Introduction To Lie algebras

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1 Lie algebras

1.1 Definition and examples

Definition 1.1. A Lie algebra is a vector space $\mathfrak{g}$ over a field F with an operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which we call a ***Lie bracket***, such that the following axioms are satisfied:

- It is bilinear.
- It is ***skew symmetric***: $[x,x] = 0$ which implies $[x,y] = -[y,x]$ for all $x, y \in \mathfrak{g}$.
- It satisfies the ***Jacobi Identitiy***: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

Definition 1.2. A ***Lie algebra Homomorphism*** is a linear map $H \in \mathrm{Hom}(\mathfrak{g},\mathfrak{h})$between to Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ such that it is compatible with the Lie bracket:

$$H : \mathfrak{g} \to \mathfrak{h} \quad \textit{and} \quad H([x,y]) = [H(x), H(y)]$$

Example 1.1. Any vector space can be made into a Lie algebra with the ***trivial bracket***:

$$[v,w] = 0$$

for all $v, w \in V$.

Example 1.2. Let $\mathfrak{g}$ be a Lie algebra over a field F. We take any nonzero element $x \in \mathfrak{g}$ and construct the space spanned by x, we denote it by Fx. This is an abelian one dimentional Lie algebra: Let $a, b \in Fx$. We compute the Lie bracket.

$$[\alpha x, \beta x] = \alpha\beta[x,x] = 0.$$

where $\alpha, \beta \in F$. Note: This shows in particular that all one dimentional Lie algebras have a trivial bracket.

Example 1.3. Any associative algebra $\mathfrak{A}$ can be made into a Lie algebra by taking commutator as the Lie bracket:

$$[x,y] = xy - yx$$

for all $x, y \in \mathfrak{A}$.

Example 1.4. Let V be any vector space. The space of $End(V)$ forms an associative algebra under function composition. It is also a Lie algebra with the commutator as the Lie bracket. Whenever we think of it as a Lie algebra we denote it by $\mathfrak{gl}(V)$. This is the General Linear Lie algebra.

Example 1.5. Let V be a finite dimentional vector space over a field F. Then we identify the Lie algebra $\mathfrak{gl}(V)$ with set of $n \times n$ matrices $\mathfrak{gl}_n(\mathbf{F})$, where n is the dimension of V. The set of all matrices with the trace zero $\mathfrak{sl}_n\mathbf{F}$ is a subalgebra of $\mathfrak{gl}_n(\mathbf{F})$.

Example 1.6. The set of anti symmetric matrices with the trace zero denoted by $\mathfrak{so}_n$ forms a Lie algebra under the commutator as the Lie bracket .

Example 1.7. Heisenberg algebra: We look at the vector space $\mathfrak{H}$ generated over F by the matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This is a linear subspace of $\mathfrak{gl}_3(F)$ and becomes a Lie algebra under the commutator bracket. The fact that $\mathfrak{H}$ is closed under the commutator bracket follows from the well-known commutator identity on the standard basis of $n \times n$ matrices:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$$

where δ_{ij} is the Kronecker delta.

1.2 Some Basic Notions

Definition 1.3. Let $\mathfrak{g}$ be a Lie algebra over F. Then a linear subspace $U \subseteq \mathfrak{g}$ is a ***Lie subalgebra*** if U is closed under the Lie bracket of $\mathfrak{g}$:

$$[x, y] \in U$$

for all $x, y \in U$.

Definition 1.4. Let I be a linear subspace of a Lie algebra $\mathfrak{g}$. Then I is an ***ideal*** of $\mathfrak{g}$ if

$$[x, y] \in I$$

whenever $x \in I$ and $y \in \mathfrak{g}$.

Definition 1.5. A Lie algebra $\mathfrak{g}$ is called ***abelian*** if the Lie bracket vanishes for all elements in $\mathfrak{g}$:

$$[x, y] = 0$$

for all $x, y \in \mathfrak{g}$.

Definition 1.6. Let U be a non empty subset of $\mathfrak{g}$, we call $\langle U \rangle$ ***the Lie subalgebra*** (ideal) ***generated by*** U, where:

$$\langle U \rangle = \bigcap \{I \subseteq \mathfrak{g} : I \;\; is \quad Lie \;\; subalgebra \;\; (ideal) \;\; containing \;\; U\}$$

2 Free Lie Algebras

Let X be a set. We define $W_X = \bigoplus_{x \in X} F$,where F is an arbitrary field. Then we denote the tensor algebra of W_X by TW_X which is as well a Lie algebra. ***The Free Lie algebra*** on X is the Lie subalgebra in TW_X generated by X. Where X can be canonically embedded into W_X via the map:

$$f : X \to W_X$$

$$x \mapsto e_x$$

.

3 Representations

A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra $End(V)$:

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

Definition 3.1. For a Lie algebra $\mathfrak{g}$ and any $x \in \mathfrak{g}$ we define a map

$$ad_x : \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x, y]$$

which is the ***adjoint action***.

Every Lie algebra has a representation on itself, the ***adjoint representation*** defined via the map:

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

$$x \mapsto ad_x$$

.

Definition 3.2. For two representations of a Lie algebra $\mathfrak{g}$, $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\phi' : \mathfrak{g} \to \mathfrak{gl}(V')$ a ***morphism*** from ϕ to ϕ' is a linear map $\psi : V \to V'$ such that it is compatible with the action of $\mathfrak{g}$ on V and V':

$$\phi'(x)\psi = \psi\phi(x)$$

For all $x \in \mathfrak{g}$. This constitutes the category $\mathfrak{g} - \mathbf{Mod}$.

4 The Universal Enveloping Algebra

For any associative algebra we construct a Lie algebra by taking the commutator as the Lie bracket. Now let us think in the reverse direction. We want to see if we can construct an associative algebra from a given Lie algebra and its consequences. With this construction, instead of non-associative scructures;Lie algebras, we can work with nicer and better developed structures: Unital associative algebras that captures the important properties of our Lie algebra.

4.1 Constructing $U(\mathfrak{g})$

Let us construct the tensor algebra of the Lie algebra $\mathfrak{g}$:

$$T\mathfrak{g} = \bigoplus_{k=0}^{\infty} T^k\mathfrak{g} = \bigoplus_{k=0}^{\infty} \underbrace{\mathfrak{g} \otimes ... \otimes \mathfrak{g}}_{k \quad times}$$

We look at the two sided ideal I generated by : $g \otimes h - h \otimes g - [g, h]$ For $g, h \in T\mathfrak{g}$. Its elements look like:

$$\sum_{i=0}^{k} c_i(x_1^{(i)} \otimes ... \otimes x_{n_i}^{(i)}) \otimes (g_i \otimes h_i - h_i \otimes g_i - [g_i, h_i]) \otimes (y_1^{(i)} \otimes ... \otimes y_{n_i}^{(i)})$$

for $g_i, h_i \in T\mathfrak{g}$.

Now the universal enveloping algebra is constructed by taking the quotient of our tensor algebra: $U(\mathfrak{g}) = T(\mathfrak{g})/I$.

Definition 4.1. For two ring homomorphisms $S : U(\mathfrak{g}) \to End(W)$ and $S' : U(\mathfrak{g}) \to End(W')$ a morphism from S to S' is a map $t : W \to W'$ such that it is compatible with the action of $U(\mathfrak{g})$ on W and W':

$$S'(x)t = tS(x).$$

This constitutes the category $U(\mathfrak{g}) - \mathbf{Mod}$.

Theorem 4.1. *To each representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ we can associate some $S_\phi : U(\mathfrak{g}) \to End(V)$ and to each ring homomorphism $S : U(\mathfrak{g}) \to End(V)$ we can associate some $\phi_S : \mathfrak{g} \to \mathfrak{gl}(V)$, such that a morphism ψ from ϕ to ϕ' is also a morphism from S_ϕ to S'_ϕ and a morphism t from S to S' is also a morphism from ϕ_s to ϕ'_S,and $\phi_{S_\phi} = \phi$ and $S_{\phi_S} = S$.*

Remark. This is equivalent as to say $\mathfrak{g}$-Mod is equivalent to $U(\mathfrak{g})$-Mod.

Proof. Suppose we are given a representation ϕ, we want to construct S_ϕ such that it will satisfy the properties of a ring homomorphism, that is:

$$\begin{aligned} &(1)\ S_\phi(1) = 1 \\ &(2)\ S_\phi(r + r') = S(r) + S(r) \\ &(3)\ S_\phi(rr') = S(r)S(r') \end{aligned}$$

We define S_ϕ as follows:

$$S_\phi(1) = 1$$

So that (1) holds. We consider an element of $x_1 \otimes ... \otimes x_n \in U(\mathfrak{g})$, we define S on $x_1 \otimes ... \otimes x_n$:

$$S_\phi(x_1 \otimes ... \otimes x_n) = \phi(x_1) \circ \phi(x_2) ... \circ \phi(x_n).$$

Now we define S on the rest of $U(\mathfrak{g})$ by linear extension:

$$S_\phi(x_1 \otimes ...x_n + y_1 \otimes ...y_m) = S_\phi(x_1 \otimes ...x_n) + S_\phi(y_1 \otimes ...y_n)$$

It is clear that (2) is also satisfied. We show that (3) is satisfied as well:

$$S_\phi(x_1 \otimes ...x_n \otimes y_1 \otimes ...y_m) = \phi(x_1) \circ ...\phi(x_n) \circ \phi(y_1)... \circ \phi(y_m) = S_\phi(x_1 \otimes ...x_n) \circ S_\phi(y_1 \otimes ...y_n)$$

Now suppose we have S and we want to define ϕ_S:

$$\phi_S(x) = S(x)$$

we want to show that its a Lie algebra representation:

$$\phi_S([x,y]) = S(x \otimes y - y \otimes x) = S(x \otimes y) - S(y \otimes x) = S(x) \circ S(y) - S(y) \circ S(x) = \phi_S(x) \circ \phi_S(y) - \phi_S(y) \circ \phi_S(x)$$

Now we will show if ψ is a morphism from ϕ to ϕ' then it is also a morphism from S_ϕ to $S_{\phi'}$. By definition $S_\phi = \phi$, Thus:

$$S_{\phi'}(x)\psi = \psi \circ S(x).$$

Taking another element $x \otimes y \in U(\mathfrak{g}) - Mod$, we will have:

$$S_{\phi'}(x \otimes y)\psi = \psi \circ S(x \otimes y)$$

$$S'(x) \circ S'(y) \circ \psi = S'(x) \circ \psi \circ S(y) = \psi \circ s(x) \circ S(y) = \psi \circ S(x \otimes y)$$

From the equation above it is obvious that this holds for any arbitrary element $x_1 \otimes ... \otimes x_n \in U(\mathfrak{g})$. That is we have:

$$S_{\phi'}(x_1 \otimes ... \otimes x_n)\psi = \psi \circ S_\phi(x_1 \otimes ... \otimes x_n)$$

The other way around to show that t is also a morphism from ϕ to ϕ' is obvious by using the one to one correspondence between ϕ and s.

The only thing left is to show that $\phi_{S_\phi}(x) = \phi(x)$ and $S_{\phi_S}(X) = S(x)$. For the first case we mean that if we start from ϕ go to ϕ_S and then to ϕ_{S_ϕ} we get the same ϕ. Let's look at the definition of ϕ_{S_ϕ}. It is a Lie algebra representation that we get from the ring homomorphism S_ϕ. by defining $\phi_{S_\phi}(x) := S_\phi(x)$, and we defined $S_\phi(x) = \phi(x)$. Therefore: $\phi_{S_\phi}(X) = \phi(x)$.

To show $S_{\phi_S}(x) = S(x)$, we use the same trick. By definition:

$$S_{\phi_S}(x_1 \otimes ... \otimes x_n) = \phi_S(x_1) \circ ... \circ \phi(x_n)$$

We insert the definition of $\phi_S(x)$:

$$\phi_S(x_1) \circ ... \circ \phi_S(x_n) = S(x_1) \circ ... \circ S(x_n)$$

but since S is a ring homomorphism, we have that it is multiplicative:

$$S(x_1) \circ .. \circ S(x_n) = S(x_1 \otimes ... \otimes x_n).$$

Therefore $S_{\phi_S}(x) = S(x)$.

□

5 Simple Lie algebras, Semisimple Lie algebras, Killing form, Cartan criterion for semisimplicity

Definition 5.1. A non abelian Lie algebra $\mathfrak{g}$ is called ***simple*** if it has no non trivial ideals.

Definition 5.2. We define a Lie algebra $\mathfrak{g}$ to be ***semisimple*** if it is the finite direct sum of simple Lie algebras $\mathfrak{g_i}$:

$$\mathfrak{g} = \mathfrak{g_1} \oplus \mathfrak{g_2} ... \oplus \mathfrak{g_n}.$$

Definition 5.3. Let $\mathfrak{g}$ be a *finite* dimentional Lie algebra over a field F. ***The Killing form*** κ is the bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to F$ defined by

$$\kappa(x, y) = Tr(ad_x \circ ad_y)$$

$\forall x, y, z \in \mathfrak{g}$. It has the following properties:

- It is bilinear.
- It is symmetric.
- It is ad invariant:

$$\kappa([y, x], z) + \kappa(x, [y, z]) = 0$$

.

Definition 5.4. The Killing form is said to be ***non degenerate*** if: $\forall y = 0 \;\; \kappa(x, y) = 0 \;\; implies \;\; x = 0$.

Theorem 5.1. ***Cartan criterion****: A Lie algebra $\mathfrak{g}$ over a field F of characteristic zero is semisimple if and only if the Killing form is non degenerate.*