The Chevalley-Eilenberg complex and abelian extensions

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Definition (Extension). *Let $\mathfrak{g}$ and $\mathfrak{a}$ be Lie algebras. An* extension of $\mathfrak{g}$ by $\mathfrak{a}$ *is a short exact sequence*

$$0 \to \mathfrak{a} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of Lie algebras.
It is called abelian, *if $\mathfrak{a}$ is abelian, i.e. if $[\cdot,\cdot]_{\mathfrak{a}} \equiv 0$.*
Two extensions of $\mathfrak{g}$ by $\mathfrak{a}$ are eqivalent, *if there exists a Lie algebra homomorphism ϕ (which by the five lemma also is an isomorphism), such that the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathfrak{a} \arrow[r] \arrow[d, equal] & \tilde{\mathfrak{g}_1} \arrow[r] \arrow[d, "\phi"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & \mathfrak{a} \arrow[r] & \tilde{\mathfrak{g}_2} \arrow[r] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Remark.
- *An extension $0 \to \mathfrak{a} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ is often denoted only by $\mathfrak{a} \to \tilde{\mathfrak{g}} \to \mathfrak{g}$ or even $\tilde{\mathfrak{g}} \to \mathfrak{g}$.*
- *$Ext(\mathfrak{g}, \mathfrak{a})$ denotes the set of eqivalence classes of extensions of $\mathfrak{g}$ by a vector space $\mathfrak{a}$.*

Lemma. *If $\tilde{\mathfrak{g}} \xrightarrow{q} \mathfrak{g}$ is an abelian extension of $\mathfrak{g}$ by $\mathfrak{a}$, there exists a $\mathfrak{g}$-module structure on $\mathfrak{a}$.*

Proof. Let $\sigma : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be a linear map with $\sigma \circ q = id_{\mathfrak{g}}$. It exists, as one can choose a basis on $\mathfrak{g}$ and lift each basis vector to a vector in $\tilde{\mathfrak{g}}$. Set $\rho : \mathfrak{g} \times \mathfrak{a} \to \mathfrak{a}$ as $\rho(x).a = [\sigma(x), a]_{\tilde{\mathfrak{g}}}$ $\forall x \in \mathfrak{g}, a \in \mathfrak{a}$, where $\mathfrak{a}$ is not distinguished from its image in $\tilde{\mathfrak{g}}$. It remains to show:

- ρ is independent of the choice of σ:
  Let $\tilde{x}, \tilde{y} \in \tilde{\mathfrak{g}}$ with $q(\tilde{x}) = q(\tilde{y})$. $\implies \tilde{x} - \tilde{y} \in ker\, q \cong \mathfrak{a} \implies [\tilde{x}, a]_{\tilde{\mathfrak{g}}} - [\tilde{y}, a]_{\tilde{\mathfrak{g}}} = [\tilde{x} - \tilde{y}, a]_{\tilde{\mathfrak{g}}} = 0$, as $\mathfrak{a}$ is abelian.
- $[\tilde{x}, a] \in ker\, q \cong \mathfrak{a}$:
  This holds, as $ker\, q$ is an ideal in $\tilde{\mathfrak{g}}$.

$\square$

Corollary. *Equivalent abelian extensions of $\mathfrak{g}$ by $\mathfrak{a}$, $\tilde{\mathfrak{g}} \to \mathfrak{g}$ lead to the same module structure on $\mathfrak{a}$.*

Proof. The following diagram commutes:

$$\begin{array}{ccccc} \mathfrak{a} & \longrightarrow & \tilde{\mathfrak{g}_1} & \xrightarrow{q} & \mathfrak{g} \\ \| & & \downarrow \phi & & \| \\ \mathfrak{a} & \longrightarrow & \tilde{\mathfrak{g}_2} & \xrightarrow{q'} & \mathfrak{g} \end{array}$$

Choose linear sections σ and σ' of q and q'. Since ϕ is the identity on $\mathfrak{a}$, it suffices to show:
$\phi\left(\left[\sigma\left(x\right), a\right]\right) = \left[\sigma'\left(x\right), a\right]\ \forall a \in \mathfrak{a},\ x \in \mathfrak{g}$ as elements of $\mathfrak{a}$.
This holds, because $\phi\left(\left[\sigma\left(x\right), a\right]\right) = \left[\phi\left(\sigma\left(x\right)\right), \phi\left(a\right)\right] = \left[\phi\left(\sigma\left(x\right)\right), a\right]$ as the diagram commutes. As shown above, the module structure does not depend on the choice of the section in a given extension, and $\phi \circ \sigma$ defines another section of $\tilde{\mathfrak{g}_2} \to \mathfrak{g}$. □

Definition (Chevalley-Eilenberg complex)**.** *Let $\mathfrak{g}$ be an Lie algebra, $U\mathfrak{g}$ its universal enveloping algebra and $\Lambda^n\mathfrak{g}$ the n-fold exterior product of $\mathfrak{g}$. Set $B_i = U\mathfrak{g} \otimes \Lambda^i\mathfrak{g}\ \forall i = 0, 1, 2\ldots$ with the $\mathfrak{g}$-module structure induced by action on the first factor, and define $d : B_n \to B_{n-1}$ by $d\left(u \otimes g_1 \wedge \ldots \wedge g_n\right) = \sum_{i=1}^{n}(-1)^{i+1}ug_i \otimes g_1 \wedge \ldots \wedge \hat{g_i} \wedge \ldots \wedge g_n + \sum_{i<j}(-1)^{i+j}u \otimes [g_i, g_j] \wedge g_1 \wedge \ldots \wedge \hat{g_i} \wedge \ldots \wedge \hat{g_j} \wedge \ldots \wedge g_n$. This leads (as $d^2 = 0$), to a complex, the* Chevalley-Eilenberg complex*:*

$$\ldots \xrightarrow{d} B_n \xrightarrow{d} B_{n-1} \xrightarrow{d} \ldots \xrightarrow{d} B_0 \xrightarrow{d} k$$

Remark. *The Chevalley-Eilenberg complex is a projective resolution of k.*

Remark (Derived functor)**.** *As in a previous talk, one can apply the left exact functor $Hom_{\mathfrak{g}}\left(-, M\right) \cong Hom_{U\mathfrak{g}}\left(-, M\right)$, for a $\mathfrak{g}$-module M to the Chevalley-Eilenberg complex. This leads to a complex of $U\mathfrak{g}$-linear maps on the B_i. As $f\left(u \otimes g_1 \wedge \ldots \wedge g_i\right) = u.f\left(1 \otimes g_1 \wedge \ldots \wedge g_i\right)\ \forall f \in Hom_{u\mathfrak{g}}\left(B_i, M\right)$, these are uniquely determined by their values on $1 \otimes \Lambda^i\mathfrak{g}$ and hence $Hom_{U\mathfrak{g}}\left(B_i, M\right) \cong Hom_k\left(\Lambda^i\mathfrak{g}, M\right)$.*

This construction leads to the following definition:

Definition (Standard complex)**.** *Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. Define $C^k\left(\mathfrak{g}, V\right)$ as the set of k-linear, alternating maps from $\mathfrak{g}^k$ to V and $C^0\left(\mathfrak{g}, V\right) = V$. Set $d : C^k\left(\mathfrak{g}, V\right) \to C^{k+1}\left(\mathfrak{g}, V\right)$ as $d\omega\left(g_1, \ldots, g_{k+1}\right) = \sum_{j=1}^{k+1}(-1)^j g_j.\omega\left(g_1, \ldots, \hat{g_j}, \ldots, g_{k+1}\right) + \sum_{i<j}(-1)^{i+j}\omega\left(\left[g_i, g_j\right], g_1, \ldots, \hat{g_i}, \ldots, \hat{g_j}, \ldots, g_{k+1}\right)$ $\forall \omega \in C^k\left(\mathfrak{g}, V\right), g_i \in \mathfrak{g}$. As $d^2 = 0$, this defines a complex, the* standard complex*.*

Example. *Let $\omega \in Z^2\left(\mathfrak{g}, V\right)$. This is equivalent to $d\omega\left(x, y, z\right) = x.\omega\left(y, z\right) - y.\omega\left(x, z\right) + z.\omega\left(x, y\right) - \omega\left(\left[x, y\right], z\right) + \omega\left(\left[x, z\right], y\right) - \omega\left(\left[y, z\right], x\right) = 0$.*

Proposition. *Let $\omega \in C^2(\mathfrak{g}, V)$. Then ω leads to an abelian extension $\mathfrak{g}_\omega = V \oplus \mathfrak{g}$ with the Lie bracket $[(v,g),(v',g')] = (g.v' - g'.v + \omega(g,g'), [g,g'])\ \forall v, v' \in V,\ g, g' \in \mathfrak{g}$ of $\mathfrak{g}$ by V if and only if $\omega \in Z^2(\mathfrak{g}, V)$.*

Proof. The Lie bracket is skew symmetric by definition, but the Jacobi identity should hold, too. Writing down the first term of the Jacobi identity one gets, with $v, v', v'' \in V$ and $g, g', g'' \in \mathfrak{g}$:

$$\begin{aligned}&\left[\left[(v,g),(v',g')\right],(v'',g'')\right] = \left[\left(g.v' - g'.v + \omega(g,g'),[g,g']\right),(v'',g'')\right]\\ &= \left(\left[g,g'\right].v'' - g''.\left(g.v' - g'.v + \omega(g,g')\right) + \omega\left(\left[g,g'\right],g''\right), \left[\left[g,g'\right],g''\right]\right)\end{aligned}$$

If one takes the sum over the permutations, as in the Jacobi identity, the second component vanishes, as the Jacobi identity holds for the Lie bracket in $\mathfrak{g}$. If one focuses at the first component, one gets:

$$\begin{aligned}&\left(\left[g,g'\right].v'' - g''.\left(g.v' - g'.v + \omega(g,g')\right) + \omega\left(\left[g,g'\right],g''\right)\right)\\ &+ \left(\left[g'',g\right].v' - g'.\left(g''.v - g.v'' + \omega(g'',g)\right) + \omega\left(\left[g'',g\right],g'\right)\right)\\ &+ \left(\left[g',g''\right].v - g.\left(g'.v'' - g''.v' + \omega(g',g'')\right) + \omega\left(\left[g',g''\right],g\right)\right)\\ &= \left[g,g'\right].v'' - g.g'.v'' + g'.g.v'' + \left[g'',g\right].v' - g''.g.v' + g.g''.v'\\ &+ \left[g',g''\right].v - g'.g''.v + g''.g'.v\\ &+ g.\omega\left(g',g''\right) - g'.\omega\left(g,g''\right) + g''.\omega\left(g,g'\right)\\ &- \omega\left(\left[g,g'\right],g''\right) + \omega\left(\left[g,g''\right],g'\right) - \omega\left(\left[g',g''\right],g\right)\end{aligned}$$

The first two rows of terms vanish by the definition of a Lie algebra action. The third and fourth row of terms are exactly the equation one gets as the derivative $d\omega$ of $\omega \in C^2(\mathfrak{g}, V)$. The Jacobi identity holds if and only if it vanishes. This eqivalent to ω being a cocycle. Then $\mathfrak{g}_\omega \to \mathfrak{g}$ defines an extension of $\mathfrak{g}$ by V. □

Theorem. *Let $\mathfrak{g}$ be a Lie algebra and V an abelian Lie algebra. Then:*

$$Ext(\mathfrak{g}, V) \cong H^2(\mathfrak{g}, V) = Z^2(\mathfrak{g}, V)/d\left(C^1(\mathfrak{g}, V)\right)$$

Proof. „$\supseteq$“ Let $\omega \in Z^2(\mathfrak{g}, V)$. As shown above, this leads to an extension of $\mathfrak{g}$ by V. It remains to show, that equivalent cocycles ω, ω' lead to equivalent extensions. Therefore one wants to define a Lie algebra homomorphism $\phi : \mathfrak{g}_\omega \to \mathfrak{g}_{\omega'}$, which has to be of the form $\phi : (v,g) \mapsto \left(v + \tilde{\phi}(g), g\right)$, as it has to be the identity on the second component, and you get the first component by linearity and Lie brackets. One computes by an similar calculation as above, that it commutes with the Lie brackets if and only if $\omega = \omega' + d\tilde{\phi}$, that means, if ω and ω' are in the same equivalence class. In this part of the proof, it is only necessary to know, that equivalent cocycles are mapped to the same equivalence class of extensions, but later also the other direction is needed, i.e. if $\mathfrak{g}_\omega$ and $\mathfrak{g}_{\omega'}$ are eqivalent, then ω and ω' are so as well via the equation between them above.

„⊆“ Let $V \xrightarrow{\mathfrak{i}} \tilde{\mathfrak{g}} \xrightarrow{q} \mathfrak{g}$ be an abelian extension. One wants to show, that it is equivalent to another abelian extension $\mathfrak{g}_\omega \to \mathfrak{g}$, where $\omega \in Z^2(\mathfrak{g}, V)$. Choose a section $\sigma : \mathfrak{g} \to \tilde{\mathfrak{g}}$, such that $q \circ \sigma = id_{\mathfrak{g}}$, as above.

Let $\omega(g,h) = [\sigma(g), \sigma(h)] - \sigma([g,h]) \ \forall g,h \in \mathfrak{g}$. To make sure it is a cocycle, one has to check:

- $im(\omega) \subset V \cong \mathfrak{i}(V)$, i.e. $q(\omega(g,h)) = 0$:
  q is a Lie algebra homomorphism, and therefore $q(\omega(g,h)) = [q(\sigma(g)), q(\sigma(h))] - q(\sigma([g,h])) = 0$
- $\omega \in Z^2(\mathfrak{g}, V)$, i.e. $d\omega = 0$ (via computation, using skew symmetry and Jacobi identity for the Lie brackets, and the definition of the $\mathfrak{g}$-action on V)

Now we want to show, that $\mathfrak{g}_\omega \to \mathfrak{g}$ is eqivalent to $\tilde{\mathfrak{g}} \to \mathfrak{g}$:
Set $\phi : \mathfrak{g}_\omega \cong V \times \mathfrak{g} \to \tilde{\mathfrak{g}}, \ (v,g) \mapsto \mathfrak{i}(v) + \sigma(g)$. ϕ is a bijective module-homomorphism. ϕ is as well a Lie algebra homomorphism from $\mathfrak{g}_\omega$ to $\tilde{\mathfrak{g}}$ as $\phi([(v,g),(v',g')]) = \mathfrak{i}(g.v') - \mathfrak{i}(g'.v) + \mathfrak{i}([\sigma g, \sigma g']) - \mathfrak{i}(\sigma[g,g']) + \sigma[g,g'] = [\mathfrak{i}(v), \sigma g'] + [\sigma g, \mathfrak{i}(v')] + [\sigma g, \sigma g']$ (use again the definition of the $\mathfrak{g}$-action on V)
ω only depends on the eqivalence class of the given extension: If two eqivalent extensions lead to two different extensions $\mathfrak{g}_\omega$ and $\mathfrak{g}_{\omega'}$, these are equivalent, too. Due to the remark in the first part of the proof, then ω and ω' are equivalent.

□

Example (Heisenberg algebra)**.** *Let* $\tilde{H} = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \middle| a,b \in \mathbb{R} \right\} \cong \mathbb{R}^2$ *and* $A = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| c \in \mathbb{R} \right\} \cong \mathbb{R}$. *To determine* $H^2\left(\tilde{H}, A\right)$ *one has to look for bilinear, antisymmetric maps from* $\mathbb{R}^2$ *to* $\mathbb{R}$. *From linear algebra or differential geometry it is known, that the determinant* ω *is the only such map, up to scalar multiplication. Therefore the only eqivalence class of extensions (up to scalar multiplication) of* $\tilde{H}$ *by* A *can be constructed via* ω, *where* $\omega\left(\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & ad-cb \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$:

$$A \to H \to \tilde{H}, \text{ where } H = A \times \tilde{H} = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \middle| a,b,c \in \mathbb{R}^3 \right\}$$

H is exactly the Heisenberg algebra*, with induced Lie bracket ω (all other components vanish). One also computes that the action of $\tilde{H}$ on A in this case is trivial.*