# INVARIANTS FROM KK-THEORY


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Hamburg, February 2015

## History and Overview

Kasparov's bivariant KK theory is an extension of $K$-theory and its dual, $K$-homology.

Topological $K$-theory–Atiyah and Hirzebruch–(1950s).
Its 'defining feature' is Bott periodicity.
So in the complex case there are two groups $K^0(X)$ and $K^1(X)$ associated to a topological space $X$.
$K$-theory is a generalised cohomology theory. A model for the dual homology theory was suggested by Atiyah.
His idea was realised by Brown, Douglas and Fillmore in a special case (1970s)
But the general definition remained elusive until Kasparov worked it out around the beginning of the 80s.

Kasparov understood that $K$-theory and its dual fitted into a more general picture in the mid 80s.

First, to go from A-H to Kasparov, we need to replace topological spaces by commutative algebras. These algebras are the continuous functions on the space vanishing at infinity (we restrict to paracompact Hausdorff spaces).

This works ultimately because vector bundles over a space $Y$ give projective modules over the continuous functions $C(Y)$ so that $K$-theory can be described algebraically.

Establishing the properties of $K$-theory of the algebra of continuous functions requires (following Atiyah's student Wood) the introduction of non-commutative algebras namely matrix algebras over $C(Y)$.

Subsequently it was realised that these proofs for topological $K$-theory 'worked also' for noncommutative $C^*$-algebras. That is, we can axiomatise $K$-theory without invoking commutativity.

This led to the Kasparov point of view via the work of many other mathematicians.

I want to start with the definition which will need some unpacking.

**Definition.** Let $A$ and $B$ be $C^*$-algebras, with $A$ separable. An odd Kasparov $A$-$B$-module $X$ consists of a countably generated ungraded right $B$-$C^*$-module $X$, with $\pi : A \to \mathrm{End}_B(X)$ a $*$-homomorphism, together with $F \in \mathrm{End}_B(X)$ such that

$$\pi(a)(F - F^*),\ \pi(a)(F^2 - 1),\ [F, \pi(a)]$$

are compact adjointable endomorphisms of $X$, for each $a \in A$.

**Additional definitions, notation...**

$X$ is a left $A$, right $B$ module.

As a $B$-module it is a $C^*$-module meaning it is equipped with a right-action and a $B$ valued inner product: $X \times X \to B$, written as $x, y \mapsto (x|y)_B \in B$. It is conjugate linear in the first variable linear in the second.

The inner product satisfies some additional axioms that generalise the notion of an inner product so that the complex numbers are replaced in a sense by the algebra (noncommutative in general) $B$.

For example in the case where $X = B$ then $(x|y)_B = x^* y$.

Rank one operators:

$$\theta_{x,y} z = x \cdot (y|z)_B \quad x, y, z \in X$$

Finite rank operators are finite linear combinations of rank one operators.

Norm on $X$ is given by $\|x\|^2 = \|(x|x)_B\|$.

$X$ must be both complete and countably generated.

$\mathrm{End}_B(X)$ consists of the $B$-linear endomorphisms of $X$.

Compact endomorphisms: close up the finite rank operators in the norm on continuous linear operators on $X$.

An adjointable operator $T$ on $X$ is one where there is an operator $T^* : X \to X$ with

$$(T^*x|y)_B = (x|Ty)_B.$$

An even Kasparov $A$-$B$-module is an odd Kasparov $A$-$B$-module, together with a $\mathbf{Z}_2$ grading operator. This means we have a self-adjoint $B$-linear endomorphism $\gamma$ with $\gamma^2 = 1$ and $\pi(a)\gamma = \gamma\pi(a)$, $F\gamma + \gamma F = 0$.

We will use the notation $({}_AX_B, F)$ or $({}_AX_B, F, \gamma)$ for Kasparov modules, generally omitting the representation $\pi$. A Kasparov module $({}_AX_B, F)$ with $\pi(a)(F - F^*) = \pi(a)(F^2 - 1) = [F, \pi(a)] = 0$, for all $a \in A$, is called *degenerate*.

We now describe the equivalence relation on Kasparov $A$-$B$-modules which defines classes in the abelian group $KK(A,B) = KK^0(A,B)$ (even case) or $KK^1(A,B)$ (odd case). Because of Bott periodicity there are only these two groups.

The relation consists of three separate equivalence relations: unitary equivalence, stable equivalence and operator homotopy.

Two Kasparov $A$-$B$-modules $({}_A(X_1)_B, F_1)$ and $({}_A(X_2)_B, F_2)$ are *unitarily equivalent* if there is an adjointable unitary $B$-module map $U : X_1 \to X_2$ such that $\pi_2(a) = U\pi_1(a)U^*$, for all $a \in A$ and $F_2 = U\,F_1\,U^*$.

Two Kasparov $A$-$B$-modules $({}_A(X_1)_B, F_1)$ and $({}_A(X_2)_B, F_2)$ are *stably equivalent* if there is a degenerate Kasparov $A$-$B$-module $({}_A(X_3)_B, F_3)$ with $({}_A(X_1)_B, F_1) = ({}_A(X_2 \oplus X_3)_B, F_2 \oplus F_3)$ and $\pi_1 = \pi_2 \oplus \pi_3$.

Two Kasparov $A$-$B$-modules $({}_A(X)_B, G)$ and $({}_A(X)_B, H)$ (with the same representation $\pi$ of $A$) are called *operator homotopic* if there is a norm continuous family $(F_t)_{t\in[0,1]} \subset \mathrm{End}_B(X)$ such that for each $t \in [0,1]$ $({}_A(X_1)_B, F_t)$ is a Kasparov module and $F_0 = G$, $F_1 = H$.

Two Kasparov modules $({}_A(X)_B, G)$ and $({}_A(X)_B, G)$ are equivalent if after the addition of degenerate modules, they are operator homotopic to unitarily equivalent Kasparov modules.

The equivalence classes of even (resp. odd) Kasparov $A$-$B$ modules form an abelian group denoted $KK^0(A, B)$ (resp. $KK^1(A, B)$). The zero element is represented by any degenerate Kasparov module, and the inverse of a class $[({}_A(X)_B, F)]$ is the class of $({}_A(X)_B, -F)$, with grading $-\gamma$ in the even case.

The deepest part of the theory developed by Kasparov is the product

$$KK^i(A,B) \times KK^j(B,C) \mapsto KK^{i+j}(A,C).$$

It remains an unresolved issue to find a more transparent proof of the existence of the Kasparov product.

The equivalence relation defining the $KK$ group, in conjunction with the Kasparov product, implies further equivalences between Kasparov modules, such as Morita equivalence.

It leads to the notion of $KK^0(A,B)$ as a 'morphism' between $A$ and $B$ because we have a 'composition rule'

$$KK^0(A,B) \times KK^0(B,C) \mapsto KK^0(A,C).$$

Special cases:

$KK^*(\mathbf{C}, C)$ is the $K$-group of $C$ and for commutative $C^*$-algebras which are of the form $C_0(Y)$, $Y$ paracompact and Hausdorff, this is the theory first introduced by Atiyah-Hirzebruch. Historically of course there was a direct definition of the K-theory groups by algebraic means before Kasparov...

$KK^*(A, \mathbf{C})$ is the $K$-homology group of $A$ due essentially to Kasparov but is based on Atiyah's Ell-theory. Representatives of this group are Hilbert spaces that are $A$ modules. They are usually called Fredholm modules.

More generally $KK(*, *)$ is a bifunctor that takes pairs of algebras to abelian groups. It is covariant in the second variable and contravariant in the first variable.

$KK(*, *)$ is 'stable' in the sense that if we tensor $A$ or $B$ by a copy of the compact operators on a separable Hilbert space then we do not change $KK^*(A, B)$.

There are a number of ways to express Bott periodicity in $KK$-theory. Kasparov exploited Clifford algebra periodicity (period 2 in the complex case and period 8 in the real case) to achieve this in his first paper.

Another way is to use the suspension of an algebra $A$ which is by definition the algebra $\Sigma B := C_0(\mathbf{R}, B)$, the continuous functions on $\mathbf{R}$ with values in $B$ and vanishing at infinity or equivalently $C_0(0,1) \otimes B$ where the subscripted zero indicates functions vanishing at the endpoints.

Note that $\Sigma$ is a functor in our category of $C^*$-algebras which introduces a shift in degree in $KK(*,*)$

Now Bott periodicity in $KK$-theory can be captured by showing that if we replace either argument in $KK(A,B)$ by its double suspension then we have an isomorphic group. The proof exploits stability.

Index theory is connected to the Kasparov product as we have

$$KK^i(\mathbf{C}, A) \times KK^j(A, \mathbf{C}) \to KK^{i+j}(\mathbf{C}, \mathbf{C})$$

Notice that when the right hand argument in $KK^*(A, B)$ is the complex numbers, then a $B$-valued inner product is just an ordinary inner product. So then ${}_A X_{\mathbf{C}}$ is a Hilbert space carrying a representation of $A$. When $A = \mathbf{C}$ it is just a Hilbert space (perhaps graded). and $KK^0(\mathbf{C}, \mathbf{C}) = \mathbf{Z}$.

Connes and Skandalis produced a proof of the Atiyah-Singer index theorem in the Kasparov framework.

The operator $F$ in that case arises from a Dirac-type operator acting on sections of a vector bundle over the underlying even dimensional manifold $M$. Using this $F$ we can produce an element of the $K$-homology of $C(M)$ that is of $KK^0(C(M), \mathbf{C})$.

Then there is a pairing, defined using the Kasparov product, with elements of $KK^0(\mathbf{C}, C(M))$, the $K$-theory of $M$. In this case one may pair with the element of $K$-theory defined by the vector bundle on which the Dirac type operator acts.

Of course in practice we want to have an explicit expression for the pairing implied by the existence of the Kasparov product and this is given by the usual Atiyah-Singer formula.

The classical Dirac type situation leads to an 'unbounded Kasparov picture' now better known via the special case of 'spectral triples' as introduced by Alain Connes.

## The unbounded version

**Definition.** Given $\mathbf{Z}_2$-graded $C^*$-algebras $A$ and $B$, an even *unbounded* Kasparov $A$-$B$-module $({}_A(X)_B, \mathcal{D})$ is given by

1. A $\mathbf{Z}_2$-graded, countably generated, right $B$ $C^*$-module $X_B$;
2. A $\mathbf{Z}_2$-graded $*$-homomorphism $\rho\colon A \to \mathrm{End}_B(X)$;
3. A self-adjoint, regular, odd operator $\mathcal{D}\colon \mathrm{Dom}(\mathcal{D}) \subset X \to X$ such that $[\mathcal{D}, \rho(a)]_\pm$ is an adjointable endomorphism, and $\rho(a)(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism for all $a$ in a dense subalgebra $\mathcal{A}$ of $A$.

If the module and algebras are trivially graded, then the Kasparov module is called odd.

**Proposition** [Baaj-Julg] If $({}_A(X)_B, \mathcal{D})$ is an unbounded Kasparov module, then $\left({}_A(X)_B, \mathcal{D}(1+\mathcal{D}^2)^{-1/2}\right)$ is a Kasparov module.

Recent work by Mesland, Kaad, Lesch, ... has shown that in many examples unbounded Kasparov modules can be used to explicitly write down the product

$$KK(A,B)\times KK(B,\mathbf{C})\to KK(A,\mathbf{C})$$

represented by explicit elements of the respective groups

$$[({}_A(X)_B,\mathcal{D}_1)]\,\hat{\otimes}_B\,[({}_B(\mathcal{H})_{\mathbf{C}},\mathcal{D}_2)]=\left[\left({}_A(X\hat{\otimes}_B\mathcal{H})_{\mathbf{C}},\mathcal{D}_1\hat{\otimes}1+1\hat{\otimes}_\nabla\mathcal{D}_2\right)\right]$$

where $[({}_A(X)_B,\mathcal{D})]$ denotes the corresponding $KK$-class from the bounded transformation of Baaj-Julg.

The full unbounded Kasparov theory and its potential applications are an area still being developed.

## Kasparov theory and the bulk-edge correspondence

The application of the general theory we are looking at is to topological insulators. These use KO-theory and hence are more complicated. We illustrate the idea with the quantum Hall effect.

In the 'tight binding' model of a quantum Hall system, we have magnetic translations $\widehat{U}$ and $\widehat{V}$ as unitary operators on $\ell^2(\mathbf{Z}^2)$. These operators commute with the unitaries $U$ and $V$ that generate the Hamiltonian $H = U + U^* + V + V^*$, where

$$(\widehat{U}\lambda)(m,n) = \lambda(m-1,n), \qquad (\widehat{V}\lambda)(m,n) = e^{-2\pi i\phi m}\lambda(m,n-1),$$
$$(U\lambda)(m,n) = e^{-2\pi i\phi n}\lambda(m-1,n), \quad (V\lambda)(m,n) = \lambda(m,n-1),$$

These operators act on $\ell^2(\mathbf{Z}^2)$. We would also like to consider a system with boundary. This uses the Hilbert space $\ell^2(\mathbf{Z}\times\mathbf{N})$. The bulk-edge correspondence is about linking the topological properties of the 'bulk' (boundary-free) system to a system with an edge.

Let $S$ be the unilateral shift operator on $\ell^2(\mathbf{N})$ with $S^*S = 1$, $SS^* = 1 - P_{n=0}$. We use the notation $\mathcal{K}(\mathcal{H})$ to denote the compact operators on a Hilbert space $\mathcal{H}$.

Kellendonk, Richter and Schulz-Baldes link 'bulk' (no boundary) and edge systems via the short exact sequence

$$0 \to C^*(\widehat{U}) \otimes \mathcal{K}[\ell^2(\mathbf{N})] \xrightarrow{\psi} C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S) \to C^*(\widehat{U}, \widehat{V}) \to 0,$$

where $\psi$ is such that

$$\psi(\widehat{U}^m \otimes e_{jk}) = (\widehat{V}^*)^j \widehat{U}^m \widehat{V}^k \otimes S^j P_{n=0} (S^*)^k$$

for matrix units $e_{jk}$ in $\mathcal{K}[\ell^2(\mathbf{N})]$ and then extended to the full algebra by linearity.

In this sequence the quotient algebra $C^*(\widehat{U}, \widehat{V})$ is the one applicable to the bulk (i.e. no boundary) system.

The algebra $C^*(\widehat{U}) \otimes \mathcal{K}[\ell^2(\mathbf{N})]$ is an ideal in $C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S)$.

We think of the ideal as operators which act on $\ell^2(\mathbf{Z} \times \mathbf{N})$ and decay sufficiently fast away from the edge $\mathbf{Z} \times \{0\}$.

## Building a Kasparov module

Abstract theory tells us that short exact sequences of the type

$$0 \to C^*(\widehat{U}) \otimes \mathcal{K}[\ell^2(\mathbf{N})] \to C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S) \to A_\phi \to 0$$

give rise to a class in $KK^1(A_\phi, C^*(\widehat{U}))$. Let's make this explicit using unbounded Kasparov theory.

We need a $C^*(\widehat{U})$-valued inner product on $C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S)$.

First, let $\Psi$ be some linear functional on $C^*(S) \subset C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S)$ and consider the formula

$$\begin{aligned}
&\left( \widehat{V}^{l_1 - l_2} \widehat{U}^{m_1} \otimes S^{l_1} (S^*)^{l_2} \,\middle|\, \widehat{V}^{n_1 - n_2} \widehat{U}^{m_2} \otimes S^{n_1} (S^*)^{n_2} \right) \\
&\qquad := \left( \widehat{V}^{l_1 - l_2} \widehat{U}^{m_1} \right)^* \widehat{V}^{n_1 - n_2} \widehat{U}^{m_2}\, \Psi \left[ \left( S^{l_1} (S^*)^{l_2} \right)^* S^{n_1} (S^*)^{n_2} \right] \\
&\qquad = \widehat{U}^{-m_1} \widehat{V}^{n_1 - n_2 - (l_1 - l_2)} \widehat{U}^{m_2}\, \Psi \left[ S^{l_2} (S^*)^{l_1} S^{n_1} (S^*)^{n_2} \right]
\end{aligned}$$

for $m_1, m_2 \in \mathbf{Z}$ and $n_1, n_2, l_1, l_2 \in \mathbf{N}$.

## Building a Kasparov module (cont.)

We want the functional $\Psi$ to have the property that $\Psi\left[S^{l_2}(S^*)^{l_1}S^{n_1}(S^*)^{n_2}\right] = \delta_{l_1-l_2,n_1-n_2}$ and that $\Psi(T) = 0$ if $T$ is compact. The functional

$$\Psi(T) = \operatorname*{res}_{s=1} \sum_{k=0}^{\infty} \langle e_k, Te_k\rangle (1+k^2)^{-s/2}$$

does the trick, where $\{e_k\}$ is any basis of $\ell^2(\mathbf{N})$. Therefore

$$\begin{aligned}\left(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\middle|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\right)& \\ = \widehat{U}^{-m_1}\widehat{V}^{n_1-n_2-(l_1-l_2)}\widehat{U}^{m_2}\,\delta_{l_1-l_2,n_1-n_2}& \\ = \widehat{U}^{m_2-m_1}\delta_{l_1-l_2,n_1-n_2}&\end{aligned}$$

so we have a $C^*(\widehat{U})$-valued inner-product.

We also introduce the right-action of $C^*(\widehat{U})$, where for any $\alpha \in \mathbf{Z}$,

$$\left(\widehat{V}^{n_1-n_2}\widehat{U}^{m}\otimes S^{n_1}(S^*)^{n_2}\right)\cdot\widehat{U}^{\alpha} = \widehat{V}^{n_1-n_2}\widehat{U}^{m+\alpha}\otimes S^{n_1}(S^*)^{n_2}.$$

We divide out the zero-length vectors of $C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S)$ in the norm induced by $(\cdot \mid \cdot)_{C^*(\widehat{U})}$ (non-trivial!) and complete to obtain the $C^*$-module $Z_{C^*(\widehat{U})}$.

Next, we need an adjointable left-action by $\mathcal{A}_\phi \cong C^*(\widehat{U}, \widehat{V})$. We define on generating elements

$$\begin{aligned}
(\widehat{U}^\alpha \widehat{V}^\beta) \cdot \left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) \\
&= (\widehat{U}^\alpha \widehat{V}^\beta \widehat{V}^{n_1-n_2}\widehat{U}^m) \otimes S^{n_1+\beta}(S^*)^{n_2} \\
&= e^{2\pi i\phi\alpha(n_1-n_2+\beta)}\widehat{V}^{\beta+n_1-n_2}\widehat{U}^{m+\alpha} \otimes S^{\beta+n_1}(S^*)^{n_2}
\end{aligned}$$

for $\alpha, \beta \in \mathbf{Z}$ with $\beta \geq 0$ and an analogous formula but with $S^{n_1}(S^*)^{n_2+|\beta|}$ for $\beta < 0$.

A computation shows that this representation is adjointable under the inner product $(\cdot \mid \cdot)_{C^*(\widehat{U})}$.

Finally, we introduce a number operator
$N : \mathrm{Dom}(N) \subset Z_{C^*(\widehat{U})} \to Z_{C^*(\widehat{U})}$ on generating elements as

$$N\left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) = (n_1-n_2)\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}.$$

**Proposition** $\left({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N\right)$ is an odd unbounded Kasparov module.

Furthermore, the corresponding class $\left[({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N)\right]$ in $KK^1(A_\phi, C^*(\widehat{U}))$ is the same as the class induced by the short-exact sequence linking the bulk and edge algebras.

Proof relies on the singular nature of the inner-product and the functional $\Psi$.

## Edge spectral triple

Next we consider our edge algebra $C^*(\widehat{U})$ acting as shift operators on the space $\ell^2(\mathbf{Z})$.

We have a natural spectral triple in this setting given by

$$\left(C^*(\widehat{U}), \ell^2(\mathbf{Z}), M\right),$$

where $M : \mathrm{Dom}(M) \to \ell^2(\mathbf{Z})$ is given by $M\lambda(m) = m\lambda(m)$.

Our spectral triple is an odd unbounded $C^*(\widehat{U})$-$\mathbf{C}$ Kasparov module and so gives a class in $KK^1(C^*(\widehat{U}), \mathbf{C})$.

## Factorisation of bulk triple

To review, we have our bulk triple giving a class in $KK(A_\phi, \mathbf{C})$, the Kasparov module representing the short exact sequence giving a class in $KK^1(A_\phi, C^*(\widehat{U}))$ and an edge spectral triple giving a class in $KK^1(C^*(\widehat{U}), \mathbf{C})$.

**Theorem** [Bourne-C-Rennie] Under the internal Kasparov product

$$KK^1(A_\phi, C^*(\widehat{U})) \times KK^1(C^*(\widehat{U}), \mathbf{C}) \to KK(A_\phi, \mathbf{C})$$

we have that

$$\Big[ ({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N) \Big] \hat{\otimes}_{C^*(\widehat{U})} \Big[ ({}_{C^*(\widehat{U})}(\ell^2(\mathbf{Z}))_{\mathbf{C}}, M) \Big]$$
$$= - \left[ ({}_{A_\phi}(\ell^2(\mathbf{Z}^2))_{\mathbf{C}}, X, \gamma) \right]$$

where $-[X]$ denotes the inverse class in the $KK$-group.

The proof of this Theorem relies on the explicit formula for the Kasparov product that the unbounded picture gives us.

## Pairings and the bulk-edge correspondence (QHE)

Recall that Bellissard's expression for the Hall conductance comes from the pairing of the $K$-theory class of the Fermi projection $[P_\mu] \in KK(\mathbf{C}, A_\phi)$ with the bulk spectral triple; that is,

$$\sigma_H = \frac{e^2}{h}\left([P_\mu]\,\hat{\otimes}_{A_\phi}\left[({}_{A_\phi}(\ell^2(\mathbf{Z}^2))_{\mathbf{C}}, X, \gamma)\right]\right).$$

We can now use the Theorem to rewrite this as

$$\sigma_H = -\frac{e^2}{h}\left([P_\mu]\,\hat{\otimes}_{A_\phi}\left[({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N)\right]\hat{\otimes}_{C^*(\widehat{U})}\left[({}_{C^*(\widehat{U})}(\ell^2(\mathbf{Z}))_{\mathbf{C}}, M)\right]\right)$$

and the bulk-edge correspondence follows immediately from the associativity of the Kasparov product.

To see this, our 'edge conductance' is

$$-\frac{e^2}{h}\left([P_\mu]\,\hat{\otimes}_{A_\phi}\left[({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N)\right]\right)\hat{\otimes}_{C^*(\widehat{U})}\left[({}_{C^*(\widehat{U})}(\ell^2(\mathbf{Z}))_{\mathbf{C}}, M)\right],$$

which is a pairing of elements in $KK^1(\mathbf{C}, C^*(\widehat{U})) \cong K_1(C^*(\widehat{U}))$ with $K^1(C^*(\widehat{U}))$. Hence this is a pairing of the $K$-theory and $K$-homology of our edge algebra.

## Other Applications

We would like use our general method study other topological phases of matter by imposing time-reversal, particle-hole or chiral symmetry. Time-reversal and particle-hole operators act anti-unitarily on our space and so introduce a Real structure into our system.

This means that instead of complex $K$-theory and ($KK$-theory) we must deal with $KO$ or $KR$-theory. Work by, among others, Thiang makes this connection explicit.

Because all the central results of Kasparov theory holds for $KKR$ and $KKO$ groups, we hope to refine our general method to establish a rigorous bulk-edge correspondence for other topological insulator systems.