On Invariance of the Projected Radon Transform

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1 Introduction

The Radon transform is a widely used technique in image processing. Most attention is paid to its inversion which plays an important role in computer tomography. The Radon transform itself was observed by Hough [1] to be useful for detecting so called "linear features" in pictures. Indeed, the Radon transform of a picture takes local maxima at lines containing isolated black segments. However, as well the calculation of the Radon transform as the search for those maxima of the transform came out to be very expensive. For that reason in [3] projected variants of the Radon transform were introduced and considered. The main result of that paper states that certain — apparently natural — requirements imply the projected Radon transform to be almost independent of the transformed pictures. So the representation of the transform contains only the integral of the image as a factor, consequently the assumed linearity, continuity together with the invariance requirements result in a useless transform which does not reflect any interesting features of a picture.
However, the papers [2, 3] contain a mistake. So it was assumed (cf. [3, page 7]) that for each pair of lines ℓ_1, ℓ_2 in the plane there exists a unique motion mapping ℓ_1 onto ℓ_2. This is obviously not true. Consequently, the definition of the sets M_a becomes unprecise, and Lemma 5.1 and Theorem 5.1, the main results, are affected, at least a new formulation is necessary.
Our attempt to "repair" the main result of [3] provided to the problem of a better understanding what kind of object the so called "accumulator" is. In [3] it was characterized as a "set of pairs of real numbers parameterizing

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all lines of the plane". In order to give improved definitions of the sets M_a and the used later on "traces" unavoidably the accumulator must be given the structure of a differentiable manifold. Its dimension is indeed two, but it does not posses any global map. So the lines L_a introduced in [3] are not proper objects for the definition of the M_a's, the traces and the formulation of the results, since they are just lines in one of the local charts. The aim of the present paper is to improve the basic definitions upon which the theory developed in [3] rests, to investigate whether the main results still hold in the new framework or if they were consequences of the mistake in "Lemma 1" and last, but not least to answer the question, if the requirements are indeed as natural as they seem to be.

2 Basic Notions

The classical Radon transform maps a function f defined on the plane $\mathbb{R}^2$ into a function $\hat{f}$ defined on the set of all straight lines of the plane which assigns to each straight line ℓ the integral $\int_\ell f\, d\sigma_\ell$. Here σ_ℓ denotes the Lebesgue measure on ℓ. So the domain of $\hat{f}$ is the set of all straight lines in $\mathbb{R}^2$, it is called the *accumulator* and denoted by A, cf. [2, 3]. In the mentioned paper A was characterized as a set of pairs of real numbers. However, it comes out to be impossible to identify A in a continuous way with a set of pairs of real numbers, i.e. a subset of $\mathbb{R}^2$. The set of all straight lines passing through a fixed point, e.g. the origin, forms the one dimensional Grassmannian manifold. In the case of all straight lines we have again the structure of a manifold, this time a two dimensional one.
Moreover, this manifold A cannot be imbedded into the plane. In [4] the Radon transform $\hat{f}$ of a function f was introduced as a function on the set $S^1 \times \mathbb{R} =: Z$, the so–called unit cylinder, but there is obviously no one–to–one correspondence between the elements of Z and the straight lines. In consequence one has to deal with certain additional identities if Z is assumed as the domain of the transforms.
In order to make the concept of projected Radon transforms precise we here introduce the manifold A by local charts. Such local charts are given by the *slope–intercept representation* or by *Hesse's normal form*, cf. [2, Examples 1 and 2]. The locality of that charts results in the fact that parallels to the y–axis cannot be represented for the first one while lines passing through the origin posses two representations for the second chart. If we add in the first case as a chart the slope–intercept representation for x being a function of y we easily show the compatibility of the charts, hence the structure of a differentiable manifold is generated. Exactly the same structure is introduced

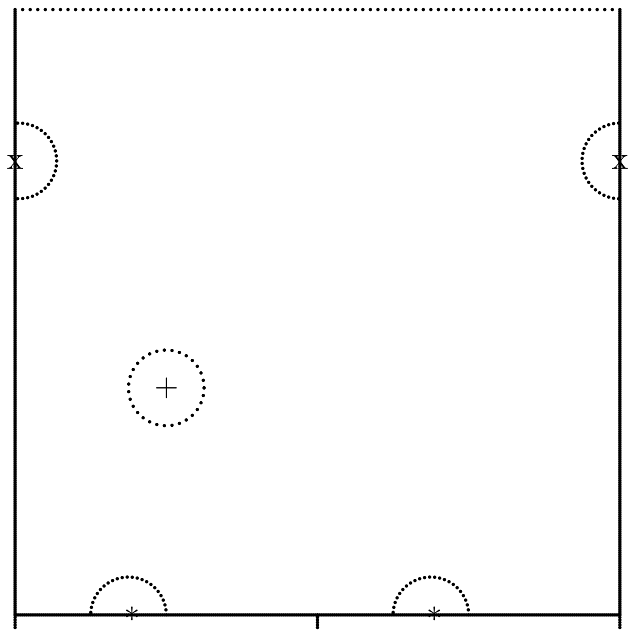

Figure 1: Topology of the accumulator A

by completing the Hesse parametrization by a second one with different origin. The manifold A can be imagined as a unit cylinder with the following identification:

$$(\zeta, \rho) \sim (-\zeta, -\rho)$$

for each pair $(\zeta, \rho) \in Z = S^1 \times \mathbb{R}$.

Remark a) A subset $V \subset A$ is open, if for sufficiently small $v \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ the condition $\ell \in V$ implies that all the lines to which ℓ moves by rotation by α around the origin and translation by v belong to V.

b) The group of all motions of the plane as well as each of its subgroups acts on A. Considering one dimensional subgroups, the orbits in the charts are in general *pairs* of rays for translations and circles for rotations, the Hesse representation, in the latter case with the center of rotation in the origin, assumed. For a line ℓ with direction $v \in \mathbb{R}^2$ belonging to the acting 1–D subgroup of translations its orbit is a single point.

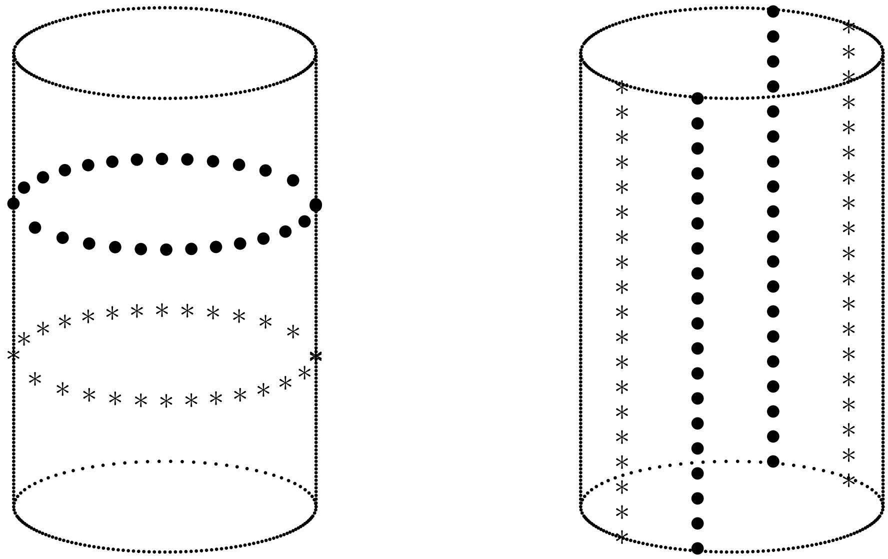

Figure 2: a) Orbits under translation. b) Orbits under rotation.

c) The graph $K_{3,3}$ can be imbedded into A. Hence, due to Kuratowski theorem, A can not be imbedded into the plane. In other words, there exists no global chart of A.

In the present context it becomes obvious that the original definition of sets (cf. [2])

$$L_a = \{(\alpha, \beta) \in A : \alpha = a\}$$

is improper since there is no global chart. As a consequence, the sets of motions M_a as sets of motions leaving L_a invariant are not acceptable. A second reason to reject the original definition ([2, Chapter 3]) is the underlying assumption of a one–to–one correspondence between pairs of lines and motions which is not true. In the next section we introduce a new consistent approach which contains a new definition of sets of motions and corresponding subsets of the accumulator playing the role of M_a and L_a, respectively.

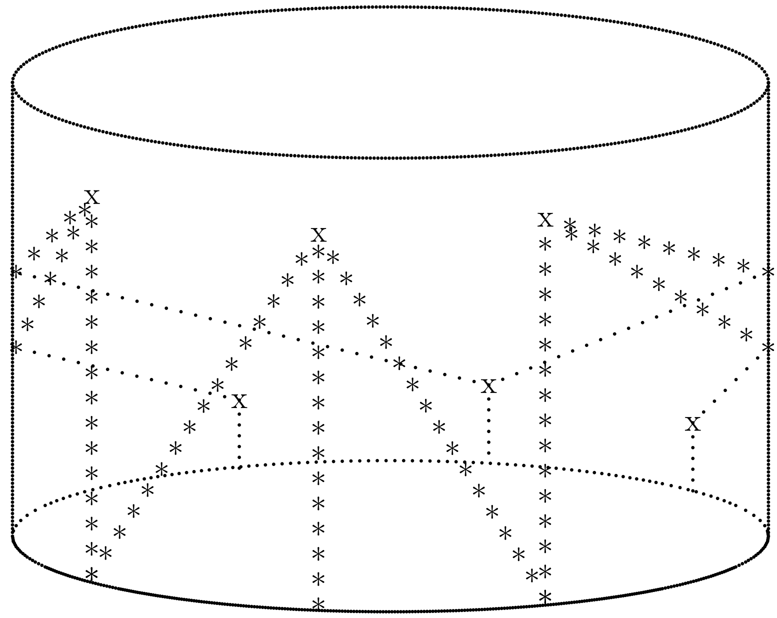

Figure 3: Imbedding of $K_{3,3}$ in A.

3 Orbits and Projections

Let us denote by G the group of all motions of the plane and by K a one dimensional Lie subgroup of G. Hence the manifold A splits into orbits under the action of K on A. We denote the action of G as well as K on A by

$$G \times A \longrightarrow A, \qquad (\kappa, \ell) \mapsto \kappa\ell,$$

thus the orbit $[\ell]$ of an element ℓ of A under K equals

$$[\ell] = \{\kappa\ell : \kappa \in K\}.$$

These orbits play the role of the traces in [2, 3], Theorem 3.1 of the former paper holds, mutatis mutandis. Each orbit is again a Lie group isomorphic to a subgroup of K. Hence, the orbits are either single points or isomorphic to a circle or to a line.
In order to introduce the concept of a projector we use functions defined on the set of all orbits $[A : K] := L$. A (trace invariant) projector Π assigns to a function defined on A a function defined on L. Usually the value $\Pi f(\lambda)$ for $\lambda = [\ell] \in L$ is some functional of the values of f on $[\ell]$, e.g. mean value or maximum.

Let now K be another Lie subgroup of G. We say that an operator T defined on a function space X on $\mathbb{R}^2$ is invariant with respect to K if for all $\kappa \in K$ and $f \in X$ it holds $T(f(\cdot)) = T(f(\kappa^{-1}\cdot))$. Here the inverse element κ^{-1} with respect to the group operation acts as a motion on the argument of f being a point of the plane. The value $f(\kappa^{-1}P)$ is for binary images the value of the characteristic function of the moved by κ image f. This moved image is denoted by $f \circ \kappa^{-1}$. Observe that in [2, 3] only translations were admissible while here rotations are included.
We introduce now the generalized Radon transform by

$$\varphi(\ell) = \hat{f}(\ell) = H(f)(\ell) = H_0(F(f,\ell))(\ell) = \int_\ell F(f,\ell)\, d\sigma_\ell .$$

Here is $F : X \times A \longrightarrow L_1(\mathbb{R}^2)$ a (generally nonlinear) filter, cf. [2, 3]. Of course, in general the Radon transform $\varphi = \hat{f} = H(f)$ is not constant on all orbits, hence it is not trivial to define a projected transform which results in a function assigning just one value to each of the orbits.
Note that in the present framework the subgroup K of motions is a primitive notion, it substitutes M_a. The sets L_a now occur as a secondary notion, i.e. as orbits $[a]$ for $a \in A$. Now, the most interesting question is whether the main result of [2, 3] remains true.

4 Invariance Properties of the Projected Radon Transform

We observe that the group G (as well as its subgroups) acts on several objects:
a) on points $P, Q, \ldots$ of the plane $\mathbb{R}^2 : P \mapsto \kappa P$,
b) on lines $\ell, \ell_1, \ell_2, \ldots$ in R^2, i.e. the elements of A:

$$\ell \mapsto \kappa\ell = \{\kappa P \ : \ P \in \ell\} \quad , \text{(for } \kappa P \text{ see a)},$$

c) on functions with domains in $\mathbb{R}^2 : f \mapsto f \circ \kappa$,

$$f \circ \kappa(P) = f(\kappa P) \quad , \text{(for } \kappa P \text{ see a)},$$

d) on functions with domain in $A : \varphi \mapsto \varphi \circ \kappa$,

$$\varphi \circ \kappa(\ell) = \varphi(\kappa\ell) \quad , \text{(for } \kappa\ell \text{ see b)},$$

e) on functions with domain in $L = [A : K] : \xi \mapsto \xi \circ \kappa$,

$$\xi \circ \kappa(\lambda) = \xi \circ \kappa([\ell]) = \xi([\kappa\ell]), \qquad \lambda = [\ell] \quad , \text{(for } \kappa\ell \text{ see b)}.$$

So the following definitions make sense.

Definition 4.1 *A set X of functions with domain in $\mathbb{R}^2$ is invariant with respect to $K \subseteq G$ if*

$$f \in X \Longrightarrow \forall \kappa \in K \quad f \circ \kappa \in X.$$

A set Y of functions with domain in A is invariant with respect to $K \subseteq G$ if

$$\varphi \in Y \Longrightarrow \forall \kappa \in K \quad \varphi \circ \kappa \in Y.$$

An operator T defined on an invariant set of functions X (resp. Y) is invariant with respect to K if

$$\forall \kappa \in K \quad T(f \circ \kappa) = Tf \qquad (\textit{resp. } T(\varphi \circ \kappa) = T\varphi).$$

A projector Π along K, $\Pi : Y \longrightarrow Y_K$, where Y_K is a set of functions on $[A : K] = L$ is called trace invariant *if it is invariant with respect to K.*

Now, the following theorem is quite obvious.

Theorem 4.1 *Let K and $\tilde{K}$ be two Lie subgroups of G and Π a projector of a function space Y on A along K. If Π is trace invariant and additionally invariant with respect to $\tilde{K}$, then it is invariant with respect to the group $[K \cup \tilde{K}]$ generated by $K \cup \tilde{K}$.*

Proof Let G_Π be the set of all motions leaving Π invariant:

$$G_\Pi = \{\kappa \in G : \forall \varphi \in Y \quad \Pi\varphi \circ \kappa = \Pi\varphi\}.$$

G_Π is obviously a subgroup of G, and by assumption it contains K and $\tilde{K}$, hence it contains $[K \cup \tilde{K}]$. ■

Corollary 4.1 *If $\tilde{K}$ differs from K, both subgroups being nontrivial groups of translations, then Π is invariant under arbitrary translations.*

Corollary 4.2 *If $\tilde{K}$ differs from K, either of them being a group of rotations, both nontrivial, then Π is invariant under arbitrary motions.*

Now, let us assume the projector Π to be defined in a linear way (i.e. some kind of mean–value over $\lambda = [\ell]$ is assigned to the orbits $\lambda \in [A : K]$):

$$\Pi : Y \longrightarrow Y_K$$

$$f \mapsto Ff \mapsto \hat{f} = \varphi \mapsto \Pi\hat{f}$$

$$X \longrightarrow L_1(\mathbb{R}^2) \longrightarrow Y \longrightarrow Y_K.$$

Theorem 4.2 *Let X be a subspace of $L_1(\mathbb{R}^2)$ and let $\Pi \circ H : X \longrightarrow Y_K$ be pointwise linear and bounded, i.e., $\forall \lambda \in L$ the assignment $f \mapsto (\Pi H f)(\lambda)$ is a linear and bounded functional on X in the L_1–norm. Then under the assumptions of Corollary 4.1 the projected Radon transform is invariant with respect to all motions.*

Proof By our assumption, for each $\lambda \in L$ there is a kernel $\Phi_\lambda \in L_\infty(R^2)$ such that

$$(\Pi H f)(\lambda) = \int\limits_{R^2} \Phi_\lambda f \, dx.$$

Since H is translation invariant we conclude

$$\forall \kappa \in G_{trans} \quad \Pi H f \circ \kappa(\lambda) = \Pi H f(\lambda)$$

$$\Longrightarrow \forall \kappa \in G_{trans} \quad \Phi_\lambda \circ \kappa = \Phi_\lambda,$$

i.e., $\Phi_\lambda(\cdot) = const = k_\lambda$. Here denotes G_{trans} the subgroup of translations in G. ■

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