Categorical Lie algebras and Lie algebra cohomology

Contents

Introductory remarks

The goal of this presentation is to explore some of the relations between "categorical Lie algebras" and the third cohomology groups $H^3(\mathfrak{g}, M)$ of Lie algebras $\mathfrak{g}$ with values in vector spaces M. Examples of categorical Lie algebras rely on elements of $H^3(\mathfrak{g}, M)$—better, on 3-cocycles $\omega \in Z^3(\mathfrak{g}, M)$. Moreover, a whole classification of categorical Lie algebras can be given, and elements of third cohomology groups feature into this classification.

Our exposition closely follows the paper [1] by Baez and Crans. In fact, this handout is a simplified reproduction of that paper.

Unfortunately, it is difficult to discuss categorical Lie algebras without mentioning categorical vector spaces; in turn, this may as well require a discussion in internal category theory. The end of this handout features some notes on these topics as well as some cursory notes on the category theoretic concepts relevant to this presentation. The curious reader is invited to consult texts by Mac Lane [3] and Jacobson [2] for a more formal introduction to category theory.

1 Categorical Lie algebras

Let k be a field. We denote by $\mathbf{Vect}_k$ the category of vector spaces over k and their linear maps.

Definition 1. *A* ***(semistrict) Lie 2-algebra*** *consists of the following data:*

1. *a 2-vector space* $L = (L_0, L_1, i, s, t, \gamma)$ *in* $\boldsymbol{Vect}_k$*;*
2. *a skew-symmetric bilinear functor called the* ***bracket****,* $[\cdot, \cdot] : L \times L \to L$*;*
3. *a completely antisymmetric, trilinear natural isomorphism called the* ***Jacobiator*** J*,*

$$J_{x,y,z} : [[x, y], z] \to [x, [y, z]] + [[x, z], y]$$

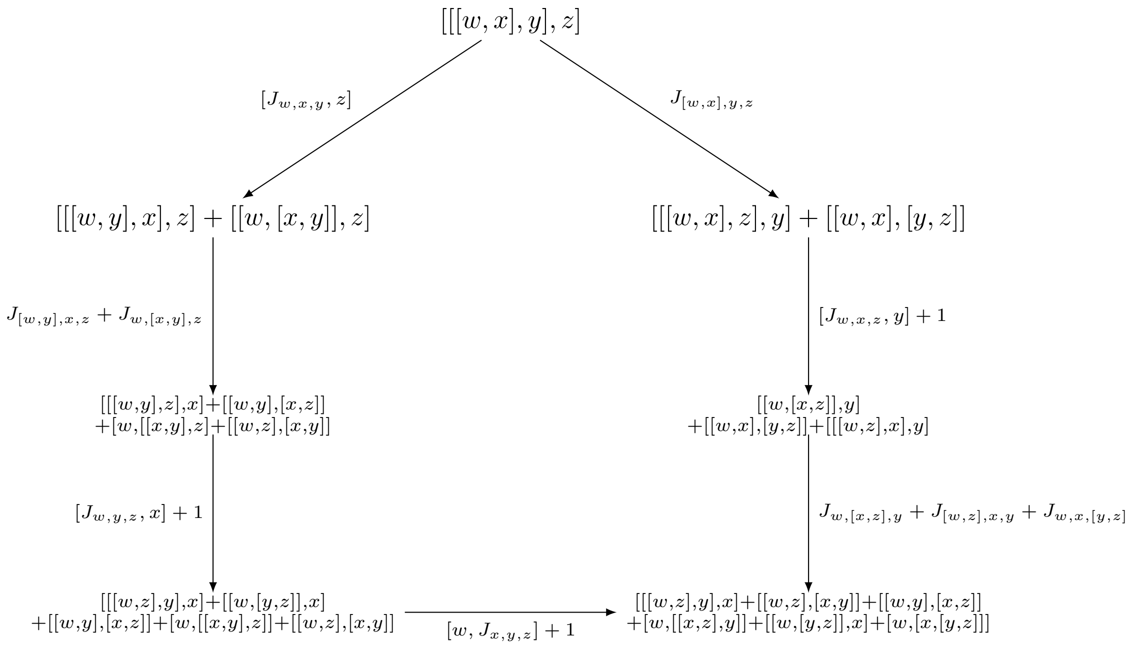
for all $x, y, z \in L_0$*.*

The Jacobiator is required to satisfy the ***Jacobiator identity****:*

$$\Big(J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}\Big)\Big([J_{w,x,z}, y] + 1\Big) J_{[w,x],y,z}$$
$$= \Big([w, J_{x,y,z}] + 1\Big)\Big([J_{w,y,z}, x] + 1\Big)\Big(J_{[w,y],x,z} + J_{w,[x,y],z}\Big)[J_{w,x,y}, z]$$

for all $w, x, y, z \in L_0$*. (There is only one choice of identity morphism 1 that can be added to each term to make the composite well-defined.)*

Here is the Jacobiator identity as a commutative diagram:



A few remarks on the Jacobiator. By definition, the Jacobiator J is a completely antisymmetric, trilinear natural transformation between linear functors in $\mathbf{Vect}_k$—in this case, J goes from the linear functor

$$[[\cdot,\cdot],\cdot] : L^3 \longrightarrow L$$

to the linear functor

$$[\cdot,[\cdot,\cdot]] + [[\cdot,\cdot],\cdot] : L^3 \longrightarrow L.$$

Explicitly, this means the Jacobiator is a trilinear map $L_0 \times L_0 \times L_0 \xrightarrow{J} L_1$, such that the appropriate diagrams in $\mathbf{Vect}_k$ commute; see Definition 18 for the precise diagrams.

Since $J_{x,y,z} \in L_1$, it follows that $J_{x,y,z}$ has a source and target for all triples $x, y, z \in L_0$. The source and target of $J_{x,y,z}$ are specified in the definition of Lie 2-algebra for all triples $x, y, z \in L_0$:

$$s\Big(J_{x,y,z}\Big) = [[x,y],z],$$

$$t\Big(J_{x,y,z}\Big) = [x,[y,z]] + [[x,z],y].$$

This explains the notation $J_{x,y,z} : [[x,y],z] \to [x,[y,z]] + [[x,z],y]$ of (3) in the definition of Lie 2-algebra. (However, this does not explain *why* we have this particular notation in the first place. The answer to this question is more fundamental, as it pertains to the basic ideas of an internal category: it is because we view $J_{x,y,z}$ as a morphism in the category L from the object $[[x,y],z]$ to the object $[x,[y,z]] + [[x,z],y]$. See the remarks following Definition 19 for an explanation of this.)

So far, we have explained the terms *trilinear* and *natural transformation* in the definition of the Jacobiator, but what about *completely antisymmetric*? Recalling definitions 26 and 27, this means

$$\begin{aligned}\text{arrow part of } J_{x,y,z} &= \overline{J}_{x,y,z} \\ &\stackrel{\text{def}}{=} J_{x,y,z} - is\Big(J_{x,y,z}\Big) \\ &= J_{x,y,z} - i\Big([[x,y],z]\Big),\end{aligned}$$

changes sign under permutations of any pair of objects x, y, z. For instance, $\overline{J}_{x,y,z} = -\overline{J}_{y,x,z}$, etc.

Definition 2. *Let L, L' be Lie 2-algebras. A* ***homomorphism*** *$F : L \to L'$ consists of:*

1. *a linear functor F from the underlying 2-vector space L to that of L';*
2. *a skew-symmetric, bilinear natural transformation*

$$F_2(x,y) : [F_0(x), F_0(y)] \to F_0[x,y]$$

such that the following diagram commutes

$$
\begin{array}{ccc}
[[F_0(x), F_0(y)], F_0(z)] & \xrightarrow{J_{F_0(x), F_0(y), F_0(z)}} & [F_0(x), [F_0(y), F_0(z)]] + [[F_0(x), F_0(z)], F_0(y)] \\
\downarrow{\scriptstyle [F_2, 1]} & & \downarrow{\scriptstyle [1, F_2] + [F_2, 1]} \\
[F_0[x, y], F_0(z)] & & [F_0(x), F_0[y, z]] + [F_0[x, z], F_0(y)] \\
\downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_2 + F_2} \\
F_0[[x, y], z] & \xrightarrow{F_1(J_{x,y,z})} & F_0[x, [y, z]] + F_0[[x, z], y]
\end{array}
$$

Given Lie 2-algebra homomorphisms $L \xrightarrow{F} L'$ and $L' \xrightarrow{G} L''$, we can suitably define their composition so that $L \xrightarrow{GF} L''$ is a Lie 2-algebra homomorphism as well. The definition goes as follows:

Definition 3. *The* ***composition*** GF *of Lie 2-algebra homomorphisms* $L \xrightarrow{F} L'$ *and* $L' \xrightarrow{G} L''$ *is defined to be the usual composite of* F *and* G *as linear functors*

$$L \xrightarrow{F} L' \xrightarrow{G} L'',$$

with $(GF)_2$ *defined as the following composite*

$$
\begin{array}{ccc}
[(FG)_0(x), (FG)_0(y)] & \xrightarrow{(GF)_2} & (FG)_0[x, y] \\
\downarrow{\scriptstyle G_2} & \nearrow{\scriptstyle F_2 \circ G} & \\
G_0[F_0(x), F_0(y)] & &
\end{array}
$$

The aim of all this is

Proposition 1. *Semistrict Lie 2-algebras in* $\boldsymbol{Vect}_k$ *and their homomorphisms form a category, which we denote* ***Lie2Alg****.*

Idea of proof. A proof of the proposition only requires straight-forward checking. One must of course verify that the composition of Lie 2-algebra homomorphisms, as defined above, indeed yields a Lie 2-algebra homomorphism, and that the composition is associative. □

2 Examples of Lie 2-algebras

Here are some examples of Lie 2-algebras.

Example 1. Let M be a vector space over k. We obtain a 2-vector space $V = (V_0, V_1, i, s, t, \gamma)$ from M as follows.

Set the trivial vector space as the object of objects, $V_0 \stackrel{\text{def}}{=} 0$, and let M be the object of arrows, $V_1 \stackrel{\text{def}}{=} M$. Source, target and identity, $M \underset{t}{\overset{s}{\rightrightarrows}} 0$ and $0 \xrightarrow{i} M$ respectively, are consequently trivial, ie. the zero map. Composition $M \times_0 M \xrightarrow{\gamma} M$ on the other hand is given by the vector space addition $+$ of M. (In this example, the pullback $M \times_0 M$ is actually the direct product $M \times M$, which is also the direct sum $M \oplus M$.) It follows readily that V is a 2-vector space in $\mathbf{Vect}_k$.

We can view V as a Lie 2-algebra with the trivial bracket functor $[\cdot,\cdot] : V \times V \to V$. That is, $[\cdot,\cdot]$ is just the trivial map on objects V_0 and arrows V_1. Notice the analogy between this example and the well-known case of turning an arbitrary vector space into an abelian Lie algebra. Indeed, we can view any 2-vector space as a Lie 2-algebra with trivial bracket structure.

Example 2. Next, we train our sights on obtaining a Lie 2-algebra L from a Lie algebra $\mathfrak{g}$. Set $L_0 = L_1 \stackrel{\text{def}}{=} \mathfrak{g}$. Define source, target, identity and composition to be the identity on $\mathfrak{g}$. (In this example, the pullback $L_1 \times_{L_0} L_1$ on which γ is defined is precisely $\mathfrak{g}$.) With this data, L is a 2-vector space, but we can say more, for the existing Lie algebra structure on $\mathfrak{g}$ endows L with further structure.

Given what we know of L, that $L_0 = L_1 = \mathfrak{g}$, there is a very natural candidate for a bracket functor $[\cdot,\cdot] : L \times L \to L$. Namely, we define

$$[x, y] \stackrel{\text{def}}{=} [x, y]_{\mathfrak{g}}$$

for all pairs of objects $x, y \in L_0$ and pairs of arrows $x, y \in L_1$, where the bracket on the right-hand side $[\cdot,\cdot]_{\mathfrak{g}}$ is the original bracket on $\mathfrak{g}$. This bracket functor is clearly skew-symmetric (in the sense of Definition 22).

There is one last set of data and axioms to define and verify in order to show that L is a Lie 2-algebra, and this involves the Jacobiator. For $x, y, z \in L_0$, the Jacobiator is given by

$$L_0 \times L_0 \times L_0 \xrightarrow{J} L_1,$$

$$J_{x,y,z} \stackrel{\text{def}}{=} [[x, y], z].$$

The Jacobiator identity holds immediately, and so it follows that L is a Lie 2-algebra.

Example 3. Let $\mathfrak{g} \xrightarrow{\varphi} \mathfrak{gl}(M)$ be a Lie algebra representation, M a vector space, and let $\omega \in Z^3(\mathfrak{g}, M)$ be a 3-cocycle. We shall construct Lie 2-algebras $\mathscr{L}$ given this data.

The underlying 2-vector space of $\mathscr{L}$ has $\mathscr{L}_0 \stackrel{\text{def}}{=} \mathfrak{g}$ and $\mathscr{L}_1 \stackrel{\text{def}}{=} \mathfrak{g} \oplus M$. Source and target are given by projecting onto $\mathfrak{g}$,

$$\mathscr{L}_1 \xrightarrow{s} \mathscr{L}_0 \qquad \mathscr{L}_1 \xrightarrow{t} \mathscr{L}_0$$

$$(x, v) \mapsto x \qquad (y, w) \mapsto y$$

while identity and composition are given by

$$\mathscr{L}_0 \xrightarrow{i} \mathscr{L}_1 \qquad \mathscr{L}_1 \times_{\mathscr{L}_0} \mathscr{L}_1 \xrightarrow{\gamma} \mathscr{L}_1$$

$$x \mapsto (x, 0) \qquad (g, f) \mapsto (x, v + w),$$

where $f, g \in \mathscr{L}_1$ with $f = (x, v)$ and $g = (x, w)$. A straight-forward calculation reveals $\mathscr{L}$ is a 2-vector space.

As for the rest of the Lie 2-algebra structure, we define the bracket functor $\mathscr{L} \times \mathscr{L} \xrightarrow{[\cdot,\cdot]} \mathscr{L}$ on objects $x, y \in \mathscr{L}_0$ by

$$[x, y] \stackrel{\text{def}}{=} [x, y]_{\mathfrak{g}},$$

where the bracket on the right hand side is the original bracket on $\mathscr{L}_0 = \mathfrak{g}$. On morphisms $f, g \in \mathscr{L}_1$, the bracket $[f, g]$ is defined by

$$[f,g] \stackrel{\text{def}}{=} \Big([x,y], \varphi(x)(w) - \varphi(y)(v)\Big),$$

where $f = (x, v)$ and $g = (y, w)$. Notice, with this last bracket, $\mathscr{L}_1$ is simply the semidirect product $\mathfrak{g} \ltimes_\varphi M$ of $\mathfrak{g}$ and M with respect to the representation φ, M being viewed as an abelian Lie algebra. In particular, $\mathscr{L}_0$ and $\mathscr{L}_1$ are Lie algebras themselves.

We would do well to define a Jacobiator by

$$J_{x,y,z} \stackrel{\text{def}}{=} \Big([[x,y],z], 0\Big)$$

for all $x, y, z \in \mathscr{L}_0$. With this data, $\mathscr{L}$ is a Lie 2-algebra.

We can define another Jacobiator on $\mathscr{L}$ by

$$J_{x,y,z} \stackrel{\text{def}}{=} \Big([[x,y],z], \omega(x,y,z)\Big),$$

for all $x, y, z \in \mathscr{L}_0$, where $\omega \in Z^3(\mathfrak{g}, V)$ is our 3-cocycle. This also defines a Lie 2-algebra $\mathscr{L}$, although it is "different" from the $\mathscr{L}$ defined immediately above. (The "difference" comes from the different Jacobiators which we have defined.) The next section is devoted to what it means for Lie 2-algebras to be different.

The reason why the Jacobiator identities hold for both examples of $\mathscr{L}$ is as follows. The Jacobiator identity is essentially a 3-cocycle identity. That is, the Jacobiator identity is describing exactly what happens when a coboundary operator δ, of a cochain complex corresponding to Lie algebra cohomology, acts on a 3-cocycle, say for instance like our $\omega \in Z^3(\mathfrak{g}, M)$. So, asking the Jacobiator identity to hold in $\mathscr{L}$ is tantamount to asking $\delta\omega = 0$, which is precisely the case by our choice of ω, ie. $\omega \in Z^3(\mathfrak{g}, M)$. Of course, zero is a perfectly good 3-cocycle as well, $0 \in Z^3(\mathfrak{g}, M)$, which explains why the Jacobiator identity holds for the first example of $\mathscr{L}$. It also helps that the brackets on $\mathscr{L}_0$ and $\mathscr{L}_1$ satisfy the Jacobi identity.

3 Classification

There is a classification of semistrict Lie 2-algebras which involves the third cohomology groups of Lie algebras $\mathfrak{g}$ with values in vector spaces M. We introduce a few more definitions before embarking on the classification.

Definition 4. *Let V be a 2-vector space, say with data $(V_0, V_1, i, s, t, \gamma)$. Then V is **skeletal** if for all $v \in V_1$, we have $t(v) = s(v)$.*

Definition 5. *Let L, L' be Lie 2-algebras. Then L and L' are **isomorphic** if there exist Lie 2-algebra homomorphisms*

$$L \underset{G}{\overset{F}{\rightleftarrows}} L'$$

*such that $GF = I$ and $FG = I$, where I denotes the identity (linear) functors on L and L' respectively. We say that L and L' are **equivalent** as Lie 2-algebras if there exist Lie 2-algebra homomorphisms*

$$L \underset{T}{\overset{S}{\rightleftarrows}} L'$$

such that $TS \simeq I$ and $ST \simeq I$, where I denotes the identity (linear) functors on L and L' respectively. Here, $\simeq$ represents linear natural isomorphism.

There are similar notions of equivalence and isomorphism of 2-vector spaces; just replace each instance of the term "Lie 2-algebra" ("Lie 2-algebra homomorphism") with "2-vector space" ("linear functor") in Definition 5.

We are now in position to introduce

Theorem 1. *There is a one-to-one correspondence between equivalence classes of Lie 2-algebras, where equivalence is given by equivalence as Lie 2-algebras, and isomorphism classes of quadruples consisting of a Lie algebra $\mathfrak{g}$, a vector space M, a representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(M)$ and $\omega \in H^3(\mathfrak{g}, M)$, an element of the third cohomology group of $\mathfrak{g}$ with values in M:*

$$\mathscr{L} \longleftrightarrow (\mathfrak{g}, M, \varphi, \omega).$$

To prove this, we make use of the following

Lemma. *Every Lie 2-algebra L is equivalent to a skeletal Lie 2-algebra L'. (Lie 2-algebras are skeletal if the underlying 2-vector spaces are skeletal.)*

Proof of lemma, sketch. It suffices to show that every 2-vector space V is equivalent to a skeletal 2-vector space. Once this is established, we can carry the result over to Lie 2-algebras L by finding a skeletal 2-vector space L' to which the underlying 2-vector space of L is equivalent; from there, we transport the Lie 2-algebra structure of L onto the skeletal 2-vector space L' and obtain the desired equivalence.

Let $V = (V_0, V_1, i, s, t, \gamma)$ be a 2-vector space. Let $d : \ker(s) \to V_0$ be the linear map defined by $d = t|_{\ker(s)}$. Since V_0 and V_1 are vector spaces, we can find subspaces $X \subset V_0$, $Y \subset V_1$ such that $V_0 = \mathrm{im}(d) \oplus X$ and $V_1 = \ker(d) \oplus Y$.

Using this, we define a 2-vector space $W = (W_0, W_1, i', s', t', \gamma')$ as follows. We have $W_0 \overset{\text{def}}{=} X$ while $W_1 \overset{\text{def}}{=} X \oplus \ker(d)$. The identity $i' : W_0 \to W_1$ is defined by $x \mapsto (x, 0)$, while source and target are given by $s(x, w) = t(x, w) = x$. Composition γ' is given by

$$\gamma'\Big((x, v), (x, w)\Big) \overset{\text{def}}{=} (x, v + w)$$

for $(x, v), (x, w) \in W_1$. This collection of data defines a 2-vector space W.

What's more, W is skeletal, and V is equivalent to it. (There are very natural choices of linear functors between the two 2-vector spaces, and their compositions are naturally isomorphic to the respective identity functors.) ☐

So much for the lemma.

Proof of Theorem 1, sketch. By the lemma, every Lie 2-algebra is equivalent to a skeletal Lie 2-algebra. The theorem will then follow if we can show the result for equivalence classes of skeletal Lie 2-algebras.

The reader may have noticed that roughly one-half of the bijective correspondence of Theorem 1 has already been calculated in Example 3. Actually, Example 3 features two Lie 2-algebras which differ by their Jacobiators. Both Lie 2-algebras are skeletal. The quadruple $(\mathfrak{g}, M, \varphi, 0)$ corresponds to the first example of $\mathscr{L}$—the one with Jacobiator given by $J_{x,y,z} = \Big([[x,y],z], 0\Big)$—while $(\mathfrak{g}, M, \varphi, \omega)$ corresponds to the second—the one with Jacobiator given by $J_{x,y,z} = \Big([[x,y],z], \omega(x,y,z)\Big)$.

At this point, we should remark 0 and ω of Example 3 are only 3-cocycles, that is, elements of $Z^3(\mathfrak{g}, M)$. However, it is not difficult to check that cohomologous cocycles yield equivalent skeletal Lie 2-algebras, so there's no real trouble here. That is, we can just pass to the cohomology classes of 0 and ω in $H^3(\mathfrak{g}, M)$ to get our precise quadruples.

On the other hand, a given skeletal Lie 2-algebra $\mathscr{L} = (\mathscr{L}_0, \mathscr{L}_1, i, s, t, \gamma)$ yields a quadruple $(\mathfrak{g}, M, \varphi, \omega)$ of our desired form. Simply take $\mathfrak{g} \stackrel{\text{def}}{=} \mathscr{L}_0$ and $M \stackrel{\text{def}}{=} \ker(s)$. We define a representation φ of $\mathfrak{g}$ on M by $\varphi(x)(m) \stackrel{\text{def}}{=} [i(x), m]$ for $x \in \mathfrak{g}$ and $m \in M$. Finally, we define $\omega : \mathfrak{g}^{\otimes 3} \to M$ by $\omega(x,y,z) \stackrel{\text{def}}{=} \overline{J}_{x,y,z}$, the arrow part of the Jacobiator. Then ω is a 3-cocycle. (Seeing that ω, when defined this way, is a 3-cocycle is the most difficult part of this proof. We take it for granted, however.) Passing to the cohomology class of ω gives us our desired element in $H^3(\mathfrak{g}, M)$.

These assignments are inverse to each other, whence the result. □

4 Additional information

The terminology we use herein is based on the Gödel-Bernays set theory.

4.1 Categories

Definition 6. *A category $\mathscr{C}$ consists of the following data:*

1. *a class $ob\mathscr{C}$, whose elements are called the* ***objects of*** $\mathscr{C}$*;*
2. *a set $hom_{\mathscr{C}}(A, B)$ for each ordered pair (A, B) of objects, whose elements are called the* ***morphisms*** *or* ***arrows from*** A ***to*** B*;*
3. *for each ordered triple (A, B, C) of objects, a function called* ***composition***

$$hom_{\mathscr{C}}(A, B) \times hom_{\mathscr{C}}(B, C) \longrightarrow hom_{\mathscr{C}}(A, C),$$

denoted $(f, g) \mapsto gf$;

4. *for every object A, a morphism $1_A \in hom_{\mathscr{C}}(A, A)$ called* ***the identity on*** A.

These data are subject to the following axioms.

A1. The sets $hom_{\mathscr{C}}(A, B)$ and $hom_{\mathscr{C}}(C, D)$ are disjoint whenever $(A, B) \neq (C, D)$.

A2. (Associativity) Given morphisms $f \in hom_{\mathscr{C}}(A, B), g \in hom_{\mathscr{C}}(B, C)$ and $h \in hom_{\mathscr{C}}(C, D)$, we have

$$(hg)f = h(gf).$$

A3. For every object A, we have

$$f1_A = f \quad \text{and} \quad 1_A g = g$$

whenever $f \in hom_{\mathscr{C}}(A, B)$ and $g \in hom_{\mathscr{C}}(B, A)$.

Remark. The axiom $A1$ is somehow a matter of convention; it is possible to avoid stating $A1$ as an axiom, but this would require slightly altering our definitions and notions. Still, the axiom is meant to ensure that functions such as $f : \mathbf{R} \to \mathbf{R}, f(x) = x^2$, and $g : \mathbf{R}_{\geq 0} \to \mathbf{R}, g(x) = x^2$, are not equal.

We denote the fact that $f \in hom_{\mathscr{C}}(A, B)$ either by $f : A \to B$ or $A \xrightarrow{f} B$. For such an arrow f, the object A is known as the ***domain of*** f while B is the ***codomain of*** f. For two arrows f, g, their composition gf is defined if and only if the codomain of f equals the domain of g, the domain of gf equals the domain of f and the codomain of gf equals the codomain of g.

Finally, the identity arrow 1_A is unique for every $A \in ob\mathscr{C}$.

Definition 7. *Let $\mathscr{C}$ be a category. An arrow $A \xrightarrow{f} B$ in $\mathscr{C}$ is an* ***isomorphism*** *if there exists an arrow $B \xrightarrow{g} A$ in $\mathscr{C}$ such that*

$$gf = 1_A \quad \text{and} \quad fg = 1_B.$$

Remark. If f is an isomorphism, then g is uniquely determined by f. So, we sometimes denote g by f^{-1} and call it the ***inverse of*** f. Two objects A, B are ***isomorphic*** if there exists an isomorphism f between them.

Example 4. The category **Set** has sets as objects and functions as morphisms. The isomorphisms of this category are the bijective functions.

Example 5. The category **Grp** has groups as objects and group homomorphisms as morphisms. Bijective group homomorphisms are the isomorphisms in this category.

Example 6. The category **Ab** has abelian groups as objects and group homomorphisms between such groups as morphisms.

Example 7. The category **Ring** has (unital) rings as objects and ring homomorphisms (which map the identity 1 of the domain to the identity $1'$ of the codomain) as morphisms. Again, the isomorphisms are precisely the bijective ring homomorphisms.

Example 8. The category **Field** has fields as objects and their respective homomorphisms as morphisms.

Example 9. Fix a ring R. The category **R-mod** has left R-modules as objects and left R-module maps as morphisms. The bijective left R-module maps are the isomorphisms of this category, too.

Example 10. Fix a field F. The category $\mathbf{Vect}_F$ has vector spaces over F as objects and linear transformations as morphisms.

Example 11. The category **Top** has topological spaces as objects and continuous functions as morphisms. The isomorphisms of **Top** are ***not*** simply the bijective continuous maps; instead, the isomorphisms are given by the bijective continuous maps whose inverses (as function) are continuous themselves—homeomorphisms.

Example 12. In any category $\mathscr{C}$ and for every $A \in ob\mathscr{C}$, the identity 1_A is always an isomorphism from A back onto A.

Example 13. Fix a topological space X. We can view X as a category $\mathscr{X}$ in the following way. The objects of $\mathscr{X}$ are the open subsets $\mathscr{U} \subseteq X$. The only morphisms are the inclusion maps; that is, for open subsets $\mathscr{U}, \mathscr{V}$ of X, we require that $hom(\mathscr{U}, \mathscr{V})$ is empty if $\mathscr{U} \nsubseteq \mathscr{V}$.

4.2 Functors

Definition 8. *Let $\mathscr{C}, \mathscr{D}$ be categories. A **(covariant) functor F from $\mathscr{C}$ to $\mathscr{D}$** consists of the following data:*

1. *an assignment $A \mapsto FA$ of $ob\mathscr{C}$ into $ob\mathscr{D}$;*
2. *for every pair of objects (A, B) of $\mathscr{C}$, an assignment $f \mapsto Ff$ of $hom_{\mathscr{C}}(A, B)$ into $hom_{\mathscr{D}}(FA, FB)$.*

The following axioms are required to hold.

B1. $F(gf) = (Fg)(Ff)$ whenever gf is defined in $\mathscr{C}$.

B2. $F1_A = 1_{FA}$ for every $A \in ob\mathscr{C}$.

Definition 9. *Assume the notation from the previous definition. A **contravariant functor** F **from $\mathscr{C}$ to $\mathscr{D}$** consists of the same data and axioms as above, except with the following modifications. For every pair of objects (A, B) in $\mathscr{C}$, we have that F maps $hom_{\mathscr{C}}(A, B)$ into $hom_{\mathscr{D}}(FB, FA)$, and $F(gf) = (Ff)(Fg)$ whenever gf is defined in $\mathscr{C}$.*

Example 14. Let $\mathscr{C}$ be any category. Then the **identity functor** I from $\mathscr{C}$ to $\mathscr{C}$ sends objects $A \mapsto A$ and arrows $f \mapsto f$.

Example 15. We have a functor ***For*** from **Grp** to **Set** which sends a group G to its underlying set G and homomorphisms φ to the underlying (set) function φ. This type of functor is called a ***forgetful functor***.

Example 16. For each positive integer n, we have a functor GL_n from **Field** to **Grp** defined as follows. Given a field F, the functor GL_n assigns F to GL_nF, the group of $n \times n$ invertible matrices over F. Given a field homomorphism $F \xrightarrow{\varphi} F'$, we have a group homomorphism $GL_nF \xrightarrow{GL_n\varphi} GL_nF'$ defined by $(a_{ij}) \mapsto (\varphi a_{ij})$.

Example 17. We have another functor $-^{\times}$ from **Field** to **Grp** defined as follows. The functor $-^{\times}$ sends a field F to its multiplicative group of units $F^{\times}$—ie. the set of nonzero elements of F. For a field homomorphism $F \xrightarrow{f} K$, the morphism $F^{\times} \xrightarrow{f^{\times}} K^{\times}$ is given by restricting f to $F^{\times}$.

Example 18. Fix a field F, and consider the category $\mathbf{Vect}_F$. Let V^* denote the dual space of a vector space V. Every linear map $V \xrightarrow{f} W$ induces a linear map $W^* \xrightarrow{f^*} V^*$. The map f^* is defined by $f^*(\varphi)(v) := \varphi(f(v))$, where $\varphi \in W^*$ and $v \in V$. Iterating this process on f^* yields a linear map $V^{**} \xrightarrow{f^{**}} W^{**}$.

One can check that the assignments $V \mapsto V^{**}$ and $f \mapsto f^{**}$ define a functor from $\mathbf{Vect}_F$ to $\mathbf{Vect}_F$. This is known as the **double dual functor**.

Example 19. Let $\mathscr{C}$ be any category, and fix an object N of $\mathscr{C}$. We have a contravariant functor $hom_{\mathscr{C}}(-, N)$ from $\mathscr{C}$ to **Set** defined as follows.

For an object M of $\mathscr{C}$, we have $M \mapsto hom_{\mathscr{C}}(M, N)$, which is a set by our axioms of a category. Given a morphism $M \xrightarrow{\varphi} M'$, the function $hom_{\mathscr{C}}(-, N)(\varphi) : hom_{\mathscr{C}}(M', N) \longrightarrow hom_{\mathscr{C}}(M, N)$ is defined by pre-composition by φ: for $f \in hom_{\mathscr{C}}(M', N)$, we have $hom_{\mathscr{C}}(-, N)(\varphi)(f) := f\varphi$, which indeed defines an arrow from M to N.

4.3 Natural transformations

Definition 10. *Let F, G be functors from a category $\mathscr{C}$ into $\mathscr{D}$. A* ***natural transformation*** *η* ***from*** *F* ***to*** *G is a function which assigns to each object A of $\mathscr{C}$ a morphism $\eta_A : FA \to GA$ of $\mathscr{D}$ such that for any objects A, B of $\mathscr{C}$ and any morphism $f \in hom_{\mathscr{C}}(A, B)$, the following rectangle commutes:*

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\eta_B]{} & GB \end{array} \qquad\qquad \begin{array}{c} A \\ {\scriptstyle f}\downarrow \\ B \end{array}$$

Moreover, if every η_A is an isomorphism, then η is called a ***natural isomorphism****. Two functors F, G are* ***naturally isomorphic*** *if there exists a natural isomorphism between them.*

Natural transformations abound. Consider for instance the following examples.

Example 20. Consider the functors GL_n, $-^{\times}$ from **Field** to **Grp**. For each field F, we have a group homomorphism $GL_n(F) \xrightarrow{det_F} F^{\times}$ which assigns each matrix $(a_{ij}) \in GL_nF$ to its determinant $det_F(a_{ij}) \in F^{\times}$. Because the determinant of a matrix is defined in the same way over any field (or any commutative ring for that matter), the following diagram commutes for all homomorphisms $f : F \to K$,

$$\begin{array}{ccc} GL_nF & \xrightarrow{\det_F} & F^{\times} \\ {\scriptstyle GL_nf}\downarrow & & \downarrow{\scriptstyle f^{\times}} \\ GL_nK & \xrightarrow[\det_K]{} & K^{\times} \end{array}$$

This tells us that there exists a natural transformation from the functor GL_n to $-^{\times}$, which we may appropriately call *det*. Note, this example can be generalised by considering GL_n, $-^{\times}$ as functors from the category of commutative rings **CRing** into **Grp**.

Example 21. Recall from linear algebra the oft-quoted statement, "V is naturally isomorphic to V^{**}," where V is a finite-dimensional vector space over a field F and V^{**} its double dual. While it is, technically speaking, an abuse of language to say that V is naturally isomorphic to V^{**} (because V, V^{**} aren't functors), there is indeed a natural isomorphism between two functors lurking about in this statement.

First let us recall a few facts from linear algebra. If V is a vector space over F, then there is a canonical way of embedding V into V^{**}, namely $v \mapsto \widehat{v}$. Here, $\widehat{v} : V^* \to F$ is defined by $\widehat{v}(\varphi) := \varphi(v)$ for $\varphi \in V^*$. This assignment is always linear and one-to-one; it is an isomorphism if and only if V is finite-dimensional.

Now fix a field F. Let $\boldsymbol{I}$ be the identity functor on $\mathbf{Vect}_F$, and let $-^{**}$ denote the double dual functor. The assignment $V \xrightarrow{\eta_V} V^{**}$, $v \mapsto \widehat{v}$, allows the following diagram to commute whenever $f : V \to W$ is linear:

$$\begin{array}{ccc} V & \xrightarrow{\eta_V} & V^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ W & \xrightarrow[\eta_W]{} & W^{**} \end{array}$$

This means there is a natural transformation η from $\boldsymbol{I}$ to $-^{**}$. If we restrict our attention to $\mathbf{FinVect}_F$, the category of finite-dimensional vector spaces over F, each η_V is an isomorphism. In this case, η is a natural isomorphism from $\boldsymbol{I}$ to $-^{**}$.

4.4 Equivalence of categories

Definition 11. *Two categories $\mathscr{C}$ and $\mathscr{D}$ are said to be **isomorphic** if there exist functors*

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

such that $GF = I_{\mathscr{C}}$ and $FG = I_{\mathscr{D}}$, the identity functors on $\mathscr{C}$ and $\mathscr{D}$ respectively.

Example 22. Let L be a Lie algebra over a field k. Then the category of L-modules is isomorphic to the category of left $U(L)$-modules, where $U(L)$ denotes the universal enveloping algebra of L.

Isomorphisms of categories are not commonly observed in practice; requiring the composition of functors F, G to equal the identity functor I turns out to be a stringent condition. This condition may be relaxed in a suitable way to yield another valuable relation between categories, namely

Definition 12. *Two categories $\mathscr{C}$ and $\mathscr{D}$ are said to be **equivalent** if there exist functors*

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

such that $GF \simeq I_{\mathscr{C}}$ and $FG \simeq I_{\mathscr{D}}$, where $\simeq$ denotes natural isomorphism of functors.

Example 23. Isomorphic categories are always equivalent.

Example 24. Let R be a ring, and let M_nR be the ring of $n \times n$ matrices with entries in R. Then the category of right R-modules **mod**-R is equivalent to the category of right M_nR-modules **mod**-M_nR.

4.5 Terminal objects

Definition 13. *An object T in a category $\mathscr{C}$ is **terminal** if for every object A in $\mathscr{C}$ there exists one and only one arrow $A \to T$.*

Remark. Terminal objects are unique up to (unique) isomorphism if they exist.

Example 25. In **Set**, every singleton $\{a\}$ is a terminal object.

Example 26. The categories **Grp**, **Ab** and R-**mod** have terminal objects, namely the trivial group, R-module, respectively.

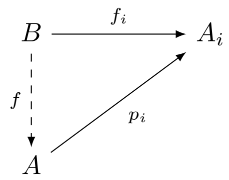
Example 27. The trivial ring (0) is terminal in **Ring**.

Example 28. However, **Field** does not have terminal objects.

Example 29. Let X be a topological space. Recall the category $\mathscr{X}$ which we discussed in Example 13. X is terminal in this category.

4.6 Products

Definition 14. *Let $\mathscr{C}$ be a category, and let $\{A_i\}_{i\in I}$ be a family of objects of $\mathscr{C}$. A* ***product*** *of the A_i consists of an object A of $\mathscr{C}$ and a family of morphisms $p_i : A \to A_i$ such that the following property holds: whenever we have a collection of morphisms $f_i : B \to A_i$, B an object of $\mathscr{C}$, there exists a unique morphism $f : B \to A$ such that the following diagram commutes for all $i \in I$,*



Remark. Using the notation from the definition, if a product of the A_i exists, then it is necessarily unique (up to isomorphism). So it makes sense to say things such as "the product" of the A_i, etc. Note, the product of the A_i is usually denoted $\prod_{i\in I} A_i$. The maps p_i of the product $\prod_{i\in I} A_i$ are sometimes called the ***projections***.

Example 30. In **Set**, the categorical product exists for any family of sets $\{A_i\}_{i\in I}$. The cartesian product $\prod_{i\in I} A_i$ along with the usual projections $\pi_j : \prod_{i\in I} A_i \to A_j$ do the trick.

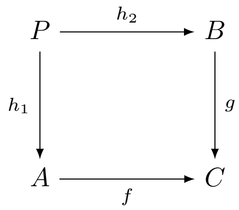
Example 31. Let R be an arbitrary ring, and consider the category R-**mod**. Then the product exists for any collection of R-modules $\{M_j\}_{j\in J}$.

For consider the direct product of the M_j, $\prod_{j\in J} M_j$. Recall that this is just the cartesian product of the M_j as a set; it is a left R-module when we consider the operations pointwise. What's more, the direct product satisfies the universal property of a categorical product as stated in definition 2.6. The projections p_i in this example are the standard projections $\pi_i : \prod_J M_j \to M_i$. Note this holds for the category of right R-modules **mod**-R as well.

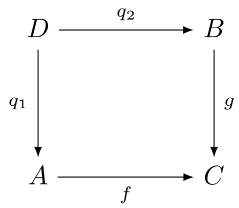
Example 32. Consider **Top**. For any family of topological spaces $\{X_i\}_{i\in I}$, the cartesian product of the X_i, denoted $\prod_{i\in I} X_i$, is also a topological space when it is given the product topology. With this topology and the usual projections $\pi_j : \prod_{i\in I} X_i \to X_j$, $\prod_{i\in I} X_i$ becomes the categorical product of the X_i in **Top**.

4.7 Pullbacks

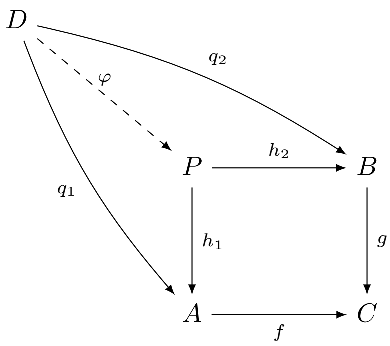
Definition 15. *Let $\mathscr{C}$ be a category. Let A, B, C be objects of $\mathscr{C}$, and let $f \in hom_{\mathscr{C}}(A, C)$ and $g \in hom_{\mathscr{C}}(B, C)$ be morphisms. A* ***pullback*** *of f, g is a pair of morphisms $h_1 : P \to A$ and $h_2 : P \to B$ such that the following (pullback) diagram commutes:*



Furthermore, if $q_1 : D \to A$ and $q_2 : D \to B$ are morphisms also allowing the square to commute,



then there exists a unique morphism $\varphi : D \to P$ such that



commutes.

Remark. With the notation of definition 2.7, a pullback of f, g is defined to be a pair of morphisms h_1, h_2. Sometimes the triple (P, h_1, h_2) is also known as a pullback; in fact P is also sometimes called a pullback.

Just as we mentioned earlier for products, pullbacks are unique if they exist; if (P', h_1', h_2') is another pullback of f, g, then there exists an isomorphism $\varphi : P \to P'$ such that $h_1'\varphi = h_1$ and $h_2'\varphi = h_2$. Because of this, we do not hesitate to say things such as *the* pullback, nor do we hesitate to denote the pullback of f, g definitively by $A \times_C B$.

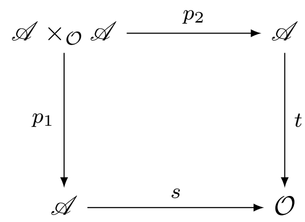
Example 33. Consider the category of groups **Grp**. Let G, G', H be groups, and let $f_1 : G \to H$ and $f_2 : G' \to H$ be homomorphisms. Then the pullback of f_1, f_2 exists.

Indeed, consider the direct product $G \times G'$. Let $P = \{(g, g') \in G \times G' \mid f_1(g) = f_2(g')\}$. Then P is a subgroup of $G \times G'$. Now, let $P \xrightarrow{p_1} G$ be the restriction of the canonical projection $G \times G' \to G$ to P, and define $P \xrightarrow{p_2} G'$ similarly. A moment's thought will reveal that (P, p_1, p_2) is a pullback of f_1, f_2.

4.8 Internal categories

Definition 16. *Let $\mathscr{C}$ be a category with pullbacks. An* ***internal category*** *or* ***category in*** $\mathscr{C}$ *consists of the following data:*

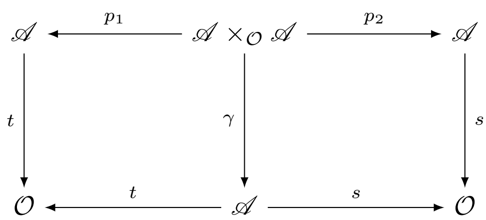
1. $\mathcal{O}, \mathscr{A} \in ob\mathscr{C}$ *respectively called the* ***object of objects*** *and* ***object of arrows (morphisms)****;*
2. $i \in hom_{\mathscr{C}}(\mathcal{O}, \mathscr{A})$ *called* ***identity****;*
3. $s, t \in hom_{\mathscr{C}}(\mathscr{A}, \mathcal{O})$ *called* ***source (domain)*** *and* ***target (codomain)*** *respectively;*
4. $\mathscr{A} \times_{\mathcal{O}} \mathscr{A} \xrightarrow{\gamma} \mathscr{A}$ *called* ***composition****, where γ is defined on the pullback*



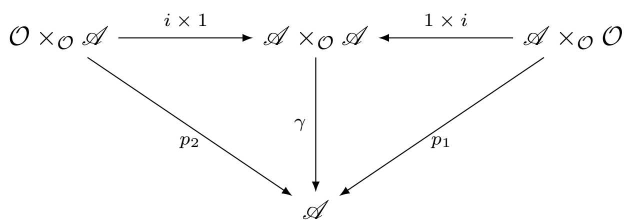
The data are subject to the following axioms. (The following diagrams are required to commute.)

A1. $si = ti = 1 \in hom_{\mathscr{C}}(\mathcal{O}, \mathcal{O})$.

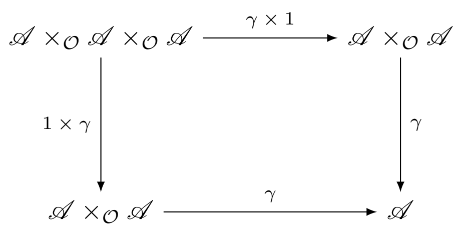
A2.



A3.



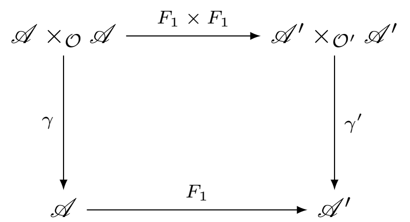
A4.

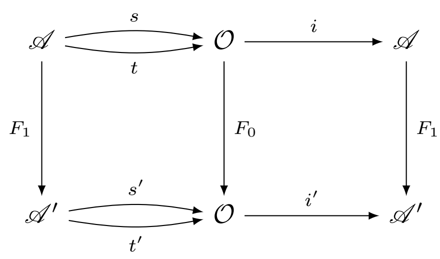


We sometimes denote an internal category in $\mathscr{C}$ by $K = (\mathcal{O}, \mathscr{A}, i, s, t, \gamma)$.

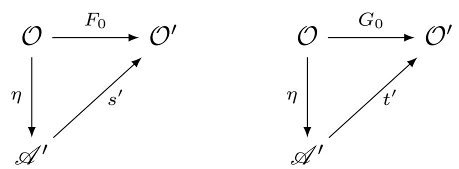
Remark. The basic notion of an internal category only requires the ambient category $\mathscr{C}$ to have pullbacks. One may ask for $\mathscr{C}$ to have more special properties, namely that $\mathscr{C}$ be finitely complete. This means $\mathscr{C}$ has a terminal object in addition to having pullbacks—in which case all finite products exist in $\mathscr{C}$. Familiar categories such as **Grp**, $\mathbf{Vect}_F$, $\mathbf{LieAlgebra}_F$, etc. are finitely complete.

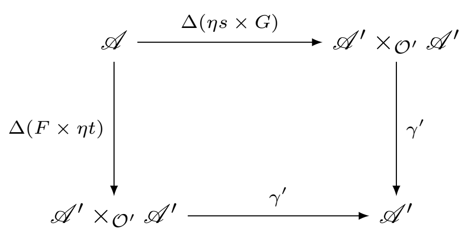
Definition 17. *Let C, K be categories in $\mathscr{C}$. Let C consist of the data $(\mathcal{O}, \mathscr{A}, i, s, t, \gamma)$, and similarly, let $K = (\mathcal{O}', \mathscr{A}', i', s', t', \gamma')$. An **internal functor** F from C to K is a pair of morphisms $\mathcal{O} \xrightarrow{F_0} \mathcal{O}'$ and $\mathscr{A} \xrightarrow{F_1} \mathscr{A}'$ which allow the following diagrams commute:*





Definition 18. *Let C, K be categories in $\mathscr{C}$, and let F, G be internal functors from C to K. An **internal natural transformation** η from F to G is a morphism $\mathcal{O} \xrightarrow{\eta} \mathscr{A}'$ for which the following diagrams commute:*





4.9 2-vector spaces

Definition 19. *Let* $\mathbf{Vect}_k$ *denote the category of vector spaces over* k *and the linear maps between them. A* ***2-vector space*** *is defined to be an internal category in* $\mathbf{Vect}_k$*.*

Let V be a 2-vector space. This means V consists of six pieces of data $V = (V_0, V_1, i, s, t, \gamma)$ subject to a few axioms (cf. Definition 16). V_0 and V_1 are vector spaces over k, while i, s, t, γ are k-linear maps. In greater detail, we have that $V_0 \xrightarrow{i} V_1$, $V_1 \underset{t}{\overset{s}{\rightrightarrows}} V_0$ and $V_1 \times_{V_0} V_1 \xrightarrow{\gamma} V_1$.

The 2-vector space V can be interpreted as a category living inside $\mathbf{Vect}_k$. This doesn't mean V is a subcategory of $\mathbf{Vect}_k$; the notion of subcategory is different and unrelated to the subject at hand. Instead, V is actually a category where the objects of V are the elements x in V_0, $x \in V_0$, and the hom-sets between objects $x, y \in V_0$ are precisely the collection of those $f \in V_1$ such that $s(f) = x$ and $t(f) = y$. In other words, we think of such an element $f \in V_1$, with $s(f) = x$ and $t(f) = y$, as a morphism from x to y in our category V.

We think of the element $i(x) \in V_1$ as the "identity morphism of the object x"—that is, $i(x) = 1_x$.

Let $f, g \in V_1$ such that

$$s(f) = x,$$
$$t(f) = y = s(g),$$
$$t(g) = z;$$

in other words, f is an arrow from the object x to y and g is an arrow from y to z. Then the composition of the arrows g, f is $h = \gamma(g, f) \in V_1$. So h is an arrow from x to z—that is, $s(h) = x$ and $t(h) = z$.

This is all just an explicit explanation of what it means to be a 2-vector space, ie. an internal category in $\mathbf{Vect}_k$.

Baez and Crans mention some "categorified linear algebra" in their paper. Here is some of that material; the proofs to the propositions follow immediately from the hypotheses.

Proposition 2. *Let* $V = (V_0, V_1, i, s, t, \gamma)$ *and* $V' = (V'_0, V'_1, i', s', t', \gamma')$ *be 2-vector spaces. Then there exists a 2-vector space* $V \oplus V'$ *which has*

1. $V_0 \oplus V'_0$ *as its vector space of objects;*
2. $V_1 \oplus V'_1$ *as its vector space of arrows;*
3. $i \oplus i'$ *as its identity map;*
4. $s \oplus s'$, $t \oplus t'$ *as its source, target maps, respectively;*
5. $\gamma \oplus \gamma'$ *as its composition.*

Proposition 3. *Let* $V = (V_0, V_1, i, s, t, \gamma)$ *and* $V' = (V'_0, V'_1, i', s', t', \gamma')$ *be 2-vector spaces. Then there exists a 2-vector space* $V \otimes V'$ *which has*

1. $V_0 \otimes V'_0$ *as its vector space of objects;*
2. $V_1 \otimes V'_1$ *as its vector space of arrows;*
3. $i \otimes i'$ *as its identity map;*
4. $s \otimes s'$, $t \oplus t'$ *as its source, target maps, respectively;*
5. $\gamma \otimes \gamma'$ *as its composition.*

We remark the usual universal properties of $\oplus$ and $\otimes$ for vector spaces hold for 2-vector spaces as well; those properties are defined in the obvious ways.

Just as categories have functors between them which preserve their category structures, there are also notions of "functors" between 2-vector spaces which preserve the 2-vector space structures.

Definition 20. *Given 2-vector spaces V and W in* $\textbf{Vect}_k$*, a* ***linear functor*** $F : V \to W$ *is an internal functor in* $\textbf{Vect}_k$ *from V to W.*

The following definitions are basic for the material on Lie 2-algebras.

Definition 21. *Let V, V' and W be 2-vector spaces. A* ***bilinear functor*** $F : V \times V' \to W$ *is a functor such that the underlying map on objects*

$$F_0 : V_0 \times V_0' \to W_0$$

and the underlying map on morphisms

$$F_1 : V_1 \times V_1' \to W_1$$

are bilinear.

Definition 22. *Let V, W be 2-vector spaces. A bilinear functor $F : V \times V \to W$ is* ***skew-symmetric*** *if $F(x, y) = -F(y, x)$ whenever (x, y) is an object or morphism of $V \times V$.*

Definition 23. *Given two linear functors F, G between 2-vector spaces, say from V to W, a* ***linear natural transformation*** *η from F to G is an internal natural transformation in* $\textbf{Vect}_k$.

Definition 24. *Let V, W be 2-vector spaces. A functor $F : V^n \to W$ is* ***n-linear*** *if $F(x_1, \ldots, x_n)$ is linear in each argument, where $(x_1, \ldots, x_n)$ is an object or morphism of V^n.*

Definition 25. *Let V, W be 2-vector spaces. Given n-linear functors $F, G : V^n \to W$, a natural transformation τ from F to G is* ***n-linear*** *if $\tau_{x_1,\ldots,x_n}$ depends linearly on each object x_i.*

Definition 26. *Let V be a 2-vector space, say $V = (V_0, V_1, i, s, t, \gamma)$. Let $f \in V_1$. The* ***arrow part of*** *f is defined to be $\overline{f} \stackrel{def}{=} f - is(f)$. Notice $\overline{f} \in V_1$.*

Definition 27. *Let V, W be 2-vector spaces; let $F, G : V^n \to W$ be n-linear functors; and let τ be an n-linear natural transformation from F to G. Then τ is* ***completely antisymmetric*** *if the arrow part of $\tau_{x_1,\ldots,x_n}$ is completely antisymmetric under permutations of the objects.*

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