Categorical tori and their representations

A report on work in progress

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Crossed modules and categorical groups

following Noohi

(Strict) categorical groups

are (strict) monoidal groupoids

$$\begin{tikzcd} G_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ G_0 \end{tikzcd}$$

with invertible objects (w.r.t. $\bullet$).

A crossed module (G, A, ψ) encodes a strict categorical group

$$\begin{tikzcd} G \ltimes A \arrow[d, shift right, "pr_1"'] \arrow[d, shift left, "pr_1 \cdot \psi"] \\ G \end{tikzcd}$$

group multiplication gives $\bullet$ and $(g\psi(b), a) \circ (g, b) = (g, ab)$.

Crossed modules

consist of a group G, a right G-module A and a homomorphism $\psi : A \longrightarrow G$ with

$$\begin{aligned} \psi(a^g) &= g^{-1}\psi(a)g \\ \psi(a) \cdot b &= a^{-1}ba. \end{aligned}$$

The crossed module of the categorical group $\mathcal{G}$ above is

$$\begin{aligned} G &= G_0 \\ A &= ker(s) \\ a^g &= g^{-1} \bullet a \bullet g \\ \psi &= t. \end{aligned}$$

Example: the crossed module of a categorical torus

Two ingredients: A lattice $\Lambda^\vee$ and a bilinear form J on $\Lambda^\vee$. From this, we form the crossed module

$$\begin{tikzcd}
\Lambda^\vee \times U(1) \arrow[r, "\psi"] & \mathfrak{t} := \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R} \\
(m, z) \arrow[r, mapsto] & m,
\end{tikzcd}$$

where the action of $x \in \mathfrak{t}$ on $\Lambda^\vee \times U(1)$ is given by

$$(m, z)^x \quad = \quad (m, z \cdot \exp(J(m, x))).$$

Categorical tori

The *categorical torus* $\mathcal{T}$ is the strict monoidal category with

objects: $\mathfrak{t}$,

arrows: $x \xrightarrow{z} x+m, \qquad x \in \mathfrak{t}, m \in \Lambda^\vee, z \in U(1)$,

composition: the obvious one,

multiplication: addition on objects and on arrows

$$(x \xrightarrow{z} x+m) \bullet (y \xrightarrow{w} y+n) = (x+y \xrightarrow{zw\,\exp(J(m,y))} x+y+m+n).$$

Classification

Schommer-Pries, Wagemann-Wockel, Carey-Johnson-Murray-Stevenson-Wang

Up to equivalence, the categorical torus $\mathcal{T}$ only depends on the even symmetric bilinear form

$$I(m,n) \;=\; J(m,n) + J(n,m).$$

More precisely,

$$-I \;\in\; Bil_{ev}(\Lambda^\vee, \mathbb{Z})^{S_2} \;=\; H^4(BT;\mathbb{Z}) \;\cong\; H^3_{gp}(T;U(1))$$

classifies the equivalence class of the extension

$$pt/\!\!/U(1) \longrightarrow \mathcal{T} \longrightarrow T.$$

Examples:

1. $T_{max} \subset G$ maximal torus of a simple and simply connected compact Lie group, $\Lambda^\vee$ coroot lattice, I_{bas} basic bilinear form,
2. (Λ_{Leech}, I) or another Niemeyer lattice.

Aussie-rules Lie group cohomology

$$H^3_{gp}(T; U(1)) \;=\; \check{H}^3(BT_\bullet; \underline{U(1)})$$

and $-I$ corresponds to the Čech-simplicial 3-cocycle

$T \times T \times T$	1			$d = 3$
$T \times T$		$\exp(-J(m, y))$		$d = 2$
T			1	$d = 1$
	$\mathfrak{t}^d$	$(\mathfrak{t} \times \Lambda^\vee)^d$	$(\mathfrak{t} \times \Lambda^\vee \times \Lambda^\vee)^d$	

where the non-trivial entry is short for

$$((x, m), (y, n)) \;\longmapsto\; \exp(-J(m, y)).$$

Autoequivalences of the category of coherent sheaves

$$\begin{array}{rcl} \widehat{T} & = & Hom(T, U(1)) \\ \\ T_{\mathbb{C}} & = & spec\, \mathbb{C}[\widehat{T}] \\ \\ \mathcal{C}oh T_{\mathbb{C}} & \simeq & \mathbb{C}[\widehat{T}] - mod^{fin} \\ \\ 1Aut(\mathcal{C}oh T_{\mathbb{C}}) & \simeq & Bimod^{fin}_{\mathbb{C}[\widehat{T}]} \qquad\qquad \text{[Deligne].} \end{array}$$

Inside $1Aut(\mathcal{C}oh(T_{\mathbb{C}}))$, we have the full subcategory of direct image functors f_* of variety automorphisms f. This categorical group belongs to the crossed module

$$\mathbb{C}[\widehat{T}]^{\times} \xrightarrow{1} Aut_{var}(T_{\mathbb{C}}),$$

where f acts on $\mathbb{C}[\widehat{T}]^{\times}$ by precomposition, $\varphi \mapsto \varphi \circ f$.

The basic representation of a categorical torus

The basic representation of $\mathcal{T}$ is the strict monoidal functor

$$\varrho_{bas} : \mathcal{T} \longrightarrow 1Aut(\mathcal{C}oh(T_{\mathbb{C}})).$$

induced by the map of crossed modules

$$r_{bas}\colon \quad \begin{tikzcd}
(m,z) \arrow[r, mapsto] & z \cdot e^{2\pi i J(m,-)} \\
\Lambda^{\vee} \times U(1) \arrow[r] \arrow[d, "\psi"'] & \mathbb{C}[\,\widehat{T}\,]^{\times} \arrow[d, "1"] \\
\mathfrak{t} \arrow[r] & Aut_{var}(T_{\mathbb{C}}) \\
x \arrow[r, mapsto] & mult_{\exp(x)}.
\end{tikzcd}$$

The involution ι

The involution ι of T, sending t to t^{-1} lifts to an involution of $\mathcal{T}$, given by the map of crossed modules

$$\begin{tikzcd}
 & (m,z) \arrow[r, mapsto] & (-m,z) \\
\iota\colon & \Lambda^\vee \times U(1) \arrow[r] \arrow[d, "\psi"'] & \Lambda^\vee \times U(1) \arrow[d, "1"] \\
 & \mathfrak{t} \arrow[r] & \mathfrak{t} \\
 & x \arrow[r, mapsto] & -x.
\end{tikzcd}$$

This gives rise to an action of the group $\{\pm 1\}$ by (strict monoidal) functors on the category $\mathcal{T}$.

Extraspecial categorical 2-groups

The fixed points of ι on T form the elementary abelian 2-group

$$T^{\{\pm 1\}} \;=\; T[2] \;\cong\; \Lambda^{\vee} / 2\Lambda^{\vee}.$$

The categorical fixed points (or equivariant objects) of ι on $\mathcal{T}$ form an extension

$$pt /\!\!/ U(1) \longrightarrow \mathcal{T}^{\{\pm 1\}} \longrightarrow \widetilde{T[2]}$$

of the extraspecial 2-group $\widetilde{T[2]}$ with Arf invariant

$$\phi(m) \;=\; \frac{1}{2} I(m, m) \quad \text{mod } 2\Lambda^{\vee}.$$

Example: In the example of the Leech lattice, $\widetilde{T[2]}$ is the subgroup of the Monster that is usually denoted 2^{1+24}.

1Automorphisms of the basic representation

Let $\mathcal{T}_{\mathbb{C}} \rtimes \{\pm 1\}$ be the categorical group of the crossed module

$$\begin{aligned} \Lambda^{\vee} \times \mathbb{C}^{\times} &\longrightarrow \mathfrak{t}_{\mathbb{C}} \rtimes \{\pm 1\} \\ (m, z) &\longmapsto (m, 1), \end{aligned}$$

where -1 acts on everything by ι.

Extend the basic representation to

$$\varrho_{bas} : \mathcal{T}_{\mathbb{C}} \rtimes \{\pm 1\} \longrightarrow 1Aut(\mathcal{C}oh(T_{\mathbb{C}})),$$

by setting $r_{bas}(-1) := \iota$. So, $\varrho_{bas}(-1) := \iota_*$.

Theorem: The 1automorphisms of this ϱ_{bas} form the extraspecial categorical 2-group $\mathcal{T}_{\mathbb{C}}^{\{\pm 1\}}$.

Normalizers

Let

$$\varrho : H \longrightarrow G = GL(V)$$

be a representation of a group H on some vector space. Then

$$Aut(\varrho) \;=\; C(\varrho) \;=\; \{g \in G \mid c_g \circ \varrho \;=\; \varrho\}$$

is the centralizer of (the image of) ϱ in G. Here c_g is conjugation by g.

Definition [Dror Farjoun, Segev]: The *injective normalizer* of ϱ is the subgroup of $Aut(H) \times G$ defined as

$$N(\varrho) \;=\; \{(f, g) \mid c_g \circ \varrho \;=\; \varrho \circ f\}.$$

If ϱ is injective, this is the normalizer of its image.

Towards the refined Monster?

(In progress)

Theorem: The 1automorphisms of $\mathcal{T}$ form an extension

$$pt /\!\!/ \Lambda \longrightarrow 1Aut(\mathcal{T}) \longrightarrow O(\Lambda^{\vee}, I).$$

Here $O(\Lambda^{\vee}, I)$ is the group of linear isometries of $(\Lambda^{\vee}, I)$.

Example: the Conway group

$$O(\Lambda^{\vee}_{Leech}, I) \;=\; Co_0.$$

In spirit, the subgroup of the Monster, known as

$$2^{1+24}.Co_1 \;=\; \widetilde{T[2]} \rtimes (Co_0 \,/\, \{\pm id\})$$

wants to parametrize the isomorphism classes of some categorical variant of normalizer of $\varrho_{bas} : \mathcal{T}_{\mathbb{C}} \rtimes \{\pm 1\} \longrightarrow 1Aut(\mathcal{C}oh(T_{\mathbb{C}}))$.