BRST in a nutshell

Katarzyna Rejzner

June 23, 2010

Contents

1 Introduction

The BRST (Becchi, Rouet, Sora, Tyutin) method is a tool used for describing physical systems with symmetries. By a physical system we mean space of configurations P (this is usually some infinite dimensional vector space) and a system of (differential) equations (so called "equations of motion") with variables in P. Solutions of those equations constitute a subset of P, denoted by Σ. As an example we can take a space of smooth functions on $\mathbb{R}^4$ ($P = \mathcal{C}^\infty(\mathbb{R}^4)$) and as the equation of motion, the wave equation: $\Box\phi(t,\boldsymbol{x}) = (\partial_t^2 - \Delta_{\boldsymbol{x}})\phi(t,\boldsymbol{x}) = 0$.

The situation starts to be more complicated if we have n equations but only $k < n$ of them are independent. We say, that the system possesses a symmetry. This is the case in many interesting physical examples:

- electrodynamics
- nonabelian gauge theories (Yang-Mills), in particular the Standard Model
- general relativity

The BRST method was originally introduced in QFT [16, 17]. It was put in a more general setting, called BV (Batalin, Vilkovinski) formalism [18, 19, 20, 21]. A very complete review of this formalism, with emphasis put on the cohomological tools is provided by [1]. General features of the BRST method, with a view towards quantization, are also well described in [2]. For more abstract view on BV formalism see for example notes of Urs Schreiber [8].

2 Toy model

2.1 Statement of the problem

In this talk I will present the general features of the BRST method using a simplified model. To avoid problems with the calculus on general locally convex vector spaces, I will assume, that the configuration space P is simply an n dimensional smooth Riemannian manifold. Let $S \in \mathcal{C}^\infty(P, \mathbb{R})$ be a functional on P. It should also satisfy certain regularity condition, which would be specified below. Let d be the exterior derivative.

Definition 1. We call a point $x \in P$ a *critical point* if $d_x S \equiv 0$.

Let Σ be the set of all critical points of S, i.e.

$$\Sigma = \{x \in P |\, dS(x) \equiv 0\}. \tag{1}$$

The condition $dS(x) = 0$ can be written in local coordinates (with respect to a chart $(U_\alpha, \varphi_\alpha)$ as a system of n equations for n variables: $\sigma_i((\varphi_\alpha^{-1})^1(x), \dots, (\varphi_\alpha^{-1})^n(x)) = 0$, $i = 1, \dots n$. In physics those correspond to "equations of motion". The surface $\Sigma \subsetneq P$ is referred to as the "space of solutions". A critical point is called nondegenerate if at this point the (local) Hessian matrix $H_S(\varphi_\alpha^{-1}(x))$ is nondegenerate. In this case we have a system of independent equations. In general only $k < n$ of them are independent. We require following regularity condition imposed on S:

Assumption 1. For each point $x \in \Sigma$ there exists an open neighborhood with the correspondng chart $(U_\alpha, \varphi_\alpha)$ such that $\sigma_i((\varphi_\alpha^{-1})^1(x), \dots, (\varphi_\alpha^{-1})^k(x)) = 0$, $i = 1, \dots k$ are independent, i.e. the Hessian matrix $H_S(\varphi_\alpha^{-1}(x))$ is of rank k for all $x \in \Sigma$.

In the following we denote the local coordinates by $x^1 \doteq (\varphi_\alpha^{-1})^1(x)$ and we keep the local chart implicit. Under the regularity condition 1 we can choose $(\sigma_1 \dots \sigma_k, x_{k+1}, \dots, x_n)$ as new local coordinates in the neighbourhood of each point of Σ. Let $\mathcal{C}^\infty(P) \doteq \mathcal{C}^\infty(P, \mathbb{R})$ denote the space of smooth functions on P. This is a vector space with addition and multiplication by scalars from $\mathbb{R}$ defined pointwise. Moreover it is a commutative algebra with multiplication also defined pointwise. Let I be an ideal of $\mathcal{C}^\infty(P)$ consisting of functions that vanish on Σ:

$$I \doteq \{f \in \mathcal{C}^\infty(P) | f(x) = 0 \;\forall x \in \Sigma\} \tag{2}$$

We have a following useful result:

Proposition 1. *Let $f \in \mathcal{C}^\infty(P)$ be a smooth function that vanishes on Σ. Then locally we have: $f(x) = \sum_{i=1}^{n} f^i(x)\sigma_i(x)$, $i = 1 \dots n$ for $f^i(x)$ smooth.*

Proof. We choose a local coordinate system $x = (z_1, \dots, z_n)$ such that $z_i = \sigma_i(x)$, $i = 1, \dots k$. In those coordinates we have: $f(0, z_{k+1}, \dots, z_n) = 0$. We can therefore write:

$$f(z_1, \dots, z_n) = \int_0^1 \frac{d}{dt} f(tz_1, \dots, tz_k, z_{k+1}, \dots, z_n) dt \tag{3}$$

This in turn is equal to:

$$f(z_1,\ldots,z_n)=\int\limits_0^1 df(0,z_{k+1},\ldots,z_n)[tz_1,\ldots,tz_k,z_{k+1},\ldots,z_n]dt \tag{4}$$

Setting $f^i(z)=\int\limits_0^1 df(0,z_{k+1},\ldots,z_n)[0,\ldots,t,\ldots,0]dt$ for $i=1,\ldots,k$, and $f^i(z)=0$ for $i=k+1,\ldots,n$ we obtain the result: $f(z)=\sum\limits_{i=1}^{n} f^i(z)\sigma_i(z)$. □

Obviously each vector field $X\in\Gamma^\infty(TP)$ acting on a function $f\in\mathcal{C}^\infty(P)$ can be written locally as: $X(f)=\sum\limits_{i=1}^{n}X^i\partial_i f$, where coefficients X^i are smooth functions. In particular: $X(S)=\sum\limits_{i=1}^{n}X^i\partial_i S=\sum\limits_{i=1}^{n}X^i\sigma_i$. Therefore every vector field $X\in\Gamma^\infty(TP)$ induces an element of I by the map: $dS(.):\Gamma^\infty(TP)\to I$. Moreover, if in addition $X(S)\equiv 0$, X induces a trivial element of I. We can define a subalgebra of the Lie algebra of vector fields $\Gamma^\infty(TP)$ by:

$$\mathfrak{g}=\{X\in\Gamma^\infty(TP)|\; X(S)=0\}\,, \tag{5}$$

Obviously $Ker(dS(.))=\mathfrak{g}$.

We now take the quotient of $\mathcal{C}^\infty(P)$ by ideal I and obtain the algebra $\mathcal{C}^\infty(\Sigma)=\mathcal{C}^\infty(P)/I$ of functions on the solution space Σ.

Digression 1. In physics we call $\mathcal{C}^\infty(P)$ the algebra of "functionals off-shell". The quotient space $\mathcal{C}^\infty(\Sigma)=\mathcal{C}^\infty(P)/I$ is referred to as the "on-shell algebra". Both concepts are crucial in QFT in the so called "functional approach". For reference see for example: [10, 12, 11]

Now let Diff(P) denote the group of diffeomorphisms of P. We define a subgroup of Diff(P) of those diffeomorphisms, that leave S invariant:

$$G\doteq\{\alpha\in \text{Diff}(P)|S(\alpha(x))=S(x)\;\forall x\in P\} \tag{6}$$

Obviously G leaves Σ invariant. We have a natural action of Diff(P) on $\mathcal{C}^\infty(P)$ by the pullback:

$$(\alpha(f))(x)\doteq\alpha^*f(x)=(f\circ\alpha)(x) \tag{7}$$

This induces also the action of G on $\mathcal{C}^\infty(P)$ and of G on $\mathcal{C}^\infty(\Sigma)$. The last one is well defined on the equivalence classes $\mathcal{C}^\infty(P)/I$ since for $\alpha\in G$, $f\in I$ we have:

$$dS(x)\equiv 0\;\forall x\in\Sigma\Rightarrow dS(\alpha(x))\equiv 0\;\forall x\in\Sigma\Rightarrow f(\alpha(x))=0\;\forall x\in\Sigma\Rightarrow\alpha(f)\in I \tag{8}$$

The action of G on $\mathcal{C}^\infty(\Sigma)$ is not faithful. Let G_0 be the subgroup of G consisting of those diffeomorphisms that act on $\mathcal{C}^\infty(\Sigma)$ trivially:

$$G_0\doteq\{\alpha\in G|\;\alpha^*f-f\in I\;\forall f\in\mathcal{C}^\infty(P)\} \tag{9}$$

It is easy to see that G_0 is a normal subgroup of G and we can take the quotient: $G_S\doteq G/G_0$. The action of G_S on $\mathcal{C}^\infty(\Sigma)$ is faithful. In general G_S is not a subgroup of G.

Digression 2. The group G_S is called in physics the group of symmetries of the action S. It maps solutions to other solutions and as a consequence it maps on-shell functionals to other on-shell functionals. Usually we are not interested in the full G_S but in it's subgroups.

Definition 2. A 1-parameter group of (smooth) transformations of P is a mapping of $\mathbb{R} \times P$ into P, $(t, p) \in \mathbb{R} \times P \to \phi_t(p) \in P$, which satisfies the following conditions:

1. For each $t \in \mathbb{R}$, $\phi_t : p \to \phi_t(p)$ is a transformation of P;
2. For all $t, s \in \mathbb{R}$ and $p \in P$, $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.

In particular we can have 1-parameter subgroups of G and G_S. Let ϕ_t be a one-parameter subgroup of G. For each point $x \in \Sigma$ we can define a curve $x(t) = \phi_t(p)$. Clearly $x(t)$ lies on Σ. We call $x(t)$ the orbit of x. Each 1-parameter group of transformations induces a vector field $X \in \Gamma^\infty(TP)$.

Definition 3. Let I_ϵ be an open interval $(-\epsilon, \epsilon)$ and U an open set of P. A local 1-parameter group of local transformations defined on $I_\epsilon \times U$ is a mapping of $I_\epsilon \times U$ into P which satisfies the following conditions:

1. For each $t \in I_\epsilon$, $\phi_t : p \to \phi_t(p)$ is a diffeomorphism of U onto the open set $\phi_t(U)$ of P;
2. If $t, s, t+s \in I_\epsilon$ and if p, $\phi_s(p) \in U$, then: $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.

Recall that we have a following result from differential geometry:

Proposition 2. *Let X be a vector field on a manifold P. For each point $p_0 \in P$, there exist a neighborhood U of p_0, a positive number ϵ and a local 1-parameter group of local transformations $\phi_t : U \to P$, $t \in I_\epsilon$, which induces the given X.*

If there exists a (global) 1-parameter group of transformations of P which induces X, then we say that X is complete. On a compact manifold every vector field X is complete. In physics we are interested in functionals on Σ that are constant along the orbits generated by local 1-parameter subgroups of G_S. Those are corresponding to certain equivalence classes of vector fields on P. To make this precise, we recall that $\mathfrak{g}$ was defined as the algebra of vector fields that annihilate the action S. Since we are interested only on the fields with flows contained in Σ we have to mode out from $\mathfrak{g}$ vector fields that vanish on Σ. Those can be also equivalently defined as:

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} |\ X(f) \in I\ \forall f \in \mathcal{C}^\infty(P)\}\,, \tag{10}$$

We define now $\mathfrak{g}_\Sigma \doteq \mathfrak{g}/\mathfrak{g}_0$. It is clear that 1-parameter subgroups of G_S generate elements of $\mathfrak{g}_\Sigma$. We can now make precise the notion of functions constant along the $\mathfrak{g}_\Sigma$-orbits on Σ. They are defined as:

$$\mathcal{C}^\infty_{\text{inv}}(\Sigma) = \{f \in \mathcal{C}^\infty(\Sigma) |\ X(f) = 0\ \forall X \in \mathfrak{g}_\Sigma\} \tag{11}$$

Later on we shall refer to this space as the space of invariant functions on Σ.

2.2 Chevalley-Eilenberg cohomology

The obvious tool for finding $\mathcal{C}^{\infty}_{\text{inv}}(\Sigma)$ is the Lie algebra cohomology. We can define the Chevalley-Eilenberg cohomology of $\mathfrak{g}_{\Sigma}$ with coefficients in the representation on $\mathcal{C}^{\infty}(\Sigma)$:

$$\begin{aligned} \gamma : \quad & \bigwedge^{q} \mathfrak{g}_{\Sigma}{}^{*} \otimes \mathcal{C}^{\infty}(\Sigma) \to \bigwedge^{q+1} \mathfrak{g}_{\Sigma}{}^{*} \otimes \mathcal{C}^{\infty}(\Sigma) \\ (\gamma\omega)(X_0, \ldots, X_{q+1}) \;\doteq\; & \sum_{i=0}^{q} (-1)^{i} X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_{q+1})) + \\ + \; & \sum_{i<j} (-1)^{i+j} \omega\left([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1}\right) \end{aligned} \tag{12}$$

The space of invariant functions on Σ corresponds now to $H^0(\gamma, \bigwedge^{\bullet} \mathfrak{g}_{\Sigma}{}^{*} \otimes \mathcal{C}^{\infty}(\Sigma))$. The grading of $\bigwedge^{\bullet} \mathfrak{g}_{\Sigma}$ will be denoted by $\tilde{r}$.

Digression 3. In physical situations one often needs to work "off-shell". This means that we do not want to deal with $\mathcal{C}^{\infty}(\Sigma)$, but with $\mathcal{C}^{\infty}(P)$. This is the case in QFT when we first want to perform the quantization, introducing a certain noncommutative product on $\mathcal{C}^{\infty}(P)$ and take the quotient by the I at the very end of the construction.

The homological interpretation of $\mathcal{C}^{\infty}(\Sigma) = \mathcal{C}^{\infty}(P)/I$ is provided by using the Koszul-Tate resolution. This will be the next step of our construction. More on Koszul-Tate complex can be found in [4, 5].

2.3 Koszul-Tate resolution

2.3.1 Koszul construction

We start with the Koszul construction. As argued before, elements of I can be locally written as $f(x) = \sum_{i=1}^{n} f^i(x)\sigma_i(x)$. We choose the local basis $\{e_1 \ldots e_n\}$ of T_xP at point x. The dual basis would be denoted by: $\{e^1 \ldots e^n\}$. Note that $\sum_{i=1}^{n} e^i\sigma_i(x)$ is an element of T^*_xP. We use this fact to construct the resolution of $\mathcal{C}^{\infty}(\Sigma)$. Let $\bigwedge^1(P)$ be the space of 1-forms on P. We define the Koszul map $\delta : \bigwedge^1(P) \to \bigwedge^0(P)$ locally by setting it's value on the basis elements and extending it by linearity to the whole $\bigwedge^1(P)$:

$$\delta(e^i)(x) = \sigma_i(x), \;\; i = 1, \ldots, n \tag{13}$$

It is now clear, that for an arbitrary $\omega \in \bigwedge^1(P)$ we have:

$$\delta(\omega)(x) = \sum_{i=1}^{n} \omega_i(x)\sigma_i(x) \in I \tag{14}$$

Therefore $Im(\delta) = I \subset \bigwedge^0(P)$. We assign to elements of this algebra grade r equal to the form degree. Now we extend δ to the whole graded algebra $\bigwedge^{\bullet}(P)$ by requirement, that it is a graded derivation.We have obviously:

$$H_0(\delta, \bigwedge^{\bullet}(P)) = \mathcal{C}^{\infty}(\Sigma) \tag{15}$$

When there exist some symmetries of the action, higher order homology can be nontrivial. To avoid this and obtain a resolution, we shall adopt the Tate construction and add further generators to the graded algebra.

2.3.2 Tate construction

The first homology of δ is, according to the definition: $H_1(\delta) = \frac{Ker(\delta)_1}{Im(\delta)_2}$. We already know that $Ker(\delta)_1$ can be characterized by elements of $\mathfrak{g}$. Now we have to find out what is $Im(\delta)_2$ in terms of elements of $\mathfrak{g}$. It is easy to check that those will be exactly vector fields from $\mathfrak{g}$ that vanish on Σ, i.e. elements of $\mathfrak{g}_0$. Therefore we can conclude that $H^1(\delta)$ can be characterized by $\mathfrak{g}/\mathfrak{g}_0 = \mathfrak{g}_\Sigma$.

We choose a local linear map $R : \mathfrak{g}_\Sigma = \mathfrak{g}/\mathfrak{g}_0 \to \mathfrak{g} \subset \Gamma^\infty(TP)$, such that $[R(X)] = X$. In other words, R chooses representant of each equivalence class. The choice of R is of course non unique. In general R is not a Lie algebra homomorphism. Let $\{f_j(x)\}$, $j = 1, \ldots, m < n$ be the local basis in $\mathfrak{g}_\Sigma$. In local coordinate system we can write R as: $R(X)^i(x) = \sum_{j=1}^{m} R^i_j(x) X^j(x)$.

Now take the map: $\mathfrak{g}_\Sigma \ni X \mapsto \sum_{i=1}^{n} R(X)^i e^i$ (defined locally). It is an element of $\bigwedge^1(P) \otimes \mathfrak{g}^*_\Sigma$. We can now write:

$$\sum_{i=1}^{n} R(X)^i e^i = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} R^i_j e^i \right) X^j , \tag{16}$$

Now we proceed analogously as with the Koszul construction. Let $S^k(\mathfrak{g}_\Sigma)$ denote the symmetrized k-th tensor power of $\mathfrak{g}_\Sigma$, $S^0(\mathfrak{g}_\Sigma) = \mathbb{R}$. Let $S^\bullet(\mathfrak{g}_\Sigma) \doteq \bigoplus_{k=0}^{\infty} S^k(\mathfrak{g}_\Sigma)$. To elements of this algebra we assign the grading $r = 2k$. We define Koszul-Tate map $\delta : S^1\mathfrak{g}_\Sigma \to \bigwedge^1(P)$ by setting it's value on the local basis:

$$(\delta f_j)(x) = \sum_{i=1}^{n} R^i_j e^i \tag{17}$$

We extend δ to be a graded derivation on the whole graded algebra $S^\bullet(\mathfrak{g}_\Sigma) \otimes \bigwedge^\bullet(P)$. If we assume that there are no further reducibility relations among the elements of $\mathfrak{g}_\Sigma$ we obtain:

$$H_0 \quad (\delta, S^\bullet(\mathfrak{g}_\Sigma) \otimes \bigwedge^\bullet(P)) = I \tag{18}$$

$$H_k \quad (\delta, S^\bullet(\mathfrak{g}_\Sigma) \otimes \bigwedge^\bullet(P)) = 0,\ k > 0 \tag{19}$$

This is the desired Koszul-Tate resolution of I

2.4 Homological perturbation theory

Now we have two graded algebras: $S^\bullet(\mathfrak{g}_\Sigma) \otimes \bigwedge^\bullet(P)$ with the differential δ and grading r and $\bigwedge^\bullet \mathfrak{g}_\Sigma{}^* \otimes \mathcal{C}^\infty(\Sigma)$ with grading $\tilde{r}$ and differential γ. We can define a joint algebra:

$$\mathcal{A} \doteq S^\bullet(\mathfrak{g}_\Sigma) \otimes \bigwedge^\bullet(P) \otimes \bigwedge^\bullet \mathfrak{g}_\Sigma{}^* \tag{20}$$

One can extend γ to the whole algebra $\mathcal{A}$ to be a differential modulo δ. We define a joint grading $N = \tilde{r} - r$. Differential δ has grade $r(\delta) = -1$ and for γ we have $\tilde{r}(\gamma) = 1$. It follows that $N(\delta) = N(\gamma) = 1$. The main theorem of homological perturbation theory (HPT) states that there exists a differential s on $\mathcal{A}$ with grade $N(s) = 1$ such that it's expansion with respect to the grading r has the form:

$$s = \delta + \gamma + \dots \tag{21}$$

Moreover we have:

$$H^0(s, \mathcal{A}) = H^0(\gamma, H_0(\delta, \mathcal{A})) = \mathcal{C}^\infty_{\text{inv}}(\Sigma) \tag{22}$$

For more on HPT see for example the notes of Birgit Richter [7].

References

[1] G. Barnich, F. Brandt and M. Henneaux, Phys. Rept. **338** (2000) 439 [arXiv:hep-th/0002245].

[2] M. Henneaux and C. Teitelboim, "Quantization of gauge systems," *Princeton, USA: Univ. Pr. (1992) 520 p.*

[3] M. Henneaux, "Lectures On The Antifield - BRST Formalism For Gauge Theories," *Lectures given at 20th GIFT Int. Seminar on Theoretical Physics, Jaca, Spain, Jun 5-9, 1989, and at CECS, Santiago, Chile, June/July 1989*, Nucl. Phys. B (Proc. Suppl.) **A18** (1990) 47.

[4] Ch. A. Weibel, *An introduction to homological algebra*, Cambridge university Press 1994

[5] D. Eisenbud, *Commutative Algebra with a view toward algebraic geometry*, Springer Verlag

[6] P.J. Hilton, U. Stammbach, *A course in homological algebra*, Springer Verlag

[7] B. Richter, *Homological perturbation theory and the existence of the BRST differential*, www.math.uni-hamburg.de/home/richter/hpthandout.pdf

[8] U. Schreiber, *On Lie ∞-modules and the BV complex*

[9] K. Fredenhagen, R. Brunetti, *Towards a Background Independent Formulation of Perturbative Quantum Gravity*, arXiv:gr-qc/0603079v3

[10] M. Dütsch, K. Fredenhagen, *The Master Ward Identity and Generalized Schwinger-Dyson Equation in Classical Field Theory*

[11] R. Brunetti, K. Fredenhagen, R. Verch, *The generally covariant locality principle - A new paradigm for local quantum field theory*, *Commun. Math. Phys.* **237** (2003) 31-68

[12] R. Brunetti, M. Duetsch, K. Fredenhagen, *Perturbative Algebraic Quantum Field Theory and the Renormalization Groups*,arXiv:math-ph/0901.2038v2

[13] L. Bonora, P. Cotta-Ramusino, *Some remarks on BRS Transformations, Anomalies and the Cohomology of the Lie Algebra of the Group of Gauge Transformations*, *Comm. Math. phys.* **87** (1983) 589-603

[14] R. Schmid *The geometry of BRS transformations*, *Illinois Journal of mathematics*, **34** (1990)

[15] R. Schmid *A Solution of the BRST Consistency Conditin and g-Symplectic Orbits*, preprint ESI # 41, Vienna, 1993

[16] C. Becchi, A. Rouet and R. Stora, *Renormalization Of The Abelian Higgs- Kibble Model*, Commun. Math. Phys. **42** (1975) 127.

[17] C. Becchi, A. Rouet and R. Stora, *Renormalization Of Gauge Theories*, Annals Phys. **98** (1976) 287.

[18] I.A. Batalin and G.A. Vilkovisky, "Relativistic S Matrix Of Dynamical Systems With Boson And Fermion Constraints," Phys. Lett. **69B** (1977) 309.

[19] I.A. Batalin and G.A. Vilkovisky, "Gauge Algebra And Quantization," Phys. Lett. **102B** (1981) 27.

[20] I.A. Batalin and G. A. Vilkovisky, "Feynman Rules For Reducible Gauge Theories," Phys. Lett. **B120** (1983) 166.

[21] I.A. Batalin and G.A. Vilkovisky, "Quantization Of Gauge Theories With Linearly Dependent Generators," Phys. Rev. **D28** (1983) 2567.