Tropicalization of the Lines on the Dwork Pencil of Quintic Threefolds

Master Thesis

June 30, 2017

Fachbereich Mathematik, Universität Hamburg

Name:	Tim Gabele
Matrikelnummer:	6346948
Studiengang:	Master of Science Mathematical Physics
Erstgutachter:	Prof. Dr. Bernd Siebert
Zweitgutachter:	Dr. Hans-Christian Graf von Bothmer

Abstract

The Dwork pencil of quintic threefolds is a 1-parameter family of quintic hypersurfaces in $\mathbb{P}^4$. A general member of the Dwork pencil contains a family of lines, parametrized by two isomorphic curves of degree 626 that can be constructed as 125:1 covers of blow ups of bidegree $(4, 4)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$, as well as 375 isolated lines that do not lie in this family. Tropical geometry is a piecewise linear shadow of algebraic geometry. The tropicalization of a variety is a polyhedral complex that can be studied combinatorically, carrying much information about the original variety. The aim of this thesis is to tropicalize the lines on the Dwork pencil of quintic threefolds. I show that the tropicalizations of the lines in the family give a family of tropical lines. The tropicalizations of the isolated lines (lying in a codimension 1 toric stratum) lie in the tropical boundary of the tropicalized Dwork pencil. At the end of the thesis I will interpret the lines on the Dwork pencils as parametrized tropical lines in the affine manifold with singularities given by the dual intersection complex of the Dwork pencil.

Contents

1 Introduction **1**

2 Tropical Geometry **2**

2.1 Tropical Hypersurfaces 3

2.2 Valuations 4

2.3 Tropicalization 5

2.4 Polyhedral Geometry 6

2.5 Gröbner Bases 7

2.6 Gröbner Complexes 9

2.7 Tropical Bases 9

2.8 The Fundamental Theorem and the Structure Theorem 10

2.9 Tropicalizing Toric Varieties 12

2.10 Parametric Tropicalization of Lines 14

3 Affine Manifolds and Parametrized Tropical Curves **16**

3.1 Affine Manifolds with Singularities 16

3.2 Tropical Curves on Tropical Manifolds 19

3.3 The Dual Intersection Complex 20

4 Example: A Family of Cubic Surfaces **23**

4.1 A Pencil of Cubic Surfaces and the Lines on it 23

4.2 Tropicalization of the Family of Cubic Surfaces 24

4.3 Tropicalization of the Lines 26

4.4 Lines in the Dual Intersection Complex 31

5 Lines on the Dwork Pencil of Quintic Threefolds **32**

5.1 Explicit parametrization 32

5.2 Implicitization 37

5.3 Plücker Coordinates 38

5.4 Symmetries and the Action of $S_5 \rtimes \mathcal{G}$ 40

6 The Parametrizing Curves **42**

6.1 The Curves $C^0_{\pm\varphi}$ in $\mathbb{C}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ 42

6.2 The quintic del Pezzo Surface dP_5 43

6.3 The Blow Ups $C_{\pm\varphi}$ of $C^0_{\pm\varphi}$ 45

6.4 The 125:1 Cover $\tilde{C}_\varphi$ of C_φ 45

6.5 The Central Fiber $\mathfrak{X}_0$ 46

7 Counting the Lines - The Number 2875 **47**

7.1 A Chern Class Computation 47

7.2 Deforming the Fermat Quintic 49

7.3 Using the Abel-Jacobi Map 50

8 Tropicalization of the Isolated Lines and the van Geemen Lines **51**

8.1 Tropicalization of the Dwork Pencil 51

8.2 Tropicalization of the Isolated Lines 54

8.3 Tropicalization of the van Geemen Lines 55

9 Tropicalization of the Parametrizing Curves **59**

9.1 Tropicalization of $C^0_{\pm\varphi}$ 59

9.2 Valuations of the Terms for the Rays of Trop($C^0_{\pm\varphi}$) 61

9.3 Valuations of the Terms for the Vertex of Trop($C^0_{\pm\varphi}$) 63

10 Tropicalization of the Family of Lines **68**

10.1 The Main Theorem 68

10.2 Tropicalization from the Defining Equations 69

10.3 Tropicalizaition from the Parametric Representation 72

10.4 Lines in the Dual Intersection Complex 75

1 Introduction

The *Dwork pencil of quintic threefolds* is the family $\mathfrak{X}$ of quintic hypersurfaces in $\mathbb{P}^4$ given by

$$t(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5) + z_0 z_1 z_2 z_3 z_4 = 0.$$

In [**5**] the mirror to a quintic threefold was constructed as a resolution of the quotient $\mathfrak{X}/\mathscr{G}$ where $\mathscr{G} \cong \left(\mathbb{Z}/5\mathbb{Z}\right)^3$. This is one of the first examples of mirror symmetry. Using period calulations on the mirror, they calculated the number of rational curves of degrees up to 10 on a quintic threefold. The number 2875 of lines was first calculated by H. Schubert in 1877. Using different methods, we will rederive this number in §7. The number of conics has been calculated by S. Katz in 1986, and the number of twisted cubics by G. Ellingsrud and S. A. Strømme in 1990, but the other numbers have not been known before.

In the 1980s, B. van Geemen showed that the members of the Dwork pencil contain 5000 special lines that are invariant under a order 3 subgroup of S_5 [**2**]. In [**15**] A. Mustaţă describes the Hilbert scheme of lines in the Dwork pencil. She shows that a general member of the pencil contains a family of lines, parametrized by two isomorphic curves of genus 626, as well as 375 lines that do not lie in this family. In [**6**] the authors give an explicit parametrization of the lines on the Dwork pencil. They describe the parametrizing curves as 125:1 covers of blow ups of singular bidegree $(4, 4)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Tropical geometry can be viewed as a combinatorial shadow of algebraic geometry. The tropicalization of the subvariety of a torus is a polyhedral complex in $\mathbb{R}^n$. This definition can be extended to subvarieties of toric varieties. The tropicalization is a piecewise linear object that can be studied combinatorially. Many enumerative problems translate to the tropical world. There are some correspondence theorems giving connections between enumerative questions in algebraic geometry and tropical geometry. For example, Mikhalkin [**13**] has shown that the number of curves in $\mathbb{P}^n$ with given degree and genus through a given number of points (if finite) coincides with the number of tropical curves of that degree if counted with right multiplicity. Further, Nishinou and Siebert [**16**] were able to show that for rational curves the same is true for any toric variety.

This thesis is separated into three parts. The first part (§2-§4) contains preliminaries of tropical and affine geometry, as well as the whole procedure of the thesis applied to a much simpler example. The second part (§5-§7) is concerned with the description of the lines on the Dwork pencil of quintic threefolds. It is mainly following [**6**], but with some statements rearranged and added. The third part (§8-§10) is dedicated to the tropicalization of the lines on the Dwork pencil, as well as the parametrized tropical curves on the dual intersection complex they describe. This is my own work and I do not know any reference for this.

2 Tropical Geometry

Tropical geometry essentially can be viewed as algebraic geometry over the *tropical semiring* $\bar{\mathbb{R}} = (\bar{\mathbb{R}}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$. Polynomials over $\bar{\mathbb{R}}$ are piecewise linear functions. While in classical algebraic geometry hypersurfaces are defined as zero sets of irreducible polynomials, this role is taken tropically by the *corner locus* of the polynomial, i.e., the locus where the polynomial is not linear. To get a connection between algebraic geometry over a field k and tropical geometry, we need a map $k \to \bar{\mathbb{R}}$ relating $+$ with $\oplus$ and $\cdot$ with $\odot$. This is done by a *valuation*. In order to get a nontrivial valuation we extend our field k to the field $K = k\{\{t\}\}$ of *Puiseux series* over k and take a valuation $v : K \to \mathbb{R} \cup \{\infty\}$. Then we can *tropicalize* a polynomial over K by replacing $+$ with $\oplus$ and $\cdot$ with $\odot$ and taking valuations of the terms. The tropicalization of a hypersurface is the corner locus of the tropicalized defining polynomial.

A *tropical variety* in the tropical torus $(\mathbb{T}^\star)^n = \mathbb{R}^n$ is defined as the tropicalization of some algebraic variety $X = V(I) \subseteq (K^\star)^n$, that is, the intersection of all tropical hypersurfaces defined by tropicalizations of Laurent polynomials in I. This definition can be extended to subvarieties of toric varieties. Note that we do not tropicalize a varietiy over k but over $K = k\{\{t\}\}$ which can be viewed as tropicalizing a family of varieties over k parametrized by t. The *Fundamental Theorem* states that the tropicalization of an algebraic variety X is equal to the image of X under the valuation map v. By the *Structure Theorem*, a tropical variety is the support of a balanced weighted polyhedral complex of pure dimension, connected through codimension 1.

While the intersection of two algebraic varieties is again an algebraic variety, this need not be true in tropical geometry. In the definition of a tropical variety, it is not enough to intersect the tropical hypersurfaces defined by the generators of the ideal I. But one can always find a finite generating set of I with this property, which we call a *tropical basis* of I. In some sense this is a generalization of a *Gröbner basis* of I. An application of Gröbner bases is the technique of *implicitization*. Given an algebraic variety in parametric representation one wants to find its defining equations. While a solution to this problem is rather simple, its computational complexity turns out to be very hard. Tropical geometry serves some help for this problem. On the other hand, using the fundamental theorem, it is possible to calculate the tropicalization of an algebraic variety directly from its parametric representation.

Many problems of algebraic geometry, especially enumerative questions, translate to the tropical world and have the same answer there. Tropically enumerative problems are of combinatorial nature, hence easier to solve. For a given problem one can hope to find the solution tropically and then lift it back to the classical world. The latter is usually the harder part. An example is the Correspondence Theorem for tropical curves mentioned in the introduction.

2.1 Tropical Hypersurfaces

Definition 2.1. The *tropical semiring* is the semiring

$$\bar{\mathbb{R}} = (\bar{\mathbb{R}}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$$

with neutral elements of addition and multiplication given by $0_{\bar{\mathbb{R}}} = \infty$ and $1_{\bar{\mathbb{R}}} = 0$, respectively. The *tropical affine n-space* is $\mathbb{T}^n = \bar{\mathbb{R}}^n$. The n-dimensional *tropical torus* is $(\mathbb{T}^\star)^n = \mathbb{R}^n$. The multiplicative inverse of $x \in \mathbb{T}^\star$ is given by $x^{\ominus 1} = -x$, but an additive inverse only exists for $0_{\bar{\mathbb{R}}}$, since $x \oplus y = 0_{\bar{\mathbb{R}}} = \infty$ implies $x = y = \infty$. We will most likely write tropical expressions with the usual operations but in quotation marks, e.g. $"x + yz" = x \oplus y \odot z = \min\{x, y + z\}$.

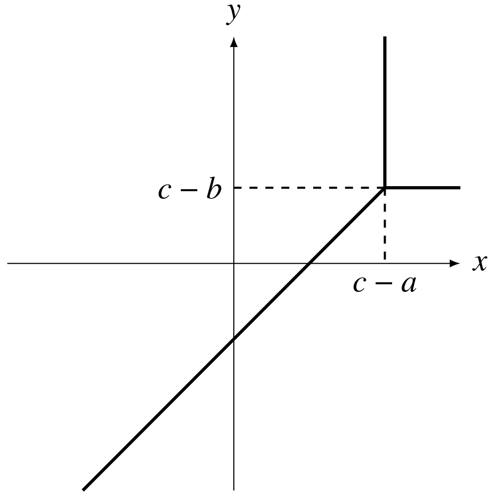
Definition 2.2. A *tropical hypersurface* defined by a tropical polynomial $f \in \bar{\mathbb{R}}[x_1, \ldots, x_n]$ is the set of points $V(f) \subseteq \mathbb{T}^n$ where the following minimum is achieved at least twice.

$$f(x) = \text{"}\sum_{u \in \mathbb{Z}^n} c_u x_1^{u_1} \cdots x_n^{u_n}\text{"} = \min\{c_u + u_1 x_1 + \ldots + u_n x_n\}$$

Remark 2.3. A tropical polynomial $f \in \bar{\mathbb{R}}[x_1, \ldots, x_n]$ defines a piecewise linear map $\bar{\mathbb{R}}^n \to \bar{\mathbb{R}}$. $V(f)$ is exactly the corner locus of this map, i.e., the set of points where it fails to be linear.

Example 2.4. (a) A general plane tropical line is a hypersurface $V(f) \subseteq \mathbb{T}^2$ defined by $f(x,y) = "ax + by + c" = \min\{a + x, b + y, c\}$. There are three cases in which the minimum is achieved at least twice, namely $a + x = b + y \leq c$, $a + x = c \leq b + y$ and $b + y = c \leq a + x$ giving the three rays of $V(f)$ as on the left hand side of Figure 2.1. Note that two points at infinity $(\infty, 0)$ and $(0, \infty)$ are contained in $V(f)$.

(b) The conic $V(f) \subseteq \mathbb{T}^2$ defined by $f(x,y) = "1x^2 + 1y^2 + 0xy + 0x + 0y + 1"$ is shown on the right hand side of Figure 2.1. Note that we can not omit the terms with coefficient 0 here.



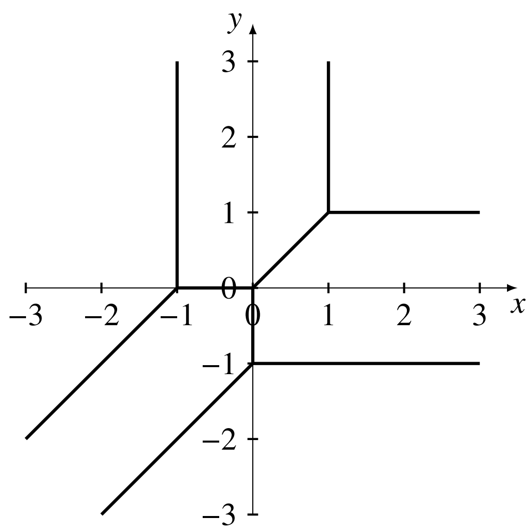


Figure 2.1: A general tropical line and a particular conic.

2.2 Valuations

Let K be a field.

Definition 2.5. A *valuation* on K is a map $v : K \to \mathbb{R} \cup \{\infty\}$ such that for all $a, b \in K$,

(1) $v(a) = \infty \Leftrightarrow a = 0$;
(2) $v(ab) = v(a) + v(b)$;
(3) $v(a + b) \geq \min\{v(a), v(b)\}$.

The image $\Gamma_v := v(K^\star)$ is an additive subgroup of $\mathbb{R}$ called the *value group* of v. The *valuation ring* $R = \{a \in K \mid v(a) \geq 0\}$ of K is a local ring with maximal ideal $\mathfrak{m}_R = \{a \in R \mid v(a) > 0\}$. The *residue field* is $k = R/\mathfrak{m}_R$.

Proposition 2.6. For all $a, b \in K$ we have the following properties.

(a) $v(1) = 0$.
(b) $v(-a) = v(a)$.
(c) If $v(a) \neq v(b)$, then $v(a + b) = \min\{v(a), v(b)\}$.

Proof. We have $v(1) = v(1^2) = v(1) + v(1)$ implying $v(1) = 0$ and $0 = v(1) = v((-1)^2) = 2v(-1)$ implying $v(-1) = 0$. Furthermore, $v(-a) = v((-1) \cdot a) = v(-1) + v(a) = v(a)$.
To prove (c), assume $v(b) > v(a)$. Then we have $v(a) \geq \min\{v(a+b), v(-b)\} = \min\{v(a+b), v(b)\}$. But we also have $v(a + b) \geq \min\{v(a), v(b)\} = v(a)$, proving equality. □

Example 2.7. (a) Every field K has a trivial valuation, defined by $v(a) = 0$ for all $a \in K^\star$.
(b) Let $p \in \mathbb{Z}$ be a prime number. The *p-adic valuation* on $\mathbb{Q}$ is defined by $v_p(q) = r$, for $q = p^r a/b \neq 0$ where $a, b \in \mathbb{Z}$ are not divisible by p.

Definition 2.8. The field of *Puiseux series* with coefficients in $\mathbb{C}$ is the field

$$\mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$$

where $\mathbb{C}((t^{1/n}))$ is the field of Laurent series in $t^{1/n}$. The elements of $\mathbb{C}\{\{t\}\}$ are formal power series $c(t) = \sum_{i=1}^{\infty} c_i t^{a_i}$ where $c_i \in \mathbb{C}^\star$ and $a_1 < a_2 < a_3 < \ldots \in \mathbb{Q}$ have a common denominator. $\mathbb{C}\{\{t\}\}$ has a valuation given by $v : \mathbb{C}\{\{t\}\}^\star \to \mathbb{R}, c(t) \mapsto a_1$.

Remark 2.9. $\mathbb{C}\{\{t\}\}$ is algebraically closed. It is the algebraic closure of $\mathbb{C}((t))$. In fact, $k\{\{t\}\}$ is the algebraic closure of $k((t))$ when k is an algebraically closed field of characteristics zero. For a proof, see [**14**], Theorem 2.1.5.

Remark 2.10. The field of rational functions $\mathbb{C}(t)$ is a subfield of $\mathbb{C}\{\{t\}\}$. The valuation on $\mathbb{C}\{\{t\}\}$ restricted to $\mathbb{C}(t)$ gives the order of the pole or zero at $t = 0$, hence is an integer.

2.3 Tropicalization

We will first define the tropicalization for subvarieties of the torus $T^n = (K^\star)^n$. The case of affine varieties will be given along with the general treatment of tropicalizing toric varieties and their subvarieties in §2.9. Recall $(K^\star)^n = V(K[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ and $(\mathbb{T}^\star)^n = \mathbb{R}^n$.

Let K be a field with valuation, e.g. $K = \mathbb{C}\{\{t\}\}$.

Definition 2.11. The *tropicalization* of a Laurent polynomial $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is the tropical Laurent polynomial $\mathrm{Trop}(f) \in \mathbb{T}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ given by

$$\mathrm{Trop}(f)(w) = \text{"}\sum_{u \in \mathbb{Z}} \nu(c_u) x^u\text{"} = \min_{u \in \mathbb{Z}^n} \left\{ \nu(c_u) + \sum_{i=1}^{n} u_i w_i \right\}.$$

The tropicalization of the hypersurface $V(f) \subset (K^\star)^n$ is the tropical variety defined by $\mathrm{Trop}(f)$, i.e.,

$$\mathrm{Trop}(V(f)) = V(\mathrm{Trop}(f)) \subset (\mathbb{T}^\star)^n.$$

Definition 2.12. Let I be an ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then the *tropicalization* of $X = V(I) \subset (K^\star)^n$ is

$$\mathrm{Trop}(X) = \bigcap_{f \in I} \mathrm{Trop}(V(f)).$$

Remark 2.13. Note that in the definition of tropicalization we have to intersect the hypersurfaces defined by all polynomials in I, not just its generators. However, in §2.7 we will show that one can always find a finite generating set $\mathcal{T}(I)$ of I, called a *tropical basis*, such that intersection of hypersurfaces defined by the finite number of polynomials in $\mathcal{T}(I)$ suffices.

Remark 2.14. In case $K = k\{\{t\}\}$ we can think of a subvariety of $(K^\star)^n$ as a family of subvarieties of $(k^\star)^n$ parametrized by t. In this case we can say we tropicalize a family of varieties.

Example 2.15. The tropical curves from §2.1, restricted to the tropical torus $\mathbb{R}^2$, can be viewed as the tropicalization of certain subvarieties of $(K^\star)^2$. This choice is not unique, since multiplication of a term in the defining equations with some $a \in \mathbb{C}^\star$ does not change the tropicalization.

(a) The tropicalization of the family of lines in $(\mathbb{C}^\star)^2$ defined by $t^a x + t^b y + t^c = 0$ is the tropical line from Example 2.15, (a), restricted to $\mathbb{R}^2$.
(b) The tropicalization of the family of conics in $(\mathbb{C}^\star)^2$ defined by $t(x^2 + y^2 + 1) + xy + x + y = 0$ is the tropical conic in Example 2.15, (b), restricted to $\mathbb{R}^2$.

Definition 2.16. A *tropical variety* in $(\mathbb{T}^\star)^n$ is a subset of $(\mathbb{T}^\star)^n = \mathbb{R}^n$ of the form $\mathrm{Trop}(X)$, where X is an algebraic subvariety of the torus $(\mathbb{K}^\star)^n$.

2.4 Polyhedral Geometry

Definition 2.17. A *polyhedron* $P \subset \mathbb{R}^n$ is the intersection of finitely many closed half-spaces:

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

where A is a $d \times n$-matrix, and $b \in \mathbb{R}^d$. Let Γ be a subgroup of $(\mathbb{R}, +)$. Then P is Γ-*rational* if A has entries in $\mathbb{Q}$, and $b \in \Gamma^d$. A *polytope* is a bounded polyhedron. If $\Gamma = \mathbb{Q}$, we simply write *rational* instead of $\mathbb{Q}$-rational.

Definition 2.18. A *face* of a polyhedron is the intersection with the boundary of one of its defining half-spaces. A face of P that is not contained in any larger proper face is called *facet*. The *dimension* of P is the dimension of the smallest affine subspace containing P.

Definition 2.19. A *polyhedral complex* is a collection Σ of polyhedra satisfying two conditions:

(1) if P is in Σ, then so is any face of both P and Q;
(2) if P and Q lie in Σ, then $P \cap Q$ is either empty or a face of both P and Q.

The polyhedra in Σ are called the *cells* of Σ. The union of the cells is called the *support* $|\Sigma|$ of Σ. The *lineality space* of a polyhedral complex is the intersection of all lineality spaces of the polyhedra in the complex. A polyhedral complex Σ is *pure* of dimension d if every facet of Σ has dimension d.

Definition 2.20. A *(polyhedral) cone* $C \subseteq \mathbb{R}^n$ is a polyhedron with exactly one vertex, which is $0 \in \mathbb{R}^n$. It is given by the positive hull of a finite subset of $\mathbb{R}^n$:

$$C = \operatorname{pos}(v_1, \ldots, v_r) = \left\{ \sum_{i=1}^{r} \lambda_i v_i \in \mathbb{R}^n \mid \lambda_i \geq 0 \text{ for all } i \right\}.$$

A polyhedral complex consisting of polyhedral cones is called a *(polyhedral) fan*.

Definition 2.21. Let $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial. The *Newton polytope* of f is

$$\operatorname{Newt}(f) = \operatorname{Conv}\{u \in \mathbb{Z}^n \mid c_u \neq 0\} \subset \mathbb{R}^n.$$

Definition 2.22. A *weight vector* $w \in \mathbb{R}^r$ induces a *regular subdivision* of a polytope $P = \operatorname{conv}\{u_1, \ldots, u_r\}$ by the projection of the lower faces of the polytope

$$P_w = \operatorname{conv}\{(u_i, w_i) \in \mathbb{R}^{n+1} \mid i = 1, \ldots, r\}$$

to P. This is a polyhedral complex with support P.

2.5 Gröbner Bases

Let k be a field. The following definitions are taken from [**12**].

Definition 2.23. A *term order* on $k[x_1, \dots, x_n]$ is a total ordering $\preceq$ on all monomials such that

(1) $1 \prec x^\alpha$ for all $\alpha \in \mathbb{N}^n \setminus \{0\}$;
(2) $x^\alpha \prec x^\beta \Rightarrow x^\alpha x^\gamma \prec x^\beta x^\gamma$ for all $\alpha, \beta, \gamma \in \mathbb{N}^n$.

A *term* is a monomial together with its coefficient. By $\prec$ we mean $x^\alpha \prec x^\beta \Leftrightarrow x^\alpha \preceq x^\beta \wedge x^\alpha \neq x^\beta$. We prefer the symbol $\prec$ over the symbol $\preceq$ when denoting term orders.

Definition 2.24. Given a *weight vector* $w \in k^n$ we can define a *refined term order* by

$$x^\alpha \prec_w x^\beta \Leftrightarrow \langle w, \alpha\rangle < \langle w, \beta\rangle \vee (\langle w, \alpha\rangle = \langle w, \beta\rangle \wedge x^\alpha \prec x^\beta).$$

Definition 2.25. The *initial term* $\mathrm{in}_\prec(f)$ of a polynomial $f = \sum_{u \in \mathbb{N}^n} c_u x^u \in k[x_1, \dots, x_n]$ is its unique maximal term with respect to $\prec$. The *initial ideal* is

$$\mathrm{in}_\prec(I) = \langle \mathrm{in}_\prec(f) \mid f \in I\rangle \subseteq k[x_1, \dots, x_n],$$

A *Gröbner basis* of I with respect to $\prec$ is a finite subset $\mathcal{G}_\prec(I) = \{g_1, \dots, g_s\} \subseteq I$ such that

$$\mathrm{in}_{\prec_w}(I) = \langle \mathrm{in}_{\prec_w}(g_1), \dots, \mathrm{in}_{\prec_w}(g_s))\rangle$$

or equivalently if for any $f \in I$ there is a polynomial $g \in \mathcal{G}$ such that $\mathrm{in}_w(g)$ divides $\mathrm{in}_w(f)$.

The latter definition of a Gröbner basis gives a way to decide wheter a polynomial $f \in k[x_1, \dots, x_n]$ is contained in an ideal $I \subset k[x_1, \dots, x_n]$ or not:

Proposition 2.26. Let $I \subset K[x_1, \dots, x_n]$ be an ideal. Let $\mathcal{G} = \{g_1, \dots, g_s\}$ be a Gröbner basis for I with respect to $w \in \Gamma_\nu^n$. Then $f \in K[x_1, \dots, x_n]$ is contained in I if and only if there are $k_1, \dots, k_s \in K$ such that

$$f = k_1 g_1 + \dots + k_s g_s.$$

This means the multivariate polynomial division of f by $\mathcal{G}$ gives the remainder 0.

Proof. Assume the multivariate polynomial division by $\mathcal{G}$ gives $f = k_1 g_1 + \dots + k_s g_s + r$. Then $r = 0$ or $\mathrm{in}_w(r)$ is not divisible by $\mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_s)$. Since $g_1, \dots, g_s \in I$ we have $f \in I$ if and only if $r \in I$. If $r = 0$, then clearly $r \in I$ and thus $f \in I$. Conversely, assume $r \in I$. Since $\mathcal{G}$ is a Gröbner basis, there is a $g \in \mathcal{G}$ such that $\mathrm{in}_w(g)$ divides $\mathrm{in}_w(r)$. But then $r = 0$. $\square$

Proposition 2.27. For any term order $\prec$ on $k[x_1, \dots, x_n]$, every ideal $I \subseteq k[x_1, \dots, x_n]$ has a Gröbner basis $\mathcal{G}_\prec(I)$.

Proof. The *Buchberger algorithm* for calculating Gröbner bases is the following.

Take a finite generating set $\mathcal{F}$ of I.

1. For each pair $f, g \in \mathcal{F}$, $f \neq g$, calculate

$$S_<(f,g) = \frac{\mathrm{lcm}\{\mathrm{in}_<(f), \mathrm{in}_<(g)\}}{\mathrm{in}_<(f)} f - \frac{\mathrm{lcm}\{\mathrm{in}_<(f), \mathrm{in}_<(g)\}}{\mathrm{in}_<(g)} g.$$

2. Let s be the remainder of the multivariate polynomial division of $S_<(f,g)$ relative to $\mathcal{F}$.
3. If $s \neq 0$, add s to $\mathcal{F}$ and go to 1.

The motivation is the following. If $f, g \in \mathcal{F}$ and $f', g' \in k[x_1, \ldots, x_n]$ it may be that $ff' - gg' \in I$ has a initial form not divisible by $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$. In this case $\mathrm{in}_<(ff') = \mathrm{in}_<(gg')$. But this case is precisely covered by $S_<(f,g)$ above, and adding the remainder of the multivariate polynomial division to $\mathcal{F}$ avoids this case. Since the algorithm above terminates after a finite number of steps, the resulting set $\mathcal{F}$ is finite, hence a Gröbner basis. □

Let K be a field with valuation ν and assume ν has a splitting denoted by $\phi : \Gamma_\nu \to K^\star, w \mapsto t^w$. If $K = \mathbb{C}\{\{t\}\}$ such a splitting is indeed given by $\phi : \mathbb{Q} \to \mathbb{C}\{\{t\}\}, w \mapsto t^w$. For a in the valuation ring $R = \{a \in K \mid \nu(a) \geq 0\}$ of K denote by $\bar{a}$ the image of a in the residue field $k = R/\mathfrak{m}_R$. The splitting ϕ induces a homomorphism of multiplicative groups $K^\star \to k^\star, a \mapsto \overline{t^{-\nu(a)}a}$. For $f \in R[x_1, \ldots, x_n]$ let $\bar{f} \in k[x_1, \ldots, x_n]$ be given by replacing every coefficient a in f by $\bar{a}$.

Definition 2.28. The *initial form* of $f = \sum_{u \in \mathbb{N}^n} c_u x^u \in K[x_1, \ldots, x_n]$ with respect to a *weight vector* $w \in \Gamma_\nu^n$ is

$$\mathrm{in}_w(f) = \sum_{\substack{u \in \mathbb{N}^n \\ \nu(c_u) + \langle w, u \rangle = trop(f)(w)}} \overline{c_u t^{-\nu(c_u)}} x^u \in k[x_1, \ldots, x_n].$$

The *initial ideal* of an ideal $I \subseteq K[x_1, \ldots, x_n]$ is

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \mid f \in I \rangle \subseteq k[x_1, \ldots, x_n].$$

A set $\mathcal{G}_w(I) = \{g_1, \ldots, g_s\} \subseteq I$ is a *Gröbner basis* for I with respect to w if

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_i), \ldots, \mathrm{in}_w(g_s) \rangle,$$

Lemma 2.29. Let $I \subset K[x_1, \ldots, x_n]$ be an ideal with homogenization $I^h \subset K[x_0, \ldots, x_n]$ and $w \in \mathbb{R}^n$. Then $\mathrm{in}_w(I)$ contains a monomial if and only if $\mathrm{in}_{(0,w)}(I^h)$ contains a monomial.

Proof. This is Lemma 2.2 in [**3**]. □

2.6 Gröbner Complexes

Let K be as in §2.5, and let $I \subseteq K[x_0, \dots, x_n]$ be a homogeneous ideal.

Definition 2.30. For $w \in \mathbb{R}^{n+1}$ define

$$C_w(I) = \{w' \in \mathbb{R}^{n+1} \mid \mathrm{in}_{w'}(I) = \mathrm{in}_w(I)\}.$$

Let $\overline{C_w(I)}$ be the closure of $C_w(I) \subset \mathbb{R}^{n+1}$ in Euclidean topology.

Proposition 2.31. The set $\overline{C_w(I)}$ is a Γ_v-rational polyhedron. If $\mathrm{in}_w(I)$ is not a monomial ideal, then there exists $w' \in \Gamma_v^{n+1}$ sucht that $\mathrm{in}_{w'}(I)$ is a monomial ideal and $\overline{C_w(I)}$ is a proper face of the polyhedron $\overline{C_{w'}(I)}$.

Proof. This is Proposition 2.5.2 in [**14**]. □

Theorem 2.32. The polyhedra $\overline{C_w(I)}$ as w varies over $\mathbb{R}^{n+1}$ form a Γ_v-rational polyhedral complex $\Sigma(I)$, called the *Gröbner complex* of I.

Proof. This is Theorem 2.5.3 in [**14**]. □

Definition 2.33. A *universal Gröbner basis* for a homogeneous ideal $I \subset K[x_0, \dots, x_n]$ is a finite subset $\mathcal{G}(I)$ of I such that, for all $w \in \mathbb{R}^{n+1}$, the set $\mathrm{in}_w(\mathcal{G}(I)) = \{\mathrm{in}_w(f) \mid f \in \mathcal{G}(I)\}$ generates the initial ideal $\mathrm{in}_w(I)$ in $k[x_0, \dots, x_n]$.

Corollary 2.34. Every homogeneous ideal $I \subset K[x_0, \dots, x_n]$ has a universal Gröbner basis.

Proof. This is Corollary 2.5.11 in [**14**]. □

2.7 Tropical Bases

A *tropical basis* is the natural analogue to the notion of a universal Gröbner basis for an ideal in the Laurent polynomial ring $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Again, assume that the valuation on K has a splitting $w \mapsto t^w$.

Definition 2.35. The *initial form* of $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with respect to a weight vector $w \in \mathbb{R}^n$ is

$$\mathrm{in}_w(f) = \sum_{\substack{u \in \mathbb{Z}^n \\ v(c_u + \langle w, u \rangle) = \mathrm{trop}(f)(w)}} \overline{c_u t^{-v(c_u)}} x^u.$$

The *initial ideal* of an ideal $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \mid f \in I \rangle \subseteq k[x_1, \dots, x_n].$$

Remark 2.36. For generic choices of w, the initial form $\mathrm{in}_w(f)$ is a unit in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and the initial ideal $\mathrm{in}_w(I)$ is the whole ring, hence contains no information. Tropical geometry is concerned with the study of those weight vectors $w \in \mathbb{R}^n$ for which the initial ideal $\mathrm{in}_w(I)$ is actually a proper ideal in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Definition 2.37. A finite generating set $\mathcal{T}(I)$ of I is a *tropical basis* if, for all vectors $w \in \mathbb{R}^n$, there is a Laurent polynomial $f \in I$ for which the minimum in $\mathrm{trop}(f)(w)$ is achieved only once if and only if there is $g \in \mathcal{T}(I)$ for which the minimum in $\mathrm{trop}(g)(w)$ is achieved only once.

This is equivalent with saying that in the definition of tropical variety it suffices to intersect over all tropical hypersurfaces defined by elements of $\mathcal{T}$:

$$\mathrm{Trop}(V(I)) = \bigcap_{f \in \mathcal{T}(I)} \mathrm{Trop}(V(f)).$$

If K has a splitting, this is equivalent with the condition that, for any $w \in \mathbb{R}^n$, the initial ideal $\mathrm{in}_w(I)$ contains a unit if and only if the finite set $\mathrm{in}_w(\mathcal{T}) = \{\mathrm{in}_w(f) \mid f \in \mathcal{T}\}$ contains a unit.

Theorem 2.38. Every ideal I in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ has a tropical basis.

Proof. This is Theorem 2.6.6 in [**14**]. □

Example 2.39. The tropical basis of a principal ideal $\langle f \rangle$ is $\{f\}$. Indeed, suppose that $\mathrm{in}_w(I)$ contains a unit. Then there exists $g \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that $\mathrm{in}_w(fg) = \mathrm{in}_w(f) \cdot \mathrm{in}_w(g)$ is a unit, and this implies that $\mathrm{in}_w(f)$ is a unit.

2.8 The Fundamental Theorem and the Structure Theorem

Now we come to the two main theorems of tropical geometry. The *Fundamental Theorem* gives three analogous characterizations of the tropicalization of an algebraic subvariety X of $(K^\star)^n$. The *Structure Theorem* states that $\mathrm{Trop}(X)$ is the support of a balanced polyhedral complex.

Theorem 2.40 (Fundamental Theorem of Tropical Algebraic Geometry)**.** Let K be an algebraically closed field with a nontrivial valuation, let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $X = V(I)$ be its variety in the algebraic torus $(K^\star)^n$. The following three subsets of $\mathbb{R}^n$ coincide:

(1) the tropical variety $\mathrm{Trop}(X)$ as in Definition 2.12;
(2) the set of all vectors $w \in \mathbb{R}^n$ with $\mathrm{in}_w(I) \neq \langle 1 \rangle$;
(3) the closure of the set of coordinate valuations of points in X,

$$\nu(X) = \{(\nu(y_1), \ldots, \nu(y_n)) \mid (y_1, \ldots, y_n) \in X\}.$$

Proof. This is Theorem 3.2.3 in [**14**]. □

Theorem 2.41 (Structure Theorem for Tropical Varieties)**.** Let X be an irreducible d-dimensional subvariety of $(K^\star)^n$. Then $\mathrm{Trop}(X)$ is the support of a balanced weighted Γ_v-rational polyhedral complex pure of dimension d. Moreover, that polyhedral complex is connected through codimension d.

Proof. This is Theorem 3.3.5 in [**14**]. □

In the special case of tropical hypersurface we can give an alternative description of this polyhedral complex.

Proposition 2.42. Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial. The tropical hypersurface $\mathrm{Trop}(V(f))$ is the support of a pure Γ_v-rational polyhedral complex of dimension $n-1$ in $\mathbb{R}^n$. It is the $(n-1)$-skeleton of the polyhedral complex Σ dual to the regular subdivision of the Newton polytope of f induced by the weights $v(c_u)$ on the lattice points in $\mathrm{Newt}(f)$.

Proof. This is Proposition 3.1.6 in [**14**]. □

Example 2.43. Let $f = t(x^2 + x^2 + 1) + xy + x + y$ as in Example 2.15, (b). Figure 2.2 shows the Newton polytope $\mathrm{Newt}(f)$ of f together with the weights given by $v(c_u)$ and the regular subdivision induced by the projection from P_w as in Definition 2.22. On the right hand side it shows the dual complex which is supported on $\mathrm{Trop}(V(f))$.

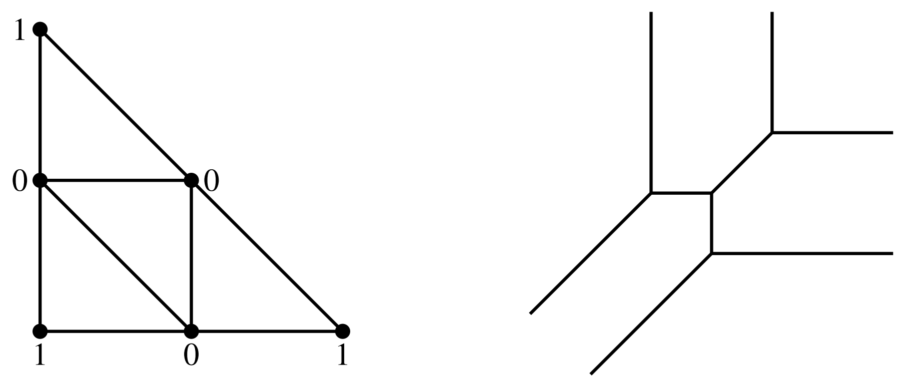


Figure 2.2: The regular subdivision of $\mathrm{Newt}(f)$ and its dual supported on $\mathrm{Trop}(V(f))$.

Using this dual description, one can define multiplicities of the maximal cells in the polyhedral complex on which $\mathrm{Trop}(V(f))$ is supported.

Definition 2.44. The multiplicity of a maximal cell σ in Σ as in Proposition 2.42 is the affine length of the edge dual to σ.

Remark 2.45. Definition 3.4.3 in [**14**] gives a more general definition of multiplicities of the cells in this polyhedral complex, but we won't need it here. Lemma 3.4.6 in [**14**] gives the compatibility with our definition.

2.9 Tropicalizing Toric Varieties

Now we want to extend the notion of tropicalization to subvarieties of toric varieties. This general definition includes the cases of affine spaces $\mathbb{A}^n$ and projective spaces $\mathbb{P}^n$.

A toric variety is an algebraic variety containing a torus as an open dense subset, such that the action of the torus on itself extends to the variety. A normal toric variety is defined by a rational fan Σ in $N_\mathbb{R} = N \otimes \mathbb{R} \cong \mathbb{R}^n$ for a lattice $N \cong \mathbb{Z}^n$. The torus T^n of a toric variety X_Σ over a field K is $N \otimes K^\star \cong (K^\star)^n$. Each cone $\sigma \in \Sigma$ determines a local chart $U_\sigma = \mathrm{Spec}(K[\sigma^\vee \cap M])$, where $M = \mathrm{Hom}(N, \mathbb{Z})$ is the dual lattice and $\sigma^\vee = \{u \in M \mid \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}$ is the dual cone. The charts are glued along the varieties defined by their common faces. For $\sigma = \{0\}$, we have $\sigma^\vee = M_\mathbb{R} = M \otimes \mathbb{R} \cong \mathbb{R}^n$, so $K[\sigma^\vee \cap M] = K[M]$ is the Laurent polynomial ring and $U_\sigma \cong T^n$.

Definition 2.46. Let Σ be a rational polyhedral fan in $N_\mathbb{R}$. For each cone $\sigma \in \Sigma$, we consider the $(n - \dim(\sigma))$-dimensional vector space $N(\sigma) = N_\mathbb{R}/\mathrm{span}(\sigma)$. As a set, the tropical toric variety X_Σ^{trop} is the disjoint union

$$X_\Sigma^{\mathrm{trop}} = \coprod_{\sigma\in\Sigma} N(\sigma).$$

To place a topology on X_Σ^{trop}, we associate to each cone $\sigma \in \Sigma$ the space

$$U_\sigma^{\mathrm{trop}} = \mathrm{Hom}(\sigma^\vee \cap M, \bar{\mathbb{R}}).$$

of semigroup homomorphisms from $(\sigma^\vee \cap M, +)$ to $(\bar{\mathbb{R}}, \odot)$. We place the pointwise convergence topology on U_σ^{trop}. This is the topology induced from the product topology on products of $\bar{\mathbb{R}}$, where we identify U_σ^{trop} as a subset of the product space $(\bar{\mathbb{R}})^{\sigma^\vee \cap M}$.

Example 2.47. Let $\sigma = \mathrm{pos}(e_1, \dots, e_n)$ in $N_\mathbb{R}$. Then $\sigma^\vee = \mathrm{pos}(e_1, \dots, e_n)$ in $M_\mathbb{R}$, $U_\sigma \cong \mathbb{A}^n$, and $U_\sigma^{\mathrm{trop}} = \mathrm{Hom}(\mathbb{N}^n, \bar{\mathbb{R}}) = \bar{\mathbb{R}}^n$. This shows that Definition 2.46 is compatible with the definition of tropical affine n-space from §1.1. The basis of its topology consists of open balls in $\mathbb{R}^n$ together with balls closed at tropical strata at infinity.

Example 2.48. The tropical projective variety $\mathbb{TP}^n$ is obtained as usually as the union of n copies of $\mathbb{T}^n$ glued along their tori $(\mathbb{T}^\star)^n = \mathbb{R}^n$, or as the quotient of $(\mathbb{T}^\star)^{n+1}$ by the diagonal (tropical) multiplicative action. For example, $\mathbb{P}^1$ consists of $\mathbb{R}$ together with two points at infinity, one on each affine chart. This is shown in Figure 2.3 which is taken from [**14**].

The Fundamental Theorem extends to subvarieties of toric varieties. For $\mathbb{A}^n$ this is:

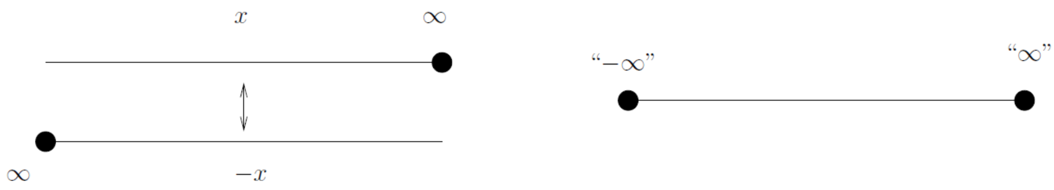


Figure 2.3: The tropical projective line $\mathbb{TP}^1$ obtained from gluing affine charts.

Theorem 2.49 (Extended Fundamental Theorem)**.** Let Y be a subvariety of $\mathbb{A}^n$, and let I be its ideal in $K[x_1, \ldots, x_n]$. The the following subsets of $\mathbb{T}^n$ coincide:

(1) $\cap_{f \in I} \mathrm{Trop}(V(I))$;
(2) the set of all vectors $w \in \bar{\mathbb{R}}^n$ for which $\mathrm{in}_w(I) \subseteq k[x_1, \ldots, x_n]$ does not contain a monomial;
(3) the set

$$\bigcup_{\sigma \subseteq \{1,\ldots,n\}} \mathrm{Trop}(Y \cup \mathbb{O}_\sigma) \times \infty^\sigma,$$

where $\mathbb{O}_\sigma = \{x \in \mathbb{A}^n \mid x_i = 0 \Leftrightarrow i \in \sigma\}$.

Proof. This is Theorem 6.2.15 in [**14**]. □

Theorem 2.50. Let $Y \subseteq (K^\star)^n$, and let $\bar{Y}$ be the closure of Y in a toric variety X_Σ. Then $\mathrm{Trop}(\bar{Y})$ is the closure of $\mathrm{Trop}(Y) \subseteq \mathbb{R}^n$ in $\mathrm{Trop}(X_\Sigma)$.

Proof. This is Theorem 6.2.18 in [**14**]. □

For the case of affine varieties, we can give an analogous definition of tropical basis as for subvarieties of a torus. Moreover, we can give a simple construction of a tropical basis in this case.

Theorem 2.51. Every ideal $I \subseteq K[x_1, \ldots, x_n]$ has a tropical basis.

Proof. This (constructive) proof is traken from [**3**]. By Lemma 2.29 we can assume I is homogeneous. Otherwise just take its homogenization and dehomogenize the tropical basis obtained by the following construction.

Let $\mathcal{F}$ be any finite generating set of I which is not a tropical basis. Pick a Gröbner cone $C_w(I)$ whose relative interior intersects $\cap_{f \in \mathcal{F}} \mathrm{Trop}(V(f))$ nontrivially and whose initial ideal $\mathrm{in}_w(I)$ contains a monomial x^u. Compute the reduced Gröbner basis $\mathcal{G}_{<_w}(I)$ for a refinement $<_w$ of w, and let h be the normal form of x^u with respect to $G_{<_w}(I)$. Let $f = x^u - h$.

Since the normal form of x^u with respect to $\mathcal{G}_<(\mathrm{in}_w(I)) = \{\mathrm{in}_w(g) \mid g \in \mathcal{G}_{<_w}(I)\}$ is 0 and h is the normal form of x^u with respect to $\mathcal{G}_{<_w}(I)$, every monomial occuring in h has higher w-weight than x^u. Moreover, h depends only on the reduced Gröbner basis $\mathcal{G}_{<_w}(I)$ and is independent of the particular choice of w in $C_w(I)$. Hence for any w' in the relative interior of $C_w(I)$, we have

$x^u = \mathrm{in}_{w'}(f)$. This implies that the polynomial $f = x^u - h$ is a *witness* for the cone $C_w(I)$ not being in the tropical variety $\mathrm{Trop}(V(I))$.

We now add the witness f to the current basis $\mathcal{F}$ and repeat the process. Since the Gröbner fan has only finitely many cones, this process will terminate after finitely many steps. It removes all cones of the Gröbner fan which violate the condition for $\mathcal{F}$ to be a tropical basis. $\square$

2.10 Parametric Tropicalization of Lines

Consider an affine line in an affine toric variety X over K given by an embedding

$$\begin{array}{rcl} \mathbb{A}^1_K & \hookrightarrow & X \\ u & \mapsto & (x_1(u,v), \ldots, x_n(u,v)) \end{array}$$

where $x_i(u,v)$ are linear functions in u. By tropicalization we get

$$\begin{array}{rcl} \mathbb{T}^1 & \hookrightarrow & \mathrm{Trop}(X) \\ \nu(u) & \mapsto & (\nu(x_1), \ldots, \nu(x_n)) \end{array}$$

where $\nu(x_i)$ depend on the valuations of terms of u.

Example 2.52. Consider the affine line in $\mathbb{A}^3_{\mathbb{C}\{\{t\}\}}$ given by

$$x_1 = u, \qquad x_2 = tu + 1, \qquad x_3 = u - t.$$

The defining equations of this line are clearly $x - z - t = 0$ and $tx - y + 1 = 0$. The valuations of the coordinates are

$$\nu(x_1) = \nu(u), \qquad \nu(x_2) = \nu(tu + 1), \qquad \nu(x_3) = \nu(u - t).$$

We have five cases, depending on the value of $\nu(u)$.

- If $\nu(u) < -1$, then $\nu(x_1) = \nu(u), \nu(x_2) = 1 + \nu(u), \nu(x_3) = \nu(u)$.
- If $\nu(u) = -1$, then $\nu(x_1) = -1, \nu(x_2) \geq 0, \nu(x_3) = -1$.
- If $-1 < \nu(u) < 1$, then $\nu(x_1) = \nu(u), \nu(x_2) = 0, \nu(x_3) = \nu(u)$.
- If $\nu(u) = 1$, then $\nu(x_1) = 1, \nu(x_2) = 0, \nu(x_3) \geq 1$.
- If $\nu(u) > 1$, then $\nu(x_1) = \nu(u), \nu(x_2) = 0, \nu(x_3) = 1$.

This gives the tropical affine line in $\mathbb{T}^3$ given by the edge connecting the vertices $(-1, 0, -1)$ and

$(1, 0, 1)$ together with the four rays

$$\begin{aligned}
&(-1, 0, -1) + (-1, -1, -1)\bar{\mathbb{R}}_{\geq 0}, \\
&(-1, 0, -1) + (0, 1, 0)\bar{\mathbb{R}}_{\geq 0}, \\
&(1, 0, 1) + (0, 0, 1)\bar{\mathbb{R}}_{\geq 0}, \\
&(1, 0, 1) + (1, 0, 0)\bar{\mathbb{R}}_{\geq 0}.
\end{aligned}$$

One can easily obtain the same tropical line from the defining equations.

Now consider a projective line in a projective toric variety X over K given by an embedding

$$\begin{array}{rcl}
\mathbb{P}^1_K & \hookrightarrow & X \\
(u, v) & \mapsto & (z_0(u, v), \ldots, z_n(u, v))
\end{array}$$

where $z_i(u, v)$ are linear functions in u and v. Here we get

$$\begin{array}{rcl}
\mathbb{TP}^1 & \hookrightarrow & \mathrm{Trop}(X) \\
(\nu(u), \nu(v)) & \mapsto & (\nu(z_0), \ldots, \nu(z_n))
\end{array}$$

where $\nu(z_i)$ depend on valuations of u and v. From the description of $\mathbb{TP}^1$ in Example 2.48 we know that if $\nu(u) \neq \infty$, we can dehomogenize by setting $\nu(u) = 0$, and get the affine chart $\mathbb{T}^1$ with coordinate $\nu(v) \in \bar{\mathbb{R}}$. Similarly, if $\nu(v) \neq \infty$, we obtain $\mathbb{T}^1$ with coordinate $\nu(u) \in \bar{\mathbb{R}}$.

We will give an example of this procedure in §4.3, and we will use it to tropicalize the lines on the Dwork pencil of quintic threefolds in §8.2, §8.3 and §10.3.

3 Affine Manifolds and Parametrized Tropical Curves

By the Structure Theorem, tropical curves are balanced 1-dimensional polyedral complexes. This suggests the description of a tropical curves as a map from a graph to some ambient space satisfying the balancing condition. The ambient space must locally look like $\mathbb{R}^n$ with transition function respecting the polyhedral structure. This is called an *affine manifold*. Under some condition for the tropical curves we can allow the affine manifold to have codimension 2 singularities. In this section we will give this more general definition of *parametrized tropical curves*. Then we show that such curves arise naturally on the *dual intersection complex* of (families with central fiber) a union of toric varieties.

For the rest of this section we fix a lattice $M \cong \mathbb{Z}^n$, its dual lattice $N = \mathrm{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^n$ and the corresponding vector spaces $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$.

3.1 Affine Manifolds with Singularities

Affine manifolds locally look like $M_{\mathbb{R}}$ with charts respecting the affine structure.

Definition 3.1. Let B be an n-dimensional manifold. An *affine structure* on B is given by an open cover $\{U_i\}$ along with coordinate charts $\psi_i : U_i \to M_{\mathbb{R}}$, whose transition functions $\psi_i \circ \psi_j^{-1}$ lie in the group of affine transformations $\mathrm{Aff}(M_{\mathbb{R}}) = M_{\mathbb{R}} \rtimes \mathrm{GL}_n(\mathbb{R})$ of $M_{\mathbb{R}}$. The affine structure is *integral* resp. *tropical* if the transition functions lie in the subgroup $M \rtimes \mathrm{GL}_n(\mathbb{Z})$ resp. $M_{\mathbb{R}} \rtimes \mathrm{GL}_n(\mathbb{Z})$ of $\mathrm{Aff}(M_{\mathbb{R}})$.

Definition 3.2. A *(tropical, integral) affine manifold with singularities* is a manifold B with an open subset $B_0 \subseteq B$ which carries a (tropical, integral) affine structure such that the *discriminant locus* $\Gamma := B \setminus B_0$ is a locally finite union of locally closed submanifolds of codimension ≥ 2.

Definition 3.3. A *polyhedral decomposition* $\mathscr{P}$ *of a topological manifold* B is a set of subsets σ of B such that $B = \cup_{\sigma \in \mathscr{P}} \sigma$ and

1. Each $\sigma \in \mathscr{P}$ is equipped with a homeomorphisms to a polyhedron in $M_{\mathbb{R}}$ with faces of rational slope and at least one vertex. In particular, we can speak of the *faces* of σ as the inverse images of faces of the polyhedron in $M_{\mathbb{R}}$.
2. If $\sigma \in \mathscr{P}$ and $\tau \subseteq \sigma$ is a face, then $\tau \in \mathscr{P}$.
3. If $\sigma_1, \sigma_2 \in \mathscr{P}$, $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

We want to define a structure of tropical affine manifold with singularities on a topological manifold B from a polyhedral decomposition $\mathscr{P}$ of B. We have a natural affine stucture on the interior *Int*(σ) of each cell $\sigma \in \mathscr{P}$ giving an affine structure on $\cup_{\sigma \in \mathscr{P}_{\max}} \mathrm{Int}(\sigma)$. We can extend this affine structure to all of B by giving a fan structure at each vertex v of $\mathscr{P}$.

Definition 3.4. Let $\mathscr{P}$ be a polyhedral decomposition of a topological manifold B. The *open star* of $\tau \in \mathscr{P}$ is

$$U_\tau = \bigcup_{\substack{\sigma \in \mathscr{P} \\ \tau \subseteq \sigma}} \mathrm{Int}(\sigma).$$

A *fan structure* along $\tau \in \mathscr{P}$ is a continuous map $S_\tau : U_\tau \to \mathbb{R}^k$ where $k = \dim \mathrm{B} - \dim \tau$, satisfying

1. $S_\tau^{-1}(0) = \mathrm{Int}(\tau)$.
2. If $\tau \subseteq \sigma \in \mathscr{P}$, then $S_\tau|_{\mathrm{Int}(\sigma)}$ is an integral affine submersion onto its image, with $\dim S_\tau(\sigma) = \dim \sigma - \dim \tau$.
3. For $\tau \subseteq \sigma$, define $K_{\tau,\sigma}$ to be the cone generated by $S_\tau(\sigma \cap U_\tau)$. Then

$$\Sigma_\tau = \{K_{\tau,\sigma} \mid \tau \subseteq \sigma \in \mathscr{P}\}$$

is a fan with $|\Sigma_\tau|$ convex.

Two fan structures $S_\tau, S_{\tau'}$ are considered *equivalent* if $S_\tau = \alpha \circ S_{\tau'}$ for some $\alpha \in \mathrm{GL}_k(\mathbb{Z})$.

Remark 3.5. A fan structure S_τ along τ can be viewed as describing an affine structure in a direction transversal to τ.

Definition 3.6. If $S_\tau : U_\tau \to \mathbb{R}^k$ is a fan structure along $\tau \in \mathscr{P}$ and $\sigma \supseteq \tau$, then $U_\sigma \subseteq U_\tau$. We then obtain a *fan structure along σ induced by S_τ* given by the composition

$$U_\sigma \hookrightarrow U_\tau \xrightarrow{S_\tau} \mathbb{R}^k \to \mathbb{R}^k/L_\sigma \cong \mathbb{R}^l$$

where $L_\sigma \subseteq \mathbb{R}^k$ is the linear span of $K_{\tau,\sigma}$. This is well-defined up to equivalence.

Remark 3.7. The most important case is when $\tau = v$ is a vertex of $\mathscr{P}$. Then a fan structure is an identification of a neighborhood of v in B with a neighborhood of the origin in $\mathbb{R}^n$, identifying $\mathscr{P}$ with a fan Σ_v in $\mathbb{R}^n$.

Construction 3.8. Let B be a topological manifold (possibly with boundary) equipped with a polyhedral decomposition $\mathscr{P}$ such that all $\sigma \in \mathscr{P}$ have faces of rational slope. We construct a structure of tropical affine manifold with singularities on B. The interior of each cell $\sigma \in \mathscr{P}$ carries a natural tropical affine structure. Indeed, σ is equipped with a homeomorphism to a polyhedron in $M_\mathbb{R}$, which is embedded in the affine space it spans in $M_\mathbb{R}$. Since the faces of σ have rational slope, this defines a tropical affine structure on

$$\bigcup_{\sigma \in \mathscr{P}_{\max}} \mathrm{Int}(\sigma) \subseteq B.$$

Given a fan structure S_v at each vertex $v \in \mathscr{P}$, we can construct a tropical structure on B as follows. First we choose a discriminant locus $\Gamma \subseteq B$ subject to the following conditions:

1. Γ does not contain any vertex of $\mathscr{P}$.
2. Γ is disjoint from the interior of any maximal cell of $\mathscr{P}$.
3. For any $\rho \in \mathscr{P}$ which is a codimension one cell not contained in ∂B, the connected components of $\rho \setminus \Gamma$ are in one-to-one correspondence with vertices of ρ, with each vertex contained in the corresponding connected component.

For a vertex v of $\mathscr{P}$, let W_v denote a choice of open neighborhood of v with $W_v \subseteq U_v$ satisfying that if $v \in \rho$ with ρ a codimension one cell then $W_v \cap \rho$ is the connected component of $\rho \setminus \Gamma$ containing v. Then

$$\{\mathrm{Int}(\sigma) \mid \sigma \in \mathscr{P}\} \cup \{W_v \mid v \in \mathscr{P}^{[0]}\}$$

form an open cover of $B_0 = B \setminus \Gamma$. We define an affine structure on B_0 via the given affine structure on $\mathrm{Int}(\sigma)$ for $\sigma \in \mathscr{P}_{\max}$,

$$\psi_\sigma : \mathrm{Int}(\sigma) \hookrightarrow M_{\mathbb{R}}$$

and the composed maps

$$\psi_v : W_v \hookrightarrow U_v \xrightarrow{S_v} \mathbb{R}^{\dim B}$$

where the first map is the inclusion.

Example 3.9. If B is compact without boundary of any dimension, we can take Γ to be the union of all simplices in the first barycentric subdivision of $\mathscr{P}$ which neither contain a vertex of $\mathscr{P}$ nor intersect the interior of a maximal cell of $\mathscr{P}$.

Proposition 3.10. Construction 3.8 gives B the structure of a tropical affine manifold with singularities. If furthermore all polyhedra in $\mathscr{P}$ are lattice polytopes, then in fact the affine structure is integral.

Proof. The crucial point is that the affine charts ψ_v induced by the choice of fan structure are compatible with the charts ψ_σ on the interior of maximal cells of $\mathscr{P}$, but this follows precisely from item (2) in the definition of a fan structure. □

Definition 3.11. A collection of fan structures $\{S_v \mid v \in \mathscr{P}^{[0]}\}$ is *compatible* if, for any two vertices v, w of $\tau \in \mathscr{P}$, the fan structures on τ induced by S_v and S_w are equivalent. Given such a compatible set of fan structures, we obtain a well-defined fan structure along every $\tau \in \mathscr{P}$.

Definition 3.12. A *tropical manifold* is a pair $(B, \mathscr{P})$ where B is a tropical affine manifold with singularities obtained from the polyhedral decomposition $\mathscr{P}$ of B and a compatible collection $\{S_v \mid v \in \mathscr{P}^{[0]}\}$ of fan structures. $(B, \mathscr{P})$ is an *integral tropical manifold* if in addition all polyhedra in $\mathscr{P}$ are lattice polyhedra.

Examples 3.13. Consider the following special cases.

(1) Any lattice polyhedron σ (bounded or not) with at least one vertex supplies an example of an integral tropical manifold with $B = \sigma$ and $\mathscr{P}$ the set of faces of σ. In this case the affine structure on $\mathrm{Int}(\sigma)$ extends to give the structure of an affine manifold on σ. Here $\Gamma = \emptyset$.

(2) Let $\Delta \subseteq M_{\mathbb{R}}$ be a reflexive lattice polytope. Then $B = \partial\Delta$ carries the obvious polyhedral decomposition $\mathscr{P}$ consisting of the proper faces of Δ. These faces are lattice polytopes. So, to specify an integral tropical manifold structure on B, we need only specify a fan structure at each vertex v of Δ. This is done via the projection $S_v : U_v \to M_{\mathbb{R}}/\mathbb{R}v$. Compatibility is easily checked, as the induced fan structure on a cell $\omega \in \mathscr{P}$ containing v is the projection $U_\omega \to M_{\mathbb{R}}/\mathbb{R}\omega$, where $\mathbb{R}\omega$ now denotes the subspace of $M_{\mathbb{R}}$ spanned by ω.

(3) In the case of (2), let $\mathscr{P}'$ be a refinement of $\mathscr{P}$ by integral lattice polytopes. Then we can use the same prescription as in (2) for the fan structure at the vertices. The discriminant locus $\Gamma' \subseteq B$ determined by $\mathscr{P}'$ may be much bigger, with $\Gamma' \cap \mathrm{Int}(\sigma) \neq \emptyset$ for some maximal proper face σ of Δ. However, the affine structure induced by $\mathscr{P}'$ on $\mathrm{Int}(\sigma)$ is compatible with the obvious affine structure on $\mathrm{Int}(\sigma)$, so it extends across points of $\Gamma' \cap \mathrm{Int}(\sigma)$. Thus we can replace Γ' with $\Gamma' \cap \bigcup_{\substack{\tau\in\mathscr{P} \\ \dim\tau=\dim\Delta-2}} \tau$.

3.2 Tropical Curves on Tropical Manifolds

Fix a tropical manifold $(B, \mathscr{P})$. Let $\bar{\Gamma}$ be a weighted connected graph with no bivalent vertices and all weights positive. Let $\bar{\Gamma}^{[0]}$ and $\bar{\Gamma}^{[1]}$ denote the sets of vertices and edges of $\bar{\Gamma}$. Let $w : \bar{\Gamma}^{[1]} \to \mathbb{N} \setminus \{0\}$ be the weight function. Let $\bar{\Gamma}^{[0]}_\infty$ be a subset of the set of univalent vertices of $\bar{\Gamma}$, and write $\Gamma = \bar{\Gamma} \setminus \Gamma^{[0]}_\infty$.

Remark 3.14. $B_0 \setminus \partial B_0$ is a tropical affine manifold, and hence carries a local system Λ. Let $u : B_0 \setminus \partial B_0 \hookrightarrow B$ be the inclusion. For U a contractible open set in B_0, $\Gamma(U, i_\star\Lambda) \cong \mathbb{Z}^n$. But if U is a small neighborhood of a point of Δ, and the affine structure can't be extended across this point, then

$$\Gamma(U, i_\star\Lambda) \cong \mathbb{Z}^k \tag{3.1}$$

with $k < n$, the monodromy invariant part of Λ on U.

Definition 3.15. A *parametrized tropical curve* is a proper continuous map $h : \Gamma \to B$ with:

(1) For each edge E of Γ, $h|_E$ is an immersion (the image can self-intersect). Furthermore, there is a section $u \in \Gamma(E, h^{-1}(i_\star\Lambda))$ which is tangent to every point of $h(E)$.

(2) For every vertex V of Γ, let $E_1, \ldots, E_m \in \Gamma^{[1]}$ be the edges adjacent to V. If $h(V) \in \Delta$, there is no further condition. Otherwise, let $u_1, \ldots, u_m$ be integral tangent vectors at $h(V)$, i.e., elements of the stalk $(i_\star\Lambda)_{h(V)}$, with u_i primitive, tangent to $h(E_i)$, and pointing away from $h(V)$. Then $\sum_{j=1}^m w(E_j)u_j = 0$.

Remark 3.16. Condition (1) means that locally $h(E)$ is a line of rational slope. If $h(E)$ contains a point of Δ with non-trivial monodromy, the tangent direction to $h(E)$ near this point is completely determined by (3.1). In particular, $h(E)$ must pass that point in a monodromy invariant direction. Condition (2) tells us that, besides the usual balancing condition, we can have edges terminating at singular points. Even if an edge terminates at a singular point, however, it still must be tangent to a monodromy invariant direction at the singular point.

3.3 The Dual Intersection Complex

We describe the dual intersection complex of a *toric degneration* $\mathfrak{X} \to D$, following [**11**], §7.

Definition 3.17. Let R be a discrete valuation ring over an algebraically closed field k. A *toric degeneration* is a normal algebraic space $\mathfrak{X}$ flat over Spec R such that

(1) The general fiber is irreducible and normal.

(2) If $\nu : \tilde{\mathfrak{X}}_0 \to \mathfrak{X}_0$ is the normalization, then $\tilde{\mathfrak{X}}_0$ is a disjoint union of toric varieties, the conductor locus $C \subseteq \tilde{\mathfrak{X}}_0$ is reduced, and the map $C \to \nu(C)$ is unramified and generically two-to-one. The square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \tilde{\mathfrak{X}}_0 \arrow[d, "\nu"] \\ \nu(C) \arrow[r] & \mathfrak{X}_0 \end{tikzcd}$$

is cartesian and cocartesian. For simplicity, we assume that every irreducible component of $\mathfrak{X}_0$ is itself normal so that $\nu : X_i \to \nu(X_i)$ is an isomorphism.

(3) $\mathfrak{X}_0$ is a reduced Gorenstein space and C restricted to each irreducible component of $\tilde{\mathfrak{X}}_0$ is the union of all toric Weil divisors of that component.

(4) There exists a closed subset $Z \subseteq \mathfrak{X}$ of relative codimension ≥ 2 such that Z satisfies the following properties: Z does not contain the image under ν of any toric stratum of $\tilde{\mathfrak{X}}_0$, and for any point $x \in \mathfrak{X} \setminus Z$, there is a neighborhood $\tilde{U}_x$ (in the analytic topology) of x, an $(n+1)$-dimensional affine toric variety Y_x, a regular function f_x on Y_x given by a monomial, and a commutative diagram

$$\begin{tikzcd} \tilde{U}_x \arrow[r, "\psi_x"] \arrow[d, "f|_{\tilde{U}_x}"] & Y_x \arrow[d, "f_x"] \\ D' \arrow[r, "\varphi_x"] & \mathbb{C} \end{tikzcd}$$

where ψ_x and φ_x are open embeddings and $D' \subseteq D$. Furthermore, f_x vanishes precisely once on each toric divisor of Y_x.

The *strata* of $\mathfrak{X}_0$ are the elements of the set

$$\mathrm{Strata}(\mathfrak{X}_0) = \{\nu(S) \mid S \text{ is a toric stratum of } X_i \text{ for some } i\}.$$

Definition 3.18. The *dual intersection complex* of $\mathfrak{X} \to D$ is the following integral tropical manifold $(B, \mathscr{P})$. We construct (1) the topological manifold B, (2) a polyhedral decomposition $\mathscr{P}$ of B, and (3) a fan structure at each vertex of $\mathscr{P}$, giving $(B, \mathscr{P})$ the structure of an integral affine manifold with singularities.

(1) Let $\{x\} \in \mathrm{Strata}(\mathfrak{X}_0)$ be a zero-dimensional stratum. Applying Definition 3.17, (4), to a neighborhood of x, there is a toric variety Y_x such that in a neighborhood of x, $f : \mathfrak{X} \to D$ is locally isomorphic to $f_x : Y_x \to \mathbb{C}$, where f_x is given by a monomial. Then there is a lattice polytope $\sigma_x \subseteq M_{\mathbb{R}}$ such that $C(\sigma_x) = \{(rm, r) \mid m \in \sigma, r \geq 0\}$ is the cone defining the toric variety Y_x. We construct B by gluing together the polytopes

$$\{\sigma_x \mid \{x\} \in \mathrm{Strata}(\mathfrak{X}_0)\}.$$

By our assumption that every irreducible component of $\mathfrak{X}_0$ is itself normal, there is a one-to-one inclusion reversing correspondence between faces of σ_x and elements of $\mathrm{Strata}(\mathfrak{X}_0)$ containing x. We can then identify faces of σ_x and $\sigma_{x'}$ if they correspond to the same strata of $\mathfrak{X}_0$.

(2) Let

$$\mathscr{P} = \{\sigma \subseteq B \mid \sigma \text{ is a face of } \sigma_x \text{ for some zero-dimensional stratum } x\}.$$

There is a one-to-one inclusion reversing correspondence between strata of $\mathfrak{X}_0$ and elements of $\mathscr{P}$.

(3) Each vertex $v \in \mathscr{P}$ corresponds to an irreducible component X_v of $\mathfrak{X}_0$ and this irreducible component is a toric variety with fan Σ_v in $\mathbb{R}^n$. Furthermore, there is a one-to-one correspondence between p-dimensional cones of Σ_v and p-dimensional cells of $\mathscr{P}$ containing v as a vertex, as thex both correspond to strata of $\mathfrak{X}_0$ contained in X_v. There is then a continuous map

$$\psi_v : U_v \to \mathbb{R}^n$$

which takes $U_v \cap \sigma$, for any $\sigma \in \mathscr{P}$ containing v as a vertex, into the corresponding cone of Σ_v integral affine linearly. These maps define fan structures at each vertex that are compatible (see Definition 3.11). This follows because there is a well-defined fan Σ_τ defining the stratum corresponding to τ.

Construction 3.8 gives $(B, \mathscr{P})$ the structure of an integral affine manifold with singularities.

Remark 3.19. One can also define the *intersection complex* of a toric degeneration. The aim of the *Gross-Siebert program* is to describe the phenomenom of *mirror symmetry* via these tropical manifolds. The dual intersection complex of a toric degeneration of Calabi-Yau varieties is the intersection complex of its mirror, and vice versa. Moreover, the intersection complex and its dual are related via the *discrete Legendre transform*. Reconstructing the toric degeneration from its (dual) intersection complex, one can hope to find the mirror partner to a given degeneration. For an overview of this program, see [**10**][**11**] and references therein.

Given a curve C in a toric variety or a toric degeneration, we can define the *associated tropical curve* in the dual intersection complex. This is a parametrized tropical curve in an affine manifold with singularities. In fact, what we need is a *torically transverse log curve*, but we will not worry about this issue here (some of our lines are not torically transverse!). Further, we restrict to the case of irreducible curves. For the general definition, see Definition 4.10 in [**10**].

Definition 3.20. Let $\mathfrak{X}$ be a toric degeneration and $f : C \to \mathfrak{X}_0$ (an embedding of) a curve in $\mathfrak{X}_0$. Let Γ_f be the weighted graph with one vertex V_C such that the set of (unbounded) edges of Γ_f is in one-to-one correspondence with the set $f^{-1}(\partial X_0)$, where ∂X_0 is the toric boundary of $\mathfrak{X}_0$, i.e., the union of codimension 1 toric strata of the X_i. Let $p \in f^{-1}(\partial X_0)$ correspond to an unbounded edge E_p. Then the weight $w(E_p)$ is the intersection multiplicity of C with ∂X_0 at $f(p)$.

We then define the *tropical curve associated to* $f : C \to \mathfrak{X}$ as $h : \Gamma_f \to M_{\mathbb{R}}$ with

(1) $h(V_C) = v \in \mathscr{P}$ if $f(C) \subseteq D_v$.
(2) $h(E_p)$ lies in cell $\sigma \in \mathscr{P}$ corresponding to the maximal-dimensional toric stratum of ∂X_0 containing $f(p)$.

Proposition 3.21. If $f : C^\dagger \to X^\dagger$ is a torically transverse log curve, then $h : \Gamma_f \to M_{\mathbb{R}}$ is a parametrized tropical curve.

Proof. This is Proposition 4.11 in [**10**]. □

4 Example: A Family of Cubic Surfaces

Before considering the Dwork pencil of quintic threefolds we will look at a simpler two-dimensional example where everything can be presented in 3-dimensional space.

4.1 A Pencil of Cubic Surfaces and the Lines on it

Consider the pencil $\mathfrak{X}$ where $\mathfrak{X}_t$ is the cubic surface in $\mathbb{P}^3$ given by

$$t(z_0^3 + z_1^3 + z_2^3 + z_2^3) + z_1 z_2 z_3 = 0. \tag{4.1}$$

The central fiber $\mathfrak{X}_0$ ist just the union of three of the four coordinate hyperplanes in $\mathbb{P}^3$, hence contains infinitely many lines. Now assume $t \neq 0$.

A line lying on $\mathfrak{X}_t$ has clearly at least two coordinates not equal to zero. Let this line be parametrized by coordinates $(u, v) \in \mathbb{P}^1$. Each coordinate is a linear combination of (u, v). At least two functions must be linearly independent as functions of u and v. Taking z_2 and z_3 to be these cooridiates, a line may be written as

$$z = (bu + qv, cu + rv, u, v), \quad (u, v) \in \mathbb{P}^1. \tag{4.2}$$

Inserting (4.2) into (4.1) we get a system of equations with the 27 solutions

$$\begin{array}{llll}
b = 0, & c = -\omega, & q = \dfrac{\gamma}{3t}, & r = \dfrac{1}{3t\omega}; \\
b = \dfrac{\gamma}{3t}, & c = \dfrac{1}{3t\omega}, & q = 0, & r = -\omega; \\
b = \gamma, & c = 3t\omega, & q = \omega\gamma, & r = 3t\omega^2;
\end{array}$$

with ω a third root of unity and γ a solution of

$$\gamma^3 = -1 - 27t^3.$$

For each row we have 9 ways to choose ω and γ giving a total number of 27 lines. Note that the second row is obtained from the first one by $b \leftrightarrow q$ and $c \leftrightarrow r$ corresponding to the change of coordinates $z_2 \leftrightarrow z_3$ and the reparametrization $u \leftrightarrow v$.

A Gröbner basis calculation leads to the defining equations for these lines. For the first row we get

$$\begin{aligned}
3tz_0 - \gamma z_3 &= 0, \\
3tz_1 + 3t\omega z_2 - \omega^2 z_3 &= 0,
\end{aligned} \tag{4.3}$$

for the second row

$$
\begin{aligned}
3tz_0 - \gamma z_2 &= 0,\\
3tz_1 - \omega^2 z_2 + 3t\omega z_3 &= 0,
\end{aligned}
\tag{4.4}
$$

and for the third row

$$
\begin{aligned}
z_0 - \gamma z_2 - \omega\gamma z_3 &= 0,\\
z_1 - 3t\omega z_2 - 3t\omega^2 z_3 &= 0.
\end{aligned}
\tag{4.5}
$$

Another way to get the defining equations is the following. A line in $\mathbb{P}^3$ is given by two linear equations

$$c_0z_0 + c_1z_1 + c_2z_2 + c_3z_3 = 0.$$

Inserting different values $(u, v) = (1, 0)$ and $(u, v) = (1, 0)$ into (4.2) gives two points on the line. For the first row this gives the system of linear equations

$$\begin{pmatrix} 0 & -\omega & 1 & 0 \\ \frac{\gamma}{3t} & \frac{1}{3t\omega} & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0.$$

The matrix has rank 2 and a basis of its kernel is given by $\{b_1, b_2\}$ where

$$b_1 = \begin{pmatrix} 3t \\ 0 \\ 0 \\ -\gamma \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 3t \\ 3t\omega \\ -\omega^2 \end{pmatrix}.$$

This gives the defining equations as in (4.3). For the other rows the calculation is similar.

4.2 Tropicalization of the Family of Cubic Surfaces

On the toric stratum of $\mathbb{P}^3$ where all coordinates are nonzero we may set $z_0 = 1$ and tropicalize (4.1) to

$$\min\{1, 3x_1 + 1, 3x_2 + 1, 3x_3 + 3, x_1 + x_2 + x_3\}.$$

The tropical hypersurface in $\mathbb{R}^3$ defined by this tropical polynomial consists of the polytope

$$\operatorname{Conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\},$$

together with the nine unbounded parts

$$
\begin{aligned}
U_i &= \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,\middle|\, x_i = 0, \sum_{k\neq i} x_k \geq 1, \forall k \neq i : x_k \geq 0\right\}, \qquad i = 1,2,3,\\
U_i' &= \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,\middle|\, 2x_i + 1 = \sum_{k\neq i} x_k \leq 1, \forall k \neq i : x_k \geq x_i\right\}, \quad i = 1,2,3,\\
U_{ij} &= \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_i = x_j \leq 0, \forall k \neq i,j : x_k \geq x_i + 1\right\}, \quad i \neq j \in \{1,2,3\}.
\end{aligned}
$$

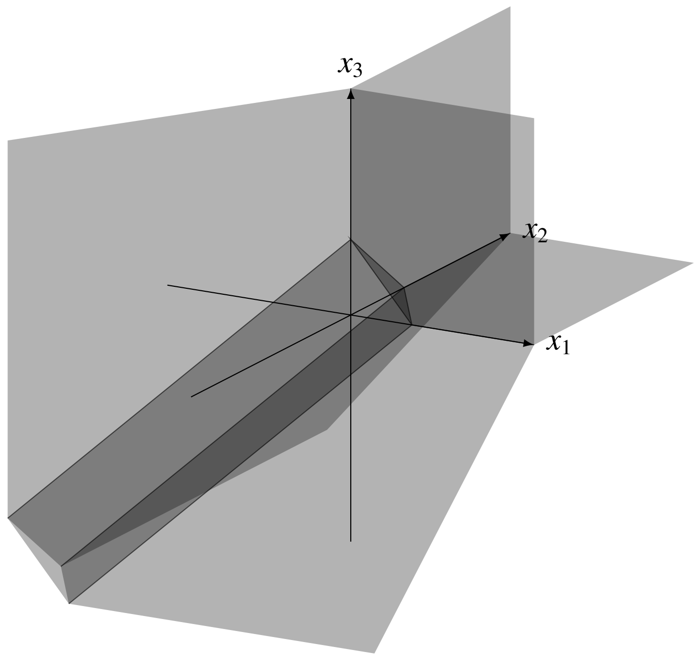


Figure 4.1: Tropicalization of the family of cubic surfaces in $\mathbb{R}^3$.

On the toric stratum where only $z_0 = 0$, we may set $z_3 = 1$ and get the tropicalization

$$\min\{3x_1 + 1, 3x_2 + 1, 1, x_1 + x_2\},$$

giving the tropical hypersurface in $\mathbb{R}^2$ shown in Figure 4.2.

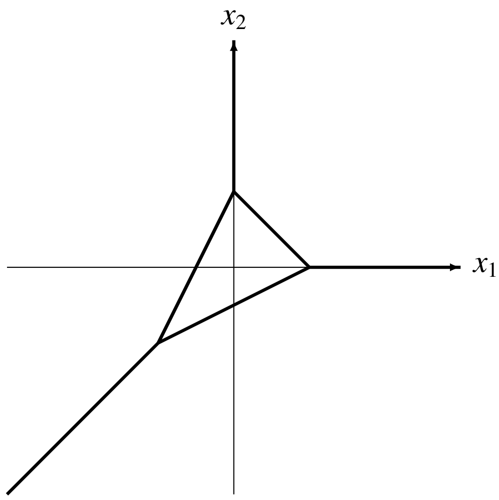


Figure 4.2: Tropicalization of the family of cubic surfaces for $z_0 = 0$.

On the stratum where only $z_3 = 0$, we may set $z_0 = 1$ and tropicalize (4.1) to

$$\min\{1, 3x + 1, 3x + 2\}.$$

This gives the tropical hypersurface in $\mathbb{R}^2$ as shown in Figure 4.3.

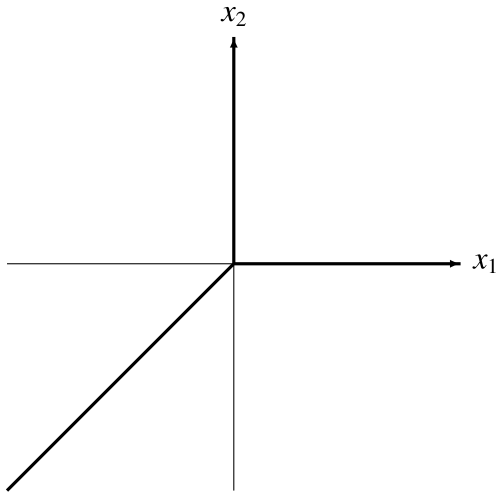


Figure 4.3: Tropicalization of a family of cubic surfaces for $z_3 = 0$.

On the strata where $z_1 = 0$ or $z_2 = 0$ we get the same picture as in Figure 4.3.

4.3 Tropicalization of the Lines

We have two different ways to tropicalize the lines on $\mathfrak{X}_t$. First, we tropicalize the polynomials in the defining equations and calculate the tropical lines they define. Then we tropicalize the lines from their parametric representation. Of course, both ways lead to the same tropical lines.

The lines do not lie in a lower-dimensional toric stratum of $\mathbb{P}^3$. Hence, we may assume $z_0 = 1$.

For $z_0 = 1$, the polynomials in (4.3) tropicalize to

$$\min\{1, x_3\}, \qquad \min\{x_1 + 1, x_2 + 1, x_3\}.$$

These tropical polynomials define the tropical line in $\mathbb{R}^2$ consisting of the three rays

$$\begin{aligned} x_1 \geq 0, \quad x_2 = 0, \quad x_3 = 1; \\ x_1 = 0, \quad x_2 \geq 0, \quad x_3 = 1; \\ x_1 = x_2 \leq 0, \qquad x_3 = 1. \end{aligned} \tag{4.6}$$

This is shown in Figure 4.4.

Note that the nine lines defined by (4.3) tropicalize to the same tropical line (4.6).

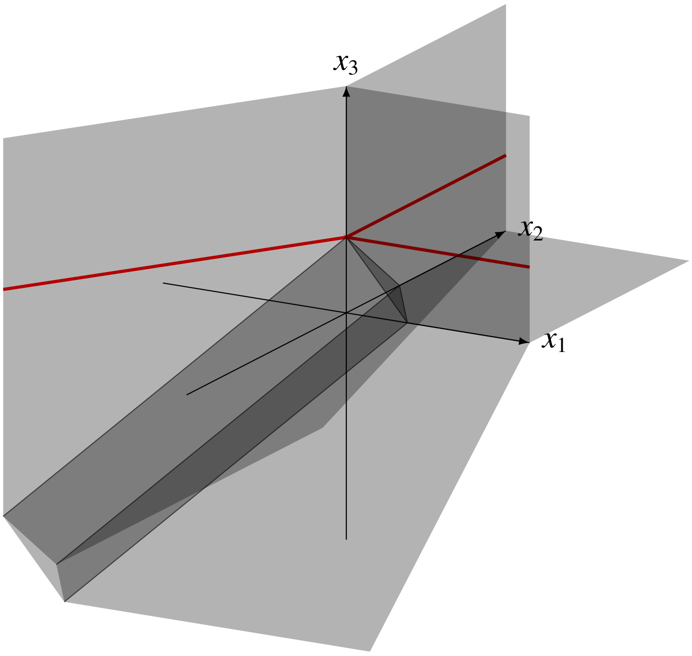


Figure 4.4: Tropicalization of nine lines on the family of cubics.

The polynomials in (4.4) tropicalize to

$$\min\{1, x_2\}, \qquad \min\{x_1 + 1, x_2, x_3 + 1\},$$

giving the tropical line in Figure 4.5.

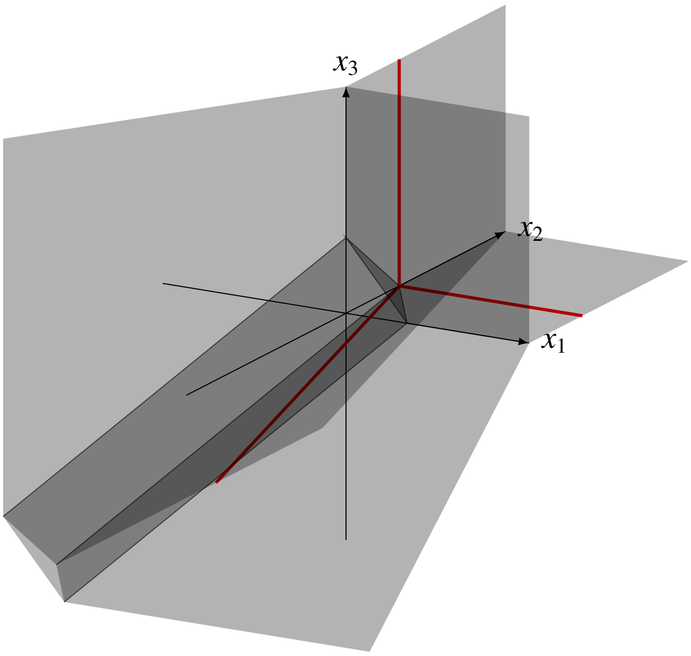


Figure 4.5: Tropicalization of nine lines on the family of cubics.

As already noted, this can be obtained from 4.4 by the change of coordinates $x_2 \leftrightarrow x_3$.

From the symmetry of (4.1) it should be clear that the tropicalization of the lines given by (4.5) should be the same as Figure 4.4 under the change $x_1 \leftrightarrow x_3$ or as Figure 4.5 under the change $x_1 \leftrightarrow x_2$. However, this tropicalization appears to be harder, since the polynomials in (4.5) do not give a tropical basis for $I = \langle z_0 - \gamma z_2 - \omega\gamma z_3, z_1 - 3t\omega z_2 - 3t\omega^2 z_3\rangle$.

Define

$$g_1 = z_0 - \gamma z_2 - \omega\gamma z_3,$$
$$g_2 = z_1 - 3t\omega z_2 - 3t\omega^2 z_3,$$

and $\mathcal{F} = \{g_1, g_2\}$. Then, for $z_0 = 1$,

$$\mathrm{Trop}(g_1) = \min\{0, x_2, x_3\},$$
$$\mathrm{Trop}(g_2) = \min\{x_1, x_2 + 1, x_3 + 1\}.$$

The finite intersection $\cup_{f\in\mathcal{F}}\mathrm{Trop}(V(f))$ of tropical hypersurfaces is shown in Figure 4.6. This is not a tropical variety, since the polyhedral complex is not of pure dimension.

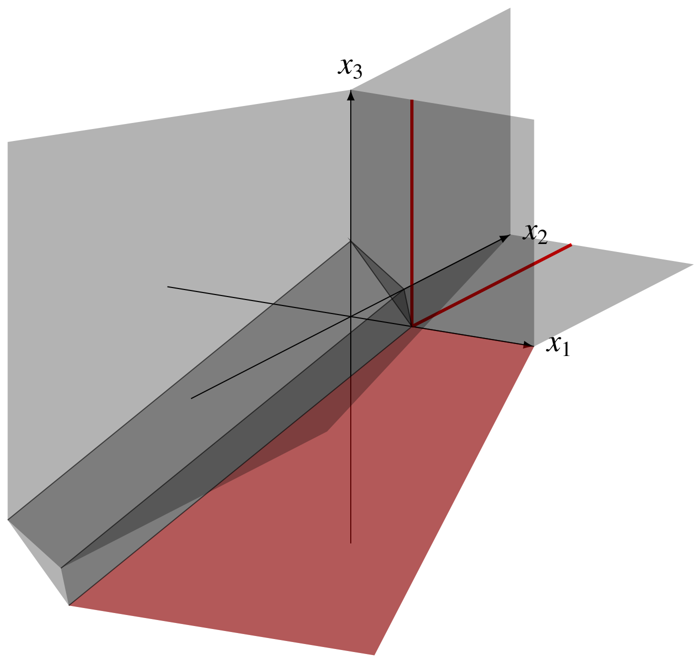


Figure 4.6: The finite intersection $\cup_{f\in\mathcal{F}}\mathrm{Trop}(V(f))$ of tropical hypersurfaces.

We will use the technique from the proof of Theorem 2.51 to calculate a tropical basis of I to be

$$\mathcal{T} = \left\{z_0 - \gamma z_2 - \omega\gamma z_3, z_1 - 3t\omega z_2 - 3t\omega^2 z_3, z_1 - \frac{3t\omega}{\gamma} z_0\right\}. \tag{4.7}$$

For $w \in \mathbb{R}^4$ in the interior of a zero- or one-dimensional polyhedron of $\cup_{f\in\mathcal{F}}\mathrm{Trop}(V(f))$, the initial ideal $\mathrm{in}_w(I)$ does not contain a monomial.

Take $w = (0, 2, 1, 1)$. Define

$$g = 3t\omega g_1 - \gamma g_2 = 3t\omega z_0 - \gamma z_1.$$

The initial ideal $\mathrm{in}_w(I)$ contains $\mathrm{in}_w(g) = -\gamma z_1$, hence the monomial z_1.

We take the lexicographical term order given by

$$z_0^\alpha \prec z_1^\beta, \quad z_1^\alpha \prec z_2^\beta, \quad z_2^\alpha \prec z_3^\beta, \qquad \text{for all} \qquad \alpha, \beta \in \mathbb{N} \setminus \{0\}.$$

Then

$$\begin{aligned} \mathrm{in}_{<_w}(g_1) &= -\omega\gamma z_3, \\ \mathrm{in}_{<_w}(g_2) &= -3t\omega^2 z_3. \end{aligned}$$

We use the Buchberger algorithm as in the proof of Proposition 2.27 to calculate $\mathcal{G}_{<_w}(I)$. We have

$$S_{<_w}(g_1, g_2) = 3t\omega z_0 - \gamma z_1.$$

This is not divisible by $\mathrm{in}_{<_w}(g_1)$ and $\mathrm{in}_{<_w}(g_2)$, so it the remainder of the multivariate polynomial division of itself relative to $\{g_1, g_2\}$, and we have to add it to $\mathcal{G}_{<_w}(I)$.

Running the Buchberger algorithm again gives nothing new, so

$$\mathcal{G}_{<_w}(I) = \{g_1, g_2, S_{<_w}(g_1, g_2)\}.$$

The normal form to z_1 with respect to $\mathcal{G}_{<_w}(I)$ is

$$h = \frac{3t\omega}{\gamma} z_0.$$

Adding $z_1 - h$ to $\mathcal{F}$ this gives the tropical basis $\mathcal{T}(I)$ as claimed in (4.7).

For $z_0 = 1$, the polynomials in $\mathcal{T}$ tropicalize to

$$\min\{0, x_2, x_3\}, \qquad \min\{x_1, x_2 + 1, x_3 + 1\}, \qquad \min\{x_1, 1\}.$$

The third minimum is not unique precisely if $x_1 = 1$. Then the first and second minimum are achieved at least twice in the same points, giving the tropical line in Figure 4.7.

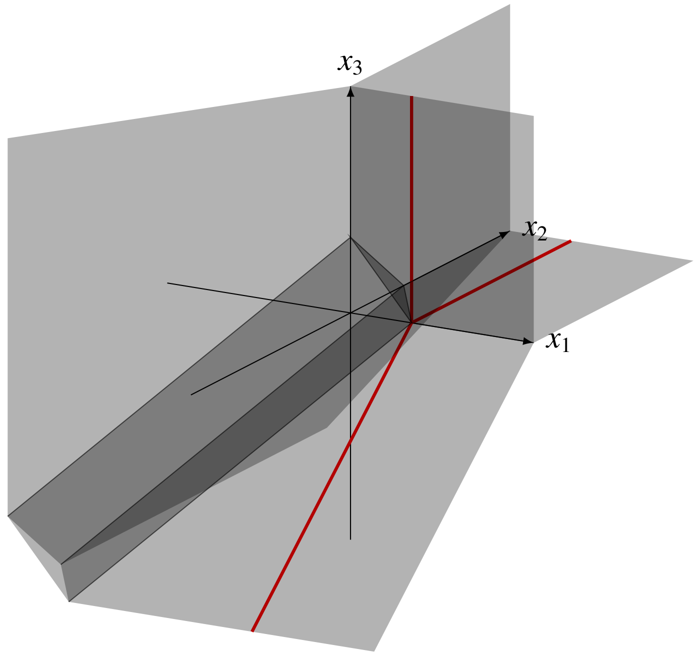


Figure 4.7: Tropicalization of nine lines on the family of cubics.

We may also calculate the tropicalizations from the parametric representation. We will do this for the first row. The valuations of the coordinates are

$$\begin{aligned}
\nu(z_0) &= -1 + \nu(v) \\
\nu(z_1) &= \nu\left(-\omega u + \frac{1}{3tw} v\right) \\
\nu(z_2) &= \nu(u) \\
\nu(z_3) &= \nu(v).
\end{aligned}$$

We dehomogenize tropically by setting $\nu(z_0) = 0$, i.e., $\nu(v) = 1$. Then

$$\begin{aligned}
\nu(z_1) &= \nu\left(-\omega u + \frac{1}{3tw} v\right) \\
\nu(z_2) &= \nu(u) \\
\nu(z_3) &= 1.
\end{aligned}$$

If $\nu(u) < 0$ then

$$\nu(z_1) = \nu(u), \quad \nu(z_2) = \nu(u), \quad \nu(z_3) = 1.$$

If $\nu(u) = 0$ then

$$\nu(z_1) \geq 0, \quad \nu(z_2) = 0, \quad \nu(z_3) = 1.$$

If $\nu(u) > 0$ then

$$\nu(z_1) = 0, \quad \nu(z_2) = \nu(u), \quad \nu(z_3) = 1.$$

This gives the same tropical lines as before, shown in Figure 4.4.

4.4 Lines in the Dual Intersection Complex

The central fiber $\mathfrak{X}_0$ consists of 3 coordinate hyperplanes of $\mathbb{P}^3$. This gives a picture roughly as in Figure 4.8. (The three unbounded edges are in fact parallel!) Furthermore, the fan structure at each vertex is the normal fan to the standard simplex, i.e., the fan for $\mathbb{P}^2$.

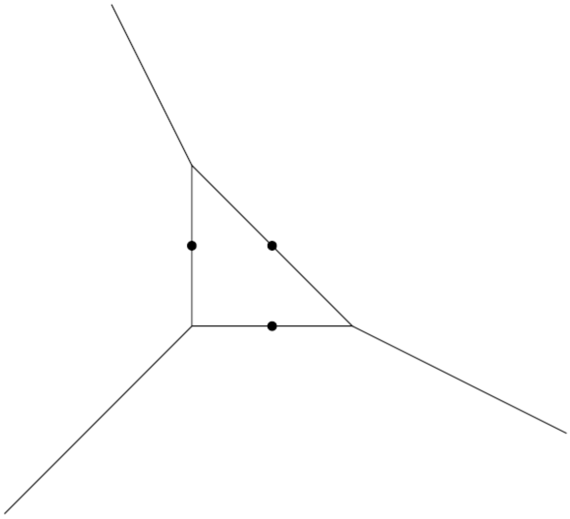


Figure 4.8: The dual intersection complex of the family of cubics.

The monodromy of Λ around the three singular points, and one finds at each point that it is given by $\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)$ in a suitable basis see [**11**] and [**9**].. As a result, it is possible to pull apart each singular point into three singular points, each with monodromy $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, as in Figure 4.9.

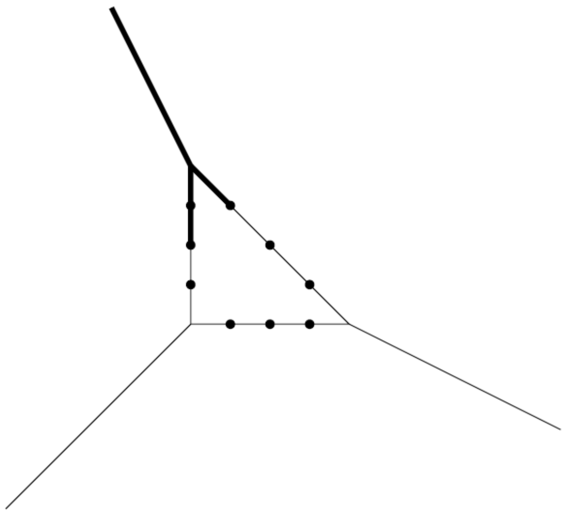


Figure 4.9: The darker lines give one of the 27 parametrized tropical curves.

Figure 4.9 also shows a parametrized tropical curve. There are 27 such tropical curves: 3 choices of unbounded edge, and 3^2 choices for the endpoints. Morally, these tropical curves correspond to the 27 lines on the cubic surface. Indeed, each line on the central fiber $\mathfrak{X}_0$ lies in a coordinate hyperplane meets two codimension 1 strata of it.

5 Lines on the Dwork Pencil of Quintic Threefolds

The Dwork pencil of quintic threefolds is the family $\mathfrak{X}$ consisting of quintic hypersurfaces in $\mathbb{P}^4$ given by

$$t(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5) + z_0 z_1 z_2 z_3 z_4 = 0 \tag{5.1}$$

where $t \in \mathbb{P}^1$ is a complex parameter. We want to find all lines lying in such a hypersurface.

Remark 5.1. Sometimes the Dwork family is defined to be parametrized by $\psi = -\frac{1}{5t}$. Note that in this case the central fiber is not a union of toric varieties.

5.1 Explicit parametrization

The central fiber $\mathfrak{X}_0$ is the union of the five coordinate hyperplanes in $\mathbb{P}^4$. Each hyperplane is a $\mathbb{P}^3$ and the space of lines lying in such a hyperplane is the Grassmannian $G(2,4) = \mathbb{G}(1,3) \cong \mathbb{P}^2$. Hence, there are infinitely many lines on $\mathfrak{X}_0$. Now assume $t \neq 0$. We want to find the lines in $\mathfrak{X}_t$.

We want to parametrize the coordinates of a line on the Dwork pencil by $(u,v) \in \mathbb{P}^1$. At least two coordinates must be linearly independent as functions of u and v. Taking u and v for the first two coordinates, the most general form of z lying on that line is

$$z = (u, v, bu + qv, cu + rv, du + sv), \qquad (u,v) \in \mathbb{P}^1, \tag{5.2}$$

with b, c, d, r, s, t complex parameters. In order for the lines to lie on the quintic $\mathfrak{X}_t$ the parameters must satisfy the equations

$$\begin{aligned} b^5 + c^5 + d^5 + 1 &= 0 \\ 5t(b^4 q + c^4 r + d^4 s) + bcd &= 0 \\ 10t(b^3 q^2 + c^3 r^2 + d^3 s^2) + cdq + bdr + bcs &= 0 \\ 10t(b^2 q^3 + c^2 r^3 + d^2 s^3) + dqr + brs + cqs &= 0 \\ 5t(bq^4 + cr^4 + ds^4) + qrs &= 0 \\ q^5 + r^5 + s^5 + 1 &= 0. \end{aligned} \tag{5.3}$$

This can be seen by inserting (5.2) into (5.1) and setting the coefficients in u and v equal to zero.

Proposition 5.2. Assume two parameters in (5.2) are zero. Then the lines in (5.2) are of the form

$$z = (u, v, -\zeta^k u, -\zeta^l v, 0), \qquad (u,v) \in \mathbb{P}^1 \tag{5.4}$$

where $1 \leq k, l \leq 5$ and ζ is a nontrivial fifth root of unity. There are 375 such *isolated lines*.

Proof. The proof is a straightforward computation. Three cases can occur.

First case: First, it may be that two parameters appearing in the same coordinate of z are zero. So let e.g. $d = s = 0$. Then the above equation simplifies to

$$z = (u, v, bu + rv, cu + sv, 0), \qquad (u, v) \in \mathbb{P}^1$$

with b, c, q, r satisfying

$$\begin{aligned} b^5 + c^5 + 1 &= 0 \\ b^4 q + c^4 r &= 0 \\ b^3 q^2 + c^3 r^2 &= 0 \\ b^2 q^3 + c^2 r^3 &= 0 \\ bq^4 + cr^4 &= 0 \\ q^5 + r^5 + 1 &= 0 \end{aligned}$$

By the third and fourth equation we have $b^3 q^3 = -bc^2 r^3 = -qc^3 r^2$ implying $br = qc$. One of our parameters must be nonzero, since otherwise we would have $z = (u, v, 0, 0, 0)$ not lying on X_t. So assume $r \neq 0$. Then

$$b = \frac{qc}{r} \tag{5.5}$$

and the first equation gives

$$\left(\frac{q}{r}\right)^5 c^5 + c^5 + 1 = 0. \tag{5.6}$$

Multiplying (5.6) with r^5 gives $(q^5 + r^5)c^5 r^5 + r^5 = 0$. By the last equation, $q^5 + r^5 = -1$, so $(1 - c^5)r^5 = 0$. But since $r \neq 0$, we have $c^5 = -1$. Now by (5.6) and (5.5) it follows that $q = 0, b = 0$ and the last equation again implies $r^5 = -1$. This leads to (5.4).

Second case: Now consider the case when two parameters of the same coordinate u or v are zero, e.g. $q = s = 0$. Then (5.4) gets

$$z = (u, v, bu, cu + rv, du), \qquad (u, v) \in \mathbb{P}^1$$

with b, c, d, r satisfying

$$\begin{aligned} b^5 + c^5 + d^5 + 1 &= 0 \\ 5tc^4 r + bcd &= 0 \\ 10tc^3 r^2 + bdr &= 0 \\ c^2 r^3 &= 0 \\ cr^4 &= 0 \\ r^5 + 1 &= 0. \end{aligned}$$

From the last equation it immediately follows that $r^5 = -1$. Then the fifth equation gives $c = 0$ and the third equation gives $b = 0$ or $d = 0$. Without loss of generality take $d = 0$. Then by the first equation $b^5 = -1$ giving (5.4).

Third case: Assume that two parameters of different coordinates in z and different variables u, v are zero, e.g. $c = s = 0$. Then

$$z = (u, v, bu + qv, rv, du), \qquad (u, v) \in \mathbb{P}^1$$

with b, d, q, r satisfying

$$\begin{aligned} b^5 + d^5 + 1 &= 0 \\ b^4 q &= 0 \\ 10tb^3q^2 + bdr &= 0 \\ 10tb^2q^3 + dqr &= 0 \\ bq^4 &= 0 \\ q^5 + r^5 + 1 &= 0 \end{aligned}$$

From the second or fifth equation it follows that $b = 0$ or $q = 0$. Without loss of generality take $q = 0$. Then the last equation gives $r^5 = -1$.The third equation gives $b = 0$ or $d = 0$. Take $d = 0$. The first equation now gives $b^5 = -1$ leading to (5.4).

Number of lines: We have 25 ways of choosing k and l. There are 5 ways to choose the zero coordinate and 3 ways of distributing u and v. This gives 15 coordinate permutations leading to different lines. Hence, there are $5 \cdot 15 = 375$ lines of this form. □

Proposition 5.3. Assume exactly one parameter in (5.2) is zero. Then up to permutations the lines in (5.2) are the 5000 *van Geemen lines* of the form

$$z = (u, v, \zeta^{-k-l} bu, \zeta^k (cu + \omega v), -\zeta^l \omega^2 (cu - v)), \qquad (u, v) \in \mathbb{P}^1 \tag{5.7}$$

where $1 \leq k, l \leq 5$ and

$$b = -\frac{3}{10t}\gamma^2, \qquad c = -\frac{1}{10t}(1 - \omega)i\gamma$$

where ζ is a nontrivial fifth root of unity, ω is a nontrivial third root of unity, and γ is a solution of

$$\gamma^{10} - \frac{1}{9}\gamma^5 - \left(\frac{10t}{3}\right)^5 = 0. \tag{5.8}$$

Proof. First we note that multiplying z_2, z_3 or z_4 with a fifth root of unity doesn't change (5.3). Moreover, the product of these fifth roots of unity must be one. Hence, by abuse of notation we can write

$$z = (u, v, \zeta^{-k-l} bu, \zeta^k (cu + rv), \zeta^l (du + sv)).$$

Write $d = c\tilde{d}$. By the second, third and fourth equation we see that we can write $b = -\frac{1}{10t}\tilde{b}\gamma^2$ and $c = -\frac{1}{10t}(1-\omega)\gamma$ with ω a nontrivial third root of unity. The factors are chosen for convenience. Note that $(1-\omega)^2 = -3\omega$. After cancellation (5.3) becomes

$$\begin{aligned} \tilde{b}^5\gamma^{10} - 27(1+\tilde{d}^5)\gamma^5 - (10t)^5 &= 0 \\ 3\omega r + 3\omega\tilde{d}^4 s + 2\tilde{b}\tilde{d} &= 0 \\ 3\omega r^2 + 3\omega\tilde{d}^3 s^2 + \tilde{b}\tilde{d}r + \tilde{b}s &= 0 \\ 3\omega r^3 + 3\omega\tilde{d}^2 s^3 + \tilde{b}rs &= 0 \\ r^4 + \tilde{d}s^4 &= 0 \\ r^5 + s^5 + 1 &= 0. \end{aligned} \tag{5.9}$$

The second and fourth equation of (5.9) together give

$$\frac{\tilde{b}}{3\omega} = \frac{r + \tilde{d}^4 s}{2\tilde{d}} = \frac{r^3 + \tilde{d}^2 s^3}{rs}. \tag{5.10}$$

Solving the second equality together with the fifth and sixth equation of (5.9) under the assumption that all parameters are nonzero gives (up to fifth roots of unity) exactly

$$\tilde{d} = -\omega^2, \quad r = \omega, \quad s = \omega^2$$

Now (5.10) gives $\tilde{b} = 3$ and the first equation of (5.9) gives the desired equation

$$\gamma^{10} - \frac{1}{9}\gamma^5 - \left(\frac{10t}{3}\right)^5 = 0.$$

We have 5 ways to choose γ, given γ^5, 2 ways to choose ω and 25 ways to choose k, l. 20 permutations lead to different lines. Hence, there are $25 \cdot 5 \cdot 2 \cdot 20 = 5000$ van Geemen lines. □

Remark 5.4. In §5 we will show that on a generic quintic threefold there are 2875 lines. Since the number of van Geemen lines exceeds this number, they have to lie in a family of lines. Indeed, the next proposition shows that the general lines on the Dwork pencil come in a family. The van Geemen lines are special lines in this family.

Proposition 5.5. Assume all parameters in (5.2) are nonzero. These *general lines* come in a *family of lines*. Up to permutations they are of the form

$$z = \Big(\alpha(\sigma,\tau)u, \alpha(\tau,\sigma)v, -\tau^{4/5}\beta(\sigma)(\sigma u + v), \beta(\sigma\tau)(\sigma u + \tau v), -\sigma^{4/5}\beta(\tau)(u + \tau v)\Big), \tag{5.11}$$

where

$$\begin{aligned} \alpha(\sigma,\tau)^5 &= \sigma^4(1-\sigma)(1-\tau)(1-\sigma\tau)\Big(1 - \tau(1+\sigma) + \tau^2(1-\sigma+\sigma^2)\Big) \\ \beta(\sigma)^5 &= (1-\sigma)(1-\sigma+\sigma^2) \end{aligned} \tag{5.12}$$

and σ, τ are complex parameters satisfying $F(\sigma, \tau) = 0$ with

$$\begin{aligned} F(\sigma,\tau) = {} & 10^5 \tilde{t}^5 \sigma^2 \tau^2 (1-\sigma)^2 (1-\tau)^2 (1-\sigma\tau)^2 \\ & - (1-\sigma+\sigma^2)(1-\tau+\tau^2)(1-\sigma\tau+\sigma^2\tau^2)\Big(1-\tau(1+\sigma)+\tau^2(1-\sigma+\sigma^2)\Big) \\ & \cdot \Big(1-\sigma(1+\tau)+\sigma^2(1-\tau+\tau^2)\Big). \end{aligned} \tag{5.13}$$

Definition 5.6. We denote the curve in $\mathbb{C}^2$ given by $F(\sigma, \tau) = 0$ by C_t^0.

Remark 5.7. In §6.1 we will show that C_t^0 consists of two irreducible components $C_{\pm\varphi}^0$ of genus 6 that are isomorphic to each other. In §6.4 we describe the curves parametrizing the lines in the family as a 125:1 cover of the desingularizations $C_{\pm\varphi}$ of these components.

Proof. Making the transformations

$$q = r\kappa, \qquad b = c\kappa\tau, \qquad d = c\kappa\tau\delta, \qquad s = r\kappa\tau\delta\sigma, \qquad t = \frac{\delta\kappa^2\tau}{cr}\tilde{t},$$

after cancellation, the equations 5.3 become

$$\begin{aligned} 1 + c^5\Big(1+\kappa^5\tau^5(1+\delta^5)\Big) &= 0 \\ 5\tilde{t}(1+\kappa^5\tau^4(1+\delta^5\sigma\tau)) + \tau &= 0 \\ 10\tilde{t}(1+\kappa^5\tau^3(1+\delta^5\sigma^2\tau^2)) + 1 + \tau + \sigma\tau &= 0 \\ 10\tilde{t}(1+\kappa^5\tau^2(1+\delta^5\sigma^3\tau^3)) + 1 + \sigma + \sigma\tau &= 0 \\ 5\tilde{t}(1+\kappa^5\tau(1+\delta^5\sigma^4\tau^4)) + \sigma &= 0 \\ 1 + s^5\Big(1+\kappa^5(1+\delta^5\sigma^5\tau^5)\Big) &= 0 \end{aligned}$$

and depend on δ and κ only through δ^5 and κ^5. Combining the second, third, fourth and fifth relations with multiplicities $(1, -1, 1, -1)$ results in the cancellation of the constant terms. The remaining equation gives

$$\delta^5 = \frac{(1-\tau)(1-\tau+\tau^2)}{\sigma\tau^4(1-\sigma)(1-\sigma+\sigma^2)}.$$

Solving the central four relations also for κ^5 and $\tilde{t}$, we find

$$\kappa^5 = -\frac{(1-\sigma)(1-\sigma-\sigma^2)}{\tau(1-\sigma\tau)(1-\sigma\tau+\sigma^2\tau^2)}, \qquad \tilde{t} = \frac{1}{10}\frac{1-\sigma\tau+\sigma^2\tau^2}{(1-\sigma)(1-\tau)}.$$

Moreover, these three relations exhaust the content of the four central relations in (5.9). The first and last relations in (5.9) now give c and s in terms of σ and τ. Finally, on substituting what we know into the relation

$$t^5 = \frac{\delta^5\kappa^{10}\tau^5}{c^5 r^5}\tilde{t}^5,$$

we obtain a constraint $F(\sigma, \tau) = 0$ with F as in (5.13) and z are as in (5.11). $\square$

5.2 Implicitization

Any line in $\mathbb{P}^4$ is a complete intersection of hypersurfaces of degree 1, i.e., is given by three linear equations

$$c_0 z_0 + c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4 = 0. \tag{5.14}$$

For the isolated lines from Proposition 5.2 this is clearly

$$z_0 + \zeta^k z_2 = 0, \quad z_1 + \zeta^l z_3 = 0, \quad z_4 = 0. \tag{5.15}$$

Proposition 5.8. The defining equations for the van Geemen line as in (5.7) are

$$\begin{aligned} b z_0 - \zeta^{k+l} z_2 &= 0 \\ c z_0 + \zeta^{-k}\omega z_3 - \zeta^{-l} z_4 &= 0 \\ \omega z_1 + \zeta^{-k}\omega^2 z_3 + \zeta^{-l} z_4 &= 0. \end{aligned} \tag{5.16}$$

Proof. Setting $(u, v) = (1, 0)$ and $(u, v) = (0, 1)$ gives two points spanning the van Geemen line. Inserting into (5.14) gives a linear system of two equations for the c_i:

$$\begin{pmatrix} 1 & 0 & \zeta^{-k-l} b & \zeta^k c & -\zeta^l \omega^2 c \\ 0 & 1 & 0 & \zeta^k \omega & \zeta^l \omega^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0.$$

The matrix has rank 2 and a basis of its kernel is given by b_1, b_2, b_3 where

$$b_1 = \begin{pmatrix} b \\ 0 \\ -\zeta^{k+l} \\ 0 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} c \\ 0 \\ 0 \\ \zeta^{-k}\omega \\ -\zeta^{-l} \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 \\ \omega \\ 0 \\ \zeta^{-k}\omega^2 \\ \zeta^{-l} \end{pmatrix}$$

This gives the coefficients in the defining equations as stated. □

Proposition 5.9. The defining equations for the familiy of lines are

$$\begin{aligned} \sigma\tau^{4/5}\beta(\sigma)\alpha(\tau,\sigma) z_0 + \tau^{4/5}\beta(\sigma)\alpha(\sigma,\tau) z_1 + \alpha(\sigma,\tau)\alpha(\tau,\sigma) z_2 &= 0 \\ \sigma\beta(\sigma\tau)\alpha(\tau,\sigma) z_0 + \tau\beta(\sigma\tau)\alpha(\sigma,\tau) z_1 - \alpha(\sigma,\tau)\alpha(\tau,\sigma) z_3 &= 0 \\ \sigma^{4/5}\beta(\tau)\alpha(\tau,\sigma) z_0 + \sigma^{4/5}\tau\beta(\tau)\alpha(\sigma,\tau) z_1 + \alpha(\sigma,\tau)\alpha(\tau,\sigma) z_4 &= 0. \end{aligned} \tag{5.17}$$

with $F(\sigma, \tau) = 0$ as in Proposition 5.5.

Proof. Fix σ, τ and t. Choosing different values $(1,0), (0,1)$ for (u,v) gives two different points lying on the curve with parameters σ, τ on X_t. Inserting into (5.14) gives a linear system of two equations for the c_i:

$$\begin{pmatrix} \alpha(\sigma,\tau) & 0 & -\sigma\tau^{4/5}\beta(\sigma) & \sigma\beta(\sigma\tau) & -\sigma^{4/5}\beta(\tau) \\ 0 & \alpha(\tau,\sigma) & -\tau^{4/5}\beta(\sigma) & \tau\beta(\sigma\tau) & -\sigma^{4/5}\tau\beta(\tau) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0.$$

The matrix has rank 2 and a basis of its kernel is given by b_1, b_2, b_3 where

$$b_1 = \begin{pmatrix} \sigma\tau^{4/5}\beta(\sigma)\alpha(\tau,\sigma) \\ \tau^{4/5}\beta(\sigma)\alpha(\sigma,\tau) \\ \alpha(\sigma,\tau)\alpha(\tau,\sigma) \\ 0 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} \sigma\beta(\sigma\tau)\alpha(\tau,\sigma) \\ \tau\beta(\sigma\tau)\alpha(\sigma,\tau) \\ 0 \\ -\alpha(\sigma,\tau)\alpha(\tau,\sigma) \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} \sigma^{4/5}\beta(\tau)\alpha(\tau,\sigma) \\ \sigma^{4/5}\tau\beta(\tau)\alpha(\sigma,\tau) \\ 0 \\ 0 \\ \alpha(\sigma,\tau)\alpha(\tau,\sigma) \end{pmatrix}.$$

This gives the coefficients in the defining equations as stated. □

5.3 Plücker Coordinates

The Grassmannian $G(2,5) = \mathbb{G}(1,4)$ parametrizing lines in $\mathbb{P}^4$ is isomorphic to the intersection of five quadric hypersurfaces in $\mathbb{P}^9$ via the Plücker map.

Let the line corresponding to a point $l \in G(2,5)$ be spanned by the rows of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{pmatrix}.$$

The *Plücker coordinates* $\pi_{ij} = x_i y_j - x_j y_i, 0 \le i < j \le 4$ are the 2×2-minors of this matrix. The *Plücker map*

$$\begin{array}{rcl} G(2,5) & \to & \mathbb{P}^9 \\ l & \mapsto & (\pi_{ij})_{0 \le i < j \le 4} \end{array}$$

is an embedding. Its image is the intersection of five quadrics in $\mathbb{P}^9$ given by the *Plücker relations*

$$\pi_{ab}\pi_{cd} - \pi_{ac}\pi_{bd} + \pi_{ad}\pi_{bc} = 0, \quad 0 \le a < b < c < d \le 4.$$

This can be found e.g. in [**7**].

We want to describe the Plücker coordinates of the lines we found in §5.1.

Proposition 5.10. Any isolated line has exactly six Plücker coordinates equal to zero.

Proof. The isolated line $(u, v, -\zeta^k u, -\zeta^l v, 0)$ is spanned by the rows of

$$\begin{pmatrix} 1 & 0 & -\zeta^k & 0 & 0 \\ 0 & 1 & 0 & -\zeta^l & 0 \end{pmatrix}.$$

The Plücker coordinates are

$$\begin{aligned}
&\pi_{01} = 1 \\
&\pi_{02} = 0 && \pi_{12} = \zeta^k \\
&\pi_{03} = -\zeta^l && \pi_{13} = 0 && \pi_{23} = \zeta^{k+l} \\
&\pi_{04} = 0 && \pi_{14} = 0 && \pi_{24} = 0 && \pi_{34} = 0.
\end{aligned}$$

The other isolated lines are obtained via other choices of k, l and permutations of the coordinates. In any case 6 Plücker coordinates are equal to zero. $\square$

Proposition 5.11. Any van Geemen line has exactly one Plücker coordinate equal to zero. For each Plücker coordinate π_{ij} there are 500 van Geemen lines with $\pi_{ij} = 0$.

Proof. The van Geemen line $(u, v, \zeta^{-k-l} bu, \zeta^k(cu + \omega v), -\zeta^l \omega^2(cu - v))$ is spanned by the rows of

$$\begin{pmatrix} 1 & 0 & \zeta^{-k-l} b & \zeta^k c & -\zeta^l \omega^2 c \\ 0 & 1 & 0 & \zeta^k \omega & \zeta^l \omega^2 \end{pmatrix}.$$

The only 2×2-minor that is zero is π_{02}. All other van Geemen lines are obtained via different choices of k, l, ω, γ and permutations. Only 10 permutations give a different zero Plücker coordinate. For each such permutation there are 25 choices for k and l and 20 choices for ω and γ. This gives 500 van Geemen lines with $\pi_{ij} = 0$. $\square$

Proposition 5.12. A general member of the family of lines has no Plücker coordinate that is equal to zero.

Proof. A line in this family is spanned by the rows of

$$\begin{pmatrix} \alpha(\sigma,\tau) & 0 & -\tau^{4/5}\beta(\sigma)\sigma & \beta(\sigma\tau)\sigma & -\sigma^{4/5}\beta(\tau) \\ 0 & \alpha(\tau,\sigma) & -\tau^{4/5}\beta(\sigma) & \beta(\sigma\tau)\tau & -\sigma^{4/5}\beta(\tau)\tau \end{pmatrix} \tag{5.18}$$

with $\sigma, \tau, \alpha(\sigma,\tau), \alpha(\tau,\sigma), \beta(\sigma), \beta(\tau)$ and $\beta(\sigma\tau)$ all nonzero for general choices of σ and τ. Hence, no 2×2-minor is equal to zero. $\square$

5.4 Symmetries and the Action of $S_5 \rtimes \mathcal{G}$

The manifolds of the Dwork pencil have a group of isomorphisms isomorphic to $\mathcal{G} \rtimes S_5$, where S_5 is the permutation group acting on the five coordinates and

$$\mathcal{G} = \left\{(n_0, n_1, n_2, n_3, n_4) \in \left(\mathbb{Z}/5\mathbb{Z}\right)^5 \mid \textstyle\sum_{i=1}^5 n_i \equiv 0 \bmod 5\right\} \Big/ \left\{(n,n,n,n,n) \mid n \in \mathbb{Z}/5\mathbb{Z}\right\} \cong \left(\mathbb{Z}/5\mathbb{Z}\right)^3 .$$

This group has order $5! \cdot 5^3 = 15000$. The product in $\mathcal{G} \rtimes S_5$ is given by

$$(\sigma, (n_0, n_1, n_2, n_3, n_3)) \cdot (\tau, (m_0, m_1, m_2, m_3, m_4)) = \left(\sigma \circ \tau, (n_{\tau(0)} + m_0, \ldots, n_{\tau(4)} + m_4\right) .$$

The action of $(\sigma, (n_0, n_1, n_2, n_3, n_4)) \in \mathcal{G} \rtimes S_5$ on $\mathbb{P}^4$ is given by

$$(z_0, z_1, z_2, z_3, z_4) \mapsto \left(\zeta^{n_0} z_{\sigma(0)}, \zeta^{n_1} z_{\sigma(1)}, \zeta^{n_2} z_{\sigma(2)}, \zeta^{n_3} z_{\sigma(3)}, \zeta^{n_4} z_{\sigma(4)}\right) .$$

In what follows we will discover the symmetries of the lines we found.

Proposition 5.13. Each isolated line is invariant under a subgroup of order 40 of $S_5 \rtimes \mathcal{G}$. This subgroup is isomorphic to $\mathbb{Z}/5\mathbb{Z} \rtimes D_4$ where D_4 is the dihedral group of the square. The other elements of $S_5 \rtimes \mathcal{G}$ give other isolated lines.

Proof. Of course, the action of $\mathcal{G} \rtimes S_5$ on an isolated line gives another isolated line. Since there are 375 isolated lines isomorphic to each other, each line must be invariant under a subgroup of order $15000/375 = 40$. Table 5.1 gives the generators of the group of isomorphisms of the isolated line $(u, v, -\zeta^k u, -\zeta^l v, 0)$ together with the reparametrization of (u, v) that yields the same parametric representation. Consider the square with vertices labeled by $0, 1, 2, 3$ in counterclockwise direction. Any of the elements $(02), (13), (01)(23)$ gives a reflection of this square. Two adjacent reflections generate D_4, hence $\langle(02), (13), (01)(23)\rangle = \langle(02), (01)(23)\rangle \cong D_4$. For each element of $\langle(02), (01)(23)\rangle \subset S_5$ there is a subgroup of $\mathcal{G}$ isomorphic to $\mathbb{Z}/5\mathbb{Z}$. The second and fourth element in Table 5.1 together generate $\mathbb{Z}/5\mathbb{Z} \rtimes D_4$. The group of isomorphisms is isomorphic to $D_4 \rtimes \mathbb{Z}/5\mathbb{Z}$ and indeed has order $|D_4| \cdot |\mathbb{Z}/5\mathbb{Z}| = 8 \cdot 5 = 40$. $\square$

Elements of $\mathbb{Z}/5\mathbb{Z} \rtimes D_4$ with $n \in \mathbb{Z}_5$	(u, v) trafnsformation
$(\mathrm{id}, (n, -n, n, -n, 0))$	$(\zeta^n u, \zeta^{-n} v)$
$((02), (n-k, -n, n+k, -n, 0))$	$(-\zeta^n u, \zeta^{-n} v)$
$((13), (n, -n-k, n, -n+k, 0))$	$(\zeta^n u, -\zeta^{-n} v)$
$((01)(23), (n, -n, n+k-l, -n-k+l, 0))$	(v, u)

Table 5.1: Elements of $\mathbb{Z}/5\mathbb{Z} \rtimes D_4$ and their corresponding reparametrizations of (u, v). The second and fourth element together generate $\mathbb{Z}/5\mathbb{Z} \rtimes D_4$.

Proposition 5.14. A van Geemen line with $\pi_{ij} = 0$ is invariant under an order 3 subgroup $\{g\} \rtimes \langle(klm)\rangle$ of $\mathcal{G} \rtimes S_5$ where $g \in \mathcal{G}$ and $\{i, j, k, l, m\} = \{0, 1, 2, 3, 4\}$.

Proof. A van Geemen line with $\pi_{02} = 0$ is$z = (u, v, \zeta^{-k-l}bu, \zeta^k(cu + \omega v), -\zeta^l\omega^2(cu - v))$. In this case $g = (0, -k, 0, -l, k + l)$. This is shown in Table 5.2. Note that $15000/3 = 5000$. □

Generator of $\{g\} \rtimes \langle(klm)\rangle$	(u, v) trafnsformation
$((134), (0, -k, 0, -l, k + l))$	$(u, -\omega^2(cu - v))$

Table 5.2: The generator of the group of isomorphisms for a van Geemen line with $\pi_{02} = 0$ and its corresponding reparametrization of (u, v).

Remark 5.15. In fact, this invariance was van Geemen's initial motivation to study these lines. It was known that a general quintic threefold contains 2875 lines. 375 of these lines are isolated lines. Now the number 2500 of missing lines is not divisible by 3, so a subgroup of order 3 has to fix some of these lines.

A general member in the family of lines of course has no stabilizer. But the action of $S_5 \rtimes \mathcal{G}$ on such a line should give another line in the family. In other words, for each permutation $\sigma \in S_5$ of the coordinates there should be a reparametrization of (σ, τ) and (u, v) yielding the same effect.

Proposition 5.16. Table 5.3 gives the reparametrizations of (σ, τ) and (u, v) yielding permutations of the coordinates of $\mathbb{P}^4$ generating S_5.

S_5 generator	(σ, τ) trafo.	(u, v) trafo.
(01)(24)	(τ, σ)	(v, u)
(01)	$\left(\frac{1}{\sigma}, \frac{1}{\tau}\right)$	$(-1)^{\frac{1}{5}}(\sigma\tau)^{\frac{8}{5}}(v, u)$
(34)	$\left(\frac{1}{\sigma}, \sigma\tau\right)$	$(-\sigma^{\frac{9}{5}}u, -\sigma^{-\frac{1}{5}}v)$
(12)	$\left(\frac{1-\sigma\tau}{1-\tau}, 1 - \tau\right)$	$\left(\frac{(1-\tau)(\sigma u+v)}{(\sigma\tau)^{\frac{1}{5}}(1-\sigma\tau)^{\frac{4}{5}}}, -\frac{(1-\sigma\tau)^{\frac{1}{5}}v}{(\sigma\tau)^{\frac{1}{5}}}\right)$

Table 5.3: Generators of S_5 and their corresponding reparametrizations of (σ, τ) and (u, v).

Proof. This can be checked easily. The table is taken from [**6**]. □

6 The Parametrizing Curves

6.1 The Curves $C^0_{\pm\varphi}$ in $\mathbb{C}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$

Proposition 6.1. The polynomial F from (5.13) factorizes to

$$F = \frac{1}{(5t)^5} F_+ F_- \quad \text{with} \quad F_\pm = G \pm \varphi H \tag{6.1}$$

where

$$\varphi^2 = -10^5 t^5 - \frac{3}{4} \tag{6.2}$$

and

$$\begin{aligned} G &= 3\sigma^2\tau^2 - \frac{1}{2}\sigma\tau(1+\sigma)(1+\tau)(1+\sigma\tau) + (1-\sigma+\sigma^2)(1-\tau+\tau^2)(1-\sigma\tau+\sigma^2\tau^2) \\ H &= \sigma\tau(1-\sigma)(1-\tau)(1-\sigma\tau). \end{aligned} \tag{6.3}$$

Proof. This is a simple calculation and can be most easily checked from the right to the left. □

This defines curves $C^0_{\pm\varphi}$ in $\mathbb{C}^2$ given by $F_\pm(\sigma, \tau) = 0$. Compactifying to $\mathbb{P}^1 \times \mathbb{P}^1$ we obtain curves

$$C^0_{\pm\varphi} = \left\{ ((\sigma_1 : \sigma_2), (\tau_1 : \tau_2)) \in \mathbb{P}^1 \times \mathbb{P}^1 \;\middle|\; \sigma_2^4 \tau_2^4 F_\pm\left(\frac{\sigma_1}{\sigma_2}, \frac{\tau_1}{\tau_2}\right) = 0 \right\}.$$

These are projective bidegree $(4, 4)$ curves. We write ∞ for $(1 : 0) \in \mathbb{P}^1$ and x for $(x : 1) \in \mathbb{P}^1$.

Proposition 6.2. For $(1/5t)^5 \neq 1, \infty$, the curves $C^0_{\pm\varphi}$ each have three singular points $(1, 1)$, $(0, \infty)$ and $(\infty, 0)$, all of them ordinary double points, and the geometric genus of $C^0_{\pm\varphi}$ is 6.

Proof. Using computer algebra one can calculate the Gröbner basis of the ideal generated by F_+, $\frac{\partial}{\partial\sigma}F_+$ and $\frac{\partial}{\partial\tau}F_+$ to be $\{t-1, s-1\}$. Hence the singular locus of F_+ is exactly the point $(1, 1)$. The same is true for F_-. Expanding $F_\pm$ as a series around $(1, 1)$ one can see

$$F_\pm = (\sigma-1)^2 + (\sigma-1)(\tau-1) + (\tau-1)^2 + R_\pm$$

where $R_\pm$ is a polynomial in σ and τ with all terms of degree at least 3. Thus $(1, 1)$ is an ordinary double point of $C^0_{\pm\varphi}$. The points at infinity lying on $C^0_{\pm\varphi}$ are

$$(\infty, -\omega), \quad (\infty, -\omega^2), \quad (\infty, 0), \quad (-\omega, \infty), \quad (-\omega^2, \infty), \quad (0, \infty).$$

Only the points $(0, \infty)$ and $(\infty, 0)$ are singular. Since the three singular points are related by the operations of Table 5.3, they all are ordinary double points. $C^0_{\pm\varphi}$ are curves of bidegree $(4, 4)$ with 3 ordinary double points. Hence, their geometric genus is $9 - 3 = 6$. □

Proposition 6.3. The curves $C^0_{+\varphi}$ and $C^0_{-\varphi}$ intersect in the 17 points

$$\begin{array}{ccccccc} & & (0,\infty), & (\infty,0), & (1,1), & & \\ (0,-\omega), & (0,-\omega^2), & (1,-\omega), & (1,-\omega^2), & (-\omega,-\omega^2), & (-\omega,\infty), & (-\omega^2,\infty), \\ (-\omega,0), & (-\omega^2,0), & (-\omega,1), & (-\omega^2,1), & (-\omega^2,-\omega), & (\infty,-\omega), & (\infty,-\omega^2). \end{array}$$

Proof. These are the solutions of (the bihomogenization) of $G = H = 0$. □

Proposition 6.4. The van Geemen lines appear precisely as limits, as we approach the points in which the curves $C^0_{\pm\varphi}$ intersect. Some of these limits are shown in Table 6.1. The other cases are obtained by acting with the elements of Table 5.3.

$(\sigma_\star, \tau_\star)$	τ for $\sigma = \sigma_\star + \epsilon$	(u, v)	Line
$(0, -\omega)$	$-\omega + 9\gamma^5\epsilon$	$\left(\frac{c\tilde{u}}{\epsilon}, -\tilde{v}\right)$	$\left(\tilde{u}, \tilde{v}, -\omega^2(c\tilde{u} - \tilde{v}), c\tilde{u} + \omega\tilde{v}, b\tilde{u}\right)$
$(1, -\omega)$	$-\omega + 9\omega\gamma^5\epsilon$	$\left(-\frac{\omega^2\tilde{v}}{\epsilon^{1/5}}, -\frac{c\tilde{u}+\omega\tilde{v}}{\epsilon^{1/5}}\right)$	$\left(\tilde{v}, c\tilde{u} + \omega\tilde{v}, -\omega^2(c\tilde{u} - \tilde{v}), b\tilde{u}, \tilde{u}\right)$
$(-\omega, -\omega^2)$	$-\omega^2 + \omega\left(\frac{10t}{3}\right)^5 \frac{\epsilon}{\gamma^{10}}$	$\left(\frac{c\tilde{u}+\omega\tilde{v}}{(1-\omega^2)^{1/5}\epsilon^{1/5}}, -\frac{\omega^2(c\tilde{u}-\tilde{v})}{(1-\omega^2)^{1/5}\epsilon^{1/5}}\right)$	$\left(c\tilde{u} + \omega\tilde{v}, -\omega^2(c\tilde{u} - \tilde{v}, b\tilde{u}, \tilde{v}, \tilde{u}\right)$
$(1, 1)$	$1 + \omega^2\epsilon + (\omega + 9\gamma^5)\epsilon^2$	$\left(\frac{\omega c\tilde{u}}{\epsilon^{6/5}} - \frac{\omega c\tilde{u}-\tilde{v}}{2\epsilon^{1/5}}, -\frac{\omega c\tilde{u}}{\epsilon^{6/5}} - \frac{\omega c\tilde{u}-\tilde{v}}{2\epsilon 1/5}\right)$	$\left(b\tilde{u}, \tilde{u}, \tilde{v}, -\omega^2(c\tilde{u} - \tilde{v}), c\tilde{u} + \omega\tilde{v}\right)$
$(1, 1)$	$1 + \omega\epsilon - (\omega + 9\gamma^5)\epsilon^2$	$\left(-\frac{\omega c\tilde{u}}{\epsilon^{6/5}} + \frac{\omega c\tilde{u}+\tilde{v}}{2\epsilon^{1/5}}, \frac{\omega c\tilde{u}}{\epsilon^{6/5}} + \frac{\omega c\tilde{u}+\tilde{v}}{2\epsilon 1/5}\right)$	$\left(\tilde{u}, b\tilde{u}, \tilde{v}, c\tilde{u} + \omega\tilde{v}, -\omega^2(c\tilde{u} - \tilde{v})\right)$

Table 6.1: The limiting process that gives rise to the van Geemen lines.

Proof. We will do this for the first row in Table 6.1. The other cases are similar. Write $\sigma = 0 + \epsilon$ and $\tau = -\omega + \tau_\epsilon\epsilon$ with τ_ϵ left to determine. Then

$$F(\sigma, \tau) = (10^5t^5 - 3(1 - \tau_\epsilon)\tau_\epsilon)\epsilon^2 + O(\epsilon^3).$$

The condition for the coefficient of ϵ^2 to vanish is precisely (5.8) for $\tau_\epsilon = 9\gamma^5$. □

Remark 6.5. Note that for the intersection in nonsingular points of $C^0_{\pm\varphi}$ we get two van Geemen lines, one for each curve, for the singular points we get four. This gives a total of 40 lines which under the action of $\mathcal{G}$ become the $40 \cdot 125 = 5000$ van Geemen lines.

6.2 The quintic del Pezzo Surface dP_5

To resolve the singularities of $C^0_{\pm\varphi} \subset \mathbb{P}^1 \times \mathbb{P}^1$ we have to blow up in the three singular points. These points do not lie on a line in $\mathbb{P}^1 \times \mathbb{P}^1$, hence are in general position. The blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at three general points is the del Pezzo surface dP_5 of degree 5, which is isomorphic to $\mathbb{P}^2$ blown up at 4 points in general position. Before giving the blow ups of the curves $C^0_{\pm\varphi}$ we will further look at dP_5 and its group of isomorphisms.

Proposition 6.6. The blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $(1, 1)$, $(0, \infty)$ and $(\infty, 0)$ can be given by

$$\begin{aligned} \Psi : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow dP_5 \subset \mathbb{P}^5 \\ (\sigma, \tau) &\mapsto (z_0, \ldots, z_5) \end{aligned} \tag{6.4}$$

with the 6 polynomials (written inhomogeneously for simplicity)

$$z_0 = \sigma^2\tau^2 - 1, \quad z_1 = \sigma\tau^2 - 1, \quad z_2 = \sigma^2\tau - 1, \quad z_3 = \sigma\tau - 1, \quad z_4 = \tau - 1, \quad z_5 = \sigma - 1.$$

Proof. The blow up is given by the polynomials of bidegree $(2, 2)$ which are zero in the three points $(1, 1)$, $(0, \infty)$ and $(\infty, 0)$. The polynomials of bidegree $(2, 2)$ are a 9-dimensional vector space with basis $\sigma_1^a\sigma_2^b\tau_1^c\tau_2^d$, $a + b = c + d = 2$. Hence, if we find 6 independent bidegree $(2, 2)$ polynomials satisfying the 3 relations of being zero at these points, we have found the blow up map. The polynomials above are clearly linearly independent. $\square$

The group of automorphisms of dP_5 is S_5. The action of S_5 on $\mathbb{P}^1 \times \mathbb{P}^1$, given by the transformations of (σ, τ) as in Table 5.3, induces these automorphisms on dP_5.

Proposition 6.7. The action of the generators of S_5 on dP_5 extend to an action on $\mathbb{C}^6$ given by the unique irreducible 6-dimensional representation of S_5. This is given in Table 6.2.

S_5 generator	action on $(z_0, z_1, z_2, z_3, z_4, z_5)$	trace
(01)(24)	$(-z_0, -z_2, -z_1, -z_3, -z_5, -z_4)$	-2
(01)	$(-z_0, -z_0 + z_5, -z_0 + z_4, -z_0 + z_3, -z_0 + z_2, -z_0 + z_1)$	0
(34)	$(-z_1 + z_5, -z_0 + z_5, -z_4 + z_5, -z_3 + z_5, -z_2 + z_5, z_5)$	0
(12)	$(-z_0 + z_2 + 2z_3 - 2z_5, -z_1 + 2z_3 + z_4 - z_5, z_2 - 2z_5, z_3 - z_5, z_4, -z_5)$	0

Table 6.2: Generators of S_5 and the image of their action on $z \in \mathbb{C}^6$.

Proof. This is a simple calculation. For example (01)(24) acts as $(\sigma, \tau) \mapsto (\tau, \sigma)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. By (6.4) this induces the action

$$z \mapsto (z_0, z_2, z_1, z_3, z_5, z_4) = (-z_0, -z_2, -z_1, -z_3, -z_5, -z_4)$$

on $dP_5 \subset \mathbb{P}^5$. Making the choices as in Table 6.2 the action extends to $\mathbb{C}^6$. All elements of S_5 act linearly, hence we get a representation of S_5 on $\mathbb{C}^6$. Comparing the traces with the character table of S_5 we see this is the unique irreducible 6-dimensional representation of S_5. $\square$

6.3 The Blow Ups $C_{\pm\varphi}$ of $C^0_{\pm\varphi}$

Proposition 6.8. The blow up $C_{\pm\varphi}$ of $C^0_{\pm\varphi}$ is defined by $G_z \pm \varphi H_z = 0$ in dP_5 where

$$\begin{aligned} G_z &= 2z_0^2 - 2z_0z_1 - 2z_0z_2 - 2z_0z_3 + z_0z_4 + z_0z_5 + 2z_1^2 + z_1z_2 - 2z_1z_3 - 2z_1z_4 \\ &\quad +2z_2^2 - 2z_2z_3 - 2z_2z_5 + 6z_3^2 - 2z_3z_4 - 2z_3z_5 + 2z_4^2 + z_4z_5 + 2z_5^2, \end{aligned}$$

and

$$H_z = \frac{1}{3}(-2z_0z_3 + z_0z_4 + z_0z_5 - z_1z_2 + 2z_1z_3 + 2z_2z_3 - 2z_3z_4 - 2z_3z_5 + z_4z_5).$$

Proof. Let $C_{\pm\varphi}$ be defined by $G_z \pm \varphi H_z = 0$ in $dP_5 \subset \mathbb{P}^5$. Then G_z has to be invariant under the S_5-action given in Table 6.2, and H_z has to be invariant up to a sign. Such polynomials can be found as $G_z = \sum_{g\in S_5} g(z_0z_1)$ and $H_z = \sum_{g\in S_5} g(z_0z_1)$. The unique S_5 invariant quadratic polynomial with variables $z_0, \dots, z_5$ is G_z as above, and the unique S_5 invariant quadratic polynomial, up to a sign, is H_z as above. One can verifiy that

$$\Psi^\star G_z = G(\sigma,\tau), \qquad \Psi^\star H_z = H(\sigma,\tau),$$

where Ψ denotes the blow up map (6.4). Hence, the curves C^0_φ are the images under the blow down $\Phi : dP_5 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of the curves $C_{\pm\varphi}$. □

Remark 6.9. The pencil of curves $\{C_\varphi\}_{\varphi\in\mathbb{P}^1}$ is known as the Wiman pencil. The curve C_0 is smooth and has automorphism group S_5. It is knows as the Wiman curve.

6.4 The 125:1 Cover $\tilde{C}_\varphi$ of C_φ

In (5.11) we have to choose of fifth root unity for each of σ, τ, $\alpha(\sigma,\tau)$ and $\beta(\sigma)$. Multiplying all the coordinates z_i by a common factor gives the same point, so there is a $\mathbb{Z}_5^3$ covering of the parametrizing curve $C_{\pm\varphi}$ and we can allow for different branches of solutions by acting with $\mathscr{G}$ on a given branch.

From (5.18) we can compute the Plücker coordinates of the lines in the family. For example,

$$\pi_{34} = \sigma^{4/5}\tau\beta(\sigma\tau)\beta(\tau)(1-\sigma).$$

Define the follwing meromorphic functions on C_φ:

$$f_{ij} := \frac{\pi_{ij}}{\pi_{34}}, \qquad g_{ij} := \left(\frac{\pi_{ij}}{\pi_{34}}\right)^5.$$

The polynomial $T^5 - g_{ij} \in \mathscr{M}(C_\varphi)[T]$ has the roots $\zeta^k f_{ij}$ for $k = 1, \dots, 5$.

From $T^5 - g_{ij}$ we can construct a 5:1 cover of C_φ as in §4.2 of [**6**]. See [**8**], Proposition 8.9, for the general statement. Choose a coordinate neighborhood U_x biholomorphic to a disc $\Delta \subset \mathbb{C}$, with $0 \in \Delta$ and local complex chart $z_x : U_x \to \Delta$ with $z_x(x) = 0$. If $x \in C_\varphi$ and g_{ij} has no poles or zeroes on U_x, this Riemann surface is locally the disjoint union of 5 copies of U_x. If x is a zero of g_{ij}, we can write

$$g_{ij} = z_x^5(1 + a_1 z_x + a_2 z_x^2 + \ldots).$$

Restricting the open subset, we may assume that $1 + a_1 z_x + \ldots = h^5$ for a holomorphic function h on U_x without zeroes and poles. On U_x the polynomial is $T^5 - (z_x h)^5 = \prod_{a=1}^5 (T - \zeta^a z_x h)$, showing that the Riemann surface is still a disjoint union of 5 copies of U_x. Thus the fact that each zero and pole of g_{ij} has multiplicity 5 guarantees that the Riemann surface $\mathfrak{X}_{ij}$ of the polynomial $T^5 - g_{ij}$ is an unramified covering of C_φ. Since the f_{ij} are meromorphic on $\tilde{C}_\varphi$, there must exist holomorphic maps

$$\tilde{C}_\varphi \to \mathfrak{X}_{ij} \to C_\varphi$$

with the first map of degree 25. The second map, of degree 5, is obtained from $T^5 - g_{ij}$.

This construction can be iterated by considering the polynomial $T^5 - g_{pq}$ on $\mathfrak{X}_{ij}$, for example, or by considering the fiber product of the Riemann surfaces $\mathfrak{X}_{ij}$ and $\mathfrak{X}_{pq}$ over C_φ. The cover $\tilde{C}_\varphi$ can be obtained with this construction from three suitably chosen g_{ij}, for example, g_{15}, g_{25}, g_{35}.

6.5 The Central Fiber $\mathfrak{X}_0$

The central fiber $\mathfrak{X}_0$ is the union of the five coordinate hyperplanes in $\mathbb{P}^4$.

Proposition 6.10. In the case $t = 0$ the polynomial F from (5.13) becomes $F = F_+ F_-$ where

$$F_+ = (\sigma + \omega^2)(\tau + \omega^2)(\sigma\tau + \omega)(\sigma\tau + \omega\sigma + \omega^2)(\sigma\tau + \omega\tau + \omega^2) \tag{6.5}$$

and F_- is obtained by $\omega \leftrightarrow \omega^2$.

Proof. The i-th factor of F_+ and $F-$ together give the i-th factor of F for $t = 0$. For example, $(\sigma + \omega^2)(\sigma + \omega) = \omega^3 + (\omega + \omega^2)\sigma + \sigma^2 = 1 - \sigma + \sigma^2$, since $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. □

This shows that the curves $C^0_{\pm\varphi}$ both become reducible. Each component of $C^0_{+\varphi}$, resp. $C^0_{-\varphi}$, parametrizes lines in one of the hyperplanes. For example, the first factor of (6.5) gives lines in the hyperplane $z_2 = 0$.

7 Counting the Lines - The Number 2875

We have seen that the general members of the Dwork pencil contain a family of lines. However, under a general deformation these family splits up into a finite number of lines. Using different methods we will show that this number is 2875.

7.1 A Chern Class Computation

Let $X \subseteq \mathbb{P}^4$ be a quintic threefold. The space parametrizing lines on X is called the *Fano scheme* $F_1(X)$ of lines on X. It is zero dimensional if the number of lines on X is finite. The lines on X in particular live in $\mathbb{P}^4$ so $F_1(X)$ is a subset of the Grassmannian $G(2,5) = \mathbb{G}(1,4)$ parametrizing lines in $\mathbb{P}^4$. If X is given by $g = 0$, then a line L lies on X if and only if $g|_L = 0$, i.e., if g is sent to 0 by the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n}(5)) \to H^0(\mathcal{O}_L(5)).$$

The following statements give a way to calculate the class of $F_1(X)$ in the Chow ring $A(\mathbb{G}(1,4))$.

Definition 7.1. The *tautological rank-k subbundle* on $G(k, V)$ is the subbundle of the trivial bundle $G(k, V) \times V$ whose fiber at a point $[\Lambda] \in G(k, V)$ is the subspace Λ itself.

Proposition 7.2. Let V be an $(n + 1)$-dimensional vector space, and let $\mathcal{S} \subset V \otimes \mathcal{O}_{\mathbb{G}}$ be the tautological rank-$(k+1)$ subbundle on the Grassmannian $G = \mathbb{G}(k, \mathbb{P}V)$ of k-planes in $\mathbb{P}V \cong \mathbb{P}^n$. A form g of degree d on $\mathbb{P}V$ gives rise to a global section σ_g of $\mathrm{Sym}^d \mathcal{S}^\star$ whose zero locus is $F_k(X)$, where X is the hypersurface $g = 0$.

Thus, when $F_k(X)$ has expected codimension $\binom{k+d}{k} = \mathrm{rank}(\mathrm{Sym}^d \mathcal{S}^\star)$ in G, we have

$$[F_k(X)] = c_{\binom{k+d}{k}}(\mathrm{Sym}^d \mathcal{S}^\star) \in A(G).$$

Proof. This is Proposition 6.4 in [**7**] □

Theorem 7.3. If $X \subset \mathbb{P}^n$ is a general hypersurface of degree $d \geq 1$, then the Fano scheme $F_1(X)$ of lines on X is reduced and has the expected dimension $2n - d - 3$.

Proof. This is Theorem 6.34 in [**7**]. □

Definition 7.4. Choose a complete flag $\mathcal{V}$ in V, that is, a nested sequence of subspaces

$$0 \subset V_1 \subset \ldots \subset V_n = V$$

with $\dim V_i = i$. For a sequence $a = (a_1, \dots, a_k)$ of integers with

$$n - k \geq a_1 \geq \dots \geq a_k \geq 0$$

define the *Schubert cycle* $\Sigma_a(\mathcal{V}) \subset G(k, n)$ to be the closed subset

$$\Sigma_a(\mathcal{V}) = \{\Lambda \in G(k, n) \mid \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i \text{ for all } i\}.$$

The *Schubert classes*

$$\sigma_a = [\Sigma_a(\mathcal{V})] \in A(G(k, n))$$

do not depend on the choice of flag.

Proposition 7.5. The class of the Fano variety $F_1(X)$ of lines on a quintic threefold X in $\mathbb{P}^4$ is

$$[F_1(X)] = 2875\sigma_{3,3}.$$

Hence, the number of lines on a quintic threefold in $\mathbb{P}^4$ is 2875.

Proof. By Theorem 7.3 and Proposition 7.2 we have

$$[F_1(X)] = c_6(\mathrm{Sym}^5 \mathcal{S}^\star)$$

where $\mathcal{S}^\star$ is the tautological rank-2 subbundle on $G(2, 5)$. The Chern class of $\mathcal{S}^\star$ is

$$c(\mathcal{S}^\star) = 1 + \sigma_1 + \sigma_{1,1}.$$

Suppose $\mathcal{S}^\star$ splits into a direct sum of two lines bundles with $c(\mathcal{L}) = 1 + \alpha$ and $c(\mathcal{M}) = 1 + \beta$. By the Whitney formula,

$$c(\mathcal{S}^\star) = (1 + \alpha)(1 + \beta),$$

hence

$$\alpha + \beta = \sigma_1 \quad \text{and} \quad \alpha \cdot \beta = \sigma_{1,1}.$$

Now $\mathrm{Sym}^3 \mathcal{S}^\star$ splits as well as

$$\mathrm{Sym}^3 \mathcal{S}^\star = \mathcal{L}^3 \oplus (\mathcal{L}^2 \otimes \mathcal{M}) \oplus (\mathcal{L} \otimes \mathcal{M}^2) \oplus \mathcal{M}^3,$$

and we have

$$c(\mathrm{Sym}^3 \mathcal{S}^\star) = (1 + 5\alpha)(1 + 4\alpha + \beta)(1 + 3\alpha + 2\beta)(1 + 2\alpha + 3\beta)(1 + \alpha + 4\beta)(1 + 5\beta).$$

The top Chern class then could be written as

$$\begin{aligned}
c_6(\mathrm{Sym}^6\mathcal{S}^\star) &= 5\alpha(4\alpha+\beta)(3\alpha+2\beta)(2\alpha+3\beta)(\alpha+4\beta)5\beta \\
&= 25\alpha\beta(24\alpha^4+154\alpha^3\beta+369\alpha^2\beta^2+154\alpha\beta^3+24\beta^4) \\
&= 25\alpha\beta(24(\alpha+\beta)^4+58\alpha\beta(\alpha+\beta)^2+9(\alpha\beta)^2 \\
&= 25\sigma_{1,1}(24\sigma_1^4+58\sigma_{1,1}\sigma_1^2+9\sigma_{1,1}^2)
\end{aligned}$$

Using Pieri's formula for lines (Proposition 4.11 of [**7**]) we calculate

$$\sigma_{1,1}\sigma_1^4 = 2\sigma_{3,3}, \quad \sigma_{1,1}^2\sigma_1^2 = \sigma_{3,3}, \quad \text{and} \quad \sigma_{1,1}^3 = \sigma_{3,3}.$$

This gives us

$$c_6(\mathrm{Sym}^6\mathcal{S}^\star) = 25(24\cdot 2\sigma_{3,3}+58\sigma_{3,3}+9\sigma_{3,3}) = 2875\sigma_{3,3}.$$

This shows that $F_1(X)$ consists of 2875 points, i.e., X contains 2875 lines. □

7.2 Deforming the Fermat Quintic

We describe the lines on a particular member of the Dwork pencil - the *Fermat quintic* X_∞ - and see which lines deform under a general deformation. This is described in [**1**].

Proposition 7.6. The lines on the *Fermat quintic* X_∞ given by $z_0^5+z_1^5+z_2^5+z_3^5+z_4^5=0$ in $\mathbb{P}^4$ are given by 50 cones of lines of the form

$$z=(u,-\zeta^k u, av, bv, cv) \quad \text{with} \quad a^5+b^5+c^5=0, \quad (u,v)\in\mathbb{P}^1.$$

The isolated lines from Proposition 5.2 are the lines in which the cones intersect. Each cone contains 15 isolated lines and meets 15 other cones in these lines.

Proof. The isolated lines do not depend on t, so they have the same form as in Proposition (5.2). A line in $\mathbb{P}^4$ has the form

$$(u, v, bu+qv, cu+rv, du+sv).$$

In order for this line to lie in X_∞ the parameters must satisfy

$$\begin{aligned}
b^5+c^5+d^5+1 &= 0 \\
b^4q+c^4r+d^4s &= 0 \\
b^3q^2+c^3r^2+d^3s^2 &= 0 \\
b^2q^3+c^2r^3+d^2s^3 &= 0 \\
bq^4+cr^4+ds^4 &= 0 \\
q^5+r^5+s^5+1 &= 0.
\end{aligned}$$

Assume one parameter is zero, e.g. $b = 0$. Then if $c = 0$ from the first equation we obtain $d = -\zeta^k$ for ζ a nontrivial fifth root of unity. Now the second equation gives $d^4 s = 0$ implying $s = 0$. The last equation gives $q^5 + r^5 + 1 = 0$ which after reparametrization of $v \mapsto av$, permutation and renaming of the parameters becomes the wanted equation. $\square$

Proposition 7.7. Let Θ_X be the tangent bundle of X and let $\rho \in H^1(\Theta_X)$ be generic. Then no line l moves in the direction of ρ, i.e., all pairs (l, X) are obstructed to first oder. There are exactly 10 lines on each cone that deform (to second oder) in the direction of ρ with monodromy $\mathbb{Z}_2$. The lines that lie on more than one cone deform in the direction of ρ with monodromy $\mathbb{Z}_5$.

Proof. This is the content of Propositions 2.1, 2.2 and 2.3 of [**1**]. $\square$

Corollary 7.8. On a generic quintic threefold there are 2875 lines.

Proof. By Proposition 7.7, on each of the 50 cones there are 10 lines splitting into 2 lines under a general deformation. The 375 lines on the intersection points split into 5 lines. $\square$

7.3 Using the Abel-Jacobi Map

The following argumentation is taken from §1.5 of [**6**].

We have seen that the family of lines is parametrized by two isomorphic curves $\tilde{C}_{\pm\varphi}$. Anca Mustaţă has shown in [**15**] that they both have genus 626. A loop $\gamma \in H_1(\tilde{C}_{\pm\varphi})$ determines a 3-cycle $T(\gamma) \in \mathfrak{X}_t$ which is the union of the lines corresponding to the points of γ. The dual map $H^3(\mathfrak{X}_t) \to H^1(\tilde{C}_{\pm\varphi})$ has a Hodge component map

$$\alpha : H^1(\Omega^2_{\mathfrak{X}_t}) \to H^0(\Omega^1_{\tilde{C}_{\pm\varphi}}).$$

The first space can be interpreted as the 101-dimensional space of infinitesimal deformations of $\mathfrak{X}_t$, thought of as the space of degree 5 polynomials P modulo the Jacobian ideal. It is shown in [**4**] that zeroes of the holomorphic 1-form $\alpha(P)$ on $\tilde{C}_{\pm\varphi}$ correspond precisely to the lines that can be infinitesimally lifted over the deformation of $\mathfrak{X}_t$ determined by P.

Since the curves $\tilde{C}_{\pm\varphi}$ both have genus 626, a differential form has $2 \cdot 626 - 2 = 1250$ zeroes. Hence, 2500 lines in the family can be infinitesimally lifted. Together with the 375 isolated lines this again gives the number 2875 of lines.

8 Tropicalization of the Isolated Lines and the van Geemen Lines

In §5 we found the lines on the Dwork pencil in parametric representation, as well as their defining equations. In §7 we calculated their number (under a genral deformation) to be 2875. In this section and the next one we want to use the methods from §2 to tropicalize these lines.

8.1 Tropicalization of the Dwork Pencil

Before we tropicalize the Dwork pencil of quintic threefolds (5.1), let us consider an analogous lower-dimensional example that can be presented in two dimensions.

Example 8.1. Consider the family of cubic curves in $\mathbb{P}^2$ defined by

$$t(z_0^3 + z_1^3 + z_2^3) + z_0 z_1 z_2 = 0.$$

To dehomogenize look at the open subset U_0 where $z_0 \neq 0$. Dividing by z_0 we may assume that $z_0 = 1$, so on U_0 the family is defined by $t(1 + z_1^3 + z_2^3) + z_1 z_2 = 0$. This polynomial tropicalizes to

$$\min\{1, 3x_1 + 1, 3x_2 + 1, x_1 + x_2\}$$

Hence, the tropicalization of the family on U_0 is the locus in $\mathbb{R}^2$ given by the six parts where

$$\begin{aligned}
1 &= 3x_1 + 1 && \leq 3x_2 + 1, x_1 + x_2\\
1 &= 3x_2 + 1 && \leq 3x_1 + 1, x_1 + x_2\\
1 &= x_1 + x_2 && \leq 3x_1 + 1, 3x_2 + 1\\
3x_1 + 1 &= 3x_2 + 1 && \leq 1, x_1 + x_2\\
3x_1 + 1 &= x_1 + x_2 && \leq 1, 3x_2 + 1\\
3x_2 + 1 &= x_1 + x_2 && \leq 1, 3x_1 + 1
\end{aligned}$$

giving

$$\begin{aligned}
x_1 &= 0, && 0 \leq x_2\\
x_2 &= 0, && 0 \leq x_1\\
x_2 &= 1 - x_1, && 0 \leq x_1 \leq 1\\
x_2 &= x_1, && x_1 \leq -1\\
x_2 &= 2x_1 + 1, && 0 \leq x_1 \leq 1\\
x_2 &= \frac{1}{2}(x_1 - 1), && 0 \leq x_1 \leq 1.
\end{aligned}$$

The tropicalization consists of the boundary of the polytope $\mathrm{Conv}\{(-1,-1),(1,0),(0,1)\}$ and three unbounded parts. It is depicted in Figure 8.1.

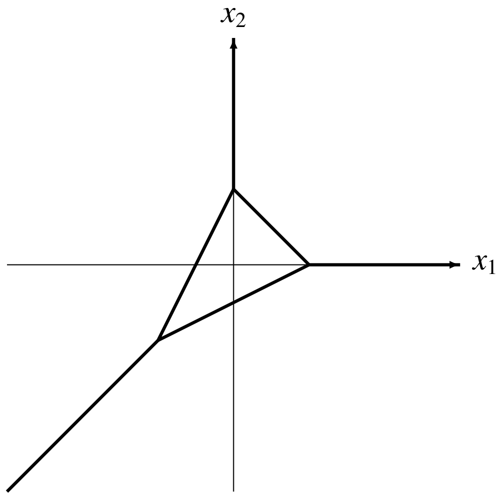


Figure 8.1: The tropicalization of a family of cubic curves in $\mathbb{R}^2$.

One the tropical stratum where $z_0 = 0$ and $z_1, z_2 \neq 0$ we can take $z_1 = 1$ and the defining equation becomes

$$1 + z_2^3 = 0.$$

The tropicalization of this part is the set where

$$\min\{0, 3x_2\}$$

is not unique, i.e., the set where $x_2 = 0$. This is the point $(0,0,0)$ in $\mathrm{Trop}(\mathbb{P}^2)$. Hence, the tropicalizations on this stratum is just one point. Similarly, one the other one-dimensional strata we get one point. Figure 8.2 shows the whole tropicalization in $\mathbb{TP}^2$.

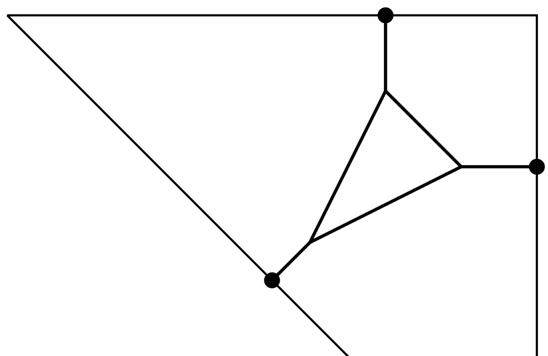


Figure 8.2: The tropicalization of a family of cubic curves in $\mathbb{P}^2$.

Example 8.2. Repeating Example 8.1 in two dimensions higher we calculate the tropicalization of the Dwork pencil

$$t(z_0^5 + z_1^5 + _2^5 + z_3^5 + z_4^5) + z_0 z_1 z_2 z_3 z_4 = 0 \tag{8.1}$$

in the tropical torus $(\mathbb{T}^\star)^4 = \mathbb{R}^4$ to be the boundary of the polytope

$$\mathrm{Conv}\{(-1,-1,-1,-1),(1,0,0,0),\ldots,(0,0,0,1)\}$$

plus the ten unbounded parts

$$U_i = \left\{x \in \mathbb{R}^4 \mid x_i = 0, x_j \geq 0 \forall j \neq i\right\}, \quad i \in \{1,2,3,4\},$$

$$U_{ij} = \left\{x \in \mathbb{R}^4 \;\middle|\; x_i = x_j \leq 0, \sum_{k \neq i,j} x_k \geq 3x_i + 1, x_k \geq x_i \forall k \neq i,j\right\}, \quad i \neq j \in \{1,2,3,4\}.$$

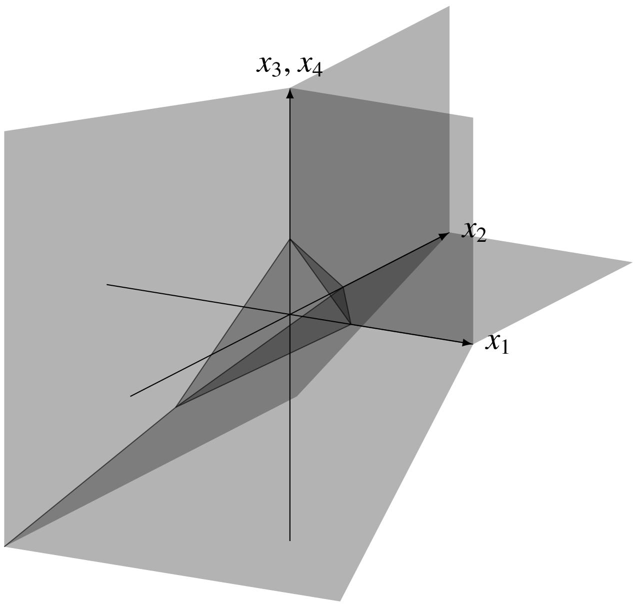


Figure 8.3: The tropicalization of the Dwork pencil in the tropical torus $\mathbb{R}^4$.

On the toric stratum where $z_0 = 0$ we can set $z_4 = 1$ and tropicalize (8.1) to

$$\min\{5x_1 + 1, 5x_2 + 1, 5x_3 + 1, 1\}$$

giving the union of the six parts

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_i = 0, x_k \geq x_i\}, \quad i \in \{1,2,3\}$$

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_i = x_j \leq 0, x_k \geq x_i\}, \quad i \neq j \in \{1,2,3\}$$

and analogously for the other strata. This is shown in Figure 8.4, together with the tropical lines lying on it. Each of the dots in Figure 8.2 is now a union of six 2-dimensional parts.

8.2 Tropicalization of the Isolated Lines

Proposition 8.3. The tropicalizations of the isolated lines are given by permutations of

$$(0, 0, 0, 0, \infty) \quad + \quad (0, 1, 0, 1, 0)\bar{\mathbb{R}}.$$

In particular, they lie in a tropical stratum with one coorinate equal to ∞.

Proof. Consider the isolated line $(u, v, -\zeta^k u, -\zeta^l v, 0)$ from (5.4). Its defining polynomials (5.15) clearly form a tropical basis. The line lies in the toric stratum where $z_4 = 0$ so its tropicalization lies in the stratum of $\mathrm{Trop}(\mathfrak{X})$ where $x_4 = \infty$. In this stratum we can set $z_0 = 1$ and the defining equations tropicalize to

$$\min\{0, x_2\}, \quad \min\{x_1, x_3\}.$$

This defines the tropical line given by $x_0 = 0$, $x_2 = 0$, $x_1 = x_3$, shown in Figure 8.4.

We can also tropicalize from the parametric representation (5.4). Setting $\nu(z_0) = \nu(u) = 0$,

$$\begin{aligned} \nu(z_1) &= \nu(v), \\ \nu(z_2) &= 0, \\ \nu(z_3) &= \nu(v), \\ \nu(z_4) &= \infty. \end{aligned}$$

All other tropical lines are given by permutations. In each of the 5 tropical strata there are 3 different lines, and for each tropical line there are 25 lines tropicalizing to it. □

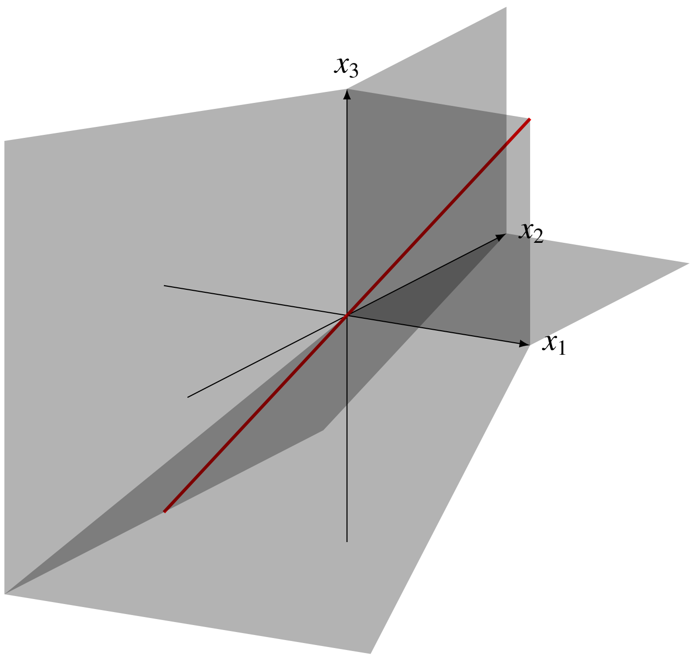


Figure 8.4: The three tropical lines in the stratum where $z_4 = 0$.

8.3 Tropicalization of the van Geemen Lines

Proposition 8.4. The tropicalizations of the van Geemen lines are permutations of the tropical line shown in Figure 8.5, consisting of the vertex $(0,0,1,0,0) \in \mathbb{TP}^4$ and the rays

$$\begin{aligned}(0,0,1,0,0) &+ (0,1,0,0,0)\bar{\mathbb{R}}_{\geq 0}\\(0,0,1,0,0) &+ (0,0,0,1,0)\bar{\mathbb{R}}_{\geq 0}\\(0,0,1,0,0) &+ (0,0,0,0,1)\bar{\mathbb{R}}_{\geq 0}\\(0,0,1,0,0) &+ (1,0,1,0,0)\bar{\mathbb{R}}_{\geq 0},\end{aligned}$$

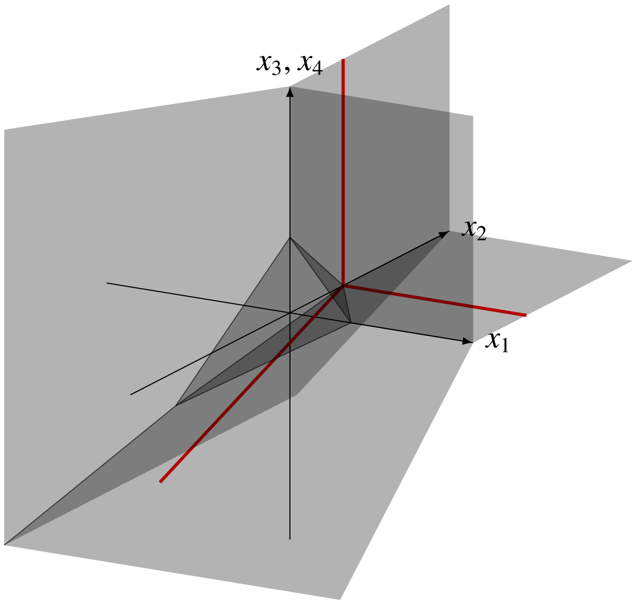


Figure 8.5: The tropicalization of a van Geemen line.

Proof. We have

$$\nu(b) = 2\nu(\gamma) - 1, \qquad \nu(c) = \nu(\gamma) - 1.$$

The equation (5.8) for γ tropicalizes to

$$\min\{10\nu(\gamma), 5\nu(\gamma), 5\}$$

giving $\nu(\gamma) = 0, 1$ and thus

$$(\nu(b), \nu(c)) \in \{(-1,-1), (1,0)\}.$$

Hence, the lines

$$(u, v, \zeta^{-k-l}bu, \zeta^k(cu + \omega v), -\zeta^l\omega^2(cu - v))$$

have two possible tropicalizations, depending on the choice of γ.

Let us consider the case $\nu(\gamma) = 1$, i.e., $\nu(b) = 1$, $\nu(c) = 0$.

Let

$$\begin{aligned} g_1 &= bz_0 - \zeta^{k+l} z_2, \\ g_2 &= cz_0 + \zeta^{-k}\omega z_3 - \zeta^{-l} z_4, \\ g_3 &= \omega z_1 + \zeta^{-k}\omega^2 z_3 + \zeta^{-l} z_4, \end{aligned}$$

be the polynomials in the defining equations (5.16) and $\mathcal{F} = \{g_1, g_2, g_3\}$. For $z_0 = 1$, they tropicalize to

$$\begin{aligned} \mathrm{Trop}(g_1) &= \min\{1, x_2\}, \\ \mathrm{Trop}(g_2) &= \min\{0, x_3, x_4\}, \\ \mathrm{Trop}(g_3) &= \min\{x_1, x_3, x_4\}. \end{aligned}$$

The finite intersection $\cap_{f\in\mathcal{F}} \mathrm{Trop}(V(f))$ of tropical hypersurfaces is shown in Figure 8.6.

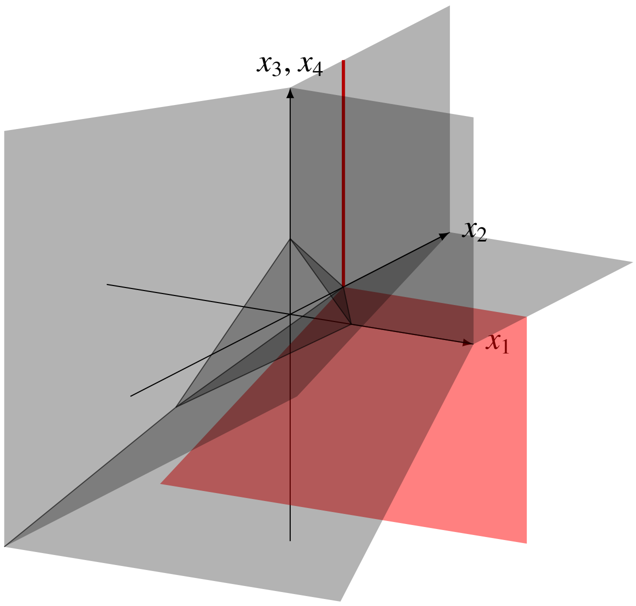


Figure 8.6: The finite intersection $\cap_{f\in\mathcal{F}} \mathrm{Trop}(V(f))$.

Let $w = (0, 0, 1, -1, -1)$. Then, with $\omega + \omega^2 = -1$,

$$g_2 + g_3 = cz_0 + \omega z_1 - \zeta^{-k} z_3 \in I$$

and $\mathrm{in}_w(g_2 + g_3) = -\zeta^{-k} z_3$. Hence, $\mathrm{in}_w(I)$ contains the monomial z_3.

To get a witness for $C_w(I)$ we have to calculate $\mathcal{G}_{\prec_w}(I)$ for a term order $\prec$. Take for example the lexicographic order

$$z_0^\alpha \prec z_1^\beta, \quad z_1^\alpha \prec z_2^\beta, \quad z_2^\alpha \prec z_3^\beta, \quad \prec z_3^\alpha \prec z_4^\beta \qquad \text{for all} \quad \alpha, \beta \in \mathbb{N}^n \setminus \{0\}.$$

Then

$$\begin{aligned}
\mathrm{in}_{<_w}(g_1) &= -\zeta^{k+l}z_2, \\
\mathrm{in}_{<_w}(g_2) &= -\zeta^{-l}z_4, \\
\mathrm{in}_{<_w}(g_3) &= \zeta^{-l}z_4.
\end{aligned}$$

and

$$S_{<_w}(g_2, g_3) = g_2 + g_3 = cz_0 + \omega z_1 - \zeta^{-k}z_3.$$

This is not divisible by any $\mathrm{in}_{<_w}(g_i)$, so $\mathcal{G}_{<_w}(I) = \{g_1, g_2, g_3, S_{<_w}(g_2, g_3)\}$.

The normal form to z_3 with respect to $\mathcal{G}_{<_w}(I)$ is

$$h = \zeta^k cz_0 + \zeta^k \omega z_1.$$

Adding $z_3 - h$ to $\mathcal{F}$ we obtain the tropical basis

$$\mathcal{T}(I) = \{g_1, g_2, g_3, z_3 - \zeta^k cz_0 + \zeta^k \omega z_1\}.$$

For $z_0 = 1$, the polynomials in $\mathcal{T}(I)$ tropicalize to

$$\min\{1, x_2\}, \quad \min\{0, x_3, x_4\}, \quad \min\{x_1, x_3, x_4\}, \qquad \min\{x_3, 0, x_1\}.$$

This gives precisely the line in the proposition.

We can also tropicalize from the parametric representation. The valuations of the coordinates in (5.7) are

$$\begin{aligned}
\nu(z_0) &= \nu(u), \\
\nu(z_1) &= \nu(v), \\
\nu(z_2) &= \nu(u), \\
\nu(z_3) &= \nu(cu + \omega v), \\
\nu(z_4) &= \nu(cu - v).
\end{aligned}$$

Dehomogenizing by setting $\nu(z_0) = \nu(u) = 0$, we get

$$\begin{aligned}
\nu(z_1) &= \nu(v), \\
\nu(z_2) &= 1, \\
\nu(z_3) &= \nu(c + \omega v), \\
\nu(z_4) &= \nu(c - v).
\end{aligned}$$

We have three cases, depending on the value of $\nu(v)$.

- If $\nu(v) < 0$ then

$$\nu(z_1) = \nu(v), \quad \nu(z_2) = 1, \quad \nu(z_3) = \nu(v), \quad \nu(z_4) = \nu(v).$$

  Note that $(0, -1, 0, -1, -1) = (1, 0, 1, 0, 0) \in \mathbb{TP}^4$, so this give the fourth ray of the claim.
- If $\nu(v) > -1$ then

$$\nu(z_1) = \nu(v), \quad \nu(z_2) = 1, \quad \nu(z_3) = 0, \quad \nu(z_4) = 0.$$

- If $\nu(v) = 0$ then $\nu(c + \omega v) \geq 0$ and $\nu(c - v) \geq 0$.

$$\nu(z_1) = 0, \quad \nu(z_2) = 1, \quad \nu(z_3) = \nu(c + \omega v) \geq 0, \quad \nu(z_4) = \nu(c - v) \geq 0.$$

  Moreover, we have $\nu(c + \omega v) = \nu((c - v) + (1 + \omega)v)$. Since $\nu(1 + \omega) = 0$, by Proposition 2.6, we have $\nu(c - v) > 0 \Rightarrow \nu(c + \omega v) = 0$, and vice versa. Hence, this part gives two rays extending in the directions of x_3 and x_4.

The second case is similar and gives the same tropical line under the permutation $x_0 \leftrightarrow x_2$. $\square$

9 Tropicalization of the Parametrizing Curves

In order to tropicalize the lines in the family, we need the valuations of the terms appearing in their defining equations (5.17), resp. their parametric representations (5.11). To find the valuations of the parameters σ and τ we will tropicalize the curves C_t^0 and $C_{\pm\varphi}^0$ in $\mathbb{C}^2$. Then, for the different cases for $\nu(\sigma)$ and $\nu(\tau)$, we will calculate the valuations of all terms we need.

9.1 Tropicalization of $C_{\pm\varphi}^0$

Proposition 9.1. The tropicalization of the curve C_t^0, shown in Figure 9.1, is

$$\mathrm{Trop}(C_t^0) = \{(s,t) \in \mathbb{T}^2 \mid s = 0 \vee t = 0 \vee s + t = 0\},$$

where each edge has weight 4.

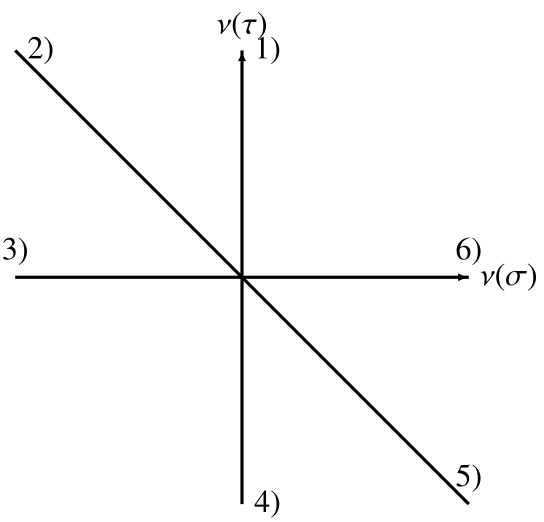


Figure 9.1: $\mathrm{Trop}(C_t^0)$ together with the labelling used in §9.2.

Proof. Using $\min\{0, x, 2x\} = 2\min\{0, x\}$ and $\min\{x + y, x + z\} = x + \min\{y + z\}$ we get

$$\begin{aligned} \mathrm{Trop}(F)(s,t) =& 2(\min\{5/2 + s + t + \min\{0, s\} + \min\{0, t\} + \min\{0, s + t\}, \\ & \min\{0, s\} + \min\{0, t\} + \min\{0, s + t\} + \min\{0, t, s + t\} + \min\{0, s, s + t\}) \\ =& 2(\min\{0, s\} + \min\{0, t\} + \min\{0, s + t\} \\ & + \min\{5/2 + s + t, \min\{0, t, s + t\} + \min\{0, s, s + t\}\}) \\ =& 2(\min\{0, s\} + \min\{0, t\} + \min\{0, s + t\} + \min\{0, t, s + t\} + \min\{0, s, s + t\}) \end{aligned} \tag{9.1}$$

The set of points where $\mathrm{Trop}(F)$ is not unique is clearly the set above.

The exponents of σ and τ in F determine the Newton polytope of F to be

$$\mathrm{Newt}(F) = \mathrm{Conv}\{(0,0),(4,0),(8,4),(8,8),(8,4),(4,0)\}.$$

Looking at $\mathrm{Trop}(C_t^0)$ we see that this is already the Newton subdivision dual to the polyhedral complex on which $\mathrm{Trop}(C_t^0$ is supported. Since all edges in this polytope have affine length 4, the weights of the rays of $\mathrm{Trop}(C_t^0)$ are all equal to 4. □

By the Fundamental Theorem this gives the possible values for $\nu(\sigma)$ and $\nu(\tau)$.

Remark 9.2. Note that $\mathrm{Trop}(C_t^0) = \mathrm{Trop}(C_0^0)$. We have

$$\mathrm{Trop}(F) = \text{"}(0+s)^2(0+t)^2(0+st)^2(0+s+st)^2(0+t+st)^2\text{"}.$$

Proposition 9.3. The tropicalizations of the curves $C_{\pm\varphi}^0$ in $\mathbb{C}^2$ are the same as $\mathrm{Trop}(C_t^0)$:

$$\mathrm{Trop}(C_{\pm\varphi}^0) = \{(s,t) \in \mathbb{T}^2 \mid s = 0 \vee t = 0 \vee s + t = 0\},$$

but with all edges of weight 2.

Proof. Since the defining equations of $C_{+\varphi}^0$ and $C_{-\varphi}^0$ only differ by a change of sign in one summand, their tropicalizations are equal. Using $\min\{2x, x+y, 2y\} = 2\min\{x,y\}$ we get

$$\begin{aligned}\mathrm{Trop}(F_+)(s,t) =&\min\{2s+2t, s+t+\min\{0,s\}+\min\{0,t\}+\min\{0,s+t\},\\ &2\min\{0,s\}+2\min\{0,t\}+2\min\{0,s+t\}\}\\ =&\min\{2s+2t, 2\min\{0,s\}+2\min\{0,t\}+2\min\{0,s+t\}\}\\ =&2(\min\{0,s\}+\min\{0,t\}+\min\{0,s+t\}).\end{aligned}$$

The set where this minimum is not unique is clearly $\mathrm{Trop}(C_{\pm\varphi}^0)$ as above. This time the Newtom polytope is

$$\mathrm{Newt}(F) = \mathrm{Conv}\{(0,0),(2,0),(4,2),(4,4),(4,2),(2,0)\}.$$

This shows that each edge has weight 2. □

Proposition 9.4. The action of S_5 on $C_{\pm\varphi}^0$ from Table 5.3 induces an action on $\mathrm{Trop}(C_{\pm\varphi}^0)$. There is a subgroup $\langle(01)(24),(01),(34)\rangle = \langle(01)(24),(34)\rangle \cong S_2 \times S_3 \cong D_6$ of S_5 which acts as the dihedral group D_6 on the rays of $\mathrm{Trop}(C_{\pm\varphi}^0)$. See Figure 9.2 for a picture of this action.

Proof. Consider Table 5.3 and Figure 9.1. The element (01)(24) acts as $(\sigma,\tau) \mapsto (\tau,\sigma)$, hence as the reflection $1 \leftrightarrow 6$, $2 \leftrightarrow 5$, $3 \leftrightarrow 4$ on the rays of $\mathrm{Trop}(C_{\pm\varphi}^0)$, where we used the labelling as in Figure 9.1. The element (34) acts as $2 \leftrightarrow 6$, $3 \leftrightarrow 5$. Together they generate the subgroup $S_2 \times S_3 \cong D_6$ where S_2 permutes $\{0,1\}$ and S_3 permutes $\{2,3,4\}$. □

As a consequence it suffices to know the tropicalization of a line parametrized by one ray of $\mathrm{Trop}(C^0_{\pm\varphi})$. Then the cases for all other rays simply follow by permutation of coordinates.

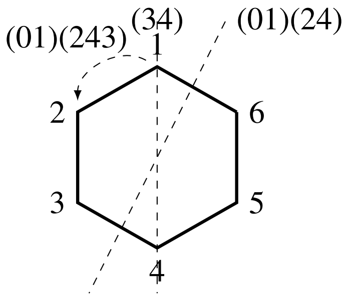


Figure 9.2: The tropicalization of a line in the family.

9.2 Valuations of the Terms for the Rays of $\mathrm{Trop}(C^0_{\pm\varphi})$

The following table shows the valuations of the terms in (5.11) and (5.17) for the rays of $\mathrm{Trop}(C^0_{\pm\varphi})$, with the labelling as in Figure 9.1.

case	$\nu(1-\sigma)$	$\nu(1-\sigma+\sigma^2)$	$\nu(1-\tau)$	$\nu(1-\tau+\tau^2)$	$\nu(1-\sigma\tau)$	$\nu(1-\sigma\tau+\sigma^2\tau^2)$
1)	0	$5+\nu(\tau)$	0	0	0	0
2)	$-\nu(\tau)$	$-2\nu(\tau)$	0	0	0	$5+\nu(\tau)$
3)	$\nu(\sigma)$	$2\nu(\sigma)$	0	$5-\nu(\sigma)$	$\nu(\sigma)$	$2\nu(\sigma)$
4)	0	$5-\nu(\tau)$	$\nu(\tau)$	$2\nu(\tau)$	$\nu(\tau)$	$2\nu(\tau)$
5)	0	0	$-\nu(\sigma)$	$-2\nu(\sigma)$	0	$5+\nu(\sigma)$
6)	0	0	0	$5+\nu(\sigma)$	0	0

case	$\nu(1-\tau(1+\sigma)+\tau^2(1-\sigma+\sigma^2))$	$\nu(1-\sigma(1+\tau)+\sigma^2(1-\tau+\tau^2))$
1)	0	$\nu(\tau)$
2)	$\nu(\tau)$	$-2\nu(\tau)$
3)	$2\nu(\sigma)$	$\nu(\sigma)$
4)	$\nu(\tau)$	$2\nu(\tau)$
5)	$-2\nu(\sigma)$	$\nu(\sigma)$
6)	$\nu(\sigma)$	0

Table 9.1: Valuations of the terms for the different rays of Figure 9.1.

Lemma 9.5. If $\nu(\sigma)=0$ and $\nu(\tau)>0$, then $\nu(1-\tau)$, $\nu(1-\tau+\tau^2)$, $\nu(1-\sigma\tau)$, $\nu(1-\sigma\tau+\sigma^2\tau^2)$ and $\nu(1-\tau(1+\sigma)+\tau^2(1-\sigma+\sigma^2))$ are all zero.

Proof. This is clear from Definition 2.5 and Proposition 2.6. $\square$

Lemma 9.6. If $\nu(\sigma) = 0$ and $\nu(\tau) > 0$, then $\nu(1 - \sigma) = 0$ and $\nu(1 - \sigma + \sigma^2) = \nu(\tau) + 5$.

Proof. Assume $\nu(\sigma) = 0, \nu(\tau) > 0$ and write

$$\begin{aligned}\sigma &= \sigma_0 + \sigma_1 t^\alpha + \tilde{\sigma}\\ \tau &= \tau_0 t^{\nu(\tau)} + \tilde{\tau}\end{aligned}$$

with $\sigma_0, \sigma_1, \tau_0 \neq 0$, $\nu(\tilde{\sigma}) > \alpha > 0$ and $\nu(\tilde{\tau}) > \nu(\tau)$. Then the t-constant term in $F(\sigma, \tau)$ is

$$F(\sigma, \tau)_0 = -(1 - \sigma_0 + \sigma_0^2)^2.$$

In order to have $F(\sigma, \tau) = 0$ we must have $F(\sigma, \tau)_0 = 0$, hence

$$1 - \sigma_0 + \sigma_0^2 = 0.$$

This implies $\nu(1 - \sigma) = 0$ and

$$1 - \sigma + \sigma^2 = \sigma_1(2\sigma_0 - 1)t^\alpha + (\text{higher order terms}), \tag{9.2}$$

showing $\nu(1 - \sigma + \sigma^2) > 0$. We show that $\nu(1 - \sigma + \sigma^2) = 5 + \nu(\tau)$.

- Assume $\alpha < 5 + \nu(\tau)$. If $\alpha \neq \nu(\tau)$, then

  $$F(\sigma, \tau)_{2\alpha} = 3\sigma_1^2$$

  implying $\sigma_1 = 0$. If $\alpha = \nu(\tau)$, then the coefficients of $t^{2\alpha}$ and $t^{3\alpha}$ in $F(\sigma, \tau)$ are

  $$\begin{aligned}F(\sigma, \tau)_{2\alpha} &= 3\sigma_1(\sigma_1 - \tau_0),\\ F(\sigma, \tau)_{3\alpha} &= \frac{1}{2}\sigma_1(-4i\sqrt{3}\sigma_1^2 - 24\sigma_1\tau_0 + 7(3 + i\sqrt{3})\tau_0^2),\end{aligned}$$

  again implying $\sigma_1 = 0$. This is a contradiction to $\sigma_1 \neq 0$.
- Assume $\alpha > 5 + \nu(\tau)$. Then the coefficient of $t^{5+2\nu(\tau)}$ in $F(\sigma, \tau)$ is

  $$F(\sigma, \tau)_{5+2\nu(\tau)} = 10^5\tau_0^2 \neq 0,$$

  a contradiction to $F(\sigma, \tau) = 0$.
- Assume $\alpha = \nu(\tau) + 5$. Then the lowest t-order term in $F(\sigma, \tau)$ is

  $$F(\sigma, \tau)_{2\nu(\tau)+5} = \tau_0(10^5\tau_0 - 3\sigma_1)$$

  implying $\sigma_1 = 10^5/3\tau_0 \neq 0$. Now by (9.2), $\nu(1 - \sigma + \sigma^2) = \alpha = 5 + \nu(\tau)$.

This proves $\nu(1 - \sigma + \sigma^2) = 5 + \nu(\tau)$. □

Lemma 9.7. If $\nu(\sigma) = 0$ and $\nu(\tau) > 0$, then $\nu(1 - \sigma(1 + \tau) + \sigma^2(1 - \tau + \tau^2)) = \nu(\tau)$.

Proof. Write

$$\nu(1 - \sigma(1 + \tau) + \sigma^2(1 - \tau + \tau^2)) = \nu((1 - \sigma + \sigma^2) - \tau(\sigma + \sigma^2) + \tau^2\sigma^2).$$

We have $\nu(\tau(\sigma+\sigma^2)) = \nu(\tau)$, which is smaller than $\nu(\tau^2\sigma^2) = 2\nu(\tau)$ and $\nu(1-\sigma+\sigma^2) = \nu(\tau)+5$. Now the claim follows from Proposition 2.6. □

Proposition 9.8. The valuations of the terms appearing in (5.11) and (5.17) are as in Table 9.1.

Proof. For the first row of Table 9.1 this is the content of the last three lemmas. All other rows follow from Proposition 9.4. For example, according to Table 5.3, we can switch from case 1) to case 2) by the transformation

$$\begin{array}{ccc} C^0_{\pm\varphi} & \to & C^0_{\pm\varphi} \\ (\sigma, \tau) & \mapsto & (\tilde{\sigma}, \tilde{\tau}) = \left(\dfrac{1}{\tau}, \sigma\tau\right) \end{array}$$

Now for $\nu(\tilde{\sigma}) = -\nu(\tilde{\tau}) < 0$, i.e. $\nu(\sigma) = 0$, $\nu(\tau) > 0$, we get for example

$$\nu(1 - \tilde{\sigma}) = \nu(1 - \frac{1}{\tau}) = \nu(-\frac{1}{\tau}(1 - \tau)) = -\nu(\tau) + \nu(1 - \tau) = -\nu(\tau),$$

$$\nu(1 - \tilde{\sigma}\tilde{\tau} + \tilde{\sigma}^2\tilde{\tau}^2) = \nu(1 - \sigma + \sigma^2) = 5 + \nu(\tau).$$

One could also do similar calculations as in the lemmas, and I did this just to check. □

9.3 Valuations of the Terms for the Vertex of Trop($C^0_{\pm\varphi}$)

Consider the vertex $\nu(\sigma) = \nu(\tau) = 0$ of Trop($C^0_{\pm\varphi}$). We can write

$$\begin{aligned} \sigma &= \sigma_0 + \sigma_1 t^{\alpha} + \tilde{\sigma}, \\ \tau &= \tau_0 + \tau_1 t^{\nu(1-\tau)} + \tilde{\tau}, \end{aligned}$$

with $\sigma_0, \sigma_1, \tau_0, \tau_1 \neq 0$, $0 < \alpha < \nu(\tilde{\sigma})$ and $0 < \beta < \nu(\tilde{\tau})$. Then the t-constant term of $F(\sigma, \tau)$ is

$$\begin{aligned} F(\sigma, \tau)_0 = &- (1 - \sigma_0 + \sigma_0^2)(1 - \tau_0 + \tau_0^2)(1 - \sigma_0\tau_0 + \sigma_0^2\tau_0^2) \\ &\cdot (1 - \tau_0(1 + \sigma_0) + \tau_0^2(1 - \sigma_0 + \sigma_0^2))(1 - \sigma_0(1 + \tau_0) + \sigma_0^2(1 - \tau_0 + \tau_0^2)). \end{aligned} \tag{9.3}$$

The factors in (9.3) are related by the transformations of (σ, τ) induced by coordinate permutations, as in Table 5.3, so it is enough to consider the factor. The lines parametrized by the vanishing of the other factors are just given by coordinate transformations.

The first factor is zero if $\sigma_0 = -\omega$ for ω a nontrivial third root of unity. This immediately implies $\nu(1-\sigma) = 0$ and $\nu(1-\sigma+\sigma^2) > 0$. The results of this subsection can be summarized in the following table, giving all possible valuations of the terms in (5.11) in this case.

case	$\nu(1-\sigma)$	$\nu(1-\sigma+\sigma^2)$	$\nu(1-\tau)$	$\nu(1-\tau+\tau^2)$	$\nu(1-\sigma\tau)$	$\nu(1-\sigma\tau+\sigma^2\tau^2)$
1)	0	λ	λ	0	0	$5+\lambda$
2)	0	$5+\lambda$	λ	0	0	λ
3)	0	λ	0	$5+\lambda$	λ	0
4)	0	$5+\lambda$	0	λ	λ	0
5)	0	μ	0	$5-\mu$	0	0

case	$\nu(1-\tau(1+\sigma)+\tau^2(1-\sigma+\sigma^2))$	$\nu(1-\sigma(1+\tau)+\sigma^2(1-\tau+\tau^2))$
1)	0	0
2)	0	0
3)	0	0
4)	0	0
5)	0	0

Table 9.2: Possible valuations for the vertex of Figure 9.1. Here $\lambda \in (0,\infty)$ and $\mu \in (0,5)$.

Lemma 9.9. If $\nu(1-\sigma) > 0$, then $\nu(1-\tau+\tau^2)$, $\nu(1-\sigma\tau)$, $\nu(1-\tau(1+\sigma)+\tau^2(1-\sigma+\sigma^2))$ and $\nu(1-\sigma(1+\tau)+\sigma^2(1-\tau+\tau^2))$ are all zero.

Proof. If $\nu(1-\tau) > 0$, then $\tau_0 = 1$. The t-constant terms are all nonzero:

$$\begin{aligned} 1-\tau_0+\tau_0^2 = 1, \quad 1-\sigma_0\tau_0 = 1+\omega, \\ 1-\tau_0(1+\sigma_0)+\tau_0^2(1-\sigma_0+\sigma_0^2) = \omega, \\ 1-\sigma_0(1+\tau_0)+\sigma_0^2(1-\tau_0+\tau_0^2) = \omega. \end{aligned}$$

This implies their valuations are zero. □

Lemma 9.10. If $\nu(1-\tau) > 0$, then either $\nu(1-\sigma+\sigma^2) = \nu(1-\tau)$ and $\nu(1-\sigma\tau+\sigma^2\tau^2) = 5+\nu(1-\tau)$ or $\nu(1-\sigma+\sigma^2) = 5+\nu(1-\tau)$ and $\nu(1-\sigma\tau+\sigma^2\tau^2) = \nu(1-\tau)$.

Proof. If $\nu(1-\tau) > 0$, then $\tau_0 = 1$. We show that $\alpha = \nu(1-\tau)$ or $\alpha = 5+\nu(1-\tau)$.

- Assume $\alpha < \nu(1-\tau)$. Then the lowest t-order term in $F(\sigma,\tau)$ is

$$F(\sigma,\tau)_{2\alpha} = -3\omega\sigma_1^2.$$

  But $\sigma_1 \neq 0$ by assumptions, so this contradicts $F(\sigma,\tau) = 0$.

- Assume $\alpha = \nu(1-\tau)$. Then $2\alpha = \alpha + \nu(1-\tau)$ and the lowest t-order term is

$$F(\sigma,\tau)_{2\alpha} = -3\sigma_1(\tau_1 - \omega\sigma_1)$$

  This is zero if $\tau_1 = \omega\sigma_1 \neq 0$, so $\alpha = \nu(1-\tau)$ gives no contradiction.
- Assume $\nu(1-\tau) < \alpha < 5 + \nu(1-\tau)$. Then the lowest t-order term is

$$F(\sigma,\tau)_{\alpha+\nu(1-\tau)} = -3\sigma_1\tau_1.$$

  But again $\sigma_1, \tau_1 \neq 0$, in contradiction to $F(\sigma,\tau) = 0$.
- Assume $\alpha = 5 + \nu(1-\tau)$. Then the lowest t-order term is

$$F(\sigma,\tau)_{\alpha+\nu(1-\tau)} = -\tau_1(3\sigma_1 - 10^5\omega\tau_1).$$

  This is zero if $\sigma_1 = \frac{10^5}{3}\omega\tau_1$, not in contradiction.
- Assume $\alpha > 5 + \nu(1-\tau)$. Then the lowest t-order term is

$$F(\sigma,\tau)_{5+2\nu(1-\tau)} = 10^5\omega\tau_1^2.$$

  But since $\tau_1 \neq 0$ this contradicts $F(\sigma,\tau) = 0$.

Now we have

$$1 - \sigma\tau + \sigma^2\tau^2 = \omega\tau_1 t^{\nu(1-\tau)} - \sigma_1 t^{\alpha} + \text{(higher order terms)}$$

For $\nu(1-\sigma+\sigma^2) = 5 + \nu(\tau)$ we have $\sigma_1 = \frac{10^5}{3}\omega\tau_1$, hence $\nu(1-\sigma\tau+\sigma^2\tau^2) = \nu(1-\tau)$. For $\nu(1-\sigma+\sigma^2) = \nu(\tau)$ we use the symmetry of $F(\sigma,\tau)$ under the transformation $(\sigma,\tau) \mapsto \left(\sigma\tau, \frac{1}{\sigma}\right)$, corresponding to $z_2 \leftrightarrow x_3$, to get $\nu(1-\sigma+\sigma^2) = 5 + \nu(\tau)$. Note that $\tau_0 = 1$, so $\frac{1}{\tau_0} = \tau_0$. $\square$

Lemma 9.11. If $\nu(1-\tau+\tau^2) > 0$, then $\nu(1-\tau)$, $\nu(1-\sigma\tau+\sigma^2\tau^2)$, $\nu(1-\tau(1+\sigma)+\tau^2(1-\sigma+\sigma^2))$ and $\nu(1-\sigma(1+\tau)+\sigma^2(1-\tau+\tau^2))$ are all zero.

Proof. If $\nu(1-\tau+\tau^2) > 0$, then $\tau_0 \in \{-\omega, -\omega^2\}$. The t-constant terms are all nonzero. For $\tau_0 = -\omega$,

$$\begin{aligned} &1 - \tau_0 = 1 + \omega, \quad 1 - \sigma_0\tau_0 + \sigma_0^2\tau_0^2 = -2\omega^2, \\ &1 - \tau_0(1+\sigma_0) + \tau_0^2(1-\sigma_0+\sigma_0^2) = -2\omega^2, \\ &1 - \sigma_0(1+\tau_0) + \sigma_0^2(1-\tau_0+\tau_0^2) = -2\omega^2. \end{aligned}$$

For $\tau_0 = -\omega^2$,

$$\begin{aligned} &1 - \tau_0 = 1 + \omega^2, \quad 1 - \sigma_0\tau_0 + \sigma_0^2 + \tau_0^2 = 1, \\ &1 - \tau_0(1+\sigma_0) + \tau_0^2(1-\sigma_0+\sigma_0^2) = \omega^2, \\ &1 - \sigma_0(1+\tau_0) + \sigma_0^2(1-\tau_0+\tau_0^2) = \omega. \end{aligned}$$

This implies their valuations are zero. $\square$

Lemma 9.12. If $\nu(1-\tau+\tau^2) > 0$, then $\nu(1-\tau) = 0$ and we have one of the three cases

$$\begin{aligned}
\nu(1-\sigma+\sigma^2) &= 5+\nu(1-\tau+\tau^2), \quad \nu(1-\sigma\tau) = \nu(1-\tau+\tau^2);\\
\nu(1-\tau+\tau^2) &= 5+\nu(1-\sigma+\sigma^2), \quad \nu(1-\sigma\tau) = \nu(1-\sigma+\sigma^2);\\
\nu(1-\sigma+\sigma^2)+\nu(1-\tau+\tau^2) &= 5, \quad \nu(1-\sigma\tau) = 0.
\end{aligned}$$

Proof. If $\nu(1-\tau+\tau^2) > 0$, then $\tau_0 \in \{-\omega, -\omega^2\}$.

Let $\tau_0 = -\omega$. Then $1-\sigma_0\tau_0 = 1+\sigma^2$, hence $\nu(1-\sigma\tau) = 0$. We show that $\alpha+\beta = 5$.

- Assume $\alpha+\beta > 5$. Then the lowest t-order term in $F(\sigma,\tau)$ is

$$F(\sigma,\tau)_0 = -3\cdot 10^5\omega.$$

  This nonzero in contradiction with $F(\sigma,\tau) = 0$.
- Assume $\alpha+\beta < 5$. Then the lowest t-order term is

$$F(\sigma,\tau)_{\alpha+\beta} = -24\sigma_1\tau_1.$$

  By the assumptions $\sigma_1,\tau_1 \neq 0$, this contradicts $F(\sigma,\tau) = 0$.
- Assume $\beta = 5-\alpha$. Then $\alpha+\beta = 5$ and the lowest t-order term is

$$F(\sigma,\tau)_5 = -24(\sigma_1\tau_1 - 5^5\omega^2).$$

  This is zero if $\sigma_1\tau_1 = 5^5\omega^2$, not contradicting the assumptions.

Let $\tau_0 = -\omega^2$, then $1-\sigma_0\tau_0 = 0$, hence $\nu(1-\sigma\tau) > 0$. We show that $\alpha = 5+\beta$ or $\beta = 5+\alpha$.

- Assume $\alpha = \beta < 5$. Then the lowest t-order term in $F(\sigma,\tau)$ is

$$F(\sigma,\tau)_{2\alpha} = -3\sigma_1\tau_1.$$

  By the assumptions $\sigma_1,\tau_1 \neq 0$ this contradicts $F(\sigma,\tau) = 0$.
- Assume $\beta < \alpha < 5+\beta$. Then the lowest t-order term is

$$F(\sigma,\tau)_{\alpha+\beta} = -3\sigma_1\tau_1.$$

  This again contradicts $F(\sigma,\tau) = 0$.
- Assume $\alpha > 5+\beta$. Then the lowest t-order term is

$$F(\sigma,\tau)_{5+2\beta} = -10^5\omega\tau_1^2.$$

  By $\tau_1 \neq 0$ this again contradicts $F(\sigma,\tau) = 0$.

- Assume $\alpha = 5 + \beta$. Then $\alpha + \beta = 5 + 2\beta$ and the lowest t-order term is

$$F(\sigma, \tau)_{\alpha+\beta} = -\tau_1(3\sigma_1 + 10^5\omega\tau_1).$$

  The solution $\sigma_1 = -\frac{10^5}{3}\omega\tau_1$ does not contradict the assumptions.
- The same arguments show that if $\beta > \alpha$, then $\beta = 5 + \alpha$.

Moreover,

$$1 - \sigma\tau = \omega^2\sigma_1 t^\alpha + \omega\tau_1 t^\beta + \text{(higher order terms)},$$

hence $\nu(1 - \sigma\tau) = \min\{\alpha, \beta\}$. □

Lemma 9.13. If $\nu(1 - \tau) = \nu(1 - \tau + \tau^2) = 0$, then one of the other factors in (9.3) is also zero. Hence, this is just a permutation of one of the former cases.

Proof. Assume $\alpha > 5$. Then the lowest t-order term in $F(\sigma, \tau)$ is

$$F(\sigma, \tau)_5 = 10^5\tau_0^2(1 - \tau_0)^2(1 + \omega\tau_0)^2.$$

By assumptions $\tau_0 \notin \{0, 1, -\omega, -\omega^2\}$, so this is nonzero in contradiction with $F(\sigma, \tau) = 0$.

Assume $\alpha < 5$. Then the lowest t-order term in $F(\sigma, \tau)$ is

$$\begin{aligned} F(\sigma, \tau)_0 = &- (1 - 2\omega)\sigma_1(1 - \tau_0 + \tau_0^2)(1 - \sigma_0\tau_0 + \sigma_0^2\tau_0^2) \\ &\cdot (1 - \tau_0(1 + \sigma_0) + \tau_0^2(1 - \sigma_0 + \sigma_0^2))(1 - \sigma_0(1 + \tau_0) + \sigma_0^2(1 - \tau_0 + \tau_0^2)). \end{aligned}$$

with $\sigma_0 = -\omega$. Hence, σ_0, τ_0 cancel one of the other factors of (9.3). □

Proposition 9.14. The first four rows of Table 9.2 are mapped to a row in Table 9.1 under the reparametrization

$$(\sigma, \tau) \mapsto (\tilde{\sigma}, \tilde{\tau}) = \left(\frac{1 - \sigma\tau}{1 - \tau}, 1 - \tau\right)$$

corresponding the the coordinate permutation $z_0 \leftrightarrow z_2$.

Proof. For the first and second row in Table 9.2 we have

$$\nu(\tilde{\sigma}) = \nu\left(\frac{1 - \sigma\tau}{1 - \tau}\right) = -\lambda, \qquad \nu(\tilde{\tau}) = \nu(1 - \tau) = \lambda.$$

This gives the second row in Table 9.1. Similarly, for the third and fourth row in Table 9.1 we have

$$\nu(\tilde{\sigma}) = \nu\left(\frac{1 - \sigma\tau}{1 - \tau}\right) = \lambda, \qquad \nu(\tilde{\tau} = \nu(1 - \tau) = 0.$$

giving the sixth row in Table 9.1. Of course, one can also check that all other entries in Table 9.1 are obtained correctly from this procedure. □

10 Tropicalization of the Family of Lines

10.1 The Main Theorem

The main theorem of this section is the following. It gives the possible tropicalizations for the lines in the family and follows from Propositions 9.4, 9.14, 10.3 and 10.5.

Theorem 10.1. Any tropicalization of a member in the family of lines is given by a permutation of coordinates from one of the following tropical lines in $\mathbb{TP}^4$, shwon in Figure 10.1.

The first one consist of the edge connecting the vectices $(0,0,1,0,0)$ and $(0,\lambda,1+\lambda,0,0)$, where $\lambda \in \overline{\mathbb{R}}$, together with the five rays

$$\begin{aligned}
(0,0,1,0,0) &+ (1,0,0,0,0)\overline{\mathbb{R}}_{\geq 0}\\
(0,0,1,0,0) &+ (0,0,0,1,0)\overline{\mathbb{R}}_{\geq 0}\\
(0,0,1,0,0) &+ (0,0,0,0,1)\overline{\mathbb{R}}_{\geq 0}\\
(0,\lambda,1+\lambda,0,0) &+ (0,1,0,0,0)\overline{\mathbb{R}}_{\geq 0}\\
(0,\lambda,1+\lambda,0,0) &+ (0,0,1,0,0)\overline{\mathbb{R}}_{\geq 0}.
\end{aligned} \tag{10.1}$$

The second one has one vertex $(0,1,1-\lambda,0,0)$, where $\lambda \in [0,1]$, and five rays

$$\begin{aligned}
(0,1,1-\lambda,0,0) &+ (1,0,0,0,0)\overline{\mathbb{R}}_{\geq 0}\\
(0,1,1-\lambda,0,0) &+ (0,1,0,0,0)\overline{\mathbb{R}}_{\geq 0}\\
(0,1,1-\lambda,0,0) &+ (0,0,1,0,0)\overline{\mathbb{R}}_{\geq 0}\\
(0,1,1-\lambda,0,0) &+ (0,0,0,1,0)\overline{\mathbb{R}}_{\geq 0}\\
(0,1,1-\lambda,0,0) &+ (0,0,0,0,1)\overline{\mathbb{R}}_{\geq 0}.
\end{aligned} \tag{10.2}$$

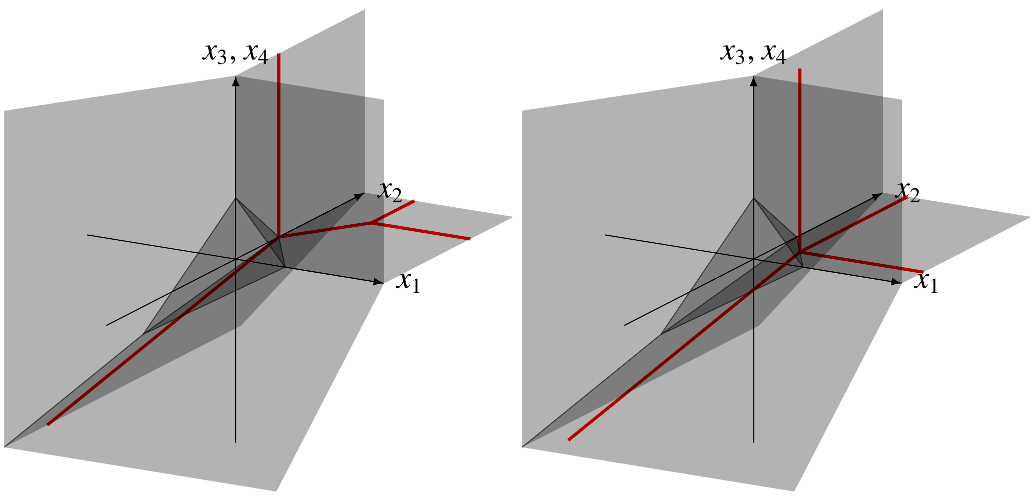


Figure 10.1: The possible tropicalizations of a line in the family.

10.2 Tropicalization from the Defining Equations

Proposition 10.2. For $\nu(\sigma) = 0$, $\nu(\tau) > 0$, a tropical basis for the ideal $I \subseteq \mathbb{C}\{\{t\}\}[z_0, z_1, z_2, z_3, z_4]$ generated by the polynomials in the defining equations (5.17) for a line in the family is

$$\begin{aligned}\mathcal{T}(I) = \{&\sigma\tau^{4/5}\beta(\sigma)\alpha(\tau,\sigma)z_0 + \tau^{4/5}\beta(\sigma)\alpha(\sigma,\tau)z_1 + \alpha(\sigma,\tau)\alpha(\tau,\sigma)z_2,\\ &\sigma\beta(\sigma\tau)\alpha(\tau,\sigma)z_0 + \tau\beta(\sigma\tau)\alpha(\sigma,\tau)z_1 - \alpha(\sigma,\tau)\alpha(\tau,\sigma)z_3,\\ &\sigma^{4/5}\beta(\tau)\alpha(\tau,\sigma)z_0 + \sigma^{4/5}\tau\beta(\tau)\alpha(\sigma,\tau)z_1 + \alpha(\sigma,\tau)\alpha(\tau,\sigma)z_4,\\ &(1-\sigma)\sigma^{4/5}\beta(\tau)\beta(\sigma\tau)z_0 - \sigma^{4/5}\beta(\tau)\alpha(\sigma,\tau)z_3 + \beta(\sigma\tau)\alpha(\sigma,\tau)z_4\}.\end{aligned} \tag{10.3}$$

Proof. Let g_1, g_2, g_3, g_4 be the polynomials in $\mathcal{T}(I)$ as above. Then $I = \langle g_1, g_2, g_3\rangle$. This ideal is homogeneous, so we can apply the construction from the proof of Theorem 2.51. Take $\mathcal{F} = \{g_1, g_2, g_3\}$. Using the valuations in the first row of Table 9.1 we tropicalize these polynomials to

$$\begin{aligned}\mathrm{Trop}(g_1) &= \min\{1 + 2\nu(\tau) + x_0, 1 + \nu(\tau) + x_1, x_2\},\\ \mathrm{Trop}(g_2) &= \min\{\nu(\tau) + x_0, \nu(\tau) + x_1, \nu(\tau) + x_3\},\\ \mathrm{Trop}(g_3) &= \min\{\nu(\tau) + x_0, \nu(\tau) + x_1, \nu(\tau) + x_4\}.\end{aligned}$$

The finite intersection $\cup_{f\in\mathcal{F}}\mathrm{Trop}(V(f))$ in the affine chart $z_0 = 1$ is shown in Figure 10.2. This is not a tropical variety, since its not a polyhedral complex of pure dimension. Thus $\mathcal{F}$ is not a tropical variety.

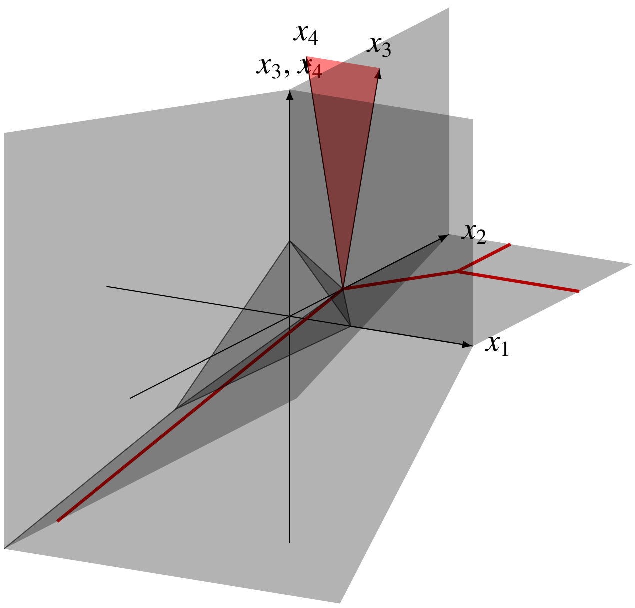


Figure 10.2: The finite intersection $\cup_{f\in\mathcal{F}}\mathrm{Trop}(V(f))$ in the affine chart $z_0 = 0$.

For w in the relative interior of a zero- or one-dimensional polyhedron of $\cup_{f\in\mathcal{F}}\mathrm{Trop}(V(f))$, the initial ideal $\mathrm{in}_w(I)$ does not contain a monomial. The crucial case is if w is in the two-dimensional polyhedron of $\cup_{f\in\mathcal{F}}\mathrm{Trop}(V(f))$.

Let $w = (0, 0, 1, 1, 1) \in \mathbb{R}^5$. Define

$$g = \beta(\sigma\tau) \cdot g_3 - \sigma^{4/5}\beta(\tau) \cdot g_2 \in I.$$

Then

$$\mathrm{in}_w(g) = \overline{(1 - \sigma)\sigma^{4/5}\beta(\tau)\beta(\sigma\tau)} z_0.$$

Hence, the ideal $\mathrm{in}_w(I)$ contains the monomial z_0.

To get a witness for $C_w(I)$ we have to calculate $\mathcal{G}_{<_w}(I)$ for a term order $<$. Take for example the lexicographic order

$$z_0^\alpha < z_1^\beta, \quad z_1^\alpha < z_2^\beta, \quad z_2^\alpha < z_3^\beta, \quad < z_3^\alpha < z_4^\beta \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}^n \setminus \{0\}.$$

Then

$$\begin{aligned}
\mathrm{in}_{<_w}(g_1) &= \alpha(\sigma, \tau)\alpha(\tau, \sigma) z_2, \\
\mathrm{in}_{<_w}(g_2) &= \tau\beta(\sigma\tau)\alpha(\sigma, \tau) z_1, \\
\mathrm{in}_{<_w}(g_3) &= \sigma^{4/5}\tau\beta(\tau)\alpha(\sigma, \tau) z_1.
\end{aligned}$$

and

$$S_{<_w}(g_2, g_3) = (1 - \sigma)\sigma^{4/5}\beta(\tau)\beta(\sigma\tau) z_0 - \sigma^{4/5}\beta(\tau)\alpha(\sigma, \tau) z_3 + \beta(\sigma\tau)\alpha(\sigma, \tau) z_4.$$

This is not divisible by any $\mathrm{in}_w(g_i)$, so $\mathcal{G}_{<_w}(I) = \{g_1, g_2, g_3, S_{<_w}(g_2, g_3)\}$.

The normal form to z_0 with respect to $\mathcal{G}_{<_w}(I)$ is

$$h = \frac{\sigma^{4/5}\beta(\tau)\alpha(\sigma, \tau)}{(1 - \sigma)\sigma^{4/5}\beta(\tau)\beta(\sigma\tau)} z_3 + \frac{\beta(\sigma\tau)\alpha(\sigma, \tau)}{(1 - \sigma)\sigma^{4/5}\beta(\tau)\beta(\sigma\tau)} z_4.$$

Adding $(1 - \sigma)\sigma^{4/5}\beta(\tau)\beta(\sigma\tau)(z_0 - h)$ to $\mathcal{F}$ we obtain $\mathcal{T}(I)$ as claimed. □

Proposition 10.3. The tropicalization of a line parametrized by σ and τ with $\nu(\sigma) = 0$ and $\nu(\tau) > 0$ consists of the edge connecting the vertices $(0, 0, 1, 0, 0)$ and $(0, \nu(\tau), 1 + \nu(\tau), 0, 0)$ together with the five rays

$$\begin{aligned}
(0, 0, 1, 0, 0) &+ (1, 0, 0, 0, 0)\bar{\mathbb{R}}_{\geq 0} \\
(0, 0, 1, 0, 0) &+ (0, 0, 0, 1, 0)\bar{\mathbb{R}}_{\geq 0} \\
(0, 0, 1, 0, 0) &+ (0, 0, 0, 0, 1)\bar{\mathbb{R}}_{\geq 0} \\
(0, \nu(\tau), 1 + \nu(\tau), 0, 0) &+ (0, 0, 1, 0, 0)\bar{\mathbb{R}}_{\geq 0} \\
(0, \nu(\tau), 1 + \nu(\tau), 0, 0) &+ (0, 1, 0, 0, 0)\bar{\mathbb{R}}_{\geq 0}
\end{aligned}$$

This is shown on the left hand side in Figure 10.1.

Proof. Dehomogenize by setting $z_0 = 1$. Using the valuations for the terms as in Table 9.1, the polynomials in the tropical basis (10.3) tropicalize to

$$\begin{aligned}
&\min\{1 + 2\nu(\tau), 1 + \nu(\tau) + x_1, \nu(\tau) + x_2\},\\
&\min\{\nu(\tau), \nu(\tau) + x_1, \nu(\tau) + x_3\},\\
&\min\{\nu(\tau), \nu(\tau) + x_1, \nu(\tau) + x_4\},\\
&\min\{0, x_3, x_4\}.
\end{aligned}$$

There are three cases in which the first minimum is achieved at least twice.

1. If $x_1 = \nu(\tau)$, $x_2 \geq 1 + \nu(\tau)$, then the other minima are not unique only if $x_3 = x_4 = 0$. This gives
$$x_1 = \nu(\tau), \quad x_2 \geq 1 + \nu(\tau), \quad x_3 = 0, \quad x_4 = 0.$$
2. If $x_2 = 1 + \nu(\tau)$, $x_1 \geq \nu(\tau)$, by the same arguments
$$x_1 \geq \nu(\tau), \quad x_2 = 1 + \nu(\tau), \quad x_3 = 0, \quad x_4 = 0.$$
3. The third case is $x_1 = x_2 - 1 \leq \nu(\tau)$. Then we again have three cases.
   a) If $0 \leq x_1 \leq \nu(\tau)$, then the second and third minimum are not unique if
   $$0 \leq x_1 \leq \nu(\tau), \quad x_2 = 1 + x_1, \quad x_3 = 0, \quad x_4 = 0.$$
   b) If $x_1 \leq 0$, the second and third minimum are not unique if
   $$x_1 \leq 0, \quad x_2 = 1 + x_1, \quad x_3 = x_1, \quad x_4 = x_1.$$
   c) If $x_1 = 0$, we fourth minimum is achieved twice in one of the following cases:
   $$x_1 = 0, \quad x_2 = 1, \quad x_3 \geq 0, \quad x_4 = 0;$$
   $$x_1 = 0, \quad x_2 = 1, \quad x_3 = 0, \quad x_4 \geq 0.$$

Each case gives precisely one edge or ray of the tropicalization as claimed. □

Proposition 10.4. For the valuations as in the fifth row of Table 9.2, a tropical basis for the ideal $I \subseteq \mathbb{C}\{\{t\}\}[z_0, z_1, z_2, z_3, z_4]$ generated by the polynomials in(5.17) is

$$\begin{aligned}
\mathcal{T}(I) = \{&\sigma\tau^{4/5}\beta(\sigma)\alpha(\tau,\sigma)z_0 + \tau^{4/5}\beta(\sigma)\alpha(\sigma,\tau)z_1 + \alpha(\sigma,\tau)\alpha(\tau,\sigma)z_2,\\
&\sigma\beta(\sigma\tau)\alpha(\tau,\sigma)z_0 + \tau\beta(\sigma\tau)\alpha(\sigma,\tau)z_1 - \alpha(\sigma,\tau)\alpha(\tau,\sigma)z_3,\\
&\sigma^{4/5}\beta(\tau)\alpha(\tau,\sigma)z_0 + \sigma^{4/5}\tau\beta(\tau)\alpha(\sigma,\tau)z_1 + \alpha(\sigma,\tau)\alpha(\tau,\sigma)z_4,\\
&(1-\sigma)\sigma^{4/5}\beta(\tau)\beta(\sigma\tau)z_0 - \sigma^{4/5}\beta(\tau)\alpha(\sigma,\tau)z_3 + \beta(\sigma\tau)\alpha(\sigma,\tau)z_4,\\
&(1-\tau)\sigma\beta(\sigma)\beta(\sigma\tau)\alpha(\tau,\sigma)z_0 + \tau^{1/5}\beta(\sigma,\tau)\alpha(\tau,\sigma)z_2 + \beta(\sigma)\alpha(\tau,\sigma)z_3\}.
\end{aligned} \tag{10.4}$$

Proof. Denote the polynomials in $\mathcal{T}(I)$ by g_1, g_2, g_3, g_4, g_5. Then g_1, g_2, g_3, g_4 are the polynomials from (10.3). Note that

$$g_5 = \frac{\tau^{1/5}\beta(\sigma\tau)}{\alpha(\sigma,\tau)} g_1 - \frac{\beta(\sigma)}{\alpha(\sigma,\tau)} g_2,$$

so this is indeed in the ideal I. The argumentation to add this to $\mathcal{T}(I)$ is the same as in the proof of Proposition 10.2. Proposition 10.5 shows that this indeed defines a tropical line. □

Proposition 10.5. The tropicalization of a line parametrized by σ and τ as in the fifth row of Table 9.2 consists of the vertex $(0, 1, 1 - \lambda, 0, 0)$, where $\lambda \in [0, 1]$, and the five rays

$$\begin{aligned}
&(0,0,1,0,1-\lambda) + (1,0,0,0,0)\overline{\mathbb{R}}_{\geq 0}\\
&(0,0,1,0,1-\lambda) + (0,1,0,0,0)\overline{\mathbb{R}}_{\geq 0}\\
&(0,0,1,0,1-\lambda) + (0,0,1,0,0)\overline{\mathbb{R}}_{\geq 0}\\
&(0,0,1,0,1-\lambda) + (0,0,0,1,0)\overline{\mathbb{R}}_{\geq 0}\\
&(0,0,1,0,1-\lambda) + (0,0,0,0,1)\overline{\mathbb{R}}_{\geq 0}.
\end{aligned} \tag{10.5}$$

Proof. Dehomogenize by setting $z_0 = 1$. Using the valuations for the terms in the fifth row of Table 9.2, writing $\lambda = \frac{1}{5}\mu$, the polynomials in the tropical basis (10.4) tropicalize to

$$\begin{aligned}
&\min\{\lambda, \lambda + x_1, x_2\},\\
&\min\{0, x_1, x_3\},\\
&\min\{1-\lambda, 1-\lambda + x_1, x_4\},\\
&\min\{1-\lambda, 1-\lambda + x_3, x_4\},\\
&\min\{\lambda, x_2, \lambda + x_3\}.
\end{aligned}$$

The cases in which all these minima are not unique give the tropicalization as claimed. □

10.3 Tropicalizaition from the Parametric Representation

Proposition 10.6. The tropicalization of a line parametrized by σ and τ with $\nu(\sigma) = 0$ and $\nu(\tau) > 0$ consists of the edge connecting the vertices $(0, 1, 0, 0)$ and $(\nu(\tau), 1 + \nu(\tau), 0, 0)$ together with the five rays

$$\begin{aligned}
(0,0,1,0,0) &\quad + \quad (1,0,0,0,0)\bar{\mathbb{R}}_{\geq 0},\\
(0,0,1,0,0) &\quad + \quad (0,0,0,1,0)\bar{\mathbb{R}}_{\geq 0},\\
(0,0,1,0,0) &\quad + \quad (0,0,0,0,1)\bar{\mathbb{R}}_{\geq 0},\\
(0,\nu(\tau),1+\nu(\tau),0,0) &\quad + \quad (0,1,0,0,0)\bar{\mathbb{R}}_{\geq 0},\\
(0,\nu(\tau),1+\nu(\tau),0,0) &\quad + \quad (0,0,1,0,0)\bar{\mathbb{R}}_{\geq 0}.
\end{aligned}$$

Proof. Using the first row of Table 9.1 we calculate the valuations of the coordinates in (5.11) to be

$$\begin{aligned} \nu(z_0) &= \nu(u), \\ \nu(z_1) &= \nu(\tau) + \nu(v), \\ \nu(z_2) &= 1 + \nu(\tau) + \nu(\sigma u + v), \\ \nu(z_3) &= \nu(\sigma u + \tau v), \\ \nu(z_4) &= \nu(u + \tau v). \end{aligned}$$

We dehomogenize by setting $\nu(z_0) = \nu(u) = 0$. Then

$$\begin{aligned} \nu(z_1) &= \nu(\tau) + \nu(v), \\ \nu(z_2) &= 1 + \nu(\tau) + \nu(\sigma + v), \\ \nu(z_3) &= \nu(\sigma + \tau v), \\ \nu(z_4) &= \nu(1 + \tau v). \end{aligned}$$

Now we have five cases for $\nu(v)$.

- If $\nu(v) < -\nu(\tau)$ then

  $$\nu(z_1) = \nu(\tau) + \nu(v), \quad \nu(z_2) = 1 + \nu(\tau) + \nu(v), \quad \nu(z_3) = \nu(\tau) + \nu(v), \quad \nu(z_4) = \nu(\tau) + \nu(v).$$

  Note that $(0, -1, -1, -1, -1) = (1, 0, 0, 0, 0) \in \mathbb{TP}^4$, so this gives the first ray in the claim.
- If $\nu(v) = -\nu(\tau)$ then $\nu(1 + \tau v) \geq$ and $\nu(\sigma + \tau v) \geq 0$.

  $$\nu(z_1) = 0, \quad \nu(z_2) = 1, \quad \nu(z_3) = \nu(\sigma + \tau v) \geq 0, \quad \nu(z_4) = \nu(1 + \tau v) \geq 0.$$

  Moreover, $\nu(1 + \tau v) = \nu((\sigma + \tau v) + (1 - \sigma))$. Since $\nu(1 - \sigma) = 0$, by Proposition 2.6 we have $\nu(\sigma + \tau v) = 0 \Rightarrow \nu(1 + \tau v) > 0$, and vice versa.
- If $\nu(v) > -\nu(\tau)$ and $\nu(v) < 0$ then

  $$\nu(z_1) = \nu(\tau) + \nu(v), \quad \nu(z_2) = 1 + \nu(\tau) + \nu(v), \quad \nu(z_3) = 0, \quad \nu(z_4) = 0.$$

- If $\nu(v) = 0$ then $\nu(\sigma + v) \geq 0$ and

  $$\nu(z_1) = \nu(\tau), \quad \nu(z_2) = 1 + \nu(\tau) + \nu(\sigma + v) \geq 0, \quad \nu(z_3) = 0, \quad \nu(z_4) = 0.$$

- If $\nu(v) > 0$ then

  $$\nu(z_1) = \nu(\tau) + \nu(v), \quad \nu(z_2) = 1 + \nu(\tau), \quad \nu(z_3) = 0, \quad \nu(z_4) = 0.$$

This gives precisely the tropical line as claimed. □

Proposition 10.7. The tropicalization of a line parametrized by σ and τ as in the fifth row of Table 9.2 consists of the vertex $(0, 1, 1 - \lambda, 0, 0)$, where $\lambda \in [0, 1]$, and the five rays

$$\begin{aligned}
&(0, 0, 1, 0, 1 - \lambda) + (1, 0, 0, 0, 0)\overline{\mathbb{R}}_{\geq 0} \\
&(0, 0, 1, 0, 1 - \lambda) + (0, 1, 0, 0, 0)\overline{\mathbb{R}}_{\geq 0} \\
&(0, 0, 1, 0, 1 - \lambda) + (0, 0, 1, 0, 0)\overline{\mathbb{R}}_{\geq 0} \\
&(0, 0, 1, 0, 1 - \lambda) + (0, 0, 0, 1, 0)\overline{\mathbb{R}}_{\geq 0} \\
&(0, 0, 1, 0, 1 - \lambda) + (0, 0, 0, 0, 1)\overline{\mathbb{R}}_{\geq 0}.
\end{aligned} \tag{10.6}$$

Proof. Using the fifth row of Table 9.2, writing $\lambda = \frac{1}{5}\mu$, we calculate the valuations of the coordinates in (5.11) to be

$$\begin{aligned}
\nu(z_0) &= \nu(u), \\
\nu(z_1) &= \nu(v), \\
\nu(z_2) &= \lambda + \nu(\sigma u + v), \\
\nu(z_3) &= \nu(\sigma u + \tau v), \\
\nu(z_4) &= 1 - \lambda + \nu(u + \tau v).
\end{aligned}$$

We dehomogenize by setting $\nu(z_0) = \nu(u) = 0$. Then

$$\begin{aligned}
\nu(z_1) &= \nu(v), \\
\nu(z_2) &= \lambda + \nu(\sigma + v), \\
\nu(z_3) &= \nu(\sigma + \tau v), \\
\nu(z_4) &= 1 - \lambda + \nu(1 + \tau v).
\end{aligned}$$

Now we have three cases for $\nu(v)$.

- If $\nu(v) < 0$ then

$$\nu(z_1) = \nu(v), \quad \nu(z_2) = \lambda + \nu(v), \quad \nu(z_3) = \nu(v), \quad \nu(z_4) = 1 - \lambda + \nu(v).$$

- If $\nu(v) > 0$ then

$$\nu(z_1) = \nu(v), \quad \nu(z_2) = \lambda, \quad \nu(z_3) = 0, \quad \nu(z_4) = 1 - \lambda.$$

- If $\nu(v) = 0$ then

$$\nu(z_1) = 0, \quad \nu(z_2) = \lambda + \nu(\sigma + v), \quad \nu(z_3) = \nu(\sigma + \tau v), \quad \nu(z_4) = 1 - \lambda + \nu(1 + \tau v),$$

  where $\nu(\sigma + v), \nu(1 + \tau v), \nu(1 + \tau v) \geq 0$. Moreover, $\nu(1 + \tau v) = \nu((\sigma + \tau v) + (1 - \sigma))$ and

This gives precisely the tropical line as claimed. $\square$

10.4 Lines in the Dual Intersection Complex

Finally, let us look at the parametrized tropical lines associated to the lines we found in §5. These lie in the affine manifold with singularities given by the dual intersection complex of $\mathfrak{X}$.

The central fiber $\mathfrak{X}_0$ is the union of the five coordinate hyperplanes in $\mathbb{P}^4$. Each of these hyperplanes is isomorphic to $\mathbb{P}^3$ and intersects all other hyperplanes in a $\mathbb{P}^2$. The dual intersection complex B of $\mathfrak{X}$ is the boundary of a 4-dimensional polyhedron with each face affine isomorphic to a standard 3-simplex. The fan structure at each vertex is given by the fan to $\mathbb{P}^3$. This gives the picture as in Figure 10.3.

The straight lines give the 1-skeleton of one of the 3-faces. The big dashed lines indicate edges going in the fourth dimension. The small dashed lines give the singular locus on the 2-faces of that 3-face. The monodromy around these singularities is calculated in [**9**] and also given in [**11**]. The monodromy invariant directions for the singular lines are perpendicular to them. For the points where they meet, we are monodromy-invariant in all directions of the 2-face.

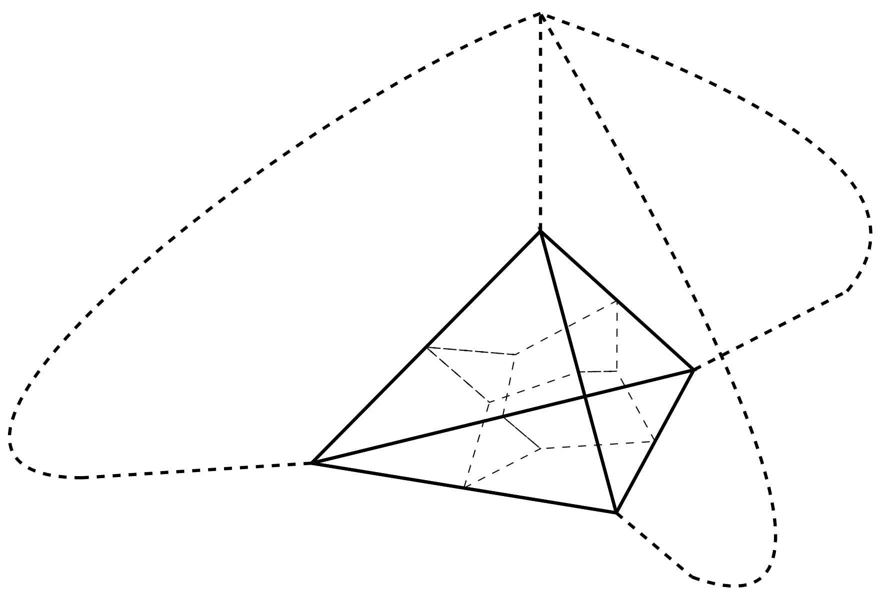


Figure 10.3: The dual intersection complex of the Dwork pencil of quintic threefolds.

Each isolated line lies in a hyperplane of $\mathfrak{X}_0$. The parametric representation $(u, v, -\zeta^k u, -\zeta^l v, 0)$ shows that such a line intersects two codimension 2 strata of its hyperplane. By Definition 3.20, the vertex of the associated tropical line is a vertex of B, and the line has two edges ending in the interiors of 2-faces not adjacent to each other. The only singular point in which they can end in a monodromy invariant direction is the center of the 2-face. This gives the picture as in Figure 10.4.

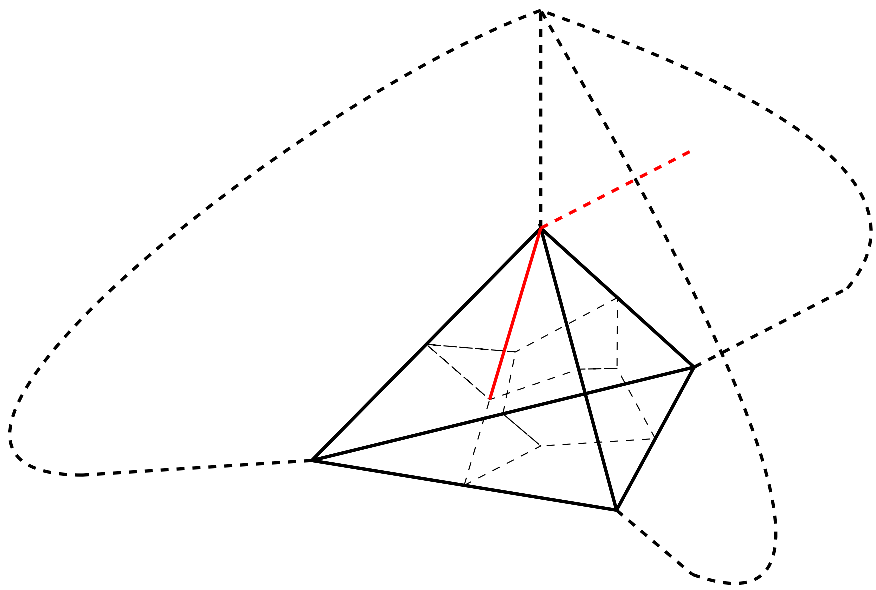


Figure 10.4: The parametrized tropical lines associated to the isolated lines.

Now consider the family of lines. As shown in §6.5, in the case $t = 0$, the polynomial F from (5.13) becomes $F = F_+F_-$ where F_+ and F_- both have five factors.

A line parametrized by σ, τ such that a factor in one of F_+ or F_- vanishes lies in a coordinate hyperplane. The parametric representation (5.11) shows that such a line intersects the four codimension 1 strata of that hyperplane and does not intersect strata of higher codimension. Hence, the parametrized tropical line associated to such a line has its vertex at a vertex of B, and four edges terminating in singular points on the edges of B. This gives a picture as in 10.5.

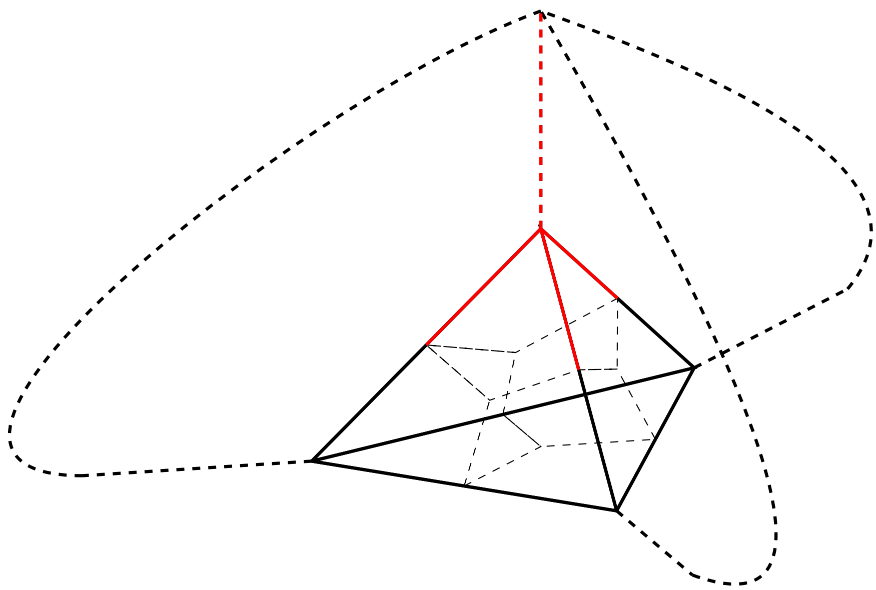


Figure 10.5: The parametrized tropical lines associated to a line in the family.

A line parametrized by σ, τ such that a factor of both F_+ and F_- vanishes lies in the intersection of two coordinate hyperplanes. Again, the parametric representation (5.11) shows that such a line intersects the three common codimension 2 strata of its hyperplanes. This gives a picture as in Figure 10.6.

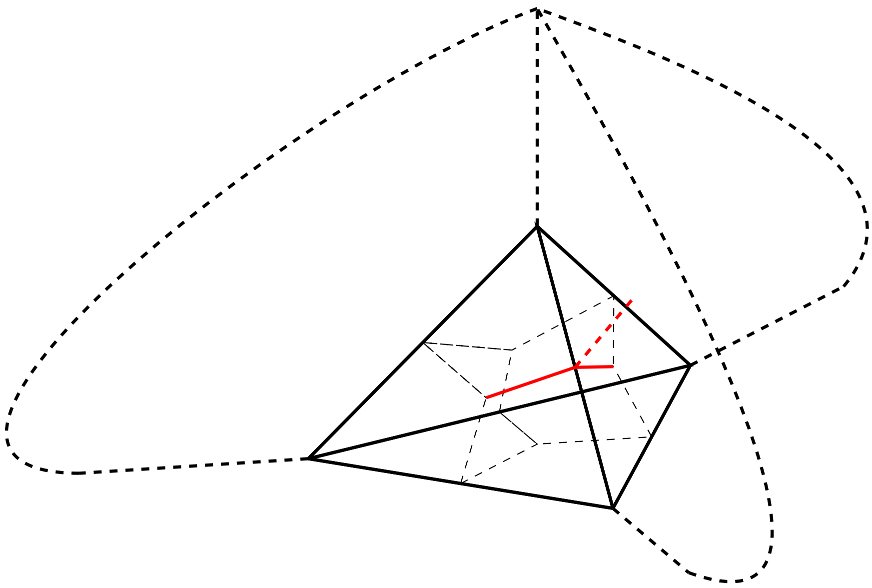


Figure 10.6: The parametrized tropical lines associated to a line in the family.

References

[1] A. Albano, S. Katz, *Lines on the Fermat Quintic Threefold and the Infinitesimal Generalized Hodge Conjecture*, Tans. Amer. Math. Soc. 324 no. 1, March 1991.

[2] A. Albano, S. Katz, *Van Geemen's Families of Lines on Special Quintic Threefolds*, Tans. Amer. Math. Soc. 324 no. 1, (1991) 353-368.

[3] T. Bogart, A. Jensen, D. Speyer, B. Strumfels, R. Thomas, *Computing Tropical Varieties*, arXiv:math/0507563, 2005.

[4] H. Clemens, *Cohomology and Obstructions III: A Variational Form of the Generalized Hodge Conjecture on K-Trivial Threefolds*, arXiv:math/9809127, 2002.

[5] P. Candelas, X. de la Ossa, P. Green, L. Parkes, *A pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nucl. Phys. B359 (1991) 21-74.

[6] P. Candelas, X. de la Ossa, B. van Geemen, D. van Straten, *Lines on the Dwork Pencil of Quintic Threefolds*, arXiv:1206.4961, 2012.

[7] D. Eisenbud, J. Harris, *3264 and All That: A Second Course in Algebraic Geometry*, Cambridge University Press, 2016.

[8] O. Forster, *Riemannsche Flächen*, Heidelberger Taschenbücher, Springer Verlag, 1977.

[9] M. Gross, *Topological Mirror Symmetry*, arXiv:math/9909015, 1999.

[10] M. Gross, *Mirror Symmetry and Tropical Geometry*, CBMS Regional Conference Series in Mathematics, 144, Amer. Math. Soc., Providencem RU, 2011.

[11] M. Gross, *Mirror Symmetry and the Strominger-Yau-Zaslow Conjecture*, arXiv:1212.4220, 2013.

[12] A. Jensen, *Algorithmic Aspects of Gröbner Fans and Tropical Varieties*, Ph.D. Dissertation, University of Aarhus, 2007.

[13] G. Mikhalkin, *Enumerative Tropical Algebraic Geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. 18 (2005), 313-377.

[14] D. Maclagan, B. Sturmfels, *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, Amer. Math. Soc., 2015.

[15] A. Mustaţă, *Degree 1 Curves in the Dwork Pencil and the Mirror Quintic*, arXiv:math/0311252, 2003.

[16] T. Nishinou, B. Siebert, *Toric Degenerations of Toric Varieties and Tropical Curves*, Duke Math. J., 135 (2006), 1-51.

Erklärung über die selbstständige Anfertigung der Arbeit

Die vorliegende Arbeit habe ich selbständig verfasst und keine anderen als die angegebenen Hilfsmittel - insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen - benutzt. Die Arbeit habe ich vorher nicht in einem anderen Prüfungsverfahren eingereicht. Die eingereichte schriftliche Fassung entspricht genau der auf dem elektronischen Speichermedium.