Topics for the seminar on Harmonic Analysis
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- **Riesz-Thorin interpolation theorem with applications and extensions to analytic families of operators**

  Given a linear operator T of strong type (p_0, q_0) and (p_1, q_1) for some $p_0, p_1, q_0, q_1 \in [1, \infty]$, the complex interpolation method, based on *Hadamard's three lines theorem* (or *Phragmén-Lindelöf theorem*), allows to obtain a logarithmically convex bound on the norm of T on the intermediate spaces. Relevant applications are Hausdorff-Young inequality for the Fourier transform and Young's inequality for the convolution. In addition, this method can be extended to the case of a family of operators depending analitically on a complex parameter z lying in a closed strip in $\mathbb{C}$, assuming strong estimates on the boundary lines.

  References: [3, Sections 1.3.2 & 1.3.3], [7, Chapter V, §1 & § 4].

- **Lorentz spaces and interpolation: main properties, normability and the off-diagonal Marcinkiewicz interpolation theorem**

  The Lorentz spaces $L^{p,q}$ are a natural generalization of the L^p spaces and their weak versions, in the sence that $L^{p,p} = L^p$ and $L^{p,\infty}$ coincides with weak L^p. These quasi-normed Banach spaces are indeed normable under certain conditions on the exponents $p, q \in [1, \infty]$. It is possible to extend Marcinkiewicz's interpolation theorem to the Lorentz spaces, obtaining an off-diagonal version of the theorem. Among its applications, there are refined versions of the Hausdorff-Young inequality and Young's inequality for weak type spaces.

  References: [3, Sections 1.4.1, 1.4.2 & 1.4.4], [7, Chapter V, § 3].

- **Convolution and approximate identities in locally compact groups**

  The most natural framework to define the convolution between locally summable functions is the context of topological groups; that is, Hausdorff topological spaces which are also groups and such that the group operation and the inverse mapping are continuous with respect to the given topology. It is relevant to notice that many properties of the convolution product do not depend on the commutativity of the group operation, so that results like the Minkowski's and Young's inequalities and the approximation through approximate identities hold true also in non abelian groups (under some suitable additional assumptions). As an interesting consequence, some classical results can be proved in an alternative way, as the Hardy inequalities (see [3, Exercises 1.2.7 & 1.2.8]).

  References: [3, Section 1.2].

- BMO **and the extension of Calderón-Zygmund operators to** L^∞

  A Calderón-Zygmund operator is a linear operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, bounded on $L^2(\mathbb{R}^n)$, and such that there exists a singular kernel K for which

  $$Tf(x) = \int_{\mathbb{R}^n} K(x,y)\, f(y)\, dy$$

  for any $f \in L^2(\mathbb{R}^n)$ with compact support and $y \notin \operatorname{supp}(f)$. As a consequence of Calderón-Zygmund decomposition, it is possible to prove that T is of weak type $(1,1)$, so that it is of strong type (p,p) for any $p \in (1,\infty)$. It is then of interest to study the behavior of T on the endpoint space $L^\infty(\mathbb{R}^n)$. However, in general T does not map $L^\infty(\mathbb{R}^n)$ into itself. This brings us to the definition of a new function space strictly larger than $L^\infty(\mathbb{R}^n)$, the space of functions with bounded mean oscillation, $BMO(\mathbb{R}^n)$, which contains all functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

  $$\|f\|_{BMO(\mathbb{R}^n)} := \sup_Q \fint_Q \left| f(x) - \fint_Q f \right| dx < \infty,$$

  where the supremum is taken over all cubes in $\mathbb{R}^n$. It is possible to prove that any Calderón-Zygmund operator can be extended to a continuous operator from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

  References: [1, Chapter 6], [4, Section 3.1], [8].

- **Oscillatory integrals, the Van der Corput lemma and the stationary phase method**

  An oscillatory integral is a function $I : \mathbb{R} \to \mathbb{C}$ of the form

  $$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(x)} \psi(x)\, dx,$$

  where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is called the *phase* and $\psi \in L^1(\mathbb{R}^n;\mathbb{C})$ (usually also smooth and with compact support). The analysis of such functions focuses on their asymptotic behaviour as $\lambda \to +\infty$. It is easy to notice that the Fourier transform provides a first example of an oscillatory integral with a nontrivial phase $\varphi(x) = x \cdot \xi$, for some ξ in the unit sphere, in which case Riemann-Lebesgue lemma implies that $I(\lambda) = o(1)$ as $\lambda \to +\infty$ for a general $\psi \in L^1(\mathbb{R}^n;\mathbb{C})$. In addition, if ψ admits some summable derivatives of order k, then $|I(\lambda)| \le C\lambda^{-k}$. In the case of a general phase, the decay of $I(\lambda)$ shall depend on the critical points of φ: in particular, Van der Corput lemma considers the one-dimensional case, while the stationary phase method is useful in the dealing with the several variables case.

  References: [3, Section 2.6], [6, Chapter VIII, Sections 1 & 2], [2, Section 4.5.3].

- **Decay estimates for Fourier transforms of measures**

  It is not difficult to show that, given a finite Radon measure μ on $\mathbb{R}^n$, then its Fourier transform is well defined and bounded, since

  $$|\hat{\mu}(\xi)| = \left| \int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot x}\, d\mu(x) \right| \le |\mu|(\mathbb{R}^n) < \infty.$$

  However, the Riemann-Lebesgue lemma does not hold: indeed, if $\mu = \delta_0$, the Dirac delta centered in the origin, then

  $$\hat{\mu}(\xi) \equiv 1.$$

Nevertheless, if Σ is a k-dimensional smooth submanifold embedded in $\mathbb{R}^n$ and $\mu = \mathscr{H}^k \llcorner \Sigma$, where $\mathscr{H}^k$ is the k-th dimensional Hausdorff measure, then it is possible to apply notions from the theory of oscillatory integrals to prove decay estimates for $\hat{\mu}$, under suitable assumptions on Σ.

References: [6, Chapter VIII, Section 3].

- **Restriction estimates for the Fourier transform**

  Thanks to the Riemann-Lebesgue lemma, we know that, if $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is a continuous function vanishing at infinity. In particular, it is clear that the restriction of $\hat{f}$ on any hypersurface Σ is well defined and (locally) integrable with respect to the surface measure $\mathscr{H}^{n-1} \llcorner \Sigma$. On the other hand, Hausdorff-Young inequality implies that, if $f \in L^p(\mathbb{R}^n)$ for some $p \in (1,2]$, then $\hat{f} \in L^{p'}(\mathbb{R}^n)$: in general, it does not seem meaningful to talk about the restriction of an $L^{p'}$-summable function on a Lebesgue negligible set. However, under some assumptions on Σ and the values of p, it is possible to prove that
  $$\|\hat{f}\|_{L^2(\Sigma;\mathscr{H}^{n-1})} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$
  for any $f \in \mathscr{S}(\mathbb{R}^n)$, so that it is possible to extend by continuity the restriction operator $f \to \hat{f}_{|\Sigma}$ to the whole $L^p(\mathbb{R}^n)$ with values in $L^2(\Sigma; \mathscr{H}^{n-1})$.

  References: [6, Chapter IX, Section 2.1].

- **Bessel Potentials and general Sobolev spaces**

  For any Schwartz function f, the Laplacian satisfies the following identity:
  $$\widehat{-\Delta f}(\xi) = 4\pi^2|\xi|^2\hat{f}(\xi).$$
  This suggests the idea of defining the fractional Laplacian operator $(-\Delta)^{z/2}$, for $z \in \mathbb{C}$, as the multiplication with $(2\pi|\xi|)^z$ on the Fourier transform. It is then known that the Fourier antitransform of $|\xi|^z$ is, up to some constant $c_{n,z}$, $|x|^{-n-z}$, and this fact leads us to the definition of Riesz potentials $\mathcal{I}_z := (-\Delta)^{-z/2}$, for $\mathrm{Re}(z) > 0$. In an analogous way, we can define the Bessel potentials $\mathcal{J}_z := (I-\Delta)^{-z/2}$ acting on a Schwartz function f as follows:
  $$\widehat{\mathcal{J}_z(f)}(\xi) = (1+4\pi^2|\xi|^2)^{-z/2}\hat{f}(\xi).$$
  Having removed the singularity in the origin on the Fourier transform generates smoothness and therefore a rapid decay at infinity for the Bessel kernel
  $$G_z(x) = \int_{\mathbb{R}^n} (1+4\pi^2|\xi|^2)^{-z/2} e^{2\pi i \xi \cdot x}\, d\xi.$$
  It can be proved that $\mathcal{J}_z$ maps continuously $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, for some suitable p, q. One of the main applications of the Bessels potentials is the definition of a general notion of Sobolev spaces with a fractional order of differentiation: for $s \in \mathbb{R}$ and $p \in (1,\infty)$, we define the *inhomogeneous Sobolev space* $L^p_s(\mathbb{R}^n)$ as the space of all the tempered distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ such that $(1+|\cdot|^2)^{s/2}\hat{u})^\vee \in L^p(\mathbb{R}^n)$, equipped with the norm
  $$\|u\|_{L^p_s(\mathbb{R}^n)} := \left\|(1+|\cdot|^2)^{s/2}\hat{u})^\vee\right\|_{L^p(\mathbb{R}^n)}.$$
  It can be proved that $L^p_k(\mathbb{R}^n)$ coincides with the classical Sobolev space $W^{k,p}(\mathbb{R}^n)$, and that some generalized versions of the Sobolev embedding theorems hold.

  References: [4, Sections 1.2.2 & 1.3.1], [5, Chapter V, Sections 2 & 3].

- **The Littlewood-Paley decomposition and an alternative characterization of general Sobolev spaces**

  Let $f_j \in L^2(\mathbb{R}^n)$ be functions with Fourier transforms $\widehat{f_j}$ supported in disjoint sets. Then these functions are orthogonal in the sense that

$$\| \sum_j f_j \|_{L^2(\mathbb{R}^n)}^2 = \sum_j \|f_j\|_{L^2(\mathbb{R}^n)}^2,$$

  thanks to Plancherel's theorem. However, if we replace the exponent 2 with $p \neq 2$, these two quantities may not even be comparable. The Littlewood-Paley decomposition provides a way to recover (to some extent) such orthogonality relation also in $L^p(\mathbb{R}^n)$. In order to construct such a decomposition, we select a Schwartz function Ψ, and we define the *Littlewood-Paley operator* Δ_j associated with Ψ by setting

$$\Delta_j(f) = f * \Psi_{2^{-j}},$$

  for any tempered distribution f, where $\Psi_{2^{-j}}(x) = 2^{jn}\Psi(2^j x)$, so that $\widehat{\Psi_{2^{-j}}}(\xi) = \widehat{\Psi}(2^{-j}\xi)$. Then, we associate to the tempered distribution f the *square function*

$$\left(\sum_{j\in\mathbb{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}}.$$

  Under some suitable assumptions on Ψ, it is possible to show that, for any $p \in (1, \infty)$, there exists $C_1, C_2 > 0$ (depending on p) such that

$$C_1 \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \le C_2 \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}$$

  for any function $f \in L^p(\mathbb{R}^n)$. This type of decomposition has been used to define many functions spaces, such as the Besov spaces, the Triebel-Lizorkin spaces, and the general inhomogeneous Sobolev spaces, in particular. Indeed, it is possible to show that, for a suitable choice of Ψ and for $\Phi(x) = \sum_{j\le 0} 2^{jn}\Psi(2^j x)$, the norm

$$\|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left\| \left(\sum_{j\in\mathbb{Z}} \left(2^{js} |\Delta_j(f)| \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}$$

  is comparable with $\|f\|_{L^p_s(\mathbb{R}^n)}$.

  References: [1, Chapter 8], [3, Section 6.1.1], [4, Section 1.3.2 & 1.3.3].

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