

Representation of Plane Curves
by Means of Descriptors
in Hough Space
I. Continuous Theory

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Abstract

The Hough (or Radon) transformation is specifically well suited for identifying lines in digital images in a robust way. It can be understood as an evidence accumulation method and it is insensitive to noise, even interrupted (dashed) lines can be detected by it. On parallel computers it can be implemented very efficiently.

There exists a very large number of papers on theory and application of the Hough transform and also on different generalizations of it. For the 'classical' transform for lines there are two different interpretations in the literature:

- The Radon transform is an integral transform mapping a function of two variables into another function of two variables.
- The Hough transform maps certain objects in the image (lines) into certain objects in Hough space (maxima).

In the present paper the Hough transform for binary images is understood as a mapping of curves in the image into curves in Hough space. This approach leads to a new understanding of the geometry in Hough space. Especially we will find a common unifying framework for different known constructions in the literature. We mention as examples:

- The descriptors of plane curves as defined by W. Scherl correspond in a quite natural way to points in Hough space. This observation will lead to a very efficient approximate method for implementing the transformation.
- When considering boundary curves of convex sets, one gets extremely simple relationships. Specifically, as already observed by Scherl, there result very simple methods for approximating the convex hull of a plane curve.
- The transformations described by Bernhardt can be understood within the framework of this theory in a very natural way. These transformations are successfully used in a practical environment.

Chapter 1

Introduction

The Radon transform [3] of a function $\beta(x, y)$ of two variables maps the plane into the cylinder $[0, 2\pi) \times (-\infty, \infty)$. For any angle $\alpha \in [0, 2\pi)$ and a parameter $p \in (-\infty, \infty)$ the value of the Radon transform is the line integral of $\beta(x, y)$ along the line given by the equation $x \sin \alpha - y \cos \alpha = p$.

Hough ([5]) observed that in binary images (i.e. the range of β consists of the values 0 and 1 only) the maxima of the Radon transform correspond to lines in the image. He used this observation for the automatic determination of lines in bubble chamber photographs. In the image processing literature the Radon transform is therefore usually denoted as Hough transform. A serious obstacle for the application of the Hough transform and its generalizations is the considerable amount of computer time which has to be spent on general purpose computers. Numerous authors attempted to reduce this amount by using suitable storage techniques, dedicated computer architectures or mathematical tricks (see [4]).

In 1983 Scherl [7] (see also [12]) proposed to characterize a two dimensional curve by a set of suitably chosen boundary points equipped with certain attributes, so-called descriptors. By means of this technique a considerable reduction of the amount of data as compared with the original image is achieved. Moreover, the descriptors carry structural information which can be used for image understanding. The descriptors can be extracted from the image in a very simple way which makes this approach extremely attractive.

The aim of this paper is to show that these descriptors can be imbedded into the image space of the Hough transform in a quite natural way.

A different approach which deserves mentioning is due to Bernhardt [1]. It also starts with the boundary curves of objects in the image. By means of a special variant of the Hough transform a very efficient approach for analyzing the content of an image is given.

The whole paper consists of three parts. The first part is devoted to Hough images of continuous curves in the original image. In the second part numerical problems arising by discretization are discussed and also some questions of practical implementation of the method are investigated. In the last part it is shown by means of realistic applications that the descriptors of Scherl can indeed be used as an efficient tool for analysis, syntactic description and understanding of images.

Notation For any set M in the plane denote by
 $\text{cl } M$ the topological closure of M ,

$\text{int } M$	the topological interior of M ,
$\text{bd } M$	the boundary of M ($\text{bd } M = \text{cl } M \setminus \text{int } M$),
$\text{conv } M$	the convex hull of M , i.e. the set of all convex combinations of points of M .

Throughout the text P is a point in the plane with coordinates x and y , and P_0 has coordinates x_0, y_0 etc.

Chapter 2

Descriptors of a Plane Curve

Given a curve K in the plane in parameter representation:

$$P(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad 0 \leq t \leq L. \quad (2.1)$$

Assumption 2.1 K is continuous and piecewise smooth, i.e. $P(t)$ is continuous and there are finitely many parameters t_i such that $\dot{P}(t)$ is continuous in each open interval containing none of the t_i .

Assumption 2.2 K contains no multiple points: $P(s) \neq P(t)$ for $s \neq t$; for closed curves we require $P(0) = P(L)$.

Let $P_0 = P(t_0)$ be a point of differentiability of K with $\dot{x}(t_0)^2 + \dot{y}(t_0)^2 > 0$. Then

$$P_0 + \omega(\dot{x}(t_0), \dot{y}(t_0)), \quad -\infty < \omega < \infty \quad (2.2)$$

is the equation of the *tangent* at K in P_0 . Define

$$\begin{aligned} \dot{x}(t) &= g(t) \cos \alpha(t), \\ \dot{y}(t) &= g(t) \sin \alpha(t) \end{aligned} \quad (2.3)$$

then the tangent can be defined as the set of all $P = (x, y)$ with

$$\begin{aligned} x \sin \alpha(t_0) - y \cos \alpha(t_0) &= p(t_0), \\ p(t_0) &:= x(t_0) \sin \alpha(t_0) - y(t_0) \cos \alpha(t_0). \end{aligned} \quad (2.4)$$

For this representation of the tangent we do not explicitly require that $P(t)$ is differentiable.

Assumption 2.3 All differentiable points of the curve are assumed to be regular points i.e. $g(t) \neq 0$.

If the parameter t is the *arc length* of the curve then $g(t) = 1$ for all t and L is the length of K .

Assumption 2.4 t is assumed to be the arc length parameter.

Definition 2.1 If K is not differentiable at $P_0 \in K$, then this point is called a vertex of K .

By requiring that K is piecewise smooth we assume that K has at most a finite number of vertices.

Definition 2.2 Let $P \in K$ and assume that a tangent in P at K is given by the equation

$$x \sin \alpha - y \cos \alpha = p \tag{2.5}$$

then the pair (p, α) is termed a Hough descriptor or H-descriptor of K at descriptor point P . The (p, α) -plane is termed the Hough space and the set of Hough descriptors is the Hough-transform of K and is denoted by $H[K]$.

Scherl [7] defines a descriptor to be a triple consisting of the coordinates of the descriptor point and the angle α . When there is no danger of confusion we understand here a descriptor to be a Hough descriptor. There exists a one-to-one correspondence between Hough descriptors and descriptors in Scherl's sense (see Chapter 9).

Remark 2.1 If K is differentiable at t then the descriptor at $P(t)$ is uniquely given by $(p(t), \alpha(t))$ according to (2.3) and (2.4). If K contains a line segment, then only a single descriptor belongs to it.

Obviously the curve $(p(t), \alpha(t))$ is continuous at each point of differentiability of K . In Chapter 5 we will show how $H[K]$ can be extended in a continuous way for all α .

In Figure 1 a plane figure is presented and in Figure 2 the transform of the boundary of it is given. Observe that the transformed curve contains a double point.

Chapter 3

Classification of Descriptors

Assumption 3.1 *The plane is partitioned by K into two topologically open regions, this means that K is a Jordan curve.*

Obviously Assumption 2.2 has to be fulfilled in order that Assumption 3.1 can be true. One of these regions we assume to be the *interior* $I(K)$ (= set of ‘black’ or foreground points) and its complement (without the boundary curve) the *exterior* $A(K)$ (= set of ‘white’ or background points).

Assumption 3.2 *K is oriented in such a way that the interior of K lies on the left-hand side when K is passed in the direction of growing t -values.*

Definition 3.1 $P_0 = P(t_0) \in K$ is said to be a *T-point* (*Top-Point*) if there exists a descriptor (p, α) at P_0 and a neighborhood U of P_0 such that for all $P \in U$ belonging to $I(K)$ one has

$$x \sin \alpha - y \cos \alpha \leq p \quad (3.1)$$

$P_0 = P(t_0) \in K$ is said to be an *S-point* (*Saddle-Point*), if there exists a descriptor (p, α) at P_0 and a neighborhood U of P_0 such that for all $P \in U$ belonging to $A(K)$ one has

$$x \sin \alpha - y \cos \alpha \geq p \quad (3.2)$$

If a point on K is a *T-point* as well as an *S-point*, it is termed an *L-point* (*Line-Point*). If a point on K is neither a *T-point* nor an *S-point*, it is termed an *I-point* (*Indifferent Point*). Denote by X the set of all *X-points* ($X = T, S, L, I$).

Theorem 3.1 $P \in K$ is an *L-point* if and only if there is a neighborhood U of P such that K is a line in U .

Proof 1. Let P be a *T-point* with corresponding neighborhood U_1 and simultaneously an *S-point* with neighborhood U_2 . In the intersection of both neighborhoods K is a line by (3.1) and (3.2).

2. If K is a line in U then obviously P is a *T-point* as well as an *S-point*. □

As a direct consequence of Theorem 3.1 we get:

Theorem 3.2 *L is an open set in the relative topology induced on K by the norm topology of the Euclidean plane.*

Definition 3.2 *A T-descriptor (p, α) is termed extreme T-descriptor if for any other T-descriptor (p', α) (with the same α) one has $p' \leq p$. Denote the set of all descriptor points of extreme T-descriptors by ET.*

An S-descriptor (p, α) is termed extreme S-descriptor if for any other S-descriptor (p', α) (with the same α) one has $p \leq p'$. Denote the set of all descriptor points of extreme S-descriptors by ES.

Theorem 3.3 *ET and ES are closed.*

Proof Given $P \in ET$ with descriptor (p, α) . Then for all $P' \in K$ one has

$$(x - x') \sin \alpha - (y - y') \cos \alpha \leq 0. \quad (3.3)$$

The set of all solutions (x, y) of this system of inequalities is closed. By intersection with the closed set K one gets ET . The proof for ES is analogous. \square

Remark 3.1 *Scherl [12] introduced two additional types of descriptors which he denoted as B- and F-descriptors. For sake of simplicity we do not introduce this refinement here.*

Chapter 4

Duality

If we interchange $I(K)$ and $A(K)$ then a duality relation holds.

Theorem 4.1 *Let the curve K defining $I(K)$ and $A(K)$ be oriented in the reverse direction i.e. $t^* = -t$, $\alpha^* = \alpha + \pi$, $p^* = -p$ (Notation: $K \rightarrow K^*$). Let $I^*(K) = I(K^*) = A(K)$ and $A^*(K) = A(K^*) = I(K)$. T^*, S^* etc. are the descriptor sets of $I^*(K)$ and $A^*(K)$. Then*

$$\begin{aligned}
 (p^*, \alpha^*) \in T^* &\iff (p, \alpha) \in S \\
 (p^*, \alpha^*) \in S^* &\iff (p, \alpha) \in T \\
 (p^*, \alpha^*) \in ET^* &\iff (p, \alpha) \in ES \\
 (p^*, \alpha^*) \in ES^* &\iff (p, \alpha) \in ET \\
 (p^*, \alpha^*) \in L^* &\iff (p, \alpha) \in L \\
 (p^*, \alpha^*) \in I^* &\iff (p, \alpha) \in I
 \end{aligned} \tag{4.1}$$

Proof Let (p^*, α^*) be a T^* -descriptor at $P(t_0)$. Then for all P in the intersection of a neighborhood of $P(t_0)$ and $I(K^*)$ is

$$x \sin \alpha^* - y \cos \alpha^* \leq p^*, \tag{4.2}$$

or

$$x \sin \alpha - y \cos \alpha \geq p \tag{4.3}$$

for all $P \in A(K)$, hence $(p, \alpha) \in S$. Let $(p^*, \alpha^*) \in ET^*$. Then $p^{*'} \leq p^*$ for all $(p^{*'}, \alpha^*) \in T^*$. We get $(p, \alpha) \in ES$ as above. The other cases are proved in an analogous way. \square

Remark 4.1 *Scherl represents this duality relation by means of allowed transitions between convex and concave cycles of descriptors ([11, page 55]).*

Chapter 5

Descriptors of Vertices

In case of vertices we formally adopt Definition 3.1 of T - and S -points.

Assumption 5.1 *Given a vertex $P_0 = P(t_0)$ of K . Assume that both the left and right limits of $\alpha(t)$ exist for $t \rightarrow t_0$.*

Definition 5.1 *Given a vertex $P_0 = P(t_0)$ of K . Let*

$$\alpha_L = \lim_{t \rightarrow t_0^-} \alpha(t), \quad p_L = x_0 \sin \alpha_L - y_0 \cos \alpha_L \quad (5.1)$$

and

$$\alpha_R = \lim_{t \rightarrow t_0^+} \alpha(t), \quad p_R = x_0 \sin \alpha_R - y_0 \cos \alpha_R. \quad (5.2)$$

the descriptors (p_L, α_L) and (p_R, α_R) are termed limit descriptors of the vertex. All descriptors $(p(\omega), \alpha(\omega))$ with

$$\alpha(\omega) = \omega \alpha_R + (1 - \omega) \alpha_L \quad (5.3)$$

$$p(\omega) = x_0 \sin \alpha(\omega) - y_0 \cos \alpha(\omega) \quad (5.4)$$

and $0 < \omega < 1$ are termed vertex descriptors.

Obviously the vertex descriptors yield a continuous continuation of the curve $H[K]$ in Hough space.

Without loss of generality we assume:

Assumption 5.2 $0 \leq \alpha_L \leq 2\pi$ and $-\pi < \alpha_R - \alpha_L \leq \pi$.

The following Theorem holds

Theorem 5.1 *For $\alpha_L < \alpha_R$ all vertex descriptors are T -descriptors, for $\alpha_R < \alpha_L$ all vertex descriptors are S -descriptors. (For $\alpha_R = \alpha_L$ we have not a vertex at all).*

Proof Consider for $0 < \omega < 1$ the function

$$F(t) = x(t) \sin \alpha(\omega) - y(t) \cos \alpha(\omega). \quad (5.5)$$

By (2.3) is

$$\begin{aligned} \dot{F}(t) &= \dot{x}(t) \sin \alpha(\omega) - \dot{y}(t) \cos \alpha(\omega) = \\ &= g(t) \sin(\alpha(\omega) - \alpha(t)), \end{aligned} \quad (5.6)$$

moreover

$$F(t_0) = x_0 \sin \alpha(\omega) - y_0 \cos \alpha(\omega) =: p(\omega) \quad (5.7)$$

and for $\alpha_L \neq \alpha_R$

$$\dot{F}(t_{0-}) \cdot \dot{F}(t_{0+}) < 0, \quad (5.8)$$

hence there exists a neighborhood U of t_0 , such that $F(t)$ is a monotone function in it. For $\alpha_L < \alpha_R$ is $\lim_{t \rightarrow t_0-} \dot{F}(t) < 0$ and $\lim_{t \rightarrow t_0+} \dot{F}(t) > 0$, consequently one has in a neighborhood of t_0

$$x(t) \sin \alpha(\omega) - y(t) \cos \alpha(\omega) < p(\omega) \quad (5.9)$$

and $(p(\omega), \alpha(\omega))$ is a T -descriptor. Analogously for $\alpha_R < \alpha_L$. \square

As a direct consequence of the proof of the Theorem we get

Corollary 5.1 *Vertex descriptors can only have type T or S . For any given vertex all descriptors have the same type.*

Hence the following Definition makes sense:

Definition 5.2 *A vertex has type T (S) if it has at least one vertex descriptor of type T (S).*

It is therefore possible to estimate the type of a vertex by investigating one single descriptor. The most natural choice is

$$\alpha_E = (\alpha_L + \alpha_R)/2. \quad (5.10)$$

The types of the limit descriptors of a vertex are not completely free:

Theorem 5.2 *Given a vertex P_0 with descriptors (p_L, α_L) and (p_R, α_R) of different types (but not L -descriptors). Then at least one of these descriptors has type I .*

Proof Let (p_L, α_L) be a T -descriptor with corresponding neighborhood U of P_0 . Let P_1 be a point with

$$\begin{aligned} x_1 &= x_0 + \varepsilon (\cos \alpha_R + \cos \alpha_L), \\ y_1 &= y_0 + \varepsilon (\sin \alpha_R + \sin \alpha_L). \end{aligned} \quad (5.11)$$

Then

$$x_1 \sin \alpha_L - y_1 \cos \alpha_L = p_L - \varepsilon \sin(\alpha_R - \alpha_L), \quad (5.12)$$

$$x_1 \sin \alpha_R - y_1 \cos \alpha_R = p_R + \varepsilon \sin(\alpha_R - \alpha_L). \quad (5.13)$$

Choosing $|\varepsilon| \neq 0$ so small that P_1 is in U and $\text{sign } \varepsilon = -\text{sign}(\sin(\alpha_R - \alpha_L))$ then P_1 belongs not to $\text{cl } I(K)$ by (5.12) and because (p_L, α_L) is a T -descriptor. By assumption (p_R, α_R) is not a T -descriptor and by (5.13) not an S -descriptor.

The remainder of the proof follows by a duality argument. \square

Chapter 6

Convex Sets

Assumption 3.1 is assumed to hold. By Definition 3.2 the following Theorem is true:

Theorem 6.1 *If $I(K)$ is a bounded set then for each α an extreme T -descriptor exists.*

Proof For given α let

$$p = \max_{p \in K} (x \sin \alpha - y \cos \alpha). \quad (6.1)$$

Then (p, α) is an extreme T -descriptor. \square

Theorem 6.2 *The convex hull of K consists of all solutions (x, y) of the linear system of inequalities*

$$x \sin \alpha - y \cos \alpha \leq p \quad (6.2)$$

for all extreme T -descriptors (p, α) .

Proof Denote by C the set of all solutions of the system of inequalities.

1. Let $P_0 \in K$ and $0 \leq \alpha \leq 2\pi$. Whenever the maximum

$$\max_{P \in K} (x \sin \alpha - y \cos \alpha) \quad (6.3)$$

exists then it defines an extreme T -descriptor belonging to α . If no maximum exists then there is in (6.2) no inequality present for the corresponding α . We conclude that P_0 fulfills all inequalities (6.2), hence $P_0 \in C$.

2. If P_0 does not belong to K then there exists by the Separation Theorem for convex sets [13, 6] an α with

$$x \sin \alpha - y \cos \alpha < x_0 \sin \alpha - y_0 \cos \alpha \quad (6.4)$$

for all $P \in K$. Then

$$p' = \max_{P \in K} (x \sin \alpha - y \cos \alpha) \quad (6.5)$$

exists and (p', α) is an extreme T -descriptor with

$$x_0 \sin \alpha - y_0 \cos \alpha > p', \quad (6.6)$$

consequently P_0 does not belong to C . \square

Theorem 6.3 *Let $I(K)$ be nonempty.*

$I(K)$ is convex if and only if all descriptors of K are extreme T -descriptors.

Proof 1. Assume that $I(K)$ is convex. For each point on K there exists by the Separation Theorem for convex sets [13, 6] an extreme T -descriptor.

2. Let $I(K)$ be nonconvex. Then there exists a point $P_0 \in K$ which is in $\text{int conv } K$. Consequently a whole neighborhood U of P_0 is in $\text{int conv } K$ and we have for each α

$$\begin{aligned} x_0 \sin \alpha - y_0 \cos \alpha &< \max_{P \in U} (x \sin \alpha - y \cos \alpha) \leq \\ &\leq \max_{P \in \text{conv } K} (x \sin \alpha - y \cos \alpha) = \\ &= \max_{P \in K} (x \sin \alpha - y \cos \alpha) =: p(\alpha). \end{aligned} \quad (6.7)$$

This means that a descriptor in P_0 is not an extreme T -descriptor. Hence we can conclude: If all descriptors of K are extreme T -descriptors then $I(K)$ is convex. \square

Theorem 6.4 *If there are exclusively T -descriptors on K then they all are extreme T -descriptors and $I(K)$ is convex.*

Proof Let $P_0 = P(t_0) \in T \setminus ET$ with descriptor (p_0, α) . Then there is a point $P_1 = P(t_1)$ with T -descriptor (p_1, α) and $p_1 > p_0$. Without loss of generality we can assume that $t_0 < t_1$. Define

$$p_m = \min_{t_0 < t < t_1} (x(t) \sin \alpha - y(t) \cos \alpha) \quad (6.8)$$

and assume that $t_m \in (t_0, t_1)$ is the minimizing argument. Then $p_m \leq p_0$ and there is a neighborhood U_m of $P(t_m)$ such that in it

$$x(t) \sin \alpha - y(t) \cos \alpha \geq p_m. \quad (6.9)$$

If $P(t_m)$ is not an L -point then in (6.9) for at least one boundary point in U_m the strict inequality sign holds, hence $P(t_m)$ is not a T -point, a contradiction.

If $P(t_m)$ is an L -point then calculate the largest value of t such that in (6.8) the minimum is attained. \square

Remark 6.1 *Theorem 6.4 was first stated in 1929 by Tietze (see [13, Theorems 4.9 and 4.10]).*

For a convex set it is easily possible to decide in the Hough space whether a given point belongs to it or not. Define

Definition 6.1 *Given a point $P = (x, y)$ in the plane. Then $H[P]$ is the set of all (p, α) with*

$$x \sin \alpha - y \cos \alpha = p. \quad (6.10)$$

Theorem 6.5 *Let $I(K)$ be convex. Then for any point P :*

(a) *If $P \in A(K)$ then $H[P]$ intersects $H[K]$ exactly twice.*

(b) If $P \in I(K)$ then $H[P]$ does not meet $H[K]$.

(c) If $P \in K$ then there is exactly one intersection point of $H[P]$ and $H[K]$.

Proof (a) Let $P \in A(K)$. The maximal and the minimal values of α with

$$x \sin \alpha - y \cos \alpha = p, \quad (p, \alpha) \in H[K] \quad (6.11)$$

define a tangent at K in P . In the plane there are no further tangents at K in P (this is certainly not true in three-dimensional space).

(b) If a line meets an interior point it cannot be a tangent at a convex set by $K = ET$ (Theorem 6.3).

(c) The line defined by the intersection point meets K in a point or in a connected line segment.
 \square

Remark 6.2 *If the assumption of convexity of $I(K)$ is not fulfilled then the situation can be very complicated (see Figures 3 and 4).*

Scherl gave very impressive examples for the application of the results of this paragraph for segmentation of objects in binary images by means of constructing convex hulls ([11, pp. 48, 64, 65, 132, 133 and 150 to 169]).

Chapter 7

Vertex Descriptors and I -Points

Theorem 7.1 *Assume that for $t \in (t_1, t_2)$ all $(p(t), \alpha(t))$ are T^- (or S^-) descriptors but not L^- -descriptors. Define*

$$\alpha_L = \lim_{t \rightarrow t_2^-} \alpha(t), \quad p_L = x_2 \sin \alpha_L - y_2 \cos \alpha_L \quad (7.1)$$

Then (p_L, α_L) is not an S^- (or T^-) descriptor.

Proof Consider the convex set of all solutions of the system of inequalities

$$x \sin \alpha(t) - y \cos \alpha(t) \leq p(t), \quad t_1 < t < t_2. \quad (7.2)$$

The inequality

$$x \sin \alpha_L - y \cos \alpha_L \leq p_L \quad (7.3)$$

is redundant with respect to (7.2), i.e. it is fulfilled for all solutions of (7.2). Specifically because of Theorems 6.2 and 6.4 for all $s \in (t_1, t_2)$

$$x(s) \sin \alpha_L - y(s) \cos \alpha_L \leq p_L. \quad (7.4)$$

If there exists in each neighborhood of t_2 an $s < t_2$ such that in (7.4) the strict inequality sign holds then (p_L, α_L) is certainly not an S^- -descriptor. Otherwise there exists a neighborhood of t_2 such that in (39) equality holds for all $s < t_2$. Then all points $P(s)$ with $s < t_2$ in this neighborhood are L^- -points. A duality argument will yield the remainder of the proof. \square

Corollary 7.1 *Assume that all points of an open differentiable piece of the boundary are T^- (S^-) points except $P_0 = P(t_0)$. Then P_0 is also a T^- (S^-) point.*

Proof Consider the corresponding system of inequalities of the form (7.2). The inequality

$$x \sin \alpha(t_0) - y \cos \alpha(t_0) \leq p(t_0) \quad (7.5)$$

is redundant with respect to the system. \square

Without the differentiability assumption the assertion of the Corollary is not necessarily true. Consider for example the third configuration in Figure 5.

Theorem 7.2 Assume that for $t \in (t_1, t_0)$ all points are T -points but not L -points and that for $t \in (t_0, t_2)$ all points are S -points but not L -points. Then $P(t_0)$ is an I -point.

Proof $P(t_0)$ is neither an S -point nor a T -point by Theorem 7.1. \square

Theorem 7.3 Assume that for $t_1 \leq t < t_0$ all points of K are T - (S -) points and that there is a T - (S -) vertex in t_0 . Then (p_L, α_L) is a T - (S -) descriptor. Analogously for the limit descriptor (p_R, α_R) .

Remark 7.1 Here L -descriptors are allowed to act as T - or S -descriptors.

Proof Let C be the set of all solutions of the system of inequalities

$$\begin{aligned} x \sin \alpha(t) - y \cos \alpha(t) &\leq p(t), & t_1 \leq t < t_0, \\ x \sin \alpha_E - y \cos \alpha_L &\leq p_E. \end{aligned} \quad (7.6)$$

(cf. (5.10) for definition of p_E). C is convex and contains by Theorem 6.1 all points $P(t)$, $t_1 \leq t \leq t_0$. With respect to this system the following inequality is redundant:

$$x \sin \alpha_L - y \cos \alpha_L \leq p_L \quad (7.7)$$

The remaining part of the proof follows from a duality argument. \square

Theorem 7.2 can be sharpened:

Theorem 7.4 Assume that for $t \in (t_1, t_0)$ all points are T - (S -) points but not L -points and that there is an S - (T -) vertex in t_0 . Then (p_L, α_L) is an I -descriptor. A similar assertion holds for (p_R, α_R) .

Proof By Theorem 7.1 the limit descriptor is not an S -descriptor. Assume that it is a T -descriptor. Then there exists a neighborhood of the vertex such that K in this neighborhood is completely contained in the set

$$x \sin \alpha_L - y \cos \alpha_L \leq p_L. \quad (7.8)$$

Since all points of K are T -points for $t_1 \leq t < t_0$, K is also contained in the set (within this neighborhood)

$$x \sin \alpha(t) - y \cos \alpha(t) \leq p(t), \quad t_1 \leq t < t_0. \quad (7.9)$$

If an S -vertex is present then

$$x \sin \alpha_E - y \cos \alpha_E \geq p_E \quad (7.10)$$

for all $P \in A(K)$ lying in a neighborhood of the vertex. Let U be the intersection of these both neighborhoods of the vertex. Assume that for an $\varepsilon > 0$

$$\begin{aligned} x &= x_0 + \varepsilon \sin \alpha_L - \varepsilon \sin \alpha_E, \\ y &= y_0 - \varepsilon \cos \alpha_R + \varepsilon \cos \alpha_E. \end{aligned} \quad (7.11)$$

Then

$$x \sin \alpha_L - y \cos \alpha_L = p_L + \varepsilon - \varepsilon \cos(\alpha_L - \alpha_E) \quad (7.12)$$

and

$$x \sin \alpha_E - y \cos \alpha_E = p_E - \varepsilon + \varepsilon \cos(\alpha_L - \alpha_E). \quad (7.13)$$

If $\alpha_L \neq \alpha_E$ then $\cos(\alpha_L - \alpha_E) < 1$. P belongs to U for suitable $\varepsilon > 0$ and there exists a neighborhood U_P of P such that all points in this neighborhood are in U and fulfill neither (7.8) nor (7.10). This means: U_P is neither contained in $I(K)$ nor in $A(K)$, a contradiction. \square

We summarize: The situation at a vertex is characterized by a triple of descriptors:

- The type of descriptor in a left neighborhood of the vertex (not to be confused with (p_L, α_L)),
- The type of the vertex,
- The type of descriptor in a right neighborhood of the vertex.

If both limit descriptors of the vertex have different types then the vertex descriptor is an I -descriptor. If both limit descriptors have the same type, then also the vertex descriptor has this type.

There are up to duality and order three possibilities for the sequence of descriptors in a vertex:

1. Both descriptor types in the neighborhood of the vertex are different. Prototype: $T - S - S$.
2. Both descriptor types in the neighborhood of the vertex are equal.
 - a) The vertex descriptor has the same type.
Prototype: $T - T - *$, ($*$ = T or L).
 - b) The vertex descriptor has different type.
Prototype: $T - S - *$, ($*$ = T or L).

In Figure 5 these three types of vertex configurations are shown.

Remark 7.2 *It is possible to deform a vertex continuously into a smooth curve such that the number of I -descriptors is preserved.*

Chapter 8

Convex Boundary Segments — Differentiability

The curve K was supposed to be piecewise differentiable. Hence it is possible to characterize S - and T -descriptors by investigating the function $\alpha(t)$.

$\alpha(t)$ is termed *increasing* at t_0 if there exists a neighborhood of t_0 such that in this neighborhood $\alpha(t) \leq \alpha(t_0)$ for all $t < t_0$ and $\alpha(t) \geq \alpha(t_0)$ for all $t > t_0$. In an analogous way a function *decreasing* at t_0 is defined.

Theorem 8.1 *Let $P(t)$ be differentiable at t_0 . If $\alpha(t)$ is increasing at t_0 then $P(t_0)$ is a T -point. If $\alpha(t)$ is decreasing at t_0 then $P(t_0)$ is an S -point.*

Proof Define

$$F(t) = x(t) \sin \alpha(t_0) - y(t) \cos \alpha(t_0) - p(t_0). \quad (8.1)$$

Then (see (2.3))

$$\dot{F}(t) = g(t) \sin(\alpha(t_0) - \alpha(t)). \quad (8.2)$$

If $\alpha(t)$ is increasing in t_0 , $\dot{F}(t)$ changes its sign at t_0 from $+$ to $-$, hence F has a local maximum in t_0 , consequently $P(t_0)$ is a T -point. A similar argument applies when $\alpha(t)$ is a decreasing function in t_0 . \square

Theorem 8.2 *Assume that the second derivative of $P(t)$ exists in a neighborhood of t . Then also the first derivative of $\alpha(t)$ exists.*

For $\dot{\alpha}(t) > 0$ is $P(t)$ a T -point.

For $\dot{\alpha}(t) < 0$ is $P(t)$ an S -point.

Proof Without loss of generality we assume $\dot{x}(t) \neq 0$ for any t . Then

$$\alpha(t) = \arctan \frac{\dot{y}(t)}{\dot{x}(t)} \quad (8.3)$$

and $\dot{\alpha}(t)$ exists. If $\dot{\alpha}(t) > 0$, α is increasing and t is parameter of a T -point by Theorem 8.1. The remainder of the proof follows by a similar argument. \square

The converse of Theorem 8.1 is not necessarily true as the following counterexample shows:

Let K be given by the function

$$y = \begin{cases} x^2(1 + \sin \frac{1}{x}) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \quad (8.4)$$

Let the interior $I(K)$ be the set of all points above K . The point $x = y = 0$ is a T -point, $\alpha(t)$ however, does not have any monotone behaviour at this point.

It is possible to guarantee under suitable assumptions that the converse of the Theorem is true:

Lemma 8.1 *Assume that for each t the set of isolated parameter values $s \in [0, L]$ with $\alpha(t) = \alpha(s)$ is finite.*

If $P(t_0)$ is a T -point, $\alpha(t)$ increases at t_0 .

If $P(t_0)$ is an S -point, $\alpha(t)$ decreases at t_0 .

If $P(t_0)$ is an L -point, $\alpha(t)$ is constant in a neighborhood of t_0 .

Proof Assume that $P(t_0)$ is not an L -point, i.e. t_0 is not in an open interval such that $\alpha(t)$ is constant in it. Then there exists an interval (t_1, t_2) such that $t_0 \in (t_1, t_2)$ and $\alpha(t)$ is different from $\alpha(t_0)$ for all t in the interval with $t \neq t_0$. Consequently there exists exactly one zero of $\dot{F}(t)$ in a neighborhood of t_0 , hence $\alpha(t)$ can only decrease or increase in t_0 .

If $\alpha(t) = \alpha(s)$ for $s < t$ and if the first derivative of α in $[s, t]$ is continuous then there exists a $t_0 \in (s, t)$ with $\dot{\alpha}(t_0) = 0$. \square

Assumption 8.1 *The set of all isolated zeroes of $\dot{\alpha}(t)$ is finite.*

If the second derivative exists in each point of differentiability of K then Theorems 7.2 and 8.2 imply that all isolated zeroes of $\dot{\alpha}(t)$ are I -points. Therefore the following Assumption makes sense:

Assumption 8.2 *The second derivative of $P(t)$ exists and is continuous at each point of differentiability of K .*

If Assumption 8.2 holds then the assumption of Theorem 8.2 is fulfilled. By (8.3) this is equivalent to the requirement that the curvature of the curve is allowed to assume zero value only in finitely many isolated points of the curve.

In addition to Theorems 3.2 and 3.3 the following Theorem can be stated:

Theorem 8.3 *If Assumption 8.2 holds, the set of all S - and T -points (including L -points) is a dense subset of K .*

Proof Let (t_1, t_2) be a subinterval of $[0, L]$ such that $P(t)$ is neither a T - nor a S -point for $t \in (t_1, t_2)$. Then $\alpha(t) = 0$ in this interval by Theorem 8.2. Consequently $\alpha(t)$ is constant in (t_1, t_2) and by (8.2) is $\dot{F}(t) = 0$, hence $F(t)$ is constant and all points of the interval are L -points. \square

Assumptions 8.1 and 8.2 together imply

Assumption 8.3 *K contains only finitely many I -points.*

We summarize:

Theorem 8.4 *Assume that Assumption 8.2 and one of the Assumptions 8.1 or 8.3 hold. Then:*

$P(t_0)$ is a T -point (S -point) if and only if $\alpha(t)$ is increasing (decreasing) at t_0 .

$P(t_0)$ is an L -point if and only if $\alpha(t)$ is constant in a neighborhood of t_0 .

Chapter 9

Reconstruction

At a first glance the following question has only theoretical value: Given the Hough image $H[K]$ of a curve K , is it possible to reconstruct K uniquely from $H[K]$?

If we assume that there are only finitely many I -points (Assumption 8.3), K consists of finitely many arcs K_i having alternatively only T - or S -points (Theorem 7.2 Corollary 7.1) it is possible that such an arc consists only of a single point (vertex of K , see the third example of Figure 5 with $* = T$).

Given such an arc K_i , we attempt to reconstruct K_i from $H[K_i]$. Without loss of generality we can assume that $H[K_i]$ is continuous (Assumption 2.1) and topologically closed. To the end points of $H[K_i]$ (if $H[K_i]$ is not defined for all α) correspond lines being tangents at the end points of K_i . Obviously K_i can only be reconstructed up to such boundary tangents. Any two arcs K_i and K_{i+1} are either connected by a common boundary point or by a common boundary tangent. If the type of descriptors does not change on the arc under investigation then the latter together with its boundary tangents is either the boundary of a convex set or the boundary of the complement of a convex set.

From the Theorems of paragraph 6 we can conclude that the reconstruction problem can be easily solved for convex sets. For a convex set $\text{cl } I(K_i)$ coincides with the set of solutions of the (semi-infinite) system of linear inequalities

$$x \sin \alpha - y \cos \alpha \leq p, \quad (p, \alpha) \in H[K_i], \quad (9.1)$$

and K_i is the boundary of the set of solutions (without the boundary tangents). Approximating $I(K_i)$ by the solution set of a finite subsystem of inequalities will yield an approximation from outside. The error occurring by this approximation will be estimated in Part II of this report.

We note that this approximation process will only work if the system of inequalities (9.1) is consistent. Figure 6 shows that this need not be always the case.

It is easily possible to identify I -points in $H[K]$. By Theorem 8.4 it is known that under Assumption 8.2 and one of the Assumptions 8.1 or 8.3 $\alpha(t)$ is an increasing function at T -points and a decreasing function at S -points. Therefore the I -points are turning points of $H[K]$:

Definition 9.1 *A point (p_0, α_0) of $H[K]$ is a turning point if there exists a neighborhood U of (p_0, α_0) such that for $\alpha > \alpha_0$ (or for $\alpha < \alpha_0$, respectively) no other points $(p, \alpha) \in H[K]$ are in U .*

A second possibility for reconstruction is given by the equation

$$p(\alpha) = x \sin \alpha - y \cos \alpha. \quad (9.2)$$

For fixed x and y there is either no value α (then P is an interior point; see Theorem 6.5), exactly two α (P is exterior point), or exactly one α (P is boundary point). For each given x it is possible to determine constructively all those α such that y as calculated by (9.2) occurs only once. Obviously these α are the arguments of extrema of the function

$$y = \frac{x \sin \alpha - p(\alpha)}{\cos \alpha}, \quad 0 < \alpha < 2\pi. \quad (9.3)$$

The easiest way of reconstruction of an arc K_i rests on the observation that K_i is the envelope of all lines with parameter $(p, \alpha) \in H[K_i]$. If we assume that p is differentiable with respect to α for all $(p, \alpha) \in H[K_i]$ we get

$$\begin{aligned} x(\alpha) &= p \sin \alpha + \frac{dp}{d\alpha} \cos \alpha, \\ y(\alpha) &= -p \cos \alpha + \frac{dp}{d\alpha} \sin \alpha. \end{aligned} \quad (9.4)$$

We summarize the results of this paragraph:

Theorem 9.1 *Let Assumption 8.2 and one of the Assumptions 8.1 or 8.3 be true. Assume further that $H[K]$ has only finitely many turning points. Then K is uniquely determined by $H[K]$ with the possible exception of boundary tangents.*

As a consequence of Theorem 9.1 we conclude that the Definition 2.2 of Hough descriptors is equivalent to Scherl's definition such that the identical notation is justified.

Note that not to each 'decent' curve in Hough space there corresponds a 'decent' curve in the image. Consider for example the curve

$$p(\alpha) = \begin{cases} 1 & \text{for } 0 \leq \alpha \leq \pi/4 \\ 2 - 4\alpha/\pi & \text{for } \pi/4 \leq \alpha \leq 3\pi/4 \\ -1 & \text{for } 3\pi/4 \leq \alpha \leq \pi \end{cases} \quad (9.5)$$

and $p(-\alpha) = p(\alpha)$ which is the transformed curve of the curve shown in Figure 6. The latter curve has two double points and therefore it does not divide the plane into two regions as required by Assumption 3.1. In particular it is not the boundary of a convex set although $p(\alpha)$ has no double points.

Chapter 10

The Hough Space

We collect some simple properties of the Hough transform of a curve:

1. Let M be any subset of the line defined by (p, α) . Then $H[M]$ consists only of the point (p, α) .
2. The image $H[P]$ of a point P is given by (6.10) (Definition 6.1).
3. The image of a circle with radius r having the origin as center is the constant function $p = r$.
4. Given a vertex P with (p_L, α_L) and (p_R, α_R) as defined by (5.1) and (5.2). The vertex descriptors (Definition 5.1) form a curve $(p(\omega), \alpha(\omega))$ (see (5.3), (5.4)). This curve passes the points (p_L, α_L) and (p_R, α_R) and satisfies (6.10).
5. Given a closed curve K without multiple points. Let (p_0, α_0) be a multiple point of $H[K]$. There is an odd number of arcs emanating from this point since it corresponds to a line in the original image and since the originals of these arcs meet this line at one point each. Only if there is an even number of arcs corresponding to these intersection points, K can be the boundary of a closed curve without multiple points (see Figures 1 to 4).
6. The components of the image of $K \setminus ET$ are connected to the image of ET at multiple points by means of an even number of arcs. If one leaves ET via a descriptor (p_0, α_0) then on the line corresponding to the descriptor there is a point of ET from which the boundary curve of the convex hull of K is continued. Stating it differently: The boundary pieces of $K \setminus ET$ have the same limit descriptors for start- and end-descriptors.

Now we collect the effects of some simple transforms of the plane in Hough space:

7. A rotation of the plane about the origin results in a translation of the Hough space parallel to the α -axis.
8. Given a fixed angle φ and consider the reflection of the plane at a line through the origin with direction φ

$$\begin{aligned}x' &= x \cos \varphi + y \sin \varphi, \\y' &= x \sin \varphi - y \cos \varphi.\end{aligned}\tag{10.1}$$

To this transformation corresponds a transformation in Hough space

$$p' = -p, \quad \alpha' = \varphi - \alpha. \quad (10.2)$$

9. Given a translation of the plane with translation vector Q . Then

$$H[K + Q] = H[K] + H[Q]. \quad (10.3)$$

10. For a fixed number a the image of the curve aK is the set of points (ap, α) with $(p, \alpha) \in H[K]$.

In the following Theorem an interesting curve is defined:

Theorem 10.1 *The image of the spiral*

$$\begin{aligned} x(\alpha) &= (a\alpha + p_0) \sin \alpha + a \cos \alpha \\ y(\alpha) &= -(a\alpha + p_0) \cos \alpha + a \sin \alpha \end{aligned} \quad (10.4)$$

is the line $p = a\alpha + p_0$ in Hough space.

In Figure 7 the upper part of this spiral for $a = 1$ and $p = 0$ is given. Approximation of K by means of pieces of the spiral amounts in an approximation of $H[K]$ by line segments.

Let $I(K)$ be convex. Then $H[K]$ is a unique 2π -periodic function defined for all α with Fourier series approximation

$$H[K](\alpha) \cong a_0/2 + \sum (a_k \cos k\alpha + b_k \sin k\alpha) \quad (10.5)$$

with

$$\begin{aligned} a_k &= 1/\pi \int_0^{2\pi} H[K](\alpha) \cos k\alpha \, d\alpha, \\ b_k &= 1/\pi \int_0^{2\pi} H[K](\alpha) \sin k\alpha \, d\alpha. \end{aligned} \quad (10.6)$$

The first summand of this expansion

$$a_1 \cos \alpha + b_1 \sin \alpha \quad (10.7)$$

can be interpreted as the image of a point $P^0 = (b_1, -a_1)$. By a translation with vector $-P^0$ the curve K is mapped into a curve K' having vanishing first coefficients of its Fourier expansion. To the constant term $\frac{a_0}{2}$ there corresponds in the original image a circle around the origin with radius $\frac{a_0}{2}$.

It is easily possible to detect symmetries of the boundary of a convex set. For this goal the function

$$p^0(\alpha) = \sum_{k=2}^{\dots} (a_k \cos kx + b_k \sin kx), \quad (10.8)$$

is defined which results from the original curve when it is translated by $-P^0$ and a circle with radius $\frac{a_0}{2}$ is subtracted. If the function

$$(p^0 * p^0)(\alpha) := \int_0^{2\pi} p^0(x)p^0(\alpha - x)dx \quad (10.9)$$

has a maximum at α_0 , then a line passing through P^0 with angle $\frac{\alpha_0}{2}$ is an axis of symmetry for K . A similar procedure applies to rotational symmetry. If the function

$$(p^0 \tilde{*} p^0)(\alpha) := \int_0^{2\pi} p^0(x) p^0(x - \alpha) dx \quad (10.10)$$

has a maximum at α_0 then K has rotational symmetry with angle α_0 .

In Figure 8 for a curve in the plane the point P^0 and the circle with radius $\frac{\alpha_0}{2}$ is indicated. The Hough transform of the curve is given in Figure 9.

Chapter 11

Connection to Other Representations

A thorough analysis of the Hough image of a curve can lead to valuable insights about properties of the underlying original curve. There is indeed in the literature a large number of proposals for the extraction of features from the Hough transform.

The most interesting approach is due to Bernhardt ([1, 2]). There is a remarkable similarity between this approach and the concept presented here. Note that the Radon- (or Hough-) transform assigns to each point (p, α) in Hough space the integral of the gray value function $\beta(x, y)$ along the line (p, α) . In the terminology of the present paper $\beta(x, y)$ is the characteristic function of the set $\text{cl } I(K)$ (binary image) and therefore the Hough transform at (p, α) is the length of the intersection of $\text{cl } I(K)$ with the line (p, α) .

When the Hough transform is interpreted as an integral transform, the boundary curves no longer play a distinguished role. In particular, an orientation of curves no longer makes sense. This means that there cannot be distinguished between K and K^* and each tangent line (p, α) can also be interpreted as $(-p, \alpha + \pi)$.

For the interpretation of unoriented curves in Hough space the assertion of the following Theorem is relevant:

Theorem 11.1 *Let K be the boundary of a convex set, $I(K)$ nonempty. The curves $H[K]$ and $H[K^*]$ are disjoint.*

Proof Let $P_0 \in I(K)$. Without loss of generality consider the curves $K' = K \setminus P_0$ and K'^* . The origin of the image is in $I(K')$. By Theorem 6.2 for each extreme T -descriptor (p, α) and for all $(x, y) \in I(K)$ is

$$x \sin \alpha - y \cos \alpha \leq p \tag{11.1}$$

For $x = y = 0$ one gets $p \geq 0$, and since a whole neighborhood of the origin is in $I(K)$, even $p > 0$. By Theorem 6.3 all descriptors are extreme T -descriptors since $I(K')$ is convex. Hence $p > 0$ for all descriptors (p, α) of K' . The descriptors of K'^* are just the $(-p, \alpha + \pi)$ with (p, α) descriptor of K' . Hence $H[K']$ and $H[K'^*]$ are disjoint, consequently the same is true for $H[K]$ and $H[K^*]$. \square

If in a gray value image there is an object which is bounded by a closed boundary curve K without multiple points, then an inclusion of the support of the Radon transform of this object can be given:

Theorem 11.2 *Assume that the gray value function $\beta(P)$ vanishes outside the set $I(K)$ with piecewise smooth boundary K having at most finitely many I -points. The Radon transform of β is defined by*

$$R[\beta](p, \alpha) := \int_{-\infty}^{\infty} \beta \begin{pmatrix} t \cos \alpha + p \sin \alpha \\ t \sin \alpha - p \cos \alpha \end{pmatrix} dt. \quad (11.2)$$

Then $R[\beta](p, \alpha) = 0$ if

$$p \geq \max \{q \mid (q, \alpha) \in H[K] \cup H[K^*]\} \quad (11.3)$$

or

$$p \leq \min \{q \mid (q, \alpha) \in H[K] \cup H[K^*]\}. \quad (11.4)$$

Proof If for given α any p lies outside the boundaries given, then the line defined by (p, α) does not meet $\text{conv } K$, and consequently it also does not meet K . \square

Bernhardt also introduced the so-called S -transform assigning to each line one half of the integral of the absolute value of the projection of the gradient of the gray value function on this line. Applied to a binary image this amounts to one half the number of black-white and white-black transitions along the line. The following Theorem holds:

Theorem 11.3 *Given a binary image, i.e. $\beta(P)$ attains only the values zero or one. Assume that $\beta(P)$ vanishes outside a set $I(K)$ with piecewise smooth boundary K and that there are at most finitely many I -points on K . The S -transform $S[\beta](p, \alpha)$ is defined to be one half the number of black-white and white-black transitions along the line (p, α) . $S[\beta](p, \alpha)$ is a piecewise constant function vanishing on the set defined by inequalities (11.3) and (11.4). The regions of constancy of S are separated from each other by pieces of $H[K]$ and $H[K^*]$.*

In Figure 10 the S -transform of the image in Figure 1 is shown.

Starting from the Radon transform and the S -transform of an object in the image Bernhardt extracted sets of descriptors which are very efficient for pattern recognition. They are used in the SIEMENS document reading system SLS which is produced since 1984 by Computer Gesellschaft Konstanz.

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Figures

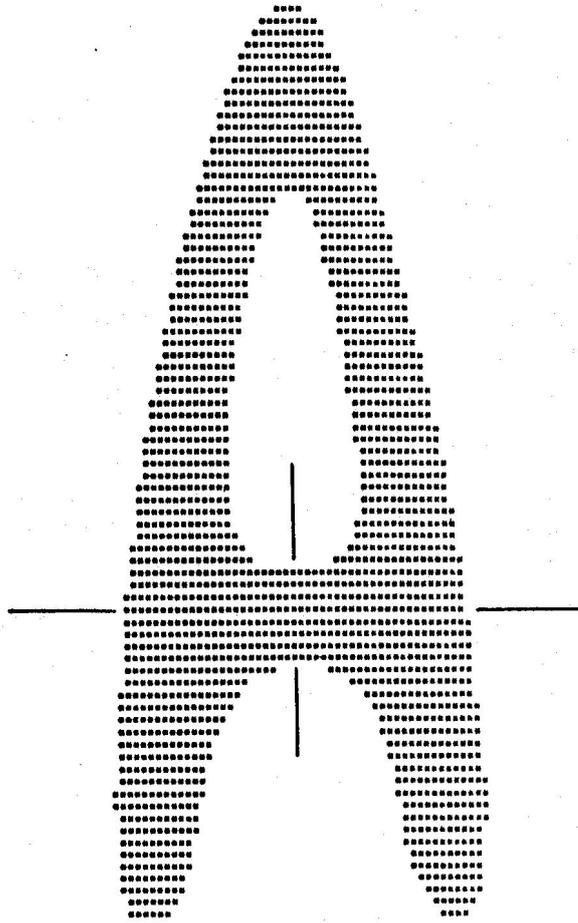


Figure 1 Digitization of letter 'A'.

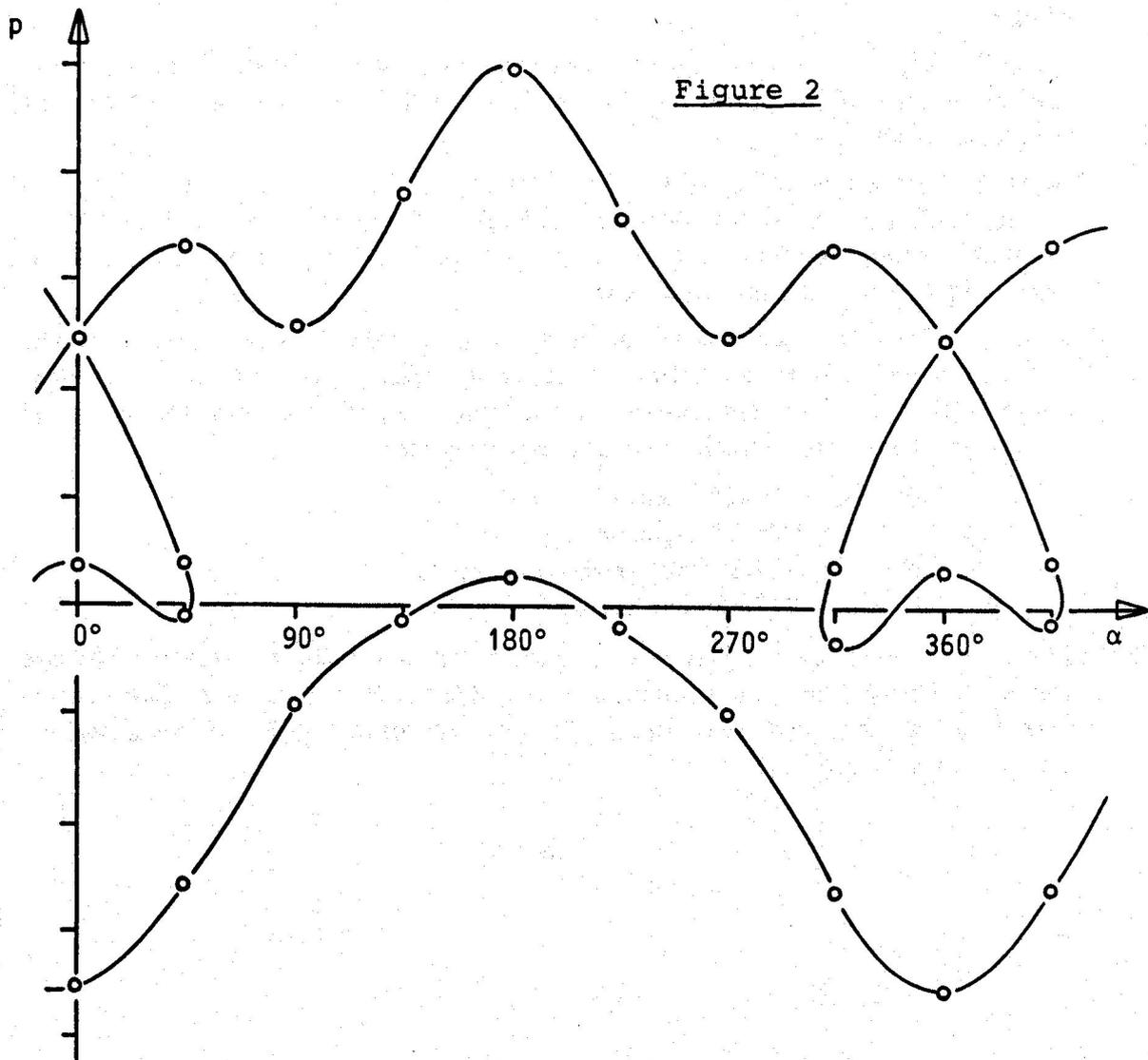


Figure 2 Image of the boundary of letter 'A' from Figure 1 in Hough space. The lower curve is the image of the boundary of the hole of the 'A'. It consists of S -points and is therefore oriented in direction of decreasing α values.

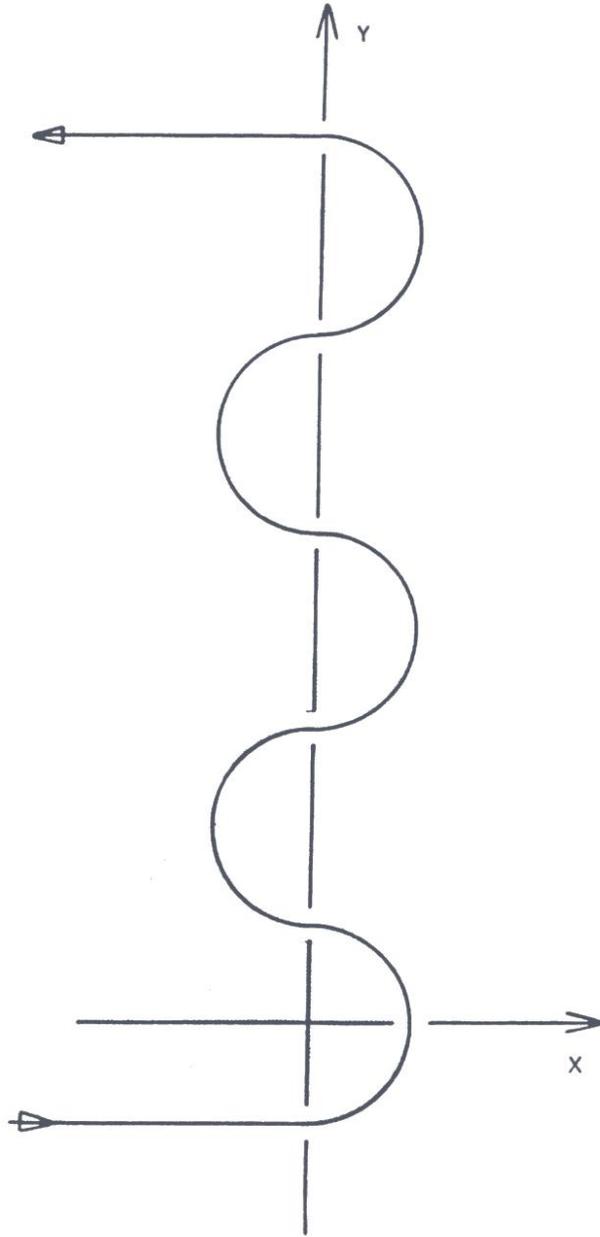


Figure 3 Boundary curve consisting of five semicircles. The interior of K under the orientation indicated by the arrows is shaded.

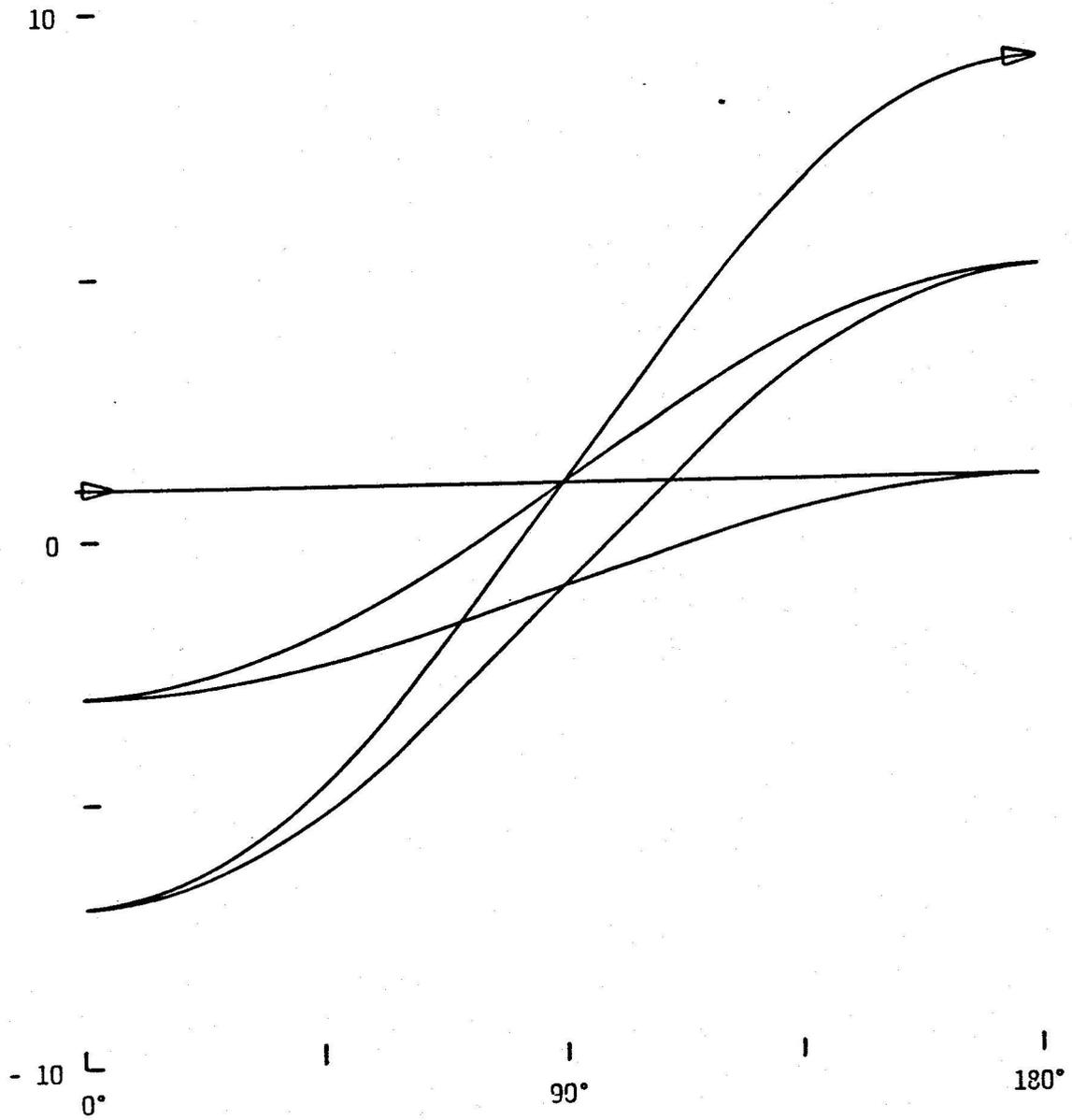


Figure 4 Hough transform of the curve from Figure 3. The curve has multiple points $p = \pm 1$ and $\alpha = \pi/2$, corresponding to the common vertical tangents at the semicircles. There are more multiple points in the Hough image.

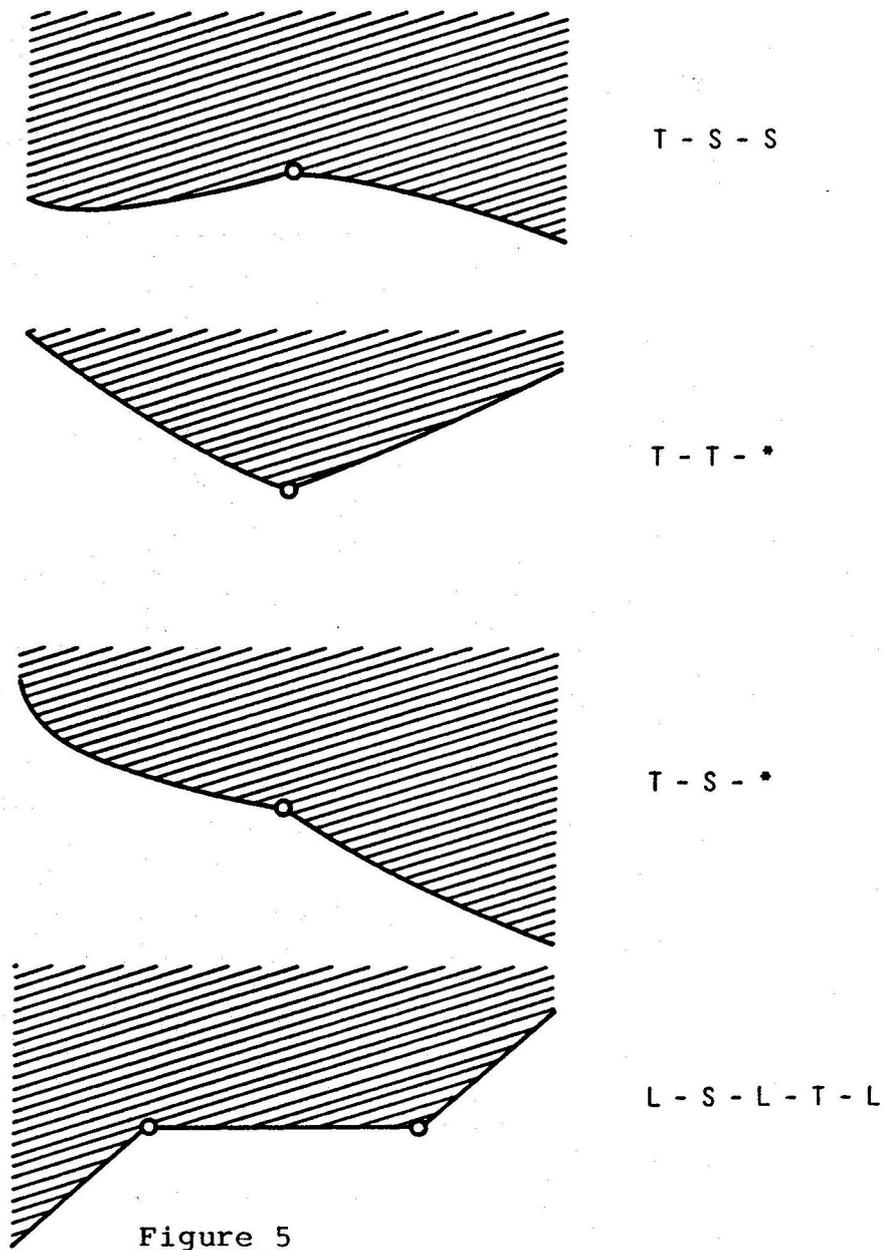


Figure 5

Figure 5 Prototypes for vertex configurations ($*$ = T or L). The fourth image shows a configuration of L -pieces having different interpretations. There are no I -points in the configuration.

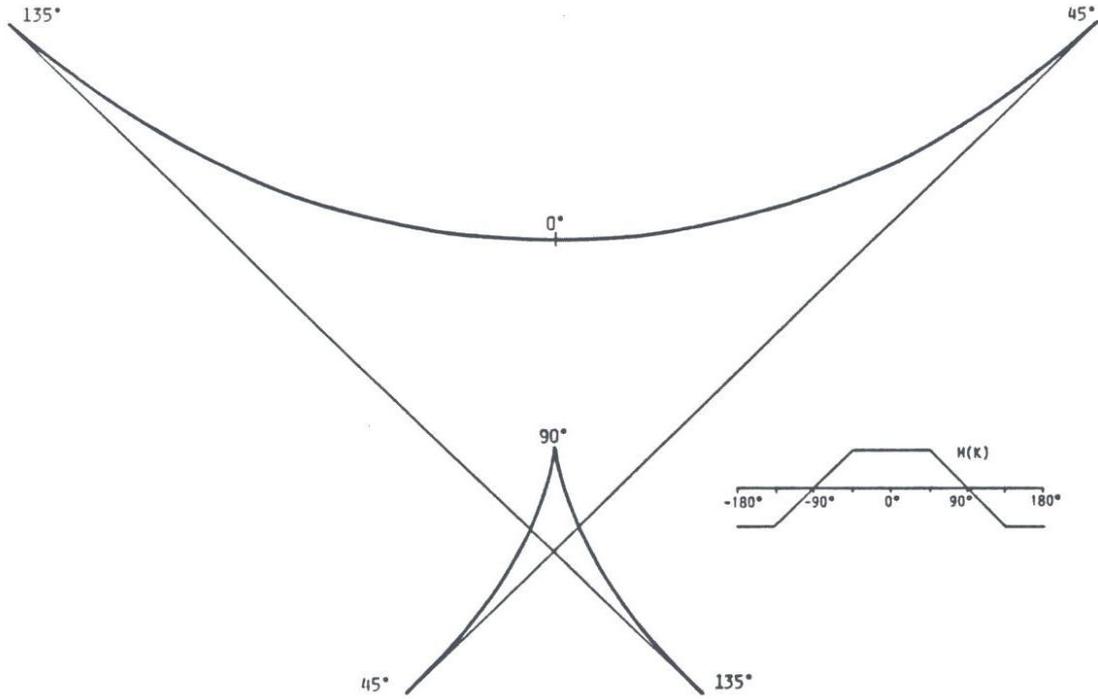


Figure 6 Original curve corresponding to the curve in Hough space given by

$$p(\alpha) = \begin{cases} 1 & \text{for } 0^\circ \leq \alpha \leq 45^\circ \\ 2 - \alpha/45^\circ & \text{for } 45^\circ \leq \alpha \leq 135^\circ \\ -1 & \text{for } 135^\circ \leq \alpha \leq 180^\circ \end{cases}$$

$p(-\alpha) = p(\alpha)$ (see (9.5)). The lines joining the 45° - and 135° -points correspond to vertices in the Hough-image.

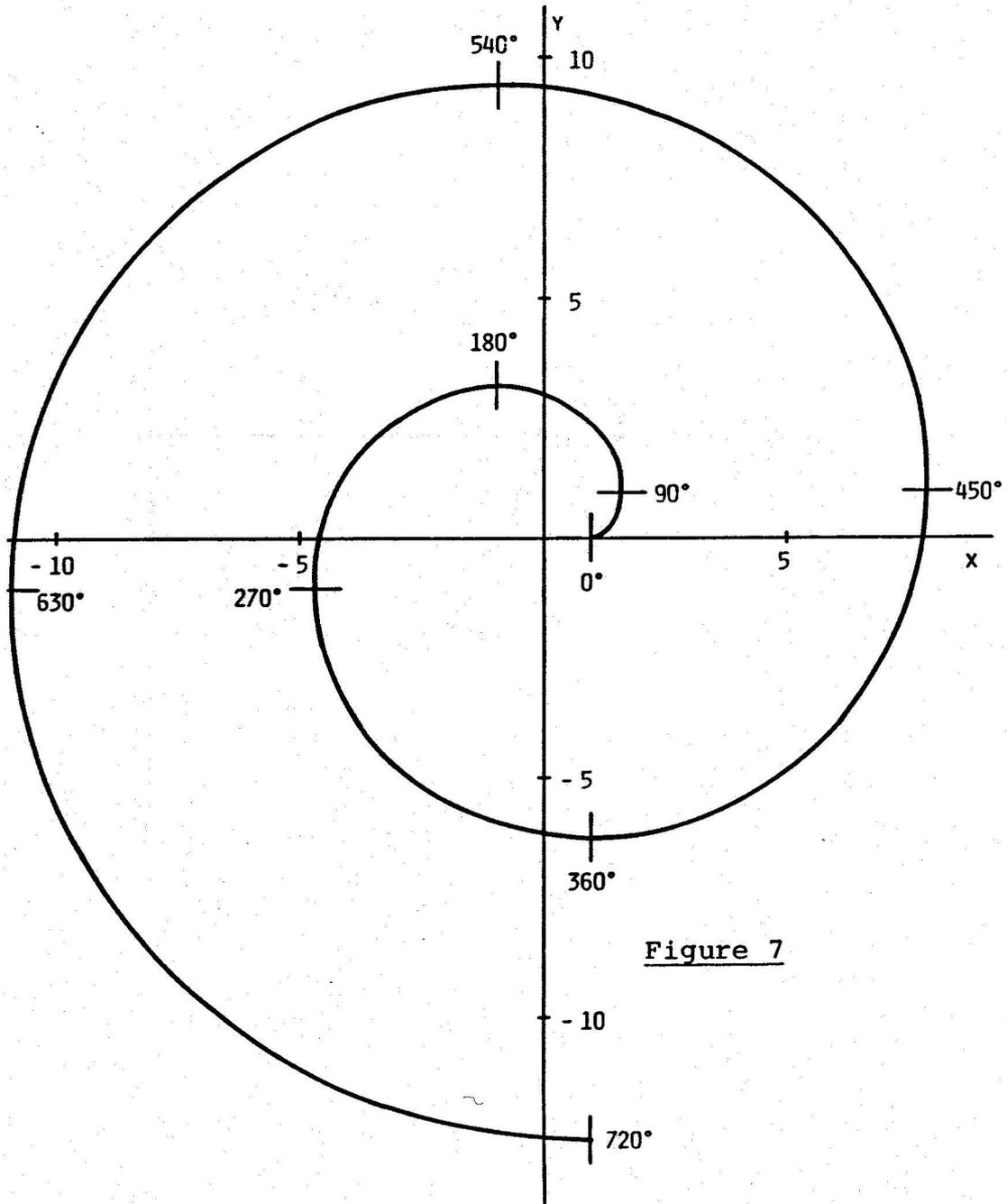


Figure 7 Original curve corresponding to the line $p = \alpha$ in Hough space. Only the upper part of the spiral ($\alpha > 0$) is shown, the lower part can be found by reflection at the x -axis.

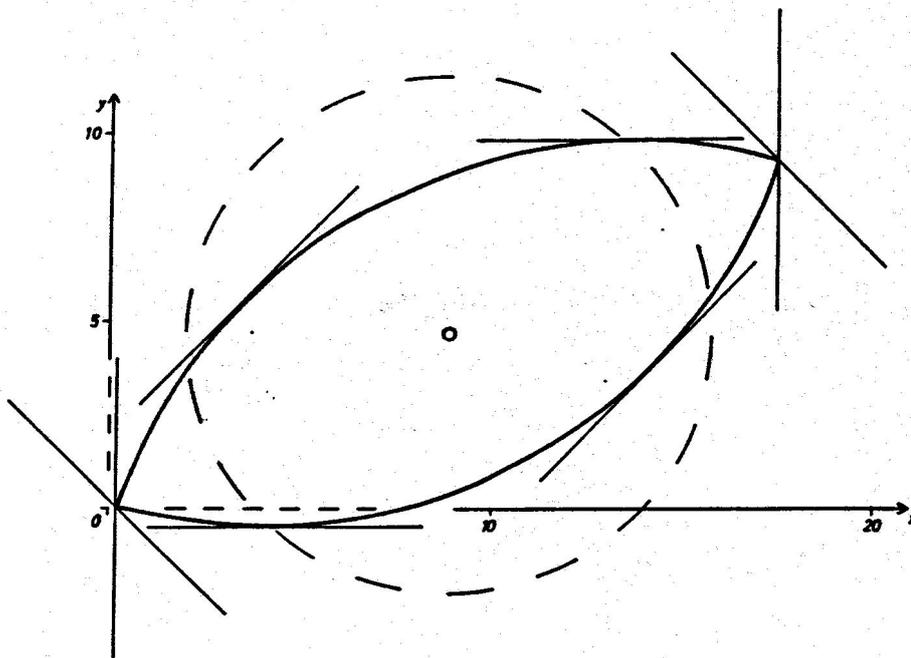


Figure 8 Figure consisting of two circle segments in the plane. The tangents corresponding to the descriptors for multiples of $\pi/4$ are indicated as well as the point belonging to the first term of the Fourier expansion and the circle belonging to the constant term of it.

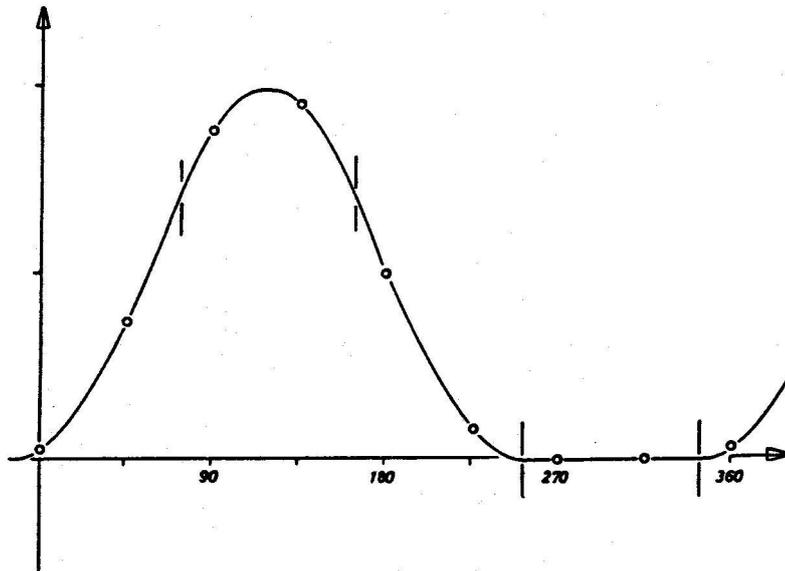


Figure 9 Hough transform of the curve from Figure 8. Those points on the curve corresponding to transitions from circle segments to vertices are marked by vertical lines. The descriptors for multiples of $\pi/4$ are indicated by circles. There are the following correspondences:

$-17.58^\circ \leq \alpha \leq 73.58^\circ$ lower circle segment

$73.58^\circ \leq \alpha \leq 162.42^\circ$ right vertex

$162.42^\circ \leq \alpha \leq 253.58^\circ$ upper circle segment

$253.58^\circ \leq \alpha \leq 342.42^\circ$ left vertex.

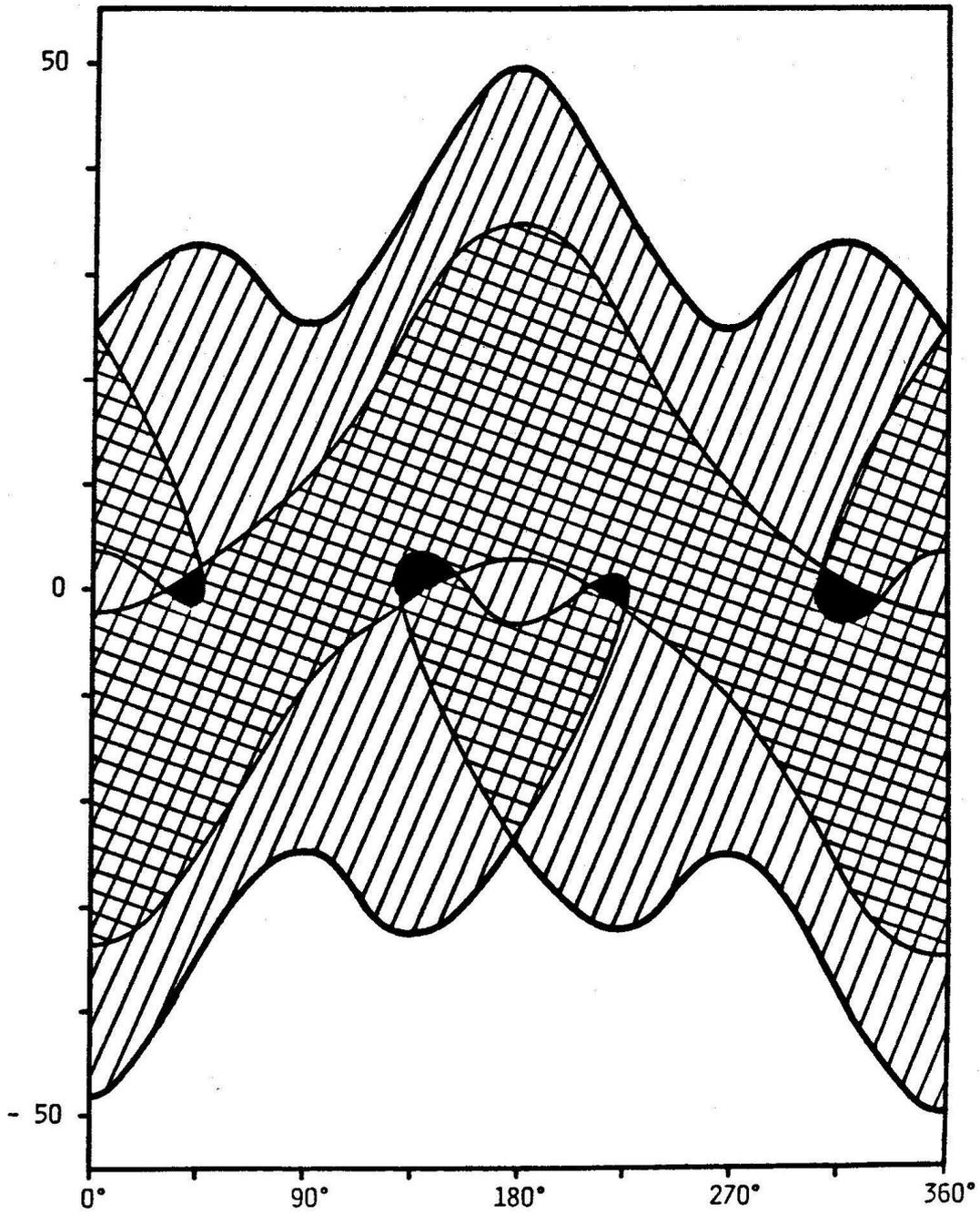


Figure 10 S-transform of Figure 1. A line can intersect the boundary of Figure 1 at least 6 times, hence the maximal value of the S-transform is 3. The regions with $S = 1$ are hatched, cross-hatched regions indicate $S = 2$ and black regions $S = 3$. See also Figure 2.