Digital Topology

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Contents

Abstract

The aim of this paper is to give an introduction into the field of digital topology. This topic of research arose in connection with image processing. It is important also in all applications of artificial intelligence dealing with spatial structures.

In this article, a simple but representative model of the Euclidean plane, called the digital plane, is studied. The problem is to introduce a satisfactory 'topology' for this essentially discrete structure. It turns out, that all known approaches, which come from different directions of applications and theory, converge to virtually one concept of 4/8– or 8/4–connectedness.

A very natural approach to problems of discrete topology is the concept of semi–topological spaces.

1 Introduction

1.1 Motivation and Scope

The only sets which can be handled on computers are discrete or *digital* sets, wich means sets that contain at most a denumerable number of elements. There are two sources for discrete sets

- The data structures of computer science are enumerable by definition. So, only discrete objects can be represented. This covers a great number of practical situations. For example in most applications of Artificial Intelligence the universe of discourse is a finite set. Another example is discrete classification e.g. of plants and animals or stars into spectral classes etc.

- Continuous objects are discretized, i.e. they are approximated by discrete objects. This is done e.g. in finite element models in engineering but also in image processing where the intrinsically continuous image is represented by a discrete set of 'pixels'.

In some cases the objects under consideration are taken from a space with certain geometric characteristics. Any useful discrete model of the situation should model the geometry faithfully in order to avoid wrong conclusions. For example, if an automatic reasoning system should interpret correctly the sentence "The policemen encircled the house", it must be able to understand that it is not possible to leave the house without meeting a policeman. Mathematically speaking, the set of policemen has some sort of 'Jordan curve property' which enables them to separate the house from the remainder of the world unless one is able to act in three dimensions.

It is clear, that no discrete model can exhibit all relevant features of a continuous original. Therefore, one has to accept compromises. The compromise chosen depends on the specific application and the questions which are typical for the application. Digital geometry is an attempt to evaluate the price one has to pay for discretization. The most simple part of geometry is topology. The digital model topology necessarily reflects only certain facets of the underlying continuous structure. It is therefore necessary to apply different approaches to digital topology.

Our visual system seems to be well adapted to cope with topological properties of the world. For example, letters in a document can be classified in a first step according to their topological homotopy types. The layout of documents frequently is based on topological predicates (e.g., a specific item must appear *within* a preassigned box). Optical checking of wiring of chips amounts in finding out whether the wiring has the homotopy type wanted.

There are mainly three approaches for defining a digital analog of the well–known 'natural' topology of the Euclidean space:

Graph–Theoretic Approach A very elementary structure which can be handled easily on computers is a graph. A graph is obtained when a neighborhood relation is introduced into the digital set. Such a structure allows to investigate connectivity of sets.

Imbedding Approach The discrete structure is imbedded into a known continuous structure, usually into an Euclidean space. Topological properties of discrete objects are then defined by means of their continuous images.

Axiomatic Approach Certain subsets of the underlying digital structure are declared to be 'open sets' and are required to fulfill certain axioms. These axioms have to be chosen in such a way that the digital structure gets properties which are as close as possible to the properties of usual topology.

All three approaches have advantages and disadvantages. The axiomatic approach is mathematically very elegant but it does not directly provide the language which is wanted in applications. The objects which are practically investigated are not open sets but rather connected sets, sets which are contained in other sets etc. The graph theoretic approach yields directly connectedness but is beomes very difficult to handle more complicated concepts of topology such as continuity, homotopy etc. The embedding approach is of course only adequate for structures which can be related to an Euclidean space. The problem is that one has to find for each question an appropriate embedding.

There are many proposals in the literature which deal with topological questions. Of course, these proposals cannot always be characterized by only one of the approaches given above.

We list some of the questions which are typically attacked by digital topology:

- Definition of connectedness and connected components of a set.
- Classification of points in a digital set as interior and boundary points, definition of boundaries of digital sets.
- Formulation of Jordan's Curve Theorem (and its higher dimensional analogs). This theorem states that a set can be represented by its boundary which leads to a reduction of dimensionality in the representation. This Theorem is, as mentioned above, essential for understanding spatial relations and is used for grouping items in spatial data structures.
- One very important topological 'invariant' of a set is its Euler number. This number is in a certain sense all what we can get by parallel working machines.
- General topology deals mainly with continuous functions. Such functions have no counterparts in discrete structures. However, it would be useful for many applications to have such a concept as 'discrete continuity'.
- By means of a suitable definition of continuity one is able to compare topological structures with each other. A very powerful concept in topology is homeomorphy. Homeomorphic topological structures are topologically the same. So, if a topological problem is investigated, one has to look for a homeomorphic structure in which the problem can be stated and solved as easily as possible.
- Homotopy theory deals with properties of sets which are invariant under continuous deformations. Translating homotopy from general topology to discrete structures raises a number of questions which are not yet solved in a satisfactory way. On the other hand, one of the most often used preprocessing methods in image processing is thinning which is the numerical realization of a 'deformation retract' from homotopy theory.

There are several practical situations other than image processing or spatial reasoning in artificial intelligence where topological spaces were used to model discrete situations, starting from Alexandrov's work in 1935 (see [1, 2]). In these publications, Alexandroff provided a theoretical basis for the topology of 'cell complexes'.

- In his book on the lambda calculus, Barendregt uses the Scott topology [10, p. 9 f.] for partially ordered sets [55], [56]. By means of this topology, concepts such as continuity of functions can be introduced. There exist numerous publications on application of discrete topologies to data structures, abstract computing models, combinatorial algorithms, complexity theory and logics.
- By using a suitable topology over the attribute set, Baik and Miller attempted to test equivalence of heterogeneous relational databases [9].

- Brissaud [12, 13, 14] defined a so–called 'pre–topology' for modelling preference structures in economics. (for other applications in economics see [4], [5], [6], [7]).

  These pre–topological spaces were later used by Arnaud et al. [8] in image processing.

In this paper, only topologies for the 2–dimensional rectangular grid $\mathbb{Z}^2$ are considered. This grid can be understood to be the set of all points of the Euclidean plane $\mathbb{R}^2$ having integer coordinates. A more general theory is possible [52] but for easier presentation this is not attempted here. The reader should be familiar with elementary topology of the Euclidean plane.

For survey articles on digital topology the reader is referred to [26, 31, 35].

Points in the digital plane $\mathbb{Z}^2$ are indicated by upper–case letters, points in the Euclidean plane $\mathbb{R}^2$ by lower–case letters. Vectors are understood to be column vectors. For easier typographic representation we often write $P = (m, n)^\top$ for the vector $P \in \mathbb{Z}^2$ having integer coordinates m and n.

For a finite set S we denote by $|S|$ the number of its elements. The notation $A := B$ means that A is defined by expression B.

1.2 Historical Remarks

In 1935 Alexandroff and Hopf published a textbook on topology [2]. In this book an axiomatic basis was given for the theory of cell complexes (so–called combinatorial topology). This theory was developed at the begin of this century in order to attack very difficult problems of topology. In 1937 Alexandroff published a paper on the same subject where the term "Discrete Topology" was explicitly used in the title [1]. E. Khalimsky investigated ordered connected topological spaces and wrote a book on this topic in 1977. Later on, in collaboration of the New York school of topology (Kopperman, Meyer, Kong and others) it turned out that the ordered connected spaces are very well suited for treating problems of digital topology. They are equivalent to Alexamndroff spaces. At the end of the eighties V. Kovalevsky gave a sound fundament for digital topology [36], which again turned out to be part of Alexandroff theory. A similar theory was provided in the same time by G. Herman [23].

In 1979 A. Rosenfeld published a paper which had the title "Digital Topology" [53]. This paper was very influential since for the first time some very difficult problems of digital topology were treated rigorously. The paper was based on results of Duda, Hart and Munson [17].

2 The Digital Plane

2.1 Basic Definitions

As a simple but nevertheless useful model we consider the *digital plane* $\mathbb{Z}^2$ which is the set of all points in the plane $\mathbb{R}^2$ having integer coordinates.

Digital Topology and Geometry are concerned with topological and geometrical properties of subsets of the digital plane, so–called *digital sets*. Digital Topology and Geometry are not new areas of mathematics. Many of the problems encountered in this connection are familiar from geometry of numbers [47, 20], from stochastic geometry [46, 57, 58, 59], integral geometry [11, 46] and from discretization of partial differential equations [44].

In image processing the digital plane is taken as a mathematical model of digitzed black–white images. In this application one usually has a given set, namely the set S of *black points* in the image and the complement $\complement S$, which is the set of *white points*.

Given a point $P = (m, n)^\top \in \mathbb{Z}^2$. The 8*–neighbors* of P are all points with integer coordinates $(k, \ell)^\top$ such that

$$\max\left(|m-k|, |n-\ell|\right) \leq 1.$$

We number the 8–neighbors of P in the following way:

row	$n-1$	n	$n+1$
		column	
$m+1$	$N_3(P)$	$N_2(P)$	$N_1(P)$
m	$N_4(P)$	P	$N_0(P)$
$m-1$	$N_5(P)$	$N_6(P)$	$N_7(P)$

Neighbors with even number are the *direct* or 4–*neighbors* of P, those with odd numbers are the *indirect neighbors.* The 8–*neighborhood* $\mathcal{N}_8(P)$ of P is the set of all 8–neighbors of P (excluding P), the 4–*neighborhood* $\mathcal{N}_4(P)$ of P is the set of all 4–neighbors of P.

Let κ be any of the numbers 4 or 8 and let $I = \{0, 1, \cdots, n\}$ be a (finite) *interval* of consecutive integers. A *digital κ–path* or simply path $\mathcal{P}$ is a sequence $\{P_i\}_{i \in I}$ of points in $\mathbb{Z}^2$ such that P_i and P_j are κ–neighbors of each other whenever $|i-j| = 1$. We note that the order induced by the numbering of the points of a path is essential. For $P \in \mathcal{P}$ we define the *κ–degree* or degree of P with respect to $\mathcal{P}$ to be the number $|\mathcal{P} \cap \mathcal{N}_\kappa(P)|$. A point of $\mathcal{P}$ having degree 1 is termed an *end point.* It is an immediate consequence of the definition that any point of a path has degree at least one. There exist at most two end points in a path. End points can only correspond to numbers 0 or n. A path with the property $P_0 = P_n$ is called a *closed path.* A closed path contains no end points.

A κ–path $\mathcal{P}$ is termed a *κ–arc* or simply an arc if it has the additional property that for any two points $P_i, P_j \in \mathcal{P}$ which are not end points $P_i \in \mathcal{N}_\kappa(P_j)$ implies $|i-j| \leq 1$. Consequently, an arc is a path which does not intersect or touch itself with the possible exception of its end points. Any point of an arc has order one or two.

We state a very simple but important Lemma:

Lemma 1 *Let $\mathcal{P}$ be a path with two end points. Then there exists an arc $\mathcal{P}_0$ which is completely contained in $\mathcal{P}$ and has the same end points.*

Proof If $\mathcal{P}$ is a κ–path but not a κ–arc then it contains at least one pair of points P_i and P_k such that $P_i \in \mathcal{N}_\kappa(P_k)$, but $|i-k| > 1$. Assume without loss of generality $k > i$. Then the path

$$\mathcal{P}' := \{P_0, P_1, \cdots, P_{i-1}, P_i, P_k, P_{k+1}, \cdots, P_n\}$$

is contained in $\mathcal{P}$, has end points P_0 and P_n, but has fewer elements as $\mathcal{P}$. Repeating the procedure we eventually arrive at an arc with the desired property. $\square$

A digital set $S \subseteq \mathbb{Z}^2$ is termed a *κ–connected set* whenever for any two points $P, Q \in S$ there exists a path $\mathcal{P}$ with the properties that it is completely contained in S and that it contains both P and Q. Due to Lemma 1 we may in this definition replace 'path' by 'arc with end points P and Q'. A *connected component* of a set $S \subseteq \mathbb{Z}^2$ is a maximal subset of S which is connected. This purely graph theoretic concept of connectedness is very fundamental for analyzing digital sets. By means of connectedness we introduce some sort of rudimentary topology in $\mathbb{Z}^2$. We therefore adopt here the familiar terminology of speaking about 4– *and* 8–*topology* for the digital plane.

2.2 Jordan's Curve Theorem

A very fundamental property — for theory as well as for applications — of the topology of the plane is that Jordan's Curve Theorem is valid. That means that any "simple closed curve" has the property of separating the plane into two parts, namely the interior with respect to the curve and the exterior. These two parts of the plane are distinguished by the fact that the latter is not bounded. For our purposes we only need this theorem for polygonal curves. A *polygonal curve* or simply curve in the plane consists of a finite number of points $\{x_0, x_1, \cdots, x_n\}$ called *vertices* such that each two consecutive vertices x_i, x_{i+1} are joined by a line segment called an *edge.* A polygonal curve is termed a *simple (polygonal) curve* if edges meet only in vertices and if for each vertex there are at most two edges meeting it. It is termed a *closed (polygonal) curve* if $x_0 = x_n$.

Since $\mathbb{Z}^2$ can be considered as a subset of $\mathbb{R}^2$, any path in $\mathbb{Z}^2$ corresponds to a polygonal curve in the plane $\mathbb{R}^2$ with vertices in $\mathbb{Z}^2$ and every arc corresponds to a simple curve in the plane. Similarly, a closed path (arc) corresponds to a closed polygonal curve (simple curve).

Theorem 1 (Jordan's Curve Theorem) *Given a simple closed polygonal curve $\mathcal{C}$ in the plane $\mathbb{R}^2$.*

Then $\mathbb{R}^2 \setminus \mathcal{C}$ consists of exactly two open connected sets (in the sense of the ordinary $\mathbb{R}^2$–topology). Exactly one of these sets is bounded and is called the interior *with respect to $\mathcal{C}$ and the other one is unbounded and is called the* exterior *with respect to $\mathcal{C}$.*

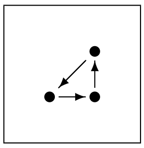
The proof of this Theorem is quite elementary but somewhat lengthy. We refer the reader to the literature (a very simple proof can be found in [3, Chapter I, §7.9]).

The 'digital analog' of this fundamental Theorem can be naively formulated in the following way:

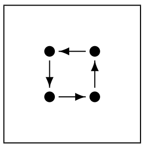
> *Given a closed simple curve $\mathcal{P}$ in the digital plane $\mathbb{Z}^2$.*
> *Then $\mathbb{Z}^2 \setminus \mathcal{P}$ consists of exactly two connected sets. Exactly one of these sets is bounded and is called the* interior *with respect to $\mathcal{P}$ and the other is unbounded and is called the* exterior *with respect to $\mathcal{P}$.*

If we interpret the term 'curve' as 'arc' then the assertion of the Theorem as formulated here is not true. The following two sets are closed arcs in the sense of the definition above but the complement (with respect to $\mathbb{Z}^2$) of each of them consists of only one connected component. Arrows in the figures indicate the order imposed by numbering the points.

8–connectivity:

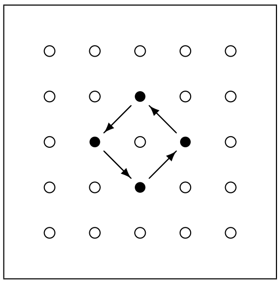


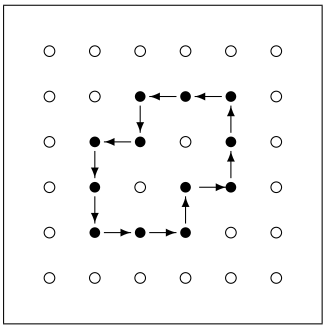
4–connectivity:



In order to rule out such singularities, we explicitly require that a closed digital 8–curve must have at least four points and a closed digital 4–curve must have at least eight points. Since there exist (up to translations, rotations by multiples of 90^o and reflections at diagonal lines of the digital plane) only finitely many arcs having fewer than four or eight points, respectively, the reader can easily verify by inspecting all possible cases that this restriction indeed makes sense. In the following, a *digital curve* is understood to be an arc which fulfills the restriction mentioned above if it is closed.

There is, however, a second problem. Consider the two following digital sets (black points ($\bullet$) are understood to belong to S, white points ($\circ$) to the complement):





The first set is an 8–curve (in the 4–sense it is not a curve). Its complement, however, consists of only one 8–connected component. Similarly the second set is a 4–curve (but not an 8–curve) and its complement consists of three 4–components. These phenomena are sometimes called 'connectivity paradoxa'.

In 1979 Rosenfeld [53] proved that Jordan's curve Theorem is indeed true for digital curves if the curve and its complement are equipped with different 'topologies'. This was observed earlier by Duda, Hart and Munson [17]. Rosenfeld was the first to give these results a sound theoretical background.

We define for $\kappa \in \{4, 8\}$

$$\bar{\kappa} := \begin{cases} 4 & \text{for } \kappa = 8 \\ 8 & \text{for } \kappa = 4. \end{cases} \tag{1}$$

The following Theorem holds:

Theorem 2 (Digital Jordan Theorem) *Given a closed simple κ–curve $\mathcal{P}$ in the digital plane $\mathbb{Z}^2$.*

Then $\mathbb{Z}^2 \setminus \mathcal{P}$ consists of exactly two $\bar{\kappa}$–connected sets. Exactly one of these sets is bounded and is called the interior *with respect to $\mathcal{P}$ and the other is unbounded and is called the* exterior *with respect to $\mathcal{P}$.*

Rosenfeld's proof of this Theorem is performed essentially by an embedding approach. We return to this subject later on.

In image processing applications the sets of black points are considered to carry the essential information of a black–white image. Therefore usually the set of all black points is equipped with the 8–topology since this topology exhibits a more complex connectivity structure. We will follow this custom here and also use the 8–topology. It is easily possible to obtain assertions about the 4–topology by investigating the *negative image* which is obtained by changing the roles of S and its complement. This has to be carried out with some caution since the situation is not quite symmetric because we sometimes assume S to be bounded.

Digital curves play a special role in image processing applications. First it is possible to represent digital curves in a storage efficient way by storing the coordinates of P_0 and then only the differences $P_{i+1} - P_i$ for $i = 0, 1, \cdots, n-1$. These differences can be coded by storing only the number in the neighborhood configuration of P_i which corresponds to P_{i+1}. This code for a digital path is named *chain code.* For coding eight neighbors of a point one needs 3 bits, hence a curve of length n can be stored using $3n$ bits if one neglects the amount of storage necessary to store the first point P_0.

For storing binary images it is sufficient to code only the boundaries of the black digital sets. The justification of representing a digital set by its boundary is given by Jordan's Curve Theorem. A binary image consisting of $n \times n$ points can be stored using $n \times n$ bits. If only the boundaries of the black sets are stored, one needs $3n_B$ bits, where n_B is the number of boundary points. Coding the boundaries pays when the number of boundary points is less than 33 % of the total number of points in the image. In usual text documents the number of boundary points amounts to less than 10 % of the number of all points, and this figure is even smaller in line-structured images such as engineering drawings. With growing resolution of scanners the number of boundary points grows roughly linear with the number of discretization points per unit length, the total number of image points (as well as the number of black points), however, grows as the square of the number of discretization points per unit length. As a consequence, boundary coding of images becomes more and more attractive with growing resolution power of scanning devices.

2.3 The graphs of 4– and 8–topologies

The approach presented in this section to 'topologize' the digital plane is mainly based on graph theoretic ideas. It uses concepts such as points of $\mathbb{Z}^2$ being the *nodes* of the graph, and a neigborhood relation which is represented by the *edges* of the (undirected) graph. This model enabeled us to speak about connectivity of sets in a graph theoretic manner. The graphs representing 4– and 8–connectivity are depicted in Figure 1.

The graph corresponding to the 4–topology is planar, which means that it can be drawn in the plane such that the lines representing the neighborhood relation meet only in vertices. In contrast, it is not possible to represent the connectivity of 8–topology by means of a planar graph. This implies that the notions "planarity" and "topology of the digital plane" are not equivalent.

In order to prove that the 8–topology cannot be modelled by a planar graph we show that the Kuratowski graph K_5 can be drawn into the graph of the digital plane equipped with the 8–topology. According to Kuratowski's theorem, a graph is not planar if and only if it has a subgraph homeomorphic to K_5 or $K_{3,3}$ (see [22, Theorem 11.13]). K_5 is the com-

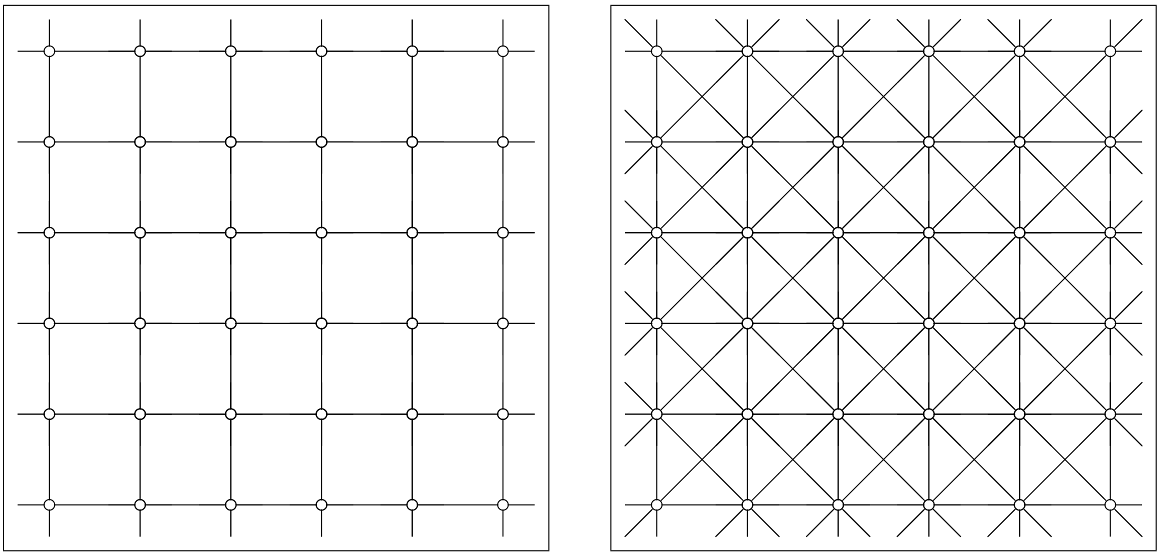


Figure 1: Graphs corresponding to the 4–topology (left) and 8–topology (right). Circles ($\circ$) denote grid points, lines indicate the appropriate neighbor relations.

plete graph having 5 nodes. In Figure 2 it is shown how to embed K_5 homeomorphically into the graph corresponding to the 8–topology.

3 Embedding the Digital Plane

The digital plane $\mathbb{Z}^2$ can be considered as a subset of $\mathbb{R}^2$. This leads immediately to the embedding approach for digital topology. We only sketch here two models for illustration purposes (see also [29, 33]).

3.1 Line Complexes

A *line complex* in the Euclidean plane consists of a finite number of points, termed *vertices*. Certain of the vertices are connected together by line segments, the *edges*. We assume that any two edges meet at most in a vertex and that any vertex belongs to at least one edge. The two points connected by an edge are termed the *end points* of this edge. A vertex is termed *incident* to an edge if it is an end point of the edge. In an analogous way, two edges are termed incident if both are incident to a common vertex. The polygonal curves introduced in section 2 are examples for line complexes. The concept of a line complex can be easily generalized to higher dimensions. On line complexes we may define in a 'naive' way topological concepts such as paths, arcs and connectivity. Moreover, since we only consider line complexes which are embedded into an Euclidean space $\mathbb{R}^d$, we can use the known topology of $\mathbb{R}^d$ for investigating topological properties of the complement of a line complex with respect to $\mathbb{R}^d$. So, a subset of the complement is termed *connected* whenever any two points of this set can be joined by a curve which is completely contained in the set. In the context of line complexes it is sufficient to consider only polygonal curves.

Given a digital set S and $\kappa \in \{4, 8\}$, $\bar{\kappa} \in \{4, 8\} \setminus \{\kappa\}$ (see (1)). For each κ–connected component of S and each $\bar{\kappa}$–connected component of the complement (with respect to $\mathbb{Z}^2$) of S we construct a line complex which will be called the *line complex associated to* S (or the complement of S, respectively). This line complex has the points of S (or of the complement of S, respectively) as vertices. If S is equipped with the 4–topology, any two directly neighboring points

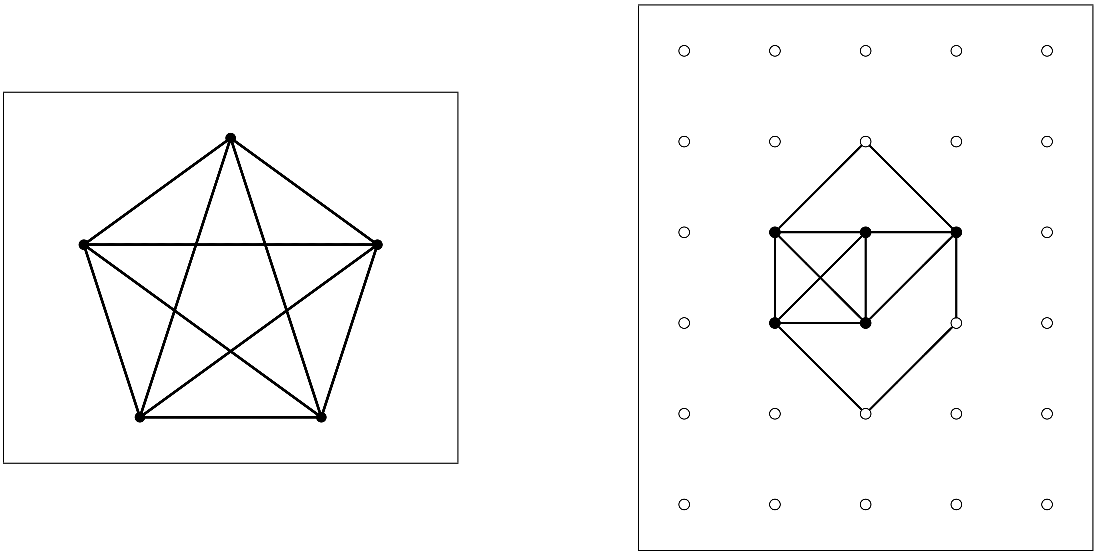


Figure 2: Homeomorphic embedding of the Kuratowski graph K_5 (left) into the digital plane equipped with the 8–topology (right). The nodes of the embedded graph K_5 are marked '•'.

in S will be joined by an edge. If S is equipped with the 8–topology, also indirectly neighboring points in S will be joined by a diagonal edge. In order to make this construction consistent with the definition of a line complex, we introduce *extra vertices* whenever two diagonal edges meet. Analogously we define for the complement of S.

One easily shows:

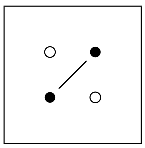
Theorem 3 *The line complexes belonging to S and to the complement of S are disjoint.*

Proof We assume that each vertex of the digital plane has color black if it belongs to S and color white otherwise. It is clear that the sets of vertices of both line complexes are disjoint.

By definition of the associated line complex, when an edge joins two points of the digital plane, these points have the same color. In this case we associate to the edge the color of its end points.

There remain the extra vertices and the edges joining points in $\mathbb{Z}^2$ with an extra vertex. However, an extra vertex is only introduced if it is on a diagonal line joining two points of the same color which are indirect neighbors of each other. Consequently, we can uniquely associate a color to each extra vertex and also to an edge joining a point in $\mathbb{Z}^2$ with an extra vertex. $\square$

Remark 1 *The meaning of the last Theorem is illustrated by the following configuration:*



If the set of black points (•) is equipped with the 8–topology, then the two black points in the picture are connected by an edge. Then, however, it is no longer possible to join both white points (○) also by an edge since both edges would intersect and thus it would be no longer possible to assign them a color in an unambiguous way.

Theorem 4 *1. A digital set is connected if and only if the line complex associated to it is connected.*

2. The line complex associated to a digital set has the same number of connected components as the digital set itself.

3. A digital set is a digital path (a closed digital path, a digital arc, a closed digital arc, respectively) if and only if the line complex associated to it is a polygonal path (a closed polygonal path, a polygonal curve, a closed polygonal curve, respectively).

The proof of this theorem is not difficult and is left to the reader. In performing the proof of Theorem 4 it becomes clear why self–contacts are not allowed for digital curves. When a digital curve touches itself in two points which are not subsequent points, these points are 'short-cut' in the corresponding line complex, thus leading to an associated polygonal set which is not a polygonal curve.

The reason for dealing with associated line complexes is that much is known about the topology of them. Line complexes (and cell complexes) are well–known objects of combinatorial topology. So we can easily reduce the proof of the Digital Jordan's Curve Theorem 2 to that of Jordan's Theorem 1 for polygonal curves. When a closed digital curve is given, then the line complex associated to it is a closed polygonal curve by Theorem 4, part 3. Since we explicitly ruled out arcs containing no points of the digital plane in the interior, the (digital) interior and the exterior with respect to this polygonal curve are not empty. By Theorem 4, part 2 both the interior and the exterior are connected digital sets. As each point in $\mathbb{Z}^2$ either belongs to the digital curve or to its complement, the Digital Jordan's Curve Theorem is proved.

Similarly, assertions concerning Euler's Theorem can be derived for digital sets by using the corresponding theorems for line complexes. Euler's Theorem gives a relation between the number of connected components of a polygonal set and the number of connected components of the complement of it.

Theorem 5 (Euler's Theorem) *For a line complex C in the plane denote by*

v the number of vertices,

e the number of edges,

r the number of connected components (in the natural topology of the plane) of the complement of C. Such components are called regions *defined by C.*

c the number of connected components of C.

Then Euler's relation *holds:*

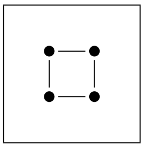
$$v - e = c - r + 1.$$

The number $E(C) := c - r + 1$ is the Euler number *(or* genus*) of the line complex C.*

The proof of this Theorem is quite elementary. Details concerning this proof and also properties of line complexes in the plane and of cell complexes in three dimensions can be found in the book of Alexandrow [3, Chapter I, §7].

Euler's theorem states that Euler's number, which characterizes a global topological property of C, can be determined by counting (locally) numbers of vertices and edges. Such locally calculable properties can be determined by means of a very simple model for parallel computation. We assume that a processor is located at each vertex. Each of the processors has only information about the neighborhood of its vertex. The processors are allowed to send messages of limited information content to a master processor which evaluates the information to find Euler's number. By means of such a model we can easily determine Euler's number for a cell complex. There are theoretical results showing that Euler's number is the only topological predicate of a cell complex which can determined in such a way [48, 43, 30].

Assume that a digital set S is equipped with the 4–topology. Euler's relation is true for the line complex associated to the set with v denoting the number of points in S and e the number of horizontal and vertical edges. In order to determine the number of regions we have to bear in mind that the associated line complex is a subset of the Euclidean plane $\mathbb{R}^2$. The regions determined by the line complex are the open sets which constitute the complement of the line complex with respect to the plane. For sake of clarity we denote the complement of a set S with respect to the Euclidean plane with $\mathbb{C}_E S := \mathbb{R}^2 \setminus S$. If S is a digital set, its complement with respect to the digital plane will be denoted $\mathbb{C}_D S := \mathbb{Z}^2 \setminus S$. There are two different types of such regions. The members of the first type are components of the complement of the line complex which do not contain any grid points belonging to $\mathbb{C}_E S$ (note that grid points in $\mathbb{C}_E S$ are points in $\mathbb{C}_D S$). These components are exactly the squares encircled by trivial closed 4–curves of the form



We denote the number of these squares by s.

The second type of connected components of the complement consists of components containing at least one grid point in $\mathbb{C}_E S$. The set of points of $\mathbb{C}_D S$ contained in such a component is 8–connected. Assume that two points in $\mathbb{C}_E S$ can be joined by a (polygonal) curve which does not meet the line complex associated to S. If this curve meets a horizontal or vertical line joining two directly neighboring grid points, then at least one of these points belongs to $\mathbb{C}_D S$. From this observation we conclude that the number of regions which contain at least one grid point is equal to the number of connected components of the line complex associated to $\mathbb{C}_D S$ and is also equal to the number of 8–components of $\mathbb{C}_D S$. We denote this number by r_8.

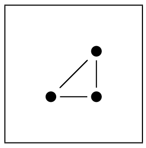
Euler's Formula (Theorem 5) now yields

$$E_{4/8}(S) := c_4 - r_8 + 1 = v - e + s.$$

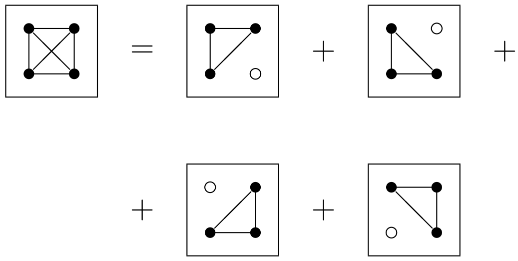
where c_4 is the number of 4–components of S. The Euler number $c_4 - r_8 + 1$ is the number of 4–components of S minus the number of *holes* in S. A hole is a bounded connected component of the complement of S (with respect to $\mathbb{Z}^2$).

The procedure is analogous in the case of the 8–topology: The number of vertices v is the number of points in S. The number of edges is the sum of the number e of horizontal and vertical edges in S and the sum d of all diagonal edges in S.

In order to calculate the number of regions defined by the line complex associated to S, we again distinguish two types of regions. The first type consists of regions containing no points from the complement of S. It consists of triangles



(and all triangles obtained by 90^o–rotations therefrom) and squares as above. Let t be the number of triangles and s be the number of squares. For the latter we observe that each square contains four triangles:



In the center of the square there is a crossing point of two lines which yields an extra vertex. Hence there are s new vertices generated, $2s$ new edges (each diagonal edge was already counted to yield the number d and is now split into two half–edges) and $4s$ new regions out of s squares. For each square, the four triangles contained in it are already counted in the number of regions. Analogously as above, we add the number r_4 of the 4–components of the complement $\mathbb{C}_E S$ corresponding to regions of second type.

We get

Total number of vertices:	$v + s$
Total number of edges:	$e + d + 2s$
Total number of regions:	$c_4 + t$.

Euler's formula yields:

$$E_{8/4}(S) := c_8 - r_4 + 1 = v - e - d + t - s.$$

From the formulae for $E_{4/8}$ and $E_{8/4}$ we conclude that Euler's number of a digital set can be calculated by (parallel) inspection of all 2×2 neighborhoods if different topologies are used for the digital set and for its complement. The latter requirement is essential, otherwise this assertion is not true [30]. It is an interesting fact that it can be made true also in this case if a simple additional condition (*well–composedness*) is imposed on the sets under consideration [38].

Line complexes may also be used for classifying the points of a digital set S into interior and boundary points. An edge of the line complex associated to S separates two regions. If both regions are of the first type as defined above (i.e. they contain no points of the complement $\mathbb{C}_D S$ of S), the edge is called an *interior edge*. Otherwise it is adjacent to at least one region of second type. In this case it is termed a *boundary edge*. A point which is incident to a boundary edge is termed a *boundary point* and all boundary points and boundary edges constitute the *boundary* of S. A point in S which is not a boundary point is termed an *interior point* of S. Using these concepts, we can define boundaries of digital sets as oriented curve–like sets.

It is possible to generalize the approach presented here to three and more dimensions [34, 41, 42].

3.2 Cellular Topology

Sometimes a different model for the digital plane is used, the so–called *cellular model*. This model was introduced by Alexandroff and Hopf [2, Erster Teil, erstes Kapitel, §1.1, Beispiel 4^o] (see Figure 3).

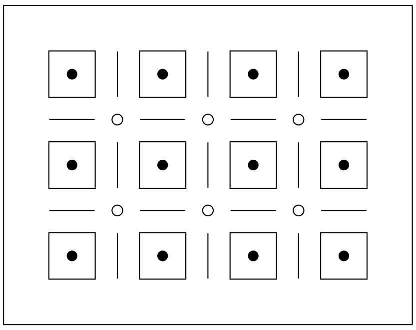


Figure 3: The cellular model. To each point ($\bullet$) in the digital plane one square, four edges and four vertices are associated.

Given a digital set $S \subseteq \mathbb{Z}^2$, we associate to it a set of certain objects in the Euclidean plane $\mathbb{R}^2$. The first type of objects are *squares* which are considered to be open in the $\mathbb{R}^2$–topology and are centered around a grid point. If $P = (m,n)^\top$ is a grid point in $\mathbb{Z}^2$, then the square corresponding to P is

$$\begin{aligned}\pi(P) \quad = \quad & \pi_{mn} := \Big\{ (\xi,\eta)^\top \in \mathbb{R}^2 \;\Big| \\ & m - \frac{1}{2} < \xi < m + \frac{1}{2}, \quad \text{and} \\ & n - \frac{1}{2} < \eta < n + \frac{1}{2} \Big\}.\end{aligned}$$

Furthermore we have *edges* which are sides of squares (without the end points) and *vertices* which are the vertices of squares. The topological closure $\bar{\pi}_{mn}$ of square π_{mn} consists of the square π_{mn} together with its vertices and edges. It is sometimes called the *pixel* belonging to $P = (m,n)^\top$.

Of course, each (open) square whose center point is in S should belong to the model of a given digital set S. In order to get in the model connectedness, we are forced to associate also edges and vertices to the model set of a digital set. We consider two different such models:

The *closed model* $\mathcal{C}(S)$ is the union of all pixels (= closed squares) whose center point is in S.

In the *open model* $\mathcal{O}(S)$, all (open) squares with center point in S belong to $\mathcal{O}(S)$. Moreover, an edge belongs to $\mathcal{O}(S)$ whenever it is boundary edge of two adjacent squares belonging to $\mathcal{O}(S)$ and a vertex belongs to $\mathcal{O}(S)$ if all edges incident to it belong to $\mathcal{O}(S)$.

We can then state the following Theorem:

Theorem 6

1.

$$\mathcal{O}(S) = \mathbb{C}_E \mathcal{C}(\mathbb{C}_D S),$$
$$\mathcal{C}(S) = \mathbb{C}_E \mathcal{O}(\mathbb{C}_D S).$$

2. $\mathcal{O}(S)$ is open and $\mathcal{C}(S)$ is closed in the $\mathbb{R}^2$–topology.

3. The number of connected components of $\mathcal{O}(S)$ (in the $\mathbb{R}^2$–topology) is equal to the number of 4–components of the digital set S.

The number of connected components of $\mathcal{C}(S)$ is equal to the number of 8–components of S.

Proof 1. Assume that there is an $x \in \mathbb{R}^2$ which belongs to both $\mathcal{O}(S)$ and $\mathcal{C}(\mathbb{C}_D S)$. Then there exists a $Q \in \mathbb{C}_D S$ such that $x \in \bar{\pi}(Q)$. Since $\pi(P) \cap \bar{\pi}(Q) = \emptyset$ for $P \neq Q$, x does not belong to any $\pi(P)$ with $P \in S$. Similarly, it is clear that x cannot be on an edge of $\mathcal{O}(S)$ since in this case both squares adjacent to this edge must belong to points in S which contradicts $x \in \bar{\pi}(Q)$. An analogous argument shows that x is not a vertex. Therefore we conclude that $\mathcal{C}(S) \cap \mathcal{O}(\mathbb{C}_D S) = \emptyset$.

Assume now that there is an $x \in \mathbb{R}^2$ which belongs neither to $\mathcal{O}(S)$ nor to $\mathcal{C}(S)$. Then x is not contained in any $\pi(P)$ since every $P \in \mathbb{Z}^2$ either belongs to S or to $\mathbb{C}_D S$. Similarly, x cannot belong to an edge for either both adjacent squares belong to points in S then the edge belongs to $\mathcal{O}(S)$ or else one of these squares belongs to a point in $\mathbb{C}_D S$ which implies $x \in \mathcal{C}(\mathbb{C}_D S)$. A similar argument holds for vertices which proves the first assertion. The second assertion is proved in an analogous manner.

2. Given a sequence $\{x_n\}$ of points in $\mathcal{C}(S)$ converging to a point x^*. Then we can find a finite number of squares $\bar{\pi}(P) \subseteq \mathcal{C}(S)$ containing all x_n. The union of these squares is closed, hence $x^* \in \mathcal{C}(S)$.

The assertion concerning $\mathcal{O}(S)$ follows by application of part 1.

3. If P and Q are 8–neighbors in $\mathbb{Z}^2$ then the line segment joining P and Q is completely contained in $\mathcal{C}(\{P, Q\})$ which yields the first assertion. An anlogous argument leads to the second assertion. $\square$

One might ask why we need different continuous models for the digital plane. The reason is that it is not possible to find a model which exhibits all topological properties needed. This is due to the obvious fact that the digital plane has a structure which is significantly different from that of the real plane. For example, in $\mathbb{Z}^2$ every bounded set is finite which is not true for the real plane. Therefore it is necessary to 'tailor' a continuous model for each specific investigation of the digital plane.

4 Axiomatic Digital Topology

4.1 Definition and Simple Properties

A *topological space* consists of a set X of points such that certain subsets of X which are called *open sets* fulfill the properties (see [25])

TO1 X and $\emptyset$ are open,

TO2 The union of any family of open sets is open,

TO3 The intersection of any finite family of open sets is open.

The system of all open sets is termed the *topology* of X and is denoted $\mathcal{T}$.

An open set which contains a point of X is called a *neighborhood* of this point.

There are two trivial topologies for a set X which play a role in the sequel. In the *discrete* topology $\mathcal{T}_d$ (in the strict sense) all subsets of X are declared to be open and in the *indiscrete* topology $\mathcal{T}_i$ the only open sets are the empty set and the whole space X.

A set $S \subseteq X$ is called a *closed* set if its complement is open.

The following classification of topological spaces according to their separation properties is due to Kolmogoroff (see [2] or [25]):

$\boldsymbol{T_0}$ For any two different points in X at least one has a neighborhood not containing the other.

$\boldsymbol{T_1}$ For any two different points P and Q in X there exists a neighborhood of P not containing Q and a neigborhood of Q not containing P.

$\boldsymbol{T_2}$ For any two different points P and Q in X there exists a neighborhood of P and a neigborhood of Q which are disjoint.

T_2–spaces are termed *Hausdorff spaces.*

A topological space is termed *discrete* if the following is true:

TO3' The intersection of any family of open sets is open.

This notion is due to Alexandroff [1]. For a T_1– or T_2–space the discrete topology as defined here coincides with the discrete topology in the strict sense. For T_0–spaces we must distinguish between the discrete topology in the strict sense and the discrete topology in the general sense. A discrete T_0–space is therefore termed an *Alexandroff space*. Axiom TO3' can also be stated using closed sets: The union of any family of closed sets is closed. In this formulation it becomes clear that Alexandroff spaces are completely symmetric with respect to open and closed sets.

In an Alexandroff space there exists for each point P a *smallest neighborhood* of P which is open:

$$OP = \bigcap_{\substack{P \in U \\ U \text{open}}} U.$$

Because of the symmetry of Alexandroff spaces there also exists a *smallest closed set* containing P:

$$CP = \bigcap_{\substack{P \in V \\ V \text{closed}}} V.$$

A point P with $CP = \{P\}$ is called a *vertex point*.

Theorem 7

1. For any two different elements P and Q of an Alexandroff space is

$$\begin{aligned} P \in OQ &\implies Q \notin OP \quad \textit{and} \\ P \in CQ &\implies Q \notin CP. \end{aligned}$$

2. $P \in CQ \iff Q \in OP$.
3. $CP \subseteq CQ \iff OQ \subseteq OP$.

Proof 1. The first assertion is an immediate consequence of the T_0–property. For the second assertion assume without loss of generality that $Q \in OP$. Then, by virtue of the the first assertion, $P \in \mathbb{C}OQ$. Hence $\mathbb{C}OQ$ is a closed set which contains P but not Q, consequently $Q \notin CP$. This means that not both $Q \in CP$ and $P \in CQ$ are possible.

2. $Q \notin OP$ means $Q \in \mathbb{C}OP$. However, $\mathbb{C}OP$ is closed and does not contain P, hence $P \notin CQ$.

Similarly, $P \notin CQ$ implies $Q \notin OP$.

3. $\mathbb{C}OP$ is closed. $Q \notin OP$ means $Q \in \mathbb{C}OP$, hence $P \in CP \supseteq CQ \subseteq \mathbb{C}OP$ which is not possible. Consequently, $Q \in OP$. If $P \neq Q$ then $Q \notin CP$ by 1. $\mathbb{C}CP$ is open and contains Q, but not P. Therefore

$$OQ \subseteq OP \cap \mathbb{C}CP \subseteq OP.$$

□

A topological space X is termed *locally finite* if each point P in X has a finite neighborhood and a finite closed set containing P.

Theorem 8 *Let X be a locally finite Alexandroff space. Then*

1. Each set CP contains at least one vertex point.

2. If $CP \neq CQ$, then there is a vertex point in one of these sets which is not contained in the other set.

Proof 1. If $CP = \{P\}$ then P is a vertex point. Otherwise there is a $Q \in CP$ with $Q \neq P$. Then $P \notin CQ \subseteq CP$ and CQ has fewer points than CP. Repeating the process with CQ we eventually arrive at a vertex point contained in CP.

2. If both sets are disjoint then the assertion follows from part 1. Otherwise $CP \cap CQ \neq \emptyset$ and we can apply the construction of part 1 to find a vertex point having the desired property. □

The Theorem states that the mapping which assigns to an element P in a locally finite Alexandroff space the set of all vertex points in CP is injective. If this mapping is bijective, the space is termed a *complete space*.

Example 1 *As an example we consider a space $X_\Box$ consisting of nine points*

$$\alpha, \beta, \gamma, \delta; a, b, c, d; Q.$$

The sets OP and CP are given in the Table in Figure 4. In the last column of this table the sets of vertex points associated to each point are given. The set of all vertex points is $\Sigma := \{\alpha, \beta, \gamma, \delta\}$. $X_\Box$ can be interpreted as a square (see Figure 4). The vertex points of $X_\Box$ are the vertices of the square, a, b, c and d correspond to the edges (without end points) and Q is the interior of the square.

This topological space is not complete since there are no points in it belonging to tree–element vertex *sets. Also the sets $\{\alpha, \gamma\}$ and $\{\beta, \delta\}$ do not belong to points in $X_\square$. We can add in an obvious way elements belonging to these sets of vertex points. There are two new edges e and f and four triangles F_1, F_2, F_3, F_4 as indicated in Figure 5. The completed topological space $\bar{X}_\square$ can be visualized as a simplex in 3–space by raising one vertex of the original square above the plane defined by the others. This is illustrated in Figure 6.*

Locally finite Alexandroff spaces which are not T_1 suffer from a very serious defect which severely limits their usage in image processing. We define in tue usual way: A function f mapping an topological space X into another topological space Y is termed *continuous* if for all sets $U \subseteq Y$ which are open in Y the inverse image

$$f^{-1}(U) := \{x \in X \mid f(x) \in U\}$$

is open in X.

The following Theorem states that the set of all continuous functions mapping a nontrivial locally finite Alexandroff space into itself is subject to severe restrictions [40].

Theorem 9 *Let X be a locally finite Alexandroff space which is not T_1.*

Then there exist two points $P \neq Q$ in X such that for every two neighborhoods $U(P)$ and $U(Q)$ of these points any injection $f : U(P) \longrightarrow U(Q)$ mapping P in Q is not continuous.

Proof X is not T_1, hence there exist points P and Q such that $Q \in OP$ and $P \notin OQ$. Thus $OQ \subset OP$ (strictly), and therefore $|OQ| < |OP|$.

OP is clearly an open subset of $U(P)$ and OQ is an open subset of $U(Q)$. Let $f : U(P) \longrightarrow U(Q)$ be any continuous injection. Since $P \in f^{-1}(OQ)$ we obtain that $OP \subseteq f^{-1}(OQ)$. Thus $|OP| \leq |f^{-1}(OQ)| = |OQ|$, and this contradicts the fact that $|OQ| < |OP|$. $\square$

A topological space is termed *homogeneous*, when any two points of it have homeomorphic neighborhoods which means that there exist a bijective function which maps one neighborhood onto the other and together with its is inverse is continuous. Then the assertion of the Theorem can be formulated as follows: Any Alexandroff space with nontrivial topology is not homogeneous.

The assertion of the Theorem is of course not completely surprising. For example, a function which maps vertices into edges in Example 1 should not be continuous.

4.2 Connectedness

Let S be a subset of X. In the *relative topology* induced in S by the topology in X the open sets are all sets $U \cap S$ with U open in X. One easily sees that this is indeed a topology in S. A set which is open in the relative topology of S is a *relatively open set.*

The set $S \subseteq X$ is termed *connected* if there is no decomposition $S = T_1 \cup T_2$ such that $T_1 \cap T_2 = \emptyset$, both $T_1, T_2 \neq \emptyset$ and relatively open with respect to S. Obviously, with respect to the indiscrete topology all sets are connected and for the strictly discrete topology only sets consisting of one point are connected. Therefore these topologies are not very interesting for investigating connectedness.

We now state a fundamental Lemma which relates the notion of connectedness defined here to the more constructive and intuitive notion of path–connectedness.

Lemma 2 *Let $X = \{P_0, P_1, \cdots, P_n\}$ be a finite set.*

There exist exactly two Alexandroff topologies on X with the property that exactly all segments of consecutive points $P_i, P_{i+1}, \cdots, P_j$ with $0 \leq i < j \leq n$ are connected sets.

These topologies are

$$\begin{aligned} \mathcal{T}_1 : \quad OP_i &= \{P_i\} \quad \textit{if } i \textit{ even,} \\ OP_i &= \{P_{i-1}, P_i, P_{i+1}\} \quad \textit{if } i \neq 1, n \textit{ odd,} \\ OP_1 &= \{P_1, P_2\}, \\ OP_n &= \{P_{n-1}, P_n\}, \quad \textit{if } n \textit{ odd.} \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}_2 : \quad OP_i &= \{P_i\} \quad \textit{if } i \textit{ odd,} \\ OP_i &= \{P_{i-1}, P_i, P_{i+1}\} \quad \textit{if } i \textit{ even and} \\ & \qquad i \neq n, n \textit{ even,} \\ OP_n &= \{P_{n-1}, P_n\}, \quad \textit{if } n \textit{ even.} \end{aligned}$$

P	OP	CP	S
α	$\{\alpha, d, a, Q\}$	$\{\alpha\}$	$\{\alpha\}$
β	$\{\beta, a, b, Q\}$	$\{\beta\}$	$\{\beta\}$
γ	$\{\gamma, b, c, Q\}$	$\{\gamma\}$	$\{\gamma\}$
δ	$\{\delta, c, d, Q\}$	$\{\delta\}$	$\{\delta\}$
a	$\{a, Q\}$	$\{a, \alpha, \beta\}$	$\{\alpha, \beta\}$
b	$\{b, Q\}$	$\{b, \beta, \gamma\}$	$\{\beta, \gamma\}$
c	$\{c, Q\}$	$\{c, \gamma, \delta\}$	$\{\gamma, \delta\}$
d	$\{d, Q\}$	$\{d, \delta, \alpha\}$	$\{\delta, \alpha\}$
Q	$\{Q\}$	X	$\{\alpha, \beta, \gamma, \delta\}$

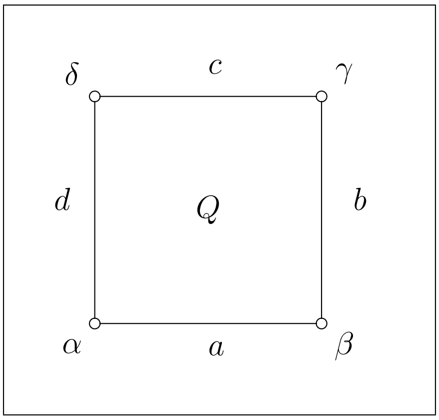


Figure 4: The topological space $X_\square$ corresponding to a square.

Proof The proof consists of three steps:
1. First we prove that OP_i does not contain any point P_k with $k \neq i-1, i, i+1$.

The set $\{P_i, P_k\}$ ist not connected by definition since i and k are not consecutive integers. Hence there exists a neighborhood of P_i not containing P_k and vice versa. This implies $P_k \notin OP_i$.

2. $|OP_i| = 2$ implies $i = 1$ or $i = n$.

Assume $OP_i = \{P_i, P_{i+1}\}$ $(i \neq n)$. If $P_i \notin OP_{i-1}$, then $\{P_{i-1}, P_i\}$ is not connected, contrary to assumption. Consequently $P_i \in OP_{i-1}$.

From part 1. we get $P_{i+1} \notin OP_{i-1}$. This implies $OP_{i-1} \cap OP_i = \{P_i\}$, hence $OP_i = \{P_i\}$ contains only one point, a contradiction.

The other possible cases are treated analogously.

3. There remain the following possibilities

$$\begin{aligned} OP_i &= \{P_i\} \quad \text{or} \\ OP_i &= \{P_{i-1}, P_i, P_{i+1}\} \end{aligned}$$

When $|OP_i| = 1$ and $|OP_{i+1}| = 1$ then the set $\{P_i, P_{i+1}\}$ is not connected. When $OP_i = \{P_{i-1}, P_i, P_{i+1}\}$ and $OP_{i+1} = \{P_i, P_{i+1}, P_{i+2}\}$ for any i with $0 < i < n$ then $OP_i \cap OP_{i+1} = \{P_i, P_{i+1}\}$ is open which contradicts part 2.

As a result, one–point neighborhoods and three–point neighborhoods alternate along X which leaves only the both possibilities of the Lemma. $\square$

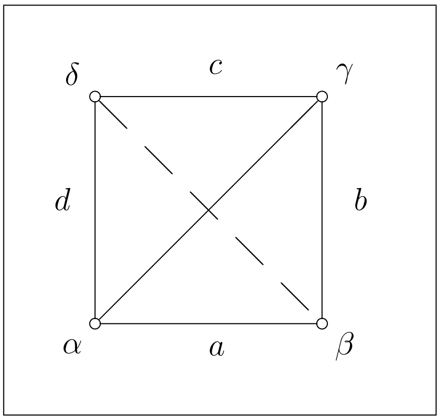
As an immediate consequence we state

Corollary 1 *If $P_0 = P_n$ then there exists an Alexandroff topology for X such that exactly all segments of consecutive points on the cyclic sequence of points (numbered modulo n) are connected if and only if n is odd.*

Example 2 *Let X be the set $\mathbb{Z}$ of all integers. We consider the topologies generated by means of intersections and unions from the systems of open sets as defined below:*

1. Open sets are all semi–infinite intervals $\{n \in \mathbb{Z} \mid n \geq n_0\}$. The topology generated by this system is an Alexandroff topology which is not locally finite.

2. The Marcus–Wyse topology *is generated by the*



New vertices of the completed space:

e has vertex points β and δ,
f has vertex points α and γ.

Faces of completed space:

F_1 has vertex points α, β, γ,
F_2 has vertex points α, γ, δ,
F_3 has vertex points β, γ, δ,
F_4 has vertex points α, β, δ.

Figure 5: Completion of $X_\Box$.

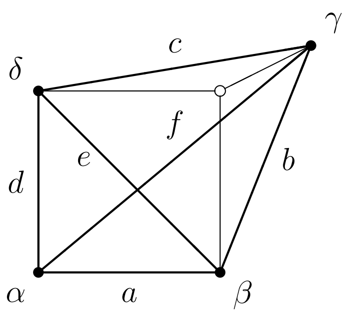


Figure 6: The completed space $\bar{X}_\Box$.

smallest open subsets of points $n \in \mathbb{Z}$

$$O\{n\} = \begin{cases} \{n\}, & \text{if } n \text{ is even,} \\ \{n-1, n, n+1\}, & \text{otherwise.} \end{cases}$$

The Marcus–Wyse topology is a locally finite Alexandroff topology [45]. As a consequence of Lemma 2 we can state that the Marcus–Wyse topology is the only locally finite Alexandroff topology for $\mathbb{Z}$ having the property that exactly the subsets consisting of consecutive numbers are connected.

We note that the Marcus–Wyse topology of the integers is not translation invariant. The translation $T : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $T(i) = i + 1$ maps even numbers into odd numbers. Consequently, this translation is not continuous (see Theorem 9).

For the digital plane one can easily prove the following Theorem.

Theorem 10 *The 2–dimensional Marcus–Wyse topology given by the smallest open set of a point $P = (m, n)^\top$*

$$OP = \begin{cases} \mathcal{N}_4(P) \cup \{P\} & \text{if } m+n \text{ is even,} \\ \{P\} & \text{otherwise.} \end{cases}$$

is (up to a translation) the only locally finite Alexandroff topology for $\mathbb{Z}^2$ with the property that a subset of $\mathbb{Z}^2$ is connected with respect to this topology if and only if it is 4–connected.

An analogous assertion holds for $\mathbb{R}^d$.

From the Corollary we get immediately the assertion that it is not possible to find a topology for $\mathbb{Z}^2$ which induces 8–connectivity. In Figure 7 a closed path with an odd number of vertices is shown which proves the assertion [15, 39, 49].

For an Alexandroff space we define: A *path* with end points P and Q is a set $\{P = P_0, P_1, \cdots, P_n =$

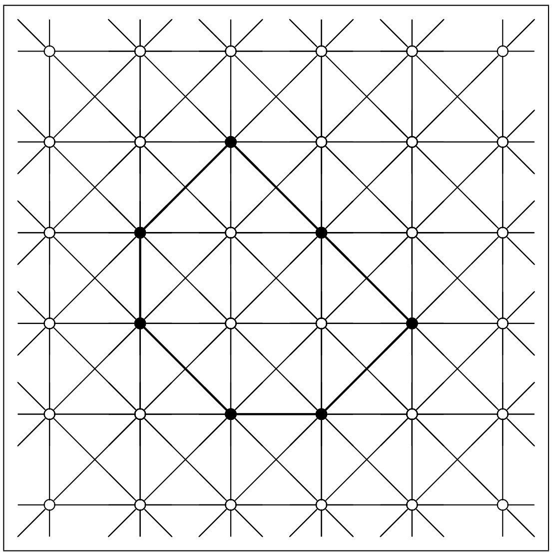


Figure 7: A closed path having an odd number of vertices can be found in the graph corresponding to 8–topology.

$Q\}$ with the property $P_i \in OP_{i+1}$ or $P_{i+1} \in OP_i$ for all $i = 0, 1, \cdots, n-1$. Obviously, by Lemma 2, a path is a connected set. This is easily seen by inspecting both possible topologies on a path. If a path were not connected, it would contain two consecutive points which belong to different (relative) neighborhoods on the path. This, however is not possible in either topology of the Lemma.

A subset S of an Alexandroff space is termed *path–connected* if for any two points $P, Q \in S$ there exists a path with end points P and Q which is completely contained in S.

Theorem 11 *A subset S of an Alexandroff space is connected if and only if it is path–connected.*

Proof 1. Assume that S is not connected. Let T_1, T_2 be the two relatively open sets in S according to the definition of connectedness. For a path from a point P in T_1 to a point $Q \in T_2$ there exists a number i such that $P_i \in T_1$ and $P_{i+1} \in T_2$. However, if $P_i \in OP_{i+1}$ then $P_{i+1} \in T_1$ since T_1 was assumed to be open which means that $OP \subseteq T_1$ for all $P \in T_1$. The same argument holds for $P_{i+1} \in OP_i$.

2. If S is connected, we define for P in S the set

$$S_P = \{Q \in S \mid \text{ there is a path from } P \text{ to } Q \text{ in } S\}.$$

S_P is an open set since it is the union of all smallest neighborhoods of points which are path–connected to P. Let $Q \in S \setminus S_P$. Then $OQ \cap S$ is a relatively open set containing Q and $OQ \cap S_P = \emptyset$, otherwise $Q \in S_P$. This implies $S \setminus S_P$ is open, contradicting the connectedness of S. $\square$

4.3 Alexandroff Topologies for the Digital Plane

It now is easily possible to define an Alexandroff topology for the digital plane. We assign a vertex to each point in the digital plane. If we interpret each smallest square of points in $\mathbb{Z}^2$ as the topological space $X_\square$ as in Example 1, we get the 4–topology in the sense that the vertices of $X_\square$ correspond to points in $\mathbb{Z}^2$ and edges symbolize connectedness of points.

In a similar way we can introduce an Alexandroff topology which is related to 8–topology into the digital plane. For this reason we interpret each smallest

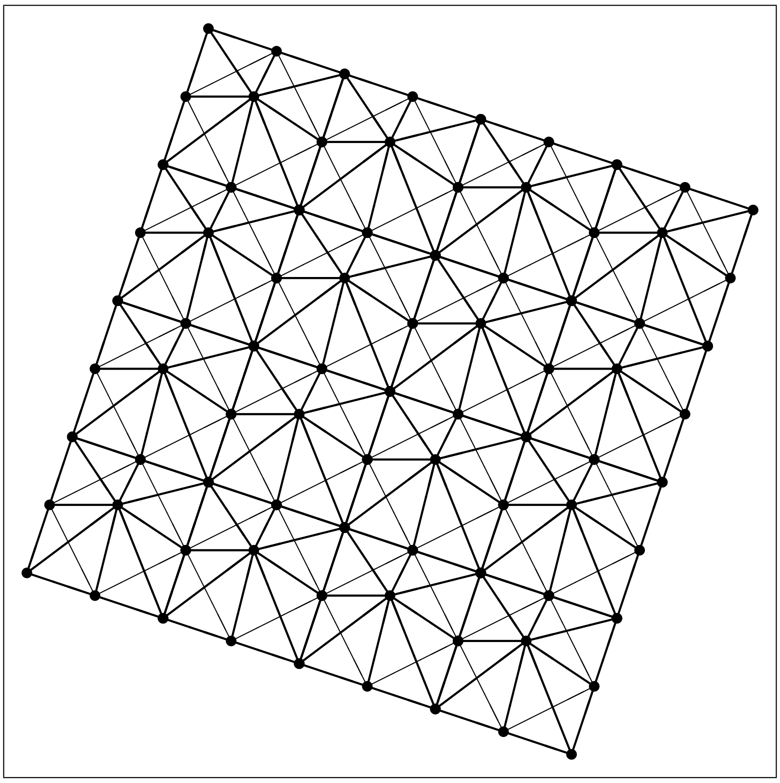
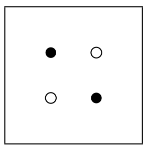


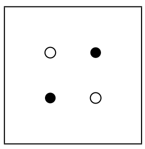
Figure 8: Modeling 8–topology by the completed space $\bar{X}_\square$. Certain points are raised above the plane to get a model without extra vertices.

square as the completed space of Example 1. We can put together the simplices of the completed space in a three–dimenisonal model as shown in Figure 8.

As an alternative we may associate to each point in the digital plane a square as in the cellular model (see Figure 3). For each point in a digital set S (or equivalently, for each square) we must decide for four vertices and four edges whether they should belong to S or not. This decision has to be done in such a way that the subset of the plane thus obtained has appropriate connectedness. In order to eliminate redundancy of these models, different authors proposed reduced models. We consider two examples.

Example 3 *Kovalevsky [36] proposed that for any square belonging to a digital set S, the left and the upper edge and the upper left vertex of the square should belong to S. The connectedness structure induces into the digital plane by this approach corresponds to the 6–neighborhood which is equivalent to a covering of the plane by hexagons (see Figure 9). Therefore, in in the following picture, the two points $\bullet$ in the left configuration are connected, in the right configuration they are not connected.*





Example 4 *In the cellular model for digital topology (see Section 3.2) we investigated two continuous models for a digital set S, the closed model $\mathcal{C}(S)$ and*

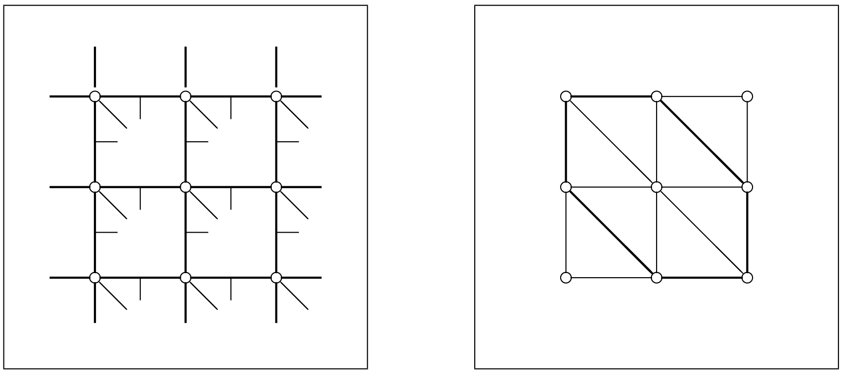


Figure 9: Kovalevsky's approach for topologizing the digital plane. If the squares are represented by points and the connectedness relation by vertices, the graph of the right picture is obtained.

the open model $\mathcal{O}(S)$. In both models it is only necessary to know which squares belong to the model of S. The edges and vertices are associated to the image of S in an consistent way.

The advantages of this convention are:

- *It can be generalized to higher dimensions,*
- *It can be generalized to sets of other shape covering the plane (hexagons, irregular sets),*
- *If we add the requirement that all objects which are not associated to S by our convention should belong to the complement of S, we automatically arrive at mixed 4/8– or 8/4–connectedness for the digital plane (Theorem 6).*

5 Semi–Topology

5.1 Motivation

From the preceeding paragraphs we can conclude that Alexandroff spaces exhibit certain defects as stated in Theorem 9. In order to resolve this problem, it is obviously necessary to modify the third axiom TO3. Alexandroff sharpened this axiom and obtained a symmetric system of axioms.

In the course of our exposition we could observe that virtually all we need from Alexandroff's topology is the knowledge of the smallest neighborhoods OP of points. So we reformulate the axiom TO3 in the following way: A set X is termed a *semi–topological space* (more precisely, a semi–topological space of character 1, see [40]) if a system of subsets, the open sets om X, fulfill axioms TO1 and TO2 from section 4.1 and in addition

TO3" For each point $P \in X$ there exists a smallest neighborhood OP such that $P \in OP$. All open sets can be represented as unions of smallest neighborhoods.

This concept was introduced in [40]. The treatment here is simplified for easier presentation. From the definition a semi–topological space it is clear that any Alexandroff space is also a semi–topological space. In order to have the property that any topological space is also semi–topological, TO3" must be modified (see [40] for details). A semi–topological space is completely determined by the neighborhoods OP of all its points P.

Example 5 *It is possible to introduce in the digital plane a semi–topology by defining for a point P the smallest neighborhood OP as the set of all points reachable by 'knight's moves' [16] which is depicted in Figure 10.*

For semi–topological spaces all assertions hold which are true in Alexandroff–spaces if they are derived without using intersections.

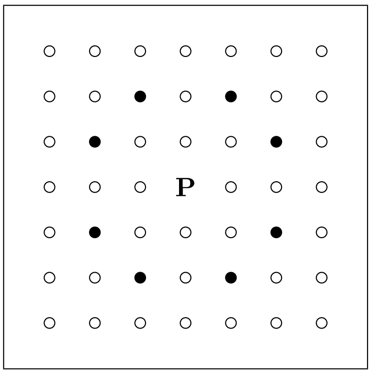


Figure 10: Smallest neighborhood of point P in the semi–topology induced by knight–moves in the digital plane.

It is for example easy to define continuity in full analogy to continuity in topological spaces:

A function $\varphi : X \longrightarrow Y$ mapping a semi–topological space X into a semi–topological space Y is *continuous*, if for each set $S \subseteq Y$ which is open in Y the set $\varphi^{-1}(S)$ is open in X. φ is termed a *homeomorphism* from X onto Y if the inverse function $\varphi^{-1} : Y \longrightarrow X$ exists and if φ and φ^{-1} are both continuous functions. X and Y are termed *homeomorphic spaces* in this case.

The following Theorem is trivial but of fundamental importance since it illustrates a typical difficulty encontered in digital topology: Homeomorphic topological spaces must have the same number of elements.

Theorem 12 *Let X be a semi–topological space consisting of finitely many elements.*

Let Y be a semi–topological space and $\varphi : X \longrightarrow Y$ a homeomorphism. Then Y has the same number of elements as X.

For the proof of this Theorem we need only the bijectivity of φ.

In other words: The number of elements in a finite set is a (semi–) topological invariant.

Analogously as for Alexandroff spaces a semi–topological space X is termed *locally finite* if OP has a finite number of elements for all $P \in X$.

Let Y be a subset of a semi–topological space X. The *relative semi–topology* in Y which is induced by the semi–topology in X is generated by the set of all neighborhoods $OP \cap Y$.

Example 6 *Consider again the set $\mathbb{Z}$. We define the neighborhood $O(n) = \{n-1, n, n+1\}$ for every $n \in \mathbb{Z}$. This yields a semi–topology in $\mathbb{Z}$ which we will call the* standard semi–topology for $\mathbb{Z}$.

5.2 The Associated Topological Space

There are some specialties with semi–topological spaces. Let S be an open subset of an Alexandroff space. Then for each $P \in S$, the set OP is contained in S. This is not necessarily true in semi–topological spaces. Therefore we define: A subset S of a semi–topological space is *strictly open* if $OP \subseteq S$ for all $P \in S$. Obviously, a strictly open set is also an open set.

Consider for example the Marcus–Wyse topology in $\mathbb{Z}$ (see Example 2). In this topology each open set is strictly open.

Let X be a semi–topological space with topology $\mathcal{T}$. We introduce in X a second topology $\mathcal{T}_A$ in which the strictly open sets are declared to be open. It is easily seen that $\mathcal{T}_A$ is indeed a topology for X.

By definition, each $\mathcal{T}_A$–open set is also $\mathcal{T}$–open which is expressed in the language of general topology by saying that $\mathcal{T}_A$ is *coarser* than $\mathcal{T}$ or $\mathcal{T}$ is *finer* than $\mathcal{T}_A$. Specifically, for the smallest open neighborhoods of a point, OP with respect to $\mathcal{T}$ and $O_A P$ with respect to $\mathcal{T}_A$, one always has $OP \subseteq O_A P$.

Theorem 13

1. $\mathcal{T}_A$ is either an Alexandroff topology or it is discrete in the strict sense or it is indiscrete.

2. Let $\mathcal{T}'$ be a topology fulfilling TO3' which is coarser than $\mathcal{T}$. Then $\mathcal{T}'$ is coarser than $\mathcal{T}_A$.

Proof 1. Given a system $\{S_\sigma\}$ of $\mathcal{T}_A$–open sets and $P \in S := \bigcap S_\sigma$. Then OP is contained in each S_σ, hence in S. This proves TO3' for $\mathcal{T}_A$.

2. For $P \in X$ let $O'P$ be the smallest $\mathcal{T}'$–neighborhood of P. Given a $\mathcal{T}'$–open set S, then $O'P \subseteq S$ for all $P \in S$, for otherwise $O'P \cap S$ would

be a neighborhood of P which is strictly contained in OP. This implies that S is $\mathcal{T}_A$–open. □

We may restate part 2 of the Theorem in saying that $\mathcal{T}_A$ is the finest topology with TO3' which is coarser than the semi–topology $\mathcal{T}$. $\mathcal{T}_A$ is termed the topology *associated* to the semi–topology $\mathcal{T}$. According to the associated topology we may classify semi–topological spaces into three classes:

AT1 $\mathcal{T}_A$ is discrete in the strict sense. Then the same holds for $\mathcal{T}$.

AT2 $\mathcal{T}_A$ is an Alexandroff topology. Then $\mathcal{T}$ is also T_0 but not T_1 since

$$\begin{aligned} Q \in OP \quad &\Longrightarrow \quad Q \in O_A P \Longrightarrow \\ &\Longrightarrow \quad P \notin O_A Q \Longrightarrow P \notin OQ. \end{aligned}$$

AT3 $\mathcal{T}_A$ is the indiscrete topology.

Spaces having property AT1 are not very interesting. Spaces with property AT2 are close relatives of Alexandroff spaces. We note that in such spaces Theorem 9 holds. The proof of this Theorem literally carries over to the semi–toplogical case when AT2 is true. The most interesting spaces for our purposes are those having property AT3.

A semi–topological space is termed *homogeneous* if for any two points P and Q the neighborhoods OP and OQ are homeomorphic. Homogeneity therefore can be considered as 'topological translation invariance'. The points of a homogeneous space cannot be distinguished topologically. We saw that a nontrivial Alexandroff space is never homogeneous. Therefore, for a homogeneous semi–topological space the associated topology is either strictly discrete (AT1) or indiscrete (AT3).

A semi–topological space X is termed *symmetric* if $Q \in OP$ implies $P \in OQ$ for all $P, Q \in X$. From Theorem 7 we know that an Alexandroff space is certainly not symmetric. Therefore, a symmetric semi–topological space belongs either to class AT1 or to class AT3.

Example 7 *The space $\mathbb{Z}$, if equipped with the standard–semi–topology, is a symmetric semi–topological space.*

The Marcus–Wyse topology was introduced in a "semi–topological" way by means of smallest open neighborhoods (see Example 2):

$$O(n) = \begin{cases} \{n\}, & \textit{if } n \textit{ is even,} \\ \{n-1, n, n+1\}, & \textit{else.} \end{cases}$$

This topology, however, does not make $\mathbb{Z}$ a symmetric space since for example $0 \in O(1) = \{0,1,2\}$ and $1 \notin O(0) = \{0\}$. Clearly, the Marcus–Wyse topology is also not homogeneous.

5.3 Related Concepts

There are some concepts in the literature which can be easily expressed in the framework of semi–topology. We consider some examples:

Example 8 *Arnaud, Lamure, Terrenoire und Tounissoux [8] proposed for image processing the pre–topological spaces of Brissaud [12].*

For a set X let $\mathcal{P}(X)$ be the set of all subsets of X. $a : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is termed the closure–mapping *and has the properties*

$$S \subseteq a(S) \qquad \textit{für alle } S \subseteq X$$

and $a(\emptyset) = \emptyset$. The set X equipped with the mapping a is termed a pre–topological space.

For a semi–topological space X we define $a(S) = \bigcup_{P \in S} OP$ which makes X a pre–topological space.

There do exist pre–topological spaces which are not semi–topological spaces. Consider for example the space $\mathbb{Z}$ with the pre–topology

$$a(S) = \begin{cases} \{n-1, n, n+1\}, & \textit{if } S = \{n\}, \\ S & \textit{else,} \end{cases}$$

then $\mathbb{Z}$ is not a semi–topological space, since it is not possible to represent all sets $a(S)$ as union of smallest neighborhoods. Consider for example a set consisting of two consecutive points in $\mathbb{Z}$.

Example 9 *Given a symmetric homogeneous semi–topological space X. Then all OP are homeomorphic to a subset B_0 in X.*

For a set $S \subseteq X$ we define the dilation *of S with* structuring element B_0 *as*

$$\mathrm{DIL}(S) = \bigcup_{P \in S} OP$$

and the erosion *of S*

$$\mathrm{ERO}(S) = \{P \mid OP \subseteq S\}.$$

We note that $\mathrm{DIL}(S)$ corresponds to the set $a(S)$ in Example 8 and $\mathrm{ERO}(S)$ corresponds to $\complement \bigcup_{P \in \complement S} OP$. One has

$$\mathrm{ERO}(S) \subseteq S \subseteq \mathrm{DIL}(S).$$

Both these sets associated to a given set are used to analyze sets. For example, one might ask, under which circumstances $\mathrm{DIL}(\mathrm{ERO}(S)) = S$ or how the points can be characterized where both these sets differ. It is also possible to investigate the iterated sets $\mathrm{DIL}^k(S)$ and so on.

These questions belong to the field of mathematical morphology *and are treated for example in Serra's books [57, 58].*

5.4 Connectedness

If we translate connectedness in semi–topological spaces verbally from Alexandroff spaces, we get a result which is not desirable:

Example 10 *In the standard semi–topology for $\mathbb{Z}$ the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are open. If we apply the definition of connectedness in a naive way, the set $\{1, 2, 3, 4, 5, 6\}$ would not be connected.*

We therefore define: The set $S \subseteq X$ is termed *connected* if there is no decomposition $S = T_1 \cup T_2$ such that $T_1 \cap T_2 = \emptyset$, both $T_1, T_2 \neq \emptyset$ and both strictly (relatively) open with respect to S.

We note, that a set is $\mathcal{T}$–connected whenever it is $\mathcal{T}_A$–connected. The converse is not necessarily true, since for example in AT3–spaces all subsets are $\mathcal{T}_A$–connected but not always $\mathcal{T}$–connected.

Example 11 *Let S be a subset of $\mathbb{Z}$, equipped with the standard semi–topology (see Example 6). Then it is not possible to have a representation $S = \bigcup\{O(n) \mid n \in S\}$, hence there exists no strictly open subset of $\mathbb{Z}$ (in the standard semi–topology). This implies that $\mathbb{Z}$ is trivially a connected space.*

We know that exactly the subsets of consecutive numbers are connected in $\mathbb{Z}$ with respect ot the Marcus–Wyse topology (see Example 2). This assertion is also true in the standard semi–topology.

Example 12 *In a pre–topological space (see Example 8) the definition of connectedness is much more complex. A subset S of a pretopological space is connected if there exist two sets F and G such that*

- $F = \bigcup F_j$, $G = \bigcup G_j$,
- $S \subseteq F \cup G$,
- $S \cap \bigcup a(F_j) \neq \emptyset$,
- $S \cap \bigcup a(G_j) \neq \emptyset$,
- $S \cap \bigcup a(F_j) \cap \bigcup a(G_j) = \emptyset$,

Obviously, in a semi–topological space both definitions of connectedness coincide.

The advantage of semi–topological spaces over pre–topologcial spaces is that the definitions in semi–topology are very close to those in general topology as we have seen for continuity and connectedness. Therefore it is possible to translate a large number of assertions from general topology almost literally into semi–topology.

Since nontrivial symmetric semi–topological spaces are always AT3–spaces, we can state that a symmetric semi–topological space is not connected if and only if it contains a nonempty strict subset which is strictly open.

Example 13 *In the digital plane we introduce the semi–topologies $\mathcal{T}_\kappa$ for $\kappa \in \{4, 8\}$ by the smallest open sets $O_\kappa(P) = \mathcal{N}_\kappa(P) \cap \{P\}$. Both these topologies are homogeneous and symmetric. $\mathcal{T}_\kappa$–connectedness is the same as κ–connectedness defined in Section 2.*

In a semi–topological space we can find a theorem which is analogous to Theorem 11. We first define a path just as in case of Alexandroff spaces. By inspection of the proof of Theorem 11 we see that it can be easily adapted to semi–topological spaces by replacing 'open' by 'strictly open'. The Theorem now reads

Theorem 14 *A subset S of an semi–topological space is connected if and only if it is path–connected.*

In a semi–topological space Lemma 2 is not true since there are more than two possibilities to (semi–) topologize a path. This generates more freedom for construction and this freedom is sufficient to allow for any closed path a topology in contrast to Corollary 1. So it is easily possible to find a semi–topology which causes the digital plane to be 8–connected (see Example 13).

There is even another advantage of semi–topology. We know that in ordinary topology curves in the definition of connectedness are introduced as homeomorphic images of a real interval. This means that we compare the topology of a given space with the well–known topology of the reals (or integers, respectively). In symmetric semi–topological spaces we can use the concept of continuity for introducing connectedness.

A subset Y of a semi–topological space X is termed an *arc* if it is a homeomorphic image of an interval in $\mathbb{Z}$, where $\mathbb{Z}$ is equipped with the standard semi–topology. A subset S of a semi–topological space is termed *arc–connected* if for any two points in S there exists an arc which is completely contained in S and which contains both points.

We now state

Lemma 3 *A subset Y of a symmetric semi–topological space is an arc if and only if it is a simple path (i.e. a path $\{P_0, P_1, \cdots, P_n\}$ such that exactly the subsets of consecutive points are connected).*

Proof 1. Assume that Y is an arc with homeomorphism $\varphi : [0,n] \longrightarrow W$, where $\varphi(i) = P_i$. Then the smallest neighborhood of a point $P_i \in W$ is the homeomorphic image of the smallest neighborhood of $i \in \mathbb{Z}$ in the standard semi–topology. Thus, OP_i contains exactly $P_j \in Y$ with $|i-j| \leq 1$.

2. Let $Y = \{P_0, P_1, \cdots, P_n\}$ be a simple path in X. $\varphi(i) = P_i$ is bijective. Since X is symmetric, by definition of a path, OP_i contains $P_j \in Y$ with $|i-j| \leq 1$. Since Y is simple, OP_i does not contain any other points in S. Hence, OP_i is homeomorphic to $O(i)$. □

We can conclude that for symmetric semi–topological spaces connectedness, path–connectedness and arc–connectedness are equivalent.

5.5 Ordered Sets

We add here a different approach to digital topology which dates back to Alexandroff. This approach uses the concept of partially ordered set ('poset') and plays a role in computer science (see e.g. [10]). Although there is a large amount of literature about this topic, we only sketch it and relate it to the structures developed here.

A set X is called a *partially ordered set* if it is equipped with a relation $\sqsubseteq$ such that

PO1 $P \sqsubseteq P$ for all $P \in X$,

PO2 $P \sqsubseteq Q$ and $Q \sqsubseteq R$ implies $P \sqsubseteq R$,

PO3 $P \sqsubseteq Q$ and $Q \sqsubseteq P$ implies $P = Q$.

A set x is called an *ordered set* if in addition

PO4 For all $P \in X$ is $P \sqsubseteq Q$ or $Q \sqsubseteq R$.

In an Alexandroff space X we introduce the partial order relation

$$Q \sqsubseteq P \qquad \Longleftrightarrow \qquad Q \in CP.$$

For the proof that this is indeed a partial order, we need Part 1 of Theorem 7. Since this assertion is not generally true in semi–topological spaces, we conclude that such an order relation does not always exist in semi–topological spaces.

Conversely, if X is a partially ordered set with order relation $\sqsubseteq$, we can define in analogy to semi–topology

$$CP = \{Q \mid Q \sqsubseteq P\}.$$

We note that this definition is indeed 'semi–topological' since we define smallest closed sets containing a point and then generate the whole topology by means of unions of such sets. In contrast to semi–topology, we do not start with open neighborhoods but it seems to be relatively clear that we arrive in this way at some sort of (semi–) topology.

When we consider Example 1 then $Q \sqsubseteq P$ means geometrically that Q is a boundary element of P. Indeed, we can interpret Alexandroff spaces as *abstract cell complexes*. This was demonstrated in [1].

As Alexandroff spaces and partially ordered spaces are equivalent, we can start our theory as well with the latter. This is usually done for modeling discrete structures in computer science [56].

Khalimsky [21, 26, 27] investigated *connected ordered topological spaces (COTS)*. Such spaces X are characterized by the properties that they are connected and that among each triple of points P, Q, R there is one, say Q, which has the property that $X \setminus \{Q\}$ consists of exactly two connected components, each containing one of the both remaining points. These spaces can be considered as homeomorphic images of intervals in $\mathbb{R}$ or $\mathbb{Z}$, hence they are ideally suited for defining paths and path–connectedness in discrete structures.

6 Applications to Image Processing

We restrict ourselves here to some applications of digital topology to image processing.

6.1 Models for Discretization

A very important and fundamental problem in image processing can be stated as follows: Given a set S in the Euclidean space $\mathbb{R}^2$ and let $\Delta : \mathbb{R}^2 \longrightarrow \mathbb{Z}^2$ be a *discretization mapping* which associates to S a discrete set $\Delta(S)$. Such a mapping can be considered as a model of a digitizer. There were different such mappings proposed in the literature, we mention Pavlidis' 'sampling theorem' [50] and Serra's book [58, Part II, Chapter VII]. More modern presentations of the subject were given in [37] and [19].

Typical questions arising in this context are:

- Which topological (geometrical) properties does the discretized set $\Delta(S)$ share with the set S?
- Is Δ or Δ^{-1} (as a point–to–set mapping) continuous?
- Which other properties has Δ?

The problem of finding a continuous model for a given discrete space is in some sense the inverse problem of discretization.

6.2 Continuity

There is a need for defining continuity in the context of digital spaces [51]. This concept has proved as a very strong and fruitful one in ordinary topology. We have seen in the definition of arc–connectedness in semi–topological spaces that by continuity it becomes possible to deduce a property of a topological space from known properties of another space. If we interpret the smallest neighborhoods of a point as a formal definition of 'nearness' (P and Q are near of each other if $P \in OQ$ or $Q \in OP$), then continuous functions are functions preserving nearness. Moreover, also connectedness is preserved under continuos mappings.

An important tool is homeomorphy which allows to solve certain problems by posing them in appropriate homeomorphic spaces. A very difficult question arising in this context is the problem of the dimension of a digital space. We saw that the 8–topology induces a graph structure which is not planar. The question is, whether one can tell from the structure of a graph whether it is a model for a plane (or more generally, a d–dimensional space) or not. This problem was investigated in [49].

Typical questions in the context of continuity are:

- The intermediate value theorem of classical analysis is a very powerful tool for solving nonlinear equations. Rosenfeld [51] was able to find a digital analog of this theorem.
- A very important constructive tool in general topology is the fixed point principle which characterizes conditions under which a continuous

function f mapping a topological space into itself has a fixed point x with $f(x) = x$. Rosenfeld [51] formulated a 'near fixed point theorem'.

- Are different models of discrete spaces homeomorphic?

6.3 Homotopy

Topological predicates are in a certain sense difficult to handle on parallel computers as mentioned in Section 3. On the other hand, the large amount of data, which is typical for image processing applications, calls for parallel processing. Therefore it was proposed very early to reduce images to a 'simple' topological equivalent before closer investigation of topological predicates. Indeed, Rosenfeld showed in 1979 [53] that the decision whether a simgle point of a digital set influences its connectedness or the connectedness of its complement, can be made locally since it can be shown to be a predicate only depending on Euler's number.

In Figure 11 the reduction process is illustrated. This procedure is known in the image processing literature as 'thinning' or 'skeletonization'. The reduced set or skeleton shown in the Figure was obtained by applying a method described in [18] which is characterized by the property of being invariant with respect to motions of the digital plane. In order to describe the thinning procedure in terms of homotopy theory, it would be desirable to interpret the reduced image as a 'deformation retract' of the original one. However, there is no such possibility within the framework of digital topology or semi–topology to deform the original image in a continuous way into the reduced image. By Theorem 12 there is no way to find a homeomorphism between finite sets having different numbers of elements.

Kong [28, 32] introduced a concept of digital homotopy theory. This approach is based on an embedding procedure which is similar to the cellular model presented in Section 3.2. The cellular images of the original image and the reduced image are indeed homeomorphic in the $\mathbb{R}^2$ topology.

6.4 Fuzzy Topology

Digital topology is the theoretical basis for understanding certain properties of sets in images, i.e. it is mainly suited to black–white images or to 'segmented' objects in images. An interesting problem is the question, whether it is possible to generalize the known concepts from black–white topology to gray–value topology. This is indeed possible by using concepts from 'fuzzy topology'. Gray–value images provide in some sense a very neat application of fuzzy set theory, since they exhibit a naturally given membership function, the gray–value function. Rosenfeld [54] presented a theory of fuzzy topology for gray–value images. He was able to generalize such concepts as connectedness, area and circumference and compactness of an object in an image. Since then, a large number of papers dealing with this subject were published.

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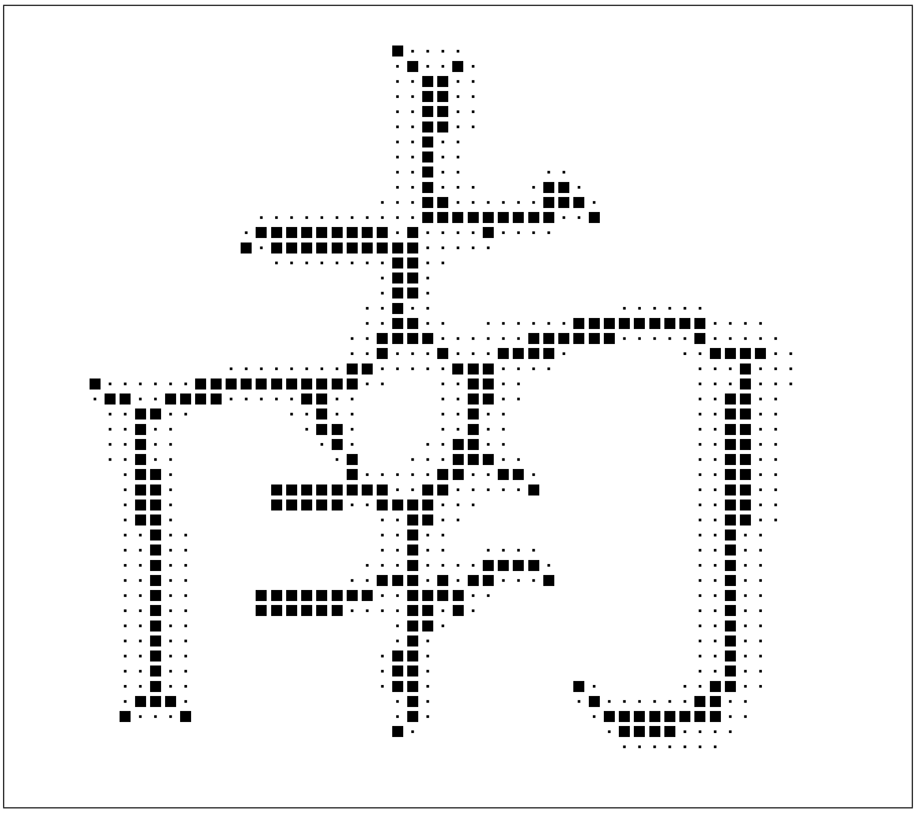

Figure 11: Reduction of an image. Points are deleted which do not affect the topology of the image. Deleted points are marked '.', remaining points are marked '■'.

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