Kac-Moody Algebras

Ana Ros Camacho

June 28, 2010

Abstract Talk for the seminar on Cohomology of Lie algebras, under the supervision of J-Prof. Christoph Wockel

Contents

1 Motivation

Kac-Moody algebras are a generalization of the finite-dimensional semi-simple Lie algebras which preserve almost the full original structure. In the last decades, they have attracted the attention of physicists as they revealed themselves as a useful tool in theoretical physics. They arise in e.g. dimensional reductions of gravity and supergravity theories, such as in dynamical summetries in string theory, conformal field theory and the theory of exactly solvable models.

2 Prerequisites

Some definitions we will need to introduce Kac-Moody algebras:

Definition 1. An *(associative) algebra* is a vector space A over $k = \mathbb{R}$ or $\mathbb{C}$ together with a bilinear map $\circ : A \times A \to A$ called *multiplication* that is associative, i.e. $(a \circ b) \circ c = a \circ (b \circ c)$, $\forall a, b, c \in A$. If, in addition, there exists an $e \in A$ such that $e \circ a = a \circ e = a$, $\forall a \in A$, e is called the *unit* and A is called an *associative unitary algebra*. A *subalgebra* of an associative unitary algebra A is a linear subspace $B \subset A$ that is closed under multiplication, i.e. $b \circ b' \in B$, $\forall b, b' \in B$, and that contains the unit ($e \in B$). An algebra is called *abelian* or *commutative* if $a \circ b = b \circ a$, $\forall a, b \in A$

Definition 2. An algebra $\mathfrak{g}$ is called a *Lie algebra* if the multiplication, called the *Lie bracket* is defined such that

1. it is antisymmetric, i.e. $[x, y] = -[y, x]$
2. it satisfies the *Jacobi identity*: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$, $\forall x, y, z \in \mathfrak{g}$

Notice that Lie algebras do not have to be associative.

Definition 3. Let $\mathfrak{g}$ be a Lie algebra and let I be a linear subspace of $\mathfrak{g}$. Then I is an *ideal* of $\mathfrak{g}$ if $[x, y] \in \mathfrak{g}$, $\forall x \in I$, $y \in \mathfrak{g}$.

Definition 4. The *center* of a Lie algebra, denoted $Z(\mathfrak{g})$ is an ideal:

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} | \, [x, y] = 0, \quad \forall y \in \mathfrak{g}\}$$

Definition 5. A *simple Lie algebra* is a non-abelian Lie algebra whose only ideals are the trivial one and itself.

Definition 6. A Lie algebra is called *semi-simple* if it is a direct sum of simple Lie algebras.

Definition 7. A *free Lie algebra*, over a given field K, is a Lie algebra generated by any set X without any imposed relations.

Definition 8. The *adjoint map* is a linear map such that, $\forall x \in \mathfrak{g}$,

$$ad_x : \mathfrak{g} \to \mathfrak{g} \tag{1}$$

$$y \mapsto ad_x(y) := [x, y] \tag{2}$$

Definition 9. The *Killing form* (also *Cartan-Killing form*) is defined as:

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) \tag{3}$$

or, given a basis $\{T^a | a = 1, \ldots, d\}$ of $\mathfrak{g}$,

$$\kappa^{ab} := \frac{1}{I_{ad}} \kappa\left(T^a, T^b\right) = \frac{1}{I_{ad}} \mathrm{tr}\left(\mathrm{ad}_{T^a} \circ \mathrm{ad}_{T^b}\right) = \frac{1}{I_{ad}} \sum_{c,e=1}^{d} f_e^{bc} f_c^{ae} \tag{4}$$

where I_{ad} is a normalization constant and f_e^{bc}, f_c^{ae} are the structure constants of the Cartan-Weyl basis.

3 Review of the finite-dimensional classification of semi-simple Lie algebras

Our aim is to classify the different types of (finite-dimensional) semi-simple Lie algebras. This reveals to be easier if we go to a special basis, the Cartan-Weyl basis.

Let $\mathfrak{g}$ be a semi-simple Lie algebra defined over $\mathbb{C}$.

Definition 10. The *ad-diagonalizable* or *semi-simple* elements of $\mathfrak{g}$ are all those $x \in \mathfrak{g}$ with the property that the map ad_x is diagonalizable.

Choose a maximal set of linearly independent elements H^i among the ad-diagonalizable elements of $\mathfrak{g}$, such that:

$$\left[H^i, H^j\right] = 0 \quad \forall i, j = 1, ..., r \tag{5}$$

Definition 11. The linear hull $\mathfrak{g}_0 \equiv span_{\mathbb{C}}\{H^i, i = 1, ...r\}$ is called a *Cartan subalgebra* of $\mathfrak{g}$.

Remark 1.
- $\mathfrak{g}$ can have many different Cartan subalgebras, but all of them are related by automorphisms of $\mathfrak{g}$ such that the freedom in choosing a Cartan subalgebra does not lead to any arbitrariness.
- all Cartan subalgebras have the same dimension r (property of $\mathfrak{g}$), called *rank* of $\mathfrak{g}$

$$r \equiv \text{rank}\,(\mathfrak{g}) = \dim\,(\mathfrak{g}_0) \tag{6}$$

A consequence of (5) is that $\mathfrak{g}$ can be spanned by any $y \in \mathfrak{g}$ such that

$$[h, y] \equiv ad_h\,(y) = \alpha_y\,(h)\,y \tag{7}$$

where $\alpha_y\,(h) \in \mathbb{C}$ and $h \in \mathfrak{g}_0^*$.

Definition 12. $\alpha_y\,(h)$ satisfying (7) is called a *root* of $\mathfrak{g}$ (relative to the chosen Cartan subalgebra $\mathfrak{g}_0$).

Remark 2. With def. 12, we can split the algebra as:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \tag{8}$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} |\, [h, x] = \alpha\,(h)\,x\}$$

Definition 13. (8) is called the *root space decomposition*

This decomposition means that there exists another basis in $\mathfrak{g}$, apart from the $\{H^i\}$ of the chosen Cartan subalgebra, which consists of elements E^α satisfying:

$$\left[H^i, E^\alpha\right] = \alpha\,(H)\,E^\alpha \quad \forall i = 1, \ldots, r \tag{9}$$

for $\alpha : \mathfrak{g}_0 \to \mathbb{C}$ linear, $H \in \mathfrak{g}_0$.

Definition 14. The r-dimensional vector $\left(\alpha^i\right)$ is called a *root vector*

Definition 15. The set of all roots of $\mathfrak{g}$ is called the *root system* of $\mathfrak{g}$ and denoted by $\phi \equiv \phi(\mathfrak{g})$

Remark 3.

- The root system is not degenerate
- $\mathfrak{g}_\alpha = \mathrm{span}_{\mathbb{C}}\{E^\alpha\}$
- $\mathfrak{g}_0 = \mathrm{span}_{\mathbb{C}}(\phi)$
- The only multiples of $\alpha \in \phi$ which are roots are $\pm\alpha$

Putting together both bases:

Definition 16.

$$B = \{H^i | i = 1, \dots, r\} \cup \{E^\alpha | \alpha \in \phi\} \tag{10}$$

such that (5) and (7) hold, is called a *Cartan-Weyl basis* of $\mathfrak{g}$

At this point, we need to define an inner product on the space of roots:

$$(\alpha, \beta) := c_\alpha c_\beta \kappa \left(H^\alpha, H^\beta\right) \tag{11}$$

where the c's are normalization constants $c_\alpha = \frac{1}{2}(\alpha, \alpha)$ and κ is the Killing form. Finite dimensional Lie algebras have only finitely many roots, and hence it is possible to find a hyperplane in the root space which does not contain any root, dividing the root space into two disjoint half spaces $V_\pm$. Then, define the sets of positive and negative as:

$$\phi_+ := \{\alpha \in \phi | \alpha > 0\} \tag{12}$$

$$\phi_- := \phi \setminus \phi_+ \tag{13}$$

and thus,

$$\{E^\alpha | \alpha \in \phi\} = \{E^\alpha | \alpha > 0\} \cup \{E^{-\alpha} | \alpha > 0\} \tag{14}$$

We will call the full set of E^α *step operators*, E^α the *raising operators* and $E^{-\alpha}$ the *lowering operators*. Given a Cartan subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$, the subspaces of $\mathfrak{g}$ that are spanned by the step operators for positive and negative roots respectively are in fact subalgebras,

$$\mathfrak{g}_\pm := \mathrm{span}_{\mathbb{C}}\{E^{\pm\alpha} | \alpha > 0\} \tag{15}$$

Putting together (8) and (15), we get:

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_- \tag{16}$$

which is called the *triangular* or *Gauss decomposition.*

Definition 17. Given ϕ_+, a *simple root* of $\mathfrak{g}$ is a positive root which cannot be obtained as a linear combination of other positive roots with positive coefficients

Remark 4.

- Denote by $E^i_\pm := E^{\pm\alpha^{(i)}}$ the step operators associated to simple roots
- Independently of the choice of the separating hyperplane, there are exactly r simple roots $\phi_s := \{\alpha^{(i)} | i = 1, \dots, r\}$

- The simple roots provide a basis for the root space, i.e. they are linearly independent and they span the whole root space
- The basis of simple roots is not orthonormal. This orthonormality is encoded in the *Cartan matrix* A of $\mathfrak{g}$ with respect to ϕ, which is defined as the $r \times r$ matrix with entries:

$$A^{ij} := 2\frac{\left(\alpha^{(i)}, \alpha^{(j)}\right)}{\left(\alpha^{(j)}, \alpha^{(j)}\right)} \tag{17}$$

The Cartan matrix summarizes the structure of a semi-simple Lie algebra completely (up to isomorphism). Its defining properties are:

1. $A^{ii} = 2$
2. $A^{ij} = 0 \leftrightarrow A^{ji} = 0$
3. $A^{ij} \in \mathbb{Z}_{\leq 0}, \forall i \neq j$
4. $\det A > 0$ [1]
5. It is indecomposable [2], i.e. not equivalent to the form

$$A = \begin{pmatrix} A_{(1)} & 0 \\ 0 & A_{(2)} \end{pmatrix} \tag{18}$$

The Cartan matrix can be of 3 kinds (and consequently the algebras generated by it):

1. If all of its principal minors are positive, then A is of *finite type*
2. If its proper principal minors are positive and $\det A = 0$, then A is of *affine type*
3. Otherwise, A is called to be of *indefinite type*

Then,

Proposition 1. *The Lie algebra algebraically generated by the* $3r$ *generators* $\{E^i_\pm, H^i | i = 1, \ldots, r\}$, *subjected to the relations*

$$\left[H^i, H^j\right] = 0 \tag{19}$$

$$\left[H^i, E^j_\pm\right] = \pm A^{ji} E^j_\pm \tag{20}$$

$$\left[E^i_+, E^j_-\right] = \delta_{ij} H^i \tag{21}$$

$$\left(ad_{E^i_+}\right)^{1-A^{ji}} E^j_\pm = 0 \tag{22}$$

(where A^{ij} *is a Cartan matrix and r is its (finite) rank) is a uniquely determined semi-simple Lie algebra associated to a set of simple roots.*

Proof 1. *Algorithm "Serre construction".*

Definition 18. For any root α, we define its *dual root* or *coroot* as:

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)} \tag{23}$$

[1] This condition implies finite dimensionality

[2] This condition is equivalent to the restriction to pass from simple algebras to semi-simple ones

4 Kac-Moody algebras

Definition 19. A matrix A is a *generalized Cartan matrix* (GCM) if it satisfies:

1. $A_{ii} = 2, \forall i$
2. $A_{ij} \in \mathbb{Z}_{\leq 0}, \forall i \neq j$
3. $A_{ij} = 0 \Rightarrow A_{ji} = 0, \forall i, j$
4. A_{ij} is indecomposable.

Definition 20. A *Kac-Moody algebra* is a complex Lie algebra generated by $3(l+1)$ generators $h_0, \ldots, h_l, e_0, \ldots, e_l, f_0, \ldots, f_l$ satisfying:

$$[h_i, h_j] = 0 \tag{24}$$
$$[e_i, f_j] = \delta_{ij} h_i \tag{25}$$
$$[h_i, e_j] = A_{ij} e_j \tag{26}$$
$$[h_i, f_j] = -A_{ij} f_j \tag{27}$$
$$\left[e_k, d_{ij}^-\right] = 0 \tag{28}$$
$$\left[f_k, d_{ij}^+\right] = 0 \tag{29}$$

where A_{ij} is a GCM and

$$d_{ij}^+ = (\mathrm{ad}_{e_i})^{1-A_{ij}}\, e_j \tag{30}$$
$$d_{ij}^- = (\mathrm{ad}_{f_i})^{1-A_{ij}}\, f_j \tag{31}$$

The first 4 relations are called the *Chevalley-Serre relations*, and the last two are the *Serre relations*.

5 Affine algebras

Among the whole class of Kac-Moody algebras, we will restrict ourselves to a special type, the symmetrizable [3] ones.

Definition 21. We say that a matrix is *degenerate positive semidefinite* if it satisfies:

$$\det A_{\{i\}} > 0 \quad \forall i = 0, \ldots, r \tag{32}$$

where $A_{\{i\}}$ denotes those matrices which are obtained from A by deleting the i-th row and the i-th column ($A_{\{i\}}$ are called the *principal minors of* A).

Definition 22. An (irreducible) *affine Cartan matrix* A is an irreducible GCM which is degenerate positive semidefinite and in addition satisfies

$$\det(A) = 0 \tag{33}$$

Definition 23. The *corank* of a matrix A is the dimension of its kernel.

[3] A Kac-Moody algebra is called *symmetrizable* if there exists a non-degenerate diagonal matrix D such that the matrix DA, with A being the GCM which generates the Kac-Moody algebra, is symmetric.

Remark 5. Affine Cartan matrices have corank 1

Definition 24. A Kac-Moody algebra whose GCM is an affine Cartan matrix is called an *affine Lie algebra*

Remark 6. Affine Lie algebras are necessarily infinite-dimensional.

Proposition 2. *Affine Lie algebras possess a non-zero center.*

Proof 2. *Consider the element:*

$$K := \sum_{i=0}^{r} a_i^{\vee} h^i \tag{34}$$

where $a_i^{\vee}$ *is defined by the equation:*

$$\sum_{j=0}^{r} A^{ij} a_j^{\vee} = 0 \tag{35}$$

Then,

$$[K, h_i] = 0 \tag{36}$$

$$[K, e_i] = 0 \tag{37}$$

(36) follows from eq. (24), (37) from (35); same for f_i*. Ergo,* K *is a central element. From (32), we know that for affine Cartan matrices* $A = A_{ij}|_{i,j=1,\ldots,n}$*, we have* $dim\,(ker\,(A)) = 1$*, so the center of* $\mathfrak{g}$ *is one dimensional, and all the central elements are multiples of* K*.* K *is called the canonical central element of* $\mathfrak{g}$*.*

6 Central extensions and loop algebras

At this point, we want to construct a sufficiently big class of examples of affine Lie algebras. We can attack this problem of constructing a so-called *central extension* of a Lie algebra $\mathfrak{g}$ from two different points of view:

1. In a basis-independent formulation, the extension of a Lie algebra by central elements is another Lie algebra $\hat{\mathfrak{g}} = V \oplus \mathfrak{g}$, where V is a vector space. The Lie bracket of this new Lie algebra is given by:

   $$[(v, x), (w, y)] := (\Omega\,(x, y), [x, y]) \tag{38}$$

   where Ω is a complex function defined on $\mathfrak{g} \times \mathfrak{g}$. This bracket must be antisymmetric, bilinear and satisfy the Jacobi identity, thus requiring $\Omega\,(x, y)$ to be also antisymmetric, bilinear and satisfy:

   $$\Omega\,(x, [y, z]) + \Omega\,(y, [z, x]) + \Omega\,(z, [x, y]) = 0 \tag{39}$$

   i.e. Ω is a cocycle of $\mathfrak{g}$. The central extension is trivial precisely if Ω is a coboundary, i.e. when $\Omega\,(x, y)$ is a linear function of $[x, y]$. The inequivalent non-trivial central extensions are then described by the vector space of 2-cocycles modulo coboundaries, i.e. the (Lie algebra) cohomology $H^2\,(\mathfrak{g}, \mathbb{C})$. An alternative way to relate $\hat{\mathfrak{g}}$ and $\mathfrak{g}$ is by requiring

   $$0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{40}$$

   to be a short exact sequence.

2. In a basis-dependent formulation, for any arbitrary Lie algebra $\mathfrak{g}$, we can construct a centrally extended Lie algebra $\hat{\mathfrak{g}}$ adding l generators K^j, $j = 1, \ldots, l$ to a basis $\{T^a\}$ of $\mathfrak{g}$ imposing the relations

$$\left[K^i, K^j\right] = 0 \tag{41}$$

$$\left[T^a, K^j\right] = 0 \tag{42}$$

$\forall i, j = 1, \ldots, l$, $\forall a = 1, \ldots, d$ while keeping the original values f^{ab}_c of those structure constants involving only the generators T^a. The most general form of the brackets among the $\hat{\mathfrak{g}}$-generators reads:

$$\left[T^a, T^b\right] = \sum_{c=1}^{d} f^{ab}_c T^c + \sum_{i=1}^{l} s^{ab}_i K^i \tag{43}$$

where s^{ab}_i are the structure constants of $\mathfrak{g}$ in the basis $\{T^a\}$. An (l-dimensional) *central extension of* $\mathfrak{g}$ is the Lie algebra $\hat{\mathfrak{g}}$ satisfying the additional relations (42) and (43).

Remark 7.
- The s^{ab}_i's can't be chosen arbitrarily, as the brackets are required to satisfy the Jacobi identity.
- A possible solution is taking $s^{ab}_i = 0$. In this case:

$$\hat{\mathfrak{g}} \sim \mathfrak{g} \oplus \mathbb{C}^l \tag{44}$$

It is not the only solution, but it is the most trivial one. $\hat{\mathfrak{g}}$ is of this form whenever there exists a choice of basis elements

$$\tilde{T}^a = T^a + \sum_{i=1}^{l} u^a_i K^i \tag{45}$$

with suitable coefficients u^a_i in which the transformed structure constants

$$\tilde{f}^{ab}_i = f^{ab}_i - \sum_{c=1}^{d} f^{ab}_c u^c_i \tag{46}$$

are zero.

Definition 25. Consider the vector space $\mathbb{C}\left[t, t^{-1}\right]$. The extended algebra of $\mathfrak{g}$ with $\mathbb{C}\left[t, t^{-1}\right]$ is called the *loop algebra* over $\mathfrak{g}$ and it is denoted by $\mathfrak{g}_{loop}$:

$$\mathfrak{g}_{loop} = \mathbb{C}\left[t, t^{-1}\right] \otimes_{\mathbb{C}} \mathfrak{g} \tag{47}$$

A basis of this vector space is given by:

$$B = \{\tilde{T}^a_n | a = 1, \ldots, d; n \in \mathbb{Z}\} \tag{48}$$

where $\tilde{T}^a_n := T^a \otimes z^n = T^a \otimes e^{2\pi i t n}$. This space inherits a natural bracket operation from $\mathfrak{g}$,

$$\begin{aligned}\left[\tilde{T}^a_m, \tilde{T}^b_n\right] &= \left[T^a \otimes z^m, T^b \otimes z^n\right] := \left[T^a, T^b\right] \otimes (z^m z^n) = \\ &= \sum_{c=1}^{d} f^{ab}_c T^c \otimes z^{m+n} = \sum_{c=1}^{d} f^{ab}_c \tilde{T}^c_{m+n}\end{aligned} \tag{49}$$

$\forall T^a \in \mathfrak{g}$, $\forall z^m, z^n \in \mathbb{C}\left[t, t^{-1}\right]$, where f_c^{ab} are the structure constants of $\mathfrak{g}$, which in the basis B, take the value: $f_c^{ab}\delta_{m+n,l}$. With this bracket, $\mathbb{C}\left[t, t^{-1}\right] \otimes_{\mathbb{C}} \mathfrak{g}$ becomes a Lie algebra.

Remark 8.
- The index n of $\tilde{T}_n^a$ is additive and provides a $\mathbb{Z}$-gradation of $\mathfrak{g}_{loop}$.
- The subset of $\mathfrak{g}_{loop}$ generated by $\tilde{T}_0^a$ is a Lie subalgebra, called the *zero mode subalgebra* of $\mathfrak{g}_{loop}$, and it is isomorphic to $\mathfrak{g}$.

Loop algebras $\mathbb{C}\left[t, t^{-1}\right] \otimes_{\mathbb{C}} \mathfrak{g}$ are not yet affine Lie algebras: the center of the loop algebra based on a simple Lie algebra is trivial. Thus, let us consider a central extension of a loop algebra of simple Lie algebras. Define a $\mathbb{C}$-valued *2-cocycle* on the loop algebra:

$$\psi(a, b) = \kappa(x, y)\,\varphi(P, Q) \tag{50}$$

where $a = P \otimes x$, $b = Q \otimes y$, $P, Q \in \mathbb{C}\left[t, t^{-1}\right]$, $x, y \in \mathfrak{g}$, and

$$\varphi(P, Q) = \mathrm{Res}\frac{dP}{dt}Q \tag{51}$$

is a bilinear $\mathbb{C}$-valued function defined on $\mathbb{C}\left[t, t^{-1}\right]$ [4] satisfying:

1. $\varphi(P, Q) = -\varphi(Q, P)$

   Proof 3.

   $$\varphi(P, Q) + \varphi(Q, P) = Res\frac{dP}{dt}Q + ResP\frac{dQ}{dt} = Res\frac{d(PQ)}{dt} = 0$$

   qed.

2. $\varphi(PQ, R) + \varphi(QR, P) + \varphi(RP, Q) = 0$

   Proof 4.

   $$\varphi(PQ, R) = Res\frac{d(PQ)}{dt}R = Res\frac{dP}{dt}QR + Res\frac{dQ}{dt}PR = \varphi(P, QR) + \varphi(Q, PR) \tag{53}$$

   Using the first property, (53) cancels the other terms of the equality, qed.

The 2-cocycle satisfies:

1. $\psi(a, b) = -\psi(b, a)$
2. $\psi([a, b], c) + \psi([b, c], a) + \psi([c, a], b) = 0$

These properties are easy to check using both the properties of $\varphi(P, Q)$ above and of the Killing form (bilinearity, symmetry and ad-invariance). This 2-cocycle is correlated to the central extension as:

[4] The residue of a Laurent polynomial $P = \sum_{k\in\mathbb{Z}} c_k t^k$ is defined by $\mathrm{Res}P = c_{-1}$, a linear functional on the algebra of polynomials in t defined by the properties:

$$\mathrm{Res}t^{-1} = 1 \quad \mathrm{Res}\frac{dP}{dt} = 0 \tag{52}$$

Proposition 3. *There exists a non-trivial extension* $\hat{\mathfrak{g}}$ *of* $\mathfrak{g}_{loop}$*:*

$$\hat{\mathfrak{g}} = \mathfrak{g}_{loop} \oplus \mathbb{C}K = \left(\mathbb{C}\left[z, z^{-1}\right] \otimes_{\mathbb{C}} \mathfrak{g}\right) \oplus \mathbb{C}K \tag{54}$$

whose bracket is given by:

$$[a + \lambda K, b + \mu K] = [a, b] + \psi(a, b) K \tag{55}$$

$\forall a, b \in \mathbb{C}\left[z, z^{-1}\right] \otimes_{\mathbb{C}} \mathfrak{g}$, $\forall \lambda, \mu \in \mathbb{C}$

A simple computation shows that the extended loop algebra $\mathbb{C}\left[t, t^{-1}\right] \otimes_{\mathbb{C}} \mathfrak{g} \oplus \mathbb{C}K$ is infinite-dimensional. In order to make the root space of the extended loop algebra finite dimensional, let us add a last generator D:

$$\bar{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}D \tag{56}$$

$\bar{\mathfrak{g}}$ is a complex vector space. To obtain a Lie algebra structure on $\bar{\mathfrak{g}}$, we define the Lie bracket of $\bar{\mathfrak{g}}$ as:

$$[(x, y), (x', y')] := (yx' - y'x + [x, y], 0) \tag{57}$$

where $x, x' \in \hat{\mathfrak{g}}$, $y, y' \in \mathbb{C}D$. A basis-dependent formulation of this bracket can be:

$$[D, T_m^a] = m T_m^a \tag{58}$$

$$[D, K] = 0 \tag{59}$$

where K is the canonical central element. More generally, $D = D_s$ can be defined as the endomorphism of $\bar{\mathfrak{g}}$ defined by

$$D_s = -t^{s+1}\frac{d}{dt} \quad D_s(K) = 0 \tag{60}$$

so that $D_0 = -D$. D is called a *derivation* of $\bar{\mathfrak{g}}$.

And finally,

Definition 26. (*Extended Cartan matrix*) Let A be a positive definite indecomposable Cartan matrix and let $\mathfrak{g} = \mathfrak{g}(A)$ be the associated simple finite-dimensional Lie algebra with generators: $H_i, E_{\pm}^i$, $\forall i = 1, \ldots, r$. There exists a unique non-zero element E_+^0 (E_-^0) in $\mathfrak{g}$ such that $\left[E_+^0, E_-^i\right]$ ($\left[E_-^0, E_+^i\right]$) vanishes for $i = 1, \ldots, r$. Then $\left[E_+^0, E_-^0\right] = H_0$, where H_0 is a linear combination of the H_i, and one normalizes E_+^0 and E_-^0 by the conditions

$$\left[H_0, E_+^0\right] = 2E_+^0 \tag{61}$$

$$\left[H_0, E_-^0\right] = -2E_-^0 \tag{62}$$

Then, $\left[H_0, E_+^i\right] = \alpha_{0i} E_+^i$, $\left[H_i, E_+^0\right] = \alpha_{i0} E_+^0$ for $i = 1, \ldots, r$ where the α's are certain non-positive integers, and one puts

$$A_{ext} = \begin{pmatrix} 2 & \alpha_{01} & \ldots & \alpha_{0r} \\ \alpha_{10} & & & \\ \vdots & & A & \\ \alpha_{r0} & & & \end{pmatrix} \tag{63}$$

This is a positive semidefinite $(r+1) \times (r+1)$ GCM, and it is called an *extended Cartan matrix*

Theorem 6.1. *Let $\mathfrak{g}$ be a simple, complex, finite-dimensional Lie algebra and let A be its extended Cartan matrix. Then, $\mathbb{C}\left[t, t^{-1}\right] \otimes_{\mathbb{C}} \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}D$ is the affine Kac-Moody algebra associated to A.*

Proof 5. *(Sketch) We want to show that the algebra $\mathbb{C}\left[t, t^{-1}\right] \otimes_{\mathbb{C}} \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}D$ is an affine Kac-Moody algebra associated to an affine Cartan matrix A. Steps to follow:*

1. *Identification of the elements candidates to be the generators of the algebra: if, associated to an affine Cartan matrix A, we have the generators $(H_i, E_i, F_i)_{i=0,\ldots,n}$, then for the algebra (56) we will have:*

$$H_i \leftrightarrow 1 \otimes h_i \, (i = 1, \ldots, n) \tag{64}$$
$$H_0 \leftrightarrow D \tag{65}$$
$$E_i \leftrightarrow 1 \otimes e_i \tag{66}$$
$$E_0 \leftrightarrow t \otimes e_0 \tag{67}$$
$$F_i \leftrightarrow 1 \otimes f_i \tag{68}$$
$$F_0 \leftrightarrow t^{-1} \otimes f_0 \tag{69}$$

2. *Show that the $3\,(l+1)$ generators from step 1 generate the Lie algebra $\bar{\mathfrak{g}}$.*

3. *Show that the Kac-Moody relations between the generators hold.*

Extended Cartan matrices comprise more than half of the affine Cartan matrices. Those affine Lie algebras whose affine Cartan matrices are extended Cartan matrices are called in the literature "*untwisted*", and those whose are not, "*twisted*". As we can see, this theorem comprises then most of the affine Lie algebras, although it is possible to extend it to the twisted case. The full classification of Kac-Moody algebras (apart from the already completed simple, affine and hyperbolic cases) is far from being completed, but at least, there are some parts which have been accomplished, and the one presented above is an important one. There is still a lot to do in this field!

To finish our exposition, we would like to give some examples of physical fields where these constructions arise:

1. WZW theories (two-dimensional conformal field theory), as current algebras.

2. In string theory, the conformal field theories based on affine Lie algebras give, for example, rise to the gauge bosons in a heterotic string compactification.

3. The Virasoro algebra: it is a bit further extended untwisted affine Lie algebra, where one introduces another central element C and infinitely many generators L_m, $m \in \mathbb{Z}$ satisfying

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{1}{12} m \left(m^2 - 1\right) \delta_{m+n,0} C \tag{70}$$
$$[L_m, C] = 0 \tag{71}$$

7 Bibliography

References

[1] Carter, R., 2005 *Lie algebras of finite and affine type* (Cambridge: Cambridge University Press)

[2] Chung, S. Lecture notes in Kac-Moody algebras, Rutgers University.

[3] Fuchs, J. and Schweigert, C., 1997 *Symmetries, Lie algebras and representations: a graduate course for physicists* (Cambridge: Cambridge University Press)

[4] Kac, V. G., 1990 *Infinite dimensional Lie algebras: an introduction* (Cambridge: Cambridge University Press)

[5] Nicolai, H. and Schlotterer, O. *Infinite dimensional symmetries* Lecture notes from the 15th Saalburg Summer School

[6] Serre, J.-P., 1966 *AlgÃ¨bres de Lie semi-simples complexes* (USA: W.A. Benjamin, Inc)

[7] Springer Online Reference Works, *Kac Moody algebras http://eom.springer.de/k/k055050.htm*

[8] Wan, Z-X. 1991 Introduction to Kac Moody algebras (Singapore: World Scientific Publishing Co. Pte. Ldt.)