# Covering the Plane by Digital Curves and Contours

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# Contents

# Chapter 1

# Contours and Curves

## 1.1 Basic Definitions

The *digital plane* is the set of all points in the Euclidean plane having integer coordinates. A *digital set* $S$ is any finite subset of the digital plane. For convenience, points in $S$ will be termed *black* or *foreground points,* while those of the complement will be termed *white* or *background points.* The *color* of a point is its property of being black or white.

Each point $P$ in the digital plane has eight *neighbors,* which are numbered 0 to 7 according to the following scheme

$$
\begin{array}{ccc}
N_3(P) & N_2(P) & N_1(P) \\
N_4(P) & P & N_0(P) \\
N_5(P) & N_6(P) & N_7(P)
\end{array}
$$

The notation $N_i(P)$ refers to the neighbor of $P$ having number $i$. $\mathcal{N}(P)$ is the set of all neighbors of $P$ (without $P$ itself) and is termed the *neighborhood* of $P$. Neighbors with even numbers are *direct neighbors* of $P$ or *4–neighbors;* neighbors having odd numbers are *indirect neighbors* of $P$. All neighbors together are sometimes called *8–neighbors.*

By means of the neighbor–relation we are able to define a topology in the digital plane [3]. As usual we assume the foreground to be equipped with the *8–topology,* i.e., two points in the set $S$ are *connected* if there exists a sequence of points in $S$ starting at one of the given points and ending at the other such that each two successive points in the sequence are 8–neighbors of each other. The background is analogously equipped with the *4–topology.* From this definition it becomes clear what is meant by foreground or background *connection components* (see [3]).

## 1.2 Contours in the Digital Plane

Let $S$ be an 8–connected subset of the digital plane and $\mathbb{C}S$ its complement. The *boundary* of $S$ is the set

$$\text{bd } S := \{P \in S \text{ and } P \text{ is 4–neighbor of } \mathbb{C}S\}.$$

On the set bd $S$ we define a binary *successor relation*: For points $P, Q \in$ bd $S$ holds suc $(P, Q)$ ($Q$ is a successor of $P$) if and only if

1. $Q$ is an 8–neighbor of $P$, viz. $Q = N_i(P)$, and

2. for all points $R \in S$ which are 8–neighbors of both $P$ and $Q$ (i.e. $R \in S \cap \mathcal{N}(P) \cap \mathcal{N}(Q)$) is $R = N_j(P)$ with $j = i + 1 \pmod 8$ or $j = i + 2 \pmod 8$.

**Example 1.1** *In the Figure 1.1, "$\bullet$" denotes an interior point of $S$, "$\circ$" a point of $\mathbb{C}S$; points on the boundary of $S$ are marked by letters. There hold the relations* suc $(N, O)$, suc $(O, P)$, suc $(P, Q)$ suc $(Q, R)$ *and* suc $(R, T)$. *However,* suc $(N, P)$ *is not true because the point $O$ belongs to $S$ and to the 8–neighborhoods of both $N$ and $P$, with $O = N_1(N)$ and $P = N_2(N)$ condition 2. is not met.*
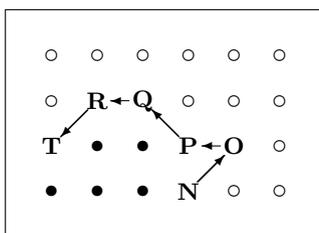


Figure 1.1: The successor relation

**Example 1.2** *Note that a point $P$ of* bd $S$ *can have more than one successor. Figure 1.2 shows a point $P$ having even four successors.*
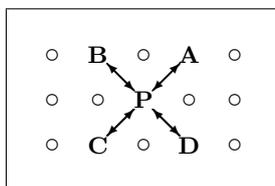


Figure 1.2: Point $P$ has 4 successors $A$, $B$, $C$ and $D$

**Proposition 1.1** *If $S$ contains more than one point, then any point $P \in \text{bd } S$ has at least one and at most four successors.*

**Proposition 1.2** *Let $P$ and $Q$ be different points on $\text{bd } S$ which are 8–neighbors of each other. Then both relations $\text{suc } (P,Q)$ and $\text{suc } (Q,P)$ are true if and only if there is no common 8–neighbor of $P$ and $Q$ which belongs to $S$.*

In Example 1.2 the point $P$ is also successor of its four successors $A$, $B$, $C$ and $D$.

As a consequence, the successor relation ist not sufficient to generate linear orderings on $\text{bd } S$. Therefore we investigate pairs of boundary points:

The pair $(P,Q)$ of points $P, Q \in \text{bd } S$ with $\text{suc } (P,Q)$ is called a *border edge*. For any two border edges $(P,Q)$ and $(P',Q')$ the relation $(P',Q') = \text{next } (P,Q)$ holds if and only if $P' = Q$, and for all points $R$ in the 8–neighborhood of $Q$ lying between $Q'$ and $P$ (when moving from $Q'$ to $P$ in counterclockwise direction) the pair $(Q,R)$ is not a border edge.

**Remark 1.1** *In Figure 1.2 is $(P,A) = \text{next } (B,P)$ but not $(P,B) = \text{next } (A,P)$.*

**Proposition 1.3** *For any border edge $(P,Q)$ there are uniquely determined border edges $(P',Q')$ and $(P'',Q'')$ with $(P',Q') = \text{next } (P,Q)$ and $(P,Q) = \text{next } (P'',Q'')$.*

As a consequence of Proposition 1.3 we get:

**Theorem 1.1** *Any border edge $(P_0, P_1)$ generates a uniquely determined sequence $\{P_i\}$ of border points with $(P_i, P_{i+1}) = \text{next } (P_{i-1}, P_i)$ for all integers $i$. This sequence $\{P_i\}$ may be cyclic, i.e. there is an integer $j$ such that $P_i = P_{i+j}$ for all $i$.*

The sequences generated by border edges are called *contours*.
The following assertions are true:

- For any border edge $(P,Q)$ there is exactly one contour $\mathcal{C}$ such that $P$ and $Q$ belong to $\mathcal{C}$.

- Any point of $\text{bd } S$ belongs to at least one contour, but it can belong to more than one contour as well.

- Any contour separates one 4–component of $\mathbb{C}S$ from $S$.

- For finite $S$ all contours are cyclic. Exactly one of them is positively (counterclockwise) oriented and separates $S$ from the only one infinite component of $\mathbb{C}S$, the others surround exactly one finite component of $S$ (the *holes* of $\mathbb{C}S$) each, and are negatively (clockwise) oriented. Clockwise oriented contours we call *outer contours*, the other cyclic contours we call *inner contours* of $S$.
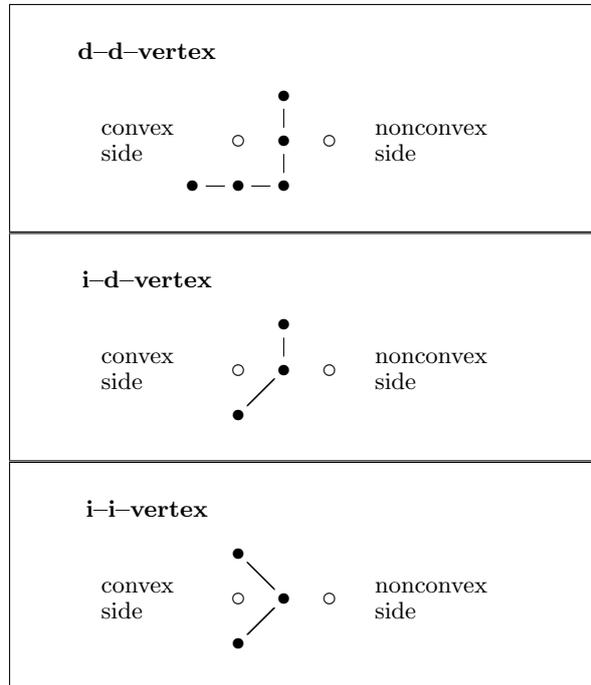
Figure 1.3: Types of vertices

A connected subset of a contour is a *d–piece* if all points on this subset are on a horizontal or vertical line. A connected subset of a contour is an *i–piece* if all points on this subset are on a diagonal line. A subset of a contour which is either a d– or an i–piece is a *piece* of the contour. A piece is *maximal* if it is not strictly contained in any other piece.

A connected subset of a contour consisting of two pieces whose union is not a piece, is a *vertex* of the contour. The components of the complement with respect to the contour are sometimes called the *convex* and the *nonconvex* side of the complement with respect to the vertex. According to the types of the pieces involved, we distinguish between d–d, d–i, i–d and i–i–vertices

A connected subset of a contour consisting of three pieces is a *transition*. A transition is a *convex transition* if the convex sides of both vertices defined by the transition coincide, otherwise the transition is an *S–transition*. In Figure 1.4 examples for all possible transitions are given.

## 1.3   Contour Curves

Some properties of contours are of interest for our further investigations:
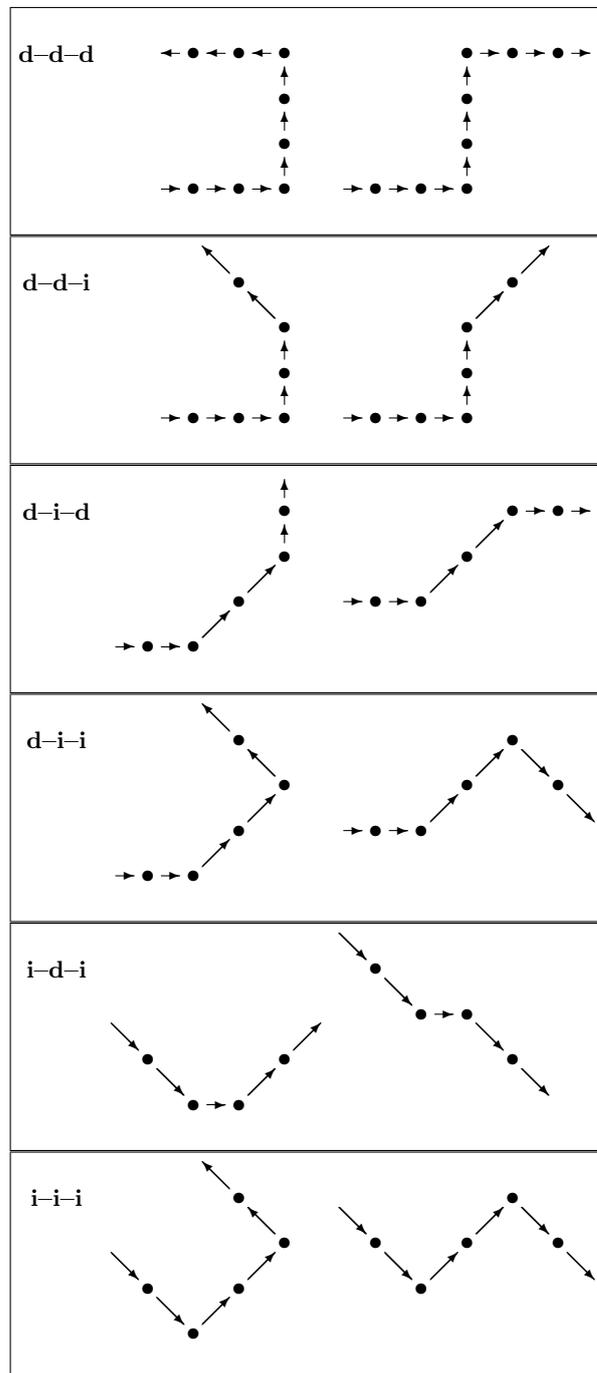
**C–1** Any contour is an 8–path in the digital plane.

Figure 1.4: Types of transitions. For each individual type the convex transition (left) and the S–transition (right) are depicted

**C–2** If for a contour $\mathcal{C} = (P_i)$ and for an index $j$ the point $P_{j-1}$ is an 8–neighbor of $P_{j+1}$, then with $P_{j+1} = N_k(P_j)$ and $P_{j-1} = N_\ell(P_j)$ it follows that $\ell = k + 1 \pmod 8$ or $\ell = k + 2 \pmod 8$.

(This means that contours have d–d–vertices or "sharp" d–i–vertices, but only such that the set $S$ lies on the convex side with respect to the vertex. See also Figure 1.5).

**C–3.a** If for a contour $\mathcal{C} = (P_i)$ there exist indices $k \neq \ell$ with $P_k = P_\ell, \cdots, P_{k+m} = P_{\ell-m}$, and $P_{k+1} \neq P_{\ell+1}$ and $P_{k+m+1} \neq P_{\ell-m-1}$ for an $m \geq 0$, then the cycle of neighbors $P_{k-1}, P_{\ell+1}, P_{k+1}$ of $P_k$ has the same orientation as the cycle of neighbors $P_{k+m+1}, P_{\ell-m-1}, P_{k+m-1}$ of $P_{k+m}$.

**C–3.b** For any contour $\mathcal{C} = (P_i)$ there are no indices $k$ and $\ell$ such that

- $P_k$ is an 8–neighbor of $P_{k+1}$, $P_\ell$ and $P_{\ell+1}$,
- with $P_{k+1} = N_j(P_k)$, $P_{\ell+1} = N_m(P_k)$ and $P_\ell = N_n(P_k)$ is $m = j + 1$ $\pmod 8$ and $n = j - 1 \pmod 8$.

**Remark 1.2** *Property (C–3) means that contours have no proper crossings. Figure 1.6 illustrates the constellations which are not possible by (C–3) for contours.*
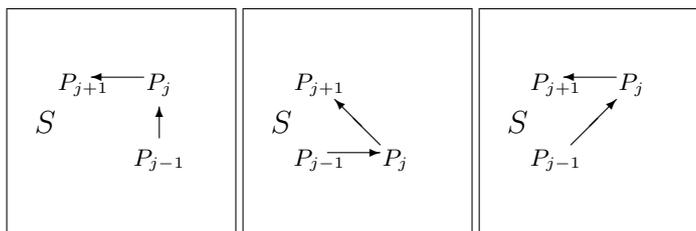


Figure 1.5: The possible constellations of a d–d–vertex and sharp i–d–vertices in a contour

We define: Any 8–path in the digital plane fulfilling conditions (C–1) — (C–3) for contours is termed a *contour curve.*

Now we are able to prove:

**Theorem 1.2** *Any contour is a contour curve. Any contour curve is a contour for a suitable set $S$. For any contour curve $\mathcal{C}$ there is a uniquely determined set $S$ such that* bd $S = \mathcal{C}$.

In the sequel we denote the unique set $S$ with bd$S = \mathcal{C}$ by $S(\mathcal{C})$.

A *generalized closed digital 8–curve* or simply *curve* is a contour curve with the property that each point on it has exactly one successor. In other words:

Figure 1.6: Constellations which are not possible for contours according to properties (C–3.a) and (C–3.b)
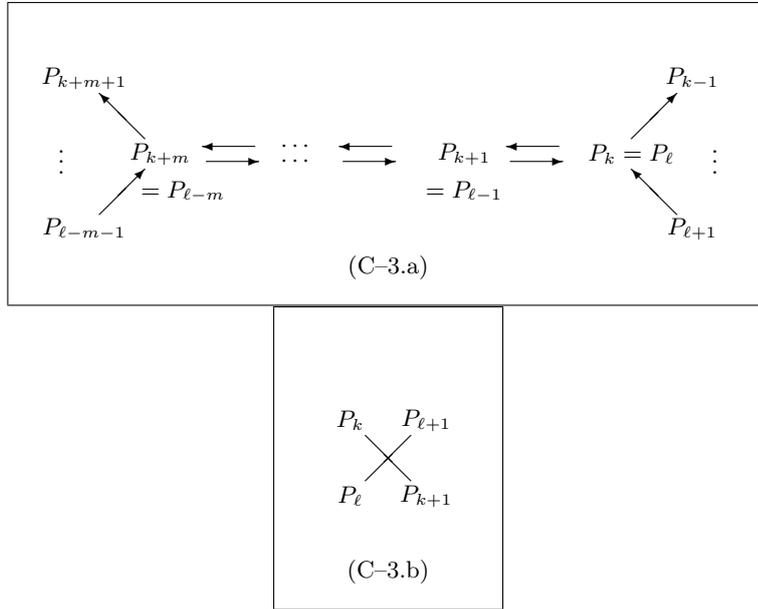
A curve is an 8–connected digital set such that each point of it has exactly two 8–neighbors belonging to the set. These neighbors are called the *curve–neighbors* of the point under consideration in order to distinguish them from the neighhbors of a point in the digital plane.

The following characterization of curves among contour curves is true:

**Theorem 1.3** *A contour curve $\mathcal{C}$ is a curve if and only if*

- *The relation 'suc ' has a unique inverse, or*

- *$\mathcal{C}$ is the boundary of $S(\mathcal{C})$ and of $\mathbb{C}S(\mathcal{C}) \cap \mathcal{C}$, or*

- *$\mathcal{C}$ has no d–d–vertices.*

As a consequence of the Theorem, if $\mathcal{C}$ is a curve then for each point $P$ on $\mathcal{C}$ its successor as well as its predecessor are uniquely determined. Therefore, curves always can be oriented in two different ways. For reasons of clarity, the orientations of contour curves are indicated by arrows in the following Figures, whenever it matters.

A generalized closed digital 8–curve is a closed digital 8–curve in the usual sense (see [1, 2, 3] for a definition) if and only if it consists of finitely many points.

7

**Theorem 1.4** *Given a generalized closed digital 8–curve $\mathcal{C}$. Then the complement of $\mathcal{C}$ consists of exactly two components.*

If $\mathcal{C}$ is a closed curve in the usual sense, the assertion of the Theorem coincides with Jordan's curve Theorem (see [3]). If $\mathcal{C}$ is not a finite set then the proof of the Theorem can easily be performed following the lines of Rosenfeld's proof.

## 1.4 Morphological Operations

We introduce four different *morphological operations* defined on contour curves. Let $\mathcal{C}$ be a contour curve.

**Dilation** DIL $(\mathcal{C})$ is the set of contours of the set $S(\mathcal{C}) \cup \text{bd} (\mathbb{C}S(\mathcal{C}))$.

**Erosion** ERO $(\mathcal{C})$ is the set of contours of the set $S(\mathcal{C}) \setminus \text{bd} \, S(\mathcal{C})$.

The iterated application of DIL and ERO starting with a contour curve $\mathcal{C}$ generates a family of contour curves:

$\text{DIL}^0(\mathcal{C}) := \{\mathcal{C}\}$.

$\text{DIL}^i(\mathcal{C})$ is the family of contours resulting from application of DIL to all contour curves of $\text{DIL}^{i-1}(\mathcal{C})$, where $i > 0$.

$\text{ERO}^0(\mathcal{C}) := \{\mathcal{C}\}$.

$\text{ERO}^i(\mathcal{C})$ is the family of contours resulting from application of ERO to all contour curves of $\text{ERO}^{i-1}(\mathcal{C})$, where $i > 0$.

With $\text{DIL}^i$ and $\text{ERO}^i$ two further operations can be defined:

**Expansion** $\text{EXP}^i(\mathcal{C})$ is the result of $\text{ERO}^i$ applied to the contour curves of the set $\mathbb{C}S(\mathcal{C}) \cup \text{bd} \, S(\mathcal{C})$ for $i > 0$.

**Shrinking** $\text{SHR}^i(\mathcal{C})$ is the result of $\text{DIL}^i$ applied to the contour curves of the set $\mathbb{C}S(\mathcal{C}) \cup \text{bd} \, S(\mathcal{C})$ for $i > 0$.

The following example shows that the operations DIL, ERO, EXP and SHR are really different:

**Example 1.3** *Given the outer contour curve of Figure 1.7.*

*Figures 1.8, 1.9, 1.10 and 1.11 show the results of the application of $\text{DIL}^1$, $\text{EXP}^1$, $\text{ERO}^1$ and $\text{SHR}^1$ to this contour curve.*

Whereas the sets generated by applying morphological operations to a contour curve are always contour curves by definition, the result of application of morphological operations to a curve need not necessarily be a curve.

Note that the operations EXP and SHR invert the character of the contour curves.

If the result of application of $\text{EXP}^1$ or $\text{SHR}^1$ to an outer (inner) contour curve is a single contour curve then this curve is an inner (outer) contour curve, i.e. the orientation of the curve has changed.

We note some properties of these operations:

Let $\mathcal{C}$ be a contour curve, $i > 0$. We define the point sets corresponding to the iterated application of morphological operations:
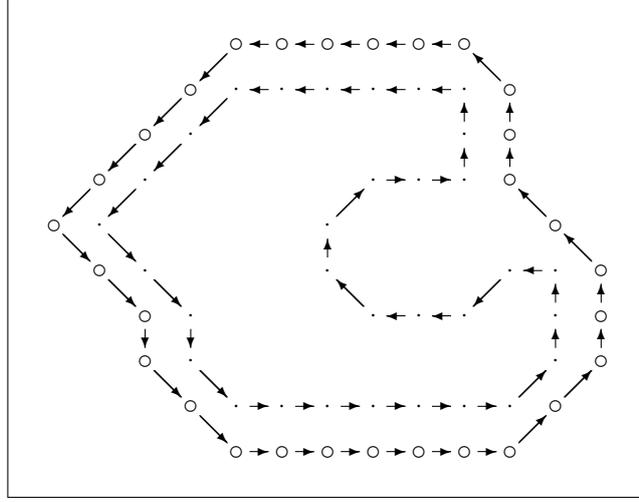
Figure 1.7: Original contour

$$S(\mathrm{DIL}^i(\mathcal{C})) := \bigcap \{S(\mathcal{C}') \mid \mathcal{C}' \in \mathrm{DIL}^i(\mathcal{C})\}.$$

If $\mathrm{DIL}^i(\mathcal{C}) = \emptyset$, we set $S(\mathrm{DIL}^i(\mathcal{C})) := \mathbb{Z}^2$, the digital plane.

$$S(\mathrm{ERO}^i(\mathcal{C})) := \bigcup \{S(\mathcal{C}') \mid \mathcal{C}' \in \mathrm{ERO}^i(\mathcal{C})\}.$$

If $\mathrm{ERO}^i(\mathcal{C}) = \emptyset$, we set $S(\mathrm{ERO}^i(\mathcal{C})) := \mathbb{Z}^2$.
    For $i > 0$:

$$S(\mathrm{EXP}^i(\mathcal{C})) := \bigcap \{\mathbb{C}S(\mathcal{C}') \cup \mathcal{C}' \mid \mathcal{C}' \in \mathrm{EXP}^i(\mathcal{C})\}.$$

If $\mathrm{EXP}^i(\mathcal{C}) = \emptyset$, we set $S(\mathrm{EXP}^i(\mathcal{C})) := \mathbb{Z}^2$.

$$S(\mathrm{SHR}^i(\mathcal{C})) := \bigcup \{\mathbb{C}S(\mathcal{C}') \cup \mathcal{C}' \mid \mathcal{C}' \in \mathrm{SHR}^i(\mathcal{C})\}.$$

If $\mathrm{SHR}^i(\mathcal{C}) = \emptyset$, we set $S(\mathrm{SHR}^i(\mathcal{C})) := \mathbb{Z}^2$.

**1. Monotonicity**    The following inclusions hold:

- $i > 0$: $S(\mathrm{DIL}^i(\mathcal{C})) \supseteq S(\mathrm{DIL}^{i-1}(\mathcal{C}))$ where $S(\mathrm{DIL}^i(\mathcal{C})) \neq S(\mathrm{DIL}^{i-1}(\mathcal{C}))$ if $\mathrm{DIL}^{i-1}(\mathcal{C}) \neq \emptyset$.

- $i > 0$: $S(\mathrm{ERO}^i(\mathcal{C})) \subseteq S(\mathrm{ERO}^{i-1}(\mathcal{C}))$ where $S(\mathrm{ERO}^i(\mathcal{C})) \neq S(\mathrm{ERO}^{i-1}(\mathcal{C}))$ if $\mathrm{ERO}^{i-1}(\mathcal{C}) \neq \emptyset$.

- $i > 1$: $S(\mathrm{EXP}^i(\mathcal{C})) \supseteq S(\mathrm{EXP}^{i-1}(\mathcal{C}))$ where $S(\mathrm{EXP}^i(\mathcal{C})) \neq S(\mathrm{EXP}^{i-1}(\mathcal{C}))$ if $\mathrm{EXP}^{i-1}(\mathcal{C}) \neq \emptyset$.

- $i > 1$: $S(\mathrm{SHR}^i(\mathcal{C})) \subseteq S(\mathrm{SHR}^{i-1}(\mathcal{C}))$ where $S(\mathrm{SHR}^i(\mathcal{C})) \neq S(\mathrm{SHR}^{i-1}(\mathcal{C}))$ if $\mathrm{SHR}^{i-1}(\mathcal{C}) \neq \emptyset$.

9

Figure 1.8: Dilation of contour

- $i > 0$: $S(\mathrm{EXP}^i(\mathcal{C})) \subseteq S(\mathrm{DIL}^i(\mathcal{C}))$ and $S(\mathrm{DIL}^i(\mathcal{C})) \neq S(\mathrm{EXP}^i(\mathcal{C}))$ is possible.

- $i > 0$: $S(\mathrm{SHR}^i(\mathcal{C})) \subseteq S(\mathrm{ERO}^i(\mathcal{C}))$ and $S(\mathrm{ERO}^i(\mathcal{C})) \neq S(\mathrm{SHR}^i(\mathcal{C}))$ is possible.

- $i \geq 0$: $S(\mathcal{C}) \subseteq S(\mathrm{DIL}^i(\mathcal{C}))$ and $S(\mathrm{DIL}^i(\mathcal{C})) \neq S(\mathcal{C})$ for $i > 0$.

- $i > 0$: $S(\mathcal{C}) \subseteq S(\mathrm{EXP}^i(\mathcal{C}))$ and $S(\mathrm{EXP}^i(\mathcal{C})) \neq S(\mathcal{C})$.

- $i \geq 0$: $S(\mathrm{ERO}^i(\mathcal{C})) \subseteq S(\mathcal{C})$ and $S(\mathrm{ERO}^i(\mathcal{C})) \neq S(\mathcal{C})$ for $i > 0$.

- $i > 0$: $S(\mathrm{SHR}^i(\mathcal{C})) \subseteq S(\mathcal{C})$ and $S(\mathrm{SHR}^i(\mathcal{C})) \neq S(\mathcal{C})$.

**2. Disjointness of boundaries**

- bd $S(\mathrm{DIL}^i(\mathcal{C})) \cap$ bd $S(\mathrm{DIL}^{i-1}(\mathcal{C})) = \emptyset$ for $i > 0$.

- bd $S(\mathrm{ERO}^i(\mathcal{C})) \cap$ bd $S(\mathrm{ERO}^{i-1}(\mathcal{C})) = \emptyset$ for $i > 0$.

- bd $S(\mathrm{EXP}^i(\mathcal{C})) \cap$ bd $S(\mathrm{EXP}^{i-1}(\mathcal{C})) = \emptyset$ for $i > 1$.

- bd $S(\mathrm{SHR}^i(\mathcal{C})) \cap$ bd $S(\mathrm{SHR}^{i-1}(\mathcal{C})) = \emptyset$ for $i > 1$.

- bd $S(\mathrm{DIL}^i(\mathcal{C})) \cap \mathcal{C} = \emptyset$.

- bd $S(\mathrm{ERO}^i(\mathcal{C})) \cap \mathcal{C} = \emptyset$.

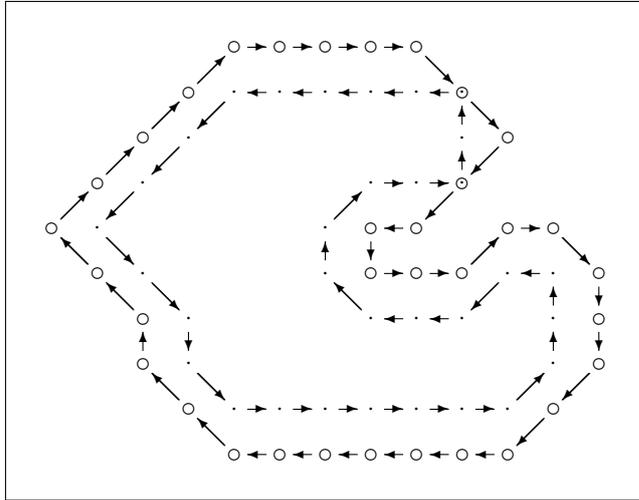- bd $S(\mathrm{EXP}^i(\mathcal{C})) \cap \mathcal{C} = \emptyset$.
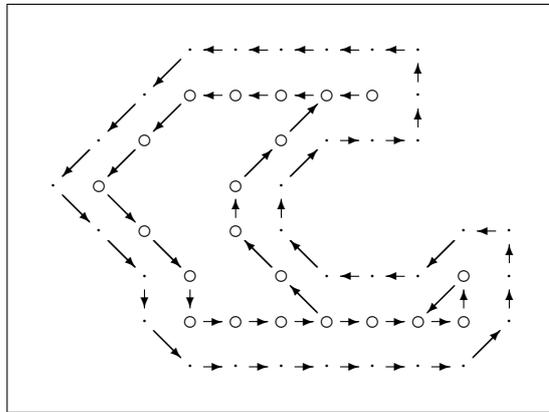
Figure 1.9: Expansion of contour



Figure 1.10: Erosion of contour

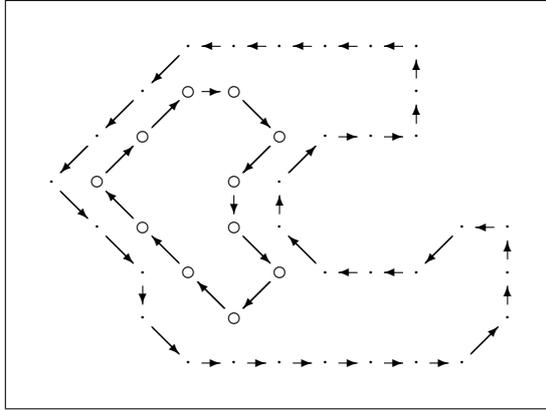Figure 1.11: Shrinking of contour

- bd $S(\mathrm{SHR}^i(\mathcal{C})) \cap \mathcal{C} = \emptyset$ for $i > 0$.

**3. Application of morphological operations**
Let $\mathcal{C}$ be an outer contour Then

- $\mathrm{DIL}^i(\mathcal{C})$ contains exactly one outer contour, and possibly several inner contours.

- $\mathrm{ERO}^i(\mathcal{C})$ contains only outer contours if $\mathrm{ERO}^i(\mathcal{C}) \neq \emptyset$.

- $\mathrm{EXP}^i(\mathcal{C})$ contains exactly one inner contour, and possibly several outer contours.

- $\mathrm{SHR}^i(\mathcal{C})$ contains only inner contours if $\mathrm{SHR}^i(\mathcal{C}) \neq \emptyset$.

Let $\mathcal{C}$ be an inner contour. Then

- $\mathrm{DIL}^i(\mathcal{C})$ contains only inner contours if $\mathrm{DIL}^i(\mathcal{C}) \neq \emptyset$.

- $\mathrm{ERO}^i(\mathcal{C})$ contains exactly one inner contour, and possibly several outer contours.

- $\mathrm{EXP}^i(\mathcal{C})$ contains only outer contours if $\mathrm{EXP}^i(\mathcal{C}) \neq \emptyset$.

- $\mathrm{SHR}^i(\mathcal{C})$ contains exactly one outer contour, and possibly several inner contours.

**4. Generation of singularities**
For $i > 0$ we define the following sets of singular points:

- $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C}) := [S(\mathrm{DIL}^i(\mathcal{C})) \setminus \mathrm{bd}\, S(\mathrm{DIL}^i(\mathcal{C}))] \setminus S(\mathrm{DIL}^{i-1}(\mathcal{C})),$

- $\mathrm{ERO}^i - \mathrm{SING}(\mathcal{C}) := [S(\mathrm{ERO}^{i-1}(\mathcal{C})) \setminus \mathrm{bd}\, S(\mathrm{ERO}^{i-1}(\mathcal{C}))] \setminus S(\mathrm{ERO}^i(\mathcal{C})),$

- $\mathrm{EXP}^i - \mathrm{SING}(\mathcal{C}) := [S(\mathrm{EXP}^i(\mathcal{C})) \backslash \mathrm{bd}\ S(\mathrm{EXP}^i(\mathcal{C}))] \backslash S(\mathrm{EXP}^{i-1}(\mathcal{C}))$, whenever $i > 1$,

   $:= [S(\mathrm{EXP}(\mathcal{C})) \backslash \mathrm{bd}\ S(\mathrm{EXP}(\mathcal{C}))] \backslash S(\mathcal{C})$, if $i = 1$,

- $\mathrm{SHR}^i - \mathrm{SING}(\mathcal{C}) := [S(\mathrm{SHR}^{i-1}(\mathcal{C})) \backslash \mathrm{bd}\ S(\mathrm{SHR}^{i-1}(\mathcal{C}))] \backslash S(\mathrm{SHR}^i(\mathcal{C}))$, whenever $i > 1$,

   $:= [S(\mathcal{C}) \backslash \mathrm{bd}\ S(\mathcal{C})] \backslash S(\mathrm{SHR}(\mathcal{C}))$ if $i = 1$.

Then the following facts hold

- There are contour curves $\mathcal{C}$ with $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C}) \neq \emptyset$ for some $i$.

- There are contour curves $\mathcal{C}$ with $\mathrm{SHR}^i - \mathrm{SING}(\mathcal{C}) \neq \emptyset$ for some $i$.

- For all contour curves $\mathcal{C}$ and for all $i$ is $\mathrm{EXP}^i - \mathrm{SING}(\mathcal{C}) = \emptyset$ and $\mathrm{ERO}^i - \mathrm{SING}(\mathcal{C}) = \emptyset$.

- The sets $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C})$ and $\mathrm{SHR}^i - \mathrm{SING}(\mathcal{C})$ contain no inner points. Therefore the contour curves of $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C})$ contain all points of $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C})$ and the contour curves of $\mathrm{SHR}^i - \mathrm{SING}(\mathcal{C})$ contain all points of $\mathrm{SHR}^i - \mathrm{SING}(\mathcal{C})$

# Chapter 2

# Covering the Plane by Curves

The aim of this Chapter is to investigate families of contour curves which are generated by applying morphological operations to a single curve.

Let $\mathcal{C}$ be a curve. Since d–d–vertices cannot occur on a curve, we only have to consider i–d– and i–i–vertices and consequently d–d–d–transitions as well as d–d–i–transitions are not possible on a curve.

One might ask under which conditions the set $\mathrm{DIL}(\mathcal{C})$ is a curve when $\mathcal{C}$ was assumed to be a curve. This question is investigated in Section 1. The subsequent Sections are concerned with families of curves generated by a single curve by iterated application of the DIL–operation to a curve.

## 2.1    Neighboring Curves

For an (oriented) digital curve $\mathcal{C}$ we denote by $-\mathcal{C}$ the curve obtained by reversing the orientation of $\mathcal{C}$. This operation corresponds to the operation of of replacing $S(\mathcal{C})$ by $\mathbb{C}S(\mathcal{C}) \cap \mathcal{C}$. From $\mathcal{C}$ we generate two families of contour curves, $\mathrm{DIL}^i(\mathcal{C})$ and $\mathrm{DIL}^i(-\mathcal{C})$. Our first objective is the question under which conditions both contour curves are again curves.

**Theorem 2.1** *Given a curve $\mathcal{C}$. Both $\mathrm{DIL}(\mathcal{C})$ and $\mathrm{DIL}(-\mathcal{C})$ are curves if for all convex transitions of $\mathcal{C}$ having a central i–piece the following is true*

- *For an i–i–i–transition the central piece has at least four points,*

- *For all other such transitions the central i–piece has at least three points.*

**Proof**  Observe that in case of a curve having a convex transition with a central i–piece the length (i.e. the number of points contained in it) of the corresponding i–piece of $\mathrm{DIL}(\mathcal{C})$ or of $\mathrm{DIL}(-\mathcal{C})$ is reduced by one. For S–transitions and
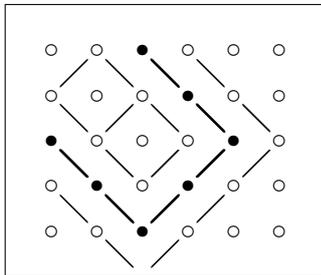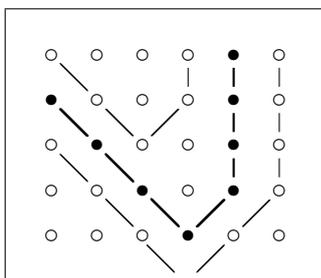
Figure 2.1: Type i–i–i, length of central piece 3



Figure 2.2: Type i–i–d, length of central piece 2

for convex i–d–i–transtitions the lengths of the corresponding central pieces of
DIL($\mathcal{C}$) and of DIL($-\mathcal{C}$) are not changed (see Figure 2.4).

In Figures 2.1, 2.2 and 2.3 the critical configurations of the Theorem are
illustrated.

As a consequence of Theorem 2.1, if a curve $\mathcal{C}$ contains a convex transition
with a central i–piece, then after a certain number of dilations either DIL($\mathcal{C}$) or
DIL($-\mathcal{C}$) is not a curve.

Given a family of sets consisting of curves and *singular sets* which are 8–
connected sets having one, two or three points each. We note that the smallest
8–curve has at least four points. The plane is said to be *covered* by the family of
sets if each point of the plane belongs to exactly one set of the family. A family
of sets is said to cover the plane *simply* if for each point in the plane which is on
a curve there exist exactly two different members of the family such that this
point is 4–adjacent, but does not belong, to them.

If we define the *multiplicity* $\mu(P)$ of a point $P$ with respect to a given covering
to be the number of different sets not containing $P$ and being 4–adjacent to it,
we can state that a family of sets covers the plane simply if each point on a curve
has multiplicity two. A point in the plane having multiplicity greater than two
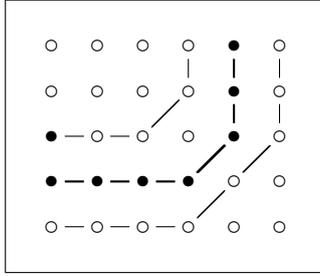is termed a *multiple point* of the covering.
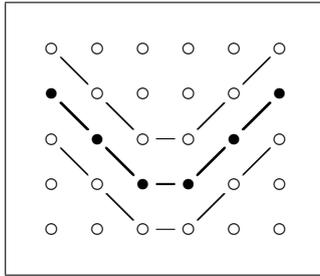
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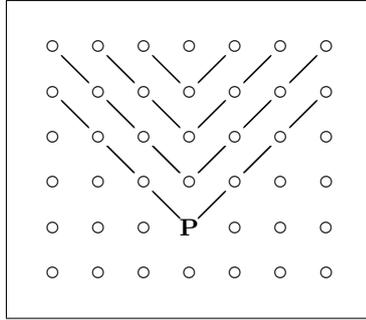Figure 2.3: Type d–i–d, length of central piece 2



Figure 2.4: Type i–d–i

Figure 2.5: Covering containing i–i–vertex

If for each curve of a family both sets DIL($\mathcal{C}$) and DIL($-\mathcal{C}$) also belong to the family, then the family covers the plane simply. The converse is also true:

**Theorem 2.2** *Given a curve $\mathcal{C}$ of a family covering the plane simply. Then DIL($\mathcal{C}$) belongs to the family if it is a curve. A similar assertion holds for DIL($-\mathcal{C}$).*

The follwoing assertion is clear:

**Theorem 2.3** *If a covering contains a finite curve then it contains at least one singular point.*

A *regular* covering of the plane by a family of curves is a covering without singular sets.

**Theorem 2.4** *Given a regular covering of the plane. If one of the curves $\mathcal{C}$ of the covering contains an i–i–vertex (or a d–i–vertex) then all curves contained in the convex side with respect to this vertex of the complement of $\mathcal{C}$ contain the same i–i–vertex (or d–i–vertex), suitably translated horizontally or vertically.*

The assertion of the Theorem is not necessarily true for coverings having singular points. Consider for example the simple covering in Figure 2.6 (the singular point is indicated by an '•'):

**Theorem 2.5** *Given a family of curves regularly and simply covering the plane. Then the convex transitions of the curves on the covering have exclusively type i–d–i.*

**Theorem 2.6** *Given a covering of the plane such that all curves of the covering family are finite. If there exists exactly one singular point then the covering is simple.*
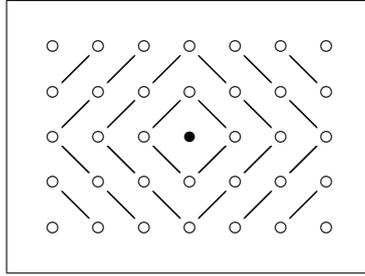
Figure 2.6: Covering containing singular point (denoted '•')

## 2.2 Multiple Points

When there exists a quadruple point in a covering of the plane without singular points then the covering has very simple properties.

**Theorem 2.7** *Given a regular covering of the plane. There are at most two quadruple points contained in the covering. Let $P$ be a quadruple point of a regular covering. Then all curves in the whole plane are determined by the curves in $N(P)$. In the latter case either two of the direct neighbors of $P$ are triple points or one direct neighbor of $P$ is a quadruple point. All other points in the plane have multiplicity two.*

**Proof** One can enumerate the following situations:
    For investigating triple points, we first prove a Lemma.

**Lemma 2.1** *Given a covering of the plane without singular points.*
    *Assume that there exists a horizontal or vertical line in the digital plane such that for each point on this line both curve–neighbors are on different sides with respect to the line.*
    *Then there exist in the whole digital plane at most four triple points. The total number of triple points in the plane is even. None of the triple points is isolated (we understand here that a quadruple point counts for two triple points).*
    *If there exists one quadruple point, then by Theorem 2.7 all points with multiplicity $> 2$ are direct neighbors of it.*

There are three possibilities for a triple point $P$:

1. $P$ has a curve–neighbor which is a direct neighbor of $P$.

2. $P$ has only indirect neighbors for curve–neighbors. Both curve–neighbors of $P$ are diametrically opposite.

3. $P$ has only indirect neighbors for curve–neighbors. Both curve–neighbors of $P$ are not diametrically opposite.

We conclude from the foregoing discussion:

**Theorem 2.8** *Whenever there is a triple point in a regular covering of the plane, then one of the direct neighbors of this point is also a triple point or a quadruple point.*

We now can prove

**Theorem 2.9** *In a regular covering of the plane there exists always an even number of triple points. Each triple point has a direct neighbor which is also a triple point. The total number of triple points is never greater than four.*
*Each quadruple point is counted for two triple points here.*

A different formulation is

**Theorem 2.10** *For a regular covering of the plane the following sum ranging over all points in the digital plane*

$$\sum (\mu(P) - 2)$$

*is always finite and attains only values 0, 2 or 4.*
*Each point having multiplicity greater than two has at least one direct neighbor of multiplicity greater than two.*

**Remark 2.1** *If the sum attains value 0, the covering is simple. Value 2 is attained for two triple points. For obtaining value 4, we need two quadruple points or one quadruple point and two triple points or else four triple points.*
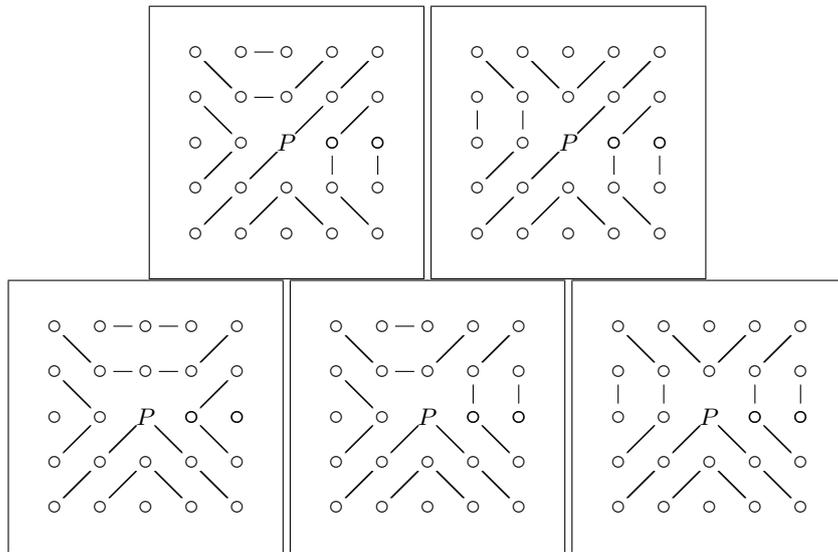
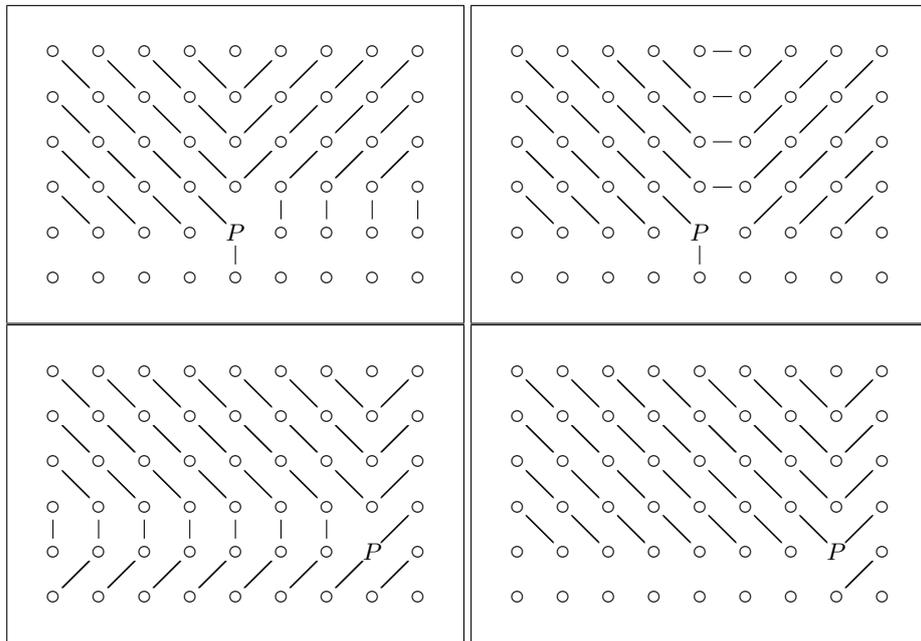Figure 2.7: Neighborhood configurations for a quadruple point



Figure 2.8: Neighborhood configurations for a triple point

# Chapter 3

# Covering the Plane by Contour Curves

## 3.1 Morphological Operations on Contour Curves

**Definition 3.1** *A family $C$ of pairwise disjoint contour curves is termed a c–covering (contour covering) of the digital plane if and only if for any point $P$ of the digital plane there exists a contour curve $\mathcal{C} \in C$ with $P \in \mathcal{C}$.*

The following Theorems follow immediately from the properties of morphological operations:

**Theorem 3.1** *The family of contour curves generated from a contour curve $\mathcal{C}$ by*
$$\text{EREX}(\mathcal{C}) := \{\mathcal{C}' \mid \mathcal{C}' \in \text{ERO}^i(\mathcal{C}) \quad \vee \quad \mathcal{C}' \in \text{EXP}^{i+1}(\mathcal{C}), \quad i \geq 0\}$$
*is a c–covering of the digital plane.*

**Theorem 3.2** *The family of contour curves generated from a contour curve $\mathcal{C}$ by*

$$\text{DISH}(\mathcal{C}) := \quad \{\mathcal{C}' \mid \mathcal{C}' \in \text{DIL}^i(\mathcal{C}) \quad \vee \quad \mathcal{C}' \text{ is a contour curve of } \text{DIL}^i - \text{SING}(\mathcal{C})$$
$$\vee \quad \mathcal{C}' \text{ is a contour curve of } \text{SHR}^{i+1} - \text{SING}(\mathcal{C}), \quad i \geq 0\}$$

*is a c–covering of the digital plane.*

If $\text{DISH}(\mathcal{C})$ contains contour curves of $\text{DIL}^i - \text{SING}(\mathcal{C})$ or of $\text{SHR}^i - \text{SING}(\mathcal{C})$ for some $i$, we call $\text{DISH}(\mathcal{C})$ a c–covering with singularities.

We now are going to characterize contour curves $\mathcal{C}$ which generate c–coverings $\text{DISH}(\mathcal{C})$ without singularities.

At first we observe that a point $P$ belongs to $\text{DIL}^i - \text{SING}(\mathcal{C})$ if and only if

1. $P \notin S(\text{DIL}^{i-1}(\mathcal{C}))$,

2. there is a direct neighbor $Q$ of $P$ with $Q \in S(\mathrm{DIL}^{i-1}(\mathcal{C}))$,

3. $P \notin \mathrm{bd}\, S(\mathrm{DIL}^i(\mathcal{C}))$.

If we analyze the properties 1., 2. and 3., we find that there exist only a few local constellations of contour points of $\mathrm{DIL}^{i-1}(\mathcal{C})$ which generate a singular point. These constellations (up to $90^o$–rotations and reflections) are shown in Figure 3.1

We call these local constellations of contour points *antipodal constellations* because they characterize the existence of opposite contour segments which generate "thin parts" of that component of $\mathbb{C}S(\mathrm{DIL}^{i-1}(\mathcal{C}))$ surrounded by the contour under consideration of $S(\mathrm{DIL}^{i-1}(\mathcal{C}))$.

**Definition 3.2** *We call a convex transition of a contour curve $\mathcal{C}$ a* convexity *of $\mathcal{C}$ if $S(\mathcal{C})$ lies on the convex side with respect to that transition.*

*Analogously a convex transition of $\mathcal{C}$ is called a* concavity *of $\mathcal{C}$ if $S(\mathcal{C})$ lies on the nonconvex side with respect to that transition.*

**Proposition 3.1** *Let $\mathcal{C}$ be a contour curve with the property that all concavities of $\mathcal{C}$ are of type i–d–i. Then $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C}) = \emptyset$ for all $i > 0$.*

**Proof** Sketch of proof: Contour curves with concavities only of type i–d–i do not contain antipodal constellations, therefore $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C}) = \emptyset$. Furthermore $\mathrm{DIL}^i(\mathcal{C})$ consists of a unique contour curve $\mathcal{C}'$ and $\mathcal{C}'$ again has only concavities of type i–d–i. Thus by induction the proof is complete. □

**Proposition 3.2** *If for a contour curve $\mathcal{C}$ and for all $i > 0$ the sets $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C})$ are empty, then $\mathcal{C}$ contains only concavities of type i–d–i.*

**Proof** Sketch of proof: Let $\mathcal{C}$ contain concavities which are not of type i–d–i. Then the central piece of such concavities is an i–piece (because contour curves cannot have concavities of type d–d–d, d–d–i, or i–d–d).

Case 1: There is a central i–piece of a concavity of $\mathcal{C}$ which has length 2. Then thrree is an antipodal constellation, and $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C})$ is not empty.

Case 2: The minimal length $\ell$ of the central i–pieces of concavities of $\mathcal{C}$ is greater than 2. Then the contour curves of $\mathrm{DIL}^i(\mathcal{C})$ contain concavities with central i–pieces of length $\ell - 1$. Thus it follows, that at least $\mathrm{DIL}^{\ell-1} - \mathrm{SING}(\mathcal{C}) \neq \emptyset$. □

**Corollary 3.1** *For any inner contour curve $\mathcal{C}$ there exists an $i > 0$ with $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C}) \neq \emptyset$.*

**Theorem 3.3** *Let $\mathcal{C}$ be a contour curve. Then $\mathrm{DISH}(\mathcal{C})$ is a c–covering without singularities if and only if*

- $\mathcal{C}$ *is an outer contour having only concavities of type i–d–i and $S(\mathcal{C})$ has no inner points, or*

23

- $\mathcal{C}$ is an infinite contour having only concavities of type i–d–i and having no concavities with a central i–piece.

The proof of this theorem uses propositions 3.1 and 3.2.

**Proposition 3.3** *If for a contour curve $\mathcal{C}$ there is no index $i$ such that $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C}) = \emptyset$ for all $j \geq i$ then $\mathcal{C}$ is an infinite contour curve.*

**Proposition 3.4** *If $\mathcal{C}$ is a contour curve having only concavities of type i–d–i then the c–covering $\mathrm{DISH}(\mathcal{C})$ contains at most a finite number of singular points.*

**Proposition 3.5** *If $\mathcal{C}$ is a finite contour curve then the c–covering $\mathrm{DISH}(\mathcal{C})$ contains at most a finite number of singular points.*

## 3.2    D–convexity

In the usual definition of convexity we call a set $M$ convex if and only if for any straight line segment $s$ (of arbitrary direction) whose end points belong to $M$ the inclusion $s \subseteq M$ is true. By restriction the line segments $s$ to those with given directions we get the notion of "directional convexity" as a generalization of the common convexity notion:

**Definition 3.3** *For a given class* $\mathrm{DIR}$ *of directions a set $M$ is called* directional convex with respect to $\mathrm{DIR}$ *or shortly* $\mathrm{DIR}$*–convex, if and only if for any straight line segment $s$ with a direction from* $\mathrm{DIR}$ *whose end points belong to $M$ the inclusion $s \subseteq M$ is true.*

Important special cases follow according to the choice of DIR

- to be the class of vertical and horizontal directions (*orthogonal convexity*),

- or to be the class of diagonal directions with slope $\pm 1$ (*diagonal convexity*).

If we restrict the above considerations in a natural way to the digital plane:

- $M$ is a set of points of the digital plane,

- $s$ is the set of all grid points which lie on a given real straight line,

we get an analogous definition for the digital plane.

In our further considerations only the diagonally convex (or shortly D–convex) digital sets will be considered. Figure 3.2 shows an example.

**Corollary 3.2** *Any 4–connected D–convex set has no holes.*

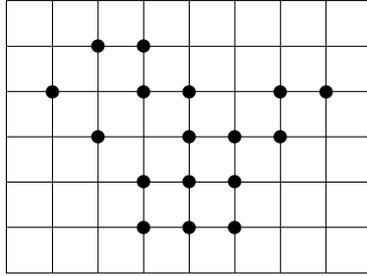Note that 8–connected D–convex sets may have holes as can be seen in Figure 3.2.

Figure 3.2: An 8–connected digital D–convex set

**Proposition 3.6** *A 4–connected set M is D–convex if and only if M has no holes and the contour $\mathcal{C}$ of M contains no concavity with a central i–piece.*

**Proof** a) Let $M$ be a 4–connected and D–convex set. Then $M$ cannot have holes. Let $\mathcal{C} = \{P_i\}$ be the contour of $M$. Assume that $\mathcal{C}$ has a concavity with a central i–piece $A = P_j, P_{j+1}, \cdots, P_k$.

We consider the diagonal line $g$ through the point $P_{j-1}$ with the same direction as the line determined by the points $P_j$ and $P_k$.

Case 1: $P_{k+1} \in g$. Then $M$ is not D–convex.

Case 2: $P_{k+1} \notin g$. Then we have one of the following two situations:
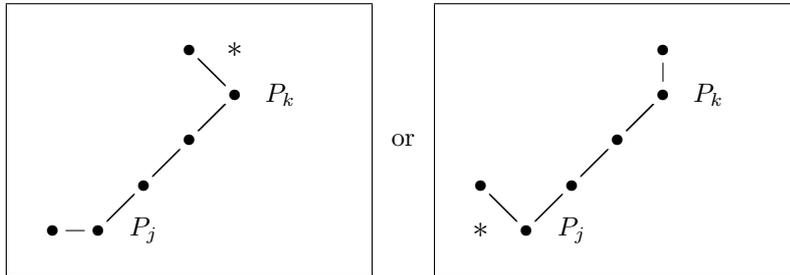


Because of 4–connectivity and because $M$ has no holes, the point marked "∗" belongs to $M$, and therefore we have a contradiction to the D–convexity of $M$.

b) Let $M$ be a 4–connected set without holes and let the contour $\mathcal{C} = \{P_i\}$ of $M$ have no concavities with a central i–piece.

Assume that $M$ is not D–convex. Then there is a diagonal line $g$ and points $P$, $Q$ and $R$ with $P, Q, R \in g$, $Q$ between $P$ and $R$, $P, R \in M$, and $Q \notin M$. Without loss of generality let $P$ and $R$ be contour points of $M$ such that all points between $P$ and $R$ do not belong to $M$. Because $M$ has no holes and is 4–connected the contour segment between $P$ and $R$ contains concavities. The assumption that all these concavities have no central i–piece contradicts the fact that the points $P$ and $R$ lie on a diagonal line, and the points between $P$ and $R$ do not belong to $M$. □

For 8–connected sets $M$ without holes this proposition is not true because it is possible to find "pathological" examples of 8–connected sets without holes which are D–convex, but whose contour has a concavity with a central i–piece. Figure 3.3 shows an example.
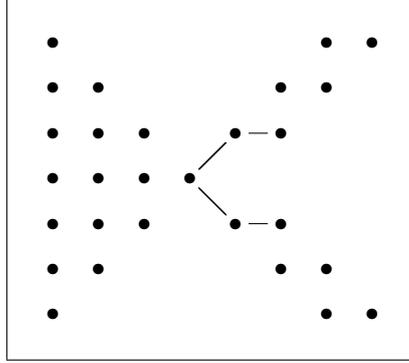


Figure 3.3: An 8–connected D–convex set without holes whose contour has a concavity with a central i–piece

**Theorem 3.4** *Let $\mathcal{C}$ be the contour curve of a 4–connected set $M$ without holes. Then $M$ is D–convex if and only if $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C}) = \emptyset$ for all $i > 0$.*

**Corollary 3.3** *Let $\mathcal{C}$ be a contour curve of an 8–connected set $M$ without holes and with $\mathrm{DIL}^i - \mathrm{SING}(\mathcal{C}) = \emptyset$ for all $i > 0$. Then $M$ is D–convex.*

**Proposition 3.7** *Let $\mathcal{C}$ be an outer contour curve. Then there is an index $i$ such that $\mathrm{DIL}^i(\mathcal{C})$ contains only one outer contour $\mathcal{C}'$ for which $S(\mathcal{C}')$ is a D–convex set.*

**Proposition 3.8** *Let $\mathcal{C}$ be an outer contour curve of a D–convex set. Then all contours in $\mathrm{ERO}^i(\mathcal{C})$ for $i > 0$ are outer contours of D–convex sets.*

**Definition 3.4** *Let $M$ be a set of points in the digital plane. The set $\mathrm{DH}(M)$ is called the* D–convex hull *of $M$ if and only if*

- *$M \subseteq \mathrm{DH}(M)$,*

- *$\mathrm{DH}(M)$ is D–convex,*

- *For all D–convex sets $M'$ with $M \subseteq M'$ is $M' \supseteq \mathrm{DH}(M)$.*

One can prove that the D–convex hull of a set $M$ always exists.

**Theorem 3.5** *Let $\mathcal{C}$ be an outer contour and let $i$ be an index such that $\mathrm{DIL}^i(\mathcal{C})$ contains only one contour $\mathcal{C}'$ which is the outer contour of a D–convex set. Then the set $\mathrm{ERO}^i(\mathcal{C}')$ contains exactly one outer contour $\mathcal{C}''$ and $\mathrm{DH}(S(\mathcal{C})) = S(\mathcal{C}'')$.*

The proof of this theorem uses propositions 3.7 and 3.8.

## 3.3 Medial Axis Transformation

Let $d$ be a digital metric. Then $d(P,Q)$ denotes the distance between the points $P$ and $Q$. For a poit set $M$ and an point $P$ we denote by $d(P,M)$ the smallest distance between $P$ and a point of $M$.

**Definition 3.5** *For a set $M$ of the digital plane the* medial axis transform $\mathrm{d}-\mathrm{MAT}(M)$ *with respect to a digital metric $d$ is defined by*

$$d - \mathrm{MAT}(M) \quad := \quad \{P \mid P \in M \wedge d(P, \mathbb{C}M) \geq d(Q, \mathbb{C}M)$$
$$\textit{for all } Q \in M \textit{ with } d(P,Q) = 1\}$$

For a contour curve $\mathcal{C}$ we set $d - \mathrm{MAT}(\mathcal{C}) := d - \mathrm{MAT}(S(\mathcal{C}))$.
For $d = d_4$ we have

$$d_4 - \mathrm{MAT}(M) \quad := \quad \{P \mid P \in M \wedge d_4(P, \mathbb{C}M) \geq d_4(Q, \mathbb{C}M)$$
$$\textit{for all direct neighbors } Q \in M \textit{ of } P\}.$$

We restrict our considerations to the $d_4 - \mathrm{MAT}$ and write shortly MAT instead of $d_4 - \mathrm{MAT}$.

**Proposition 3.9** *For a contour curve $\mathcal{C}$ and for $i > 0$ holds*

$$S(\mathrm{SHR}^i(\mathcal{C})) \subseteq \{P \mid P \in S(\mathcal{C}) \wedge d_4(P, \mathbb{C}S(\mathcal{C})) > i\}.$$

**Theorem 3.6** *For any contour curve $\mathcal{C}$ holds*

$$\mathrm{MAT}(\mathcal{C}) \quad = \quad \bigcup_{i > 0} \mathrm{SHR}^i - \mathrm{SING}(\mathcal{C})$$
$$\cup \{P \mid P \textit{ is a d–d–vertex or a sharp d–i–vertex of } \mathcal{C}\}$$

The proof of this theorem uses Proposition 3.9 and the fact that $\mathrm{SHR}^i - \mathrm{SING}(\mathcal{C})$ contains exactly those points of $S(\mathcal{C})$ which belong to $S(\mathrm{SHR}^{i-1}(\mathcal{C}))$ but not to bd $\mathrm{SHR}^{i-1}(\mathcal{C})$ and not to $S(\mathrm{SHR}^i(\mathcal{C}))$.
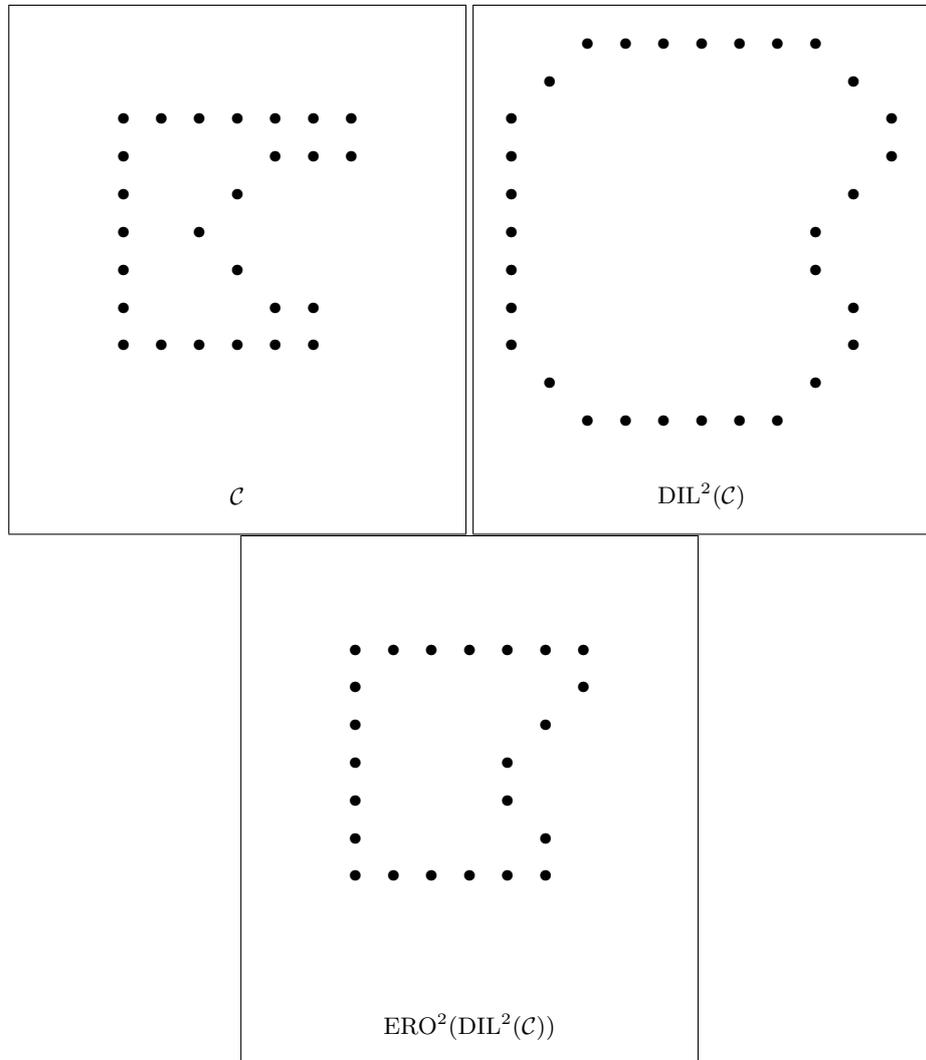
Figure 3.4: Example

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