Fourier Transform and one of its applications

The main goal of this script is to present a relatively self-contained introduction of the Fourier Transform and as well one of its applications. This material is intented to be covered in a talk of about 90 minutes. Since this topic is indeed dense I should apologise for the omisions I should make during the talk in order to make it suitable for the time I have. Nevertheless, The motivated reader can consult about this topic either way here or in one the bibliographical sources.

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1 Notation and first formulas

From now on we are going to consider always complex valued functions with n variables.

Definition 1. *A n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of non negative integeres is said to be a multiindex. The set of all such a tuples is going to be denoted by $\mathbb{N}_0^n$.*
For $\alpha, \beta \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$ we define

- *(length)* $|\alpha| := |\alpha|_1$
- *(Factorial)* $\alpha! := \alpha_1! ... \alpha_n!$
- *(Weak-order)* $\alpha \leq \beta \quad :\Leftrightarrow \quad \forall j: \quad \alpha_j \leq \beta_j$
- *(Monomial)* $x^\alpha := x_1^{\alpha_1} ... x_n^{\alpha_n}$
- *(Derivatives)* $\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$ *and* $D^\alpha := (-i)^{|\alpha|} \partial^\alpha$
- *(Binomial coeficient)* $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \prod_j \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}$ *where* $\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} := \frac{\alpha_j!}{\beta_j!(\alpha_j - \beta_j)!}$

Remark 1. *For $\alpha, \beta \in \mathbb{N}_0^n$, it holds that*

$$\begin{pmatrix} \alpha \\ \beta - e_j \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + e_j \\ \beta \end{pmatrix}$$

It basically follows from the well known identity for natural numbers .

$$\begin{pmatrix} a \\ b - 1 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + 1 \\ b \end{pmatrix} \quad a, b \in \mathbb{N}$$

Since stimations with multiindexes are common and indeed tedious, in this script just fundamental staff is going to be verified in order to keep the patient of the reader.

Theorem 1. *For $\alpha \in \mathbb{N}_0^n$, one has*

1. *(Binomial formula)*

$$x, y \in \mathbb{R}^n \quad : \quad (x + y)^\alpha = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} x^\beta y^{\alpha - \beta}$$

2. *(Leibniz formula) For* $u, v \in C^{|\alpha|}$ *one has*

$$\partial^\alpha(uv) = \sum_{\beta\leq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \partial^\beta u \partial^{\alpha-\beta} v$$

Proof. It is enough to prove (1) by induction on $|\alpha|$.

- The case $|\alpha| = 0$ is indeed trivial.
- Suppose the proposition holds for α multiindex. By calculation it is straightforward to notice that

$$(x+y)^{\alpha+e_j} = (x+y)^\alpha \; . \; (x+y)^{e_j} = \left[\sum_{\beta\leq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} x^\beta y^{\alpha-\beta}\right] .(x^{e_j} + y^{e_j})$$

$$= \underbrace{\sum_{\beta\leq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} x^{\beta+e_j} y^{\alpha-\beta}}_{:=\mathcal{B}} + \sum_{\beta\leq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} x^\beta y^{\alpha-\beta+e_j}$$

making the change of multiindexes $\theta := \beta + e_j$ in $\mathcal{B}$, one gets

$$\mathcal{B} = \sum_{\theta\leq\alpha+e_j} \begin{pmatrix} \alpha \\ \theta - e_j \end{pmatrix} x^\theta y^{\alpha-\theta+e_j}$$

therefore, using remark 1

$$(x+y)^{\alpha+e_j} = \sum_{\theta\leq\alpha+e_j} \left[\begin{pmatrix} \alpha \\ \theta \end{pmatrix} + \begin{pmatrix} \alpha \\ \theta - e_j \end{pmatrix}\right] x^\theta y^{\alpha-\theta+e_j}$$

$$= \sum_{\theta\leq\alpha+e_j} \begin{pmatrix} \alpha + e_j \\ \theta \end{pmatrix} x^\theta y^{\alpha-\theta+e_j} \quad \blacksquare$$

The following integral appears often in calculations, so that we need a sufficient condition to assure its convergence.

Lemma 1. *Define*

$$< x > := (1 + |x|^2)^{\frac{1}{2}} \qquad x \in \mathbb{R}^n$$

Let $1 \leq p < \infty$. *Then* $s > n/p$ *implies*

$$\int_{\mathbb{R}^n} < x >^{-sp} \, dx < \infty$$

Proof. Let $s > \frac{n}{p}$. We recall that

1.
$$\underbrace{\sqrt[n]{\prod_i (1 + x_i^2)}}_{Geometric \quad average} \leq \underbrace{\sum_j \frac{1 + x_i^2}{n}}_{(Arithmetic) \quad average} \leq 1 + |x|^2$$

2.
$$\int_{\mathbb{R}} (1 + x^2)^{-1} dx = \pi$$

it follows by simple calculation

$$(1+|x|)^{-sp} = (1+2|x|+|x|^2)^{-\frac{ps}{2}} \leq_{(1)} \left[\prod_i \left(1+x_i^2\right)\right]^{-\frac{sp}{2n}} \leq_{Hip} \left[\prod_i \left(1+x_i^2\right)\right]^{-1}$$

Now integrate over the space and conclude usying (2). ∎

Remark 2. *The following relation is sometimes useful when dealing with monomials. You may see [1] A.9. for its proof*

$$\sum_{|\alpha|\leq m} x^{2\alpha} \leq < x >^{2m} = \sum_{|\alpha|\leq m} C_{m,\alpha} x^{2\alpha} \leq C_m \sum_{|\alpha|\leq m} x^{2\alpha}$$

For some positive constants $C_{m,\alpha}, C_m$.'

2 The Schwartz space

One space in which one is interested when doing Fourier transform is the well known Schwartz space $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ - or the space of rapidly decreasing functions -. This space consists in all the smooth mappings $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{C})$ such that

$$|\varphi|_k := sup\{|x^\alpha \partial^\beta \varphi| \quad : \quad x \in \mathbb{R}^n \quad |\alpha+\beta| \leq k \ \} < \infty \qquad k \in \mathbb{N}_0$$

Remark 3. 1. *We recall that each $|.|_k$ for $k \in \mathbb{N}_0$ defines a semminorm and also that.*

$$\varphi \in \mathcal{S} \quad \Leftrightarrow \quad p_k(\varphi) := sup\{| < x >^k \partial^\alpha \varphi(x)| \quad : \quad x \in \mathbb{R}^n \ |\alpha| \leq k\} < \infty$$

moreover, the family $(p_k)_k$ in view of remark 2 induces the same topology as $(|.|_k)_k$

2. *Consider a linear mapping $f : \mathcal{S} \longrightarrow \mathcal{S}$. Then*

$$f \ continuous \quad \Leftrightarrow \quad \forall l \in \mathbb{N}_0 \ \exists m \in \mathbb{N}_0 \ \exists C > 0 : \ |f(x)|_l \leq C|x|_m \quad x \in \mathcal{S}$$

for the proof of this statement we should work with some topology. You may have a glance at the last part of this script

3. *Recalling that $|x^\alpha| \leq |x|^{|\alpha|}$, One notices that*

$$|x^\alpha \partial^\beta \varphi(x)| \leq |x|^{|\alpha|}|\partial^\beta \varphi(x)| \leq \frac{(|x|^{|\alpha|+1} + |x|^{|\alpha|})}{1+|x|}|\partial^\beta \varphi(x)| \leq \frac{< x >^l + < x >^m}{1+|x|}|\partial^\beta \varphi(x)|$$

$$\leq \frac{C_{l,m}}{1+|x|} \to 0 \quad as \quad |x| \to \infty$$

which leads to

$$\varphi \in \mathcal{S} \quad \Leftrightarrow \quad lim_{|x|\to\infty} x^\alpha \partial^\beta \varphi(x) = 0 \quad \forall \alpha, \beta \in \mathbb{N}_0^n$$

There is one subspace of $\mathcal{S}$ which is particullary important. This space consists in all the smooth functions with compact support. However it is not obvious that such a space is non-trivial.

Example 1. *Set $\varphi(x) := f(|x|^2 - 1)$, where*

$$f(t) := \begin{cases} e^{1/t} & if\ t < 0; \\ 0 & if\ t \geq 0. \end{cases}$$

In analysis courses one shows that $\varphi \in C^\infty$ and $supp(\varphi) = \overline{B_1(0)}$. Which means that this function belongs to the mentioned space.

Proposition 1. *(Immersion property) One has that $\mathcal{S} \hookrightarrow L^p$ for all $1 \leq p \leq \infty$.*

Proof. For each $\varphi \in \mathcal{S}$ we remark that

- $p = \infty \quad : \; ||\varphi||_\infty = |\varphi|_0$
- $1 \leq p < \infty \; :$

$$|\varphi(x)|^p \leq |\varphi(x)|^p \underbrace{\left[\prod_j (1+x_j^2)\right]^{p-1}}_{>1}$$

$$\leq sup \left\{ \underbrace{\left| \varphi(x) \prod_j (1+x_j^2) \right|}_{\leq \, |\varphi(x)| C_n <x>^{2n}} \; : x \in \mathbb{R}^n \right\}^p \left(\prod_j (1+x_j^2) \right)^{-1} \leq C_{n,p} |\varphi|_{2n}^p \left(\prod_j (1+x_j^2) \right)^{-1}$$

Integrating over $\mathbb{R}^n$ and recalling the above identity

$$\int_{\mathbb{R}} (1+x^2)^{-1} = \pi$$

one obtains

$$||\varphi||_p \leq \overline{C}_{n,p} |\varphi|_{2n}$$

Since the functions in $\mathcal{S}$ are smooth - in particular continuous- , one has that the natural inclusion is injective. ■

Remark 4. *The last proposition allows us to treat $\mathcal{S}$ as a subspace of L^p for each p. Moreover, it can be proved that such an immersion is <u>dense</u>. You may see [1] Lemma 5.2(3) .*

3 Fourier Transform

The Fourier transform of a function $u(x) \in L^1$ with new variable ξ is defined as

$$\hat{u}(\xi) := \mathcal{F}_\xi(u) := \int_{\mathbb{R}^n} e^{-i<x,\xi>} u(x) dx \qquad \xi \in \mathbb{R}^n$$

The following theorem states the most basic properties of the Fourier Transform. We recall here that there are important ones which are not discussed in this script - for instance, Plancheler's Theorem-. The interested reader might have a glance at [1].

Theorem 2. *(Properties)*

1. *$\mathcal{F} : L^1(\mathbb{R}^n) \longrightarrow C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a well defined continuous map, such that $||\hat{u}||_\infty \leq ||u||_1$ and $\hat{u}(\xi) \to 0$ as $|\xi| \to \infty$*
2. *$\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$ is a continuous linear map and also*

$$\mathcal{F}\left(x^\alpha D_x^\beta f(x)\right) = -(D_\xi)^\alpha \left(\xi^\beta \hat{f}(\xi)\right) \qquad \alpha, \beta \in \mathbb{N}_0^n$$

3. *Define the Co-Fourier transform as the mapping*

$$\overline{\mathcal{F}}_{\xi}(u) := \int_{\mathbb{R}^n} e^{i<x,\xi>} u(x)dx \qquad \xi \in \mathbb{R}^n$$

By saying $\overline{\mathcal{F}f}$ is meant $\overline{\mathcal{F}}\,\overline{f}$. Then $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$ is a bijection - in fact an Homeomorphism - with inverse $\mathcal{F}^{-1} := (2\pi)^{-1}\overline{\mathcal{F}}$.

Proof.

1. It is clear that $||\hat{f}||_{\infty} \leq ||f||_1$. For the continuity just notice that

$$|e^{-i<x,\xi>} f(x)| \leq |f(x)| \in L^1$$

Using standart resoults concerning integration depending on parameteres, one knows then that $\hat{f}$ is continuous - as function of ξ-. The fact that $\hat{u}(\xi) \to 0$ as $|\xi| \to \infty$ will be proved later.

2. For the identity it is enough to show that

- For $f(x) \in L^1$ such that $x_j f(x) \in L^1$, one has

$$\partial_{\xi_j}\mathcal{F}(f(x)) = \mathcal{F}_{\xi}(-ix_j f(x))$$

In fact, one notices

$$|\partial_{\xi_j}(e^{-i<x,\xi>} f(x))| = |-ix_j e^{-i<x,\xi>} f(x)| = |x_j f(x)| \in L^1$$

therefore, applying the same standart result we already used, one obtains

$$\partial_{\xi_j} \int_{\mathbb{R}^n} e^{-i<x,\xi>} f(x)dx = \int_{\mathbb{R}^n} -ix_j e^{-i<x,\xi>} f(x)dx$$

which proves the point.

- $$\mathcal{F}(\partial_j f(x)) = -i\xi_j \mathcal{F}(f(x))$$

Using integration by parts:

$$\mathcal{F}_{\xi}(\partial_j f(x)) = lim_{R\to\infty} \int_{B_R(0)} e^{-i<x,\xi>} \partial_j f(x)dx$$

$$= lim_{R\to\infty} \left(\int_{B_R(0)} -i\xi_j e^{-i<x,\xi>} f(x)dx + \int_{\partial B_R(0)} e^{-i<x,\xi>} f(x) \frac{x_j}{|x|} dx \right)$$

However

$$\left| \int_{\partial B_R(0)} e^{-i<x,\xi>} f(x) \frac{x_j}{|x|} \right| \leq \int_{\partial B_R(0)} |f(x)| \frac{x_j}{|x|} dx \leq \int_{\partial B_R(0)} |f(x)| dx$$

$$\leq \mu(\partial B_R(0))\ sup\{|f(x)| \ \ : \ x \in \partial B_R(0)\} \to 0 \quad as \quad R \to \infty$$

Since $f \in \mathcal{S}$. Now replace and conclude the identity.

The right side of this identity in view of Leibniz formula shows that $\hat{f} \in \mathcal{S}$ - again after some calculations with multiindexes- . Now let $\varepsilon > 0$ and $f \in L^1$. Recall a couple of facts

(a) There is a $g \in L^1$ such that $||f-g||_1 < \frac{\varepsilon}{2}$. By (1), one obtains

$$|\hat{f}-\hat{g}| \leq ||\hat{f}-\hat{g}||_\infty \leq ||f-g||_1 < \frac{\varepsilon}{2} \qquad \xi \in \mathbb{R}^n$$

(b) Since $\hat{g} \in \mathcal{S}$, there is an $R > 0$ such that $|\xi| \geq R$ implies $|\hat{g}(\xi)| \leq \frac{\varepsilon}{2}$

Combining those, one has

$$|\hat{f}(\xi)| \leq |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)| < \varepsilon \qquad |\xi| \geq R$$

which is the missing fact in (1). For the *continuity* notice that

$$\mathcal{F}((1-\Delta)f) = \left(1 + \sum \xi_j^2\right)\hat{f} =< \xi >^2 \hat{f}$$

for each $k \in \mathbb{N}_0$ put

$$l := \begin{cases} \frac{k}{2} & \text{if } k \text{ even;} \\ \frac{k+1}{2} & \text{if } k \text{ odd.} \end{cases}$$

It can be proved (cf. [1] theorem 5.4) that

$$|\hat{f}|_0 \leq || < x >^{-n-1} ||_1 |f|_{n+1}$$

$$|\hat{f}|_k \leq C_{k,l} |f|_{k+n+1}$$

where $C_{k,l} := sup\{< x >^{n+1} \ (1-\Delta)^l (x^\alpha f(x)) \ : \ x \in \mathbb{R}^n \ |\alpha| \leq k\}$. It implies the continuity of $\mathcal{F}$.

3. The proof of this proposition carries lots of calculation, it is convenient to sketch it

(a) The first fact we shall remark is that

$$\overline{\mathcal{F}\overline{f}} = \overline{\mathcal{F}}(f)$$

Which means that $\overline{\mathcal{F}}$ has the same properties that $\mathcal{F}$ has – in particular it maps $\mathcal{S}$ onto itself continuously –.

(b) We shall try to calculate

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{i<x,\xi>} \left(\int_{\mathbb{R}^n} e^{-i<y,\xi>} f(y) dy \right) d\xi$$

but the function $e^{-i<\xi,x-y>} f(y)$ is NOT intgrable on $\mathbb{R}^{2n}$. It means that one can not just change the integration's order.

(c) To overcome this difficulty we shall introduce a function $\psi(\xi) \in \mathcal{S}$ which will be removed afterwords passing to the limit. In more detail; for $\varepsilon > 0$ insert $\psi(\varepsilon\xi)$ with $\psi \in \mathcal{S}$ and use the change of varibables $(\eta, z) = (\varepsilon\xi, (y-x)/\varepsilon))$. Some calculations lead to

$$\int_{\mathbb{R}^n} e^{i<x,\xi>} \psi(\varepsilon\xi) \hat{f}(\xi) d\xi = \int_{\mathbb{R}^n} \hat{\psi}(z) f(x+\varepsilon z) dz$$

(d) Let $\varepsilon \to 0$ and use the theorem of Lebesgue to show

$$\psi(0) \int_{\mathbb{R}^n} e^{i<x,\xi>} \hat{f}(\xi) d\xi = f(x) \int_{\mathbb{R}^n} \hat{\psi}(z) dz$$

(e) We need the following lemma

Lemma 2. *It holds for $\varphi(x) := e^{-|x|^2/2}$ that $\hat{\varphi}(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}$.*

Proof. First recall that

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-i<x,\xi>} e^{-|x|^2/2} dx = \prod \int_{\mathbb{R}} e^{-ix_j\xi_j} e^{-x_j^2/2} dx_j$$

and so, it is enough to consider the case n=1. Notice φ satisfies the following ODE

$$y' + xy = 0$$

with initial condition $y(0) = 1$. Put $g := (2\pi)^{-1/2}\hat{\varphi}$ and recall that

$$g(0) = (2\pi)^{-1/2}\hat{\varphi}(0) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-x^2/2} dx = 1$$

Using the last property of the Fourier transform, and the fact that φ satisfies the mentioned ODE one obtains

$$0 = \hat{\varphi'} + \widehat{x\varphi} = i\xi\hat{\varphi} + \left(\frac{1}{-i}\hat{\varphi}\right)' \quad \Leftrightarrow \quad \xi g + g' = 0$$

by Picard's theorem, one concludes

$$\varphi = g \quad \Leftrightarrow \quad \hat{\varphi} = (2\pi)^{1/2}\varphi$$

and so, the n- dimensional case reduces simply to

$$\hat{\varphi}(\xi) = \prod \int_{\mathbb{R}} e^{-ix_j\xi_j} e^{-x_j^2/2} dx_j = (2\pi)^{n/2} \prod e^{-\xi_j^2/2} = (2\pi)^{n/2} e^{-|\xi|^2/2} \quad \blacksquare$$

choose $\psi(\xi) := e^{-|\xi|^2/2}$ and apply such a lemma to obtain

$$\hat{\psi}(z) = (2\pi)^{n/2} e^{-|z|^2/2} \qquad \psi(0) = 1 \qquad \int_{\mathbb{R}^n} \hat{\psi}(x) dx = (2\pi)^n$$

one concludes the result from the indentity of the last item $\blacksquare$

Definition 2. *(Convolution)*

1. *For $f \in L^1(\mathbb{R}^n)$ and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. The convolution*

$$(f \star g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

Is defined for all $x \in \mathbb{R}^n$, and satisfies $||f \star g||_\infty \leq ||f||_1 ||g||_\infty$

2. *In the case $f, g \in L^1(\mathbb{R}^n)$, the convolution is defined a.e. and it holds that $||f \star g||_1 \leq ||f||_1 ||g||_1$*

Theorem 3. *(Convolution's properties)*

1. *For $f, g \in L^1(\mathbb{R}^n)$ one has that $\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g)$.*
2. *For $f, g \in \mathcal{S}$. It holds that $f \star g \in \mathcal{S}$*

Proof.

1. *For ξ a.e*

$$\mathcal{F}((f\star g)(x))(\xi)=\mathcal{F}\left(\int_{\mathbb{R}^n} f(x-y)g(y)dy\right)(\xi)=\int_{\mathbb{R}^n} e^{-i<x,\xi>}\left[\int_{\mathbb{R}^n} f(x-y)g(y)dy\right]dx$$
$$=\int_{\mathbb{R}^n} g(y)\left[\int_{\mathbb{R}^n} e^{-i<x,\xi>}f(x-y)dx\right]dy$$
$$=\int_{\mathbb{R}^n} g(y)\left[\int_{\mathbb{R}^n} e^{-i<x+y,\xi>}f(x)dx\right]dy$$
$$=\int_{\mathbb{R}^n} e^{-i<x,\xi>}f(x)dx\int_{\mathbb{R}^n} e^{-i<y,\xi>}g(y)dy=(\mathcal{F}(f).\mathcal{F}(g))(\xi)$$

2. *Notice $\mathcal{F}(\varphi\star\psi)=\mathcal{F}(\varphi)\mathcal{F}(\psi)\in\mathcal{S}$. Now apply inverse Fourier transform and conclude.*

■

It is convenient at least to state the well known result of Plancheler, which allows us to consider Fourier transform defined on L^2 with some aditional considerations (for a complete proof of this statement you may see [2] or [1])

Theorem 4. *(Parseval-Plancheler theorem)*

1. *The Fourier transform $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ extends in a unique way to an isometric isomorphism $\mathcal{F}_2$ of $L^2(\mathbb{R}^n,dx)$ onto $L^2(\mathbb{R}^n,(2\pi)^{-n},dx)$ which satisfies the following identities*

$$\int f(x)\overline{g(x)}dx=(2\pi)^{-n}\int\mathcal{F}_2(f)(\xi)\overline{\mathcal{F}_2(g)(\xi)}d\xi$$
$$\int|f(x)|^2dx=(2\pi)^{-n}\int|\mathcal{F}_2(f)(\xi)|^2d\xi$$

for all $f,g\in L^2(\mathbb{R}^n)$

2. *There is an identification*

$$\mathcal{F}_2(f)=\mathcal{F}(f)\quad for\quad f\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$$

4 Application

As an example of aplication we will solved the $n-$ dimensional heat equation with an special initial condition. The following example is made with the aim of setting ideas before solving the mentioned equation

Example 2. *(a non-offensive guy) For $k\in\mathbb{R}$ fixed consider the following problem*

$$\begin{cases}\frac{dg}{dt}(t)-kg(t)=0 & t>0\\ g(0)=c & c\in\mathbb{R}.\end{cases}$$

as we did in school we calculate the roots of the characteristic polynomial of owr ODE

$$P(\lambda):=\lambda-k=0\quad\Rightarrow\lambda=k$$

therefore the solution is given by

$$g(t)=g(0)e^{-kt}=ce^{-kt}\qquad t>0$$

Now look at $P(\lambda)$ and at Theorem 2 (2). Is it SCREAMING something to you?.

Excercise 1. *(Wave equation) Consider the homogeneous n-dimensional heat equation.*

$$\begin{cases} u_t(x,t) - \Delta_x u(x,t) = 0 & t{>}0; \\ u(x,0) := \varphi(x) & x \in \mathbb{R}^n. \end{cases}$$

for some $\varphi \in \mathcal{S}$.

We are going to show that this problem has one solution in $\mathcal{S}$. By taking Fourier transform one obtains

$$\mathcal{F}(u_t(x,t)) = \int_{\mathbb{R}^n} e^{-i<x,\xi>} \frac{\partial}{\partial t} u(x,t)dx = lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}^n} e^{-i<x,\xi>} \left[u(x,t+h) - u(x,t)\right] dx$$

$$= lim_{h\to 0} \frac{1}{h} \left[\int_{\mathbb{R}^n} e^{-i<x,\xi>} u(x,t+h)dx - \int_{\mathbb{R}^n} e^{-i<x,\xi>} u(x,t)dx \right]$$

$$= lim_{h\to 0} \frac{1}{h} [\hat{u}(\xi,t+h) - \hat{u}(\xi,t)] = \frac{\partial}{\partial t} \hat{u}(\xi,t)$$

on the other hand

$$\mathcal{F}(\Delta_x u) = \sum \xi_j^2 \hat{u}(\xi,t) = |\xi|^2 \hat{u}(\xi,t) \quad .$$

For a fixed $\xi \in \mathbb{R}^n$ and named $g(t) := \hat{u}(\xi,t)$, our Cauchy's problem turns out to be way easier. More in detail, it becomes

$$\begin{cases} \frac{dg}{dt}(t) - |\xi|^2 g(t) = 0 & t{>}0 \\ g_\xi(0) = \hat{\varphi}(\xi) & \xi \in \mathbb{R}^n. \end{cases}$$

from the last example, for a fixed ξ one knows that

$$\hat{u}(\xi,t) = \hat{\varphi}(\xi) e^{-|\xi|^2 t} \quad t > 0$$

by Theorem 3 one obtains

$$u(x,t) = \varphi(x) \star \mathcal{F}^{-1}(e^{-|\xi|^2 t})(x) \quad t > 0 \quad x \in \mathbb{R}^n$$

In our context, the last property is telling us that the solution of our Cauchy's problem is unique -if it lives in $\mathcal{S}$; since the Gaussian function is a rapidly decreasing function- . Similarly as we did when calculating the Fourier transform of the Gaussian function, one can find that

$$\mathcal{F}_x^{-1}(e^{-|\xi|^2 t}) = (2\pi)^{-n/2} t^{-n/2} e^{-|x|^2/4t} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$$

thus, one solution - the unique rapidly decreasing one -of this PDE is given by

$$u(x,t) = (2\pi)^{-n/2} t^{-n/2} e^{-|x|^2/4t} \star \varphi(x) \qquad x \in \mathbb{R}^n \quad t > 0$$

$$u(x,0) = \varphi(x) \quad x \in \mathbb{R}^n$$

5 Some used topology

This section is devoted to summarize the most important topological facts that were used in the talk, for a detailed proof of those statements which are not shown here please check [1] Appendix B.

Definition 3. *A topological vector space over $\mathbb{K}$ is a vector space X provided with a topology τ with respect to which the vector space operations are continuous i.e. The mappings*

$$(x, y) \to x + y \quad (\lambda, x) \to \lambda x \quad (\lambda, x, y) \in \mathbb{K} \times X \times X$$

are continuous

Remark 5. *The topology of every topological vector space can be described by a neighborhood system at 0.*

Definition 4. *1. A locally convex space is a topological vector space which has a local basis at 0 consisting of convex sets.*

2. A topological vector space X is called a Frechet space when X is metrizable with a translation invariant metric d, (X, d) is complete and X is locally convex.

Example 3. *$\mathcal{S}$ provided with the family of semminorms $\mathcal{P} := (|.|_k)_k$ defined in definition 2 - or remark 3 - is metrizable (see excercise sheet 1 (3)) and it is complete with respect to this metric. Notice $\mathcal{P}$ separates points.*

Having a vector space X and a separating family of semminorms, we are interested in endowing X with a topology which makes it become a topological space.

Theorem 5. *Let X be a vector space and $\mathcal{P}$ be a separating family of semminorms. One topology on X can be defined by taking as a local system of neighborhoods at 0 the sets*

$$V(p, \varepsilon) := \{x \in X \ : \ p(x) < \varepsilon\} \qquad p \in \mathcal{P} \quad (x, \varepsilon) \in X \times \mathbb{R}^+$$

together with their finite interesections

$$W(p_1, ..., p_N) := \bigcap_{i \leq N} V(p_i, \varepsilon_i)$$

with this topology X is a topological vector space and each semminorm is a continuous mapping.

Definition 5. *A separating family of semminorms $\mathcal{P}$ is said to satisfy the max-property if and only if*

$$\forall p_1, p_2 \in \mathcal{P} \ \exists p \in \mathcal{P} \ \exists C > 0 \ : \ p \geq C max\{p_1, p_2\}$$

Example 4. *The family of semminorms $(p_k)_k$ defined in remark 3 (1), satisfies the max-property*

Remark 6. *Each countable separating family of semminorms $\mathcal{P}$ can be replaced by a new countable separating family $\mathcal{P}'$ which induces the same topology and satisfies the max-property. In fact one can simply define $\mathcal{P}'$ as the set of semminorms $p \in \mathcal{P}$ with $p = max\{p_1, ..., p_N\}$ for $p_j \in \mathcal{P}$ and $N \in \mathbb{N}$.*

Lemma 3. *Let X, Y be topological vector spaces with topologies given by a separating family of semminorms $\mathcal{P}, \mathcal{Q}$ - respectively- having the max-property, usying the method set in theorem 5. Then for a linear mapping $T : X \to Y$ one has*

$$T \ continuous \quad \Leftrightarrow \quad (\forall q \in \mathcal{Q})(\exists p \in \mathcal{P})(\exists C > 0) \quad : \ |q(T(x))| \leq Cp(x) \quad x \in X$$

Proof. $\Leftarrow$ is clear.
$\Rightarrow$. Since $\mathcal{P}, \mathcal{Q}$ satisfy the max-property and the sets W in theorem 5 constitute a local basis at 0, the notion of continuity reduces to

$$\forall \varepsilon \; \exists \delta > 0 \quad : \quad \forall q \in \mathcal{Q} \quad , \exists p \in \mathcal{P} \quad s.t. \; p(x) < \delta \; \Rightarrow |q(T(x))| < \varepsilon$$

Our claim holds for $c := \frac{\varepsilon}{\delta}$. In fact one has

1. p(x)=0 implies $q(T(x)) = 0$ otherwise $|q(T(tx))| = t|q(T(x))| \to \infty$ for $t \to \infty$ whereas $p(tx) = 0$ for t>0. It contradicts the continuity of T.

2. $p(x) > 0$. Let $0 < \delta' < \delta$. Then

$$p\left(\frac{\delta'}{p(x)}x\right) = \delta' < \delta \quad \Rightarrow \quad q\left(T\left(\frac{\delta'}{p(x)}x\right)\right) < \varepsilon \quad \Rightarrow \quad q(T(x)) < \frac{\varepsilon}{\delta'}p(x)$$

   now let $\delta' \to \delta$ and conclude. ■

References

[1] "Distributions and operators: Gerd Grubb"

[2] "Funktionalanalysis: Werner.D"