1 Categorical Background

1.1 Categories and Functors

Definition 1.1.1 A category $\mathcal{C}$ is given by a class of objects, often denoted by $ob\,\mathcal{C}$, and for any two objects A, B of $\mathcal{C}$ a proper set of morphisms $\mathcal{C}(A, B)$, such that

1. there is an associative composition of morphisms
$$\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C),\ (f, g) \mapsto g \circ f$$
2. for any object A of $\mathcal{C}$, there is an identity morphism $\mathrm{id}_A \in \mathcal{C}(A, A)$ which is a unit for the composition law.

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ consists of a subclass of objects of $\mathcal{C}$, for every $A, B \in ob\,\mathcal{C}'$ we have $\mathcal{C}'(A, B) \subseteq \mathcal{C}(A, B)$, and composition law and identity element are inherited from $\mathcal{C}$.

This definition comprises the basic mathematical philosophy that mathematical objects can be understood by investigating their transformations. Almost every mathematical structure a student will encounter during his first years will be rooted in a category. Some examples are given below.

1. The category $\mathcal{SET}$ of sets, together with maps between sets as morphisms. This category is of particular importance, since many categories are defined by adding extra structure to sets and morphisms of sets.
2. The category $\mathcal{VECT}_k$ of vector spaces and linear maps between vector spaces as morphisms. More generally, for a ring R, there is the category $R - \mathcal{MOD}$ of modules over R together with R-linear maps as morphisms.
3. The category $\mathcal{TOP}$ of topological spaces and continuous maps as morphisms. This is one of the main categories we will be working with.
4. The category $h\mathcal{TOP}$ of topological spaces and homotopy classes of continuous maps between them as morphisms. That is, for any two topological spaces X, Y, a morphism $[f] \in h\mathcal{TOP}(X, Y)$ is an equivalence class of continuous maps from X to Y, where two such maps f_0, f_1 are equivalent, if there is a continuous homotopy $H : X \times [0, 1] \to Y$ such that $H_0 = f_0$, $H_1 = f_1$. This is the first example of a category whose morphisms are not given as concrete maps.
5. A group G can be regarded as a category with a single object G with morphisms the elements of G.

Applying the basic philosophy to the definition of a category itself, we have to specify morphisms of categories. These are so called functors.

Definition 1.1.2 Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is given by

(i) a map $ob\,\mathcal{C} \to ob\,\mathcal{D}$ (denoted by F as well)

(ii) for any pair A, B of objects of $\mathcal{C}$ a map

$$\mathcal{C}(A,B) \to \mathcal{D}(F(A), F(B))$$

(again denoted by F).

Furthermore, the map on morphism level is required to be compatible with identity and composition, that is, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Again there are many examples that will be well known to the student, whereas the abstract interpretation may be unfamiliar.

Example 1.1.3 1. The functor $\mathcal{P} : \mathcal{SET} \to \mathcal{SET}$ assigns to a set X its power set $\mathcal{P}(X)$. On morphisms, if $f : X \to Y$ is a map, $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ maps a subset $A \subseteq X$ to the subset $f(A) \subseteq Y$.

2. The functor $T : \mathcal{MAN} \to \mathcal{MAN}$ of the category of (smooth) manifolds assigns to a manifold M its tangential bundle TM. For a smooth map $f : M \to N$, Tf is the usual tangential map.

3. The functor $\pi_0 : \mathcal{TOP} \to \mathcal{SET}$ assigns to a topological space X the set $\pi_0(X)$ of its connected components. For a continuous map $f : X \to Y$, $\pi_0(f)$ maps a connected component C of X to the connected component of Y which contains $f(C)$ (since images of connected subsets are connected, this is well defined).

A functor is called faithful, if the induced map $\mathcal{C}(A,B) \to \mathcal{D}(F(A), F(B))$ is injective for all objects A, B of $\mathcal{C}$. It is called full, if this map is surjective and fully faithful, if it is bijective. In the above examples, the power set functor is faithful, but not full. A map $\mathcal{P}(X) \to \mathcal{P}(Y)$ having as value on a one-point set a more-point set can not lie in the image of the functor. The tangential bundle functor similarly is faithful but not full. A tangential map projects to its underlying map via the bundle projection, hence, the functor is faithfull. But a tangential map is always linear when restricted to tangential spaces, so it can not be full. Finally, the π_0-functor is full but not faithful. Any map $\pi_0(X) \to \pi_0(Y)$ can be realized as a map of spaces by taking maps that are constant on connected components. And obviously, any continuous map of connected spaces induces the same map in π_0.

One can carry on and ask for transformations of functors and transformations of these transformations etc. We will only need the concept of a natural transformation between functors.

Definition 1.1.4 Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors. A natural transformation $\eta : F \to G$ is defined by a collection of morphisms $\eta_A : F(A) \to G(A)$ for every object A of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"'] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

commutes for any two objects A, B of $\mathcal{C}$ and every morphism $f : A \to B$.

Example 1.1.5 The above definition clarifies the meaning of the word natural in many mathematical contexts. As an example, let V be a k-vector space and let V'' be its second dual space. This specifies a functor

$$D : \mathcal{VECT}_k \to \mathcal{VECT}_k,\ V \mapsto V'',\ (f : V \to W) \mapsto (f'' : V'' \to W'').$$

Recall that the dual map of $f : V \to W$ is defined by $f' : W' \to V',\ f'(w')(v) = w'(f(v))$, so the double dual is defined as $f'' : V'' \to W'',\ f''(v'')(w') = v''(f'(w'))$. We now define a natural transformation between the identity functor and D, namely we define

$$\eta_V : V \to V'',\ v \mapsto (v' \mapsto v'(v)).$$

It is easy to see that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"] \\ V'' \arrow[r, "f''"] & W'' \end{tikzcd}$$

commutes, so indeed η is a natural transformation. It is well-known that η_V is a bijection if and only if V is finite dimensional.

1.2 Adjoint Functors

When one is interested in a certain functor $F : \mathcal{C} \to \mathcal{D}$ between two categories, from an abstract point of view there is not much else one can do than to investigate the morphism sets $\mathcal{D}(F(A), B)$ for objects A of $\mathcal{C}$ and B of $\mathcal{D}$. Under this aspect it is often extremely useful, and in general has severe implications on the structure of the functor, that this morphism set can be identified with a morphism set $\mathcal{C}(A, G(B))$, where $G : \mathcal{D} \to \mathcal{C}$ is another functor. This amounts to the following definition.

Definition 1.2.1 Let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. If there is a natural equivalence

$$\mathcal{C}(G(A), B) \to \mathcal{D}(A, F(B))$$

for any objects A of $\mathcal{D}$, B of $\mathcal{C}$, then F is called a right adjoint functor of G and G is called a left adjoint functor of F.

As we already pointed out, being an adjoint functor has severe categorical implications, but it would lead us too much afar to discuss these here. Instead, we will give some examples.

Definition 1.2.2 1. A very important case of adjoint functors arises in connection with forgetful functors. A forgetful functor is a functor $f : \mathcal{C} \to \mathcal{D}$ which, rather imprecisely, "forgets" some of the structure of the category $\mathcal{C}$. The functor $F : \mathcal{TOP} \to \mathcal{SET}$, assigning to a topological space its underlying set and to a continuous map its underlying set map is a typical example of a forgetful functor. Similar examples arise as forgetful functors $\mathcal{VECT}_k \to \mathcal{SET}$, $\mathcal{MAN} \to \mathcal{SET}$, $\mathcal{MAN} \to \mathcal{TOP}$.

We claim that the functor $F : \mathcal{TOP} \to \mathcal{SET}$ has a right adjoint $i : \mathcal{SET} \to \mathcal{TOP}$, assinging to the set X the topological space X with the discrete topology. Indeed, if we define

$$\eta : \mathcal{TOP}(i(X), Y) \to \mathcal{SET}(X, F(Y)),\ f \mapsto f,$$

this is a natural equivalence. Every set map $X \to F(Y)$ is continuous when X carries the discrete topology, no matter which topology is imposed on $F(Y)$.

There is also a right adjoint funtor for the forgetful functor $F : \mathcal{VECT}_k \to \mathcal{SET}$. For a set X, we define $i(X)$ to be the free vector space with basis X, that is,

$$i(X) = \{\sum_{i\in I} \lambda_i x_i \mid I \text{ finite},\ \lambda_i \in k,\ x_i \in X\}.$$

For a map $f : X \to Y$, we define

$$i(f) : i(X) \to i(Y),\ \sum \lambda_i x_i \mapsto \sum \lambda_i f(x_i),$$

which is obviously well-defined and linear. To see that this functor is right adjoint to F, we define

$$\eta : \mathcal{SET}(X, F(V)) \to \mathcal{VECT}_k(i(X), V),\ f \mapsto \left(\sum \lambda_i x_i \mapsto \sum \lambda_i f(x_i)\right).$$

This is a natural equivalence, since every linear map $i(X) \to V$ is completely determined by the images of a base of $i(X)$, which is X, and the images of elements of the base carry no further restriction.

2. To see that adjoint functors can also occur in different contexts than with forgetful functors, we give another example. For a set X, consider the functor

$$\mathcal{SET}(X, \cdot) : \mathcal{SET} \to \mathcal{SET},\ Y \mapsto \mathcal{SET}(X, Y).$$

For a morphism $f : Y \to Z$, we define

$$\mathcal{SET}(X, f) : \mathcal{SET}(X, Y) \to \mathcal{SET}(X, Z),\ h \mapsto f \circ h.$$

We claim that this functor has a left adjoint, namely the functor

$$\cdot \times X : \mathcal{SET} \to \mathcal{SET},\ Y \mapsto Y \times X$$

and for $f : Y \to Z$,

$$f \times X : Y \times X \to Z \times X,\ (y, x) \mapsto (f(y), x).$$

We define a map

$$\eta : \mathcal{SET}(Y, \mathcal{SET}(X, Z)) \to \mathcal{SET}(Y \times X, Z),\ \eta(f)(y, x) = f(y)(x).$$

This map has an inverse, assigning to $g : Y \times X \to Z$ the map $\hat{g}$ such that $\hat{g}(y)(x) = g(y, x)$, hence, η is bijective. Naturality is easily checked, so the two functors are indeed adjoints.

1.3 Monoidal Categories

In this section we want to introduce an additional structure on a category. The basic idea is that we want to add structure to morphism sets. For example, in the category of topological space, we want a topology on sets of continuous maps. To be able to do so in a not completely arbitrary way, we first need the notion of a monoidal category. This is, roughly speaking, a category together with a functor from the category of pairs to the category itself, satisfying properties similar to those of a monoid, as in linear algebra.

Definition 1.3.1 A monoidal category is a category $\mathcal{M}$ together with an object I of $\mathcal{M}$ and a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, such that the following axioms are satsified.

1. There is a natural equivalence α between the functors $(\otimes, \mathrm{id}_{\mathcal{M}}) : \mathcal{M}^3 \to \mathcal{M}$ and $(\mathrm{id}_{\mathcal{M}}, \otimes) : \mathcal{M}^3 \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \arrow[d, "\alpha_{A \otimes B, C, D}"'] & A \otimes (B \otimes C) \otimes D \arrow[d, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A,B,C \otimes D}"] & A \otimes (B \otimes (C \otimes D)))
\end{tikzcd}$$

   commutes.

2. There are natural equivalences λ and ρ between the functors $A \mapsto I \otimes A$ and the identity and $I \otimes A \mapsto A$ and the identity, respectively, such that the diagram

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda"] \\
& A \otimes B &
\end{tikzcd}$$

   commutes.

One should think of the functor $\otimes$ as an associative, unital product on the category $\mathcal{M}$. There are many examples for monoidal categories.

Example 1.3.2 1. The category $\mathcal{SET}$ together with the cartesian product and the one point set as unit is a monoidal category.

2. More generally, the categories of topological spaces, topological Hausdorff spaces or smooth manifolds are monoidal categories with the cartesian product and the one point space as unit.

3. The category $\mathcal{VECT}_k$ is a monoidal category with the tensor product as monoidal product and k as unit.

4. The category $\mathcal{TOP}_*$ of pointed topological spaces is a monoidal category with product the smash product and unit the zero sphere $\mathbb{S}^0$.

5. Every monoid M is a monoidal category when considered as a category with a single object M and morphisms the elements of M.

When dealing with monoidal categories, we should also specify functors that preserve the monoidal structure

Definition 1.3.3 Let $\mathcal{C}$, $\mathcal{D}$ be two monoidal categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. F is called a monoidal functor, if there is a natural equivalence

$$F(A \otimes B) \cong F(A) \otimes F(B)$$

and an isomorphism

$$F(I_{\mathcal{C}}) \cong I_{\mathcal{D}}.$$

We can now try to put the structure of objects of a monoidal category on the morphism sets of an arbitrary category $\mathcal{C}$.

Definition 1.3.4 Let $\mathcal{D}$ be a monoidal category. A $\mathcal{D}$-category or a category over $\mathcal{D}$ is a monoidal category $\mathcal{C}$ together with a functor $F : \mathcal{C} \to \mathcal{D}$ such that the following holds.

1. F is monoidal.

2. F is faithful.

3. For every object A of $\mathcal{C}$, the functor $\bullet \otimes F(A) : \mathcal{D} \to \mathcal{D}$ has a right adjoint, denoted by $\mathcal{C}_F(A, \bullet)$. Thus, for objects X, Y of $\mathcal{D}$ there is a natural isomorphism

$$\mathcal{D}(X, \mathcal{C}_F(A, Y)) \cong \mathcal{D}(X \otimes F(A), Y).$$

4. For any two objects A, B of $\mathcal{C}$, there is the image $\varepsilon = \varepsilon_{AB}$ of the identity under the adjunction

$$\mathcal{D}(\mathcal{C}_F(A, B), \mathcal{C}_F(A, B)) \cong \mathcal{D}(\mathcal{C}_F(A, B) \otimes A, B).$$

For three objects A, B, C of $\mathcal{C}$, the image of $\varepsilon_{BC} \circ (\mathrm{id} \otimes \varepsilon_{AB})$ under the adjunction

$$\mathcal{D}(\mathcal{C}_F(B, C) \otimes \mathcal{C}_F(A, B) \otimes A, C) \cong \mathcal{D}(\mathcal{C}_F(B, C) \otimes \mathcal{C}_F(A, B), \mathcal{C}_F(A, C))$$

is called the composition morphism, denoted $\circ$, and is required to make the diagram

$$\begin{tikzcd}
(\mathcal{C}_F(C,D)\otimes\mathcal{C}_F(B,C))\otimes\mathcal{C}_F(A,B) \arrow[r, "\alpha"] \arrow[dd, "\circ\otimes\mathrm{id}"'] & \mathcal{C}_F(C,D)\otimes(\mathcal{C}_F(B,C)\otimes\mathcal{C}_F(A,B)) \arrow[d, "\mathrm{id}\otimes\circ"] \\
 & \mathcal{C}_F(B,D)\otimes\mathcal{C}_F(A,B) \arrow[d, "\circ"] \\
\mathcal{C}_F(B,D)\otimes\mathcal{C}_F(A,B) \arrow[r, "\circ"] & \mathcal{C}_F(A,D)
\end{tikzcd}$$

commutative.

5. For any object C of $\mathcal{C}$, the image of the identity id_C under the equivalences $\mathcal{D}(C,C)\cong\mathcal{D}(I\otimes C,C)\cong\mathcal{D}(I,\mathcal{C}_F(C,C))$, denoted by $\mathbb{1}_C$, fits into the commutative diagram

$$\begin{tikzcd}
I\otimes\mathcal{C}_F(B,C) \arrow[r, "\mathbb{1}_C\otimes\mathrm{id}"] \arrow[d, "\lambda"'] & \mathcal{C}_F(C,C)\otimes\mathcal{C}_F(B,C) \arrow[dl, "\circ"] \\
\mathcal{C}_F(B,C) &
\end{tikzcd}.$$

By definition, if $\mathcal{C}$ is a category over $\mathcal{D}$, the map $\mathcal{C}(A,B)\to\mathcal{D}(F(A),F(B)),\ f\mapsto F(f)$ is injective. Therefore we can regard $\mathcal{C}(A,B)$ as a subset of $\mathcal{D}(F(A),F(B))$. So it makes sense to require a $\mathcal{D}$-morphism $F(A)\to F(B)$ to be a $\mathcal{C}$-morphism.

Note also that we have written $\mathcal{C}_F(B,C)$ with objects B,C of $\mathcal{C}$, whereas it should be $\mathcal{C}_F(B,F(C))$ by definition. However, the short-hand notation is much more convenient.

We provide some examples for categories over another category.

i) The category $\mathcal{SET}$ is a category over itself. The functor F is taken to be the identity. The adjoint functor of $\bullet\times X$ is $\mathcal{SET}(X,\bullet)$, therefore composition and unit morphism are induced by the composition law and the identity morphisms in SET.

ii) Similarly, every monoidal category where the monoidal product is the usual cartesian product, such as $\mathcal{TOP}$, $\mathcal{MAN}$, or suitable subcategories of these, is a category over $\mathcal{SET}$. The functor F is the forgetful functor assigning to a space, manifold etc. its underlying set. The adjoint is given as in i) and composition and unit are again coming from actual composition.

iii) The category $\mathcal{VECT}_k$ is not a category over $\mathcal{SET}$, since the obvious forgetful functor is not monoidal. But even better, $\mathcal{VECT}_k$ is a category over itself. Every set $L(V,W)$ carries the obvious structure of a k-vector space and the functor $L(V,\bullet)$ is a right adjoint to the functor $\bullet\otimes V$. Composition is induced by actual composition. Note that the composition map $L(V,W)\times L(U,V)\to L(U,W)$ is not linear, but bilinear. However, $\times$ is not the monoidal product in $\mathcal{VECT}_k$, but $\otimes$ is.

iv) The category $\mathcal{LCH}$ of locally compact Hausdorff spaces is a category over $\mathcal{HAUS}$, the category of Hausdorff spaces. The functor F is the obvious forgetful functor. Sets $\mathcal{C}(X, Y)$ of continuous maps are provided with the compact open topology. We will investigate this topology in detail in a subsequent chapter. The adjoint of the functor $\bullet \times X$ is the functor $\mathcal{C}(X, \bullet)$, composition is induced by actual composition.

v) Similarly, the category $h\mathcal{LCH}_*$ of pointed locally compact Hausdorff spaces with homotopy classes of maps as morphisms is a category over $h\mathcal{HAUS}_*$. The functor F is again the obvious forgetful functor. The adjoint functor of $\bullet \wedge X$ is the functor $h\mathcal{TOP}_*(X, \bullet)$, composition is induced by actual composition.

In all the above examples, the composition morphism came from actual composition of maps. This is indeed alwas true in the cases we will consider, and examples violating this rule are rather artificial.

1.4 $\mathcal{C}$-Groups

Another important feature of monoidal categories is that we can define group objects in a monoidal category. For this, we recall that a group in the classical sense is completely determined by a multiplication map, an inversion map and the inclusion of a unit element, satsifying certain conditions. In a monoidal category, we can require similar maps to exist and call the resulting object a group object.

Definition 1.4.1 Let $\mathcal{D}$ be a monoidal category. A $\mathcal{D}$-monoid or a monoid object in $\mathcal{D}$ is an object G of $\mathcal{D}$ together with two morphisms

$$\mu : G \otimes G \to G, \; \eta : I \to G,$$

such that the two diagrams

$$\begin{tikzcd} G \otimes G \otimes G \arrow[r, "\mu \otimes \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \otimes \mu"'] & G \otimes G \arrow[d, "\mu"] \\ G \otimes G \arrow[r, "\mu"] & G \end{tikzcd}$$

and

$$\begin{tikzcd} I \otimes G \arrow[r, "\lambda_G"] \arrow[d, "\eta \otimes \mathrm{id}_G"'] & G \arrow[d, "\mathrm{id}_G"'] & G \otimes I \arrow[l, "\rho_G"'] \arrow[d, "\mathrm{id}_G \otimes \eta"] \\ G \otimes G \arrow[r, "\mu"] & G & G \otimes G \arrow[l, "\mu"'] \end{tikzcd}$$

commute. μ is called the multiplication and η is called the unit of G.

A $\mathcal{D}$-group is a $\mathcal{D}$-monoid together with three additional morphisms

$$\Delta : G \to G \otimes G, \; \varepsilon : G \to I, \; i : G \to G$$

such that the diagram

$$\begin{tikzcd}
 & G \arrow[dl, "\Delta"'] \arrow[d, "\varepsilon"'] \arrow[dr, "\Delta"] & \\
G \otimes G \arrow[d, "\mathrm{id}_G \otimes i"'] & I \arrow[d, "\eta"'] & G \otimes G \arrow[d, "i \otimes \mathrm{id}_G"] \\
G \otimes G \arrow[r, "\mu"] & G & G \otimes G \arrow[l, "\mu"']
\end{tikzcd}$$

commutes. Δ is called the comultiplication, ε is called the counit and i is called the inversion of G.

The abundance of morphisms involved in the definition may look frightening at first sight. Therefore it is essential to keep the most common examples in mind and to know where the respective morphisms have their theoretical origin.

1. If G is a group in the sense of algebra, we can identify G with a $\mathcal{SET}$-group. μ and i are multiplication and inversion, η is the inclusion of the identity element, ε is the unique map into the one-point space and Δ is the diagonal map $g \mapsto (g, g)$.

2. In the category of topological spaces, a $\mathcal{TOP}$-group is specified by the same maps as a $\mathcal{SET}$-group. The additional requirement is that all the involved maps are continuous. Since Δ, η and ε are automatically continuous, this amounts to check multiplication and inversion for continuity.

3. In the category of smooth manifolds, again a $\mathcal{MAN}$-group is specified by the same maps as a $\mathcal{SET}$-group, now required to be smooth. Again, the maps Δ, η and ε are automatically smooth. In addition one can show, using the implicit function theorem, that smoothness of μ implies smoothness of i. Hence now we just have to check that μ is smooth. A $\mathcal{MAN}$-group is usually called a *Lie*-group.

4. In the category $\mathcal{VECT}_k$ of k-vector spaces, the definition of $\mathcal{VECT}_k$-groups is a bit more involved. The product in $\mathcal{VECT}_k$ is the actual tensor product of vector spaces. Let G be an ordinary group and let kG be the free k-vector space with basis G. We define the following maps.

$$\begin{aligned} \mu : kG \otimes kG \to kG,\ g \otimes h \mapsto g \circ h, &\qquad \Delta : kG \to kG \otimes kG,\ g \mapsto g \otimes g \\ \varepsilon : kG \to k,\ g \mapsto 1, &\qquad \eta : k \to kG,\ \lambda \mapsto \lambda e \\ i : kG \to kG,\ g \mapsto g^{-1} & \end{aligned}$$

   We extend these maps linearly, where necessary. One checks readily that these maps indeed define the structure of a $\mathcal{VECT}_k$-group on kG.

5. As a final example, consider the category $h\mathcal{TOP}_*$ of pointed topological spaces with pointed homotopy classes of G-maps as morphisms. The monoidal product is the smash product. We have the obvious comultiplication, unit and counit as in

the category of topological spaces. Therefore, the structure of an $h\mathcal{TOP}_*$-group is completely determined by continuous maps

$$\mu : G \wedge G \to G,\ i : G \to G$$

such that the diagrams from the definition commute "up to homotopy".

Part 4 of the preceeding examples is of particular interest. We have already defined the functor $\mathcal{SET} \to \mathcal{VECT}_k$, mapping a set to the free vector space with basis X and we see that a $\mathcal{SET}$-group is mapped to a $\mathcal{VECT}_k$-group by this functor. This is true in general.

Proposition 1.4.2 *Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories and $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. If G is a $\mathcal{C}$-group, then $F(G)$ is a $\mathcal{D}$-group.*

Proof. There is almost nothing to prove. We define the structural morphism for $F(G)$ by the image of the respective morphisms for G under F. Since F is monoidal, all the diagrams involved in the definition are preserved by F, showing that $F(G)$ indeed becomes a $\mathcal{D}$-group. $\square$

It is obvious that the $\mathcal{C}$-groups in a category $\mathcal{C}$ form a category themselves. Morphisms are taken to be $\mathcal{C}$-group homomorphisms. These are $\mathcal{C}$-morphisms compatible with all possible structure maps. To be precise, we have a $\mathcal{C}$-morphism $\varphi : G \to H$ between $\mathcal{C}$-groups G, H, and the following diagrams commute.

$$\begin{tikzcd} G \otimes G \arrow[r, "\mu_G"] \arrow[d, "\varphi\otimes\varphi"'] & G \arrow[d, "\varphi"] \\ H \otimes H \arrow[r, "\mu_G"] & H \end{tikzcd} \qquad \begin{tikzcd} G \otimes G \arrow[d, "\varphi\otimes\varphi"'] & G \arrow[l, "\Delta_G"] \arrow[d, "\varphi"] \\ H \otimes H & H \arrow[l, "\Delta_H"] \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "\eta_G"] \arrow[dr, "\eta_H"'] & G \arrow[d, "\varphi"] \\ & H \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\varepsilon_G"] \arrow[d, "\varphi"'] & I \\ H \arrow[ur, "\varepsilon_H"'] & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\varphi"'] & G \arrow[d, "\varphi"] \\ H \arrow[r, "i_H"] & H \end{tikzcd} .$$

In practice, G will often be a group in the ordinary sense and checking for a $\mathcal{C}$-group homomorphism will just amount to checking whether a given group homomorphism is a morphism in $\mathcal{C}$.

We close this section with an important class of examples of $\mathcal{D}$-monoids, rather than $\mathcal{D}$-groups. Let $\mathcal{C}$ be a category over $\mathcal{D}$. Then the composition law restricts to a $\mathcal{D}$-morphism

$$\mathcal{C}_F(C, C) \otimes \mathcal{C}_F(C, C) \to \mathcal{C}_F(C, C)$$

and the axioms ensure that this map, together with the unit morphism $I \to \mathcal{C}_F(C, C)$, determine the structure of a $\mathcal{D}$-monoid on $\mathcal{C}_F(C, C)$.

2 Basic Theory of Transformation Groups

2.1 Definition and Examples of Group Actions

In the literature there are several ways to define a group action and our aim is to give a unified treatment. This is the reason why it was necessary to introduce monoidal categories, $\mathcal{D}$-categories and group objects in the first place.

Definition 2.1.1 Let $\mathcal{D}$ be a monoidal category, let $\mathcal{C}$ be a category over $\mathcal{D}$ and G be a $\mathcal{C}$-group. An action of G on an object C of $\mathcal{C}$ is given by a $\mathcal{D}$-monoid homomorphism

$$\rho : F(G) \to \mathcal{C}_F(C, C)$$

such that the adjoint morphism $\alpha : G \otimes C \to C$ is a $\mathcal{C}$-morphism. ρ is called the action homomorphism. The triple (G, C, ρ) is called a G-object of $\mathcal{C}$, and if no confusion is possible, we speak of C as a G-object.

Of course, if $\mathcal{C}$ is a specific category, we will speak of G-spaces, G-sets or G-vector spaces, according to the given situation.

Translating the property that ρ is a monoid homomorphism into properties of its adjoint α yields that the diagrams

$$\begin{tikzcd} G \otimes G \otimes C \arrow[r, "\mathrm{id}_G \otimes \alpha"] \arrow[d, "\mu \otimes \mathrm{id}_C"'] & G \otimes C \arrow[d, "\alpha"] \\ G \otimes C \arrow[r, "\alpha"] & G \end{tikzcd} \qquad \begin{tikzcd} I \otimes C \arrow[d, "\eta \otimes \mathrm{id}_C"'] \arrow[dr, "\lambda"] & \\ G \otimes C \arrow[r, "\alpha"] & C \end{tikzcd}$$

commute, where μ is the group multiplication and $\eta : I \to G$ the unit morphism of G. On the other hand, any morphism $\alpha : G \otimes C \to C$ fitting in two commutative diagrams as above specifies a monoid homomorphism $\rho : F(G) \to \mathcal{C}_F(C, C)$ via the exponential law. We will therefore not distinguish whether an action is given by ρ or by its adjoint α and we will call ρ the action homomorphism, whereas α is called the action map.

We can now define the category of symmetric objects in a category $\mathcal{C}$.

Definition 2.1.2 Let $\mathcal{C}$ be a category over $\mathcal{D}$. The category $\mathit{Sym}_{\mathcal{D}}\, \mathcal{C}$ has as objects triples (G, A, ρ), where G is a $\mathcal{C}$-group and ρ an action of G on A. A morphism between (G, A, ρ) and (H, B, σ) is given by a pair (φ, f), with a $\mathcal{C}$-group homomorphism $\varphi : G \to H$ and a morphism $f : A \to B$. These morphisms are required to fit into the commutative diagram

$$\begin{tikzcd} G \otimes A \arrow[r, "\alpha"] \arrow[d, "\varphi \otimes f"'] & A \arrow[d, "f"] \\ H \otimes B \arrow[r, "\beta"] & B \end{tikzcd},$$

where α, β are the adjoints of ρ, σ, respectively. We have the subcategory of $\mathit{Sym}\, \mathcal{C}$ of G-objects in $\mathcal{C}$, objects being triples (G, A, ρ) as above with G fixed and morphisms of the form (id_G, f). In this case, the morphism $f : A \to B$ is called G-equivariant.

Sometimes one does not want a $\mathcal{C}$-group to act on an object C of $\mathcal{C}$, but rather a $\mathcal{C}'$-group for some other category $\mathcal{C}'$. This can be done via a monoidal functor $i : \mathcal{C}' \to \mathcal{C}$, because, as we have seen, $i(G)$ is a $\mathcal{C}$-group if G is a $\mathcal{C}'$-group. An action of G on C is then an action of $i(G)$ on C.

Example 2.1.3 1. Let V be a k-vector space. Finding a subgroup of the linear automorphisms of V amounts to specifying a group homomorphism $G \to GL_k(V)$. Let $i : \mathcal{SET} \to \mathcal{VECT}_k$ be the induction functor. Then an action of the $\mathcal{VECT}_k$-group kG on V is a monoid homomorphism $kG \to L(V, V)$ (we take $\mathcal{VECT}_k$ as a category over itself via the identity functor). Every element $g \in G$ must map to an automorphism of V, since $i(\mu)(g \otimes g^{-1}) = e$ and e maps to $\mathbb{1}_V$ under the action map. Hence we see that the actions of kG on $L(V)$ are in one-to-one correspondence with the group homomorphisms $G \to GL_k(V)$.

2. Let X be a topological space and G a topological group. There is the obvious forgetful functor $\mathcal{TOP} \to \mathcal{SET}$ and it is monoidal. It turns $\mathcal{TOP}$ into a category over $\mathcal{SET}$. An action of G on X is a monoid homomorphism

$$\rho : G \to \mathcal{TOP}(X, X)$$

such that the adjoint map $G \times X \to X$ is continuous. $\mathcal{TOP}(X, X)$ in this case is equipped with the compact open topology. Details are postponed to the next section. Note that, since G is a group, the image of ρ must necessarily be contained in the set of self-homeomorphisms of X.

3. Let X be a topological space and G a group. The functor i from sets to $\mathcal{TOP}$ assigning the discrete topology to a set is monoidal. Hence, an action of G on X is defined as a continuous monoid homomorphism

$$\rho : i(G) \to \mathcal{TOP}(X, X)$$

such that the adjoint map $i(G) \times X \to X$ is continuous.

4. Let M be a manifold, G a Lie group. We have the obvious forgetful functor $\mathcal{MAN} \to \mathcal{SET}$, which is monoidal. Therefore, an action of G on M is defined as a monoid homomorphism

$$G \to \mathcal{C}^{\infty}(M, M)$$

such that the adjoint map $G \times M \to M$ is smooth.

5. Let X be a locally compact pointed Hausdorff space and G a $h\mathcal{LCH}_*$-group. There is the forgetful functor from the category $h\mathcal{LCH}_*$ to the category $h\mathcal{HAUS}_*$. The functor $\bullet \wedge X$ for X locally compact Hausdorff has as adjoint the functor $\mathcal{HAUS}_*(X, \bullet)$. So an action of G on a pointed locally compact Hausdorff space is given by a homotopy class of a continuous map

$$G \to \mathcal{HAUS}_*(X, X).$$

The requirement that the adjoint is a $h\mathcal{LCH}_*$-morphism is fulfilled automatically, because it is just the homotopy class of a continuous pointed map. Altogether we see that an action of G on X is determined by a continuous pointed map $\alpha : G \wedge X \to X$ such that the diagrams

$$\begin{tikzcd}
G \wedge G \wedge X \arrow[r, "\mathrm{id}_G \wedge \alpha"] \arrow[d, "\alpha \wedge \mathrm{id}_X"'] & G \wedge X \arrow[d, "\alpha"] \\
G \wedge X \arrow[r, "\alpha"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}^0 \wedge X \arrow[d, "\eta \wedge \mathrm{id}_X"'] \arrow[dr] & \\
G \wedge X \arrow[r, "\alpha"] & X
\end{tikzcd}$$

are commutative up to pointed homotopy, where η is the unit of G.