Dr. Philipp Wruck 24.11.2011

Exercise Sheet 3
Transformation Groups

Exercise 1: Let G be a topological group, acting on itself via conjugation. For a G-space A, consider the set of G-maps $A \to G$.

a) Show that there is a map

$$TOP_G(A, G) \to Homeo_G(A),\ \vartheta \mapsto (a \mapsto \vartheta(a).a)$$

and conclude that there is an equivariant homeomorphism $A \times A \to A \times A,\ (a, b) \mapsto (\vartheta(a).b, \vartheta(a).a)$.

b) In the case $A = \mathbb{S}^{n-1}$, $G = O(n)$, determine all possible homeomorphisms that can arise from a).

c) Take a non-trivial homeomorphism $\varphi : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ obtained from b). Denote the space

$$\mathbb{S}^{n-1} \times \mathbb{B}^n \cup_{\varphi^k} \mathbb{S}^{n-1} \times \mathbb{B}^n$$

by Σ_k^{2n-1}. Show that Σ_1^{2n-1} is homeomorphic to $\mathbb{S}^{2n-1}$.

Exercise 2: The Klein bottle K is defined as the quotient of $[0, 1]^2$, obtained by identifying $(x, 0)$ with $(1 - x, 1)$ and $(0, y)$ with $(1, y)$. Show that K is a fibre bundle with typical fibre $\mathbb{S}^1$ and structure group $\mathbb{Z}_2$.

Exercise 3: Let $p : E \to B$ be a fibre bundle with typical fibre F and structure group K. Let $\{U_i\}_{i\in I}$ be an atlas of bundle charts and ϑ_{ij} the transition function for U_i and U_j. Define a G-space E' as the quotient space of the space (disjoint union)

$$\coprod_{i\in I} U_i \times K$$

by the relation $U_i \times K \ni (x, k) \sim (x, \vartheta_{ij}(x) \circ k) \in U_j \times K$. Show that the projection $p' : E' \to B$ onto the first factor is a principal K-bundle and that p is the F-bundle associated to p'.

Handing in: 08.12.11 in exercise class.

Action of the Week

Group:	$\mathbb{Z}_p$, p prime
Space:	$\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$
Action:	$1.(z_1, \ldots, z_n) = (\mathrm{e}^{\frac{2\pi i q_1}{p}} \cdot z_1, \ldots, \mathrm{e}^{\frac{2\pi i q_n}{p}} \cdot z_n)$ $q_1, \ldots, q_n$ relatively prime to p
Isotropies:	$\{e\}$
Further properties:	Quotient spaces are the lens spaces