A new String group model from LG

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Let us recall the definition of the String 2-group for a compact, simple simply connected Lie group G as given by BSCS. Start with the principal bundle (and group extension)

$$\begin{array}{ccc} \Omega G & \longrightarrow & PG \\ & & \downarrow{\scriptstyle ev_1} \\ & & G \end{array} \tag{1}$$

The mapping spaces are as follows:

$$PG = \{I \xrightarrow{\gamma} G \mid \gamma(0) = e_G\}$$

and

$$\Omega G = \{\gamma \in PG \mid \gamma(1) = e_G\}.$$

Note that the loops in ΩG are not necessarily smooth at the identity! We will also use the universal central extension

$$U(1) \to \widehat{\Omega G} \to \Omega G$$

which is a nontrivial $U(1)$-bundle (there are a collection of such extensions, for various spaces of loops – we shall see a couple more later). There is an obvious action of $\widehat{\Omega G}$ on PG, namely $(\widehat{\alpha}, \gamma) \mapsto \alpha \cdot \gamma$, where the latter is pointwise multiplication. We thus get a Fréchet Lie groupoid (an action groupoid)

$$\left(\widehat{\Omega G} \times PG \rightrightarrows PG\right)$$

which I shall denote by $S(PG)$.

However, there is more going on, in that the adjoint action of PG on ΩG lifts to an action on $\widehat{\Omega G}$ (this makes $\widehat{\Omega G} \to PG$ a crossed module) We thus get a semidirect group structure on $\widehat{\Omega G} \times PG$, and

FACT $S(PG)$ is a (strict) group object in the category of Lie groupoids. The multiplication functor is given by the group structure on the objects and arrows, and so.

Thus $S(PG)$ is what we call a *strict 2-group.* However:

1. Smoothness at e_G is an issue because smooth functions $S^1 \to G$ are better.

2. There is no S^1-action on ΩG by rotation (needed for various applications/constructions: Witten genus, Freed-Hopkins-Teleman theorem, positive energy representations and so on)

We could deal with the first point by passing to the bundle over G given by the space $\mathcal{A}_{G\times S^1}$ of connections on the trivial G-bundle on S^1 (this works as G is connected). But this doesn't help with the second point. What we want is an analogue of (1) of the form

$$\begin{array}{ccc} LG & \longrightarrow & ? \\ & & \downarrow \\ & & G \end{array}$$

where here LG is the Fréchet-Lie group $C^\infty(S^1, G)$, and, one might hope, this bundle is somehow 'group like'.

First, note that $LG \simeq \{\mathbb{R} \xrightarrow{p} G \mid p(t+1)p(t)^{-1} = e_G \ \forall t \in \mathbb{R}\}$. We thus make the following

DEFINITION The space of *quasiperiodic paths* is

$$QG := \{\mathbb{R} \xrightarrow{p} G \mid p(t+1)p(t)^{-1} = \text{ const } \forall t \in \mathbb{R}\},$$

where we give QG the topology of uniform convergence of all derivatives on compact subsets of $\mathbb{R}$.

One might picture this as an infinite 'helix' in G such that translation by a constant element takes one from any point on the path to the corresponding point on the next loop around. There are otherwise no constraints on such paths. There is a continuous map $\Theta\colon QG \to G$, $\Theta(p) = p(1)p(0)^{-1}$.

PROPOSITION 1 $QG \to G$ has the structure of a nontrivial smooth principal right LG-bundle, with action

$$\begin{aligned} QG \times LG &\to QG \\ (p, \alpha) &\mapsto p \cdot \alpha. \end{aligned}$$

The definition of QG and the proposition in fact work for arbitrary locally convex Lie goups, but we won't need this generality. Note that if G is Milnor regular, then QG/G, where G is considered as the space of constant paths, is diffeomorphic to $\mathcal{A}_{G\times S^1}$.
FACT Pointwise mulitplication of paths does not preserve quasiperiodicity, hence cannot give a group structure on QG. In fact the situation is even worse:
PROPOSITION 2 For G compact and simply-connected, QG admits *no* Lie group structure making $QG \to G$ a homomorphism with kernel LG.

The proof is via classifying, using Lie theory, LG extensions on simply-connected compact Lie groups.

Thus the naïve guess that QG could simply replace PG fails. But we can proceed! Note that we still have the central extension $U(1) \to \widehat{LG} \to LG$, so can form the Lie groupoid

$$S(QG) := (\widehat{LG} \times QG \rightrightarrows QG)$$

arising from the action of $\widehat{LG}$ on QG. Note that this is a bundle gerbe, just as $S(PG)$ was, but we won't be needing that structure for the present. The maps $\widehat{LG} \to QG$ is certainly not a crossed module, but one might hope that even if QG doesn't fit the bill, some other construction might.
PROPOSITION 3 For G compact, simple, simply-connected, any crossed module $\widehat{LG} \xrightarrow{t} H$ with $kertsimeqU(1)$, $cokert \simeq G$ gives a trivial 2-group extension of G by the 2-group $pt//U(1)$.
COROLLARY There is no strict 2-group model for $String_G$ that uses $\widehat{LG}$.

So we must do something else. To proceed, we need to make several subsidiary definitions.
DEFINITION Let $Q_\flat G = \{p \in QG \mid p^{(n)}(0) = p^{(n)}(1) = 0\ \forall n \geq 1\}$. For $p, q \in Q_\flat G$, define $p \diamond q$ by $p \diamond q(t) = p(t)q(t)$ for $t \in [0,1]$ and then extend to a quasiperiodic path. Finally, let $L_\flat G = LG \cap Q_\flat G$.

Note that there is a map $\Theta_\flat \colon Q_\flat G \to G$ given by restricting Θ.
LEMMA The inclusion $Q_\flat G \hookrightarrow QG$ makes $Q_\flat G \to G$ a reduction of QG to an $L_\flat G$-bundle.

We can then show that $\Omega_\flat G := L_\flat G/G \to Q_\flat G/G \to G$ is an extension of Lie groups. Denote $Q_\flat G/G$ by $Q_{\flat,*}G$. We have, using the central extension $\widehat{\Omega_\flat G}$, an action groupoid

$$S(Q_{\flat,*}G) := (\widehat{\Omega_\flat G} \times Q_{\flat,*}G \rightrightarrows Q_{\flat,*}G)$$

and in fact this is a strict 2-group, using the group structure on $Q_{\flat,*}G$.
PROPOSITION 4 The inclusion functor $\iota\colon S(Q_{\flat,*}G) \to S(QG)$ satisfies:

1. ι is a weak equivalence of Fréchet-Lie groupoids (using the pretopology of surjective submersions of Fréchet manifolds);

2. The submersion $Q_{\flat,*}G \times_{\iota,QG,s} (\widehat{LG} \times QG) \xrightarrow{t\circ pr_2} QG$ witnessing this fact has a smooth section.

COROLLARY There is an adjoint equivalence $S(Q_{\flat,*}G) \leftrightarrows S(QG)$ of Lie groupoids, and hence a *coherent* Lie 2-group structure on $S(QG)$.

One application of this is as follows. Let $R^+(\widehat{LG})$ denote the category of positive energy representations of $\widehat{LG}$ (these are unitary representations on separable infinite-dimensional Hilbert spaces satisfying extra conditions). If $Mod_{Hilb,\mathcal{G}}(S(QG))$ is some category of bundle gerbe Hilbert modules (with structure group $\mathcal{G}$; various options are available: $\mathcal{U}_{res}$, $U_{1+\mathcal{K}}$ etc), then there is a functor

$$\begin{gathered} R^+(\widehat{LG}) \to Mod_{Hilb,\mathcal{G}}(S(QG)) \\ \left(\widehat{LG} \xrightarrow{\rho} U(\mathcal{H}_\rho)\right) \mapsto (QG \times \mathcal{H}_\rho \to QG)\,. \end{gathered}$$

We can be agnostic at present as to the structure group the Hilbert bundles have—actually a nontrivial decision—since the underlying bundles arising using this functor are trivial.