ASYMPTOTIC DIMENSION OF MULTI-ENDED QUASI-TRANSITIVE GRAPHS

MATTHIAS HAMANN

Abstract. We prove the existence of an upper bound on the asymptotic dimension of tree amalgamations of locally finite quasi-transitive connected graphs. This generalises a result of Dranishnikov for free products with amalgamation and a result of Tselekidis for HNN-extensions of groups to tree amalgamations of graphs. As a corollary, we obtain an upper bound on the asymptotic dimension of a multi-ended quasi-transitive locally finite graph based on any of their factorisations.

1. Introduction

Asymptotic dimension of metric spaces was introduced by Gromov [11]. It is a quasi-isometry invariant and hence an invariant of finitely generated groups. Thus, it is interesting to see how the asymptotic dimension behaves with respect to free products with amalgamations and HNN-extensions. Bell and Dranishnikov [2] proved for these products that the asymptotic dimension is finite provided that the asymptotic dimension of the factors is finite. The best upper bound for the free product with amalgamation $A *_C B$ of finitely generated groups A and B is given by Dranishnikov [7]:

$$\operatorname{asdim}(A *_C B) \leq \max\{\operatorname{asdim}(A), \operatorname{asdim}(B), \operatorname{asdim}(C) + 1\}.$$

Recently, Tselekidis [13] obtained a similar upper bound for the HNN-extension $A *_C$ of a finitely generated group A:

$$\operatorname{asdim}(A *_C) \leq \max\{\operatorname{asdim}(A), \operatorname{asdim}(C) + 1\}.$$

We generalise these two results in our main theorem. For that, a tree amalgamation can be thought of as a graph theoretic analogue of free products with amalgamation and HNN-extensions of groups. We refer to Section 2 for the precise definitions.

Theorem 1.1. *Let G_1, G_2 be locally finite connected graphs and let Γ_1, Γ_2 be groups acting quasi-transitively on G_1, G_2, respectively. Let $G = G_1 * G_2$ be the tree amalgamation of bounded identification respecting the actions of Γ_1 and Γ_2. Then*

$$\operatorname{asdim}(G) \leq \max\{\operatorname{asdim}(G_1), \operatorname{asdim}(G_2), \operatorname{asdim}(C) + 1\},$$

where C is an arbitrary adhesion set of $G_1 * G_2$.

We note that neither in the results of Dranishnikov and of Tselekidis the group C has to be finite nor in our result the adhesion set C has to be finite.

Supported by the Heisenberg-Programme of the Deutsche Forschungsgemeinschaft (DFG Grant HA 8257/1-1).

The bound in Theorem 1.1 is sharp as some tree amalgamations of finite graphs are quasi-isometric to trees by [12, Theorem 7.4] and thus have asymptotic dimension 1 but finite graphs have asymptotic dimension 0.

Let G be a locally finite quasi-transitive connected graph with more than one end. A tuple $(G_1, \ldots, G_n)$ of locally finite quasi-transitive connected graphs is a *factorisation* of G if G is obtained by iterated non-trivial tree amalgamations of all G_i of finite adhesion and bounded identification respecting the group actions and distinguishing ends.

By [12], all multi-ended quasi-transitive locally finite connected graphs have a *non-trivial* factorisation, i. e. a factorisation with more than one factor. But there are examples of Dunwoody [8, 9] that show that not every such graph has a *terminal* factorisation, i. e. a factorisation all of whose factors have at most one end.

The following result is an immediate corollary of Theorem 1.1.

Theorem 1.2. *Let $(G_1, \ldots, G_n)$ be a factorisation of a locally finite quasi-transitive connected graph G. Then* $\operatorname{asdim}(G) \leq \max\{1, \operatorname{asdim}(G_i) \mid 1 \leq i \leq n\}$. □

Aside from our motivation from the results of Bell and Dranishnikov and of Tselekidis, asymptotic dimensions got more attention in graph theory in general. E. g. this notion has been studied in connection with minor-closed families of graphs, see the results of Fujiwara and Papasoglu [10] for planar graphs and the subsequent results by Bonamy et al. [5, 6] on general proper minor-closed families of graphs. For graphs, the most general result among those three mentioned above, is the one in [5], which says that proper minor-closed families of graphs have asymptotic dimension at most 2.

2. Tree amalgamations

In this section, we will define all notations and state all results that we need in the context of tree amalgamations.

A tree is (p_1, p_2)-*semiregular* if for the canonical bipartition $\{V_1, V_2\}$ of its vertex set all vertices in V_i have degree p_i for $i = 1, 2$.

Let I_1 and I_2 be disjoint sets and let G_1 and G_2 be graphs. Let $(S^i_k)_{k \in I_i}$ be families of subsets of $V(G_i)$ for $i = 1, 2$ such that all sets S^i_k have the same cardinality. For all $k \in I_1$ and $\ell \in I_2$, let $\phi_{k\ell}\colon S^1_k \to S^2_\ell$ be a bijection. Set $\phi_{\ell k} = \phi^{-1}_{k\ell}$. The maps $\phi_{k\ell}$ and $\phi_{\ell k}$ are the *bonding maps*.

Let T be a $(|I_1|, |I_2|)$-*semiregular* tree with canonical bipartition $\{V_1, V_2\}$ such that the vertices in V_i have degree $|I_i|$. Let $D(T)$ be the set obtained from the edge set of T by replacing every $xy \in E(T)$ by two directed edges $\vec{xy}$ and $\vec{yx}$. For a directed edge $\vec{e} = \vec{xy} \in D(T)$, we denote by $\overleftarrow{e}$ the edge with the reversed orientation, i. e. $\overleftarrow{e} = \vec{yx}$. Let $f\colon D(T) \to I_1 \cup I_2$ be a labelling such that for every $i \in \{1, 2\}$ and every $v \in V_i$ each $k \in I_i$ occurs in the set of labels of edges starting at v precisely once.

For every $i \in \{1, 2\}$ and every $v \in V_i$, let G_v be a copy of G_i. Denote by S^v_k the corresponding copies of S^i_k in $V(G_v)$. Let $G_1 + G_2$ be the graph obtained from the disjoint union of all graphs G_v with $v \in V(T)$ by adding new edges as follows: for every edge $\vec{e} = \vec{uv}$ with $f(\vec{e}) = k$ and $f(\overleftarrow{e}) = \ell$ we add an edge between each $x \in S^u_k$ and $\phi_{k\ell}(x) \in S^v_\ell$. Note that this does not depend on the orientation we pick for e, since $\phi_{\ell k} = \phi^{-1}_{k\ell}$. If we contract all edges of $G_1 + G_2$ that lie outside of the copies G_v we obtain the *tree amalgamation* $G_1 *_T G_2$ of the graphs G_1 and G_2

over the *connecting tree* T. If T is clear from the context, we just write $G_1 * G_2$. Let $\pi\colon V(G_1 + G_2) \to V(G_1 * G_2)$ be the canonical map that maps each vertex of $G_1 + G_2$ to the vertex obtained from it after all the contractions.

The sets S_k^i and their images under π in $G_1 * G_2$ are the *adhesion sets* of the tree amalgamation. A tree amalgamation has *finite adhesion* if all its adhesion sets are finite. We call a tree amalgamation $G_1 *_T G_2$ *trivial* if for some $v \in V(T)$ the restriction of π to $V(G_v)$ is a bijection.

For a vertex $x \in V(G_1 *_T G_2)$ let T_x be the maximal subtree of T such that every node of T_x contains a vertex y with $\pi(y) = x$. The *identification size* of x is the cardinality of $V(T_x)$. The tree amalgamation has *bounded identification* if the identification sizes of its vertices are bounded.

A tree amalgamation *distinguishes ends* if there is some adhesion set $S_k^v = S_\ell^u$ for adjacent vertices u, v of T such that for every component C of $T - uv$ the graph induced by $\bigcup_{w\in C} G_i^w$ contains a one-way infinite path.

Let G and H be graphs. A map $f\colon V(G) \to V(H)$ is a *quasi-isometry* if there are constants $\gamma \geq 1$ and $c \geq 0$ such that

$$\gamma^{-1} d_G(x,y) - c \leq d_H(f(x), f(y)) \leq \gamma d_G(x,y) + c$$

for all $x, y \in V(G)$.

Remark 2.1. Let G and H be graphs. It is obvious from the construction that the map $\pi\colon V(G + H) \to V(G * H)$ is a quasi-isometry.

So far, the tree amalgamation is independent from any group action. In the following, we describe some conditions on tree amalgamations that ensure that tree amalgamations of quasi-transitive graphs are again quasi-transitive, see [12, Lemma 5.3].

For $i = 1, 2$, let Γ_i be a group acting on G_i. Let $i \in \{1, 2\}$. The tree amalgamation *respects* $\gamma \in \Gamma_i$ if there is a permutation π of I_i such that for every $k \in I_i$ there exist $\ell \in I_j$ and τ in the setwise stabiliser of S_ℓ in Γ_j such that

$$\phi_{k\ell} = \tau \circ \phi_{\pi(k)\ell} \circ \gamma \restriction_{S_k} .$$

The tree amalgamation *respects* Γ_i if it respects every $\gamma \in \Gamma_i$.

Let $k \in I_i$ and let $\ell, \ell' \in I_j$. We call the bonding maps from k to ℓ and ℓ' *consistent* if there exists $\gamma \in \Gamma_j$ such that

$$\phi_{k\ell} = \gamma \circ \phi_{k\ell'}.$$

The bonding maps between $J_i \subseteq I_i$ and $J_j \subseteq I_j$ are *consistent* if they are consistent for all $k \in J_i$ and $\ell, \ell' \in J_j$.

The tree amalgamation $G_1 * G_2$ is of *Type 1 respecting the actions of* Γ_1 *and* Γ_2 if the following holds:

(i) The tree amalgamation respects Γ_1 and Γ_2.
(ii) The bonding maps between I_1 and I_2 are consistent.

The tree amalgamation $G_1 * G_2$ is of *Type 2 respecting the actions of* Γ_1 *and* Γ_2 if the following holds:

(o) $G_1 = G_2 =: G$, $\Gamma_1 = \Gamma_2 =: \Gamma$ and $I_1 = I_2 =: I$,[1] and there exists $J \subseteq I$ such that $f(\vec{e}) \in J$ if and only if $f(\overleftarrow{e}) \notin J$.

[1] Formally we would have to work with a bijective map $I_1 \to I_2$ since we asked I_1 and I_2 to be disjoint.

(i) The tree amalgamation respects Γ.
(ii) The bonding maps between J and $I \setminus J$ are consistent.

The tree amalgamation $G_1 * G_2$ *respects the actions (of* Γ_1 *and* Γ_2*)* if it is of either Type 1 or Type 2 respecting the actions Γ_1 and Γ_2.

3. Asymptotic dimension

In this section, we state the definitions and cite the results that we need regarding the asymptotic dimension. We keep this section in the general setting of metric spaces instead of restricting it to graphs since the cited results are all for metric spaces.

Let X be a metric space. A cover $\mathcal{V}$ of X is *uniformly bounded* if $\sup\{\operatorname{diam}(V) \mid V \in \mathcal{V}\}$ is finite. The *multiplicity* $\operatorname{mult}(\mathcal{V})$ of $\mathcal{V}$ of X is the largest number of elements of $\mathcal{V}$ that contain a common point of X or, equivalently, it is the smallest number n such that every $x \in X$ belongs to at most n elements of $\mathcal{V}$. A cover $\mathcal{U}$ *refines* $\mathcal{V}$ if for every $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ with $U \subseteq V$.

The space X has *asymptotic dimension at most* n if for every uniformly bounded open cover $\mathcal{V}$ of X there is a uniformly bounded open cover $\mathcal{U}$ of X of multiplicity at most $n+1$ so that $\mathcal{V}$ refines $\mathcal{U}$. It has *asymptotic dimension* n and we write $\operatorname{asdim}(X) = n$ if it has asymptotic dimension at most n but not at most $n-1$. A family $\mathcal{V}$ of subsets of a metric space X is r*-disjoint* for $r > 0$ if $d(V, V') \geq r$ for all $V \neq V' \in \mathcal{V}$.

Proposition 3.1. [4, Theorem 1] *Let X be a metric space. Then* $\operatorname{asdim}(X) \leq n$ *if and only if for every $r > 0$ there exist r-disjoint uniformly bounded families $\mathcal{V}_0, \ldots, \mathcal{V}_n$ of subsets of X such that $\bigcup_{0 \leq i \leq n} \mathcal{V}_i$ is a cover of X.* □

Let$Y$ be a metric space. A map $f\colon X \to Y$ is a *coarse equivalence* if there are non-decreasing unbounded functions $\varrho_1, \varrho_2 \colon \mathbb{R}_+ \cup \{\infty\} \to \mathbb{R}_+ \cup \{\infty\}$ such that

$$\varrho_1(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \varrho_2(d_X(x, x'))$$

for all $x, x' \in X$.

By [1, Proposition 22], the asymptotic dimension is invariant under coarse equivalence. As quasi-isometries are coarse equivalences, we directly have the following.

Proposition 3.2. *The asymptotic dimension is a quasi-isometry invariance.* □

The next lemma says that restricting ourselves to subspaces does not increase the asymptotic dimension.

Lemma 3.3. [3, Proposition 23] *Let X be a metric space and $Y \subseteq X$. Then* $\operatorname{asdim}(Y) \leq \operatorname{asdim}(X)$. □

A family $(X_i)_{i \in I}$ of subsets of X satisfies the inequality $\operatorname{asdim}(X_i) \leq n$ *uniformly* if for every $r > 0$ there exists a $R \in \mathbb{N}$ such that for every $i \in I$ there exist r-disjoint families $\mathcal{V}_i^0, \ldots, \mathcal{V}_i^n$ of R-bounded subsets of X_i such that $\bigcup_{0 \leq j \leq n} \mathcal{V}_i^j$ is a cover of X_i.

Theorem 3.4 is the Infinite Union Theorem for the asymptotic dimension.

Theorem 3.4. [1, Theorem 1] *Let X be a metric space and let $n \in \mathbb{N}$ such that $X = \bigcup_{i \in I} X_i$ for some family $(X_i)_{i \in I}$ with* $\operatorname{asdim}(X_i) \leq n$ *uniformly for $(X_i)_{i \in I}$. For every $r > 0$, let $Y_r \subseteq X$ with* $\operatorname{asdim}(Y_r) \leq n$ *such that $d(X_i \smallsetminus Y_r, X_j \smallsetminus Y_r) \geq r$ for all $X_i \neq X_j$. Then* $\operatorname{asdim}(X) \leq n$. □

A consequence of Theorem 3.4 is the Finite Union Theorem, Theorem 3.5.

Theorem 3.5. [1, Finite Union Theorem] *Let $X = A \cup B$ be a metric space. Then* $\operatorname{asdim}(X) \leq \max\{\operatorname{asdim}(A), \operatorname{asdim}(B)\}$. □

Let $\mathcal{U}$ be a covering of a metric space X. We denote by

$$L(\mathcal{U}) := \inf_{U \in \mathcal{U}}\{\sup_{x \in X}\{d(x, X \setminus U)\}\}$$

the *Lebesgue number* of $\mathcal{U}$.

Let $r > 0$, $d > 0$ and $n \in \mathbb{N}$. We write $(r, d) - \dim(X) \leq n$ if there exists a d-bounded cover $\mathcal{V}$ of X with $\operatorname{mult}(\mathcal{V}) \leq n + 1$ and with $L(\mathcal{V}) > r$. We call such a cover an *(r, d)-cover* of X.

Proposition 3.6. [7, Proposition 2.1] *Let X be a metric space. Then X has symptotic dimension at most n if and only if there exists a function $d\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $(r, d(r)) - \dim(X) \leq n$ for all $r > 0$.* □

For a subset A of X we denote by ∂A its boundary and by $\operatorname{Int}(A)$ its interior, i. e. $A \setminus \partial A$. A *partition* of a metric space X is a presentation as a union $X = \bigcup_{i \in I} W_i$ such that $Int(W_i) \cap Int(W_j) = \emptyset$ for all $i \neq j$.

The last result that we need for our proof of Theorem 1.1 is the Partition Theorem, Theorem 3.7.

Theorem 3.7. [7, Theorem 2.10] *Let X be a geodesic metric space and let $n \in \mathbb{N}$. If for every $R > 0$ there exists $d > 0$ and a cover $\mathcal{V}$ of X with $\operatorname{Int}(U) \cap \operatorname{Int}(V) = \emptyset$ for all $U, V \in \mathcal{V}$, with $\operatorname{asdim}(V) \leq n$ uniformly for $\mathcal{V}$ and such that $(R, d) - \dim(\bigcup_{\mathcal{V}} \partial V) \leq n - 1$, where ∂V is taken with the metric restricted from X, then $\operatorname{asdim}(X) \leq n$.* □

4. Proof of Theorem 1.1

In this section, we will prove our main result, Theorem 1.1. Before we do that, we need some notations and we prove a lemma.

Let $G_1 * G_2$ be a tree amalgamation of locally finite quasi-transitive connected graphs over the connecting tree T. Let $t \in V(T)$. For every $m \in \mathbb{N}$, let O_m be the subgraph of $G + H$ induced by the vertex set

$$\{x \in V(G_1^u), y \in V(G_2^v) \mid d(t, u) = m, d(t, v) = m\}$$

and let Q_m be the subgraph of $G + H$ induced by the vertex set

$$\bigcup_{n \leq m} V(O_n).$$

Lemma 4.1. *Let G_1 and G_2 be quasi-transitive locally finite graphs with asymptotic dimension at most n. Then the following holds.*

(i) *For every $m \in \mathbb{N}$ we have $\operatorname{asdim}(O_m) \leq n$.*
(ii) *For every $m \in \mathbb{N}$ we have $\operatorname{asdim}(Q_m) \leq n$.*

Proof. We prove (i) inductively. Since G_1 and G_2 have asymptotic dimension at most n, it follows from the definition that each G_i^u with the metric inherited by $G_1 + G_2$ has asymptotic dimension at most n and that the family of all G_i^u satisfies $\operatorname{asdim}(G_i^u)$ uniformly.

Since $O_0 = G_i^t$, we have $\operatorname{asdim}(G_i^t) \leq n$. We are going to apply Theorem 3.4. Let $r > 0$ and let U_m be the graph induced by the vertices in O_m of distance at most r to O_{m-1}. Since G_i is quasi-transitive and there is a unique adhesion set in each G_j^u with $d(t, u) = m - 1$ that is not adjacent to O_m, we conclude that O_{m-1}

is either empty or quasi-isometric to U_m. In particular, we have $\mathrm{asdim}(U_m) \leq n$ by induction and by Proposition 3.2.

Let u, v be distinct vertices of T of distance m to t. Any path P from $V(G_j^u) \smallsetminus V(U_m)$ to $V(G_j^v) \smallsetminus V(U_m)$ must pass through G_{3-j}^w, where w is the neighbour of u on the unique t-u path in T. Thus, P has length at least $d(V(G_j^u) \smallsetminus V(U_m), V(G_{3-j}^w)) > r$ and hence Theorem 3.4 implies $\mathrm{asdim}(O_m) \leq n$.

By Theorem 3.5, $\mathrm{asdim}(Q_m) \leq n$ follows directly from $\mathrm{asdim}(O_m) \leq n$. □

Now we are ready to prove our main theorem.

Proof of Theorem 1.1. By Remark 2.1 and Proposition 3.2 it suffices to prove the assertion for $H := G_1 + G_2$ instead of G. Let T be the connecting tree of our tree amalgamation. Let $t_1 t_2 \in E(T)$ such that the graph associated to t_i is G_i. Let $\mathcal{S}_i$ be a set of representatives of the orbits of adhesion sets in $G_i^{t_i}$ under the action of $\Gamma_i^{t_i}$ and let $\mathcal{S}_i^u$ be its image in G_i^u. Let

$$n := \max\{\mathrm{asdim}(G_1), \mathrm{asdim}(G_2), \mathrm{asdim}(S) + 1 \mid S \in \mathcal{S}_1 \cup \mathcal{S}_2\}.$$

Let $\pi_T \colon V(H) \to V(T)$ be the canonical map that maps $x \in G_i^u$ to u. For $t \in V(T)$ let T^t be the subtree induced by all $t' \in V(T)$ that are separated by t from t_1. Let $r, R \in \mathbb{N}$ such that $r > 4R$ and r is even.

Let U_r be the subgraph of H induced by

$$\begin{aligned} &\left(\pi_T^{-1}(B_{r-1}(t_1)) \cap \{v \in V(H) \mid d(v, \bigcup \mathcal{S}_1^{t_1}) \geq R\}\right) \\ \cup &\bigcup \left\{ \left(B_r\left(\bigcup \mathcal{S}_j^v\right) \cap G_j^v\right) \mid d(v, t_1) = r \right\} \end{aligned}$$

and set

$$M_R := \{v \in V(H) \mid d(v, \bigcup \mathcal{S}_1^{t_1}) = R\}.$$

Let W_r be U_r without those edges that have both their incident vertices in M_R.

Let us extend the definitions of W_r and M_r to all vertices t of T of even distance to t_1: we set

$$W_r^t := f_t(W_r) \cap H[\pi_T^{-1}(V(T^t))]$$

and

$$M_R^t := f_t(M_R) \cap \pi_T^{-1}(V(T^t)),$$

where $H[\pi_T^{-1}(V(T^t))]$ is the subgraph of H induced by $\pi_T^{-1}(V(T^t))$ and where $f_t \in \mathrm{Aut}(H)$ maps t_1 to t such that the adhesion set separating G_1^t from $G_1^{t_1}$ lies in $f_t(\mathcal{S}_1)$. Note that this definition does not depend on the particular choice of f_t.

We consider the set

$$\mathcal{W} := \{W_r^t \mid d(t, t_1) \in r\mathbb{N}\} \cup \{W^0\},$$

where W^0 is the graph induced by $B_R(\bigcup \mathcal{S}_1^{t_1})$. Our aim is to show that we can apply Theorem 3.7 for the set $\mathcal{W}$. It follows directly from its construction that $\mathcal{W}$ is a cover of H. The elements of $\mathcal{W}$ are edge-disjoint by construction but they may share vertices that lie in M_R or its images M_R^t. Thus, $\mathrm{Int}(U) \cap \mathrm{Int}(W) = \emptyset$ for all $U, W \in \mathcal{W}$.

Since W^0 is quasi-isometric to $\bigcup \mathcal{S}_1^{t_1}$, their asymptotic dimensions coincide. By Theorem 3.5, we conclude that it is at most $n - 1$. Thus, in order to show that $\mathrm{asdim}(W) \leq n - 1$ uniformly for $\mathcal{W}$, it suffices to show it for $\mathcal{W} \smallsetminus \{W^0\}$. But this follows from Lemmas 3.3 and 4.1 since $W_r \subseteq Q_r$ and W_r^t is isomorphic to a subgraph of W_r.

Let us prove $(R,d)-dim(Z)\leq n-1$ for $Z:=\bigcup_{W\in\mathcal{W}}\partial W$. We have

$$Z=\bigcup\left\{\partial M_R^t\mid t\in V(T),d(t,t_1)\in r\mathbb{N}\right\}.$$

Note that M_R is quasi-isometric to $\bigcup\mathcal{S}_1$. Since $\mathcal{S}_1$ has at most two elements each of which has asymptotic dimension at most $n-1$, we conclude $\operatorname{asdim}(\bigcup\mathcal{S}_1)\leq n-1$ by Theorem 3.5. Thus, we have $\operatorname{asdim}(M_R)\leq n-1$. By Proposition 3.6, there exists a $d>0$ and an (R,d)-cover $\mathcal{U}$ of M_R with multiplicity at most n.

Set $\mathcal{V}:=\mathcal{U}\cup\bigcup\{g_t(U)\cap M_R^t\mid U\in\mathcal{U},d(t,t_1)\in r\mathbb{N}_{>0}\}$, where g_t is an automorphism of H that maps $G_1^{t_1}$ to G_1^t. Obviously, $\mathcal{V}$ is a d-bounded cover of Z. Since the sets M_R^t are pairwise disjoint, the multiplicity of $\mathcal{V}$ is at most $n-1$.

To prove $(R,d)-dim(Z)\leq n-1$, it remains to prove $L(\mathcal{V})>R$. For that, we just have to show $d(M_R^t,M_R^{t'})>R$ for all $t,t'\in V(T)$ with $d(t,t_1),d(t',t_1)\in r\mathbb{N}$. Let v be the vertex on the t-t' path in T closest to t_1. If v is distinct from t and t', then every path with one end vertex in M_R^t and the other in $M_R^{t'}$ must pass through G_j^v. So we have $d(M_R^{t'},M_R^t)>d(G_j^v,M_R^t)>R$. If v is not distinct from t and t', the let us assume $v=t$. Let S_t, $S_{t'}$ be the adhesion set in G_1^t, in $G_1^{t'}$ separating $G_1^{t_1}$ from G_1^t, from $G_1^{t'}$, respectively. Let w be the neighbour of t on the unique t-t' path and let S_v be the adhesion set in G_2^v separating G_2^v from G_1^t. Then $d(M_R^t,M_R^{t'})\geq d(S_v,S_t')\geq 3R>R$. So we have $d(M_R^t,M_R^{t'})>R$ in both cases.

Now Theorem 3.7 implies $\operatorname{asdim}(H)\leq n$ and thus we have $\operatorname{asdim}(G)\leq n$ by Remark 2.1 and Proposition 3.2. □

As mentioned in [12], there are Cayley graphs of free products with amalgamation or HNN-extensions that are tree amalgamations of Cayley graphs of their factors. Thus, the results of Dranishnikov [7] and of Tselekidis [13] follow from Theorem 1.1.

References

1. G. Bell and A. Dranishnikov, *On asymptotic dimension of groups*, Algebr. Geom. Topol. **1** (2001), 57–71.
2. ———, *On Asymptotic Dimension of Groups Acting on Trees*, Geom. Dedicata **103** (2004), 89–101.
3. ———, *Asymptotic dimension*, Topology Appl. **155** (2008), 1265–1296.
4. ———, *Asymptotic dimension in Będlewo*, Topology Proc. **38** (2011), 209–236.
5. M. Bonamy, N. Bousquet, L. Esperet, C. Groenland, C. Liu, F. Pirot, and A. Scott, *Asymptotic dimension of minor-closed families and Assouad-Nagata dimension of surfaces*, J. Eur. Math. Soc (2023).
6. M. Bonamy, N. Bousquet, L. Esperet, C. Groenland, F. Pirot, and A. Scott, *Surfaces have (asymptotic) dimension 2*, arXiv:2007.03582.
7. A. Dranishnikov, *On asymptotic dimension of amalgamated products and right-angeled Coxeter groups*, Algebr. Geom. Topol. **8** (2008), 1281–1293.
8. M.J. Dunwoody, *An inaccessible group*, Geometric Group Theory (G.A. Niblo and M.A. Roller, eds.), L.M.S. Lecture Note Ser., vol. 181, Cambridge University Press, 1992, pp. 75–78.
9. ———, *An Inaccessible Graph*, Random walks, boundaries and spectra, Progr. Probab., vol. 64, Birkhäuser/Springer Basel AG, 2011, pp. 1–14.
10. K. Fujiwara and P. Papasoglu, *Asymptotic dimension of planes and planar graphs*, Trans. Amer. Math. Soc. **374** (2021), 8887–8901.
11. M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, 1993, pp. 1–295.
12. M. Hamann, F. Lehner, B. Miraftab, and T. Rühmann, *A Stallings type theorem for quasi-transitive graphs*, J. Combin. Theory (Series B) **157** (2022), 40–69.
13. P. Tselekidis, *Asymptotic Dimension of Graph of Groups and One Relator Groups*, Algebr. Geom. Topol. **23** (2023), 3587–3613.

Matthias Hamann, Department of Mathematics, University of Hamburg, Hamburg, Germany