BRST in a nutshell

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June 23, 2010

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1 Introduction

The BRST (Becchi, Rouet, Sora, Tyutin) method is a tool used for describing physical systems with symmetries. By a physical system we mean space of configurations $P$ (this is usually some infinite dimensional vector space) and a system of (differential) equations (so called "equations of motion") with variables in $P$. Solutions of those equations constitute a subset of $P$, denoted by $\Sigma$. As an example we can take a space of smooth functions on $\mathbb{R}^4$ ($P = C^\infty(\mathbb{R}^4)$) and as the equation of motion, the wave equation:

$$\Box \phi(t, x) = (\partial^2_t - \Delta_x)\phi(t, x) = 0.$$ 

The situation starts to be more complicated if we have $n$ equations but only $k < n$ of them are independent. We say, that the system possesses a symmetry. This is the case in many interesting physical examples:

- electrodynamics
- nonabelian gauge theories (Yang-Mills), in particular the Standard Model
- general relativity

The BRST method was originally introduced in QFT [16, 17]. It was put in a more general setting, called BV (Batalin, Vilkovinski) formalism [18, 19, 20, 21]. A very complete review of this formalism, with emphasis put on the cohomological tools is provided by [1]. General features of the BRST method, with a view towards quantization, are also well described in [2]. For more abstract view on BV formalism see for example notes of Urs Schreiber [8].
2 Toy model

2.1 Statement of the problem

In this talk I will present the general features of the BRST method using a simplified model. To avoid problems with the calculus on general locally convex vector spaces, I will assume, that the configuration space $P$ is simply an $n$ dimensional smooth Riemannian manifold. Let $S \in C^\infty(P, \mathbb{R})$ be a functional on $P$. It should also satisfy certain regularity condition, which would be specified below. Let $d$ be the exterior derivative.

**Definition 1.** We call a point $x \in P$ a critical point if $dS(x) \equiv 0$.

Let $\Sigma$ be the set of all critical points of $S$, i.e.

$$\Sigma = \{x \in P| dS(x) \equiv 0\}. \tag{1}$$

The condition $dS(x) = 0$ can be written in local coordinates (with respect to a chart $(U_\alpha, \varphi_\alpha)$ as a system of $n$ equations for $n$ variables: $\sigma_i((\varphi_\alpha^{-1})^i(x), \ldots, (\varphi_\alpha^{-1})^n(x)) = 0$, $i = 1, \ldots n$. In physics those correspond to "equations of motion". The surface $\Sigma \subseteq P$ is referred to as the "space of solutions". A critical point is called nondegenerate if at this point the (local) Hessian matrix $H_S(\varphi_\alpha^{-1}(x))$ is nondegenerate. In this case we have a system of independent equations. In general only $k < n$ of them are independent. We require following regularity condition imposed on $S$:

**Assumption 1.** For each point $x \in \Sigma$ there exists an open neighborhood with the corresponding chart $(U_\alpha, \varphi_\alpha)$ such that $\sigma_i((\varphi_\alpha^{-1})^1(x), \ldots, (\varphi_\alpha^{-1})^k(x)) = 0$, $i = 1, \ldots k$ are independent, i.e. the Hessian matrix $H_S(\varphi_\alpha^{-1}(x))$ is of rank $k$ for all $x \in \Sigma$.

In the following we denote the local coordinates by $x^1 \doteq (\varphi_\alpha^{-1})^1(x)$ and we keep the local chart implicit. Under the regularity condition 1 we can choose $(\sigma_1, \ldots, \sigma_k, x_{k+1}, \ldots, x_n)$ as new local coordinates in the neighbourhood of each point of $\Sigma$. Let $C^\infty(P) \doteq C^\infty(P, \mathbb{R})$ denote the space of smooth functions on $P$. This is a vector space with addition and multiplication by scalars from $\mathbb{R}$ defined pointwise. Moreover it is a commutative algebra with multiplication also defined pointwise. Let $I$ be an ideal of $C^\infty(P)$ consisting of functions that vanish on $\Sigma$:

$$I \doteq \{f \in C^\infty(P)| f(x) = 0 \ \forall x \in \Sigma\} \tag{2}$$

We have a following useful result:

**Proposition 1.** Let $f \in C^\infty(P)$ be a smooth function that vanishes on $\Sigma$. Then locally we have:

$$f(x) = \sum_{i=1}^n f^i(x)\sigma_i(x), \text{ for } f^i(x) \text{ smooth.} \tag{3}$$

**Proof.** We choose a local coordinate system $x = (z_1, \ldots, z_n)$ such that $z_i = \sigma_i(x), i = 1, \ldots k$. In those coordinates we have: $f(0, z_{k+1}, \ldots, z_n) = 0$. We can therefore write:

$$f(z_1, \ldots, z_n) = \int_0^1 \frac{d}{dt} f(tz_1, \ldots, tz_k, z_{k+1}, \ldots, z_n)dt$$
This in turn is equal to:

\[
f(z_1, \ldots, z_n) = \int_0^1 df(0, z_{k+1}, \ldots, z_n)[tz_1, \ldots, tz_k, z_{k+1}, \ldots, z_n] dt\tag{4}
\]

Setting \( f^i(z) = \int_0^1 df(0, z_{k+1}, \ldots, z_n)[0, \ldots, t, \ldots, 0] dt \) for \( i = 1, \ldots, k \), and \( f^i(z) = 0 \) for \( i = k + 1, \ldots, n \) we obtain the result: \( f(z) = \sum_{i=1}^n f^i(z)\sigma_i(z). \)

Obviously each vector field \( X \in \Gamma^\infty(TP) \) acting on a function \( f \in C^\infty(P) \) can be written locally as: \( X(f) = \sum_{i=1}^n X^i\partial_i f \), where coefficients \( X^i \) are smooth functions. In particular:

\[
X(S) = \sum_{i=1}^n X^i\partial_i S = \sum_{i=1}^n X^i\sigma_i. \tag{5}
\]

Therefore every vector field \( X \in \Gamma^\infty(TP) \) induces an element of \( I \) by the map: \( dS(.) : \Gamma^\infty(TP) \to I \). Moreover, if in addition \( X(S) \equiv 0 \), \( X \) induces a trivial element of \( I \). We can define a subalgebra of the Lie algebra of vector fields \( \Gamma^\infty(TP) \) by:

\[
g = \{ X \in \Gamma^\infty(TP) \mid X(S) = 0 \}. \tag{6}
\]

Obviously \( \text{Ker}(dS(.)) = g \).

We now take the quotient of \( C^\infty(P) \) by ideal \( I \) and obtain the algebra \( C^\infty(\Sigma) = C^\infty(P)/I \) of functions on the solution space \( \Sigma \).

*Digression* 1. In physics we call \( C^\infty(P) \) the algebra of "functionals off-shell". The quotient space \( C^\infty(\Sigma) = C^\infty(P)/I \) is referred to as the "on-shell algebra". Both concepts are crucial in QFT in the so called "functional approach". For reference see for example: [10, 12, 11]

Now let \( \text{Diff}(P) \) denote the group of diffeomorphisms of \( P \). We define a subgroup of \( \text{Diff}(P) \) of those diffeomorphisms, that leave \( S \) invariant:

\[
G \doteq \{ \alpha \in \text{Diff}(P) \mid S(\alpha(x)) = S(x) \ \forall x \in P \}. \tag{6}
\]

Obviously \( G \) leaves \( \Sigma \) invariant. We have a natural action of \( \text{Diff}(P) \) on \( C^\infty(P) \) by the pullback:

\[
(\alpha(f))(x) \doteq \alpha^* f(x) = (f \circ \alpha)(x) \tag{7}
\]

This induces also the action of \( G \) on \( C^\infty(P) \) and of \( G \) on \( C^\infty(\Sigma) \). The last one is well defined on the equivalence classes \( C^\infty(P)/I \) since for \( \alpha \in G, f \in I \) we have:

\[
dS(x) \equiv 0 \ \forall x \in \Sigma \Rightarrow dS(\alpha(x)) \equiv 0 \ \forall x \in \Sigma \Rightarrow f(\alpha(x)) = 0 \ \forall x \in \Sigma \Rightarrow \alpha(f) \in I \tag{8}
\]

The action of \( G \) on \( C^\infty(\Sigma) \) is not faithful. Let \( G_0 \) be the subgroup of \( G \) consisting of those diffeomorphisms that act on \( C^\infty(\Sigma) \) trivially:

\[
G_0 \doteq \{ \alpha \in G \mid \alpha^* f - f \in I \ \forall f \in C^\infty(P) \}. \tag{9}
\]

It is easy to see that \( G_0 \) is a normal subgroup of \( G \) and we can take the quotient: \( G_S \doteq G/G_0 \). The action of \( G_S \) on \( C^\infty(\Sigma) \) is faithful. In general \( G_S \) is not a subgroup of \( G \).
Digression 2. The group $G_S$ is called in physics the group of symmetries of the action $S$. It maps solutions to other solutions and as a consequence it maps on-shell functionals to other on-shell functionals. Usually we are not interested in the full $G_S$ but in its subgroups.

**Definition 2.** A 1-parameter group of (smooth) transformations of $P$ is a mapping of $\mathbb{R} \times P$ into $P$, $(t, p) \in \mathbb{R} \times P \rightarrow \phi_t(p) \in P$, which satisfies the following conditions:

1. For each $t \in \mathbb{R}$, $\phi_t : p \rightarrow \phi_t(p)$ is a transformation of $P$;

2. For all $t, s \in \mathbb{R}$ and $p \in P$, $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.

In particular we can have 1-parameter subgroups of $G$ and $G_S$. Let $\phi_t$ be a one-parameter subgroup of $G$. For each point $x \in \Sigma$ we can define a curve $x(t) = \phi_t(p)$. Clearly $x(t)$ lies on $\Sigma$. We call $x(t)$ the orbit of $x$. Each 1-parameter group of transformations induces a vector field $X \in \Gamma^\infty(TP)$.

**Definition 3.** Let $I_\epsilon$ be an open interval $(-\epsilon, \epsilon)$ and $U$ an open set of $P$. A local 1-parameter group of local transformations defined on $I_\epsilon \times U$ is a mapping of $I_\epsilon \times U$ into $P$ which satisfies the following conditions:

1. For each $t \in I_\epsilon$, $\phi_t : p \rightarrow \phi_t(p)$ is a diffeomorphism of $U$ onto the open set $\phi_t(U)$ of $P$;

2. If $t, s, t + s \in I_\epsilon$ and if $p, \phi_s(p) \in U$, then: $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.

Recall that we have a following result from differential geometry:

**Proposition 2.** Let $X$ be a vector field on a manifold $P$. For each point $p_0 \in P$, there exist a neighborhood $U$ of $p_0$, a positive number $\epsilon$ and a local 1-parameter group of local transformations $\phi_t : U \rightarrow P$, $t \in I_\epsilon$, which induces the given $X$.

If there exists a (global) 1-parameter group of transformations of $P$ which induces $X$, then we say that $X$ is complete. On a compact manifold every vector field $X$ is complete. In physics we are interested in functionals on $\Sigma$ that are constant along the orbits generated by local 1-parameter subgroups of $G_S$. Those are corresponding to certain equivalence classes of vector fields on $P$. To make this precise, we recall that $g$ was defined as the algebra of vector fields that annihilate the action $S$. Since we are interested only on the fields with flows contained in $\Sigma$ we have to mode out from $g$ vector fields that vanish on $\Sigma$. Those can be also equivalently defined as:

$$g_0 = \{X \in g \mid X(f) \in I \forall f \in C^\infty(P)\},$$

We define now $g\Sigma = g/g_0$. It is clear that 1-parameter subgroups of $G_S$ generate elements of $g\Sigma$. We can now make precise the notion of functions constant along the $g\Sigma$-orbits on $\Sigma$. They are defined as:

$$C^\infty_{\text{inv}}(\Sigma) = \{f \in C^\infty(\Sigma) \mid X(f) = 0 \forall X \in g\Sigma\}$$

Later on we shall refer to this space as the space of invariant functions on $\Sigma$. 

2.2 Chevalley-Eilenberg cohomology

The obvious tool for finding $C^\infty_{\text{inv}}(\Sigma)$ is the Lie algebra cohomology. We can define the Chevalley-Eilenberg cohomology of $g_{\Sigma}$ with coefficients in the representation on $C^\infty(\Sigma)$:

$$
\gamma : \bigwedge^q g_{\Sigma}^* \otimes C^\infty(\Sigma) \rightarrow \bigwedge^{q+1} g_{\Sigma}^* \otimes C^\infty(\Sigma)
$$

$$(\gamma \omega)(X_0, \ldots, X_{q+1}) = \sum_{i=0}^q (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_{q+1})) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1}) \quad (12)$$

The space of invariant functions on $\Sigma$ corresponds now to $H_0(\gamma, \bigwedge \cdot g_{\Sigma}^* \otimes C^\infty(\Sigma))$. The grading of $\bigwedge \cdot g_{\Sigma}$ will be denoted by $\tilde{r}$.

**Digression 3.** In physical situations one often needs to work "off-shell". This means that we do not want to deal with $C^\infty(\Sigma)$, but with $C^\infty(P)$. This is the case in QFT when we first want to perform the quantization, introducing a certain noncommutative product on $C^\infty(P)$ and take the quotient by the $I$ at the very end of the construction.

The homological interpretation of $C^\infty(\Sigma) = C^\infty(P)/I$ is provided by using the Koszul-Tate resolution. This will be the next step of our construction. More on Koszul-Tate complex can be found in [4, 5].

2.3 Koszul-Tate resolution

2.3.1 Koszul construction

We start with the Koszul construction. As argued before, elements of $I$ can be locally written as $f(x) = \sum_{i=1}^n f_i(x)\sigma_i(x)$. We choose the local basis $\{e_1 \ldots e_n\}$ of $T_x P$ at point $x$. The dual basis would be denoted by: $\{e^1 \ldots e^n\}$. Note that $\sum_{i=1}^n e^i\sigma_i(x)$ is an element of $T_x^* P$. We use this fact to construct the resolution of $C^\infty(\Sigma)$. Let $\Lambda^1(P)$ be the space of 1-forms on $P$. We define the Koszul map $\delta : \Lambda^1(P) \rightarrow \Lambda^0(P)$ locally by setting it’s value on the basis elements and extending it by linearity to the whole $\Lambda^1(P)$:

$$
\delta(e^i)(x) = \sigma_i(x), \quad i = 1, \ldots, n \quad (13)
$$

It is now clear, that for an arbitrary $\omega \in \Lambda^1(P)$ we have:

$$
\delta(\omega)(x) = \sum_{i=1}^n \omega_i(x)\sigma_i(x) \in I \quad (14)
$$

Therefore $\text{Im}(\delta) = I \subset \Lambda^0(P)$. We assign to elements of this algebra grade $r$ equal to the form degree. Now we extend $\delta$ to the whole graded algebra $\Lambda^\cdot(P)$ by requirement, that it is a graded derivation.We have obviously:

$$H_0(\delta, \Lambda^\cdot(P)) = C^\infty(\Sigma) \quad (15)$$
When there exist some symmetries of the action, higher order homology can be nontrivial. To avoid this and obtain a resolution, we shall adopt the Tate construction and add further generators to the graded algebra.

### 2.3.2 Tate construction

The first homology of $\delta$ is, according to the definition:

$$H_1(\delta) = \frac{\text{Ker}(\delta)}{\text{Im}(\delta)}_1.$$

We already know that $\text{Ker}(\delta)_1$ can be characterized by elements of $g$. Now we have to find out what is $\text{Im}(\delta)_2$ in terms of elements of $g$. It is easy to check that those will be exactly vector fields that vanish on $\Sigma$, i.e. elements of $g_0$. Therefore we can conclude that $H^1(\delta)$ can be characterized by $g/g_0 = g_\Sigma$.

We choose a local linear map $R : g_\Sigma = g/g_0 \to g \subset \Gamma^\infty(TP)$, such that $[R(X)] = X$. In other words, $R$ chooses representant of each equivalence class. The choice of $R$ is of course non unique. In general $R$ is not a Lie algebra homomorphism. Let $\{f_j(x)\}$, $j = 1, \ldots, m < n$ be the local basis in $g_\Sigma$. In local coordinate system we can write $R$ as:

$$R(X)_i(x) = \sum_{j=1}^m R_{ij}(x) X_j(x).$$

Now we proceed analogously as with the Koszul construction. Let $S^k(g_\Sigma)$ denote the symmetrized $k$-th tensor power of $g_\Sigma$, $S^0(g_\Sigma) = \mathbb{R}$. Let $S^*(g_\Sigma) = \bigoplus_{k=0}^\infty S^k(g_\Sigma)$. To elements of this algebra we assign the grading $r = 2k$. We define Koszul-Tate map $\delta : S^1g_\Sigma \to \bigwedge^1(P)$ by setting it’s value on the local basis:

$$(\delta f_j)(x) = \sum_{i=1}^n R_{ij}^i e^i$$

We extend $\delta$ to be a graded derivation on the whole graded algebra $S^*(g_\Sigma) \otimes \bigwedge^*(P)$. If we assume that there are no further reducibility relations among the elements of $g_\Sigma$ we obtain:

$$H_0 (\delta, S^*(g_\Sigma) \otimes \bigwedge^*(P)) = I$$

$$H_k (\delta, S^*(g_\Sigma) \otimes \bigwedge^*(P)) = 0, \; k > 0$$

This is the desired Koszul-Tate resolution of $I$.

### 2.4 Homological perturbation theory

Now we have two graded algebras: $S^*(g_\Sigma) \otimes \bigwedge^*(P)$ with the differential $\delta$ and grading $r$ and $\bigwedge^* g_\Sigma^* \otimes \mathcal{C}^\infty(\Sigma)$ with grading $\tilde{r}$ and differential $\gamma$. We can define a joint algebra:

$$\mathcal{A} = S^*(g_\Sigma) \otimes \bigwedge(P) \otimes \bigwedge g_\Sigma^*$$
One can extend $\gamma$ to the whole algebra $\mathcal{A}$ to be a differential modulo $\delta$. We define a joint grading $N = \tilde{r} - r$. Differential $\delta$ has grade $r(\delta) = -1$ and for $\gamma$ we have $\tilde{r}(\gamma) = 1$. It follows that $N(\delta) = N(\gamma) = 1$. The main theorem of homological perturbation theory (HPT) states that there exists a differential $s$ on $\mathcal{A}$ with grade $N(s) = 1$ such that it's expansion with respect to the grading $r$ has the form:

$$s = \delta + \gamma + \ldots$$

Moreover we have:

$$H^0(s, \mathcal{A}) = H^0(\gamma, H_0(\delta, \mathcal{A})) = C^\infty_{inv}(\Sigma)$$

For more on HPT see for example the notes of Birgit Richter [7].

References

[8] U. Schreiber, *On Lie $\infty$-modules and the BV complex*


