices of this subsegment, which is impossible since M is a $\mathcal{C}$-block.

Considering an arbitrary nontrivial component of $P \cap M$, its two edges have a common vertex which we denote by m. With an analogous argument as above, both edges are directed away from m. Let us denote their heads by u and v, respectively. By construction, u and v lie both in the separator $S_M \subset M$ that lies on the unique shortest path between M and S in $\mathcal{T}$. Consider an arbitrary cut in $\mathcal{C}$ with separator S_M. Then u has a neighbor u' in the wing not containing m. Let ψ be an automorphism with $(mu)^\psi = by$ and either $(uu')^\psi = yz$, if $uu' \in ED$ or $(u'u)^\psi = xy$, if $u'u \in ED$. Since $\mathcal{C}$ is nested we have $S_M^\psi \subset B$ which means that x and z are separated from b by S_M^ψ. By relabeling $S := S_M^\psi$ and $a := v^\psi$, if neccessary, we may assume that ba is an edge.

If a has no other neighbor $z' \neq b$ in the component of $B \setminus A$ containing b, then $(S \setminus \{y, a\}) \cup \{b\}$ is a separator in D that separates ends and has smaller cardinality than S, contradicting the fact that $\mathcal{C}$ is basic. So there is such a neighbor z'. Since D is triangle-free, we can find an automorphism γ with $(by)^\gamma = ba$ and either $x^\gamma = z'$ or $z^\gamma = z'$, depending on the orientation of the edge between b and z'. By the nestedness of $\mathcal{C}$ and since S^γ separates b from z' we have $S^\gamma \subset B$ and also $B^\gamma \subseteq B$. Now x lies in $A \subset A^\gamma$ and b lies in B^γ, thus x is separated from b by S^γ and we have $y \in S^\gamma$. But that implies that y and a both have b as their unique neighbor in B^γ. Hence, $(S^\gamma \setminus \{y, a\}) \cup \{b\}$ contradicts the minimality of the cardinality of S as before. □

Lemma 3.6. *Let D be a connected C-homogeneous triangle-free digraph that is not a tree and that has more than one end, and let $\mathcal{C}$ be a basic cut system of D. Let S be a $\mathcal{C}$-separator and let $s \in S$. Then there is precisely one $\mathcal{C}$-block that contains s and all edges directed away from s, and there is precisely one $\mathcal{C}$-block that contains s and all edges directed towards s. Furthermore $d^+(s) > 1$ and $d^-(s) > 1$.*

Proof. By Lemma 3.5 there is at most one kind of neighbors in each $\mathcal{C}$-block. Suppose first that there is a $\mathcal{C}$-block Z with only one neighbor a of s. We may assume that $as \in ED$. By C-homogeneity, we can map each edge xs onto as. As there is by Lemma 3.1 precisely one $\mathcal{C}$-block Y that contains xs, Y contains no other neighbor of s, because the same holds for s and Z. Thus every component of each $\mathcal{C}$-block is either a single vertex or a star the edges of which are directed towards the leaves of the star. If each $\mathcal{C}$-block is a tree and every $\mathcal{C}$-separator consists of one vertex, then the digraph D has to be a tree. Since we excluded this case, there is a second vertex $t \in S$. For every component C of $D - S$, there is an (undirected) s-t-path P with all its vertices but s and t in C. Let X be a $\mathcal{C}$-block with maximal distance to S in the structure tree of G and $\mathcal{C}$ such that there are at least two edges from P in X. This $\mathcal{C}$-block exists by Lemma 2.3. As each component of X that contains edges is a star, the longest subpath of P that lies completely in X has length 2. Let xyz be such a subpath. Then due to Lemma 3.3 we have $xy, zy \in ED$ and y is the only neighbor in X of both x and z. Let S' be the $\mathcal{C}$-separator in X that separates X from S. Then, S' contains x and z. But, as in the previous lemma, $(S' \setminus \{x, z\}) \cup \{y\}$ would be a separator of smaller cardinality than S that separates two ends, a contradiction.

Thus a $\mathcal{C}$-block cannot contain s together with a single neighbor of s and by C-homogeneity there has to be one $\mathcal{C}$-block that contains all in-neighbors of s and one that contains all out-neighbors of s. □

Lemma 3.7. *Let D be a connected C-homogeneous triangle-free digraph that is not a tree and that has more than one end, and let $\mathcal{C}$ be a basic cut system of D. Then every $\mathcal{C}$-separator has degree two in the structure tree $\mathcal{T}$ of D and $\mathcal{C}$.*

Proof. Let S be a C-separator. Then for each component X of $\mathcal{T} - S$ the vertex set $(\bigcup X) \setminus S$ is the union of components of $D - S$. Since each $s \in S$ has a neighbor in each component of $D - S$, it also has at least one neighbor in each component of $\mathcal{T} - S$. With Lemma 3.6 we have $d_{\mathcal{T}}(S) = 2$. □

If we combine Lemma 3.6 and Lemma 3.7 we get the following corollary.

Corollary 3.8. *Let D be a connected C-homogeneous triangle-free digraph that is not a tree and that has more than one end, and let $\mathcal{C}$ be a basic cut system of D. Let B be a $\mathcal{C}$-block, $S \subset B$ a $\mathcal{C}$-separator and $s \in S$. If s has no neighbor in B, then there is exactly one $\mathcal{C}$-separator $S' \subset B$ such that $s \in S' \cap S$. If s has a neighbor in B, then S is the only $\mathcal{C}$-separator in B that contains s.* □

Lemma 3.9. *Let D be a connected C-homogeneous digraph with more than one end that embeds a triangle, and let $\mathcal{C}$ be a basic cut system of D. Then every $\mathcal{C}$-block that contains edges is a tournament and D has connectivity 1.*

Proof. Let S be a $\mathcal{C}$-separator and let $x \in S$. Then x has adjacent vertices in both wings of each cut $(A, B) \in \mathcal{C}$ with $A \cap B = S$. As D contains triangles, each edge lies on a triangle. We know that each wing of (A, B) contains both an in- and an out-neighbor of x, as any triangle contains a 2-arc and D is edge-transitive. Thus every induced path of length 2 in D can be mapped on a path crossing S, i.e. a path both end vertices of which lie in distinct wings of (A, B). Hence no two vertices in the same $\mathcal{C}$-block can have distance 2 from each other and, in particular, every component of every $\mathcal{C}$-block has diameter 1.

To prove that each $\mathcal{C}$-block has diameter 1 we just have to show that each $\mathcal{C}$-block is connected. So let us suppose that this is not the case. Let X be a $\mathcal{C}$-block and let P be a minimal (undirected) path in D from one component of X to another. Let Y be a $\mathcal{C}$-block with maximal distance in the structure tree of D and $\mathcal{C}$ to X that contains edges of P. By Lemma 3.1 the block Y has to contain at least two edges and there are two non-adjacent vertices in the same component of Y. This contradicts the fact that these components are complete graphs. Hence each $\mathcal{C}$-block that contains edges has precisely one component which has diameter 1.

For any $\mathcal{C}$-block X, there is a $\mathcal{C}$-separator S with $S \subseteq X$. By Lemma 3.1, S contains no edge and thus precisely one vertex. □

4. C-homogeneous digraphs of Type I

In this section we shall completely classify the countable connected C-homogeneous digraphs of Type I with more than one end and give—apart from the classification of infinite uncountable homogeneous tournaments—a classification of uncountable such digraphs. As a part of the countable classification we apply a theorem of Lachlan [21], see also [2], on countable homogeneous tournaments. Lachlan proved that there are precisely 5 such tournaments. Three of them are infinite, one is the digraph on one vertex with no edge and one is the directed triangle. For the uncountable case there is up to now no such classification of homogeneous tournaments.

To state Lachlan's theorem let us first define the countable tournament $\mathcal{P}$ to be the digraph with the rationals in the interval $[-\pi, \pi]$ as vertex set and direct the edge from x to y if

$$x - y \leq \pi \mod 2\pi$$

and from y to x otherwise. The generic countable tournament is the unique (cp. [5, p. 213] and [10, p. 98]) countable homogeneous tournament that embeds all finite tournaments.

Theorem 4.1. [21, Theorem 3.6] *There are up to isomorphism only* 5 *countable homogeneous tournaments: the trivial tournament on one vertex, the directed triangle, the generic tournament on ω vertices, the tournament that is isomorphic to the rationals with the usual order, and the tournament $\mathcal{P}$ described above.* □

For a homogeneous tournament T let $X_\lambda(T)$ denote the digraph where each vertex is a cut vertex and lies in λ distinct copies of T. Thus the underlying undirected graph is a distance-transitive graph as described in [17, 23, 27].

Theorem 4.2. *Let D be a connected digraph with more than one end. Then D is C-homogeneous of Type I if and only if one of the following statements holds:*

(1) D is a tree with constant in- and out-degree;
(2) D is isomorphic to a $X_\lambda(T^\kappa)$, where κ and λ are cardinals with $\lambda \geq 2$ and κ either 3 *or infinite and T^κ is a homogeneous tournament on κ vertices.*

Proof. Let us first assume that D is a C-homogeneous digraph of Type I. Then the underlying undirected graph is isomorphic to a $X_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$ by Theorem 2.4. If $\kappa = 2$, then D is a tree with constant in- and out-degree, so we may assume $\kappa \geq 3$. As each block is a complete digraph, it is homogeneous and, thus, we conclude from Theorem 4.1 that the cardinal κ has to be either 3 or infinite. This proves the necessity-part of the statement.

Since the digraphs of part (1) are obviously C-homogeneous of Type I, we just have to assume for the remaining part that D is isomorphic to $X_\lambda(T^\kappa)$ for a cardinal $\lambda \geq 2$ and a homogeneous tournament T^κ on κ vertices for a cardinal κ that is either 3 or infinite. Let $\mathcal{C}$ be a basic cut system of D. Let X and Y be two connected induced finite and isomorphic subdigraphs of D. Let φ be the isomorphism from X to Y. If X has no cut vertex, then X lies in a subgraph of D that is a homogeneous tournament and the same is true for Y, so φ extends to an automorphism of D. So let $x \in VX$ be a cut vertex of X. Hence x^φ is a cut vertex of Y. It is straight forward to see that for any $\mathcal{C}$-block B the image of $X \cap B$ in Y is precisely the intersection of Y with a $\mathcal{C}$-block A. Since the $\mathcal{C}$-blocks are all isomorphic homogeneous tournaments, the isomorphism from $X \cap B$ to $Y \cap A$ extends to an isomorphism from B to A. Thus the isomorphism from X to Y easily extends to an automorphism of D. Since the underlying undirected graph is C-homogeneous by Theorem 2.4, D is C-homogeneous of Type I. □

Lachlan's theorem together with Theorem 4.2 enables us to give a complete classification of countable connected C-homogeneous digraphs of Type I and with more than one end:

Corollary 4.3. *Let D be a countable connected digraph with more than one end. Then D is C-homogeneous of Type I if and only if one of the following assertions holds:*

(1) D is a tree with constant countable in- and out-degree;
(2) D is isomorphic to a $X_\lambda(Y)$, where κ is a countable cardinal greater or equal to 2 and Y is one of the four non-trivial homogeneous tournaments of Theorem 4.1. □

5. Reachability and descendant digraphs

In this section we prove that, if a connected C-homogeneous digraph D with more than one end contains no triangles, then D is highly-arc-transitive, every reachability digraph of D is bipartite, and, if furthermore D has infinitely many ends, then the descendants of each vertex in D induce a tree. All these properties were proved to be true in the case that D is locally finite, see [14, Theorem 4.1].

Theorem 5.1. *Let D be a connected C-homogeneous triangle-free digraph with more than one end. Then D is highly-arc-transitive.*

Proof. Let $\mathcal{C}$ be a basic cut system. It suffices to show that each directed path is induced. Suppose this is not the case. Then there is a smallest k such that there is a k-arc $A = x_0 \dots x_k$ that is not induced. Hence there is an edge between x_0 and x_k. Consider a $\mathcal{C}$-separator S that contains x_1. By Lemma 3.4 we have $x_i \notin S$ for all $1 \neq i \leq k$. As $A - x_1$ is connected, all x_i with $i \neq 1$ lie on the same side of S. But that x_0 and x_2 lie on the same side of S is a contradiction to Lemma 3.5. □

In an edge-transitive digraph all *reachability digraphs* $\Delta_e := D[\mathcal{A}(e)]$ with $e \in ED$ are isomorphic, so we may denote a representative of their isomorphism type by $\Delta(D)$. Furthermore Cameron, Praeger and Wormald [1, Proposition 1.1] proved that the reachability relation in such a digraph is either universal or the corresponding reachability digraph is bipartite. We shall now prove that the reachability relation is not universal in our case.

Theorem 5.2. *Let D be a connected C-homogeneous triangle-free digraph with more than one end. Then $\Delta(D)$ is bipartite and, if D is not a tree, then each Δ_e with $e \in ED$ is a component of a $\mathcal{C}$-block. Furthermore, if D has infinitely many ends, then every descendant digraph* $\mathrm{desc}(x)$ *with $x \in VD$ is a tree.*

Proof. Let $\mathcal{C}$ be a basic cut system. We first show that either D is a tree or any Δ_e with $e \in ED$ is a component of a $\mathcal{C}$-block. Let us assume that D is not a tree. Lemma 3.6 immediately implies that, for any $e \in ED$, Δ_e cannot be separated by any $\mathcal{C}$-separator and, thus, each Δ_e lies in a $\mathcal{C}$-block. As there are induced paths of length 2 crossing some $\mathcal{C}$-separator and as D contains no triangle, a component of a $\mathcal{C}$-block X cannot contain more vertices than Δ_e with $e \in E(D[X])$ contains. Thus Δ_e is a component of a $\mathcal{C}$-block.

Suppose that $\Delta(D)$ is not bipartite. Then there is a cycle of odd length in $\Delta(D)$. Thus there has to be a directed path of length at least 2 on that cycle. By Lemma 3.3 this path lies in distinct $\mathcal{C}$-blocks. This is not possible as shown above and thus $\Delta(D)$ has to be bipartite.

Now suppose that there is $x \in VD$ such that $\mathrm{desc}(x)$ contains a cycle. So by transitivity there is a descendant y of x such that there are two x-y-arcs that are apart from x and y totally disjoint. Thus, since D is C-homogeneous, any two out-neighbors of x have a common descendant. Assume that there are two distinct $\mathcal{C}$-separators S, S' such that both $Y := S \setminus S'$ and $Y' := S' \setminus S$ contain an out-neighbor of x. Then there exists a vertex z in D with Y-z- and Y'-z-arcs. But by the Lemmas 3.4 and 3.5 the vertices x and z cannot lie on the same side of S and S', respectively, hence S and S' meet on both sides, a contradiction to the nestedness of $\mathcal{C}$. Thus there is a $\mathcal{C}$-separator S_{+1} that contains the whole out-neighborhood of x. This implies that all descendants of distance k are contained in a common $\mathcal{C}$-separator S_{+k}, since either all distinct k-arcs originated at x are disjoint, and we can apply the same argument as above, or each two of those k-arcs intersect in a vertex x' in D that has the same distance to x on both arcs by Lemma 3.4, and we are home by induction.

With a symmetric argument we get that every k-arc that ends in x has to start in a common $\mathcal{C}$-separator S_{-k}. For a path P in D that starts in x, let $\sigma(P)$ denote the difference of the number of edges in P that are directed away from x (with respect to P) minus the number of edges of the other type. Then one easily checks that the endvertex of P lies in $S_{\sigma(P)}$. Since all $\mathcal{C}$-separators have the same finite order s, say, there can be at most $2s$ rays that are eventually pairwise disjoint. Hence D has finitely many ends, which proves the last statement of the theorem. $\square$

Lemma 5.3. *Let D be a connected C-homogeneous triangle-free digraph with more than one end and let $\mathcal{C}$ be a basic cut system of D. Then for every $\mathcal{C}$-separator S of order at least 2 there is a reachability digraph Δ_e and a $\mathcal{C}$-block K such that $|S \cap \Delta_e| \geq 2$, $\Delta_e \subseteq K$, and $S \subseteq K$.*

Proof. Let S be a $\mathcal{C}$-separator with $|S| \geq 2$. Suppose that there is no reachability digraph Δ_e with $|S \cap \Delta_e| \geq 2$. Let $x, y \in S$ and let P be an x-y-path in a component of $D - S$. Let B be a $\mathcal{C}$-block that contains edges of P and such that $d_{\mathcal{T}}(S, B)$ is maximal with this property. This $\mathcal{C}$-block exists by Lemma 2.3. Then the $\mathcal{C}$-separator $S_B \subseteq B$ that separates S and B in $\mathcal{T}$ has the desired property and thus every $\mathcal{C}$-separator has it, in contradiction to the assumption. $\square$

We have roughly described the global structure of C-homogeneous digraphs. To investigate the local structure of these graphs, we show that the underlying undirected graph of each reachability digraph is a connected C-homogeneous bipartite graph. Such graphs will be described in the next section.

Lemma 5.4. *Let D be a triangle-free connected C-homogeneous digraph with more than one end. Then the underlying undirected graph of $\Delta(D)$ is a connected C-homogeneous bipartite graph.*

Proof. By Theorem 5.2 $\Delta(D)$ is bipartite. The remainder of the proof is the same as the proof of the locally finite case in [14, Lemma 4.3]. $\square$

6. C-homogeneous bipartite graphs

In this chapter we complete the classification of connected C-homogeneous bipartite graphs, which was already done for locally finite graphs, by Gray and Möller [14]. They already mentioned that their work should be extendable with not too much effort—and indeed this section has essentially the same structure.

The proof of the locally finite analog [14, Lemma 4.4] of Lemma 6.1 is self contained and does not use the local finiteness of the graph. Thus we can omit the proof here.

Lemma 6.1. *Let G be a connected C-homogeneous bipartite graph with bipartition $X \cup Y$. If G is not a tree and has at least one vertex with degree greater than 2 then G embeds C_4 as an induced subgraph.* $\square$

Let G be a bipartite graph with bipartition $X \cup Y$. Then for every edge $\{x, y\} \in EG$ we define the neighborhood graph to be:

$$\Omega(x, y) := G[N(x) + N(y) - \{x, y\}]$$

A C-homogeneous graph G is, in particular, edge-transitive, hence there is a unique neighborhood graph $\Omega(G)$.

Lemma 6.2. *Let G be a connected C-homogeneous bipartite graph. Then $\Omega(G)$ is a homogeneous bipartite graph, and therefore is one of: an edgeless bipartite graph, a complete bipartite graph, a complement of a perfect matching, a perfect matching, or a homogeneous generic bipartite graph.*

Proof. If we do not ask $\Omega(G)$ to be finite, the proof of the locally finite analogue [14, Lemma 4.5] carries over. Compared to the locally finite case, we only have to deal with one other 'type' of graph, due to [15, Remark 1.3]. $\square$

Lemma 6.3. *Let G be a C-homogeneous generic bipartite graph. Then G is homogeneous bipartite.*

Proof. Let $VG = A \cup B$ be the natural bipartition of G, let X and Y be two isomorphic induced finite subgraphs of G, and let $\varphi : X \to Y$ be an isomorphism. Let $a \in A \setminus X$ be a vertex adjacent to all the vertices of $X \cap B$ and let $b \in B \setminus X$ be a vertex adjacent to all the vertices of $X \cap A$ and to a. Let a', b' be the corresponding vertices for Y. Since G is bipartite, both $G[X+a+b]$ and $G[Y+a'+b']$ are connected induced subgraphs of G that are isomorphic to each other. Furthermore there is an isomorphism $\psi : G[X + a + b] \to G[Y + a' + b']$ such that the restriction of ψ to X is φ. As there is an automorphism of G that extends ψ, this automorphism also extends φ and G is homogeneous. $\square$

Theorem 6.4. *A connected graph is a C-homogeneous bipartite graph if and only if it belongs to one of the following classes:*

(i) $T_{\kappa,\lambda}$ *for cardinals* κ, λ*;*
(ii) C_{2m} *for* $m \in \mathbb{N}$*;*
(iii) $K_{\kappa,\lambda}$ *for cardinals* κ, λ*;*
(iv) CP_κ *for a cardinal* κ*;*
(v) homogeneous generic bipartite graphs.

Proof. The nontrivial part is to show that this list is complete. So consider an arbitrary connected C-homogeneous bipartite graph G with bipartition $X \cup Y$. If

G is a tree then it is obviously semi-regular and hence a $T_{\kappa,\lambda}$. So suppose G contains a cycle. Then, since G is C-homogeneous, each vertex lies on a cycle. Now G is either a cycle, which is even since G is bipartite, or at least one vertex in G has a degree greater than 2 and G embeds a C_4, due to Lemma 6.1. Thus $\Omega(G)$ contains at least one edge and by Lemma 6.2 we have to consider the following cases:

Case 1: *$\Omega(G)$ is complete bipartite.* Suppose that there is an induced path $P = uxyv$ in G. Then $\Omega(x, y)$ gives rise to an edge between u and v, a contradiction. Hence G is complete bipartite.

Case 2: *$\Omega(G)$ is the complement of a perfect matching.* Consider $x \in X$ and $y \in Y$ such that $\{x, y\}$ is an edge of G. Since $\Omega(x, y)$ is the complement of a perfect matching and G is not a cycle, there is an index set $I \supseteq \{1, 2\}$ such that $N(x) = \{y\} \cup \{y_i | i \in I\}$, $N(y) = \{x\} \cup \{x_i | i \in I\}$ and for $i \in I$ the vertex x_i is nonadjacent to y_i but adjacent to all y_j with $j \in I \setminus \{i\}$. Since $\Omega(x, y_1)$ is also the complement of a perfect matching there is a unique vertex $a \in N(y_1) \setminus N(y)$. Since x_i with $i \neq 1$ is adjacent to y_1, it is contained in $\Omega(x, y_1)$ and therefore y_i is adjacent to a. Thus for all $i \in I$ we have $N(y_i) = N(y) - x_i + a$. Now by symmetry there is a unique vertex b adjacent to all x_i with $i \in I$ but non-adjacent to x and for all $i \in I$ there is $N(x_i) = N(x) - y_i + b$. If we look at $\Omega(x_1, y_2)$ we have $x, a \in N(y_2)$ and $y, b \in N(x_1)$ which implies $\{a, b\} \in EG$ and hence $N(a) = N(x) - y + b$ and $N(b) = N(y) - x + a$. Because G is connected we have $X = N(y) + a$ and $Y = N(x) + b$ which means that G is itself the complement of a perfect matching.

Case 3: *$\Omega(G)$ is a perfect matching.* For the same reason as for locally finite graphs this case cannot occur (cp. [14, Theorem 4.6]).

Case 4: *$\Omega(G)$ is homogeneous generic bipartite.* Let U and W be two disjoint finite subsets of X (of Y). Since G is connected there is a finite connected induced subgraph $H \subset G$ that contains both U and W. By genericity, we find an isomorphic copy H_Ω of H in $\Omega(G)$. Because G is C-homogeneous there is an automorphism φ of G with $H_\Omega^\varphi = H$. Now there is a vertex v in Y (in X) that is adjacent to all vertices in $U^{\varphi^{-1}}$ and non-adjacent to all vertices in $W^{\varphi^{-1}}$. Hence v^φ is adjacent to all vertices in U and none in W which implies that G is generic bipartite. Furthermore G is homogeneous bipartite by Lemma 6.3, as it is C-homogeneous. □

7. C-homogeneous digraphs of Type II

It is well known that a transitive locally finite graph either contains one, two, or infinitely many ends. For arbitrary transitive infinite graphs, this was proved by Diestel, Jung and Möller [6]. Since the underlying undirected graph of a transitive digraph is also transitive, the same holds for infinite transitive digraphs. The two-ended C-homogeneous digraphs have a very simple structure which we could easily derive from the results of the previous sections. But since two-ended connected transitive digraphs are locally finite [6, Theorem 7] we refer to Gray and Möller [14, Theorem 6.2] instead. Consequently, this section only deals with digraphs that have infinitely many ends.

As a first result we prove that no connected C-homogeneous digraph of Type II with more than one end contains any triangle.

Lemma 7.1. *Let D be a connected C-homogeneous digraph of Type II with more than one end. Then D contains no triangle.*

Proof. Let $\mathcal{C}$ be a basic cut system. Suppose that D contains a triangle. By Lemma 3.9 every $\mathcal{C}$-block of D that contains an edge is a tournament and D has connectivity 1. So each $\mathcal{C}$-block contains edges, and the $\mathcal{C}$-blocks have to be homogeneous tournaments. Thus D is of Type I in contradiction to the assumption. □

In preparation of the next lemma we introduce the following well-known construction: Given an edge-transitive bipartite digraph Δ with bipartition $A \cup B$ such that every edge is directed from A to B we define $DL(\Delta)$ to be the unique connected digraph such that each vertex separates the digraph, lies in exactly two copies of Δ, and has both in- and out-neighbors (cp. [1, 14]).

Lemma 7.2. *Let D be a connected C-homogeneous digraph of Type II with more than one end. If D has connectivity* 1*, then D is isomorphic to $DL(\Delta(D))$.*

Proof. This is direct consequence of Lemma 7.1 and Lemma 3.6. □

In the next two theorems we prove that in the cases that the reachability digraph is either isomorphic to CP_κ or to $K_{2,2}$ the digraph has connectivity at most 2. Thus, in these case it remains to determine those with connectivity exactly 2.

We first define a class of digraphs with connectivity 2 and reachability digraph CP_κ. Given $2 \leq m \in \mathbb{N}$ and a cardinal $\kappa \geq 3$ consider the tree $T_{\kappa,m}$ and let $U \cup W$ be its natural bipartition such that the vertices in U have degree m. Now subdivide each edge once and endow the neighborhood of each $u \in U$ with a cyclic order. Then for each new vertex y let u_y be its unique neighbor in U and denote by $\sigma(y)$ the successor of y in $N(u_y)$. Then for each $w \in W$ and each $x \in N(w)$ we add an edge directed from x to all $\sigma(y)$ with $y \in N(w) - x$. Finally we delete the vertices of the $T_{\kappa,m}$ together with their incident edges to obtain the digraph $M(\kappa, m)$. The locally finite subclass of this class of digraphs coincides with those digraphs $M(k, n)$ for $k, n \in \mathbb{N}$ that are described in [14, Section 5]. In Figure 1 the digraph $M(3, 3)$ is shown: once with its construction tree and once with its set of $\mathcal{C}$-separators.

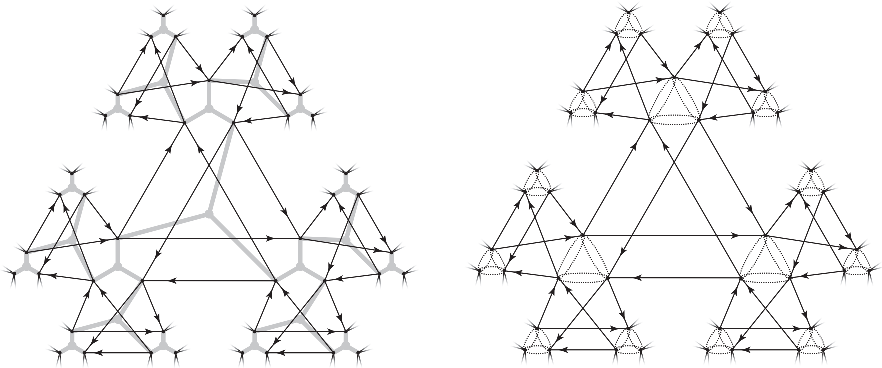

FIGURE 1. The digraph $M(3, 3)$

Theorem 7.3. *Let D be a connected C-homogeneous digraph of Type II with infinitely many ends and with $\Delta(D) \cong CP_\kappa$ for a cardinal $\kappa \geq 3$. If D has connectivity more than one, then D is isomorphic to $M(\kappa, m)$ for an $m \in \mathbb{N}$ with $m \geq 2$.*

Proof. By Lemma 7.1 the digraph D contains no triangle. Let $\mathcal{C}$ be a basic cut system and let $\mathcal{T}$ be the structure tree of D and $\mathcal{C}$. Let S^0 be a $\mathcal{C}$-separator, let $X^0 = \Delta_e$ for an $e \in ED$ such that $|S^0 \cap X^0| \geq 2$, and let K^0 be a $\mathcal{C}$-block with $S^0 \subseteq K^0$ and $\Delta_e \subseteq K^0$, which all exist by Lemma 5.3. Let $A \cup B$ be the natural bipartition of X^0 such that its edges are directed from A to B. For each $a \in A$ let us denote with b_a the unique vertex in B such that ab_a is not an edge in X^0. By symmetry we may assume that $A \cap S^0 \neq \emptyset$, so let $a \in A \cap S^0$.

First we shall show that $X^0 \cap S^0 = \{a, b_a\}$. Since S^0 contains no edges by Lemma 3.1, it suffices to show that $A \cap S^0 = \{a\}$. So let us suppose that there is another vertex $a' \neq a$ in $A \cap S^0$. Since any two vertices in A have a common successor in B, we have $A \subseteq S^0$ by C-homogeneity. Let $a' \in A$ be distinct from a and P an induced a-a'-path whose interior is contained in $D - K^0$. Denote the unique neighbor of a on P by c. Taking into account that X^0 is a CP_κ, there is a common successor for each pair of A-vertices; let b be such a common successor of a and a'. Since S^0 separates both, b and b_a, from the interior of P, the paths cPb and cPb_a are isomorphic and, by C-homogeneity, we can map cPb onto cPb_a by an automorphism φ of D. Then a^φ is a successor of c that sends an edge to b_a. Hence a^φ lies in A and is distinct from a, contradicting the fact that $\operatorname{desc}(c)$ is a tree by Theorem 5.2. Thus we know that $X^0 \cap S^0 = \{a, b_a\}$ for a vertex $a \in A$.

For the remainder let $X^0 \cap S^0 = \{x_0, x_1\}$. Because each vertex clearly lies in exactly two distinct reachability digraphs, there is a unique reachability digraph $X^1 \neq X^0$ that contains x_1. If $x_0 \in X^1$ then it is straight forward to see that $D \cong M(\kappa, 2)$. So assume $x_0 \notin X^1$ and let ψ be an automorphism of D mapping X^0 onto X^1 and x_0 to x_1. Let S^1, K^1 denote the image under ψ of S^0, K^0, respectively, and let $x_2 = x_1^\psi$. Since $\mathcal{C}$ is basic, there is an induced x_0-x_1-path P the interior of which lies in $D - K^0$. We shall show that P contains x_2.

Suppose that P does not contain x_2 and has minimal length with this property. Let u be the neighbor of x_1 on P, which clearly lies in X^1, and let v be a neighbor of u in X^1 distinct from x_1. If v does not lie on P, then Puv is a path of the same length as P which is induced by the minimality of P and Theorem 5.2, contradicting the fact that x_0 and v cannot lie in a common reachability digraph. On the other hand, if v lies on P then consider a neighbor w of x_2 in X^1 distinct from v. Remark that since X^1 is a CP_κ there is an edge between v and x_2. Thus by the choice of P the path Pvx_2w is induced and of the same length as P, which is impossible since x_0 and w do not belong to a common reachability digraph. Hence P contains x_2.

We have just proved that $\{x_1, x_2\}$ separates x_0 from any neighbor of x_1 in X^1. Hence all $\mathcal{C}$-separators have order 2 and thus the blocks which contain edges consist each of a single reachability digraph. Now we repeat the previous construction to continue the sequences $(X^i)_{i\in\mathbb{N}}$, $(S^i)_{i\in\mathbb{N}}$, $(K^i)_{i\in\mathbb{N}}$ and $(x_i)_{i\in\mathbb{N}}$, respectively. Since Px_2 is an induced x_0-x_2-path the interior of which lies in $D - K^1$, we can apply the same argument as above to assure that P contains x_3. Hence by induction we have $x_i \in P$ for all $i \in \mathbb{N}$, and since P is finite there is an $m \in \mathbb{N}$ such that $x_m = x_0$. Furthermore we have $X^m = X^0$, $S^m = S^0$ and $K^m = K^0$. One can verify that $\{x_0, x_1, \ldots, x_{m-1}\}$ forms a maximal $\mathcal{C}$-inseparable set—a $\mathcal{C}$-block—which means that D is isomorphic to $M(\kappa, m)$. $\square$

In preparation of the next theorem we define a class of digraphs with connectivity 2 and reachability digraph $K_{2,2}$. For $2 \leq m \in \mathbb{N}$ consider the tree $T_{2,2m}$ and let $U \cup W$ be its natural bipartition such that the vertices in U have degree $2m$. Now subdivide every edge once and enumerate the neighborhood of each $u \in U$ from 1 to $2m$ in a such way that the two neighbors of each $w \in W$ have distinct parity. For every new vertex x let u_x be its unique neighbor in U and define $\sigma(x)$ to be the successor of x in the cyclic order of $N(u_x)$. For any $w \in W$ we have a neighbor a_w with even index, and a neighbor b_w with odd index. Then we add edges from both a_w and $\sigma(a_w)$ to both b_w and $\sigma(b_w)$. Finally we delete the vertices of the $T_{2,2m}$ together with their incident edges. With $M'(2m)$ we denote the resulting digraph. Figure 2 shows the digraph $M'(6)$: on the left side with its construction tree and on the right side with the separators of the two possible basic cut systems.

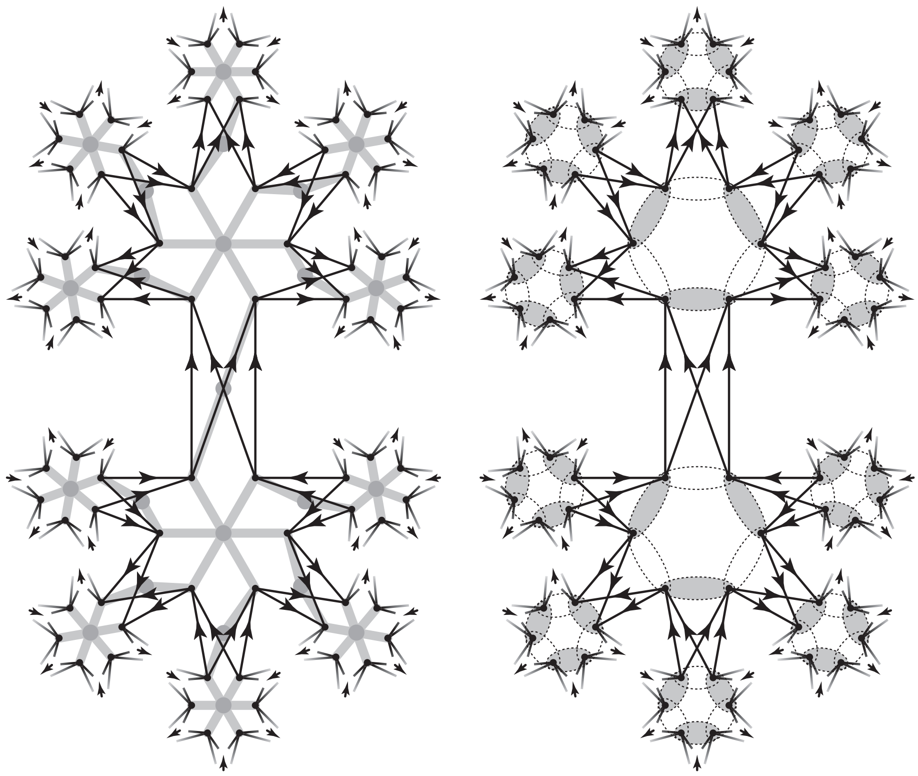

FIGURE 2. The digraph $M'(6)$

Theorem 7.4. *Let D be a connected C-homogeneous digraph of Type II with infinitely many ends and with $\Delta(D) \cong K_{2,2}$. If D has connectivity more than one, then D is isomorphic to $M'(2m)$ for $2 \leq m \in \mathbb{N}$.*

Proof. Lemma 7.1 implies that D contains no triangle. Let $\mathcal{C}$ be a basic cut system of D. Let S^0 be a $\mathcal{C}$-separator and let $X^0 = \Delta_e$ for an $e \in ED$ such that $|S^0 \cap X^0| \geq 2$. Such an X^0 exists by Lemma 5.3. As $\Delta(D) \cong K_{2,2}$ and as no $\mathcal{C}$-separator contains any edge by Lemma 3.1, there is $|S^0 \cap X^0| = 2$. So let x_0, x_1 be the two vertices in $X^0 \cap S^0$. Let X^1 be the other reachability digraph that contains x_1 and let x_2 be the unique vertex in X^1 that is not adjacent to x_1. Let ψ be an automorphism of D that maps X^0 onto X^1 and let S^1 be the image of S^0 under ψ.

With the same technique as in the previous proof, we can verify that $\{x_1, x_2\}$ separates D and so $S^0 = \{x_0, x_1\}$. We can continue the sequences $(x_i)_{i\in\mathbb{N}}$ and $(S^i)_{i\in\mathbb{N}}$ so that $S^1 = \{x_1, x_2\}$ and $S^i = \{x_i, x_{i+1}\}$, and there is an $n \in \mathbb{N}$ such that $x_n = x_0$. Since D has infinitely many ends we have $n \geq 3$, and as $x_i \in S^i$ only holds for all even integers i we have $n = 2m$ with $m \geq 2$. Now analog as in the proof of Theorem 7.3 $\bigcup_i S^i$ forms a $\mathcal{C}$-block that contains no edges. Hence there are precisely two $\mathrm{Aut}(D)$-orbits on the $\mathcal{C}$-blocks and D is isomorphic to $M'(2m)$. $\square$

If we assume $\Delta(D)$ to be one of the other possibilities as described in Theorem 6.4, then the C-homogeneous digraphs have—in contrast to the other two cases—connectivity 1.

Lemma 7.5. *Let D be a connected C-homogeneous digraph of Type II with infinitely many ends and such that $\Delta(D)$ is isomorphic to a $T_{\kappa,\lambda}$ for cardinals κ, λ, a C_{2m} with $4 \leq m \in \mathbb{N}$, a $K_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$, or an infinite homogeneous generic bipartite digraph. Then D has connectivity 1.*

Proof. Since D is of Type II, it contains no triangle by Lemma 7.1. Let us suppose that D has connectivity at least 2 and let $\mathcal{C}$ be a basic cut system of D. Let S be a $\mathcal{C}$-separator and let X be a reachability digraph with $|S \cap X| \geq 2$ as in Lemma 5.3. We investigate the given reachability digraphs one by one and get in each case a contradiction and, thereby, we get a contradiction in general to the assumption that D has connectivity at least 2. So let us assume that $X \cong T_{\kappa,\lambda}$ for cardinals κ, λ. By Lemma 3.6 we know that $\kappa, \lambda \geq 2$, as D is not a tree. Let $x, y \in S \cap X$ such that $d_X(x, y)$ is maximal. Such vertices exist as S is finite. Let e_1 be the first edge on the path from x to y in X and let e_2 be another edge incident with x. There is an $\alpha \in \mathrm{Aut}(D)$ with $e_1^\alpha = e_2$. But then y^α lies in a common separator with x, as $x^\alpha = x$. By Corollary 3.8 the separator S^α has to be the same as S. But this contradicts the maximality of $d_X(x, y)$, as $d_X(y^\alpha, y) > d_X(x, y)$.

Let us now assume that $X \cong C_{2m}$ for a $4 \leq m \in \mathbb{N}$ and let x, y be distinct vertices in $S \cap X$. Then there is an induced path P from x to y that lies apart from x and y in a component of $D - S$ that intersects trivially with X. We first show that we may assume that $d_X(x, y) \geq 4$. Let e_1, e_2 be the two edges in $D[X]$ that are incident with x. If $d_X(x, y) = k \leq 3$, then let $\alpha \in \mathrm{Aut}(D)$ with $e_1^\alpha = e_2$. Then there is $d_X(y, y^\alpha) = 2k$, as $m \geq 4$. Thus we have shown that there are $x, y \in S \cap X$ with $d_X(x, y) \geq 4$. Let s_1 and s_2 be the vertices in X that are adjacent to y and let t be a vertex in X that is adjacent to x. Since $d_X(x, y) \geq 4$, the graphs $txPys_i$ for $i = 1, 2$ are induced paths. Hence there is an automorphism α of D that maps $txPys_1$ onto $txPys_2$ and thus $d_X(s_1, x) = d_X(s_2, x)$ and $d_X(s_1, t) = d_X(s_2, t)$, a contradiction as X is a cycle.

For the next case let us assume that $X \cong K_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$. Let $A \cup B$ be the natural bipartition of X. Since $|S \cap X| \geq 2$, the vertices in $S \cap X$ lie in the same partition set, A say. By the C-homogeneity it is an immediate consequence that $A \subseteq S$. As the $\mathcal{C}$-separators have minimal cardinality with respect to separating ends, there is $|A| \leq |B|$. If there is a $\mathcal{C}$-separator S' with $|S' \cap B| \geq 2$, then $B \subseteq S'$. If in addition the intersection of B with another reachability digraph distinct from X is B, then it is a direct consequence that $\kappa = \lambda$ is finite and that D has two ends. Thus there are two distinct reachability digraphs X_1, X_2 that intersect with B non-trivially and that are distinct from X. Let A_1, B_1, A_2, B_2 be the natural bipartitions of X_1, X_2, respectively. Let P be an induced path from

$A_1 \cap B$ to $A_2 \cap B$ in a component of $D - S'$ that intersects non-trivially with X. Let a be the vertex on P that is adjacent to the vertex in $P \cap A_1$ and let b be a vertex in $B \cap A_1$ not on P. Then there is an automorphism α of D that maps P onto baP. But this contradicts the fact that the endvertices of P lie both in B but the endvertices of baP do not lie in in any common reachability digraph as $|A_1 \cap B| = 1$. Thus we conclude that $|B \cap S'| = 1$. So let $x, y, z \in B$ be three distinct vertices. There is a shortest induced path P from x to y in that component of $D - S$ that contains B. Let $a \in A$ and let b be the vertex on P with distance 2 to y. Then there is an automorphism α of D that maps $zaxPb$ onto $yaxPb$. Thus we conclude that $d(b, z) = 2$. But then z has to have incident edges that are directed both towards or both from distinct $\mathcal{C}$-blocks. This contradicts Lemma 3.6.

Let us finally assume that X is isomorphic to an infinite homogeneous generic bipartite digraph. Let again $A \cup B$ be the natural bipartition of X. Since X is homogeneous, all vertices in the same set A or B have distance 2 to each other. We conclude that $|S \cap A| \geq 2$ immediately implies $A \subseteq S$ which contradicts the finiteness of S. Conversely we also know $|B \cap S| \leq 1$. Since D has connectivity at least 2, there is $|A \cap S| = 1 = |B \cap S|$. Let a, b be the vertices in $A \cap S, B \cap S$, respectively, and let $ab'a'b$ be a path of length 3 from a to b. This path exists because each two vertices in the same set A or B have distance 2 to each other as before. Since there are infinitely many vertices in A that are adjacent to b' but not to b, all these vertices have to lie in S, a contradiction. Thus we conclude that D has connectivity 1. $\square$

Let us summarize the conclusions of this section in the following theorem. In its proof we will finally prove that all the candidates for C-homogeneous digraphs are really C-homogeneous.

Theorem 7.6. *Let D be a connected digraph of Type II with infinitely many ends. Then D is C-homogeneous if and only if one of the following holds:*

(1) $\Delta(D) \cong CP_\kappa$ for a cardinal $\kappa \geq 3$ and $D \cong DL(\Delta(D))$.
(2) $\Delta(D) \cong C_{2m}$ for $2 \leq m \in \mathbb{N}$ and $D \cong DL(\Delta(D))$.
(3) $\Delta(D) \cong K_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$ and $D \cong DL(\Delta(D))$.
(4) $\Delta(D)$ is isomorphic to an infinite homogeneous generic bipartite digraph and $D \cong DL(\Delta(D))$.
(5) $\Delta(D) = CP_\kappa$ and $D \cong M(\kappa, m)$ for a cardinal $\kappa \geq 3$ and $2 \leq m \in \mathbb{N}$.
(6) $\Delta(D) = K_{2,2}$ and $D \cong M'(2m)$ for $2 \leq m \in \mathbb{N}$.

Proof. By the Lemmas 7.1, 7.2, and 7.5 and by the Theorems 7.3 and 7.4, it remains to show that the described digraphs are indeed C-homogeneous. Remark that the underlying undirected graph of $DL(T_{\kappa,\lambda})$ is a regular tree and thus $DL(T_{\kappa,\lambda})$ is not of Type II. It is straight forward to see that the graphs of the part (1)-(4) are C-homogeneous. So let $D \cong M(\kappa, m)$ for an $m \in \mathbb{N}$ with $m \geq 2$ and a cardinal κ. Let $\mathcal{C}$ be a basic cut system of D. Let A and B be two connected induced finite and isomorphic subdigraphs of D and let φ be an isomorphism from A to B. Let us first consider the case that A contains no 2-arc. Then both A and B lie in a reachability digraph, each. Without loss of generality we may assume that they lie in the same reachability digraph Δ of D. But, as the reachability-digraphs are obviously C-homogeneous, it is straight forward to see that the isomorphism φ from A to B first extends to an automorphism of Δ and then also to an automorphism of D. So let us assume that A contains a 2-arc. Let us consider the case that A

is a k-arc for some $k \geq 2$. Let A_1, A_2 be two induced subdigraphs of A that have one common vertex, are both connected, and whose union is A. Then both are shorter arcs and, by induction, we can extend both restrictions, $\varphi|_{A_1}$ and $\varphi|_{A_2}$, to automorphisms ψ_1, ψ_2 of D, respectively. Let S be a $\mathcal{C}$-separator that contains the common vertex of A_1 and A_2. There are two possibilities for S if $m \geq 3$, and one possibility if $m = 2$. If $m = 2$, then it is an immediate consequence that $S^{\psi_1} = S^{\psi_2}$ and that we can combine the two automorphisms to one that extends φ by setting $\varphi|_{K_i} = \psi_i|_{K_i}$, where K_i is the component of $D - S$ that contains vertices of A_i, and $\varphi|_S = \psi_1|_S$. So we assume that $m \geq 3$. Then S lies in a common $\mathcal{C}$-block either with an edge of A_1 or with an edge of A_2, but not both. Since φ preserves this distinction we have $S^{\psi_1} = S^{\psi_2}$. In the same way as above, we can combine appropriate restrictions of ψ_1 and ψ_2 to an automorphism of D that extends φ.

Now let us assume that A is no k-arc. Then there is a $\mathcal{C}$-block X that contains two edges of A that have a common vertex. Let us first assume that X contains three edges of A. Then, since $\Delta \cong CP_\kappa$, we know that $X \cap A$ is connected. Thus, $(X \cap A)^\varphi$ lies in a $\mathcal{C}$-block Y of B and we have $(X \cap A)^\varphi = B \cap Y$. We have already shown that we can extend $\varphi|_{A \cap X}$ to an automorphism ψ_X of D. If each component of $D - X$ contains at most one component of A, then we have the extensions of the restriction of φ to these components and we can construct, as in the case of k-arcs, an automorphism of D. So we assume that there is at least one component C of $D - X$ such that, for the $\mathcal{C}$-separator $S \subseteq X$ that separates X and C, the digraph $A' = A \cap (C \cup S)$ contains at least two components. As the $\mathcal{C}$-separators have cardinality 2, A' consists of precisely two components. Let $Z \neq X$ be the second $\mathcal{C}$-block that contains S. If Z contains edges, that means $m = 2$, then $A \cap Z$ consists of precisely two edges that have their other incident vertices again in a common separator. Since the same must be true for $Z^{\psi_X} \cap B$, we may assume inductively that we have extended $\varphi|_{A \cap X}$ so that ψ_X coincides with $\varphi|_{Z \cap A}$ on $Z \cap A$. Thus, we can consider the case that Z does not contain any edge. There is an enumeration $z_1, \ldots, z_m$ of the vertices of Z such that $\{z_m, z_1\}$ and for all $i \leq m$ also $\{z_i, z_{i+1}\}$ are all the $\mathcal{C}$-separators in X. We may assume that $S = \{z_1, z_m\}$. Let C_i be the subdigraph induced by z_i, z_{i+1} and that component of $D - \{z_i, z_{i+1}\}$ that contains no other z_j. If $C_i \cap A$ consists of one component and contains z_i and z_{i+1}, then we can extend the restriction of φ to that component to an automorphism ψ_i of D and we may suppose that we have chosen ψ_X so that they are equal on C_i. If there is one C_i that has at least two components of $A \cap C_i$, then it is unique and we can suppose that $\psi_X|_{C_i} = \psi_i|_{C_i}$ on all C_i such that $A \cap C_i$ is connected. By induction, we can assume that the same holds also for a component C_i such that $A \cap C_i$ is not connected. So the only remaining situation in this case is if $C_i \cap A$ is connected but contains only one of the vertices z_i, z_{i+1}. But in this case we know that this situation occurs in at most one other C_j with $i \neq j$. Then $\varphi|_{A \cap C_k}$ with $k \in \{i, j\}$ extends to an automorphism ψ_k of D by induction. Because these two automorphisms exist, we know that $S_i^{\psi_k}$ contains only one vertex of B, and hence we can assume that ψ_X and ψ_k coincide on C_k. Thus, if we extend this to all the components of $D - X$, we know that ψ_X extends φ.

The final case that remains is when the block X contains only two edges. Then it might be the case that $X \cap A$ is not connected. If it is not connected, then there has to be a $\mathcal{C}$-block that contains at least three edges, so we assume that $X \cap A$ is connected. If, for the $\mathcal{C}$-block Y that contains $(X \cap A)^\varphi$, we have that $Y \cap B$ is connected, then we can construct an automorphism that extends φ as in the case where X contained three edges of A. On the other hand, if $Y \cap B$ is not connected, there has to be a $\mathcal{C}$-block that contains three edges of B, and the same must be true for a $\mathcal{C}$-block and A. Since we know that in this case there is an automorphism of D that extends φ, we have proved that $M(\kappa, m)$ is C-homogeneous.

In the case that $D \cong M'(2m)$ for an $m \in \mathbb{N}$ the arguments used are analog ones as in the case $D \cong M(\kappa, m)$ and therefore we omit that proof here. $\square$

It is well known (see [1]) that line digraphs of highly-arc-transitive digraphs are again highly-arc-transitive. In some cases also C-homogeneity is preserved under taking the line digraph: Gray and Möller [14] stated that the line digraph of a $DL(C_{2m})$ is C-homogeneous. In terms of our classification:

Remark 7.7. For every $m \in \mathbb{N}$ we have $L(DL(C_{2m})) \cong M'(2m)$.

Proof. Consider the digraph $D = DL(C_{2m})$ for an $m \in \mathbb{N}$. By construction the deletion of each single vertex v of D splits the digraph into two components such that v has two out-neighbors in the one and two in-neighbors in the other component. Thus the four edges that are incident with v form a $K_{2,2}$ in $L(D)$ whose independent vertex sets separate $L(D)$. Furthermore the edges of each C_{2m} in D form an independent set in $L(D)$ so that any two adjacent edges lie in a common $K_{2,2}$ in $L(D)$. One can easily verify that this digraph is indeed isomorphic to $M'(2m)$. $\square$

Interestingly, our classification implies that C-homogeneity is not generally preserved under taking line digraphs. Indeed, for all $m \in \mathbb{N}$ the line digraph of $M'(2m)$ is triangle-free, has infinitely many ends, and has connectivity 4, hence it is not of Type II. Thus, by Theorem 7.6, we know that $L(M'(2m)) \cong L(L(DL(C_{2m})))$ is not C-homogeneous. This had remained an open question in [14].

8. Final remarks

Let us take a closer look at two specific kinds of digraphs that occur as 'building blocks' in our classification. The first kind are the homogeneous tournaments, which feature in our classification of the connected C-homogeneous digraphs of Type I. While Lachlan [21] classified the countable homogeneous tournaments, no characterization is known for the uncountable ones. The second kind of building blocks that deserve a closer look are the generic homogeneous bipartite graphs, which occur in the classification of the connected C-homogeneous digraphs of Type II. There is exactly one countable such digraph [15, Fact 1.2], but it is shown in [15] that the number of isomorphism types of homogeneous generic bipartite graphs with $\aleph_0$ vertices on the one side of the bipartition and $2^{\aleph_0}$ vertices on the other side is independent of ZFC. Hence, classifying the uncountable generic homogeneous bipartite graphs remains an undecidable problem.

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