Lie algebra cohomology

Relation to the de Rham cohomology of Lie groups

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Abstract

We define the complex of invariant forms on a left G-manifold M, and prove that the cohomology of this complex is isomorphic to the cohomology of M if the manifold M is compact and the Lie group G compact and connected.

1 Foundational Material

In this section, we give a crash course on manifolds, Lie groups, their associated Lie algebras and de Rham cohomology. General references are [1], [2], [3], [4], [5], [6] and [7] for instance.

Definition 1.1 (**Topological manifold**)**.** Let $\mathcal{M}$ be a topological space. We say $\mathcal{M}$ is a *topological manifold of dimension* n if it has the following properties:

- $\mathcal{M}$ *is a Hausdorff space:* For every pair of points $p, q \in \mathcal{M}$, there are disjoint open subsets $U, V \subset \mathcal{M}$ such that $p \in U$ and $q \in V$.
- $\mathcal{M}$ *is second countable:* There exists a countable basis for the topology of $\mathcal{M}$.
- $\mathcal{M}$ *is locally Euclidean of dimension* n*:* Every point p in $\mathcal{M}$ has a neighborhood U such that there is a homeomorphism[1] ϕ from U onto an open subset of $\mathbb{R}^n$.

Let U be an open subset of $\mathcal{M}$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a *chart* of $\mathcal{M}$, U a *coordinate neighborhood* of $\mathcal{M}$, and ϕ a *coordinate map* on U.

Definition 1.2. Two charts $(U, \phi : U \to \mathbb{R}^n)$, $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are C^∞*-compatible* if the two maps

$$\phi \circ \psi^{-1} : \psi\left(U \cap V\right) \to \phi\left(U \cap V\right), \quad \psi \circ \phi^{-1} : \phi\left(U \cap V\right) \to \psi\left(U \cap V\right)$$

are C^∞. These two maps are called the *transition functions* between the charts.

Definition 1.3. A C^∞ *atlas* or simply an *atlas* on a locally Euclidean space $\mathcal{M}$ is a collection $\{(U_i, \phi_i)\}_{i \in I}$ of compatible charts that cover $\mathcal{M}$ such that $\mathcal{M} = \bigcup_{i \in I} U_i$.

An altas $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ on a locally Euclidean space is said to be *maximal* if it is not contained in a larger atlas.

[1] A homeomorphism is a bijective continuous map with continuous inverse.

Definition 1.4 (**Smooth manifold**). A *smooth* or C^∞ *manifold* is a topological manifold $\mathcal{M}$ together with a maximal atlas.
The maximal altas is also called a *differentiable structure* on $\mathcal{M}$. A monifold is said to have a dimension n if all of its connected components have dimension n. A manifold of dimension n is also called n*-manifold*.

For further information about tangent space, tangent bundle, 1-forms,. . ., see Appendix!

Definition 1.5. A map $f : M \to N$ between two smooth manifolds is said to be smooth (or differentiable or C^∞) if, for any chart ϕ on M and ψ on N, the function $\psi \circ f \circ \phi^{-1}$ is smooth as soon as it is defined.
The map f is a diffeomorphism if it is a bijection and both f and f^{-1} are smooth.

Definition 1.6 (**Lie group**). A *Lie group* is a C^∞ manifold G which is also a group such that the two group operations, multiplication

$$\mu : G \times G \to G, \qquad \mu(a, b) = ab$$

and inverse

$$\iota : G \to G, \qquad \iota(a) = a^{-1}$$

are C^∞.

A *homomorphism of Lie groups* is a homomorphism of groups which is also a smooth map. An *isomorphism of Lie groups* is a homomorphism f which admits an inverse (also C^∞) f^{-1} as maps and such that f^{-1} is also a homomorphism of Lie groups.

We introduce here the notion of Lie algebras and the example of main interest for us, the tangent space $T_e(G)$ of a Lie group G at the identity e.

Definition 1.7 (**Lie algebra**). A *Lie algebra* over $\mathbb{R}$ is a *vector space* $\mathfrak{g}$ together with a bilinear homomorphism, called the *Lie bracket*,

$$[-, -] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that, for any $x \in \mathfrak{g}$, $y \in \mathfrak{g}$, $z \in \mathfrak{g}$, one has:

$$\begin{aligned} [x, y] &= -[y, x] && (\textit{skew symmetry}) \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 && (\textit{Jacobi identity}). \end{aligned}$$

A *homomorphism of Lie algebras* is a linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ with $\phi([x,y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$.

Definition 1.8. A fixed element $g \in G$ gives the *left translation* $L_g : G \to G$ with $L_g(h) = g \cdot h$ for all $h \in G$. Similarly, we define *right translations* R_g by $R_g(h) = h \cdot g$.

Recall first that, if $p : T(G) \to G$ is the tangent bundle of the manifold G, a *vector field* X on G is a smooth section of p.

Definition 1.9. Denote by $DL_g : T(G) \to T(G)$ the map induced by the left translation L_g. A vector field X on G is called left invariant if $DL_g(X) = X$, for any $g \in G$. This means $DL_g(X_h) = X_{gh}$, for any $h \in G$.
In other words, a vector field X is left-invariant if and only if it is L_g-related to itselft for all $g \in G$.

If G is a Lie group, we denote by $\mathfrak{g}$ the vector space of left invariant vector fields on G. If X and Y are vector fields, then their *bracket* is defined to be the vector field $[X,Y]f = X(Yf) - Y(Xf)$ for all functions f. The bracket is anti-commutative and satisfies a Jacobi identity. If X and Y are left invariant vector fields, their bracket $[X,Y]$ is also left invariant. Therefore, the vector space $\mathfrak{g}$ has the structure of a Lie algebra, called *the Lie algebra associated to the Lie group G*.

X is a vector field on a Lie group G, we see directly from the definition that X is left invariant if and only if

$$X_g = (DL_g)(X_e).$$

The vector space $\mathfrak{g}$ is isomorphic to the tangent space, $T_e(G)$, at the identity e of G.

Proposition 1.10. There is a morphism of Lie groups $\mathrm{Ad} : G \to Gl(\mathfrak{g})$ given by $\mathrm{Ad}(g)(X) = ((DR_g)^{-1} \circ (DL_g))(X)$, where $Gl(\mathfrak{g})$ is the group of linear isomorphisms of the Lie algebra $\mathfrak{g}$.

Definition 1.11. The map $\mathrm{Ad} : G \to Gl(\mathfrak{g})$ is called the *adjoint representation* of the Lie group G.

Let G be a Lie group and M a differentiable manifold (always of class C^∞). A *differentiable action* of G on M is an action $G \times M \to M$ which is a C^∞-differentiable map. A manifold M together with a differentiable action of G on it is called a *differentiable G-manifold.*

Definition 1.12. A Lie group G acts on a manifold M, on the left, if there is a smooth map $G \times M \to M$, $(g, x) \mapsto gx$, such that $(g \cdot g')x = g(g' \cdot x)$ and $ex = x$ for any $x \in M$, $g \in G$, $g' \in G$. Such data endows M with the appellation of a left *G-manifold.* A left action is called

- *effective* if $gx = x$ for all $x \in M$ implies $g = e$;
- *free* if $gx = x$ for any $x \in M$ implies $g = e$.

Example 1.13. Let G be a Lie group. The Lie group multiplication gives to G the structure of a

- left G-manifold, with $L : G \times G \to G$, $\quad L(g, g') = L_g(g') = g \cdot g'$,
- right G-manifold, with $R : G \times G \to G$, $\quad R(g', g) = R_g(g') = g' \cdot g$.

We denote the p-forms on M by $A^p_{DR}(M)$ and the entire graded algebra of forms by $A_{DR}(M)$.

Let G be a Lie group. If M is a left G-manifold, we denote by

$$g^* : A_{DR}(M) \to A_{DR}(M)$$

the "pullback" map induced on differential forms by the action of $g \in G$. More specifically, for vector fields $X_1, \ldots, X_k$ and a k-form ω, we define at $m \in M$,

$$g^*\omega(X_1, \ldots, X_k)(m) = \omega_{g \cdot m}(Dg_m X_1(m), \ldots, Dg_m X_k(m)).$$

We sometimes write $\omega_x(X_1, \ldots, X_k) = \omega(X_1, \ldots, X_k)(x)$, $L^*_g = g^*\omega$ and $Dg = DL_g$.

Definition 1.14. An *invariant form on a left G-manifold M* is a differential form $\omega \in A_{DR}(M)$ such that $g^*\omega = \omega$ for any $g \in G$. We denote the *set of invariant forms* by $\Omega_L(M)$.

In the case of a Lie group G, we note that the left invariant forms (right invariant forms) correspond to the left (right) translation action. We denote these sets by

$\Omega_L(G)$ and $\Omega_R(G)$ respectively. A form on G that is left and right invariant is called bi-invariant (or invariant if there is no confusion). The corresponding set is denoted by $\Omega_I(G)$.

The aim of this section is to prove that these different sets of invariant forms allow the determination of the cohomology of G-manifolds and Lie groups.
There are several important operations that are performed on forms to create new ones illuminating certain geometric properties. Let M^n be a manifold with a vector field X and a $(p+1)$-form α defined on it, then we define a p-form $i(X)\alpha$ by

$$(i(X)\alpha)(Y_0, \ldots, Y_{p-1}) = \alpha(X, Y_0, \ldots, Y_{p-1}),$$

where $Y_0, \ldots, Y_{p-1}$ are vector fields. The operation on the $(p+1)$-form α that produces the p-form $i(X)\alpha$ is called *interior multiplication* by X.

We defined the differential of a function f to be the 1-form df satisfying $df(X) = Xf$ for a vector field X. The operator d is called the *exterior derivative* and is defined on all forms. For a $(p-1)$-form α and vector fields $Y_0, \ldots, Y_{p-1}$, the p-form $d\alpha$ is defined by

$$\begin{aligned} d\alpha(Y_0, \ldots, Y_{p-1}) = &\sum_{j=0}^{p-1} (-1)^j Y_j(\alpha((Y_0, \ldots, \hat{Y_j}, \ldots, Y_{p-1})) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([Y_i, Y_j], Y_0, \ldots, \hat{Y_i}, \ldots, \hat{Y_j}, \ldots, Y_{p-1}), \end{aligned}$$

where "hats" above $Y_i's$ indicate that they are missing. We denote this operation on forms by $d:\ A^p_{DR}(M) \to A^{p+1}_{DR}(M)$. For instance, from the formula, we recover the definition $df(Y) = Yf$ and, for a 1-form α, we find

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).$$

A collection of vector spaces $\{V^k\}_{k=0}^{\infty}$ with linear maps $d_k :\ V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a *differential complex* or a *cochain complex.* For any open subset U of $\mathbb{R}^n$, the exterior derivative d makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the *de Rham complex of* U:

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \ldots .$$

The closed forms ($d\omega = 0$) are precisely the elements of the kernel of d and the exact forms are the elements of the image of d.

The exterior derivative $d: A^p_{DR}(M) \to A^{p+1}_{DR}(M)$ makes $A^*_{DR}(M)$ a cochain complex. Its cohomology is called the *de Rham cohomology* $H^*_{DR}(M)$,

$$H^p_{DR}(M) = \frac{\operatorname{Ker}\left(d: \ A^p_{DR}(M) \to A^{p+1}_{DR}(M)\right)}{\operatorname{Im}\left(d: \ A^{p-1}_{DR}(M) \to A^p_{DR}(M)\right)} \quad (p\text{-th de Rham cohomology}).$$

The de Rham cohomology is the quotient of the closed forms modulo the exact forms. For instance, if $f: M \to \mathbb{R}$ is a smooth function on a connected manifold M, then $df = 0$ only if f is a constant function. Since there are no exact 0-forms, we have $H^0_{DR}(M) = \mathbb{R}$. If M is not connected, then a function could take different constant values on different components. If there are k components, $H^0_{DR}(M) = \mathbb{R}^k$.

2 Lie algebra and de Rham cohomology

We will typically write $H^*(M;\mathbb{R})$ when we refer to cohomology, even though it may be coming from forms.
The de Rham cohomology of the compact Lie group G is isomorphic to the Lie algebra cohomology of the Lie algebra $\mathfrak{g}$ (Theorem 2.3). Actually, the Lie algebra cohomology can be defined as the cohomology of the complex of left-invariant differential forms on the corresponding group G. Note that cohomology classes of the compact group G can be represented by left-invariant forms by averaging over the group any closed form from a given cohomology class.

A reference for this section is [2] (section 1.6) and [8] (chapter 10 and 12):

Proposition 2.1. Let G be a Lie group and M be a left (or a right) G-manifold. Then the set of invariant forms of M is stable under d. Then the sets of left invariant forms on G are invariant under $i(X)$, for X a left invariant vector field.

Proof. Suppose ω is a left invariant form on G and X is a left invariant vector field on G. We have, using the left invariance of X and ω,

$$\begin{aligned} L_g^* i(X)\omega(Y_1,\dots,Y_k)(x) &= L_g^*\omega(X,Y_1,\dots,Y_k)(x) \\ &= \omega_{gx}(DL_gX(x), DL_gY_1(x),\dots,DL_gY_k(x)) \\ &= i(DL_gX)\omega(DL_g(Y_1),\dots,DL_g(Y_k))(gx) \\ &= i(X)\omega(Y_1,\dots,Y_k)(gx). \end{aligned}$$

Hence, $i(X)\omega$ is left invariant. □

The previous result justifies the following definition.

Definition 2.2. Let G be a Lie group and M be a left G-manifold. The invariant cohomology of M is the homology of the cochain complex $(\Omega_L(M), d)$. We denote it by $H_L^*(M)$.

The main result is the following theorem.

Theorem 2.3. Let G be a compact connected Lie group and M be a compact left G-manifold. Then

$$H_L^*(M) \cong H^*(M; \mathbb{R}).$$

We will prove that the injection map $\Omega_L(M) \to A_{DR}(M)$ induces an isomorphism in cohomology. For that, we need some results concerning integration on a compact connected Lie group.

Proposition 2.4. On a compact connected Lie group, there exists a bi-invariant volume form.

Proof. Recall from the Foundational Material that the tangent bundle of G trivializes as $T(G) \cong G \times \mathfrak{g}$. If $\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g}$, we therefore have a trivialization of the cotangent bundle $T^*(G) \cong G \times \mathfrak{g}^*$ and of the differential forms bundle. Exactly as for vector fields, we observe that left (right) invariant forms are totally determined by their value at the unit e and that we have isomorphisms of vector spaces

$$\Omega_L(G) \cong \Omega_R(G) \cong \wedge \mathfrak{g}^*,$$

where $\wedge \mathfrak{g}^*$ is the exterior algebra on the vector space $\mathfrak{g}^*$. To make this space precise, recall that the elements of $\mathfrak{g}^*$ are left invariant 1-forms dual to left invariant vector fields. If we choose a basis $\{\omega_1, \ldots, \omega_n\}$ dual to a basis of left invariant vector fields, an element of $\wedge \mathfrak{g}^*$ may be written

$$\alpha = \sum a_{i_1 \cdots i_p} \, \omega_{i_1} \cdots \omega_{i_p}$$

where the $a_{i_1 \cdots i_p}$'s are constant. Choose such an α of degree n equal to the dimension of G. We associate to α a unique left invariant form α_L such that $(\alpha_L)_e = \alpha$ and a unique right invariant form α_R such that $(\alpha_R)_e = \alpha$. More precisely, we set:

$$\begin{aligned}
(\alpha_L)_e(X_1, \ldots, X_n) &= \alpha((DL_g)^{-1}X_1, \ldots, (DL_g)^{-1}X_n), \\
(\alpha_R)_e(X_1, \ldots, X_n) &= \alpha((DR_g)^{-1}X_1, \ldots, (DR_g)^{-1}X_n).
\end{aligned}$$

Recall, from Definition 1.11, the homomorphism of Lie groups $Ad\colon G \to Gl(\mathfrak{g})$. As direct consequences of the definitions, we have

$$\begin{aligned}(L_g^*\alpha_R)_h(X_1,\dots,X_n) &= (\alpha_R)_{gh}(DL_g(X_1),\dots,DL_g(X_n))\\ &= \alpha((DR_{gh})^{-1}\circ(DL_g)(X_1),\dots)\\ &= \alpha((DR_h)^{-1}\circ(DR_g)^{-1}\circ DL_g(X_1),\dots)\\ &= (\alpha_R)_h((DR_g)^{-1}\circ DL_g(X_1),\dots)\\ &= (\det(\mathrm{Ad(g)})(\alpha_R)_h(X_1,\dots,X_n).\end{aligned}$$

The composition $\det\circ\mathrm{Ad}:\ G\to\mathbb{R}$ has for image a compact subgroup of $\mathbb{R}$; that is, $\{1\}$ or $\{-1,1\}$. Since the group G is connected, we get $\det(\mathrm{Ad(g)})=1$, for any $g\in G$, and α_R is a bi-invariant volume form. □

Proof of Theorem 2.3

Denote by $\iota:\ \Omega_L(M)\hookrightarrow A_{DR}(M)$ the canonical injection of the set of left invariant forms. We choose the bi-invariant volume form on G such that the total volume of G is 1, $\int_G dg=1$ ($dg=$ measure on G). This volume form allows the definition of $\int_G dg\in\mathbb{R}^k$ for any smooth function $f:\ G\to\mathbb{R}^k$.

Let $\omega\in A^k_{DR}(M)$ and $x\in M$ be fixed. As a function f, we take $G\to\wedge T_x(M)^*$, $g\mapsto g^*\omega(x)$. We get a differential form $\rho(\omega)$ on M defined by:

$$\begin{aligned}\rho(\omega)(X_1,\dots,X_k)(x) &= \int_G g^*\omega(X_1,\dots,X_k)(x)dg\\ &= \int_G (L_g)^*\omega(X_1,\dots,X_k)(x)dg.\end{aligned}$$

We have thus built a map $\rho:\ A_{DR}(M)\to A_{DR}(M)$ and we now analyze its properties:

<u>**Fact 1:** $\rho(\omega)\in\Omega_L(M)$.</u>

Proof. Let $g'\in G$ be fixed. The map $(DL_{g'}):\ T_x(M)\to T_{g'x}(M)$ induces a map $\wedge(DL_{g'})^*:\ \wedge T_{g'x}(M)^*\to\wedge T_x(M)^*$. Therefore, one has (in convenient shorthand):

$$\begin{aligned}(DL_{g'})^*\rho(\omega)(x) &= \wedge(DL_{g'})^*\int_G (Lg)^*\omega(x)dg\\ &= \int_G (L_{g'\cdot g})^*\omega(x)dg\\ &= \int_G (L_g)^*\omega(x)dg\\ &= \rho(\omega)(x).\end{aligned}$$

□

Fact 2: If $\omega \in \Omega_L(M)$ then $\rho(\omega) = \omega$.

Proof. If $(L_g)^*\omega(x) = \omega(x)$, then $\rho(\omega)(x) = \int_G (L_g)^*\omega(x)dg = \omega(x)\int_G dg = \omega(x)$. $\square$

Fact 3: $\rho \circ d = d \circ \rho$.

Proof. This is an easy verification from the definitions of d and ρ. $\square$

From Facts $1-3$, we deduce that $H(\rho) \circ H(\iota) = \mathrm{id}$ and $H(\iota)$ is injective.

Fact 4: The integration can be reduced to a nighborhood of e.

Proof. Let U be a neighborhood of e. We choose a smooth function $\varphi : G \to \mathbb{R}$, with compact support included in U, such that $\int_G \varphi \, dg = 1$. Now we denote the bi-invariant volume form dg by ω_{vol}. By classical differential calculus on manifolds, the replacement of ω_{vol} by $\varphi\omega_{vol}$ leaves the integral unchanged. The fact that $\varphi\omega_{vol}$ has its support in U allows the reduction of the domain of integration to U. $\square$

Our construction process can now be seen in the following light.
Let $L : G \times M \to M$ be the action of G on M. Denote by $\pi_G^*(\varphi\omega_{vol})$ the pullback of $\varphi\omega_{vol}$ to $A_{DR}(U \times M)$ by the projection $\pi_G : G \times M \to G$ and by $L^* : A_{DR}(M) \to A_{DR}(U \times M)$ the map induced by L. If α is a form on $U \times M$, we denote by $I(\alpha)$ the integration of $\alpha \wedge \pi_G^*(\varphi\omega_{vol})$ over the U-variables, considering the variables in M as parameters. We then have a map $I : A_{DR}(U \times M) \to A_{DR}(M)$ which is compatible with the coboundary d and which induces $H(I)$ in cohomology.

To any $\omega \in A_{DR}(M)$ we associate the form $L^*(\omega) \wedge \pi_G^*(\varphi\omega_{vol})$ on $U \times M$ and check easily (see [9], page 150):

$$\rho(\omega) = I(L^*(\omega)).$$

In other words, the following diagram is commutative:

$$\begin{tikzcd} A_{DR}(M) \arrow[r, "L^*"] \arrow[dr, "\rho"'] & A_{DR}(U \times M) \arrow[r, "I"] & A_{DR}(M) \\ & \Omega_L(M) \arrow[ur, "\iota"'] & \end{tikzcd}$$

For U, we now choose a contractible neighborhood of e. The identity map on $U \times M$ is therefore homotopic to the composition

$$U \times M \xrightarrow{\pi} M \xrightarrow{j} U \times M,$$

where π is the projection and j sends x to (e, x).
By using $I \circ \pi^* = \mathrm{id}$ and the compatibility of de Rham cohomology with homotopic

maps, we get:

$$\begin{aligned} H(I) \circ H(L^*) &= H(I) \circ \mathrm{id}_{H(M \times U)} \circ H(L^*) \\ &= H(I) \circ H(\pi^*) \circ H(j^*) \circ H(L^*) \\ &= H(j^*) \circ H(L^*) = H((L \circ j)^*) = \mathrm{id}. \end{aligned}$$

This implies $\mathrm{id} = H(\iota) \circ H(\rho)$ and $H(\iota)$ is surjective. $\square$

3 Appendix

General references for the whole appendix are [1], [2], [5], [6], [7], [8] and [10].

The dual of a vector space V over $\mathbb{K}$, denoted V^*, is the space $\mathrm{Hom}(V, \mathbb{K})$ of linear forms on V, i.e. of linear maps

$$\alpha : V \to \mathbb{K} \qquad \alpha(tx + sy) = t\alpha(x) + s\alpha(y) \quad \forall t, s \in \mathbb{K},\ x, y \in V.$$

We have $\dim(V) = \dim(V^*)$ and $(V^*)^* \cong V$ for any infinite dimensional vector space V. The elements of V^* are called *covectors* or 1-*covectors* on V.

In calculus we visualize the tangent space $T_p(\mathbb{R}^n)$ at p in $\mathbb{R}^n$ as the vector space of all arrows emanating from p. Elements of $T_p(\mathbb{R}^n)$ are called tangent vectors at p in $\mathbb{R}^n$.
As for $\mathbb{R}^n$, the tangent vectors at p form a vector space $T_p(M)$, called the tangent space of manifold M at p. We also write T_pM instead of $T_p(M)$.
The set $TM = \bigcup\limits_{p \in M} T_pM$ is called the *tangent bundle*, the set $T^*M = \bigcup\limits_{p \in M} (T_pM)^*$ the *cotangent bundle* of M.

A *vector field* X on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector X_p in $T_p(\mathbb{R}^n)$.

A *section* of a vector bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = 1_M$. This condition means precisely that for each p in M, $s(p) \in E_p$. Pictorially we visualize a section as a cross-section of the bundle (Figure 3.1). We say that a section is smooth if it is smooth as a map from M to E.

A *vector field* X on a manifold M is a function that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi : TM \to M$ and the vector field is *smooth* if it is smooth as a map from M to TM.

The *cotangent space* to $\mathbb{R}^n$ at p, denoted by $T_p^*(\mathbb{R}^n)$ or $T_p^*\mathbb{R}^n$, is defined to be the dual space $(T_p\mathbb{R}^n)^*$ of the tangent space $T_p(\mathbb{R}^n)$.

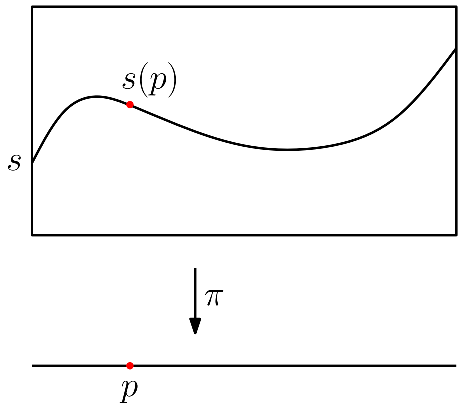

Fig. 3.1: A section of a vector bundle.

Thus, an element of the cotangent space $T_p^*(\mathbb{R}^n)$ is a covector or a linear functional on the tangent space $T_p(\mathbb{R}^n)$. In parallel with the definition of a vector field, a *covector field* or a *differential* 1-*form* ω on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a covector $\omega_p \in T_p^*(\mathbb{R}^n)$. We call a differential 1-*form* a 1-*form* for short.

More generally, a *differential form* ω of degree k or a k-*form* on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U an alternating k-linear function on the tangent space $T_p(\mathbb{R}^n)$.
At each point p in U, ω_p is a linear combination

$$\omega_p = \sum a_I(p) dx_p^I, \text{ with } dx_p^I = dx_p^{i_1} \wedge \cdots \wedge dx_p^{i_k}, \quad 1 \leq i_1 < \cdots < i_k \leq n,$$

and a k-form ω on U is a linear combination

$$\omega = \sum a_I dx^I,$$

with function coefficients $a_I : U \to \mathbb{R}$. We say that a k-form ω is C^∞ on U if all the coefficients a_I are C^∞ on U. Denote by $\Omega^k(U)$ the vector space of C^∞ k-forms on U.

Let M be a smooth manifold. A *differential* 0-*form* on M is a smooth function $\omega : M \to \mathbb{R}$. If $k \geq 1$ is an integer, a *differential* k-*form* on M is a rule ω that to every point $p \in M$ assigns an alternating k-linear form ω_p on T_pM such that the family $\{\omega_p\}_{p \in M}$ depends smoothly on p in the following sense. If $U \subset M$ is any open subset and $X_1, \ldots, X_k$ are smooth vector fields on U, then the function

$$\omega(X_1, \ldots, X_k) : U \to \mathbb{R}, \quad p \mapsto \omega_p(X_1(p), \ldots, X_k(p)),$$

is also smooth.

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