Infinite-dimensional and higher structures in differential geometry

–preliminary draft—

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Introduction

These are lecture notes for a course held at the university of Hamburg in the spring of 2013. Two major topics were treated in this course, which seem at first glance unrelated: infinite-dimensional manifolds (in particular mapping spaces) and geometric (or smooth) stacks. However, the perspective to infinite-dimensional manifolds was chosen to be a topos theoretic one, which means that one first constructs mapping "spaces" in a category of sheaves, and then enhances them in a second step to a honest infinite-dimensional manifold. This approach has many benefits, amongst others that it makes the reader familiar with notions that make the passage from sheaves to stacks a more or less natural one. Moreover, the notion of a geometric stack then also is quite natural. In the end, the chapter on string group models shows how these two concepts interact nicely. This is due to the fact that the most conceptual picture on the string group is to view it as a certain geometric stack, while explicit (in particular strict) models of it are constructed from certain mapping spaces.

Requirements and Notation

The general pattern that we will use is as follows. At first we assume that the reader is familiar with the basic concepts mentioned below. Based on this we will develop all concepts and give most of the proofs explicitly. From time to time we take the liberty of referring to textbooks for facts that we do not prove and defer easy arguments to exercises. This comprises the part of the lecture that addresses the novice. In addition to this, we sometimes give remarks on the extendability and greater validity of the results proven in the lecture. These extended results will not be used later on, but are for readers with a broader interest. In these remarks we then also refer to more advanced texts.

The basic concepts that we assume familiarity with are some notions from topology (e.g., continuous functions, open covers, metrisable spaces, compact, locally compact and paracompact spaces and some basics of the compact-open topology) and knows what submanifolds of $\mathbb{R}^n$ are. We also assume some familiarity with the basic notions from category theory, i.e., that the reader knows what a category is and what functors and natural transformations are. In particular, we will use the categories **Set** (of sets[1]), **Ab** (of abelian groups), **R-Mod** (of R-modules for a fixed ring R), **G-Mod** ($=\mathbb{Z}$**[G]-Mod** for $\mathbb{Z}[G]$ the group ring of some group G) and **Top** of topological spaces. Moreover, we will assume in Section 7 that the reader is familiar with the basic properties of the exponential map of a (finite-dimensional) Riemannian manifold.

Some notation and abbreviations that we use throughout:

pr_n	projection the the n-th factor in a (cartesian) product $\prod_{i\in I} X_i$
$\mathrm{pr}_{\widehat{n}}$	projection that omits the n-th factor in a cartesian product
tvs	topological vector space
lcs	locally convex (vector) space
$U \Subset X$	U is an *open* subset of the topological space X
$x \in U \Subset X$	U is an open neighbourhood of x
$B_x(\varepsilon)$	open ε-ball around x
$\widehat{f}$	defined by $\widehat{f}(x)(y) := f(x,y)$ for $f\colon X \times Y \to Z$
$\check{f}$	defined by $\check{f}(x,y) := f(x)(y)$ for $f\colon X \to \mathbf{Set}(Y,Z)$

[1]We will throughout assume that out sets have a cardinality bound, so that **Set** is a small category, cf. Remark 2.6.

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1 Categorical Preliminaries

In this section we give the necessary categorical background for the lecture. This comprises

- limits and colimits,
- adjoint functos and preservation of (co)limits and
- some examples.

We now introduce the parts of the language of categories that we will use throughout. For instance, it will be necessary to perform constructions of objects in a category from other prescribed objects. A first example is the product $X \times Y$ of two sets (groups, R-modules, topological spaces or (sub)manifolds). While the construction of finite products is often pretty natural, the construction of *infinite* products is not always possible and sometimes ambiguous. For instance in **Top**, where the infinite product of topological spaces $\prod_{i \in I} X_i$ may be endowed with different topologies (i.e., the box topology or the product topology). However, only the product topology turns $\prod_{i \in I} X_i$ into a (categorical) product in **Top** (cf. Example 1.3 c)) and the following definition provides the framework for this.

Definition 1.1. Let $\mathcal{C}$ be a category and J be a small category, i.e., the objects of J form a set. Then the category of functors from J to $\mathcal{C}$ (and natural transformations between them, cf. Exercise 2.35) is abbreviated $\mathcal{C}^J$. Then the diagonal functor $\Delta_J\colon \mathcal{C} \to \mathcal{C}^J$ sends each object C to the constant functor that maps each object of J to C and each morphisms of J to id_C. A morphism $f\colon C \to D$ is sent to the natural transformation given by map each object of J to f.

Call an object A of $\mathcal{C}^J$ a *diagram* of type J (in $\mathcal{C}$). Then a natural transformation π from the constant diagram $\Delta_J(C)$ to some other diagram A of $\mathcal{C}^J$ then consists of a family of morphisms $(\pi_j\colon C \to A_j)_{j \in \mathrm{Ob}(J)}$ such that

$$\begin{tikzcd} & C \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_k"] & \\ A_j \arrow[rr, "A_u"] & & A_k \end{tikzcd}$$

commutes for each morphism $u\colon j \to k$ of J. Such a natural transformation π is called a *cone* $\pi\colon C \to A$ over the diagram A with vertex C.

A cone $\pi\colon L \to A$ over A with vertex L is *universal* if for each cone $\varphi\colon C \to A$ there is a unique morphism $f\colon C \to A$ in $\mathcal{C}$ such that $\varphi_j = \pi_j \circ f$ for each object j of J, as depicted by the diagram

$$\begin{tikzcd}
 & C \arrow[dl, "\pi_j"'] \arrow[d, dashed, "!f"] \arrow[dr, "\pi_k"] & \\
 & L \arrow[dl, "\varphi_k"] \arrow[dr, "\varphi_j"'] & \\
A_j \arrow[rr, "A_u"] & & A_k
\end{tikzcd}$$

This universal cone (if it exists) is called the *limit* of the diagram A in the category $\mathcal{C}$. It is the also denoted by $(\varprojlim_J A \xrightarrow{\pi_j} A_j)_{j\in \mathrm{Ob}(J)}$ or simply $\varprojlim_J A$

One similarly defines *colimits* as universal cocones, where a cocone for a diagram A of type J in $\mathcal{C}$ with vertex C is a natural transformation from A to $\Delta_J(C)$. Colimits are denoted by $(A_j \xrightarrow{\pi_j} \varinjlim_J A)_{j\in \mathrm{Ob}(J)}$ or simply $\varinjlim_J A$. ■

Remark 1.2. The usual reasoning shows that limits are unique up to isomorphism, provided that they exist (cf. Exercise 1.11). ■

Example 1.3. a) Set $J = \bullet \rightrightarrows \bullet$ (the identity morphisms in J will always be omitted). Then a diagram of type J in a category $\mathcal{C}$ is given by two objects A, B of $\mathcal{C}$ and two morphisms $f\colon A \to B$ and $g\colon A \to B$:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B\ . \tag{1}$$

A cone with vertex C is an object C with morphisms $C \to A$ and $C \to B$ such that

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d] & C \arrow[d] \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

commutes, i.e., it is a morphism $i\colon C \to A$ such that $f \circ i = g \circ i$. A limit L of (1) is thus an object L with a morphism $\iota\colon L \to A$ such that $f \circ \iota = g \circ \iota$ and that if $i\colon C \to A$ satisfies $f \circ i = g \circ i$, then there exists a unique $\varphi\colon L \to C$ with $i = \iota \circ \varphi$:

$$\begin{tikzcd}
L \arrow[r, "\iota"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
C \arrow[u, dashed, "\exists!\varphi"] \arrow[ur, "i"'] & &
\end{tikzcd}\ .$$

This limit is also called *equaliser* of the diagram (1). In **Set** and **Ab** equalisers always exist (cf. Exercise 1.12).

b) If $J = \emptyset$, then for each category there exists a unique diagram of type J in each category $\mathcal{C}$, namely the empty diagram $\emptyset$. A limit of $\emptyset$ in $\mathcal{C}$ is called *terminal object* of $\mathcal{C}$. It is an object of $\mathcal{C}$ that has a unique morphism from each other object of $\mathcal{C}$. A colimit of $\emptyset$ is called *initial object* of $\mathcal{C}$, it hat a unique morphism into each other object of $\mathcal{C}$ (cf. Exercise 1.13).

c) The product $\prod_{i\in I} X_i$ (in **Set**, **Ab**, **R-Mod** or **Top**) is a limit of a diagram of type J, where J has $\mathrm{Ob}(J) = I$ and no non-identity morphisms and the diagram is given by the values X_i on the objects $i \in I$. Likewise, the disjoint union (in **Set**) or direct sum (in **Ab** or **R-Mod**) is a colimit.

d) Particularly important will be the case where $J = \bullet \longrightarrow \bullet \longleftarrow \bullet$. In this case, a diagram A of type J consists of two morphisms $p\colon X \to Z$ and $q\colon Y \to Z$. Then a limit of A consists of an object P of $\mathcal{C}$, together with morphisms $a\colon P \to X$ and $b\colon P \to Y$ such that $p \circ a = q \circ b$ and that whenever $c\colon Q \to X$ and $d\colon Q \to Y$ satisfy $p \circ c = q \circ d$, then there exists a unique $f\colon Q \to P$ such that $c = a \circ f$ and $d = b \circ f$:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists! f"] \arrow[drr, bend left, "d"] \arrow[ddr, bend right, "c"'] & & \\
& P \arrow[r, "b"] \arrow[d, "a"'] & Y \arrow[d, "q"] \\
& X \arrow[r, "p"] & Z
\end{tikzcd}$$

In this case, the limit is also called *pullback* or *fibre product* of p and q. It is also denoted by $X \times_Z Y$ (if the morphisms $X \to Z$ and $Y \to Z$ are understood).

e) In $\mathcal{C} = \mathbf{Set}$, the pullback is given by

$$P = \{(x, y) \in X \times Y \mid p(x) = q(y)\}, \quad a(x, y) = x, \quad b(x, y) = y.$$

For $Z = \mathrm{pt}$ a single point one sees that this notion generalises the cartesian product of the sets X and Y.

f) In $\mathcal{C} = \mathbf{Top}$, the pullback is also given by

$$P = \{(x, y) \in X \times Y \mid p(x) = q(y)\}, \quad a(x, y) = x, \quad b(x, y) = y,$$

endowed with the subspace topology of the product topology. In particular, if X is a topological space and $U \subseteq X$ and $V \subseteq X$ are subsets (endowed with the subspace topology), then we have that the pullback of the inclusions $U \to X$ and $V \to X$ is given by the intersection $U \cap V \cong P$ (for P as above), cf. Exercise 1.14. ■

Definition 1.4. Let $\mathcal{C}$ be a category and A be a diagram of type J. Then we say that $\mathcal{C}$ has

$$\left.\begin{array}{r}\text{arbitrary limits}\\ \text{finite limits}\\ \text{arbitrary products}\\ \text{finite products}\end{array}\right\} \text{ if } \varprojlim_J A \text{ exists for } \left\{\begin{array}{l}\text{arbitrary } J\\ J \text{ with } \mathrm{Ob}(J) \text{ finite}\\ J \text{ with only identity morphisms}\\ J \text{ with only identity morphisms and } \mathrm{Ob}(J) \text{ finite}\end{array}\right. .$$

Analogously, one says that $\mathcal{C}$ has arbitrary (finite) colimits (coproducts) if $\varinjlim_J A$ for arbitrary (finite) J (with only identity morphisms). ■

Limits behave nicely under (right) adjoint functors, i.e., we will see that (right) adjoint functors are *continuous*.

Definition 1.5. Let $\mathcal{C}, \mathcal{D}$ be categories and $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. Then F is *left adjoint* to G and G is *right adjoint* to F if there exists a natural isomorphism

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left, "\mathcal{D}(F(\,\cdot\,),\,\cdot\,)"] \arrow[r, bend right, "\mathcal{C}(\,\cdot\,, G(\,\cdot\,))"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{Set} \end{tikzcd}$$

between the functors $(C, D) \mapsto \mathcal{D}(F(C), D)$ and $(C, D) \mapsto \mathcal{C}(C, G(D))$ (more precisely one also says that η is an adjunction between F and G). The natural transformation η is also called *adjunction* (between F and G) and is denoted by $\eta\colon F \dashv G$. ■

Example 1.6. a) The standard example of an adjunction is that the functor that creates a free R-module F_X on a set X is left adjoint to the forgetful functor that assigns to an R-module the underlying set.

b) Let $\mathcal{C}$ have arbitrary products. Then the product functor $\prod\colon \prod_{i\in I} \mathcal{C} \to \mathcal{C}$ is right adjoint to the diagonal functor $\Delta\colon \mathcal{C} \to \prod_{i\in I} \mathcal{C}$. Likewise, if $\mathcal{C}$ has arbitrary codproducts, then coproduct $\coprod\colon \prod_{i\in I} \mathcal{C} \to \mathcal{C}$ is left adjoint to Δ.

c) The forgetful functor $F\colon \mathbf{Top} \to \mathbf{Set}$ that sends each topological space to its underlying set has a left adjoint the functor $\delta\colon \mathbf{Set} \to \mathbf{Top}$ $X \mapsto \delta(X)$, where $\delta(X)$ is the set X equipped with the discrete topology. Since $\delta(X)$ has the property that each map $\delta(X) \to Y$ to a topological space Y is continuous we have

$$\mathbf{Top}(\delta(X), Y) = \mathbf{Set}(X, Y)$$

and thus $\delta \dashv F$. Moreover, F also has a right adjoint. If $\chi(X)$ denotes X, equipped with the chaotic topology $\{\emptyset, X\}$, then each map $Y \to \chi(X)$ is continuous and thus

$$\mathbf{Top}(Y, \chi(X)) = \mathbf{Set}(Y, X)$$

shows $F \dashv \chi$ ■

Proposition 1.7. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be two functors. Then the following are equivalent:*

a) There exists an adjunction $\eta\colon F \dashv G$.

b) There exists natural transformations $\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\delta\colon FG \to \mathrm{id}_{\mathcal{D}}$ such that the compositions

$$F \xrightarrow{C \mapsto F(\varepsilon_C)} FGF \xrightarrow{C \mapsto \delta_{F(C)}} F \quad \text{and} \quad G \xrightarrow{D \mapsto \varepsilon_{G(D)}} GFG \xrightarrow{D \mapsto G(\delta_D)} G \tag{2}$$

are the identity transformations on F and G respectively.

Proof. a)⇒b): If natural bijections $\eta_{(C,D)}\colon \mathcal{D}(F(C), D) \to \mathcal{C}(C, G(D))$ are given (we will omit the indices of η in the sequel), then we set

$$\varepsilon_C := \eta(\mathrm{id}_{F(C)})\colon C \to G(F(C)). \quad \text{and} \quad \delta_D := \eta^{-1}(\mathrm{id}_D)\colon F(G(D)) \to D$$

The naturality of η means that for arbitrary $g\colon D \to D'$, $\varphi\colon F(C) \to D$ and $f\colon C' \to C$ we have

$$\eta(g \circ \varphi \circ F(f)) = G(g) \circ \eta(\varphi) \circ f. \tag{3}$$

If we fix $\xi\colon A \to B$ and evaluate (3) for, $g = \mathrm{id}_{F(B)}$, $\varphi = \mathrm{id}_{F(B)}$ and $f = \xi$, then we get

$$\eta(F(\xi)) = \eta(\mathrm{id}_{F(B)}) \circ \xi = \varepsilon_B \circ \xi.$$

If we evaluate (3) for $g = F(\xi)$, $\varphi = \mathrm{id}_{F(A)}$ and $f = \mathrm{id}_A$, then we get

$$\eta(F(\xi)) = G(F(\xi)) \circ \eta(\mathrm{id}_{F(A)}) = G(F(\xi)) \circ \varepsilon_A.$$

Thus ε is natural. A similar argument shows that δ is natural as well. Moreover, if we evaluate (3) for $g = \mathrm{id}_{F(A)}$, $\varphi = \delta_{F(A)}$ and $f = \varepsilon_A$, then we get

$$\eta(\delta_{F(A)} \circ F(\varepsilon_A)) = \eta(\delta_{F(A)}) \circ \varepsilon_A = \varepsilon_A = \eta(\mathrm{id}_{F(A)})$$

and thus $\delta_{F(A)} \circ F(\varepsilon_A) = \mathrm{id}_{F(A)}$ since η is bijective. The other identity of (2) is shown similarly.

b)⇒a): This is left as Exercise 1.15. ■

Definition 1.8. The natural transformation ε in part b) of the preceding proposition is also called the *unit* (or front adjunction) of the adjunction $\eta\colon F \dashv G$. The natural transformation δ is also called *counit* (or rear adjunction). ■

The importance of adjoint functors is that they preserves (co)limits in the following sense.

Proposition 1.9. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$.*

a) If A is a diagram of type J in $\mathcal{C}$ that has a colimit $(A_j \xrightarrow{\iota_j} \varinjlim_J A)_{j\in\mathrm{Ob}(J)}$ in $\mathcal{C}$, then $F \circ A$ has a colimit $(F(\varprojlim_J A) \xrightarrow{F(\iota_j)} F(A_j))_{j\in\mathrm{Ob}(J)}$ in $\mathcal{D}$.

b) If B is a diagram of type K in $\mathcal{D}$ that has a limit $(\varprojlim_K B \xrightarrow{\pi_k} B_k)_{k\in\mathrm{Ob}(K)}$ in $\mathcal{D}$, then $G \circ B$ has a limit $(G(\varprojlim_K B) \xrightarrow{G(\pi_k)} G(B_k))_{k\in\mathrm{Ob}(K)}$ in $\mathcal{C}$.

Proof. We only show part b), part a) follows similarly. Let $\eta_{(C,D)}\colon \mathcal{D}(F(C), D) \to \mathcal{C}(C, G(D))$ be the adjunction. Since G commutes with compositions it is clear that $(G(\varprojlim_K B) \xrightarrow{G(\pi_k)} G(B_k))_{k\in\mathrm{Ob}(K)}$ is a cone over $G \circ B$ in $\mathcal{C}$. Suppose $(C \xrightarrow{\varphi_k} G(B_k))_{k\in\mathrm{Ob}(K)}$ is another cone over $G \circ B$, then we obtain a cone $(F(C) \xrightarrow{\psi_k} B_k)_{k\in\mathrm{Ob}(K)}$ with $\psi_k := \eta^{-1}(\varphi_k)$ (check this!) and thus a unique morphism $f\colon F(C) \to \varprojlim_K B$ such that $\psi_k = \pi_k \circ f$ for all k. Since

$$\psi_k = \pi_k \circ f \Leftrightarrow \varphi_k = \eta(\psi_k) = \eta(\pi_k \circ f) = G(\pi_k) \circ \eta(f)$$

we have that $g := \eta(f)\colon C \to G(\varprojlim_K B)$ is unique with $\varphi_k = G(\pi_k) \circ g$ for all k. Thus $(G(\varprojlim_K B) \xrightarrow{G(\pi_k)} G(B_k))_{k\in\mathrm{Ob}(K)}$ is universal. ■

Definition 1.10. If a functor $F\colon \mathcal{C} \to \mathcal{D}$ (not necessarily one for which there exists an adjoint) satisfies Property a) of Proposition 1.9, then we say that F *preserves colimits*. Likewise, if $G\colon \mathcal{D} \to \mathcal{D}$ has Property b), then G *preserves limits*. ■

Exercise 1.11. Show that any two limits of the same diagram A are isomorphic. Moreover, show that if A, B are diagrams of type J in $\mathcal{C}$ with limits $\varprojlim_J A$ and $\varprojlim_J B$ and $A \to B$ is a morphism of diagrams (i.e., a natural transformation of functors $A, B\colon J \to \mathcal{C}$), then there exists a unique morphism $\varprojlim_J A \to \varprojlim_J B$ such that

$$\begin{tikzcd}
\varprojlim_J A \arrow[r] \arrow[d, "\pi_j"'] & \varprojlim_J B \arrow[d, "\pi'_j"] \\
A_j \arrow[r] & B_j
\end{tikzcd}$$

commutes and that the assignment $(A \to B) \mapsto (\varprojlim_J A \to \varprojlim_J B)$ is functorial. ■

Exercises for Section 1

Exercise 1.12. Show that in **Ab** the equaliser of two arbitrary morphisms f and g always exists and is given by the kernel of $f - g$. Then show that also in **Set** equalisers always exist. ■

Exercise 1.13. Determine the initial and terminal objects in **Set** and **Ab**. Conclude that right adjoints do not preserve colimits in general. ■

Exercise 1.14. If X is a topological space and $U, V \subseteq X$ are open subsets, equipped with the subspace topology, then show that the pullback of the inclusions $U \hookrightarrow X$ and $V \hookrightarrow X$ is given by $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$. ■

Exercise 1.15. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be two functors and let $\varepsilon\colon \operatorname{id}_{\mathcal{C}} \to GF$ and $\delta\colon FG \to \operatorname{id}_{\mathcal{D}}$ be natural transformations such that the compositions

$$F \xrightarrow{C \mapsto F(\varepsilon(C))} FGF \xrightarrow{C \mapsto \delta(F(C))} F \quad \text{and} \quad G \xrightarrow{D \mapsto \varepsilon(G(D))} GFG \xrightarrow{D \mapsto G(\delta(D))} G$$

are the identity transformations on F and G respectively. Then show that $\eta(C, D)\colon \mathcal{D}(F(C), D) \to \mathcal{C}(C, G(D))$, $\varphi \mapsto G(\varphi) \circ \varepsilon(C)$ is an adjunction $\eta\colon F \dashv G$. ■

Exercise 1.16. Show that in **Top** limits and colimits always exist. In contrast to this, show that in the category $\mathbf{Top_{Haus}}$ of topological Hausdorff spaces, limits do always exist, but colimits do not (**Hint:** The colimit of the embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$ would have to have the property that each continuous map $\mathbb{Q} \to X$ to an arbitrary Hausdroff space extends to a continuous map on $\mathbb{R}$). ■

Exercise 1.17. Show that the forgetful functor **Man** $\to$ **Top** does in general not preserve pull-backs. **Hint:** If $f\colon \mathbb{R} \to \mathbb{R}$ is smooth such that $f^{-1}(0) = \{\frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{0\}$, then the pull-back of f and $\{0\} \hookrightarrow \mathbb{R}$ in **Man** is $\{\frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{0\}$ with the *discrete* smooth structure. ■

2 Presheaves and Sheaves

In this section we give the basic definitions of presheaves and sheaves (on a Grothendieck site). By doing so, we try to minimise the technical notation, i.e., do not treat Grothendieck (pre)topologies, but only coverages. The exposition tries to make the passage to stacks later quite natural. The section contains roughly

- definitions of presheaves, sites and sheaves thereon,
- proof that (pre)sheaves are complete and cocomplete,
- discussion of exponential objects (mapping objects) and of cartesian closed categories and
- examples (in particular that manifolds do in general not have exponential objects, pull-backs and push-outs).

Sheaves serve many purposes in mathematics, they are very useful in Algebraic and Differential Geometry, Algebraic Topology, Representation Theory and are the foundational concept in Topos Theory. The perspective that we will take in this lecture is that a sheaf is the concept that endows a topological space with additional geometric structure.

To illustrate this a bit more, note that a topological spaces provides us with the notion of locality[2]. For example, we can say what it means for a function to be continuous at a point. Note that there are concepts that can be defined in local terms (like continuity, i.e., a function is continuous if and only if it is continuous on a neighbourhood of each point), but that there are notions that cannot be defined in local terms (like boundedness, i.e., a function can be bounded on some neighbourhood of each point, but might not be bounded on the whole space). On the other hand, we have all learned in the beginners classes that geometry is conveniently expressed in algebraic terms (Linear Algebra in most cases, but not exclusively). The idea of a sheaf is to combine these two concepts in order to endow a topological space with "local geometric structure". References for this section are [Mac98, MLM94].

The difference between presheaves and sheaves is something quite subtle and perhaps kind of mysterious at first reading. It will become clearer in Section 12.

Definition 2.1. If X is a topological space, then a *presheaf* F in the category of abelian groups on X is an assignment $U \mapsto F(U)$, where $U \subseteq X$ is open and $F(U)$ is an abelian group, and to each inclusion $U \hookrightarrow V$ of open subsets a morphism $\rho_{UV} \colon F(V) \to F(U)$ of abelian groups such that $\rho_{UU} = \mathrm{id}_U$ and $\rho_{UV} \circ \rho_{VW} = \rho_{UW}$ whenever $U \hookrightarrow V \hookrightarrow W$. We call the elements of $F(U)$ the *sections* (of F) over U and the elements of $F(X)$ the *global sections*. For $U \hookrightarrow V$ as above and $f \in F(V)$ we sometimes denote by $f|_U := \rho_{UV}(f)$ and call this the *restriction* of f to U (see the examples below for an illustration). ■

This definition is a specialisation of the following more general notion.

[2] As first approximation, one can think of locality as being a notion of "closeness". However, the mathematically correct incarnation of "closeness" is given by a uniform structure, which is related to locality, but is strictly speaking not the same. Note that for metric spaces or topological groups one always also has a canonically associated uniform structure, so that the difference between these notions is subtle (recall the confusion that many beginners have when introducing uniform continuity).

Definition 2.2. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a presheaf on $\mathcal{C}$ in $\mathcal{D}$ is a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. This covers the above definition by setting $\mathcal{C}$ to be the category $\mathbf{Open}_X$ whose objects are the open subsets of X and[3]

$$\mathbf{Open}_X(U,V) = \begin{cases} \{U \hookrightarrow V\} & \text{if } U \subseteq V \\ \emptyset & \text{else} \end{cases}$$

with the obvious composition and identity morphisms, and setting $\mathcal{D}$ to be the category $\mathbf{Ab}$ of abelian groups. If $\mathcal{D}$ is omitted, then it is taken to be the category $\mathbf{Set}$ of sets, i.e., by a presheaf on $\mathcal{C}$ we will always mean a presheaf in $\mathbf{Set}$. ■

Example 2.3. a) To each topological space X there is associated the presheaf $\mathcal{O}_X$ of continuous functions $U \mapsto C(U,\mathbb{R})$, where ρ_{UV} is the restriction $f \mapsto f|_U$. This is also called the *structure sheaf* of X. More generally, if Y is another fixed topological space, then $\mathcal{O}_X^Y$ is the presheaf $U \mapsto C(U,Y)$ on X with the same restriction morphisms.

There is an associated "big" example as a presheaf on the category $\mathbf{Top}$ of all topological spaces. If we fix some topological space Z (which was $\mathbb{R}$ above), then we have a presheaf $h^Z\colon \mathbf{Top} \to \mathbf{Set}$ given by $X \mapsto C(X,Z)$ and for a continuous function $f\colon X \to Y$ the "restriction" morphisms are given by

$$\rho_f\colon C(Y,Z) \to C(X,Z), \quad \varphi \mapsto \varphi \circ f.$$

b) To each topological space X there is associated the presheaf $\mathcal{B}_X$ of bounded continuous functions $U \mapsto \{f\colon U \to \mathbb{R} \mid f \text{ is bounded }\}$, again with ρ_{UV} the restriction $f \mapsto f|_U$.

c) To each submanifold $M \subseteq \mathbb{R}^n$ there are associated the presheaf $\mathcal{O}_M$ of smooth functions $U \mapsto C^\infty(U,\mathbb{R})$[4], as well as the presheaf $\mathcal{X}_M$ of (smooth) vector fields $U \mapsto \mathcal{V}(U)$, in both cases with ρ_{UV} given by restrictions.

d) Let A be an abelian abelian group, X be a topological space and $x_0 \in X$. Then we define the *skyscraper presheaf* $\mathcal{S}_{x_0}^A$ by

$$U \mapsto \begin{cases} A & \text{if } x_0 \in U \\ 0 & \text{else} \end{cases}$$

and

$$\rho_{UV} = \begin{cases} \mathrm{id}_A & \text{if } x_0 \in U \cap V \\ 0 & \text{else} \end{cases}.$$

This example easily generalises to a presheaf in $\mathcal{D}$ on X, where $\mathcal{D}$ is a category with terminal object.

e) Let X be a topological space and Y be a set. Then the *constant presheaf* with values in Y is given by

$$U \mapsto \{f\colon U \to Y \mid \text{ there exists } y \in Y \text{ such that } f \text{ is the constant function with value } y\} \cong Y$$

[3]We will follow the convention to denote the morphisms from x to y for objects x, y of a category $\mathcal{C}$ by $\mathcal{C}(x,y)$.

[4]Note that each open subset of a submanifold is again a submanifold, so that it makes sense to talk about smooth functions on U.

and $\rho_{U,V}$ the restriction $f \mapsto f|_U$. There is a variant of this, called the *locally constant* presheaf with values in Y, which is most conveniently defined as $U \mapsto C(U,Y)$, where Y is endowed with the discrete topology.

f) If $\mathcal{C}$ is an arbitrary category and Z is an object of $\mathcal{C}$, then a presheaf h^Z in **Set** on $\mathcal{C}$ is given as in part a) by $X \mapsto \mathcal{C}(X,Z)$ and for $f \in \mathcal{C}(X,Y)$ the "restriction" morphism if given by

$$\rho_f \colon \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z), \quad \varphi \mapsto \varphi \circ f. \tag{4}$$

We will frequently denote the functor h^Z also by $\mathcal{C}(\,\cdot\,,Z)$. Note that $X \mapsto \mathcal{C}(X,X)$ cannot be made into a functor in a natural way, if one wants to mimic the construction to involve both arguments then one has to consider $(X,Y) \mapsto \mathcal{C}(X,Y)$ as a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$. ∎

Note: We frequently denote the restriction homomorphism ρ_{UV} also by F_{UV} (for $U \hookrightarrow V$ an inclusion of open subsets) and ρ_f also as $F(f)$ (for f a morphism in $\mathcal{C}$).

Remark 2.4. Note that $\mathcal{X}_M$ has additional algebraic structure, i.e., that for each $U \subseteq M$ open we have that $\mathcal{O}_M(U)$ is a ring and $\mathcal{X}_M(U)$ is a module over $\mathcal{O}_M(U)$, since we can multiply vector fields point-wise with smooth functions. In this case one says that $\mathcal{X}_M$ is a module (pre)sheaf over the (pre)sheaf of rings $\mathcal{O}_M$. This is the general idea in Algebraic (and also in Differential) Geometry, that one first specifies a certain (pre)sheaf of rings on a topological space X (called the *structure sheaf*), and then the geometric notions are expressed in terms of module (pre)sheaves over the structure (pre)sheaf. ∎

Taking Definition 2.2 into account, the notion of morphism of presheaves is immediate.

Definition 2.5. If $F, G \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ are presheaves, then a *morphism* from F to G (denoted either by $F \to G$ or also by $F \Rightarrow G$) is a natural transformation of functors.

If $\mathcal{C}$ is a small category (i.e., the class of objects forms a set), then functors and natural transformations between them form a category (cf. Exercise 2.35). This defines for each small category $\mathcal{C}$ the category $\mathbf{PSh}_{\mathcal{C}}^{\mathcal{D}}$ of presheaves in $\mathcal{D}$ on $\mathcal{C}$. If $\mathcal{D}$ is omitted, then it is considered to be **Set**. If $\mathcal{C} = \mathbf{Open}_X$, then we also set $\mathbf{PSh}_X := \mathbf{PSh}(\mathbf{Open}_X)$. ∎

Remark 2.6. The preceding definition requires $\mathcal{C}$ to be a small category in order to turn functors and natural transformations (shortly called *functor category*) into a category. The subtle issue here is that for two functors $F, G \colon \mathcal{C} \to \mathcal{D}$, the natural transformations should form a set (and not a proper class). This is an important requirement in category theory.

The problem now is, that this assumption would spoil the treatment of functors $\mathbf{Set} \to \mathbf{Set}$ or $\mathbf{Top} \to \mathbf{Set}$ as a functor category. The remedy for this is to put a cardinality bound on all sets (and thus also topological spaces), which is large enough to make all constructions from set theory work *below* this cardinal. Technically, this is implemented by assuming the existence of a *Grothendieck universe*, in which all these operations take place. We will throughout assume (without any further mentioning) that all categories (like **Set**, **Ab**, **Top**) only contain sets below this cardinality bound and thus are small (cf. [Mac98, Chap. I] or [Bor94, Sect. 1.1] for a more detailed treatment). ∎

Remark 2.7. By Example 2.3 f) we can assign to each object Z of $\mathcal{C}$ the presheaf h^Z on $\mathcal{C}$. If $g \in \mathcal{C}(Z,W)$, then we obtain a morphism of presheaves $g_* \colon h^Z \to h^W$, given by

$$h^Z(X) \to h^W(X), \quad \left(X \xrightarrow{\varphi} Z\right) \mapsto \left(X \xrightarrow{\varphi} Z \xrightarrow{g} W\right).$$

From the associativity in $\mathcal{C}$ and (4) it follows immediately that this defines in fact a natural transformation. If $\mathcal{C}$ is small, then this results in a functor $\mathcal{C} \to \mathbf{PSh}_{\mathcal{C}}$, which is also called the *Yoneda embedding*. That this is in fact an embedding is content of the following lemma. ■

Lemma 2.8 (Yoneda Lemma). *For any small category $\mathcal{C}$, the assignments*

$$Z \mapsto h^Z \quad \text{and} \quad (Z \xrightarrow{g} W) \mapsto (g_* \colon h^Z \to h^W) \tag{5}$$

is a functor $\mathcal{C} \to \mathbf{PSh}_{\mathcal{C}}$. Moreover, for each presgeaf F on $\mathcal{C}$ we have that

$$\mathbf{PSh}(h^Z, F) \to F(Z), \quad \alpha \mapsto \alpha(Z)(\mathrm{id}_Z) \tag{6}$$

is a bijection. In particular, (5) is fully faithful, i.e., $\mathcal{C}(Z,W) \to \mathbf{PSh}_{\mathcal{C}}(h^Z, h^W)$, $g \mapsto g_*$ is bijective for each two objects Z, W of $\mathcal{C}$.

Proof. For composable morphisms $Z \xrightarrow{g} W \xrightarrow{f} Q$ we have

$$(f \circ g)_*(\varphi) = (f \circ g) \circ \varphi = f \circ (g \circ \varphi) = f_*(g_*(\varphi))$$

and clearly $(\mathrm{id}_Z)_* = \mathrm{id}_{h^Z}$. Thus (5) defines a functor. By definition, a natural transformation $\alpha \colon h^Z \to F$ is the same thing as for each X an assignment of a map $\alpha(X) \colon \mathcal{C}(X, Z) \to F(X)$ such that for each $f \colon X \to Y$ the diagram

$$\begin{tikzcd}
\mathcal{C}(Y,Z) \arrow[r, "h^Z(f)"] \arrow[d, "\alpha(Y)"'] & \mathcal{C}(X,Z) \arrow[d, "\alpha(X)"] \\
F(Y) \arrow[r, "F(f)"] & F(X)
\end{tikzcd}$$

commutes. For $Y = Z$ we have that $h^Z(\mathrm{id}_Z) = f$ and thus

$$\alpha(X)(f) = \alpha(X)(h^Z(f)(\mathrm{id}_Z)) = F(f)(\alpha(Z)(\mathrm{id}_Z)).$$

Thus α is uniquely determined by $\alpha(Z)(\mathrm{id}_Z)$ and if $u \in F(Z)$, then $\alpha(X)(f) := F(f)(u)$ defines an inverse to (6). If we apply this to $F = h^W$, then we get

$$\alpha(X)(f) = h^W(f)(\alpha(Z)(\mathrm{id}_Z)) = (\alpha(Z)(\mathrm{id}_Z)) \circ f$$

and thus $\alpha(X) = g_*$ for $g = \alpha(Z)(\mathrm{id}_Z) \in \mathcal{C}(Z, W)$. ■

Definition 2.9. A presheaf F on a category $\mathcal{C}$ is called *representable* if F is isomorphic to the presheaf h^Z for some object Z of $\mathcal{C}$. One then says that Z represents the presheaf F. ■

Corollary 2.10. *If F is representable, then the object of $\mathcal{C}$ that represents F is uniquely determined up to isomorphism.*

Proof. This is Exercise 2.36 ■

Constructions in $\mathbf{PSh}_{\mathcal{C}}$ are "easy" (at least compared to constructions on $\mathcal{C}$), as the following example and proposition is supposed to illustrate.

Example 2.11. Let $\mathcal{C}$ be a small category and $(F_i)_{i\in I}$ be a family of presheaves on $\mathcal{C}$. Then we may define the *product* $\prod_{i\in I} F_i$ in $\mathbf{PSh}_{\mathcal{C}}$ by

$$(\prod_{i\in I} F_i)(C) := \prod_{i\in I} F_i(C) \quad \text{and} \quad \prod_{i\in I} F_i(C \xrightarrow{f} D) := \prod_{i\in I} F_i(f), \tag{7}$$

together with the canonical projections (recall that $\prod$ is a functor from the product category $\prod_{i\in I}\mathbf{Set} \to \mathbf{Set}$). Note that this is possible since $\mathbf{Set}$ has products, but no products in $\mathcal{C}$ are involved. Note that for instance the category of (sub)manifolds (or varieties) does not have arbitrary products (cf. Remark 2.19). ■

We take the preceding as a motivating example for showing that even more is true: $\mathbf{PSh}_{\mathcal{C}}$ not only has arbitrary products, but also arbitrary limits and colimits. As in (7), this will be done for each object C on $\mathcal{C}$ separately, so we first consider the target category $\mathbf{Set}$.

Proposition 2.12. *If A is a diagram of type J (with J small by definition) in* $\mathbf{Set}$*, then A has a limit and colimit in* $\mathbf{Set}$*. Shortly:* $\mathbf{Set}$ *has arbitrary small limits and colimits.*

Proof. Set $I = \mathrm{Ob}(J)$ and consider the product $\prod_{i\in i} A_i$ of the sets A_i. Then we set

$$\varprojlim_J A := \{(a_i)_{i\in I} \in \prod_{i\in I} A_i \mid A_u(a_j) = a_k \text{ if } u\colon j \to k \text{ in } J\}.$$

Then $\pi_j\colon \varprojlim_J A \to A_j$, $(a_i)_{i\in I} \mapsto a_j$ is a map satisfying $A_u \circ \pi_j = \pi_k$. If $\varphi_j\colon C \to A_j$ satisfies $A_u \circ \varphi_j = \varphi_k$, then $f\colon C \to \varprojlim_J A$, $c \mapsto (\varphi_i(c))_{i\in I}$ is uniquely determined by satisfying $\varphi_j = \pi_j \circ f$. Thus $\varprojlim_J A$ is a limit of A.

The colimit is slightly more involved. We consider the disjoint union $\coprod_{i\in I} A_i$ of the sets A_i and set

$$\varinjlim_J A := \coprod_{i\in I} A_i / \sim,$$

where $\sim$ is the equivalence relation generated by the reflexive and transitive relation

$$a_j \sim a_k :\Leftrightarrow \exists\, u\colon j \to k \text{ in } J \text{ with } A_u(a_j) = a_k.$$

Then $\iota_j\colon A_j \to \varinjlim_J A$, $a_j \mapsto [a_j]$ satisfies $\iota_j \circ A_j = \iota_k$ and if $\varphi_j\colon A_j \to C$ satisfies $\varphi_j \circ A_u = \varphi_k$, then $f\colon \varinjlim_J A \to C$, $[a_j] \mapsto \varphi_j(a_j)$ is well-defined and uniquely determined by satisfying $f \circ \iota_j = \varphi_j$. ■

Remark 2.13. The following proposition generalises the preceding one to arbitrary presheaves. For this we observe that a functor $A\colon J \to \mathbf{PSh}_{\mathcal{C}}$ is the same thing as a functor $A\colon J \times \mathcal{C} \to \mathcal{D}$. In particular, we get for each object C of $\mathcal{C}$ a functor

$$A(\,\cdot\,, C)\colon J \to \mathbf{Set}, \quad j \mapsto A(j, C) \quad \text{and} \quad \left(j \xrightarrow{u} k\right) \mapsto \left(A(j,C) \xrightarrow{A(u,\mathrm{id}_C)} A(u,C)\right) \tag{8}$$

from precomposing A with the functor $J \times C \to J \times \mathcal{C}$, where C denotes also the subcategory of $\mathcal{C}$ with one object C and sole morphism id_C. Then $A(\,\cdot\,, C)$ is called the *evaluation* of A at C. This is to be spelled out in detail Exercise 2.37. ■

Proposition 2.14. *If A is a diagram of type J (with J small by definition) in the presheaf category* $\mathbf{PSh}_{\mathcal{C}}$*, then A has a limit and colimit in* $\mathbf{PSh}_{\mathcal{C}}$*. Shortly:* $\mathbf{PSh}_{\mathcal{C}}$ *has arbitrary small limits and colimits.*

Proof. We shall only show the statment for limits, colimits are treated similarly. For each object C of $\mathcal{C}$, consider the diagram A_C and its limit

$$L_C := \varprojlim_J A(\,\cdot\,, C)$$

in **Set**. For each morphism $f\colon C \to D$ in $\mathcal{C}$ and each $u\colon j \to k$ in J we have that

$$\begin{tikzcd}
 & & L_C \arrow[dddll, "\pi_C(j)"'] \arrow[dddrr, "\pi_C(k)"] & & \\
 & & L_D \arrow[dl, "\pi_D(j)"'] \arrow[dr, "\pi_D(k)"] & & \\
 & A(j,D) \arrow[rr, "A(u{,}\mathrm{id}_D)"] & & A(k,D) & \\
A(j,C) \arrow[ur, "A(\mathrm{id}_j{,}f)"'] \arrow[rrrr, "A(u{,}\mathrm{id}_C)"] & & & & A(k,C) \arrow[ul, "A(\mathrm{id}_k{,}f)"]
\end{tikzcd}$$

commutes since A is a functor and $(u, \mathrm{id}_D) \circ (\mathrm{id}_j, f) = (u, f) = (\mathrm{id}_k, f) \circ (u, \mathrm{id}_C)$. Thus we have a unique morphism $L_f\colon L_C \to L_D$ making the resulting diagram

$$\begin{tikzcd}
 & & L_C \arrow[dddll, "\pi_C(j)"'] \arrow[dddrr, "\pi_C(k)"] & & \\
 & & L_D \arrow[u, dashed, "\exists! L_f"] \arrow[dl, "\pi_D(j)"'] \arrow[dr, "\pi_D(k)"] & & \\
 & A(j,D) \arrow[rr, "A(u{,}\mathrm{id}_D)"] & & A(k,D) & \\
A(j,C) \arrow[ur, "A(\mathrm{id}_j{,}f)"'] \arrow[rrrr, "A(u{,}\mathrm{id}_C)"] & & & & A(k,C) \arrow[ul, "A(\mathrm{id}_k{,}f)"]
\end{tikzcd}$$

commute. From the uniqueness it also follows that $L_{f\circ g} = L_g \circ L_f$ for $g\colon D \to E$ another morphism in $\mathcal{C}$ (simply put yet another cone around and observe that both $L_g \circ L_f$ and $L_{f\circ g}$ make it commute). Thus $C \mapsto L_C$ is a presheaf, also denoted by L, and $L_C \xrightarrow{\pi_C(j)} A(j, C)$ is a morphism of functors $\Delta_J(L) \to A$, and whence a cone over A (with vertex L).

Finally, this cone is universal, for if $\varphi\colon \Delta_J(M) \to A$ is any other cone over A, then $\varphi_j(C)\colon M(C) \to A(j, C)$ satisfies $A(u, \mathrm{id}_C) \circ \varphi_j(C) = \varphi_k(C)$ for each $j \xrightarrow{u} k$ and thus there exist unique $g_C\colon M_C \to L_C$ with $\varphi_j = g_C \circ \pi_C(j)$. These combine to give a natural transformation (check this!) and thus a morphism $M \to L$ with the desired properties. ■

Remark 2.15. The conclusion of the proof may also be rephrased by saying that (small) limits and colimits in $\mathbf{PSh}_{\mathcal{C}}$ exist and may be computed point-wise:

$$(\varprojlim_J A)(C) := \varprojlim_J A(C) \quad \text{and} \quad (\varinjlim_J A)(C) := \varinjlim_J A(C).$$ ■

Remark 2.16. Proposition 2.14 might look as abstract nonsense, but has the following remarkable consequence. In the categories **Top** and **Man** we consider the morphisms

$$f\colon \mathbb{R} \to \mathbb{R}, x \mapsto x^2(x+1) \quad \text{and} \quad g\colon \mathbb{R} \to \mathbb{R}, y \mapsto y^2.$$

We are interested in the limits of

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xleftarrow{g} \mathbb{R} \tag{9}$$

in **Top** and **Man**. Let T be the limit in **Top** (which exists, cf. Example 1.3 f)) and M the one in **Man** (if it exists). We want to show that M cannot exist in **Man**, so we assume that it exists and derive from this a contradiction.

The forgetful functors **Man** $\to$ **Set** and **Top** $\to$ **Set** have left adjoint (cf. Example 1.6 c) and B.3 a)) as left adjoints the functors that send sets to manifolds and topological spaces endowed with the discrete smooth structure and topology. Thus they preserve limits by Proposition 1.9, which means that the limits in **Man** and **Top** all have as underlying set the limit $\{(x,y) \in \mathbb{R}\times\mathbb{R} \mid y^2 = x^2(x+1)\}$ in **Set** (cf. Figure 1).

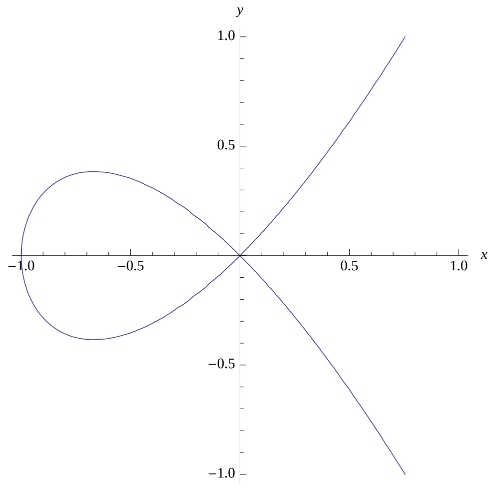

Figure 1: Limit in **Top**, but not in **Man**

In **Top**, the limit of f and g is given by $T = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid y^2 = x^2(x+1)\}$ with the subspace topology of $\mathbb{R}^2$ and the canonical projections to $\mathbb{R}$.

Likewise, if M existed in **Man**, then this would amount to endowing $\{(x,y) \in \mathbb{R}\times\mathbb{R} \mid y^2 = x^2(x+1)\}$ with a manifold structure such that the projections to $\mathbb{R}$ are smooth. Moreover, the structure morphisms from the limit $M \to \mathbb{R}$ are in particular continuous and thus yield a continuous map $M \to T$, which is the identity on the underlying sets (why?). If we now consider the maps $\mathbb{R} \to \mathbb{R}$, given by $x \mapsto x^2 - 1$ and $x \mapsto x(x^2-1)$, then they give rise to a unique continuous map $\varphi\colon \mathbb{R} \to T$ and a unique smooth map $\varphi\colon \mathbb{R} \to M$, both given by $x \mapsto (x^2-1, x(x^2-1))$. Both, φ and ψ map $\{-1,1\}$ to the singular point $(0,0)$ (cf. Figure 1). If $U = [-1-\varepsilon, -1+\varepsilon] \cup [1-\varepsilon, 1+\varepsilon]$, then φ maps U to a neighbourhood of $(0,0)$ in T and ψ maps U to a compact set containing $(0,0)$. Since $\mathrm{id}\colon M \to T$ is continuous, $\psi(U)$ is a compact neighbourhood of $(0,0)$ in M, and thus $\mathrm{id}|_{\psi(U)}$ is a homeomorphism[5]. This of course cannot be the case, since no neighbourhood of $(0,0)$ in T (cf.

[5]Recall from topology or prove as an exercise that a continuous bijection $f\colon X \to Y$ with X compact and Y Hausdorff is automatically a homeomorphism.

Figure 1) is homeomorphic to an open interval (why?). This shows that **Man** does not have arbitrary limits (even not finite ones!). Moreover, one can also show that **Man** does not have finite colimits.■

We now turn to another categorical construction that is, in general, only possible in $\mathbf{PSh}_{\mathcal{C}}$, but not in $\mathcal{C}$ itself.

Remark 2.17. Recall that in **Set** we have the natural isomorphism

$$\mathbf{Set}(X \times Y, Z) \xrightarrow{\cong} \mathbf{Set}(X, \mathbf{Set}(Y, Z)), \quad f \mapsto \widehat{f} \quad \text{with} \quad \widehat{f}(x)(y) := f(x, y) \tag{10}$$

where $\mathbf{Set}(Y, Z)$ denotes, of course, the set of functions from Y to Z (this is important to note in order to give $\mathbf{Set}(X, \mathbf{Set}(Y, Z))$ a meaning). Natural means here that (10) is an adjunction $(\,\cdot\, \times Y) \dashv \mathbf{Set}(Y, \,\cdot\,)$ for each fixed Y. ■

Definition 2.18. Let A be a category with finite products. An object Y of $\mathcal{C}$ is called *exponentiable* if $\,\cdot\, \times Y$ has a right adjoint, usually denoted by either $\underline{\mathcal{C}}(Y, \,\cdot\,)$ or $(\,\cdot\,)^Y$. For each other object Z, the object Z^Y is also called the *exponential* of Y and Z. The category $\mathcal{C}$ is called *cartesian closed* if each object is exponentiable. ■

Remark 2.19. We take a look at the exponentiable objects Y in **Man**. If $Y = \{y_1, ..., y_n\}$ is a finite collection of points, then we have $X \times Y \cong X \sqcup ... \sqcup X$ and thus

$$\mathbf{Man}(X \times Y, Z) \cong \mathbf{Man}(X \sqcup ... \sqcup X, Z) \cong \prod_{i=1}^{n} \mathbf{Man}(X, Z) \cong \mathbf{Man}(X, \prod_{i=1}^{n} Z).$$

Thus $Z \mapsto \prod_{i=1}^{n} Z$ is the exponential of Y and Z. However, $\mathbb{N}$ is not exponentiable, as one sees as follows. Suppose that for each Z we have a manifold structure on $\mathbf{Man}(\mathbb{N}, Z)$. We denote this manifold by $\underline{\mathbf{Man}}(\mathbb{N}, Z)$. If this would satisfy $(X \times \,\cdot\,) \dashv \mathbf{Man}(\,\cdot\,, Z)$ for suitable natural isomorphisms

$$\mathbf{Man}(X \times \mathbb{N}, Z) \xrightarrow{\cong} \mathbf{Man}(X, \underline{\mathbf{Man}}(\mathbb{N}, Z)), \tag{11}$$

then we would have in particular natural isomorphisms

$$\mathbf{Man}(X, \underline{\mathbf{Man}}(\mathbb{N}, Z)) \cong \mathbf{Man}(X \times \mathbb{N}, Z) \cong \prod_{\mathbb{N}} \mathbf{Man}(X, Z).$$

Thus the product $\prod_{\mathbb{N}} h^Z$ of the representable presheaves h^Z on **Man** would be represented by $\underline{\mathbf{Man}}(\mathbb{N}, Z)$. As in Remark 2.16, we see that the products in Man and Top all have as underlying set the cartesian product $\prod_{\mathbb{N}} Z$ and that the identity map $\mathrm{id}\colon \underline{\mathbf{Man}}(\mathbb{N}, Z) \to \prod_{\mathbb{N}} Z$ is continuous with respect to the topology underlying $\underline{\mathbf{Man}}(\mathbb{N}, Z)$.

We now set $Z = S^1 = \mathbb{R}/\mathbb{Z}$. Then we have the smooth manifold $\prod_{\mathbb{N}} \mathbb{R}$ and the projections $\mathrm{pr}_n\colon \prod_{\mathbb{N}} \mathbb{R} \to \mathbb{R}$ are continuous, linear and thus smooth (cf. Example A.2 e)). Thus they give rise to smooth maps $\overline{\mathrm{pr}}_n\colon \prod_{\mathbb{N}} \mathbb{R} \to S^1$, and thus induce a smooth map $\varphi\colon \prod_{\mathbb{N}} \mathbb{R} \to \underline{\mathbf{Man}}(\mathbb{N}, Z)$, giving rise to the composition

$$\prod_{\mathbb{N}} \mathbb{R} \xrightarrow{\varphi} \underline{\mathbf{Man}}(\mathbb{N}, S^1) \xrightarrow{\mathrm{id}} \prod_{\mathbb{N}} S^1.$$

of continuous maps, which equals the projection $\prod_{\mathbb{N}} \mathbb{R} \to \prod_{\mathbb{N}} S^1$ (by the uniqueness of the map into the product). Since the projection is open and φ is continuous, it follows that id is open, and thus that id is a homeomorphism. But by the definition of the product topology, each open neighbourhood of $(1, 1, ...)$ in $\prod_{\mathbb{N}} S^1$ is of the form

$$V_1 \times ... \times V_m \times S^1 \times S^1 \times ... \tag{12}$$

for only finitely many open neighbourhoods $V_1, ..., V_m$ of $1 \in S^1$. Thus (12) is not contractible, but in a manifold each point has a contractible neighbourhood (homeomorphic to an open convex set in a lcs). This contradicts the existence of a manifold structure on $\mathbf{Man}(\mathbb{N}, S^1)$ satisfying (11) (cf. Exercise 2.38).

One can phrase the difference between $\{y_1, ..., y_n\}$ and $\mathbb{N}$ by noting that the former space is compact (in the discrete topology) and the latter is not. We will see later on that the compactness of a manifold Y will be essential for endowing $\mathbf{Man}(Y, Z) = C^\infty(Y, Z)$ with meaningful smooth structures. ■

Remark 2.20. We can remedy the above failure by passing from $\mathbf{Man}$ to $\mathbf{PSh}_{\mathbf{Man}}$. More generally, for presheaves G, H on an arbitrary category $\mathcal{C}$, we obtain a new presheaf $\underline{\mathbf{PSh}}_{\mathcal{C}}(G, H)$ which has the property that we have natural isomorphisms

$$\mathbf{PSh}_{\mathcal{C}}(F, \underline{\mathbf{PSh}}_{\mathcal{C}}(G, H)) \cong \mathbf{PSh}_{\mathcal{C}}(F \times G, H) \tag{13}$$

If we assume that $\underline{\mathbf{PSh}}_{\mathcal{C}}(G, H)$ exists and evaluate (13) for $F = h^Z$ a representable functor, then we obtain by the Yoneda Lemma 2.8

$$\underline{\mathbf{PSh}}_{\mathcal{C}}(G, H)(Z) \cong \mathbf{PSh}_{\mathcal{C}}(h^Z, \underline{\mathbf{PSh}}_{\mathcal{C}}(G, H)) \cong \mathbf{PSh}_{\mathcal{C}}(h^Z \times G, H).$$

We now take this as a definition of the presheaf $\underline{\mathbf{PSh}}_{\mathcal{C}}(G, H)$, i.e., we set

$$\underline{\mathbf{PSh}}_{\mathcal{C}}(G, H)(Z) := \mathbf{PSh}_{\mathcal{C}}(h^Z \times G, H).$$

This is a presheaf if we set

$$\underline{\mathbf{PSh}}_{\mathcal{C}}(G, H)(Z \xrightarrow{f} W) \colon \mathbf{PSh}_{\mathcal{C}}(h^W \times G, H) \to \mathbf{PSh}_{\mathcal{C}}(h^Z \times G, H), \quad \alpha \mapsto \alpha \circ (f_* \times \mathrm{id}_G)\,. \qquad ■$$

We have thus shown the following proposition:

Proposition 2.21. *For each category $\mathcal{C}$ the category $\mathbf{PSh}_{\mathcal{C}}$ is cartesian closed.*

From the perspective of sheaf theory, the objects one is interested in most of the time are the representable presheaves. However, many presheaves that are of significant importance are not representable, as the preceding examples and propositions are supposed to illustrate. One can nevertheless take the presheaves there as a "first approximation" to an object that one would like to consider (for instance the "manifold" of all smooth functions $C^\infty(M, N)$ between manifolds M, N), and then try to enhance the properties of these presheaves as far as it gets to a representable functor.

The first step in this enhancement this is to check whether one actually has a sheaf (rather than a presheaf), since representable presheaves are (almost[6]) always so. This is what we now turn to

As we have seen in the introductory remarks, not all of the above examples really qualify as a local notion, since functions are not bounded (respectively constant) if they are locally bounded

[6] At least in the cases that we will treat in this lecture, representable functors will always be sheaves, cf. Proposition 12.2.

(respectively locally constant). The following notions subsume that in a sheaf we know sections if (and only if) we know them locally.

As above, we fist do the case of sheaves on topological spaces. We will follow throughout the conventions that multiple indices on sets of a covering denotes multiple intersections, i.e.,

$$U_{i_0\dots i_m} := U_{i_0} \cap \dots \cap U_{i_m}.$$

Definition 2.22. Suppose that F is a presheaf (in **Set**) on the topological space X. Then F is a *sheaf* if the following condition holds:

Let $U \subseteq X$ be open and $(U_i)_{i\in I}$ be an open cover of U. If $f_i \in F(U_i)$ are sections such that $f_i\|_{U_{ij}} = f_j\|_{U_{ij}}$ for each $i, j \in I$, then there exists a *unique* $f \in F(U)$ such that $f\|_{U_i} = f_i$.

A morphism of sheaves is a morphism of presheaves, i.e., presheaves for a full subcategory of the categories of sheaves (this is just a fancy word for saying that sheaves are presheaves with a certain additional property). This defines the category $\mathbf{Sh}(X)$ of sheaves on X. ■

Remark 2.23. Suppose X is a topological space, $U \subseteq X$ is open and $(U_i)_{i\in I}$ is an open cover of U. If F is a presheaf in abelian groups on X, then we have for each $i \in I$ a morphism $F(U) \to F(U_i)$ of abelian groups and thus (by the universal property of the product) a morphism

$$F(U) \to \prod_{i\in I} F(U_i).$$

Now for each pair of elements $j, k \in I$, we have two morphisms

$$\prod_{i\in I} F(U_i) \xrightarrow{\mathrm{pr}_j} F(U_j) \xrightarrow{\rho_{U_{jk},U_j}} F(U_{jk}) \quad \text{and} \quad \prod_{i\in I} F(U_i) \xrightarrow{\mathrm{pr}_k} F(U_k) \xrightarrow{\rho_{U_{jk},U_k}} F(U_{jk}) \quad .$$

This results in in two morphisms

$$\prod_{i\in I} F(U_i) \xrightarrow{p} \prod_{j,k\in I} F(U_{jk}) \quad \text{and} \quad \prod_{i\in I} F(U_i) \xrightarrow{q} \prod_{j,k\in I} F(U_{jk})$$

If now F is a presheaf in abelian groups, then F is a sheaf if and only if the sequence

$$0 \to F(U) \to \prod_{i\in I} F(U_i) \xrightarrow{p-q} \prod_{j,k\in I} F(U_{jk}) \tag{14}$$

is exact. The notion of exactness also makes sense for presheaves in **Set**, in this case one requires that $i\colon F(U) \to \prod_{i\in I} F(U_i)$ is the equaliser of

$$\prod_{i\in I} F(U_i) \xrightarrow{p} \prod_{j,k\in I} F(U_{jk}) \quad \text{and} \quad \prod_{i\in I} F(U_i) \xrightarrow{q} \prod_{j,k\in I} F(U_{jk}).$$ ■

Remark 2.24. The condition on a presheaf F on X for being a sheaf may be phrased by saying:

Local sections glue uniquely to global sections if the gluing condition $f_i\|_{U_{ij}} = f_j\|_{U_{ij}}$ is satisfied.

There exists two weakened versions of this. One is by demanding that glueing is perhaps not always possible, but if it is, it is unique:

A presheaf F on X is *separated* (or a *monopresheaf*) if for each open $U \subseteq X$, open cover $(U_i)_{i\in I}$ of U and $f_i \in F(U_i)$ there exists at most one $f \in F(U)$ such that $f|_{U_i} = f$.

The other is by demanding that gluing is always possible, but perhaps not unique:

A presheaf F on X is an *epipresheaf* if for each open $U \subseteq X$, open cover $(U_i)_{i\in I}$ of U and $f_i \in F(U_i)$ such that $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ there exists at least one $f \in F(U)$ such that $f|_{U_i} = f_i$.

Clearly, a sheaf is a presheaf that is a separated epipresheaf. ■

We now go through Example 2.3 and look whose of the above conditions are satisfied:

Example 2.25. a) $\mathcal{O}_X^Y$ is a sheaf. It is a monopresheaf since a function is uniquely determined by its restrictions to an open cover. Moreover, if $(U_i)_{i\in I}$ is an open cover of $U \subseteq X$ open, and $f_i \in C(U_i, Y)$ satisfy $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, then we may define

$$f\colon U \to Y, \quad x \mapsto f_i(x) \text{ if } x \in U_i.$$

Since $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ we have that $f_i(x) = f_j(x)$ for each $x \in U_{ij}$ and f is well-defined. Moreover, f is continuous since on U_i it coincides with the continuous function f_i and is thus continuous on a neighbourhood of each point of U.

b) $\mathcal{B}_X$ is a separated presheaf that does not satisfy gluing. For instance, take $X = \mathbb{R}_{\geq 0}$, $I = \mathbb{N}^+$ and $U_i = [0, n)$. Then $f_i(x) = x$ is bounded on each U_i, but the function $f(x) = x$ is not bounded on $\mathbb{R}_{\geq 0}$.

c) The presheaf $\mathcal{O}_M$ is actually a sheaf, for the same reason as for $\mathcal{O}_X$ above.

d) A quick and direct check shows that the skyscraper presheaf $\mathcal{S}_{x_0}^A$ is a sheaf.

e) The constant presheaf is not a sheaf. Take for instance $X = \mathbb{R} \setminus \{0\}$, $U_1 = \mathbb{R}_{<0}$ and $U_2 = \mathbb{R}_{>0}$ and $f_1 = 0$, $f_2 = 1$. Then there exists no constant $f\colon X \to \mathbb{R}$ with $f|_{U_1} = f_1$ and $f|_{U_2} = f_2$. However, the locally constant presheaf is a sheaf, since it is a special case of a). ■

If we want to generalise the notion of a sheaf to presheaves on arbitrary categories, then we first have to give the intersection $U \cap V$ of open subsets $U, V \subseteq X$ a meaning. The most important feature of $U \cap V$ is that is comes along with natural morphisms $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ that were used in Definition 2.22 and Remark 2.23. Recalling that $U \cap V \cong U \times_X V$, it is now natural to replace intersections by pull-backs and simply demanding that the necessary pull-backs exist.

Definition 2.26. If $\mathcal{C}$ is a category and C is an object of $\mathcal{C}$, then a C-*family* (of morphisms in $\mathcal{C}$) is a family of morphisms $\{f_i\colon D_i \to C \mid i \in I\}$ with codomain C. A *coverage* on $\mathcal{C}$ is a function K that assigns to each object C of $\mathcal{C}$ a collection $K(C)$ of C-families of morphisms, called *covers* of C, such that

If $\{f_i\colon D_i \to C \mid i \in I\} \in K(C)$, then for each $i \in I$ and each morphism $g\colon X \to C$ in $\mathcal{C}$ the pullbacks $X \times_C D_i$ exist and $\{X \times_C D_i \to X \mid i \in I\} \in K(X)$.

The pair $(\mathcal{C}, K)$ is also called *site*. ■

This now allows for the following natural generalisation of Definition 2.22:

Definition 2.27. Let $(\mathcal{C}, K)$ be a site and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on $\mathcal{C}$. If $R = \{f_i\colon D_i \to C \mid i \in I\}$ is a cover in $K(C)$, then a *matching family* (of F with respect to R) is an element

$$(x_i)_{i\in I} \in \prod_{i\in I} F(D_i) \quad \text{such that} \quad F(\pi_{ij})(x_i) = F(\rho_{ij})(x_j) \text{ for all } i, j \in I \tag{15}$$

(shortly denoted by (x_i)) where $\pi_{ij}\colon D_i \times_C D_j \to D_i$ and $\rho_{ij}\colon D_i \times_C D_j \to D_j$ are the projections from the pull-back

$$\begin{tikzcd} D_i \times_C D_j \arrow[r, "\rho_{ij}"] \arrow[d, "\pi_{ij}"'] & D_j \arrow[d, "f_j"] \\ D_i \arrow[r, "f_i"] & C \end{tikzcd} .$$

An *amalgamation* for a matching family (x_i) is an object $x \in F(C)$ such that $F(f_i)(x) = x_i$ for all i. Finally, F is a *sheaf* (with respect to K) if each matching family has a unique amalgamation.

A morphism of sheaves is a morphism of presheaves, i.e., sheaves form a full subcategory of the categories of presheaves. This defines the category $\mathbf{Sh}_{(\mathcal{C},K)}$ of sheaves on $(\mathcal{C}, K)$. If K is understood, then we omit it from the notation. ■

Remark 2.28.
a) As in Remark 2.24 we say that a general presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is *separated* (or a monopresheaf) if amalgamations are unique (whenever they exist) and that F is an *epipresheaf* if amalgamations always exist (but are possibly not unique).

b) Note that the property on F being a sheaf can also be rephrased as follows: for each cover $\{f_i\colon D_i \to C \mid i \in I\}$ of an object of C, the morphism

$$F(C) \xrightarrow{e} \prod_{i\in I} F(D_i), \quad x \mapsto F(f_i)(x)$$

is the equaliser of the morphisms

$$\prod_{i\in I} F(D_i) \xrightarrow{p} \prod_{i,j\in I} F(D_i \times_C D_j), \quad p((x_i)_{i\in I})_{ij} = F(\pi_{ij})(x_i)$$

and

$$\prod_{i\in I} F(D_i) \xrightarrow{q} \prod_{i,j\in I} F(D_i \times_C D_j), \quad q((x_i)_{i\in I})_{ij} = F(\rho_{ij})(x_j).$$ ■

Example 2.29.
a) If X is a topological space, then we obtain a coverage of $\mathbf{Open}_X$ if we let covers $\{f_i\colon U_i \to U \mid i \in I\}$ be all families of morphisms (aka collection of open subsets of U) for which $U = \cup_{i\in I} U_i$. With this we obtain to notion of sheaf from 2.22.

b) We also obtain a coverage of $\mathcal{C} = \mathbf{Top}$ if we define $\{f_i\colon U_i \to X \mid i \in I\}$ to be a cover of a topological space X if each U_i is an open subset of X, f_i is the inclusion $U_i \hookrightarrow X$ and if $X = \cup_{i\in I} U_i$. This is referred to as the *open cover* coverage of $\mathbf{Top}$. This also defines a coverage on the subcategory $\mathbf{Top_{Haus}}$ of topological Hausdorff spaces. All the representable functors h^X are sheaves with respect to this coverage. In fact, if h^Z, $X \mapsto C(X, Z)$ is representable and $(U_i)_{i\in I}$ is an open cover of X, then $(f_i) \in \prod_{i\in I} C(U_i, Z)$ is a matching family if and only if $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ for all $i, i \in I$. Thus there exists a unique map $f\colon X \to Z$ such that $f|_{U_i} = f_i$, which is continuous since f_i is so.

c) On **Top** and **Top$_{\mathbf{Haus}}$** we also have the following coverage: the covers of C are the singletons $\{f\colon D \to C\}$, where f is surjective and admits local sections through each point, i.e., for each $d \in D$ there exists an open neighbourhood V of $f(d)$ and a continuous map $\sigma\colon V \to D$ such that $\sigma(f(d)) = d$ and $f \circ \sigma = \mathrm{id}_V$. Indeed, if $f\colon D \to C$ admits local sections and $g\colon X \to C$ is continuous, then the pull-back

$$X \times_C D = \{(x,d) \in X \times D \mid g(x) = f(d)\}$$

also admits local sections: if $(x,d) \in X \times_C D$, V is an open neighbourhood of $f(d)$ and $\sigma\colon V \to D$ satsifies $f \circ \sigma = \mathrm{id}_V$, then $g^{-1}(V)$ is an open neighbourhood of x (since $g(x) = f(d)$) and $x \mapsto (x, \sigma(g(x)))$ is a local section of the projection $X \times_C D \to X$ (since $g(x) = f(\sigma(g(x))) = \gamma(x)$). We call this coverage the *local section* coverage. ■

The following two sites will be of big importance to us, thus we give them a separate definition.

Lemma 2.30. *On* **Man** *we obtain a coverage if we set*

$$K(M) := \bigcup_{N \in \mathrm{Ob}(\mathbf{Man})} \{f\colon N \to M \mid f \text{ is surjective submersion}\}. \tag{16}$$

Proof. If $g\colon N \to P$ is a surjective submersion and $f\colon M \to P$ is arbitrary, then for each $m \in M$ there exists $n \in N$ with $f(m) = g(n)$, and thus $\mathrm{pr}_1\colon M \times_P N \to M$ is again surjective and a submersion by Proposition C.8. ■

Definition 2.31. We call the coverage on **Man** given by (16) the surjective submersion coverage. ■

Definition 2.32. Let **Euc** be the category whose objects are all open subsets of all $\mathbb{R}^n$ and $\mathbf{Euc}(U,V) = C^\infty(U,V)$ with the usual composition of smooth maps. A cover of an object U of **Euc** is given by an open cover in the usual sense, i.e., $\{f_i\colon U_i \to U \mid i \in I\}$ is a cover if $U_i \subseteq U$ are open, f_i are the inclusion maps and $U = \cup_{i \in I} U_i$. ■

It is remarkable that each presheaf may be turned into a sheaf by a more or less canonical procedure. We will not treat this procedure here, but only state the corresponding result.

Theorem 2.33. *The inclusion functor* $i\colon \mathbf{Sh}_{(\mathcal{C},K)} \to \mathbf{PSh}_{\mathcal{C}}$ *has a left adjoint*

$$a\colon \mathbf{PSh}_{\mathcal{C}} \to \mathbf{Sh}_{(\mathcal{C},K)},$$

called the sheafification *functor, and the composition* $a \circ i$ *is naturally isomorphic to* $\mathrm{id}_{\mathbf{Sh}_{(\mathcal{C},K)}}$. *Moreover,* $\mathbf{Sh}_{(\mathcal{C},K)}$ *has arbitrary (small) limits and a preserves finite limits.*

Proof. This is [MLM94, Proposition III.4.4, Theorem III.5.1 and Corollary III.5.6]. ■

Likewise, we only cite the following fact on the cartesian closedness of $\mathbf{Sh}_{(\mathcal{C},K)}$. We will give a more elementary proof for the site **Euc** later on.

Proposition 2.34. *Let* $(\mathcal{C},K)$ *be a site and* $F,P \in \mathbf{PSh}_{\mathcal{C}}$ *be presheaves. If* F *is a sheaf, then so is the exponential* $\underline{\mathbf{PSh}}_{\mathcal{C}}(F,P)$. *In particular,* $\mathbf{Sh}_{\mathcal{C},K}$ *is cartesian closed.*

Proof. The first assertion is [MLM94, Proposition III.6.1]. Since $\mathbf{Sh}_{(\mathcal{C},K)}$ is in particular closed under finite products, we have

$$\mathbf{Sh}_{(\mathcal{C},K)}(G, \underline{\mathbf{PSh}}_{\mathcal{C}}(F,P)) = \mathbf{PSh}_{\mathcal{C}}(G, \underline{\mathbf{PSh}}_{\mathcal{C}}(F,P)) \cong \mathbf{PSh}_{\mathcal{C}}(G \times F, P) = \mathbf{Sh}_{(\mathcal{C},K)}(G \times F, P)$$

and thus that $\underline{\mathbf{PSh}}_{\mathcal{C}}(F,P)$ is also an exponential in $\mathbf{Sh}_{(\mathcal{C},K)}$. ■

Exercises for Section 2

Exercise 2.35. If $\mathcal{C}$ and $\mathcal{D}$ are categories and $\mathcal{C}$ is small, show that functors $F, G\colon \mathcal{C} \to \mathcal{D}$, together with natural transformations $\alpha\colon F \Rightarrow G$ form a category. It is part of the exercise to determine the composition and identity morphism. We denote this functor category by $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ or simply $\mathcal{D}^{\mathcal{C}}$. ■

Exercise 2.36. Show that if $F\colon \mathcal{C} \to \mathbf{Set}$ is representable, then the object of $\mathcal{C}$ that represents F is uniquely determined up to isomorphism. ■

Exercise 2.37. a) Show that

$$\mathbf{Set}(X \times Y, Z) \to \mathbf{Set}(X, \mathbf{Set}(Y, Z)), \quad f \mapsto \widehat{f} \quad \text{with} \quad \widehat{f}(x)(y) := f(x, y)$$

is in fact a natural bijection (natural in the sense that is gives an adjunction $(\,\cdot \times Y) \dashv \mathbf{Set}(Y, \,\cdot\,)$ for each fixed Y).

b) Suppose that $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are small categories. Show that we have natural isomorphism of categories

$$\mathbf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathbf{Fun}(\mathcal{C}, \mathbf{Fun}(\mathcal{D}, \mathcal{E})). \tag{17}$$

Conclude from this that (8) indeed defines indeed a functor $J \to \mathbf{Set}$. ■

Exercise 2.38. Let $\prod_{\mathbb{N}} S^1$ be the product of $\mathbb{N}$ copies of S^1 in **Top**, i.e., the cartesian product endowed with the product topology. Show that $(1, 1, ...)$ (or equivalently each point) does not have an open neighbourhood which is homeomorphic to an open subset of a lcs. ■

Exercise 2.39. Show that open covers (as in Example 2.29 b) and Definition 2.32) yield indeed a coverage on **Top** and **Euc**. Moreover, show that these sites are subcanonical. ■

Exercise 2.40. Show that a sequence $A \to B \to C$ of abelian group, such that the composition $A \to C$ is 0, is short exact if and only if the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

is cartesian and cocartesian. ■

Exercise 2.41. More generally than in Definition 2.26 one defines a coverage to be a function K that assigns to each object C of $\mathcal{C}$ a collection $K(C)$ of C-families of morphisms, called *covers* of C, such that

> If $\{f_i\colon D_i \to C \mid i \in I\} \in K(C)$ and $\varphi\colon X \to C$ is any morphism, then there exists a cover $\{g_j\colon Y_j \to X \mid j \in J\}$ of X such that each for each $j \in J$ there exists some $i \in I$ and $\psi\colon Y_j \to D_i$ such that $\varphi \circ g_j = f_i \circ \psi$.

A matching family for a presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ with respect to $R \in K(C)$ is then an element $(x_i) \in \prod_{i \in I} F(D_i)$ such that for each pair $\pi\colon E \to D_i$ and $\rho\colon E \to D_j$ satisfying $f_i \circ \pi = f_j \circ \rho$ we have $F(\pi)(x_i) = F(\rho)(x_j)$. With this notion of matching family, F is defined to be a sheaf if for each object C and each cover $R \in K(C)$ each matching family has a unique amalgamation.

a) Let K be a coverage of $\mathcal{C}$ in the sense of Definition 2.26. For each object C and $R = \{f_i\colon D_i \to C \mid i \in I\}$ in $K(C)$ set

$$\overline{R} := \{f\colon D \to C \mid \text{there exists } i \in I \text{ and } \varphi\colon D \to D_i \text{ such that } f = f_i \circ \varphi\}$$

Then show that $\overline{K}(C) := \{\overline{R} \mid R \in K(C)\}$ defines a coverage in the above sense.

b) Show that $F \in \mathbf{PSh}_{\mathcal{C}}$ is a sheaf (in the sense of Definition 2.27) with respect to K if and only if it is a sheaf (in the sense of the above definition) with respect to $\overline{K}$. ■

3 Diffeological Spaces

This section introduces the category **Diff** of diffeological spaces and discusses their basic properties. In particular, it contains

- the definition of diffeological spaces and their morphisms,
- the natural functor **Man** $\to$ **Diff** (and shows that it is fully faithful on finite-dimensional manifolds) and
- the (co)completeness and cartesian closedness of of **Diff**

We now introduce a type of sheaf that will help us in understanding infinite-dimensional manifolds. The basic idea is to generalise manifolds by a concept of space that only has smooth maps *into* it[7]. Recall the site **Euc** from Definition 2.32. A good reference for this section is [IZ13].

Definition 3.1. a) If $G\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf, then a *subpresheaf* is a presheaf F on $\mathcal{C}$ such that $F(C) \subseteq G(C)$ for each object C of $\mathcal{C}$ and $F(f) = G(f)|_{F(C)}$ for each morphism $f\colon C \to D$ of $\mathcal{C}$. We shortly denote this by $F \subseteq G$. If G is a sheaf, then F is called *subsheaf* F is itself a sheaf.

b) A *diffeolocigal space* is a pair (D, X) consisting of a set X and a sheaf $D\colon \mathbf{Euc}^{\mathrm{op}} \to \mathbf{Set}$ such that $D(*) = X$ (where $* := \mathbb{R}^0$) and D is a subsheaf of $U \mapsto \mathbf{Set}(U, X)$. The elements $\varphi\colon U \to X$ of $D(U) \subseteq \mathbf{Set}(U, X)$ are called *plots* (from U into X). We then also say that D is a *diffeology* on X and simply denote the diffeological space by X if the diffeology is understood.

c) A morphism of diffeological spaces is a morphism of the underlying presheaves. This defines the category **Diff** of diffeological spaces. ■

Remark 3.2. a) A diffeological space is a set X together with a sheaf of smooth functions from open subsets of all $\mathbb{R}^n$'s into it. Although this looks very similar to the notion of a topological space, endowed with a structure sheaf (cf. Remark 2.4), this differs significantly from it (mainly because the structure maps are *into* the space and not *out of* it, see [Sta10] for a comparison of the different possible concepts of smooth spaces that arise this way).

[7] One can also study a space with a notion of smooth maps *out of* it, see [Sta10] for a comparison of the different possible concepts of smooth spaces that arise this way

b) Note that constant maps (aka maps $* \to X$) are plots by requiring $D(*) = X$. Moreover, if $\varphi\colon U \to X$ is locally constant and $(U_i)_{i\in I}$ is a cover of U such that $\varphi|_{U_i}$ is constant, then $\varphi|_{U_i}$ is a plot of D and thus is φ by the sheaf property of D.

c) Be aware that some authors call morphisms of diffeological spaces smooth maps, while we reserve the term *smooth map* for a morphisms of manifolds. It will turn out in Section 4 that for locally metrisable manifolds these notion coincide anyway.

d) A morphism $D \to D'$ of diffeological spaces gives rise to a map $f\colon D(*) = X \to D'(*) = X'$. If $\alpha_u\colon * \to U$ represents an element $u \in U$, then we have that $D(U) \to D(*)$ is given by restriction of the map $\mathbf{Set}(U,X) \to \mathbf{Set}(*,X) \cong X$, and thus given by $\varphi \mapsto \varphi(u)$. Since

$$\begin{tikzcd} D(U) \arrow[r, "D(\alpha_u)"] \arrow[d] & D(*) \arrow[d, "f"] \\ D'(U) \arrow[r, "D'(\alpha_u)"] & D'(*) \end{tikzcd}$$

commutes we thus have that the morphism $D(U) \to D'(U)$ is given by $\varphi \mapsto \varphi \circ f$. Thus $D \to D'$ is determined entirely by $f\colon X \to X'$. We will thus identify a morphism of diffeological spaces $f\colon (X,D) \to (X',D')$ with the associated map $f\colon X \to X'$. ■

Example 3.3. a) Each set X has two canonical diffeologies. Let $X^\delta(U) := \{\varphi\colon U \to X \mid \varphi \text{ is locally constant}\}$. This clearly has $X^\delta(*) \cong X$ (canonically). Moreover, $f_i\colon U_i \to X$ locally constant with $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ give rise to a unique and locally constant $f\colon U = \cup_{i\in I} U_i \to X$. Thus X^δ is a sheaf. Observe that $f\colon X \to Y$ induces naturally a morphism $f\colon X^\delta \to Y^\delta$ and thus $X \mapsto X^\delta$ is a functor $\delta\colon \mathbf{Set} \to \mathbf{Diff}$. Moreover, each map $X \to D(*)$ maps locally constant maps under postcomposition to locally constant maps and since locally constant maps are plots we have $\mathbf{Diff}(X^\delta, D) = \mathbf{Set}(X, D(*))$. Thus $\delta \dashv F$ for $F\colon \mathbf{Diff} \to \mathbf{Set}$, $D \mapsto D(*)$ (the forgetful functor).

On the other hand $D^c(U) := \mathbf{Set}(U,X)$ clearly is also a diffeology on X and each map $X \to Y$ gives rise to a morphism $X^c \to Y^c$. Moreover, each map $D(*) \to X$ is a morphism of diffeological spaces $D \to X^c$. Thus $\mathbf{Diff}(D, X^c) = \mathbf{Set}(D(*), X)$ and $F \dashv c$ (for $c\colon \mathbf{Set} \to \mathbf{Diff}$, $X \mapsto X^c$ and F as above).

b) Each manifold M gives rise to a diffeology D_M on the set underlying M, defined by $D_M(U) := C^\infty(U,M)$. This clearly is a sheaf, called the *canonical diffeology* on M. Moreover, a smooth map $f\colon M \to N$ induces a morphism $D_M \to D_N$ of diffeological spaces, since composites of smooth maps are again smooth. We thus obtain a functor $\mathbf{Man} \to \mathbf{Diff}$. We will show in Section 4 that this is actually an embedding (i.e., full and faithful) if we restrict to manifolds modelled on metrisable spaces.

c) If X is a diffeological space and $A \subseteq X$ is a subspace, then A inherits a diffeology $D|_A$ form the one on X by setting

$$D|_A(U) := \{\varphi \in D(U) \mid \varphi(U) \subseteq A\}.$$

It is clear that $D|_A(*) = A$ and that $D|_A$ is a sheaf since gluing maps with values in A will result in a map with values in A.

d) If $(X_i, D_i)_{i\in I}$ is a family of diffeological space, then we obtain a diffeology on $\prod_{i\in I} X_i$ be setting $(\prod_{i\in I} D_i)(U) := \prod_{i\in I} D_i(U)$. It is clear that $\prod_{i\in I} D_i(*) = \prod_{i\in I} X_i$ (with $X_i = D_i(*)$) and that $\prod_{i\in I} D_i(U) \subseteq \prod_{i\in I} \mathbf{Set}(U, X_i)$ since these equalities hold for each $i \in I$ separately. Moreover, $\prod_{i\in I} D_i$ is in fact a sheaf, since a matching family for a cover $(U_k)_{k\in K}$ of $\prod_{i\in I} D_i$ consists of a tuple $\varphi_{i,k} \in D_i(U_k)$ such that $\varphi_{i,k}|_{U_{ij}} = \varphi_{j,k}|_{U_{ij}}$ for all $i, j \in I$ and $k \in K$. Thus there exists for each $k \in K$ a unique $\varphi_k \in D_k(U)$ with $\varphi_k|_{U_i} = \varphi_{i,k}$. Thus $\prod_{i\in I} D_i$ is a sheaf. Combining this with the construction of limits in **Set** as in Proposition 2.12 and the subspace diffeology one sees that **Diff** has arbitrary (small) limits.

e) A very similar argument shows that **Diff** also has arbitrary (small) colimits, cf. Exercise 3.9.■

We will not treat diffeological spaces as spaces of interest per se, but more as a tool for studying manifolds in a very convenient way.

Definition 3.4. If Y, Z are diffeological spaces, then we define

$$\underline{\mathbf{Diff}}(Y,Z)(U) := \{\varphi\colon U \to \mathbf{Diff}(Y,Z) \mid \check{\varphi} \in \mathbf{Diff}(D_U \times Y, Z)\}, \tag{18}$$

where we have used the natural bijection $\mathbf{Set}(U, \mathbf{Set}(Y,Z)) \to \mathbf{Set}(U \times Y, Z)$, $f \mapsto \check{f}$ with $\check{f}(u,y) := f(u)(y)$. ■

Proposition 3.5. *Let X, Y, Z be diffeological spaces. Then (18) defines a diffeology on $\mathbf{Diff}(Y,Z)$ and we have natural isomorphisms*

$$\mathbf{Diff}(X \times Y, Z) \cong \mathbf{Diff}(X, \underline{\mathbf{Diff}}(Y,Z)), \tag{19}$$

given by the identity map on the underlying set. In particular, **Diff** *is cartesian closed.*

Proof. Since we have omitted the proof for the general fact that exponentials of sheaves are sheaves we will give it here in this more basic case. Let D, E, F be the diffeologies on X, Y, Z respectively, and let us first observe that by (18) we have $\underline{\mathbf{Diff}}(Y,Z)(*) \cong \mathbf{Diff}(Y,Z)$ canonically. Moreover, we have

$$\begin{aligned}
&\varphi\colon U \to \mathbf{Diff}(Y,Z) \in \underline{\mathbf{Diff}}(Y,Z)(U)\\
\Leftrightarrow\ &(u,v) \mapsto \varphi(\xi(u))(\psi(v)) \in F(U' \times V) \text{ for all } \psi\colon V \to Y \in E(V) \text{ and all } \xi \in D_U(U') = C^\infty(U',U)\\
\Leftrightarrow\ &(u,v) \mapsto \varphi(u)(\psi(v)) \in F(U \times V) \text{ for all } \psi\colon V \to Y \in E(V),
\end{aligned} \tag{20}$$

where the first equivalence is simply the definition and the second follows from the presheaf property of F, applied to $\xi \times \mathrm{id}_V$ for each $\xi \in C^\infty(U',U)$.

To check that (18) actually defines a sheaf, let

$$(\varphi_i) \in \prod_{i\in I} \mathbf{Diff}(D_{U_i} \times Y, Z) \subseteq \prod_{i\in I} \mathbf{Set}(U_i \times Y, Z)$$

be a matching family, i.e., satisfy $\varphi_i|_{U_{ij}\times Y} = \varphi_j|_{U_{ij}\times Y}$ for all $i, j \in I$. Then there exists a unique $\varphi \in \mathbf{Set}(U \times Y, Z)$ such that $\varphi|_{U_i\times Y} = \varphi_i$ for all $i \in I$. In order to verify (20) for φ, let $\psi\colon V \to Y$ be in $D(V)$. Since $\varphi_i \in \underline{\mathbf{Diff}}(Y,Z)(U_i)$, we have for each $i \in I$ that $U_i \times V \ni (u,v) \mapsto \varphi(u, \psi(v)) \in Z$ is an element of $F(U_i \times V)$, again by (20). Thus $U \times V \ni (u,v) \mapsto \varphi(u,\psi(v)) \in Z$ is in $F(U \times V)$ by the sheaf property of F, applied to the open cover $(U_i \times V)_{i\in I}$ of $U \times V$. Thus $\varphi \in \underline{\mathbf{Diff}}(Y,Z)(U)$, and it is an amalgamation of (φ_i).

We now claim that a map $f\colon X \times Y \to Z$ is a morphism in **Diff** if and only if the map $\widehat{f}\colon X \to \mathbf{Diff}(Y,Z)$ is a morphism in **Diff**. In fact, we have

$$\begin{aligned} f\colon X\times Y\to Z\in \mathbf{Diff}(X\times Y,Y) &\Leftrightarrow f\circ(\varphi\times\psi)\in F(U\times V) \text{ for all } \varphi\in D(U),\psi\in E(V)\\ &\Leftrightarrow \hat{f}\circ\varphi\in \underline{\mathbf{Diff}}(Y,Z)(U) \text{ for all } \varphi\in D(U)\\ &\Leftrightarrow \hat{f}\in \mathbf{Diff}(X,\underline{\mathbf{Diff}}(Y,Z)). \end{aligned}$$

Thus the isomorphisms in (19) are in fact the identity maps, which are of course natural. This finishes the proof. ∎

Corollary 3.6. *Let Y, Y', Z, Z' be diffeological spaces and $f\colon Y' \to Y$, $g\colon Z \to Z'$ be morphisms in* **Diff**. *Then*

$$g_* f^*\colon \underline{\mathbf{Diff}}(Y,Z)\to\underline{\mathbf{Diff}}(Y',Z'),\quad \varphi\mapsto g\circ\varphi\circ f$$

is a morphism in **Diff**.

Proof. This is left as Exercise 3.8. ∎

Exercises for Section 3

Exercise 3.7. Show that the restriction of the functor $\mathbf{Man}\to\mathbf{Diff}$, $M\mapsto D_M$ to the category $\mathbf{Man}_{\mathbf{fin}}$ of finite-dimensional manifolds is fully faithful. ∎

Exercise 3.8. Let Y, Y', Z, Z' be diffeological spaces and $f\colon Y' \to Y$, $g\colon Z \to Z'$ be morphisms in **Diff**. Show that

$$\underline{\mathbf{Diff}}(Y,Z)\to\underline{\mathbf{Diff}}(Y',Z'),\quad \varphi\mapsto g\circ\varphi\circ f$$

is a morphism in **Diff**. ∎

Exercise 3.9. Show that the category **Diff** of diffeological spaces has arbitrary (small) colimits. ∎

4 Comparison of Diffeological Spaces and Manifolds

This section contains the first main result of the lecture: the functor $\mathbf{Man}\to\mathbf{Diff}$ is fully faithful on locally metrisable manifolds.

We now turn towards the first main result. It is a powerful tool for constructing smooth maps between certain infinite-dimensional manifolds. In a first (an less general) approximation it says that a map between certain infinite-dimensional manifolds is smooth it if maps smooth curves (into the space) to smooth curves.

Remark 4.1. In Appendix A the notions of differentiability of functions $f\colon U \Subset X \to Y$ for X, Y locally convex spaces (lcs) have been laid out. In particular, we have that a curve $\gamma\colon I \Subset \mathbb{R} \to Y$ is differentiable (in the sense of Definition A.9) if the ordinary difference quotient

$$\gamma'(t) := \frac{\partial}{\partial t}\gamma(t) := \lim_{s\to 0}\frac{1}{s}(\gamma(t+s)-\gamma(t)) = d\gamma(x,1)$$

exists for all $t \in I$, since

$$d\gamma(x,v) = \lim_{s\to 0} \frac{1}{s}(\gamma(t+sv)-\gamma(t)) = \lim_{\frac{s}{v}\to 0} \frac{v}{s}(\gamma(t+s)-\gamma(t)) = v\gamma'(t). \tag{21}$$

If γ is differentiable, then it is also continuous, since then $(t_n) \to p$ implies

$$\lim_{n\to\infty} \gamma(t_n) = \lim_{n\to\infty} \frac{\gamma(t_n)-\gamma(p)}{t_n - p}(t_n - p) + \gamma(p) = \gamma'(t)0 + \gamma(p) = \gamma(p)$$

by the continuity of the scalar multiplication in Y. Moreover, (21) shows that if γ' is continuous, then so is $d\gamma$ and thus γ is then a C^1-map. Likewise, γ is smooth if and only if $\gamma^{[k]} := (\gamma^{[k-1]})'$ with $\gamma^{[0]} = \gamma$ is continuous for all k. ∎

We recall the following elementary Lemma.

Lemma 4.2. *Let X, Y be topological spaces, X be metrisable, $f\colon X \to Y$ be a map and $p \in X$ and (t_n) be a sequence in $\mathbb{R}_{>0}$ with $t_n \to 0$. If for each sequence (x_n) in X with $d(x_n, p) < t_n$ it follows that $f(x_n) \to f(p)$, then f is continuous in p.*

Proof. Let d be a metric inducing the topology on X and suppose that f is not continuous in p. Then there exists a neighbourhood $U_{f(p)}$ of $f(p)$ such that for each $\delta > 0$ there exists some $x \in X$ with $d(x,p) < \delta$ and $f(x) \notin U_{f(p)}$. In particular, we find with $\delta = t_n$ a sequence (x_n) with $d(x_n, p) < t_n$ and $f(x_n) \nrightarrow f(p)$. This contradicts the assumption. ∎

Metrisability of a lcs will be an important property in the sequel. We first recall some equivalent conditions for this.

Theorem 4.3. *For a lcs X the following are equivalent*

a) The topology of X is metrisable.

b) The topology of X is locally metrisable.

c) The topology of X is induced by an at most countable family of point-separating semi-norms.

Proof. a) $\Leftrightarrow$ b) is [Rud91, Theorem 1.24] and b) $\Leftrightarrow$ c) is [Rud91, Remark 1.38 b) and c)]. ∎

Lemma 4.4 (Special Curve Lemma). *Suppose that X is a metrisable locally convex space and that the topology on X is induced by a countable family $(p_n)_{n\in\mathbb{N}}$ of point-separating semi-norms on X that satisfy $p_{n+1} \geq p_n$ for all n. If $U \Subset X$ is convex with $0 \in U$ and (x_n) is a sequence in U such that $2^{nk} p_n(x_n) \to 0$ for each $k \in \mathbb{N}$, then there exists a smooth function $\gamma\colon \mathbb{R} \to U$ such that $\gamma(\frac{1}{2^n}) = x_n$ and $\gamma(0) = 0$.*

Proof. The idea is to take a convex combination between x_{n+1} and x_n on the interval $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$, put them together to obtain a continuous function and then to "smoothen out" the function at the points $\{\frac{1}{2_n} \mid n \in \mathbb{N}\} \cup \{0\}$ by a reparametrisation that kills all derivatives.

Let $\theta \in C^\infty(\mathbb{R},\mathbb{R})$ be such that $\theta|_{(-\infty,0]} \equiv 0$ and $\theta|_{[1,\infty)} \equiv 1$ and define

$$\gamma(t) := \begin{cases} 0 & \text{if } t \leq 0 \\ (1-\theta(2^{n+1}t-1))x_{n+1} + \theta(2^{n+1}t-1)x_n & \text{if } t \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \\ x_1 & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Then γ is continuous on $\mathbb{R}\setminus\{0\}$ since

$$((1-\theta(2^{n+1}t-1))x_{n+1}+\theta(2^{n+1}t-1)x_n)\big|_{\frac{1}{2^{n+1}}} = ((1-\theta(2^{n+2}t-1))x_{n+2}+\theta(2^{n+2}t-1)x_{n+1})\big|_{\frac{1}{2^{n+1}}}$$

for all n. Moreover, γ is also continuous in 0: Let (h_n) be a sequence with $0\leq h_n\leq \frac{1}{2^n}$ for all n. Then $h_n\in\left[\frac{1}{2^{n+m_n+1}},\frac{1}{2^{n+m_n}}\right]$ for some m_n and if we set $s_n=2^{n+m_n+1}h_n-1$, then we have

$$\begin{aligned} p_n(\gamma(h_n)) = p_n((1-\theta(s))x_{n+m_n+1}+\theta(s)x_{n+m_n}) \leq \\ p_n((1-\theta(s))x_{n+m_n+1})+p_n(\theta(s)x_{n+m_n}) \leq p_{n+m_n+1}(x_{n+m_n+1})+p_{n+m_n}(x_{n+m_n}) \xrightarrow{n\to\infty} 0 \end{aligned}$$

since $p_n\leq p_{n+m_n}\leq p_{n+m_n+1}$ and $\theta(s)\in[0,1]$. Since (h_n) was arbitrary, this implies $\lim_{h\searrow 0}\gamma(h)=0$ by Lemma 4.2 (se also Exercise 4.18 b)).

We can use a similar argument to show inductively that γ is smooth: If $k\in\mathbb{N}^+$ and γ in C^{k-1}, then γ is a C^k-map on $\mathbb{R}\setminus\{0\}$ for each $k\in\mathbb{N}_0$, since $\gamma|_{\left[\frac{1}{2^{n+1}},\frac{1}{2^n}\right]}=(1-\theta(2^{n+1}t-1))x_{n+1}+\theta(2^{n+1}t-1)x_n$ is so and

$$\begin{aligned} (2^{n+1})^k\left(-\theta^{[k]}(2^{n+1}t-1)x_{n+1}+\theta^{[k]}(2^{n+1}t-1)x_n\right)\Big|_{\frac{1}{2^{n+1}}} = 0 = \\ (2^{n+1})^k\left(-\theta^{[k]}(2^{n+2}t-1)x_{n+2}+\theta^{[k]}(2^{n+2}t-1)x_{n+1}\right)\Big|_{\frac{1}{2^{n+1}}} \end{aligned}$$

is implied by $\theta^{[k]}(0)=\theta^{[k]}(1)=0$. Thus $\gamma^{[k]}$ exists and is continuous on $\mathbb{R}\setminus\{0\}$. In order to check that γ is C^k also in 0 it suffices to check that $\gamma^{[k]}$ extends continuously to $\mathbb{R}$, since then

$$t\mapsto \gamma^{[k-1]}(t_0)+\int_{t_0}^{t}\gamma^{[k]}dt$$

defines a C^1-map coinciding with the continuous map $\gamma^{[k-1]}$ on the dense subset $\mathbb{R}\setminus\{0\}$ and thus equals $\gamma^{[k-1]}$. Let (h_n) be a sequence with $0\leq h_n\leq\frac{1}{2^n}$ for all n. Then $h_n\in\left[\frac{1}{2^{n+m_n+1}},\frac{1}{2^{n+m_n}}\right]$ for some m_n and if we set $s_n=2^{n+m_n+1}h_n-1$, then we have

$$\begin{aligned} p_n(\gamma^{[k]}(h_n)) = (2^{n+m_n+1})^k p_n(-\theta^{[k]}(s_n)x_{n+m_n+1}+\theta^{[k]}(s_n)x_{n+m}) \leq \\ (2^{m+m_n+1})^k|\theta^{[k]}(s_n)|\big(p_{n+m_n+1}(x_{n+m_n+1})+p_{n+m_n}(x_{n+m_n})\big)\to 0 \end{aligned}$$

since $p_n\leq p_{n+m_n}\leq p_{n+m_n+1}$, $\theta^{[k]}$ is bounded (since continuous on $[0,1]$ and constantly 0 everywhere else), and since

$$(2^{n+m_n+1})^k p_{n+m_n+1}(x_{n+m_n+1})\to 0 \quad\text{and}\quad (2^{n+m_n+1})^k p_{n+m_n}(x_{n+m_n})\to 0$$

follow from $(2^n)^k p_n(x_n)\to 0$. As above, this implies $\lim_{h\searrow 0}\gamma^{[k]}(h)=0$. ∎

Corollary 4.5. *Let X be a metrisable locally convex space, Y be a topological space, $U\subseteq X$ be open and $f\colon U\to Y$ be a function. If $f\circ\gamma$ is continuous for each $\gamma\in C^\infty(\mathbb{R},U)$, then f is continuous. In particular, the topology on U is the final topology for all $\gamma\in C^\infty(\mathbb{R},U)$.*

Proof. We show that for each $p\in U$ there exists an open convex neighbourhood V_p of p such that $f|_{V_p}$ is continuous in p. Without loss of generality we may assume that $p=0$, since we can translate U to $U-p$. Let V_p be open an convex with $p\in V_p$ and $V_p\subseteq U$. By Theorem 4.3 there exists a countable family $(p'_n)_{n\in\mathbb{N}}$ of point-separating semi-norms on X that induce the topology. By setting $p_n := p'_1 + ... + p'_n$ we may assume without loss of generality that $p_{n+1} \geq p_n$ for all n.

If (x_n) is a sequence in V_p with $d(x_n, 0) < \left(\frac{1}{2^n}\right)^n$, then we have for each fixed $k \in \mathbb{N}$ and $n \geq 2$ that

$$\sum_{i\in\mathbb{N}} 2^{-i} \min\{p_i(x_n), 1\} < \left(\frac{1}{2^n}\right)^n \Rightarrow \min\{p_n(x_n), 1\} < \left(\frac{1}{2^{n-1}}\right)^n \Rightarrow p_n(x_n) < \left(\frac{1}{2^{n-1}}\right)^n$$

and this implies

$$2^{kn} p_n(x_n) < \left(\frac{2^k}{2^{n-1}}\right)^n \xrightarrow{n\to\infty} 0$$

for each $k \in \mathbb{N}$. Thus there exists by Lemma 4.4 some $\gamma \in C^\infty(\mathbb{R}, V_p)$ such that $\gamma(\frac{1}{2^n}) = x_n$ and $\gamma(0) = 0$. Since $f \circ \gamma$ is continuous we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(\gamma(\frac{1}{2^n})) = f(\gamma(\lim_{n\to\infty} \frac{1}{2^n})) = f(\gamma(0)) = f(0)$$

and thus that f is continuous by Lemma 4.2. ■

Remark 4.6. Note that the assumptions of the preceding corollary are in particular satisfied if $f\colon U \Subset X \to Y$ maps either smooth curves to smooth curves or continuous curves to continuous curves. ■

Proposition 4.7. *Let X, Y be lcs, X metrisable and $f\colon U \Subset X \to Y$ be such that for each $n \in \mathbb{N}$ and $\gamma \in C^\infty(\mathbb{R}^n, U)$ the composite $f \circ \gamma$ is also smooth. Then f is smooth (where throughout smoothness refers to Definition A.9).*

Proof. We show that for each k and $(x, v_1, ..., v_k) \in U \times X^k$ the higher differentials $d^k f(x)(v_1, ..., v_k)$ exist and that $(x, v_1, ..., v_k) \mapsto d^k f(x)(v_1, ..., v_k)$ is continuous.

Consider the map $\gamma\colon V \to Y$, $\gamma(s_1, ..., s_k) = x + s_1 v_1 + ... + s_k v_k$, which is defined on some open zero neighbourhood V, diffeomorphic to $\mathbb{R}^k$. We have that $f \circ \gamma$ is smooth by assumption and thus

$$\frac{\partial}{\partial s_k} \cdots \frac{\partial}{\partial s_k} f \circ \gamma \bigg|_{s_1=...=s_k=0} = d^k f(x)(v_1, ..., v_k) \tag{22}$$

exists. We now show that $(x, v_1, ..., v_k) \mapsto d^k f(x)(v_1, ..., v_k)$ has the property that whenever $I \Subset \mathbb{R} \to U \times X^k$, $t \mapsto (x(t), v_1(t), ..., v_k(t))$ is smooth, then $t \mapsto d^k f(x(t))(v_1(t), ..., v_k(t))$ is continuous. Indeed, the map

$$(t, s_1, ..., s_k) \mapsto x(t) + s_1 v_1(t) + ... + s_k v_k(t)$$

is smooth and thus $(t, s_1, ..., s_k) \mapsto \widetilde{\gamma}(t, s_1, ..., s_k) := f(x(t) + s_1 v_1(t) + ... + s_k v_k(t))$ is smooth. Thus

$$\frac{\partial}{\partial t} d^k f(x(t))(v_1(t), ..., v_k(t)) = \frac{\partial}{\partial t} \frac{\partial}{\partial s_k} \cdots \frac{\partial}{\partial s_k} f \circ \widetilde{\gamma} \bigg|_{s_1=...=s_k=0}$$

exists for all t, implying that $t \mapsto d^k f(x(t))(v_1(t), ..., v_k(t))$ is continuous (cf. Remark 4.1). By Corollary 4.5, $d^k f$ is continuous. Since k was arbitrary, this shows the claim. ■

Remark 4.8. Proposition 4.7 is valid also for not necessarily locally convex (but still metrisable) X [BGN04, Theorem 12.4]. Moreover, a slightly more detailed analysis shows that it suffices in

Proposition 4.7 to check the case $n = 1$ [Glö04b, Proposition E.3], even if X is not necessarily locally convex (but still metrisable). This implies in particular that what is called *conveniently smooth maps* between locally convex metrisable spaces (cf. [KM97, Definition 3.11]) are smooth maps in the sense of Definition A.9. However, this is only a slightly stronger statement, since in practice it is usually not harder to show that $f \circ \gamma$ is smooth for each $\gamma \in C^\infty(\mathbb{R}^n, U)$ than just for each $\gamma \in C^\infty(\mathbb{R}, U)$. ■

Remark 4.9. Recall that a topological space X is locally metrisable if each point has a metrisable open neighbourhood. In particular, for a locally metrisable manifold the charts have as targets open subsets of metrisable lcs by Theorem 4.3. ■

The following theorem is the main result of this section.

Theorem 4.10. *Let* $\mathbf{Man}_{lm}$ *denote the category of locally metrisable manifolds. Then the functor* $\mathbf{Man}_{lm} \to \mathbf{Diff}$ *from Example 3.3 b) is fully faithful, i.e., a map* $f\colon M \to N$ *between locally metrisable manifolds* M, N *is smooth if and only if* $f \circ \gamma \in C^\infty(O, N)$ *for each* $O \Subset \mathbb{R}^n$ *and* $\gamma \in C^\infty(O, M)$*. Moreover, the last statement remains true if restricted to* $O = \mathbb{R}^n$*.*

Proof. Let M, N be locally metrisable manifolds an $f\colon M \to N$ be a map that induces a morphism of the diffeological spaces $D_M \to D_N$. We show that each coordinate representation of f is smooth under these assumptions. To this end, suppose that $\varphi\colon U \to \varphi(U) \Subset X$ and $\psi\colon V \to \psi(V) \Subset Y$ are charts of M and N (respectively) with $f(U) \subseteq V$. Then in the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \varphi(U) \arrow[r, "\widetilde{f}"] & \psi(U) \end{tikzcd}$$

we may interpret φ and ψ as morphisms between the induced diffeological spaces. Thus $\widetilde{f} = \psi \circ f \circ \varphi^{-1}$ is a morphism of diffeological spaces, which is smooth by Proposition 4.7.

The last assertion follow from applying Proposition 4.7 to what we have just shown. ■

We now turn to comparing the topologies on locally metrisable manifolds and the natural topologies induced from their diffeologies.

Definition 4.11. If (X, D) is a diffeological space, then the d-topology on X is the final topology for all plots, i.e., $O \subseteq X$ is d-open if and only if $\varphi^{-1}(O)$ is open for each plot $\varphi \in D(U)$ and each $U \Subset \mathbb{R}^n$. This is the finest topology that makes each plot continuous. ■

Lemma 4.12. *A morphism of diffeological spaces is continuous for the d-topology.*

Proof. This is left as Exercise 4.19. ■

Lemma 4.13. *If* M *is a locally metrisable manifold, then the d-topology on* M *is the topology of the underlying topological space of* M*.*

Proof. Let $(\varphi_i\colon U_i \to \varphi(U_i) \Subset X_i)_{i \in I}$ be an atlas of M. Then

$$\begin{aligned} f\colon M \to Y \text{ is continuous } &\Leftrightarrow f|_{U_i} : U_i \to Y \text{ is continuous for all } i \in I \\ &\Leftrightarrow f \circ \varphi_i^{-1} \colon \varphi_i(U_i) \to Y \text{ is continuous for all } i \in I \\ &\Leftrightarrow f \circ \varphi_i \circ \gamma \colon \mathbb{R} \to Y \text{ is continuous for all } i \in I, \gamma \in C^\infty(\mathbb{R}, \varphi_i(U_i)) \\ &\Leftrightarrow f \circ \gamma \colon \mathbb{R} \to Y \text{ is continuous for all } i \in I, \gamma \in C^\infty(\mathbb{R}, U_i), \end{aligned}$$

where the last but one equivalence follows from Corollary 4.5. Thus the topology of M is final for the set of all smooth curves that have range in the domain of some chart. Adding functions to this set makes the induced final topology finer, and does not change it if the added functions already are continuous. Since all plots of M are continuous, the topology is thus final for all plots. ■

For later reference, we record the following lemma. Note that we endow throughout the set of smooth functions $C^\infty(M,N)$ between locally metrisable manifolds with the diffeology from the identifications $C^\infty(M,N) \cong \mathbf{Diff}(D_M, D_N)$ (see Definition 3.4 and Proposition 3.5).

Lemma 4.14. *If M,N are manifolds, M is compact, N is locally metrisable and $O \subseteq N$ is open, then $C^\infty(M,O)$ is open in the d-topology on $C^\infty(M,N)$.*

Proof. The proof is left as Exercise 4.20. ■

We have seen that metrisability is an important property for us. Unfortunately, the question whether or not a space is metrisable is quite delicate and one of the fundamental questions in general topology. We will not go into detail here but only list some properties under which spaces are metrisable (this will only shorten some arguments in the sequel, the examples that we will treat often come along with natural metrics).

Theorem 4.15. *([Mun75, Theorem 6.5.1]) A topological space is metrisable if and only if it is locally metrisabe and paracompact.*

Corollary 4.16. *Each finite-dimensional paracompact manifold is metrisable. In particular, each closed submanifold of $\mathbb{R}^n$ is metrisable.*

Theorem 4.17. *([EG54]) If $\pi\colon Y \to Z$ is a locally trivial bundle with fibre X such that Z and Y are metrisable, then Y is metrisable.*

Exercises for Section 4

Exercise 4.18. Let X be a lcs such that the topology of X is induced by a countable family of semi-norms $(p_n)_{n\in\mathbb{N}}$.

a) Show that if we set $p'_n := \sum_{i\leq n} p_i$, we obtain another family of semi-norms inducing the same topology on X. We may thus w.l.o.g. assume that the family satisfies $p_n \leq p_{n+1}$.

b) Show that the following statements are equivalent conditions for a sequence (x_k) in X and $p \in X$:

   i) $(x_k) \xrightarrow{k\to\infty} p$ in the topology of X.

   ii) $(d(x_k,p)) \xrightarrow{k\to\infty} 0$, where d is the metric $d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n}\frac{p_n(x-y)}{1+p_n(x-y)}$.

   iii) $(p_n(x_k - p)) \xrightarrow{k\to\infty} 0$ for each n.

   Moreover, if the family satisfies $p_n \leq p_{n+1}$, then show that any of these conditions is implied by

   iv) $(p_k(x_k - p)) \xrightarrow{k\to\infty} 0$. ■

Exercise 4.19. Show that a morphism of diffeological spaces is continuous for the d-topology. ■

Exercise 4.20. Show that if M,N are manifolds, M is compact, N is locally metrisable and $O \subseteq N$ is open, then $C^\infty(M,O)$ is open in the d-topology on $C^\infty(M,N)$. ■

5 Spaces of Smooth Maps: The Topology

In this section we treat the topology structure on $C^\infty(M,N)$ for M a compact manifold and N. In particular, the section

- introduces the smooth compact-open topology (topology of uniform convergence of all derivatives on compact subsets) on $C^\infty(M,N)$,
- discusses the metrisability of the topology on $C^\infty(M,N)$ and
- discusses completeness issues for $C^\infty(M,Y)$.

We will now finally turn to understanding the smooth structure on spaces of smooth maps $C^\infty(M,N)$. From Proposition 3.5 we already know how to treat them as diffeological spaces and the d-topology from Definition 4.11 turns them into topological spaces. What we need to know in order to understand $C^\infty(M,N)$ as manifolds is to see that it is locally homeomorphic to open subsets of locally convex spaces.

For the main results of this section our approach only works for compact M. This applies in particular to the smooth structure on $C^\infty(M,N)$ and $\mathrm{Diff}(M)$. However, some more basic results are also valid for non-compact M and we stick to this more general setting whenever this is necessary or it is possible without major changes. The more general assertions on $C^\infty(M,N)$ and $\mathrm{Diff}(M)$ for non-compact (but still finite-dimensinoal) M are not treated, since the modelling spaces are then spaces of compactly supported maps, which are in general not metrisable.

Definition 5.1. If M,N are manifolds modelled on lcs X and Y, then we endow the set $C^\infty(M,N)$ with the initial topology with respect to

$$C^\infty(M,N) \hookrightarrow \prod_{k\in\mathbb{N}_0} C(T^kM,T^kN)_{c.o.}, \quad f\mapsto (T^kf)_{k\in\mathbb{N}_0},$$

where $C(X,Y)_{c.o.}$ denotes for two Hausdorff spaces X,Y the space of continuous functions endowed with the compact-open topology (cf. [Mun75, Section 7.5]). ■

Remark 5.2. a) The topology on $C^\infty(M,N)$ is also the initial topology for

$$T^k\colon C^\infty(M,N)\to C(T^kM,T^kN)_{c.o.}, \quad f\mapsto T^kf \quad \text{with} \quad k\in\mathbb{N}_0,$$

but it sometimes is more convenient to consider $C^\infty(M,N)$ as a subspace of $\prod_{k\in\mathbb{N}_0} C(T^kM,T^kN)_{c.o.}$.

b) Note that the topology on $C^\infty(M,N)$ is designed to make $C^\infty(M,N)\to C^\infty(T^kM,T^kN)$, $f\mapsto T^kf$ continuous. In particular, if M,N are open subsets of lcs' X,Y, then

$$C^\infty(M,N)\to C^\infty(M\times X^k,Y), \quad f\mapsto d^kf$$

is continuous, since d^kf only puts some of the arguments of T^kf to zero and takes the last component (cf. Exercise 5.15).

c) In case that the target $N=X$ is a lcs, then for $f\colon M\to X$ smooth there exists an intermediate differential D^kf between d^kf and T^kf. Since $TX=X\times X$ we have $T^kX=T^{2^k}X$ and we set

$D^k f := \mathrm{pr}_2 \circ T^k f$ (where we follow the convention that locally we have $Tf = f \times df$, i.e., we put the map in the first and the differential in the second component). Then we clearly have that $Tf = f \times Df$ (where we have used the canonical isomorphism $TX \cong X \times X$). Moreover, $f \mapsto D^k f$ is also continuous.

d) Besides the properties that we will show in this lecture, the topology on $C^\infty(M,N)$ carries some very important properties. For instance, continuous homotopies in $C^\infty(M,N)$ coincide with smooth homotopies and the inclusion $C^\infty(M,N) \hookrightarrow C(M,N)$ is a homotopy equivalence (both provided that M is finite-dimensional). Similar statements also hold for spaces of smooth sections [Woc09]. ■

Lemma 5.3. *If X, Y are lcs, $U \Subset X$ and $V \Subset Y$, then the topology on $C^\infty(U,V)$ is initial for*

$$d^k\colon C^\infty(U,V) \to C(U \times X^k, Y)_{c.o.}, \quad f \mapsto d^k f \quad \text{with} \quad k \in \mathbb{N}_0.$$

Proof. We have

$$d^k f(x)(v_1, ..., v_k) = \mathrm{pr}_{2^k}\left(Tf^k(x, w_1, ..., w_{2^k-1})\right)$$

with $w_{2^i+1} = v_{i+1}$ for $0 \leq i \leq k-1$ and $w_i = 0$ else (cf. Exercise 5.15). Thus if Z is an arbitrary topological space and $\varphi\colon Z \to C^\infty(U,V)$ is continuous, then $d^k \circ \varphi$ is continuous for each $k \in \mathbb{N}_0$.

Conversely, let $\psi\colon Z \to C^\infty(U,V)$. Then we claim that for each U, V, each $\psi\colon Z \to C^\infty(U,V)$ we have

$$d^k \circ \psi \text{ continuous for all } k \in \mathbb{N}_0 \Rightarrow T^n \circ \psi \text{ continuous for all } n \in \mathbb{N}_0.$$

Since $T^0\psi = \psi$ this shows the statement for $n = 0$. Moreover,

$$T^n(\psi(z)) = T(T^{n-1}\psi(z)) = T^{n-1}\psi(z) \times dT^{n-1}\psi(z)$$

shows inductively that $T^n \circ \psi$ is continuous if $T^{n-1} \circ \psi$ is so (since $T^{n-1}\psi(z) \in C^\infty(T^{n-1}U, T^{n-1}V)$).■

Lemma 5.4. *If M is a manifold and Y a lcs, then the topology on $C^\infty(M,Y)$ is initial for*

$$D^k\colon C^\infty(M,Y) \to C(T^k M, Y)_{c.o.}, \quad f \mapsto D^k f \quad \text{with} \quad k \in \mathbb{N}_0.$$

Proof. The proof is entirely analogous to the previous one. ■

Lemma 5.5. *If $f\colon M' \to M$ and $g\colon N \to N'$ are smooth, then*

$$g_* f^*\colon C^\infty(M,N) \to C^\infty(M',N'), \quad \gamma \mapsto g \circ \gamma \circ f$$

is continuous. In particular, the restriction map $f \mapsto f|_Q$ for $Q \subseteq M$ a submanifold is continuous.

Proof. For $n \in \mathbb{N}_0$, $K \subseteq T^n M$ compact and $O \subseteq T^n N$ open, we set

$$\lfloor n, K, O \rfloor := \{\eta \in C^\infty(M,N) \mid T^n\eta(K) \subseteq O\}.$$

Suppose $\gamma \in (g_*f^*)^{-1}(V)$ for some $V \Subset C^\infty(M',N')$. Then there exists some $l \in \mathbb{N}$ and for each $1 \leq i \leq l$ some $n_i \in \mathbb{N}_0$, $K_i \subseteq T^{n_i}M$ compact and $O_i \subseteq T^{n_i}N$ open such that

$$g \circ \gamma \circ f \in \lfloor n_1, K_1, O_1 \rfloor \cap \ldots \cap \lfloor n_l, K_l, O_l \rfloor.$$

and

$$\lfloor n_1, K_1, O_1 \rfloor \cap \ldots \cap \lfloor n_l, K_l, O_l \rfloor \subseteq V.$$

Now

$$\begin{aligned} g \circ \eta \circ f &\in \lfloor n_1, K_1, O_1 \rfloor \cap \ldots \cap \lfloor n_l, K_l, O_l \rfloor \\ &\Leftrightarrow \eta \in \lfloor n_1, T^{n_1} f(K_1), (T^{n_1} g)^{-1}(O_1) \rfloor \cap \ldots \cap \lfloor n_l, T^{n_l} f(K_l), (T^{n_l} g)^{-1} O_l \rfloor, \end{aligned}$$

and thus

$$\lfloor n_1, T^{n_1} f(K_1), (T^{n_1} g)^{-1}(O_1) \rfloor \cap \ldots \cap \lfloor n_l, T^{n_l} f(K_l), (T^{n_l} g)^{-1} O_l \rfloor \subseteq (g_* f^*)^{-1}(V). \qquad \blacksquare$$

Lemma 5.6. *Let M, N be manifolds and M be finite-dimensional. Then the evaluation map*

$$\mathrm{ev}\colon C^\infty(M,N) \times M \to N, \quad (\gamma, x) \mapsto \gamma(x)$$

is continuous.

Proof. Since M is locally compact (i.e., each point has a compact neighbourhood) and the topology on $C^\infty(M,N)$ is finer than the compact-open topology, this follows from the the fact that the evaluation

$$\mathrm{ev}\colon C(M,N)_{c.o.} \times M \to N, \quad (\gamma, x) \mapsto \gamma(x)$$

is continuous in this case (cf. Exercise 5.17 and [Mun75, Theorem 7.5.3]). ■

The definition of the topology on $C^\infty(M,N)$ is quite general and becomes useful only in more specialised cases, as in the following proposition.

Lemma 5.7. *Let X be a Hausdorff space and (Y,d) be a metric space. The for $f \in C(X,Y)$ the sets*

$$O_f(K,\varepsilon) := \{g \in C(X,Y) \mid d(f(x), g(x)) < \varepsilon \text{ for all } x \in K\}$$

form a basis for a topology on $C(X,Y)$, where f runs through $C(X,Y)$, K through the compact subsets of X and ε over $\mathbb{R}_{>0}$.

Proof. We show that for each two $O_f(K,\varepsilon), O_g(K',\varepsilon')$ and each $h \in O_f(K,\varepsilon) \cap O_g(K',\varepsilon')$ there exists K'', ε'' such that $O_h(K'',\varepsilon) \subseteq O_f(K,\varepsilon) \cap O_g(K',\varepsilon')$ (cf. [Mun75, Section 2-2]). Since the image of $K \ni x \mapsto d(f(x),h(x)) \in [0,\varepsilon)$ is compact there exist $\widetilde{\varepsilon}$ such that $d(f(x),h(x)) + \widetilde{\varepsilon} < \varepsilon$ for all $x \in K$. Likewise, there exists $\widetilde{\varepsilon}'$ such that $d(g(x),h(x)) + \widetilde{\varepsilon}' < \varepsilon'$ for all $x \in K'$. Thus $\gamma \in O_h(K \cup K', \min(\widetilde{\varepsilon}, \widetilde{\varepsilon}'))$ implies

$$d(\gamma(x), f(x)) \leq d(\gamma(x), h(x)) + d(h(x), f(x)) < \widetilde{\varepsilon} + d(h(x), f(x)) < \varepsilon \text{ for all } x \in K$$

and

$$d(\gamma(x), g(x)) \leq d(\gamma(x), h(x)) + d(h(x), g(x)) < \widetilde{\varepsilon}' + d(h(x), g(x)) < \varepsilon' \text{ for all } x \in K'$$

and hence $\gamma \in O_f(K,\varepsilon) \cap O_g(K',\varepsilon')$. ■

Definition 5.8. The topology described in Lemma 5.7 is called the *topology of compact convergence* and is denoted by $C(X,Y)_c$. ■

We do not indicate the dependence of the topology $C(X,Y)_c$ on the metric d. This is justified by the first part of the following proposition. In there and in the sequel we will frequently use spaces that admit a compact exhaustion in the following sense.

Remark 5.9. A Hausdorff space X is *σ-compact* if there exists a sequence $K_1, K_2, \dots$ of compact subspaces such that $\cup_{n\in\mathbb{N}} K_n = X$. We will in the following always assume that the K_i's are increasing (if necessary then we replace K_n by $\cup_{i\leq n} K_i$). Note that a compact space is σ-compact. Some further properties of σ-compact spaces are established in Exercise 5.19. ■

Proposition 5.10. *Suppose X is a Hausdorff space and Y is a metrisable space.*

a) The topology $C(X,Y)_c$ equals the compact-open topology. In particular, it is independent of the metric on Y.

b) If X' is another Hausdorff space, Y' is another metrisable space and $f\colon X' \to X$, $g\colon Y \to Y'$ are continuous, then $C(X,Y)_c \to C(X',Y')_c$, $\gamma \mapsto g\circ\gamma\circ f$ is continuous.

c) Suppose d is a bounded metric on Y and $K_1, K_2, \dots$ is a sequence of compact subsets of X with $\cup_{n\in\mathbb{N}} K_n = X$. Then

$$d(f,g) := \sum_{i\in\mathbb{N}} 2^{-i} d^{K_i}(f,g)$$

with $d^{K_i}(f,g) := \sup\{d(f(x),g(x)) \mid x \in K_i\}$ defines a metric on $C(X,Y)$ whose induced topology is $C(X,Y)_c$.

d) If X is compact and d is a metric on Y, then the topology $C(X,Y)_c$ is induced by the metric

$$d(f,g) = \sup\{d(f(x,g(x)) \mid x \in X\}.$$

e) If X is σ-compact and Y is metrisable, then $C(X,Y)_c$ is metrisable.

Proof. a) This is left as Exercise 5.18, see also [Mun75, Theorem 7.5.1].

b) Follows from the previous part since the corresponding statement is true for the compact-open topology.

c) We assume without loss of generality that d is bounded by 1. If K is an arbitrary compact space, then we consider on $C(K,Y)$ the metric

$$d^K(f,g) := \sup\{d^Y(f(x),g(x)) \mid x \in K\}.$$

If $K' \subseteq K$ is compact, then $B_f(\varepsilon) \subseteq O_f(K',\varepsilon)$ and thus $O_f(K',\varepsilon)$ is open in the topology induced by d^K. Conversely, $O_f(K,\varepsilon) \subseteq B_f(\varepsilon)$ by definition, and thus $B_f(\varepsilon)$ is open in $C(K,Y)_c$. Thus $C(K,Y)_c$ is metrisable.

Now we have by definition

$$d(f,g) := \sum_{i\in\mathbb{N}} 2^{-i} d^{K_i}(f|_{K_i}, g|_{K_i}),$$

which clearly defines a metric on $C(X,Y)$ (cf. Example A.2 e) and f)).

In order to check that $C(X,Y)_c$ is finer than the metric topology take $B_f(\varepsilon)$ arbitrary. Then there exists $n \in \mathbb{N}$ such that $\sum_{i=n}^{\infty} 2^{-i} < \frac{\varepsilon}{2}$. We claim that $O_f(K_n, \frac{\varepsilon}{2}) \subseteq B_f(\varepsilon)$. In fact, if $\gamma \in O_f(K_n, \frac{\varepsilon}{2})$, then

$$d(\gamma(x), f(x)) < \frac{\varepsilon}{2} \text{ if } x \in K_n$$

and $d(\gamma(x), f(x)) < 1$ if $x \notin K_n$. Thus

$$\begin{aligned} d(\gamma, f) = \sum_{i\in\mathbb{N}} 2^i \sup\{d(\gamma(x), f(x)) \mid x \in K_i\} \leq \\ \sum_{i=1}^{n} 2^i \sup\{d(\gamma(x), f(x)) \mid x \in K_n\} + \sum_{i=n+1}^{\infty} 2^i \leq \sum_{i=1}^{n} 2^i \frac{\varepsilon}{2} + \sum_{i=n+1}^{\infty} 2^i \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned} \tag{23}$$

since $K_i \subseteq K_n$ if $i \leq n$ and $d(\gamma(x), f(x))$ is bounded by 1. Thus $B_f(\varepsilon)$ is open in $C(X,Y)_c$.

In order to show that the metric topology is finer than $C(X,Y)_c$, we first note that if $K \subseteq X$ is compact, then it is contained in some K_n. In fact, $(\mathrm{int}(K_i))_{i\in\mathbb{N}}$ is an open cover of X and thus $K \subseteq \mathrm{int}(K_1) \cup \ldots \cup \mathrm{int}(K_n) = \mathrm{int}(K_n) \subseteq K_n$ for some n. Now take an arbitrary $O_f(K,\varepsilon)$ and assume $K \subseteq K_n$. Then $B_f(\varepsilon) \subseteq O_f(K,\varepsilon)$, since

$$\begin{aligned} \gamma \in B_f(\varepsilon) \Rightarrow \sum_{i\in\mathbb{N}} 2^{-i} d^{K_i}(f|_{K_i}, \gamma|_{K_i}) < \varepsilon \Rightarrow d^{K_n}(f|_{K_n}, \gamma|_{K_n}) < \varepsilon \\ \Rightarrow d(f(x), \gamma(x)) < \varepsilon \text{ for all } x \in K_n \\ \Rightarrow d(f(x), \gamma(x)) < \varepsilon \text{ for all } x \in K \\ \Rightarrow \gamma \in O_f(K,\varepsilon). \end{aligned}$$

Thus $O_f(K,\varepsilon)$ is open in the metric topology.

d) There exists a bounded metric defining the topology on Y (replace an arbitrary metric with the equivalent metric $\min\{d,1\}$ if necessary). Hence the claim follows from part c) if we set $K_1 = X$ and $K_n = \emptyset$ for $n \geq 2$.

e) As before, there exists a bounded metric. Thus the claim follows from part c). ■

Corollary 5.11. *The topology on $C^\infty(M,N)$ from Definition 5.1 is metrisable if M is finite-dimensional and σ-compact and N is metrisable.*

Proof. The topology on $C^\infty(M,N)$ is the initial topology with respect to

$$C^\infty(M,N) \hookrightarrow \prod_{k\in\mathbb{N}_0} C(T^kM, T^kN)_{c.o.}, \quad f \mapsto (T^kf)_{k\in\mathbb{N}_0}.$$

If M is σ-compact then so is T^kM (see Exercise 5.18) and if N is metrisable then so is T^kN (see Theorem 4.17). Thus $C(T^kM, T^kN)_{c.o.} = C(T^kM, T^kN)_c$ is metrisable for each $k \in \mathbb{N}_0$ by Proposition 5.10 c) and thus $\prod_{k\in\mathbb{N}_0} C(T^kM, T^kN)_{c.o.}$ is metrisable (see Exercise 5.20). Consequently, $C^\infty(M,N)$ is metrisable as a subspace of a metrisable space. ■

We now consider the more special case in which Y is a locally convex space.

Lemma 5.12. *a) Suppose X is a σ-compact space and Y is a metrisable lcs. Then the topology induced by the semi-norms from Example A.2 g) coincides with $C(X,Y)_c$. In particular, $C(X,Y)_c$ is then again a metrisable lcs.*

b) Suppose M is a finite-dimensional σ-compact space and Y is a metrisable lcs. Then the topology from Definition 5.1 turns $C^\infty(M,Y)$ into a metrisable lcs.

Proof. a) Let for the moment $C(X,Y)_{lc}$ denote the topology on $C(X,Y)$ from Example A.2 g). Since both topologies are vector topologies we only have to check that the bases of zero neighbourhoods agree. Let $K_1 \subseteq K_2 \subseteq ...$ be an increasing sequence of compact subsets with $\cup_n K_n = X$. Then we first observe that $C(X,Y)_c$ carries the initial topology of the maps $\operatorname{res}_n \colon C(X,Y) \to C(K_n,Y)_c$, $f \mapsto f|_{K_n}$ for $n \in \mathbb{N}$. Indeed, by Proposition 5.10 d) the topology on $C(K_n,Y)_c$ is induced by the metric $d_n(f,g) = \sup\{d(f(x,g(x)) \mid x \in K_n\}$. Thus a basis for the zero neighbourhoods in the initial topology of $(\operatorname{res}_n)_{n\in\mathbb{N}}$ is (in the notation of Lemma 5.7)

$$O_0(K_1,\varepsilon_1) \cap ... \cap O_0(K_m,\varepsilon_m)$$

where m runs through $\mathbb{N}$ and $\varepsilon_1,...,\varepsilon_m$ through the positive reals. These are clearly open in $C(X,Y)_c$. This implies

$$\langle\{\operatorname{res}_n^{-1}(C(K_n,Y)) \mid n \in \mathbb{N}\}\rangle \subseteq C(X,Y)_c.$$

Conversely, if $K \subseteq X$ is compact, then $K \subseteq K_N$ for some $n \in \mathbb{N}$ and thus $O_0(K_N,\varepsilon) \subseteq O_0(K,\varepsilon)$. This implies

$$C(X,Y)_c \subseteq \langle\{\operatorname{res}_n^{-1}(C(K_n,Y)) \mid n \in \mathbb{N}\}\rangle.$$

A similar argument also shows that the topology of $C(X,Y)_{lc}$ coincides with the initial topology of the maps $\operatorname{res}_n \colon C(X,Y) \to C(K_n,Y)_{lc}$.

By the above it suffices to check the claim in the case where X is compact. A basis for the zero neighbourhoods in $C(X,Y)_c$ are given by

$$O_\varepsilon = \{f \in C(X,Y) \mid d(f(x),0) < \varepsilon \text{ for all } x \in X\},$$

where ε runs through to positive reals. Let $(p'_k)_{n\in\mathbb{N}}$ is a countable point-separating family of semi-norms defining the topology on Y. By setting $p_n := p'_1 + ... + p'_n$ we may assume without loss of generality that $p_{n+1} \geq p_n$ for all n. A basis for the zero neighbourhoods of $C(X,Y)_{lc}$ is given by

$$P_{n,\varepsilon} = \{f \in C(X,Y) \mid p_n(f(x)) < \varepsilon \text{ for all } x \in K_n\},$$

where n runs through $\mathbb{N}$ and ε through the positive reals (we don't have to take multiple intersections here, since we assumed $K_n \subseteq K_{n+1}$ and $p_n \leq p_{n+1}$). By Exercise A.15 we may assume that

$$d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n} \min\{p_n(x-y),1\}.$$

If ε is fixed, then choose N such that $\sum_{i=N}^{\infty} 2^{-i} < \frac{\varepsilon}{2}$. This implies $P_{N,\varepsilon 2^{-N}} \subseteq O_\varepsilon$ and thus $C(X,Y)_c \subseteq C(X,Y)_{lc}$. On the other hand, we have for each n and each $0 < \varepsilon < 1$ that $O_{\varepsilon 2^{-n}} \subseteq P_{n,\varepsilon}$ and thus $C(X,Y)_{lc} \subseteq C(X,Y)_c$.

b) The topology on $C^\infty(M,Y)$ is metrisable by Corollary 5.11. From Lemma 5.4 and Proposition 5.10 a) it follows that the topology on $C^\infty(M,Y)$ is the initial topology for

$$D^k\colon C^\infty(M,Y)\to C(T^kM,Y)_c,\quad f\mapsto D^kf \quad\text{with}\quad k\in\mathbb{N}_0.$$

Since the topology on each $C(T^kM,Y)_c$ is locally convex this also holds for the initial topology.■

The following fact is probably well-known, we repeat it here for convenience.

Lemma 5.13. *Suppose Y is a Fréchet space. If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C^1(\mathbb{R}^m,F)$ such that $f_n\to f$ in $C(\mathbb{R}^m,F)_c$ and $Tf_n\to g$ in $C(T\mathbb{R}^m,TF)_c$, then $f\in C^1(\mathbb{R}^m,F)$ and $Tf=g$.*

Proof. For each $x,v\in\mathbb{R}^m$ and $s>0$ we have by the Fundamental Theorem, Proposition A.8 and Exercise 5.21

$$f(x+sv)-f(x)=\lim_{n\to\infty}(f_n(x+sv)-f_n(x))=\lim_{n\to\infty}\int_0^1 df_n(x+tsv)(sv)dt=s\int_0^1 \mathrm{pr}_2(g(x+tsv)(v))dt.$$

Thus f is differentiable and $df(x,v)=\mathrm{pr}_2(g(x,v))$ (cf. Exercise A.18), which implies $Tf=g$. ■

Although completeness is not an important property of lcs for our purposes, we record the following fact.

Theorem 5.14. *If M is a finite-dimensional σ-compact manifold and Y is a Fréchet space, then the topology from Definition 5.1 turns $C^\infty(M,Y)$ into a Fréchet space.*

Proof. The topology on $C^\infty(M,Y)$ from Definition 5.1 is the initial topology with respect to

$$C^\infty(M,Y)\hookrightarrow\prod_{k\in\mathbb{N}_0}C(T^kM,Y)_c,\quad f\mapsto(T^kf)_{k\in\mathbb{N}_0}.$$

(cf. Proposition 5.10 a)). A metric on $\prod_{k\in\mathbb{N}_0}C(T^kM,Y)_c$ is given by

$$(f,g)\mapsto\sum_{k\in\mathbb{N}_0}2^{-k}d_k(T^kf,T^kg),$$

where d_k denotes a bounded invariant metric on $C^k(T^kM,Y)$ (see Proposition 5.10 c) and Exercise 5.20). In particular, we may assume without loss of generality that $f\mapsto T^kf$ is a contraction.

From this it follows that if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then $(T^kf_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence for all $k\in\mathbb{N}_0$. Since $C(T^kM,F)_c$ is complete (see Example g)), (T^kf_n) converges in $C(T^kM,F)_c$ for each k to a function $g_k:=\lim_{n\to\infty}T^kf_n$. We set $f:=g_0$ and claim that g is C^k and that $T^kg=g_k$ for each k. This implies that $g\in C^\infty(M,Y)$ and $f_n\to f$ in $C^\infty(M,Y)$ and thus finishes the proof.

In order to verify the claim we first observe that it suffices to show the claim for $M=\mathbb{R}^m$. Indeed, $T^kg=g_k$ holds if and only if we find an open cover $(U_i)_{i\in I}$ with each U_i diffeomorphic to $\mathbb{R}^m$ such that $T^k(g|_{U_i})=g_k|_{T^kU_i}$ for each i. Moreover, $\gamma\mapsto\gamma|_{T^kU_i}$ is continuous by Lemma 5.5 and thus we have that $T^kf_n\big|_{U_i}$ converges to $g_k|_{T^kU_i}$ for each k and each i.

On $\mathbb{R}^n$ we proceed inductively: Lemma 5.13 shows directly that $Tf=g_1$. If we know that $T^ig=g_i$ for $i\leq k-1$, then

$$g_k=\lim_{n\to\infty}T^kf_n=\lim_{n\to\infty}T(T^{k-1}f_n)=T(\lim_{n\to\infty}T^{k-1}f_n)=T(g_{k-1})=T(T^{k-1}g)=T^kg$$

follows again from Lemma 5.13 and the continuity of $\gamma\mapsto T\gamma$. This finishes the proof. ■

Exercises for Section 5

Exercise 5.15. Show inductively that if $U \Subset X$ and $V \Subset Y$, then

$$d^k f(x)(v_1, ..., v_k) = \mathrm{pr}_{2^k}\left(T^k(x, w_1, ..., w_{2^k-1})\right)$$

with $w_{2^i+1} = v_{i+1}$ for $0 \leq i \leq k-1$ and $w_i = 0$ else. ■

Exercise 5.16. Show that if X, Y, Z are Hausdorff spaces and $f\colon X \times Y \to Z$ is continuous, then for each $x \in X$ we have $\widehat{f}(x) \in C(Y, Z)$ and $\widehat{f}\colon X \to C(Y, Z)_{c.o.}$ is continuous. ■

Exercise 5.17. Show that if X, Y are topological spaces and X is Hausdorff and locally compact (i.e., each point in X has a compact neighbourhood), then the evaluation map

$$\mathrm{ev}\colon C(X, Y) \times X \to Y, \quad (\gamma, x) \mapsto \gamma(x)$$

is continuous. ■

Exercise 5.18. Show that for a Hausdorff space X and a metrisable space Y the topology of compact convergence equals the compact-open topology. **Hint:** This involves various typical compactness arguments. ■

Exercise 5.19. Show the following:

a) Finite products of σ-compact spaces are again σ-compact (with respect to the product topology).

b) Open subsets of $\mathbb{R}^n$ are σ-compact.

c) A σ-compact space X is also Lindelöf, i.e., each open cover of Xhas a countable subcover.

d) Countable disjoint unions of σ-compact spaces are σ-compact (with respect to the disjoint union or colimit topology).

e) Closed subsets of σ-compact spaces are again σ-compact (with respect to the subspace topology).

Conclude that if M is a σ-compact manifold, then so is its tangent bundle TM. ■

Exercise 5.20. Suppose that $(X_n)_{n\in\mathbb{N}}$ is a countable family of metrisable spaces. Let d_n be a metric on X_n that is bounded by 1 and that induces the topology on x_n. Then show that

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) := \sum_{n\in\mathbb{N}} 2^{-n} d_n(x_n, y_n)$$

defines a metric on $\prod_{n\in\mathbb{N}} X_n$ that induces the product topology. ■

Exercise 5.21. Show that one may interchange limits and integration in the following sense: Suppose that X is a lcs and $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C([0,1], X)_c$ with $\lim_{n\to\infty} f_n = f$. If $\int_0^1 f_n dt$ exists for each n and $\lim_{n\to\infty} \int_0^1 f_n dt$ exists, then $\int_0^1 f(t)dt$ exists and equals $\lim_{n\to\infty} \int_0^1 f_n(t)dt$. ■

6 Spaces of Smooth Maps: The Exponential Law

In this section we prove the exponential law

$$C^\infty(U, C^\infty(M,O)) \cong C^\infty(U \times M, O)$$

for $U \subseteq \mathbb{R}^n$ open and O an open subset of some locally convex space.

In order to prove the results of the next section the following result. Its proof will occupy this entire section.

Theorem 6.1. *Suppose $U \subseteq \mathbb{R}^n$ is open, M is a compact manifold and $O \subseteq X$ is open for some lcs X. In the following, smoothness refers to Definitions B.1 and A.9.*

a) If $f\colon U \times M \to O$ is smooth, then so is $\widehat{f}\colon U \to C^\infty(M,O)$, $\widehat{f}(x)(y) := f(x,y)$.

b) The evaluation map

$$\mathrm{ev}\colon C^\infty(M,O) \times M \to O, (\gamma, x) \mapsto \gamma(x)$$

is smooth. Moreover, the map

$$C^\infty(U \times M, O) \to C^\infty(U, C^\infty(M,O)), \quad f \mapsto \widehat{f}$$

(which is well-defined by a)) is a bijection.

We will split the proof up into several lemmas. Throughout this section, smoothness refers to Definitions B.1 and A.9 and $C^\infty(M,N)$ is endowed with the topology from Definition 5.1. Recall that $C^\infty(M,X)$ is a lcs if M is finite-dimensional and σ-compact and X is a lcs (cf. Lemma 5.12 b)).

Lemma 6.2. *Suppose X, Z are lcs's, $U \Subset X$ and $V \Subset \mathbb{R}^n$. It $f\colon U \times V \to Z$ is smooth, then so is*

$$\widehat{f}\colon U \to C^\infty(V,Z).$$

Note that the following proof goes through also for $V \Subset Y$ for an arbitrary lcs Y, but then $C^\infty(V,Z)$ is not a lcs any more (cf. Lemma 5.12) and we are working here only with smooth maps into locally convex spaces (be aware that then the notion of smoothness that we have chosen is not the appropriate one in non-locally convex spaces, cf. [Glö04a]).

Proof. To simplify notation we set $Y := \mathbb{R}^n$. We first show that $\widehat{f}$ is C^0 (aka continuous). To this end it suffices to show by Lemma 5.3 that $d^k \circ \widehat{f}\colon U \to C(V \times Y^k, X)$ is continuous for each $k \in \mathbb{N}_0$. For $k = 0$ this is true by the basic properties of the compact-open topolgoy (cf. Exercise 5.16). Moreover, we have inductively

$$d^k(\widehat{f}(u)) = \widehat{d_2^k f}(u)$$

for each $u \in U$, where d_2^k denotes the differential with respect to the second variable:

$$\widehat{d_2^k f}(u)(x, v_1, ..., v_k) := d_2^k f(u,x)(v_1, ..., v_k) := d^k f(u,x)((0,v_1), ..., (0,v_k)).$$

In fact, the case $k = 0$ is trivial and we have

$$\begin{aligned} d^k(\widehat{f}(u))(x, v_1, ..., v_k) &= \lim_{s\to 0} \frac{1}{s}\Big((d^{k-1}\widehat{f}(u))(x+sv_k)(v_1, ..., v_{k-1}) - (d^{k-1}\widehat{f}(u))(x)(v_1, ..., v_{k-1})\Big) \\ &= \lim_{s\to 0} \frac{1}{s}\Big(d^{k-1}f(u, x+sv_k)((0,v_1), ..., (0,v_{k-1})) - d^{k-1}f(u,x)((0,v_1), ..., (0,v_{k-1}))\Big) \\ &= d^k f(u,x)((0,v_1), ..., (0,v_k)). \end{aligned}$$

Thus $d^k \circ \widehat{f}$ is continuous by induction and hence $\widehat{f}$ is C^0.

We now show that $\widehat{f}$ is C^1. We claim that for fixed $u \in U$ and $v \in X$ we have

$$d\widehat{f}(u)(v) = \lim_{s\to 0} \frac{1}{s}\Big(\widehat{f}(u+sv) - \widehat{f}(u)\Big) = \widehat{d_1 f}(u,v), \tag{24}$$

where d_1 denotes the differential with respect to the first variable:

$$\widehat{d_1^k f}(u, v_1, ..., v_k)(x) := d_1^k f(u,x)(v_1, ..., v_k) := d^k f(u,x)((v_1,0), ..., (v_k,0)).$$

To this end, let $(s_n)_{n\in\mathbb{N}}$ be a null sequence and let

$$\lfloor n_1, K_1, O_1 \rfloor \cap ... \cap \lfloor n_l, K_l, O_l \rfloor$$

be a basic open neighbourhood of $\widehat{d_1 f}(u,v)$ in the topology of $C^\infty(V, X)$ (cf. Lemma 5.5), i.e.,

$$d^{n_i}(\widehat{d_1 f}(u,v))(y) \in O_i \text{ for each } y \in K_i$$

for $K_i \subseteq V \times Y^{n_i}$ compact and $O_i \subseteq X$ is open. Observe that

$$d^{n_i}(\widehat{d_1 f}(u,v))(y) = d^{n_i+1}f(u,y_0)((v,0),(0,y_1), ..., (0,y_{n_i})) = d^{n_i+1}f(u,y_0)((0,y_1), ..., (0,y_{n_i}),(v,0))$$

since the higher differentials are symmetric. Since the difference quotient is extended continuously to $s = 0$ by the differential (cf. Exercise A.18), there exists for each $y \in K_i$ some $y \in P_y \Subset V \times Y^{n_i}$ and some $\varepsilon_y > 0$ such that the function

$$\begin{aligned} &P_y \times (-\varepsilon_y, \varepsilon_y) \setminus \{0\} \to X, \\ &\qquad (z,s) \mapsto \frac{1}{s}\left(d^{n_i}f(u+sv, z_0)((0,z_1), ..., (0,z_{n_i})) - d^{n_i}f(u,z_0)((0,z_1), ..., (0,z_{n_i}))\right), \end{aligned}$$

takes values in O_i and extends continuously to some function $P_y \times (-\varepsilon_y, \varepsilon_y) \to O_i$. Since K_i is compact it is covered by finitely many $P_{y_1}, ..., P_{y_q}$ and thus there exists some $N \in \mathbb{N}$ with $|s_n| < \varepsilon_{y_j}$ for each $n \geq N$ and each $1 \leq j \leq q$. Thus if $n \geq N$ and $z \in K_i$, then $z \in P_{y_j}$ and $|s_n| < \varepsilon_{y_j}$ for some j and thus

$$\frac{1}{s_n}\left(d^{n_i}f(u+s_n v, z_0)((0,z_1), ..., (0,z_{n_i})) - d^{n_i}f(u,z_0)((0,z_1), ..., (0,z_{n_i}))\right) \in O_i.$$

This means that
$$\frac{1}{s_n}\left(\widehat{f}(u+s_n v)-\widehat{f}(u)\right)\in \lfloor n_1,K_1,O_1\rfloor\cap\ldots\cap\lfloor n_l,K_l,O_l\rfloor$$
if $n\geq N$, and thus (24) holds. Since $\widehat{d_1 f}(u,v)$ is C^0 by what we have already shown we conclude that $\widehat{f}$ is C^1.

We are now ready to show inductively that $\widehat{f}$ is C^k for $k\geq 2$. In fact,
$$d^k\widehat{f}(u)(v_1,...,v_k)=\widehat{d_1^k f}(u,v_1,...,v_n)$$
holds by the following induction:
$$d^k\widehat{f}(u)(v_1,...,v_k)=d(d^{k-1}\widehat{f})(u,v_1,...,v_{k-1})(v_k,0,...,0)=$$
$$d(\widehat{d_1^{k-1}f})(u,v_1,...,v_{k-1})(v_k,0,...,0)=d_1(\widehat{d_1^{k-1}f})(u,v_1,...,v_{k-1})(v_k)=\widehat{d_1^k f}(u,v_1,...,v_k),$$
where we have used (24) for the third equality. Thus $d^k\widehat{f}(u)(v_1,...,v_k)$ exists and is continuous by what what we have shown for $k=0$. This shows that $\widehat{f}$ is C^k for each $k\in\mathbb{N}_0$. ∎

Lemma 6.3. *If $V\Subset\mathbb{R}^n$ and X is a lcs, then the evaluation map*
$$\mathrm{ev}\colon C^\infty(V,X)\times V\to X,\quad (\gamma,x)\mapsto\gamma(x)$$
is smooth.

Proof. We show ev is a C^k-map for each $k\in\mathbb{N}^+$ (which then also implies that ev is a C^0-map). We claim that if $(\gamma,x)\in C^\infty(V,O)\times V$ and $(\eta_1,y_1),...,(\eta_k,y_k)\in C^\infty(V,X)\times\mathbb{R}^n$ are given, then
$$d^k\,\mathrm{ev}((\gamma,x))((\eta_1,y_1),...,(\eta_k,y_k))=d^k\gamma(x)(y_1,...,y_k)+\sum_{i=1}^{k}d^{k-1}\eta_i(x)(y_1,...,\widehat{y_i},...,y_k)\tag{25}$$
From this it follows that ev is a C^k-map, since $\gamma\mapsto d^k\gamma$, $\eta_i\mapsto d^{k-1}\eta_i$ and the evaluations are continuous linear by Remark 5.2 and Lemma 5.6 and thus smooth (hence in particular C^k). In order to establish (25), we first observe that $f\mapsto d^k f$ is linear (which can be shown by a trivial induction). Now (25) holds by the following induction: If $k=1$, then
$$d\,\mathrm{ev}(\gamma,x)(\eta,y)=\lim_{s\to 0}\frac{1}{s}(\mathrm{ev}(\gamma+s\eta)(x+sy)-\mathrm{ev}(\gamma)(x))=$$
$$\lim_{s\to 0}\frac{1}{s}(\gamma(x+sy)+s\eta(x+sy)-\gamma(x))=d\gamma(x)(y)+\eta(x)$$

by the continuity of η. If (25) holds for $k-1$, then we have

$$\lim_{s\to 0}\frac{1}{s}\Big(d^{k-1}\operatorname{ev}(\gamma+s\eta_k,x+sy_k)\big((\eta_1,y_1),...,(\eta_{k-1},y_{k-1})\big)-d^{k-1}\operatorname{ev}(\gamma,x)\big((\eta_1,y_1),...,(\eta_{k-1},y_{k-1})\big)\Big)=$$
$$\lim_{s\to 0}\frac{1}{s}\Big(d^{k-1}(\gamma+s\eta_k)(x+sy_k)(y_1,...,y_{k-1})+\sum_{i=1}^{k-1}d^{k-2}\eta_i(x+sy_k)(y_1,...,\widehat{y_i},...,y_{k-1})-$$
$$d^{k-1}\gamma(x)(y_1,...,y_{k-1})+\sum_{i=1}^{k-1}d^{k-2}\eta_i(x)(y_1,...,\widehat{y_i},...,y_{k-1})\Big)=$$
$$\lim_{s\to 0}\frac{1}{s}\Big(d^{k-1}(\gamma)(x+sy_k)(y_1,...,y_{k-1})+sd^{k-1}(\eta_k)(x+sy_k)(y_1,...,y_{k-1})-d^{k-1}\gamma(x)(y_1,...,y_{k-1})+$$
$$\sum_{i=1}^{k-1}d^{k-1}\eta_i(x)(y_1,...,\widehat{y_i},...,y_{k-1},y_k)\Big)=$$
$$d^k\gamma(x)(y_1,...,y_k)+\sum_{i=1}^{k}d^{k-1}\eta_i(x)(y_1,...,\widehat{y_i},...,y_{k-1},y_k),$$

where the last equality follows from the continuity of $d^{k-1}\eta_k$. ■

Lemma 6.4. *If M,N are manifolds, M is compact and $U_1,...,U_n$ is a finite open cover of M and X is a lcs, then the map*

$$\varphi\colon C^\infty(M,N)\to\{(\gamma_i)_{i=1,...,n}\in\prod_{i=1}^{n}C^\infty(U_i,N)\mid\gamma_i|_{U_{ij}}=\gamma_j|_{U_{ij}}\},\quad\gamma\mapsto(\gamma|_{U_i})_{i=1,...,n}$$

is a homeomorphism onto its image (endowed with the subspace topology of $\prod_{i=1,...,n}C^\infty(U_i,N)$). Moreover, the image is closed in $\prod_{i=1,...,n}C^\infty(U_i,N)$.

Proof. Since smooth functions are a sheaf, the image of φ is given by

$$\{(\gamma_i)_{i=1,...,n}\in\prod_{i=1}^{n}C^\infty(U_i,N)\mid\gamma_i|_{U_{ij}}=\gamma_j|_{U_{ij}}\}=\bigcap_{x_{ij}\in\coprod_{i,j=1}^{n}U_{ij}}\operatorname{ev}_{x_{ij}}^{-1}(\Delta N).\tag{26}$$

Here $\operatorname{ev}_{x_{ij}}$ denotes the map $\prod_{i=1}^{n}C^\infty(U_i,N)\to N\times N$, $(\gamma_k)_{k=1,...,n}\mapsto(\gamma_i(x_{ij}),\gamma_j(x_{ij}))$ and $\Delta N\subseteq N\times N$ the diagonal. This map is continuous since the projection $(\gamma_k)_{k=1,...,n}\mapsto\gamma_i$ and evaluation $\gamma_i\mapsto\gamma_i(x_{ij})$ is continuous (cf. Lemma 5.5). Since ΔN is closed so is $\operatorname{ev}_{x_{ij}}^{-1}(\Delta N)$ and thus (26) is also closed.

Since $\gamma\mapsto\gamma|_{U_i}$ is continuous by Lemma 5.5 it follows that φ is continuous. It is clearly injective, so it remains to show that φ is open onto its image. Since M is compact and regular (cf. [Mun75, Section 4-2]), there exist an open cover $V_1,...,V_n$ such that

$$U_i\setminus(\bigcup_{j\neq i}U_j)\subseteq V_i\subseteq\overline{V_i}\subseteq U_i$$

(cf. Exercise 6.6). If $\gamma\in\lfloor n_1,K_1,O_1\rfloor\cap...\cap\lfloor n_l,K_l,O_l\rfloor$, then

$$\gamma|_{U_i}\in\lfloor n_1,K_1\cap T^{n_1}\overline{V_i},O_1\rfloor\cap...\cap\lfloor n_l,K_l\cap T^{n_l}\overline{V_i},O_l\rfloor,$$

where $T^{n_j}\overline{V_i} := \pi^{-1}(\overline{V_i}) \subseteq T^{n_j}U_i$ for $\pi\colon T^{n_j}U_i \to U_i$ the bundle projection (note that $T^{n_j}\overline{V_i}$ is in particular closed in $T^{n_j}U_i$). Moreover, if $(\eta_i)_{i=1,\dots,n} \in \prod_{i=1}^{n} C^\infty(U_i, N)$ satisfies $\eta_i|_{U_{ij}} = \eta_j|_{U_{ij}}$ and

$$\eta_i \in \lfloor n_1, K_1 \cap T^{n_1}\overline{V_i}, O_1 \rfloor \cap \dots \cap \lfloor n_l, K_l \cap T^{n_l}\overline{V_i}, O_l \rfloor,$$

then $\eta \in \lfloor n_1, K_1, O_1 \rfloor \cap \dots \cap \lfloor n_l, K_l, O_l \rfloor$ for the amalgamation $\eta \in C^\infty(M, N)$ with $\eta|_{U_i} = \eta_i$, since K_j is covered by $T^{n_j}\overline{V_1}, \dots, T^{n_j}\overline{V_1}$. Thus

$$\prod_{i=1}^{n} \lfloor n_1, K_1 \cap T^{n_1}\overline{V_i}, O_1 \rfloor \cap \dots \cap \lfloor n_l, K_l \cap T^{n_l}\overline{V_i}, O_l \rfloor \subseteq \varphi(\lfloor n_1, K_1, O_1 \rfloor \cap \dots \cap \lfloor n_l, K_l, O_l \rfloor).$$

Since $\gamma \in \lfloor n_1, K_1, O_1 \rfloor \cap \dots \cap \lfloor n_l, K_l, O_l \rfloor$ was arbitrary, this chows that φ is open onto its image. ■

We now eventually proof Theorem 6.1: Suppose $U \Subset \mathbb{R}^m$, M is a compact manifold and $O \Subset X$ for some lcs X.

a) If $f\colon U \times M \to O$ is smooth, then so is $\widehat{f}\colon U \to C^\infty(M, O)$, $\widehat{f}(x)(y) := f(x, y)$.

b) The evaluation map

$$\mathrm{ev}\colon C^\infty(M, O) \times M \to O, (\gamma, x) \mapsto \gamma(x)$$

is smooth and the map

$$C^\infty(U \times M, O) \to C^\infty(U, C^\infty(M, O)), \quad f \mapsto \widehat{f} \tag{27}$$

(which is well-defined by a)) is a bijection.

Proof. We first note that $C^\infty(M, O) = C^\infty(M, X) \cap \lfloor M, O \rfloor$ is open in $C^\infty(M, X)$, since M is compact.

a) By the preceding it clearly suffices to show the assertion if $O = X$. Together with Lemma 6.4 this implies that $\widehat{f}$ is smooth if and only if

$$U \ni x \mapsto (\widehat{f}(u)\Big|_{U_i})_{i=1,\dots,n} \in \{(\gamma_i)_{i=1,\dots,n} \in \prod_{i=1}^{n} C^\infty(U_i, X) \mid \gamma_i|_{U_{ij}} = \gamma_j|_{U_{ij}}\}$$

is smooth, where $(\varphi\colon U_i \to \varphi(U_i))_{i=1,\dots,n}$ is a finite atlas of M (the latter exists since M is compact). Since

$$\{(\gamma_i)_{i=1,\dots,n} \in \prod_{i=1}^{n} C^\infty(U_i, X) \mid \gamma_i|_{U_{ij}} = \gamma_j|_{U_{ij}}\}$$

is closed in $\prod_{i=1}^{n} C^\infty(U_i, X)$[8] it thus suffices to show that each $U \ni x \mapsto \widehat{f}(x)\Big|_{U_i} \in C^\infty(U_i, X)$ is smooth, which is by Lemma 5.5 the case if and only if $U \ni x \mapsto \widehat{f}(x)\Big|_{U_i} \circ \varphi_i^{-1} \in C^\infty(\varphi(U_i), X)$ is smooth. Now the latter map is smooth by Lemma 6.2.

[8]Note that if $f\colon O \Subset X \to Y$ and Y is a closed subspace of Y', then f is smooth if and only $\iota \circ f\colon O \Subset X \to Y'$ is smooth (the problem that could occur here is that the limit in the difference quotient exists in Y' but not in Y).

b) We first note in showing that ev is smooth we may assume that $O = X$ since $C^\infty(M, O)$ is open in $C^\infty(M, X)$. Let us consider for the moment the evaluation map

$$\mathrm{ev}_N : C^\infty(N, X) \times N \to X, \quad (\gamma, x) \mapsto \gamma(x)$$

for N an arbitrary finite-dimensional manifold. Now ev_M is smooth if an only if it is so on an open neighbourhood of each point $(\gamma, x) \in C^\infty(M, O) \times M$. If $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a chart with $x \in U$, then $\mathrm{ev}_M(\eta, \varphi^{-1}(y)) = \mathrm{ev}_{\varphi(U)}((\varphi^{-1})^*(\eta), y)$ for all $y \in \varphi(U)$, where $(\varphi^{-1})^* : C^\infty(M, O) \to C^\infty(\varphi(U), O)$ is the restriction of the linear and continuous map $C^\infty(M, X) \to C^\infty(\varphi($ $\eta \mapsto \eta \circ \varphi^{-1}$ to $C^\infty(M, O) \Subset C^\infty(M, X)$. Thus $(\varphi^{-1})^*$ is smooth and ev_M is smooth if $\mathrm{ev}_{\varphi(U)}$ is smooth for each chart $\varphi : U \to \varphi(U)$ of M. Since $\mathrm{ev}_{\varphi(U)}$ is smooth by Lemma 6.3, this shows that ev_M is smooth.

In order to check that (27) is bijective, it remains to check that it is surjective since it clearly is injective. If $f : U \to C^\infty(M, O)$ is smooth, then $\check{f}$ is given by the composite of

$$U \times M \to C^\infty(M, O) \times M, \quad (u, m) \mapsto (f(u), m)$$

and ev. Since these maps are smooth, so is $\check{f}$. Thus (27) is surjective. ■

Remark 6.5. Along the same lines one can also show that a) and b) hold if M is only locally compact (or equivalently finite-dimensional) and $O = X$ is the whole lcs [Glö04b, Proposition 12.2]. Note that if M is not compact, then $C^\infty(M, O)$ is (in general) not open in $C^\infty(M, X)$, so the corresponding statement for locally compact M and arbitrary $O \Subset X$ is not even well-defined. A more subtle question is what the correct exponential law is in case of finite differentiability order, see [AS12]. ■

Exercises for Section 6

Exercise 6.6. Show that if X is a compact space and $(U_i)_{i \in I}$ is an open cover of X, then there exists an open cover $(V_i)_{i \in I}$ such that $\overline{V_i} \subseteq U_i$. Does this also hold if X is not assumed to be compact? ■

7 Spaces of Smooth Maps: The Smooth Structure

In this section we treat the smooth structure on $C^\infty(M, N)$. In particular, the section

- derives the smooth structure on $C^\infty(M, N)$ with an explicit description of its charts and establishes its basic properties (exponential law, smoothness of pull-back, push-forward and composition map) and
- discusses the Lie group structure on $\mathrm{Diff}(M)$ and mapping groups.

All these results are making heavy use of the results of Sections 4, 5, 6 and the results for diffeological spaces that correspond to the above properties of $C^\infty(M, N)$.

We now are now almost ready to describe the manifold structure on $C^\infty(M, N)$. We only have to explain the ideas and terms occurring in its description. The idea is that for a fixed map $f : M \to N$ one can construct a chart on the set U_f of maps that "differ from f" be a vector field with small values: In order to make this idea precise, we will have to "add" two smooth functions with values in a manifold. This does not work pre se, but is subject to the following additional structure.

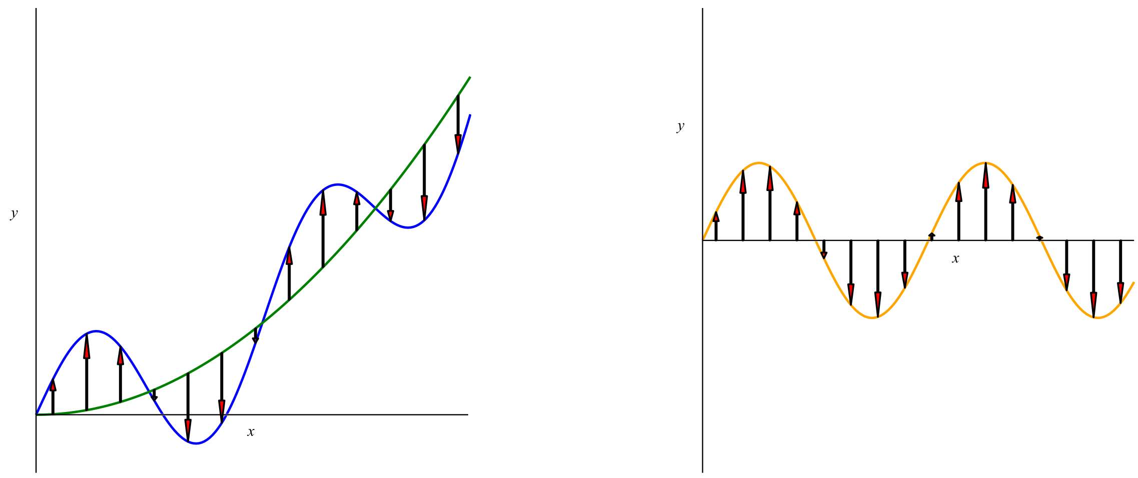

Figure 2: A function (green line) differing from another (blue line) by a vector field (orange line).

Definition 7.1. Let M be a manifold and $\pi\colon TM \to M$ its tangent bundle. We denote by σM the zero section $\{0_m \mid m \in M\} \subseteq TM$ (note that $m \mapsto 0_m$ is in particular a section of $TM \to M$). Then a *local addition* on M is a smooth map $\alpha\colon U \Subset TM \to M$, defined on an open neighbourhood U of σM such that

a) $\pi \times \alpha\colon U \to M \times M$, $v \mapsto (\pi(v), \alpha(v))$ is a diffeomorphism onto an open neighbourhood of the diagonal $\Delta M \subseteq M \times M$ and

b) $\alpha(0_m) = m$ for all $m \in M$.

If there exists a local addition on M, then M is said to *admit a local addition*. ■

Example 7.2. a) If M is a finite-dimensional manifold, then M admits a local addition. In fact, on M there exists a Riemannian metric and an associated exponential function $\exp\colon TM \to M$ such that $\exp(0_m) = m$ and $T\exp|_{T_mM} = \mathrm{id}_{T_m}$ [Lan95, Section IV.4]. Thus $(\pi \times \exp)|_{T_mM \times T_mM} = \mathrm{id}_{T_m}$ and by the Inverse Functions Theorem there exists $0_m \in U_m \Subset TM$ such that $(\pi \times \exp)|_{U_m}$ is a diffeomorphism onto its image. Thus $\pi \times \exp$ restricts to a diffeomorphism of $U := \cup_{m \in M} U_m$ onto its image.

b) If G is a Lie group, then G admits a local addition. Namely, let $\varphi\colon U \to \varphi(U) \Subset X$ be a chart around e such that $\varphi(e) = 0$. Then $\widetilde{U} := T_e\varphi^{-1}(\varphi(U))$ is open in T_eG and we set $\alpha_e\colon \widetilde{U} \to U$, $v_e \mapsto \varphi^{-1}(T_e\varphi(v_e))$. In order to extend this to all of G we observe that $TG \cong X \times G$ is trivial and thus $V := \cup_{g \in G} g \cdot \widetilde{U} \cong U \times G$ is open in TG. Then

$$\alpha\colon V \to G, v \mapsto \pi(v) \cdot \varphi^{-1}(T_e\varphi(\pi(v)^{-1} \cdot v))$$

is a local addition on G. ■

The modelling spaces of $C^\infty(M, N)$ will be spaces of sections in certain vector bundles. These we already know to be lcs.

Proposition 7.3. *Let $\pi\colon E \to M$ be a vector bundle with fibre X and compact base M. Then the space of sections*

$$\Gamma(E \to M) = \{\sigma \in C^\infty(M, E) \mid \pi \circ \sigma = \mathrm{id}_M\}$$

is a closed subspace of $C^\infty(M, E)$. The point-wise application of the addition $E \times_M E \to E$ and scalar multiplication $\mathbb{R} \times E \to E$ turn $\Gamma(E \to M)$ into a topological vector space. Moreover, if $U_1, ..., U_n$ is a cover of M and $\Phi_i\colon \pi^{-1}(U_i) \to U_i \times X$ are local trivialisations for $1 \leq i \leq n$, then

$$\Gamma(E \to M) \to \{(\xi_i)_{i=1,..,n} \in \prod_{i=1}^{n} C^\infty(U_i, X) \mid \Phi_i^{-1}(x, \xi_i(x)) = \Phi_j^{-1}(x, \xi_j(x)) \text{ for all } x \in U_{ij}\},$$
$$\xi \mapsto \left(\mathrm{pr}_2 \circ \Phi_i \circ \xi|_{U_i}\right)_{i=1,\dots,n} \quad (28)$$

is a homeomorphism onto its image (endowed with the subspace topology) and this image is a closed subspace of $\prod_{i=1}^{n} C^\infty(U_i, X)$. In particular, $\Gamma(E \to M)$ is a lcs and is metrisable if X is so.

Proof. This follows from Lemma 5.5 and Lemma 6.4. The details are left as Exercise 7.12. ∎

Corollary 7.4. *If M is a compact manifold and $E \to M$ is a vector bundle with fibre X a Fréchet space, then $\Gamma(E \to M)$ is a Fréchet space.*

Proof. By the previous theorem, $\Gamma(E \to M)$ is isomorphic to a closed subset of a finite product of complete metrisable spaces (cf. Theorem 5.14) and thus in particular complete and metrisable. ∎

Corollary 7.5. *Let $V \Subset \mathbb{R}^n$ and $E \to M$ be a vector bundle with compact base. Then the map*

$$C^\infty(V, \Gamma(E \to M)) \to \{g \in C^\infty(V \times M, E) \mid \widehat{g}(u) \in \Gamma(E \to M) \text{ for each } u \in V\}, \quad f \mapsto \check{f}$$

is a bijection.

Proof. By (28) and Theorem 6.1, we have the bijections

$$\begin{aligned} C^\infty(V, \Gamma(E \to M)) &\cong \{(f_i) \in \prod C^\infty(V, C^\infty(U_i, E)) \mid \Phi_i^{-1}(x, f_i(v)(x)) = \Phi_j^{-1}(x, f_j(v)(x)) \text{ for all } x \in U_{ij}, v \in V\} \\ &\cong \{(\check{f}_i) \in \prod C^\infty(V \times U_i, E)) \mid \Phi_i^{-1}(x, \check{f}_i(v, x)) = \Phi_j^{-1}(x, \check{f}_j(v, x)) \text{ for all } x \in U_{ij}, v \in V\} \\ &\cong \{\check{g} \in C^\infty(V \times M, E) \mid g(v) \in \Gamma(E \to M) \text{ for each } v \in V\}. \end{aligned}$$
∎

We now can prove the central result of this chapter:

Theorem 7.6. *Let M be a compact manifold and N be a locally metrisable that admits a local addition $\alpha\colon U \Subset TN \to N$. Set $V := (\pi \times \alpha)(U)$, which is an open neighbourhood of the diagonal ΔN in $N \times N$. For each $f \in C^\infty(M, N)$ we set*

$$O_f := \{g \in C^\infty(M, N) \mid (f(x), g(x)) \in V\}.$$

Then the following assertions hold.

a) The set O_f contains f, is open in $C^\infty(M, N)$ and the formula $(f(x), g(x)) = (f(x), \alpha(\varphi_f(g)(m)))$ determines a homeomorphism

$$\varphi_f\colon O_f \to \{h \in C^\infty(M, TN) \mid \pi(h(x)) = f(x)\} \cong \Gamma(f^*(TN))$$

from O_f onto the open subset $\{h \in C^\infty(M, TN) \mid \pi(h(x)) = f(x)\} \cap C^\infty(M, U)$ of $\Gamma(f^*(TN))$.

b) *The family* $(\varphi_f : O_f \to \varphi_f(O_f))_{f \in C^\infty(M,N)}$ *is an atlas, turning* $C^\infty(M,N)$ *into a smooth locally metrisable manifold.*

c) *We have* $C^\infty(M,N) = \mathbf{Diff}(D_M, D_N)$ *and the diffeology associated to the manifold structure from b) on* $C^\infty(M,N)$ *by Example 3.3 b) equals the diffeology* $\underline{\mathbf{Diff}}(DM, DN)$ *from Definition 3.4. In short, we have* $D_{C^\infty(M,N)} = \underline{\mathbf{Diff}}(D_M, D_N)$.

d) *The manifold structure on* $C^\infty(M,N)$ *from b) is independent of the choice of the local addition* α.

e) *If* L *is another locally metrisable manifold, then a map* $f : L \times M \to N$ *is smooth if and only if* $\widehat{f} : L \to C^\infty(M,N)$ *is smooth. In other words,*

$$C^\infty(L \times M, N) \to C^\infty(L, C^\infty(M,N)), \quad f \mapsto \widehat{f}$$

is a bijection (which is even natural).

f) *If* M' *is compact,* N' *is locally metrisable and admits a local addition and* $\mu : M' \to M$ $\nu : N \to N'$ *are smooth, then*

$$\nu_* \mu^* : C^\infty(M,N) \to C^\infty(M', N'), \quad \gamma \mapsto \nu \circ \gamma \circ \mu$$

is smooth.

g) *If* M' *is another compact manifold, then the composition map*

$$\circ : C^\infty(M', N) \times C^\infty(M, M') \to C^\infty(M,N), \quad (\gamma, \eta) \mapsto \gamma \circ \eta$$

is smooth.

h) *The d-topology on* $C^\infty(M,N)$*, induced from* $\underline{\mathbf{Diff}}(M,N)$ *equals the topology on* $C^\infty(M,N)$ *from Definition 5.1.*

Proof. To avoid confusion let us recall that “smooth” refers for us to a smooth map between manifolds (a morphism in **Man**).

a) Since $(f(x), f(x)) \in \Delta N \subseteq V$ for each $x \in M$ we have $f \in O_f$. Since $\iota_f : C^\infty(M,N) \to C^\infty(M,N) \times C^\infty(M,N) \cong C^\infty(M, N \times N)$, $g \mapsto (f,g)$ is continuous we have that $O_f = \iota_f^{-1}(C^\infty(M,V))$ is open.

   Set $\beta := (\pi \times \alpha)^{-1} : V \to U \subseteq TN$. If $g \in O_f$, then $\beta \circ (f \times g)$ defines a smooth map $M \to TN$ mapping each $x \in M$ into $T_{f(x)}N$. Thus we have $\varphi_f(g) = \beta \circ (f \times g)$ and thus φ_f is continuous by Lemma 5.5. Since $\varphi_f^{-1}(h) = \alpha \circ h$ we also have that φ_f^{-1} is continuous and thus φ_f is a homeomorphism. Since TN has a metrisable fibre, $\Gamma(f^*TN)$ is metrisable and thus $\varphi_f(O_f)$ is metrisable.

b) We first note that by Corollary 7.5 the diffeology associated to the manifold structure $\Gamma(f^*TN)$ and the subspace diffeology of $\underline{\mathbf{Diff}}(DM, DTN)$ coincide (where we have identified $\{h \in C^\infty(M, TN) \mid \pi(h(x)) = f(x)\}$ with $\Gamma(f^*(TN))$). The coordinate change $\varphi_{f'} \circ \varphi_f^{-1}$ is given by

$$\varphi_f(O_f \cap O_{f'}) \to \varphi_{f'}(O_f \cap O_{f'}), \quad h \mapsto \beta \circ (f' \times \alpha \circ h), \tag{29}$$

which is a morphism of diffeological spaces by Corollary 3.6. Since $\Gamma(f^*TN)$ and $\Gamma((f')^*TN)$ are metrisable by Proposition 7.3, Theorem 4.10 shows that (29) is smooth. The same argument also applies to the inverse of (29), showing that it is a diffeomorphism.

c) That $C^\infty(M,N)$ equals $\mathbf{Diff}(DM,DN)$ is Theorem 4.10. In order to show that the diffeologies agree we have to check by definition for each $U \subseteq \mathbb{R}^n$ that $\psi\colon U \to C^\infty(M,N)$ is smooth if and only if $\check{\psi}\colon U \times M \to N$ is a morphism in $\mathbf{Diff}$. Since $U \times M$ and N are locally metrisable the latter is by Theorem 4.10 equivalent to $\check{\psi}$ being smooth. Since this is a local statement we may assume that $\psi(U) \subseteq O_f$ for some $f \in C^\infty(M,N)$. Then ψ is smooth if and only if $\varphi_f \circ \psi$ is smooth. This is by Corollary 7.5 the case if and only if $\widecheck{\varphi_f \circ \psi}\colon U \times M \to f^*TN$, which is turn is equivalent to $(\mathrm{id}_U \times \varphi_f^{-1}) \circ \widecheck{\varphi_f \circ \psi} = \check{\psi}$ being smooth.

d) Since by Theorem 4.10 the manifold structure is for a locally metrisable manifold uniquely determined by the underlying diffeological space and since the diffeology on $\underline{\mathbf{Diff}}(D_M, D_N)$ is independent of α, the assertion follows from part b) and part c).

e) By Theorem 4.10 $f\colon L \times M \to N$ is smooth if and only if f is a morphism in $\mathbf{Diff}$, which is by Proposition 3.5 the case if and only if $\widehat{f}\colon L \to C^\infty(M,N)$ is a morphism in $\mathbf{Diff}$, which is by Theorem 4.10 the case if and only if $\widehat{f}$ is smooth. That $f \mapsto \widehat{f}$ is natural follows from the naturality of $f \mapsto \widehat{f}$ in $\mathbf{Set}$.

f) This is left as Exercise 7.10

g) This is left as Exercise 7.11

h) Since on $C^\infty(M,N)$ we have $D_{C^\infty(M,N)} = \underline{\mathbf{Diff}}(D_M, D_N)$ it follows that the d-topology on $C^\infty(M,N)$ from $\underline{\mathbf{Diff}}(D_M, D_N)$ is the d-topology from $D_{C^\infty(M,N)}$. The latter equals the topology on $C^\infty(M,N)$ by part b) and Lemma 4.13. ■

Remark 7.7. Note that we have refrained from denoting the manifold structure on $C^\infty(M,N)$ by $\underline{\mathbf{Man}}(M,N)$, although Theorem 7.6 e) suggests this. The problem here is that $C^\infty(M,\,\cdot\,)$ is a functor between *different* categories, i.e., the category of manifolds admitting a local addition and the category of all manifolds. Thus a compact manifold M is strictly speaking not exponentiable in the sense of Definition 2.18. The remedy for this would be to show that if N admits a local addition, then so does $C^\infty(M,N)$, but since $C^\infty(M,N)$ does not have a tangent bundle in our setting (since the tangent spaces are not all isomorphic), this goes beyond the scope of this lecture. ■

Corollary 7.8. *If G is a locally metrisable Lie group modelled on the lcs X and M is a compact manifold, then $C^\infty(M,G)$, equipped with the manifold structure from Theorem 7.6 b) and the pointwise group operations is a locally metrisable Lie group with modelling space $C^\infty(M,X)$.*

Proof. By Theorem 7.6 f) the multiplication map $\mu(f,g)(x) := f(x)\cdot g(x) = \mu_G \circ (f \times g) \circ \Delta$ and inversion map $\iota(f)(x) := \iota_G(f(x)) = \iota \circ f$ are smooth. Since $TG \cong G \times X$ (the trivial bundle, see Example D.12) and sectinos of trivial bundles are maps to the fibre, we have that $\Gamma(f^*TG) \cong C^\infty(M,X)$ for all $f \in C^\infty(M,G)$. ■

Theorem 7.9. *If M is a compact manifold, then the group of diffeomorphisms* $\mathrm{Diff}(M)$ *is an open submanifold of $C^\infty(M,M)$. Moreover, the inversion map* $\mathrm{inv}\colon \mathrm{Diff}(M) \to \mathrm{Diff}(M)$, $f \mapsto f^{-1}$ *is smooth. Consequently,* $\mathrm{Diff}(M)$ *is a Lie group with modelling space* $\Gamma(TM \to M)$.

Proof. In order to show that $\mathrm{Diff}(M)$ is open in $C^\infty(M,M)$ we will show that if $\gamma\colon \mathbb{R}\to C^\infty(M,M)$ is smooth and $c(0)$ is a diffeomorphism, then there exists some $\varepsilon>0$ such that $c(t)$ is a diffeomorphism if $|t|<\varepsilon$. To this end we first observe that if $f\colon U \Subset \mathbb{R}^n \to \mathbb{R}^n$ is such that $\widehat{df}(x_0)$ is an invertible linear map, then there exists a neighbourhood V_x of x such that $\widehat{df}(y)$ is invertible for each $y\in V$. In fact, the invertible linear maps $\mathrm{GL}_n(\mathbb{R})$ are open in the space of all linear maps $\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)$, equipped with the subspace topology of $C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ (since the subspace topology of $C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ on $\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)$ is locally convex it has to be the usual one). Since $\widehat{df}\in C(U,\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n))$ and $\widehat{df}(x)\in \mathrm{GL}_n(\mathbb{R})$, V_x exists as claimed.

The above implies that there exists $\varepsilon>0$ such that $T_m c(t)$ is an isomorphism if $|t|<\varepsilon$. In fact, for each $m\in M$ and chart $\varphi\colon U\to\varphi(U)\Subset\mathbb{R}^n$ around m there exists some $\delta>0$ and a chart $\psi\colon V\to\psi(V)\Subset\mathbb{R}^n$ such that $c(t)(U)\subseteq V$ if $|t|<\delta$. Then

$$(-\delta,\delta)\xrightarrow{c} C^\infty(M,M)\xrightarrow{|_U^V} C^\infty(U,V)\cong C^\infty(\varphi(U),\psi(V))\xrightarrow{d} C(\varphi(U)\times\mathbb{R}^n,\mathbb{R}^n)$$

is continuous, and thus is its adjoint

$$C^\infty((-\delta,\delta)\times M,M)\to C((-\delta,\delta)\times\varphi(U)\times\mathbb{R}^n,\mathbb{R}^n)$$

and since $T_m(\,\cdot\,)(\,\cdot\,)$ equals $d(\,\cdot\,)(\varphi(m),\,\cdot\,)$ in local coordinates and since $T_m c(0)$ is invertible, this show that there exists ε_m and $m\in V_m\Subset M$ such that $T_n c(t)$ is invertible if $|t|<\varepsilon$ and $n\in V_m$. Since M is compact, ε exists as desired.

Thus $c(t)$ is a local diffeomorphism if $|t|<\varepsilon$. In order to check that it is a diffeomorphism we first check that it is surjective. In fact, then $\check{c}(\{t\}\times M)$ is compact since $\check{c}$ is continuous and thus closed. It is open since $c(t)$ is a local diffeomorphism. Since $c(0)(M)$ meets each component of M so does $c(t)(M)$, and thus $c(t)$ is surjective if $|t|<\varepsilon$.

If $c(t)$ was not injective for small t, then there exists a null sequence (t_n) and sequences (x_n), (y_n) such that $x_n\neq y_n$ and $c(t_n)(x_n)=c(t_n(y_n))$. Since M is compact we may replace (x_n) and (y_n) by convergent subsequences. If $(x_n)\to x$ and $(y_n)\to y$, then $c(t_n)(x_n)=c(t_n)(y_n)$ implies that $c(0)(x)=c(0)(y)$ and thus $x=y$. Since there exists $x\in W_x\Subset M$ such that $c(t)|_{W_x}$ is injective for t small enough this implies $c(t_n)(x_n)\neq c(t_n)(y_n)$ for n large enough. This contradicts $c(t_n)(x_n)=c(t_n(y_n))$.

To show that inv is smooth, let $\varphi\colon U\to\mathrm{Diff}(M)$ be smooth. Then $\widetilde{\mathrm{inv}\circ\varphi}$ fulfils the implicit equation

$$\check{\varphi}(u,\widetilde{\mathrm{inv}\circ\varphi}(u,m))=m$$

and is thus smooth by the implicit function theorem. Thus inv is a morphism of diffeological spaces and thus smooth by Theorem 4.10. ■

Exercises for Section 7

Exercise 7.10. Show that if M,M' are compact manifolds, N,N' are locally metrisable manifolds and $\mu\colon M'\to M$ $\nu\colon N\to N'$ are smooth, then

$$\nu_*\mu^*\colon C^\infty(M,N)\to C^\infty(M',N'),\quad \gamma\mapsto\nu\circ\gamma\circ\mu$$

is smooth. ■

Exercise 7.11. Show that if M, M' are compact manifolds and N is a locally metrisable manifold, then the composition map

$$\circ\colon C^{\infty}(M', N) \times C^{\infty}(M, M') \to C^{\infty}(M, N), \quad (\gamma, \eta) \mapsto \gamma \circ \eta$$

is smooth. ■

Exercise 7.12. Fill in the details of the proof of Proposition 7.3. ■

Exercise 7.13. Suppose M is a compact manifold, N is a locally metrisable manifold and $L \subseteq M$ be a subset and $n_0 \in N$. Show that

$$C^{\infty}_{L}(M, N) := \{f \in C^{\infty}(M, N) \mid f|_L = n_0\},$$

where n_0 is identified with the constant map with values n_0 is a closed submanifold of $C^{\infty}(M, N)$.■

8 Some Basic 2-Category Theory

From this section on we turn to the discussion of higher geometric objects. To this end, we give in this introductory section the necessary categorical background. In order not to overwhelm the lecture with too much category theory we have decided to shorten the exposition and give some textbook references. The following topics are discussed

- strict 2-categories, 2-functors and examples thereof
- pseudo functors and pseudo natural transformations
- bicategories and morphisms thereof (motivated my the bicategory of bimodules over a ring)

In this section we will provide the background on 2-category theory that we use throughout. The treatment will not be exhaustive, we will mainly provide explicit references to the literature that should help in collecting the necessary material as quickly and directly as possible.

Definition 8.1. (cf. [Bor94, Definition 7.1.1] and [Mac98, Section XII.3]) A (strict) 2-category $\mathcal{C}$ consists of

a) a class $\mathrm{Ob}(\mathcal{C})$, whose elements are called objects,

b) for each pair X, Y of objects a small category $\mathcal{C}(X, Y)$,

c) for each triple X, Y, Z of objects a composition functor

$$c_{X,Y,Z}\colon \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$$

(for $\times$ the cartesian product of categories) and

d) for each object X an identity object i_X of $\mathcal{C}(X, X)$ (which we shall frequently identify with its identity morphism in $\mathcal{C}(X, X)$).

These data are required to satisfy the following relations: for each quadruple of objects X, Y, Z, W we have that the composition functor is associative

$$c_{X,Z,W} \circ (c_{X,Y,Z} \times \mathrm{id}) = c_{X,Y,W} \circ (\mathrm{id} \times c_{Y,Z,W})$$

and for each pair of objects X, Y the identity is a left and right unit

$$c_{X,X,Y} \circ (i_X \times \mathrm{id}) = \mathrm{id} = c_{X,Y,Y} \circ (\mathrm{id} \times i_Y).$$

■

Remark 8.2. In comparison to ordinary categories, one has two types of morphisms. For each two objects X, Y one has objects of $\mathcal{C}(X, Y)$, which we call *1-morphisms* (or also 1-arrows). We denote 1-morphisms usually by $f\colon X \to Y$ and depict them by

$$X \xrightarrow{f} Y .$$

But we also have for each two objects $f, g\colon X \to Y$ of $\mathcal{C}(X, Y)$ morphisms in $\mathcal{C}(X, Y)$ between them. Those are called *2-morphisms* (or also 2-arrows). We denote them usually by $\alpha\colon f \Rightarrow g$ and depict them by

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd} .$$

We also usually simply say "$\alpha\colon f \Rightarrow g$ is a 2-morphsim (in $\mathcal{C}$)" instead of "α is a 2-morphism between the 1-morphisms f and g".

There are two types of compositions for different types of morphisms. If we have 2-morphisms $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$, then composition in $\mathcal{C}(X, Y)$ yields another morphisms $\beta \circ \alpha\colon f \Rightarrow h$ in $\mathcal{C}(X, Y)$. If $\alpha\colon f \Rightarrow g$ in $\mathcal{C}(X, Y)$ and $\beta\colon h \Rightarrow k$ is a 2-morphism in $\mathcal{C}(Y, Z)$, then the composition functor yields 1-morphisms $h \circ f := c(f, h)$ and $k \circ g := c(g, k)$ and a 2-morphism $\beta * \alpha\colon h \circ f \Rightarrow k \circ g$. One usually calls $\beta \circ \alpha$ the *vertical composition* and $\beta * \alpha$ the *horzontal composition* of 2-morphisms.■

The following examples are given in somewhat more detail in [Bor94, Example 7.1.4].

Example 8.3. a) Each category $\mathcal{C}$ is a 2-category if we consider for each two objects X, Y of $\mathcal{C}$ the set of morphisms $\mathcal{C}(X, Y)$ as a category with only identity morphisms (thus the resulting 2-category has only identity 2-morphisms). Functors between such categories are the same thing as maps of the underlying objects, thus the composition map $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ provides a composition functor.

b) We obtain a 2-category **Cat** of small categories whose objects are given by the class of small categories and $\mathbf{Cat}(X, Y) := \mathbf{Fun}(X, Y)$ is the functor category of X and Y. The composition functor is given on 1-morphisms by the usual composition of functors and on 2-morphisms by

$$c\colon \mathbf{Fun}(X, Y) \times \mathbf{Fun}(Y, Z) \to \mathbf{Fun}(X, Z), (\alpha, \beta) \mapsto \beta * \alpha,$$

where $\beta * \alpha$ is the natural transformation

$$C \mapsto \beta(g(C)) \circ h(\alpha(C)) = k(\alpha(C)) \circ \beta(f(C)).$$

That this defines indeed a functor follows from $(\delta * \gamma) \circ (\beta * \alpha) = (\gamma \circ \alpha) * (\delta \circ \beta)$ (cf. Exercise 8.12). Together with the identity natural transformation this defines a 2-category (since the composition functor is built out of the associative compositions it is automatically associative and likewise has the identity natural transformation as unit.)

If we consider not arbitrary categories X but only those that are groupoids (and leave the remaining structure unchanged), then we obtain the 2-category **Grpd** of small groupoids.

c) For each two topological spaces let $\mathbf{2\text{-}Top}(X, Y)$ denote the category with objects continuous functions and morphisms the homotopy classes of homotopies (fixing the boundaries) between continuous functions. Then horizontal and vertical composition of homotopies endows this with the structure of a 2-category (and the associativity and unit conditions are satisfied since we have taken homotopies up to homotopies).

d) If G, H are groups, then denote by $\mathbf{2\text{-}Grp}(G, H)$ the category in which an object is a group homomorphism $\phi\colon G \to H$ and a morphism from φ to ψ is an element $\alpha \in H$ such that $\varphi(g) \cdot \alpha = \alpha \cdot \psi(g)$ for all $g \in G$. Then composition of group homomorphisms, multiplication in H and $\beta * \alpha := h(\alpha) \cdot \beta$ turn this into a 2-category. ■

Remark 8.4. There is an obvious notion of morphism of 2-categories, namely that of a (strict) 2-functor. If $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, then a 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ assigns an object $F(X)$ of $\mathcal{D}$ to each object X of $\mathcal{C}$ and to each pair X, Y of objects a functor $F(X, Y)\colon \mathcal{C}(X, Y) \to \mathcal{D}(X, Y)$. This is required to be compatible with the composition functors and identities of $\mathcal{C}$ and $\mathcal{D}$ (see [Bor94, Definition 7.2.1] for details).

However, life is in general not that nice in the sense that many 2-functors are not strict. One example is the "2-functor"

$$\mathbf{B}G\colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grpd}, \quad M \mapsto \mathbf{Bun}(M, G), \quad (f\colon M \to L) \mapsto (f^*\colon \mathbf{Bun}(L, G) \to \mathbf{Bun}(M, G)),$$

from Example 9.4 b). Since $\mathbf{B}G(f \circ g) \neq \mathbf{B}G(g) \circ \mathbf{B}G(f)$ it noes not even make sense to require $\mathbf{B}G$ to be compatible with the composition in $\mathbf{Man}^{\mathrm{op}}$. However, this assignment gives what is called a weak 2-functor (which even preserves identities in this case). ■

Example 8.5. Let $\mathcal{C}$, $\mathcal{D}$ be 2-categories. Then a *pseudo functor* $F\colon \mathcal{C} \to \mathcal{D}$ also assigns an object $F(X)$ of $\mathcal{D}$ to each object X of $\mathcal{C}$ and to each pair X, Y of objects a functor

$$F(X, Y)\colon \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y)).$$

However, it also has as part of its data for each three objects X, Y, Z a natural isomorphism

$$\gamma_{X,Y,Z}\colon c_{F(X),F(Y),F(Z)} \circ (F(X, Y) \times F(Y, Z)) \Rightarrow F(X, Z) \circ c_{X,Y,Z}$$

and for each object X an isomorphism

$$i_{F(X)} \Rightarrow F(X, X)(i_X)$$

in $F(X)$. These are then required to obey the coherence conditions from [Bor94, Definition 7.5.1] (note that the concept of "lax 2-functor" introduces in [Bor94, Definition 7.5.1] agrees with the concept of pseudo functor if all categories $\mathcal{D}(X,Y)$ are groupoids, and thus in all cases that we are interested in).

Likewise, there is a notion of transformation between pseudo functors, which is the one of *pseudo natural transformation*, see [Bor94, Definition 7.5.2]. In the same way like a category has one additional layer of morphisms than sets do have (namely functors and natural transformations), there is an additional layer of morphism for 2-categories, the *modifications* (cf. [Bor94, Definition 7.5.3]). For fixed 2-categories $\mathcal{C}, \mathcal{D}$ (with $\mathcal{C}$ small), the pseudo functors, pseudo natural transformations and modifications constitute a 2-category [Bor94, Proposition 7.5.4]. ■

Definition 8.6. If $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, then a weak 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ is a pseudo functor $F\colon \mathcal{C} \to \mathcal{D}$ (in the sense of [Bor94, Definition 7.5.1]) that preserves identities (i.e. satisfies $F(i_X) = i_{F(X)}$). A morphism of weak 2-functors is a pseudonatural transformation and a 2-morphism of weak 2-functors is a modification. ■

Corollary 8.7. *If $\mathcal{C}$ is a small 2-category, then weak 2-functors, pseudonatural transformations and modifications form a 2-category $\mathcal{D}^{\mathcal{C}}$.*

Remark 8.8. Of course one should in general not spoil the generality that one has obtained by allowing a weak 2-functor not to preserve the composition with forcing it to preserve identities. However, one can show that one does not lose any generality when assuming preservation of identities, and thus we will do so in order to keep the technicalities as simple as possible. ■

Remark 8.9. When we said before that life is in general not that nice that functors are strict, then this also applies to the concept of a 2-category. There we required functors to be the same, namely the functors $c_{X,Z,W} \circ (c_{X,Y,Z} \times \mathrm{id})$ and $c_{X,Y,W} \circ (\mathrm{id} \times c_{Y,Z,W})$ are the *same*, as well as the functors $c_{X,X,Y} \circ (i_X \times \mathrm{id})$, id and $c_{X,Y,Y} \circ (\mathrm{id} \times i_Y)$. However, in many examples it is more natural to require these functors not to be the same but only to differ by a natural equivalence. This the leads to the notion of a bicategory.

A bicategory $\mathcal{C}$ consists of

a) a class $\mathrm{Ob}(\mathcal{C})$, whose elements are called objects,

b) for each pair X, Y of objects a small category $\mathcal{C}(X,Y)$,

c) for each triple X, Y, Z of objects a composition functor

$$c_{X,Y,Z}\colon \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$$

(for $\times$ the cartesian product of categories) and

d) for each object X an identity object i_X of $\mathcal{C}(X,X)$ (which we shall frequently identify with its identity morphism in $\mathcal{C}(X,X)$).

These data are *not* required to satisfy the associativity and unit relations as in Definition 8.1, we rather have the additional data of

e) for each quadruple X, Y, Z, W oj objects a natural isomorphism

$$\alpha_{X,Y,Z,W}\colon c_{X,Z,W} \circ (c_{X,Y,Z} \times \mathrm{id}) \Rightarrow c_{X,Y,W} \circ (\mathrm{id} \times c_{Y,Z,W})$$

f) for each pair of objects X, Y natural isomorphisms

$$\lambda_{X,Y}\colon \operatorname{id} \Rightarrow c_{X,X,Y} \circ (i_X \times \operatorname{id}) \quad \text{and} \quad \rho_{X,Y}\colon \operatorname{id} \Rightarrow c_{X,Y,Y} \circ (\operatorname{id} \times i_Y).$$

These natural isomorphisms are then required to satisfy themselves coherence conditions that we do not spell out here, the entire definition can be found in [Bor94, Section 7.7], [Mac98, Section XII.6] or [Lei98, Section 1.0]. The following example and Exercise 8.13 should be illustrative enough to see the point of the concept. ∎

Example 8.10. Let R be some commutative ground ring and A, B be R-algebras[9] (all with unit). Recall that an A-B bimodules is an R-modules M together with the structure of a left A-module and a right B-module such that $(a.m).b = a.(m.b)$ for all $a \in A$, $m \in M$ and $b \in B$. We denote such an A-B bimodule shortly by ${}_AM_B$. A morphism $\varphi\colon {}_AM_B \to {}_AN_B$ of A-B bimodules is an R-linear map $\varphi\colon M \to N$ that is a morphism of left A-modules and right B-modules. Clearly, the identity and the composition of two such A-B bimodule morphisms is again one and we thus obtain the category $\mathbf{Bim}_R(A, B)$ of A-B bimodules.

If we have an A-B bimodule ${}_AM_B$ and an B-C bimodule ${}_BN_C$, then we can form the relative tensor product $M \otimes_B N$ as follows: Since R and B are unital, we have in particular an R-module structure on M and N so that we can take the tensor product $M \otimes_R N$. So far this is an R-module. Each $a \in A$ induces an R-linear map $a\colon M \to M$ that induces a map $a \otimes \operatorname{id}\colon M \otimes_R N \to M \otimes_R N$. This way $M \otimes_R N$ becomes a left A-modules and in a similar way a right C-module. Now the subset

$$\{m \otimes b.n - m.b \otimes n \mid m \in M, n \in N, b \in B\}$$

generates an A-C-submodule I_B so that $M \otimes_B N := M \otimes_R N/I_B$ carries the structure of an A-C bimodule that we also denote by $M \otimes_B N$ (or ${}_A(M \otimes_B N)_C$ if we want to highlight the bimodule structure). The equivalence class of $m \otimes_R n$ in $M \otimes_B N$ is denoted $m \otimes_B n$. The usual reasoning shows that if ${}_AP_C$ is an A-C bimodule and $\varphi\colon M \times N \to P$ is R-bilinear, left A-linear, right C-linear and B-invariant (i.e., $\varphi(m.b, n) = \varphi(m, b.n)$), then there exists a unique A-C linear map $\overline{\varphi}\colon M \otimes_B N \to P$ such that $\overline{\varphi}(m \otimes_B n) = \varphi(m, n)$.

From the uniqueness assertion it follows that for an A-B linear map $\varphi\colon {}_AM_B \to {}_AM'_B$ and an B-C linear map $\psi\colon {}_BN_C \to {}_BN'_C$ there exists a unique A-C bilinear map

$$\varphi \otimes_B \psi\colon {}_AM \otimes_B N_B \to {}_AM' \otimes_B N'_B.$$

This gives rise to a functor

$$\otimes_B\colon \mathbf{Bim}_R(A, B) \times \mathbf{Bim}_R(B, C) \to \mathbf{Bim}_R(A, C)$$

(the properties for $\otimes_B$ to define a functor follow easily from the above uniqueness assertion).

Note that for bimodules ${}_AM_B$, ${}_BN_C$ and ${}_CP_D$ we do in general have

$$(M \otimes_B N) \otimes_C P \neq M \otimes_B (N \otimes_C P),$$

simply since the underlying sets are not the same. Thus the functors

$$c_{X,Z,W} \circ (c_{X,Y,Z} \times \operatorname{id}) \quad \text{and} \quad c_{X,Y,W} \circ (\operatorname{id} \times c_{Y,Z,W})$$

[9] Note that if $R = \mathbb{Z}$, then rings are precisely R-algebras.

cannot be the *same*, the most one can ask for is that they are *isomorphic*.

Something that one can do at this point is to make *choices*, that means a representative in the isomorphism class of each bimodule. However, this spoils for instance the construction of natural morphisms of bimodules, in the case where $A = B = C$ is a filed this would for instance amount to choosing a basis in each vector space and then expressing every liner map in this basis. ∎

Remark 8.11. Suppose $\mathcal{C}$ and $\mathcal{D}$ are bicategories. Then a homomorphism of bicategories $\mathcal{C} \to \mathcal{D}$ assigns an object $F(X)$ to each object X of $\mathcal{C}$ and to each pair of objects X, Y of $\mathcal{C}$ a functor $F(X,Y)\colon \mathcal{C}(X,Y) \to \mathcal{D}(X,Y)$. Moreover, part of the data are also for each three objects X, Y, Z of $\mathcal{C}$ a natural isomorphism

$$\gamma_{X,Y,Z}\colon c_{F(X),F(Y),F(Z)} \circ F(X,Y) \times F(Y,Z) \Rightarrow F(X,Z) \circ c_{X,Y,Z}$$

and for each object X of $\mathcal{C}$ an isomorphism

$$i_{F(X)} \to F(X,X)(i_X)$$

in $\mathcal{D}(F(X), F(X))$. These are then required to obey the coherence conditions from [Lei98, Section 1.1].

Likewise, there is a notion of transformation between homomorphisms of bicategories (what is called a strong transformation in [Lei98, Section 1.2]) and the one of a modification of transformations (see [Lei98, Section 1.3]). For fixed bicategories $\mathcal{C}, \mathcal{D}$ (with $\mathcal{C}$ small), the homomorphisms, strong transformations and modifications constitute themselves a bicategory [Lei98, Section 2.0]. ∎

Exercises for Appendix 8

Exercise 8.12. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, f, g, h, k be functors, α, β be natural transformations such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd} .$$

Then show that

$$\beta(g(X)) \circ h(\alpha(X)) = k(\alpha(X)) \circ \beta(f(X)) \tag{30}$$

for all objects X of $\mathcal{C}$ and that (30) defines a natural transformation $\beta * \alpha\colon h \circ f \Rightarrow k \circ g$. Moreover, if

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "p"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma"] & \mathcal{D} \arrow[r, bend left=60, "h"] \arrow[r, "k" description] \arrow[r, bend right=60, "q"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathcal{E} \end{tikzcd} ,$$

then show that $(\delta * \gamma) \circ (\beta * \alpha) = (\gamma \circ \alpha) * (\delta \circ \beta)$. ∎

Exercise 8.13. Let $(\mathcal{C}, K)$ be a site such that each $K(C) = \{f\colon D \to C\}$ is a singleton. Recall that then for each morphism $f\colon X \to C$ the pull-back $X \times_C D$ exists. For objects X, Y of $\mathcal{C}$ we the category $\mathbf{Span}(X,Y)$ to be given by objects

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

with $f \in K(X)$ and a morphism from $(X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y)$ to $(X \overset{f'}{\longleftarrow} Z' \overset{g'}{\longrightarrow} Y)$ to be given by a morphism $\alpha \colon Z \to Z'$ such that $f' \circ \alpha = f$ and $g' \circ \alpha = g$:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\ X & & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}.$$

(the composition and identity in $\mathbf{Span}(X,Y)$ is induced from the one in $\mathcal{C}$). We define a "composition functor"

$$\mathbf{Span}(X,Y) \times \mathbf{Span}(Y,Z) \to \mathbf{Span}(X,Z).$$

as follows. If $(X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y, Y \overset{h}{\leftarrow} W \overset{k}{\rightarrow} Z)$ is an object in $\mathbf{Span}(X,Y) \times \mathbf{Span}(Y,Z)$, then we define the composite object to be

$$\begin{tikzcd} & & Z \times_Y W \arrow[dl] \arrow[dr] & & \\ & Z \arrow[dl, "f"'] \arrow[dr, "g"] & & W \arrow[dl, "h"'] \arrow[dr, "k"] & \\ X & & Y & & Z \end{tikzcd}.$$

If (α, α') is a morphism in $\mathbf{Span}(X,Y) \times \mathbf{Span}(Y,Z)$, then this induces a unique morphism on the pull-backs, which we take to be the composite morphism.

Show that this way we obtain a bicategory **Span** and nail down why we do (in general) *not* obtain a 2-category (or under which conditions we obtain a 2-category rather than a bicategory). ■

9 Weak Presheaves in Groupoids and Stacks

This section is the analogue to Section 2 for higher objects. In particular, we give

- the definition of weak presheaves in groupoids,
- the definition of stack and
- the fundamental example (in terms of principal bundles over manifolds).

The above suggest that we first lay down the foundations of presheaves in categories and groupoids. As in the preceding section, we will use these rather abstract category theoretic concepts to justify the geometric constructions later on.

Recall from the introduction that the theme that we follow is that of *groupoidification*, i.e., replace elements in a set by objects of a groupoid, relations (equations) between the elements by morphisms and require that the morphisms are "'coherent"' (in a certain sense). Moreover, we get an additional layer of information, namely the morphisms between the objects we define this way.

Definition 9.1. A category $\mathcal{G}$ is a groupoid if all morphisms are actually isomorphisms[10]. If $\mathcal{C}$ is a category, then a *weak presheaf in groupoids* on $\mathcal{C}$ is a weak 2-functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Grpd}$ into the category of groupoids (where we have regarded $\mathcal{C}^{\mathrm{op}}$ as a 2-category with only identity 2-morphisms as in Example 8.3 a)). More precisely, F assigns

a) to each object C of $\mathcal{C}$ a groupoid $F(C)$,

b) to each morphism $f\colon D \to C$ in $\mathcal{C}$ a functor $F(f)\colon F(C) \to F(D)$ and

c) to each pair of composable morphisms $g\colon E \to D$ and $f\colon D \to C$ a natural transformation[11]

$$\begin{tikzcd} & F(D) \arrow[dr, "F(g)"] \arrow[d, phantom, "{\overset{F(f,g)}{\Downarrow}}"] & \\ F(C) \arrow[ur, "F(f)"] \arrow[rr, "F(f\circ g)"'] & {} & F(E) \end{tikzcd}$$

such that $F(\mathrm{id}_c) = \mathrm{id}_{F(C)}$ (the identity natural transformation) for each object C and for each triple of composable morphisms $h\colon F \to E$, $g\colon E \to D$ and $f\colon D \to C$ the diagram

$$\begin{tikzcd} (F(h)F(g))F(f) \arrow[rr, equal] \arrow[d, Rightarrow, "F(g,h)\,\mathrm{id}_{F(f)}"'] & & F(h)(F(g)F(f)) \arrow[d, Rightarrow, "\mathrm{id}_{F(h)}\,F(f,g)"] \\ F(gh)F(f) \arrow[dr, Rightarrow, "F(f,gh)"'] & & F(h)F(fg) \arrow[dl, Rightarrow, "F(fg,h)"] \\ & F(fgh) & \end{tikzcd} \tag{31}$$

commutes, where $F(g,h)\,\mathrm{id}_{F(f)}$ and $\mathrm{id}_{F(h)}\,F(f,g)$ denote the natural transformations

$$C \mapsto F(g,h)(F(f)(C)) \quad \text{and} \quad C \mapsto F(h)(F(f,g)(C)).$$

A *morphism* $\alpha\colon F \to G$ between the weak presheaves in groupoids F and G is a pseudonatural transformation of weak 2-functors. More precisely, it assigns

a) to each object C of $\mathcal{C}$ a functor $\alpha(C)\colon F(C) \to G(C)$ and

b) to each morphism $f\colon D \to C$ a natural transformation

$$\begin{tikzcd} & F(D) \arrow[dr, "\alpha(D)"] \arrow[dd, phantom, "\Downarrow \alpha(f)"] & \\ F(C) \arrow[ur, "F(f)"] \arrow[dr, "\alpha(C)"'] & & G(D) \\ & G(C) \arrow[ur, "G(f)"'] & \end{tikzcd}$$

[10] Note that there is another related definition of groupoid as generalisation of a group in which the multiplication is only partially defined. this is **not** what we mean here (although the concepts are related).

[11] Note that our natural transformations are automatically natural equivalences since we consider functors and natural transformations with target in the 2-category of *groupoids*, rather than arbitrary categories.

such that for each pair of composable morphisms $g\colon E \to D$ and $f\colon D \to C$ the diagram

$$\begin{tikzcd}
 & G(g)\alpha(D)F(f) \arrow[rr, Rightarrow, "G(g)\alpha(f)"] & & G(g)G(f)\alpha(C) \arrow[dr, Rightarrow, "G(f{,}g)\alpha(C)"] & \\
\alpha(E)F(g)F(f) \arrow[ur, Rightarrow, "\alpha(g)F(f)"] \arrow[drr, Rightarrow, "\alpha(E)F(f{,}g)"'] & & & & G(fg)\alpha(C) \\
 & & \alpha(E)F(fg) \arrow[urr, Rightarrow, "\alpha(fg)"'] & &
\end{tikzcd}$$

commutes. ■

Remark 9.2. a) In Definition 9.1 we allowed the weak 2-functor F not to preserve the composition of morphisms, but to spoil this by a "coherent" choice of natural transofromation. One can also allow F not to preserve the identity in a similar fashion (this is what is also called a morphism of bicategories). However, while the first relaxion is crucial in order to obtain interesting structure, one can show that one may replace a morphism of bicategories always by an isomorphic weak 2-functor. Thus the second relaxion would not bring in additional generaliry (and makes things considerably more complicated.)

b) Morphisms of weak presheaves in groupoids compose in an obvious way and build a category this way. One may also define 2-morphisms between morphisms as *modifications* between pseudonatural transformations (again, when regarding $\mathcal{C}^{\mathrm{op}}$ as a 2-category with only identity 2-morphisms). This then gives a (strict) 2-category of weak presheaves in groupoids on $\mathcal{C}$, denoted $\mathbf{Grpd}^{\mathcal{C}^{\mathrm{op}}}$ (cf. Corollary 8.7). ■

We first give a couple of examples of groupoids.

Example 9.3. a) Each set X can be seen as a groupoid if we define X to be the objects of the groupoid and morphisms to be only identity morphisms. In this way **Set** embeds into **Grpd**.

b) Each category $\mathcal{D}$ gives rise to a groupoid $\mathcal{D}^{\times}$ we take the same objects but only invertible morphisms. Since functors preserve invertible morphisms the assignment $\mathcal{D} \mapsto \mathcal{D}^{\times}$ actually describes a 2-functor $\mathbf{Cat} \to \mathbf{Grpd}$.

c) If $(\mathcal{C}, K)$ is a site and $\mathcal{C}$ has arbitrary coproducts, then for each $R = \{f_i\colon D_i \to C \mid i \in I\} \in K(C)$ we obtain a groupoid $\check{C}(R)$ with set of objects $\coprod_{i\in I} D_i$, morphisms $\coprod_{i,j\in I} D_i \times_C D_j$. The identity morphism is induced by the morphisms $D_i \to D_i \times_C D_i$, the source by the morphisms $\pi_{ij}\colon D_i \times_C D_j \to D_i$ and the target by $\rho_{ij}\colon D_i \times_C D_j \to D_j$ for π_{ij} and ρ_{ij} as in

$$\begin{tikzcd}
D_i \times_C D_j \arrow[r, "\rho_{ij}"] \arrow[d, "\pi_{ij}"'] & D_j \arrow[d, "f_j"] \\
D_i \arrow[r, "f_i"] & C
\end{tikzcd} .$$

The composition is given by

$$\Big(\coprod_{i,j} D_i \times_C D_j\Big) \times_{\coprod_i D_i} \Big(\Big(\coprod_{k,l} D_k \times_C D_l\Big)\Big) \cong \coprod_{D_{i,j,k}} D_i \times_C D_j \times_C D_k \xrightarrow{\coprod \mathrm{pr}_1 \times \mathrm{pr}_3} \coprod_{i,k} D_i \times_C D_k.$$

The flip $D_i \times_C D_j \to D_j \times_C D_i$ endows $\check{C}(R)$ with an inversion and thus turns it into a groupoid. Note that we strictly speaking do not have constructed a category $\check{C}(R)$, but rather a *category object in $\mathcal{C}$*.

d) If G is a Lie group and M is a manifold, then the category $\mathbf{Bun}(M,G)$ is a gropuoid (cf. Exercise D.14). ■

Example 9.4. a) By Example 9.3 a), each presheaf in **Set** can be turned into a weak presheaf in groupoids by composing it with the embedding $\mathbf{Set} \to \mathbf{Grpd}$.

b) The fundamental example of a weak presheaf in groupoids is the following. Fix some Lie group G. Then we set

$$\mathbf{B}G \colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grpd}, \quad M \mapsto \mathbf{Bun}(M,G),$$
$$(f\colon M \to L) \mapsto (f^*\colon \mathbf{Bun}(L,G) \to \mathbf{Bun}(M,G)),$$

where we fix

$$f^*(P) := f^*(\pi\colon P \to L) := \{(m,p) \in M \times P \mid f(m) = \pi(p)\},$$

for $f\colon M \to L$ smooth and $\pi\colon P \to L$ a principal G-bundle (cf. Lemma D.9) and define for a morphism $\varphi\colon P \to Q$ of principal G-bundles over L the morphism $f^*(\varphi)\colon f^*(P \to L) \to f^*(Q \to L)$ to be given by $(m,p) \mapsto (m,\varphi(p))$. Then f^* is a functor (cf. Remark D.10). If $g\colon N \to M$ and $f\colon M \to L$ are smooth, then we have $(f \circ g)^*(P) \neq g^*(f^*(P))$ in general, but rather

$$(f \circ g)^*(P) = \{(n,p) \in N \times P \mid f(g(n)) = \pi(p)\}$$

and

$$g^*(f^*(P)) = \{(n,m,p) \in N \times M \times P \mid g(n) = m \text{ and } f(m) = \pi(p)\}.$$

Thus $\varphi(f,g)\colon g^*(f^*(P)) \to (f \circ g)^*(P)$, $(n,m,p) \mapsto (n,p)$ is an isomorphism of principal G-bundles (with inverse $(n,p) \mapsto (n,g(n),p)$) that is obviously natural. Since the diagram (31) commutes (check this!), this defines the weak presheaf $\mathbf{B}G$ in groupoids.

c) Consider the assignment

$$\mathbf{B}G_{triv}\colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grpd}, \quad M \mapsto B(C^\infty(M,G)), \quad f\colon M \to L \mapsto (f^*\colon B(C^\infty(L,G)) \to B(C^\infty(M,G)))$$

where $B(C^\infty(M,G))$ denotes the groupoid with one object and morphisms the smooth functions $M \to G$. This forms a groupoid if we define the composition of morphisms to be the point-wise multiplication of functions. Clearly, $f^*\colon C^\infty(L,G) \to C^\infty(M,G)$ is a group homomorphism and thus describes a functor. If $g\colon N \to M$ is another smooth function, then we have that the functors $(f \circ g)^*$ and $g^* \circ f^*$ are identically the same (check this!). We thus may take $\varphi(f,g)$ to be the identity natural transformation. Note that this is the variant of $\mathbf{B}G$, where we assign to M not the category of *all* principal G-bundles over M, but the full subcategory with object the *trivial* principal bundle (this subcategory is isomorphic to $B(C^\infty(M,G))$). ■

We now turn to the definition of a stack. For this recall that Section 2 emphasised that manifolds (or more generally speaking objects of an arbitrary category) may be viewed as representable presheaves. Moreover, we saw in Section 3 and 7 how to first consider smooth structures on $C^\infty(M,N)$ as sheaves $\underline{\mathbf{Diff}}(M,N)$ and then identifying them as actually coming from differential geometric structure (more precisely a manifold structure). Notice also that Theorem 4.10 tells us that the prescription of smooth functions from opens in $\mathbb{R}^n$ is sufficient in order to describe the (global) manifold structure on locally metrisable manifolds.

Although we try to promote the perspective to stacks given here as natural, there exist many different perspectives to them. They may also be defined in terms of fibered categories, which are equivalent to weak presheaves in groupoids by the Grothendieck construction. Sometimes smooth stacks are introduced in a rather ad-hoc fashion as “Lie groupoids modulo Morita equivalence”. This highlights the differential geometric structure that one has on a smooth stack, and we will discuss the equivalence of this approach and ours in length later on. Likewise, smooth stacks can equivalently be defined as the category of (left principal) Lie gorupoid bibundles, we will also come back to this in the sequel.

However, the underlying idea is always that a stack is something that is built out of objects that are parametrised (by open subsets of a topological space or by some arbitrary category) that behave well under gluing.

In order to define what a stack is we first have to define what a matching object is. Following the idea of groupoidification, we have to replace elements in a set by objects of a groupoid, relations (equations) between the elements by morphisms and require that the morphisms are “‘coherent”’ (in a certain sense). Moreover, we get an additional layer of information, namely the morphisms between the objects we define this way.

Definition 9.5. Let $(\mathcal{C},K)$ be a site and let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Grpd}$ be a weak presheaf in groupoids. If $R = \{f_i\colon D_i \to C \mid i \in I\} \in K(C)$ is a cover in $K(C)$, then an *object* in $\mathbf{Match}(R,F)$ consists of elements

$$(X_i)_{i\in I} \in \prod_{i\in I} \mathrm{Ob}(F(D_i)) \quad \text{ and } \quad (\varphi_{ij})_{i,j\in I} \in \prod_{i,j\in I} \mathrm{Mor}(F(D_i \times_C D_j))$$

such that

$$\varphi_{ij}\colon F(\pi_{ij})(X_i) \to F(\rho_{ij})(X_j) \quad \text{ and } \quad F(\mathrm{pr}_{\widehat{k}})(\varphi_{ij}) \circ F(\mathrm{pr}_{\widehat{i}})(\varphi_{jk}) = F(\mathrm{pr}_{\widehat{j}})(\varphi_{ik}) \tag{32}$$

where $\pi_{ij}\colon D_i \times_C D_j \to D_i$ and $\rho_{ij}\colon D_i \times_C D_j \to D_j$ are the projections from the pull-back and $\mathrm{pr}_{\widehat{l}}$ is the morphism from the triple pull-back to the double pull-back that omits the l-th component[12]. We will often abbreviate $D_i \times_C D_j$ by D_{ij} and $F(\pi_{ij})(X_i)$ by $X_i|_{D_{ij}}$ (and similarly for multiple pull-backs and other obvious morphisms between them). Then (32) reads

$$\varphi_{ij}\colon\ X_i|_{D_{ij}} \to X_j|_{D_{ij}} \quad \text{ and } \quad \varphi_{ij}|_{D_{ijk}} \circ \varphi_{jk}|_{Dijk} = \varphi_{ik}|_{D_{ijk}}\,.$$

A *morphism* in $\mathbf{Match}(R,F)$ from $((X_i)_{i\in I},(\varphi_{ij})_{i,j\in I})$ to $((Y_i)_{i\in I},(\psi_{ij})_{i,j\in I})$ is an element

$$(\alpha_i)_{i\in I} \in \prod_{i\in I} \mathrm{Mor}(F(D_i))$$

[12]Note that we have canonical isomorphisms $D_i \times_C (D_j \times_C D_k) \cong (D_i \times_C D_j) \times_C D_k$, so we may omit parentheses in multiple pull-backs.

such that $\alpha_i \colon X_i \to Y_i$ and $\alpha_j|_{D_{ij}} \circ \varphi_{ij} = \psi_{ij} \circ \alpha_i|_{D_{ij}}$ for all $i, j \in I$, i.e., each diagram

$$\begin{tikzcd} X_i|_{D_{ij}} \arrow[r, "\alpha_i|_{D_{ij}}"] \arrow[d, "\varphi_{ij}"'] & Y_i|_{D_{ij}} \arrow[d, "\psi_{ij}"] \\ X_j|_{D_{ij}} \arrow[r, "\alpha_j|_{D_{ij}}"] & Y_j|_{D_{ij}} \end{tikzcd}$$

commutes. ∎

Lemma 9.6. *If $(\mathcal{C}, K)$ is a site and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Grpd}$ is a weak presheaf in groupoids, then $\mathbf{Match}(R, F)$ is for each $R = \{f_i \colon D_i \to C \mid i \in I\} \in K(C)$ a category with respect to $\mathrm{id}_{(X_i, \varphi_{ij})} = (\mathrm{id}_{X_i})$ and $(\alpha_i) \circ (\beta_i) = (\alpha_i \circ \beta_i)$. Moreover,*

$$F(C) \to \mathbf{Match}(R, F), \quad (X \mapsto ((X|_{D_i})_{i \in I}, (\varphi_{ij}(F, X))_{i,j \in I})), \ (\alpha \mapsto (\alpha|_{D_i})_{i \in I}), \tag{33}$$

where $\varphi_{ij}(F, X) := F(f_i, \pi_{ij})(X)^{-1} \circ F(f_j, \rho_{ij})(X)$, is a functor.

Proof. This is straight forward and left as Exercise 9.16. ∎

According to the idea of groupoidification, we have to replace the equality in the definition of a sheaf (cf. Definition 2.27) by the corresponding notion for categories. The groupoidified version of a bijection is clearly an equivalence of categories.

Definition 9.7. If $(\mathcal{C}, K)$ is a site and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Grpd}$ is a weak presheaf in groupoids, then F is a *stack* if for each $R \in K(C)$ the functor $F(C) \to \mathbf{Match}(R, F)$ from (33) is an equivalence of categories. The 2-category $\mathbf{St}_{(\mathcal{C}, K)}$ of stacks on $(\mathcal{C}, K)$ is the full 2-subcategory of $\mathbf{Grpd}^{\mathcal{C}^{\mathrm{op}}}$ whose objects are stacks. ∎

Remark 9.8. There exist weakened versions of the notion of a stack: if for each $R \in K(C)$ the functor $F(C) \to \mathbf{Match}(R, F)$ from (33) is fully faithful, then F is called *separated* (or a prestack). ∎

Example 9.9. If $(\mathcal{C}, K)$ is a subcanonical site, then each representable sheaf h^C is a stack if we regard h^C as a (weak) presheaf of groupoids as in Example 9.4 a). Indeed, there are by definition no 2-morphisms (on neither the source, nor the target of the weak presheaf). Thus each object in $\mathbf{Match}(R, F)$ is indeed a matching family and there are no non-identity morphisms in $\mathbf{Match}(R, F)$. Thus $F(C) \to \mathbf{Match}(R, F)$ is an equivalence of categories if and only if it a bijection on the sets of objects, which is the same as the sheaf property. A stack that is isomorphic to some h^C is then called *representable*. ∎

The following is our key example.

Proposition 9.10. *The weak presheaf in groupoids*

$$\mathrm{B}G \colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grpd}, \quad M \mapsto \mathbf{Bun}(M, G), \quad (f \colon M \to N) \mapsto (f^* \colon \mathbf{Bun}(N, G) \to \mathbf{Bun}(M, G)),$$

from Example 9.4 b) is a stack for the open cover topology on $\mathbf{Man}$.

Proof. Let $R = \{U_i \hookrightarrow M \mid i \in I\}$ be an open cover of some manifold M. We first show that the functor $\mathbf{B}G(M) \to \mathbf{Match}(R, \mathbf{B}G)$ is essentially surjective.

An object in $\mathbf{Match}(R, \mathbf{B}G)$ is given by principal G-bundles $P_i \to U_i$ and isomorphisms $\varphi_{ij}\colon\ P_i|_{U_{ij}} \to P_j|_{U_{ij}}$s such that $\varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}}$. For the sake of simplicity let us assume that $\varphi_{ii} = \mathrm{id}$[13]. Then we set

$$P := \coprod_{i\in I} P_i / \sim$$

where we denote the elements of P_i by p_i and define $p_i \sim q_j$ if $\pi_i(p_i) = \pi_j(q_j)$ (in $U_{ij} \subseteq M$) and $\varphi_{ij}(p_i) = q_j$. Then we have a well-defined map $\pi\colon P \to M$, $[p_i] \mapsto \pi_i(p_i)$. We have for each $i \in I$ a bijection

$$\pi^{-1}(U_i) \to P_i, \quad [q_j] \mapsto \varphi_{ij}^{-1}(q_j)$$

that we declare to be a diffeomorphism. Since the coordinate changes are then given by

$$P_i \ni p_i \mapsto \varphi_{ij}(p_i) \in P_j$$

and since these are diffeomorphisms, this gives rise to a well-defined manifold structure on P. Moreover, if $(U_{i,j})_{j\in J_i}$ is a cover of U_i and $\Phi_{i,j}\colon \pi_i^{-1}(U_{i,j}) \to U_{i,j} \times G$ are local trivialisations of P_i, then $(U_{i,j})_{j\in J_i, i\in I}$ is a cover of M and $\Phi_{i,j}\colon \pi^{-1}(U_{i,j}) = \pi_i^{-1}(U_{i,j}) \to U_{i,j} \times G$ are local trivialisation of P. Since the trivialisation changes of P_i are compatible with the G-action so are the ones of P and thus P is a principal G-bundle. Clearly, $P|_{U_i} \to P_i$, $[q_j] \to \varphi_{ij}^{-1}(q_j)$ provide an isomorphism in $\mathbf{Match}(R, \mathbf{B}G)$ to $(P_i)_{i\in I}$ and thus $\mathbf{B}G(M) \to \mathbf{Match}(R, \mathbf{B}G)$ is essentially surjective.

To check that $\mathbf{B}G(M) \to \mathbf{Match}(R, \mathbf{B}G)$ is fully faithful, let $P, Q \to M$ be a principal G-bundles and let $\alpha_i\colon\ P|_{U_i} \to Q|_{U_i}$ be isomorphisms such that $\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$. Then there exists a unique $\alpha\colon P \to Q$ such that $\alpha|_{U_i} = \alpha_i$, given by $p \mapsto \alpha_i(p)$ if $p \in P|_{U_i}$. Thus $\mathbf{B}G(M) \to \mathbf{Match}(R, \mathbf{B}G)$ is fully faithful. ∎

The following is our key not-example.

Proposition 9.11. *The weak presheaf in groupoids*

$$\mathbf{B}G_{triv}\colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grpd}, \quad M \mapsto B(C^\infty(M,G)), \quad f\colon M \to L \mapsto (f^*\colon B(C^\infty(L,G)) \to B(C^\infty(M,G)))$$

from Example 9.4 c) is not *a stack for the open cover topology on* $\mathbf{Man}$.

Proof. Let $R = (U_i)_{i\in I}$ be an open cover of M. An object in $\mathbf{Match}(R, \mathbf{B}G_{triv})$ is a family of smooth functions $(g_{ij}\colon U_{ij} \to G)_{i,j\in I}$ (recall that $U_i \times_M U_j = U_i \cap U_j = U_{ij}$) such that

$$g_{ij}|_{U_{ijk}} \cdot g_{jk}|_{U_{ijk}} = g_{ik}|_{U_{ijk}}. \tag{34}$$

A morphism from (g_{ij}) to (h_{ij}) is given by a family of smooth functions $(f_i\colon U_i \to G)_{i\in I}$ such that

$$f_i|_{U_{ij}} \cdot g_{ij} = h_{ij} \cdot f|_{U_{ij}}.$$

[13] This is not always the case but can be assumed without loss of generality. The more general case will also follow from Theorem 11.16 below.

In order to see that

$$\mathbf{B}G_{triv}(M) \to \mathbf{Match}(R, \mathbf{B}G_{triv}), \quad (* \mapsto (h_{ij} \equiv e), [f\colon M \to G] \mapsto ((f|_{U_i} : U_i \to G)_{i \in I}))$$

is in general not essentially surjective, we have to give an example of a group G and a manifold M, an open covering $(U_i)_{i \in I}$ of M and $(g_{ij}\colon U_{ij} \to G)_{i,j \in I}$ satisfying (34) such that there does not exist $(f_i\colon U_i \to G)_{i \in I}$ with

$$g_{ij} = f_i|_{U_{ij}}^{-1} \cdot f|_{U_{ij}}.$$

This is the case for instance if $G = \{\pm 1\}$, $M = S^1$, (U_1, U_2) is a cover by two open arcs, $g_{11} = g_{22} \equiv 1$, $g_{12} \equiv 1$ on one component and $g_{12} \equiv -1$ on the other component of $U_1 \cap U_2$. ■

Remark 9.12. In the preceding proofs, we have seen that the category $\mathbf{Match}(R, \mathbf{B}G_{triv})$ is just the subcategory of $\mathbf{Match}(R, \mathbf{B}G)$ where all bundles $(P_i)_{i \in I}$ are assumed to be trivial. The proof of Proposition 9.10 then shows that we can "glue" the trivial bundles in order to obtain a new bundle. But this bundles does not have to be trivial, and thus is not isomorphic to an object of $\mathbf{B}G_{triv}(M)$. The upshot of this is:

> Principal bundles form a stack since one can "glue" them along compatible isomorphisms, whereas trivial principal bundles do not form a stack since one can glue trivial bundles and obtain a possibly non-trivial bundle.

Exercises for Section 9

Exercise 9.13. Let $\mathbb{Z}_n$ act on $\mathbb{C}$ (from the right) by $z.[s] := \rho(z, [s]) := e^{\frac{2\pi i}{s}} \cdot z$.

a) Show that $\mathbb{C}/\mathbb{Z}_n$, together with the quotient topology and the quotient map $\mathbb{C} \to \mathbb{C}/\mathbb{Z}_n$ is the limit of the diagram
$$\mathbb{C} \times \mathbb{Z}_n \xrightarrow{\mathrm{pr}_1} \mathbb{C} \xleftarrow{\rho} \mathbb{C} \times \mathbb{Z}_n \tag{35}$$
in **Top**.

b) Show that $\mathbb{C}/\mathbb{Z}_n$ does not possess a manifold structure such that $\mathbb{C} \to \mathbb{C}/\mathbb{Z}_n$ is smooth and the limit of in **Man**. **Hint:** The corresponding statement is true if one considers the action of $\mathbb{Z}_n$ on $\mathbb{C}\backslash\{0\}$. Assuming that there also exists a chart around $0 \cdot \mathbb{Z}_n$ in $\mathbb{C}/\mathbb{Z}_n$, consider the smooth curve $\mathbb{R} \hookrightarrow \mathbb{C}$ and show that this cannot be mapped to a smooth curve in the quotient (for instance if $n = 2$). ■

Exercise 9.14. a) Let $\mathcal{C}$ be a category with finite products. Define the notion of a *group object* in $\mathcal{C}$ (your definition should yield that group objects in **Man** are Lie groups), along with morphisms of group objects. Assure yourself that the definition also works if $\mathcal{C}$ does not have arbitrary products but only the products occurring in the definition and generalise the notion to groupoid objects (and morphisms of them).

b) Show that group objects (respectively morphisms between those) in $\mathbf{PSh}_{\mathcal{C}}$ are those functors $\mathcal{C} \to \mathbf{Set}$ (respectively natural transformations) that have values in **Grp** (respectively in group homomorphisms). Conclude that the category of group objects in $\mathbf{PSh}_{\mathcal{C}}$ is $\mathbf{Grp}^{\mathcal{C}^{\mathrm{op}}}$.

c) Does the same also work in the same manner for algebra objects? ■

Exercise 9.15. Work out Example 9.3 c) in the case of the open covering topology of **Top** explicitly. Even more explicitly, describe $\check{C}(R)$ for R the open covering of $\mathbb{S}^1$ by two (respectively three) open connected arcs such that each $e^{\frac{2\pi i}{n}}$ is for $n = 2$ (respectively $n = 3$) contained in only one arc. ■

Exercise 9.16. If $(\mathcal{C}, K)$ is a site and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Grpd}$ is a weak presheaf in groupoids, show that $\mathbf{Match}(R, F)$ is for each $R = \{f_i\colon D_i \to C \mid i \in I\} \in K(C)$ a category with respect to $\mathrm{id}_{(X_i,\varphi_{ij})} = (\mathrm{id}_{X_i})$ and $(\alpha_i) \circ (\beta_i) = (\alpha_i \circ \beta_i)$. ■

10 Lie Groupoids as Generalisations of Manifolds

This short section contains the following:

- definitions of Lie groupoids (in particular infinite-dimensional), smooth functors and smooth natural transformations between them
- examples
- an illustration of why smooth functors are not general enough (for instance for describing weak equivalences of Lie groupoids appropriately).

As stacks are groupoid versions of sheafs we need a groupoid version of manifolds if we want to investigate "stacks with smooth structure". This rôle will be played by Lie groupoids.

Note: In the same way as groupoids should be thought of as a generalisation of sets, Lie groupoids should be thought of as a generalisation of manifolds (sets with additional smooth structure). Lie groupoids can also be seen as a generalisation of Lie groups (an more generally as a generalisation of Lie group bundles), but this will *not* be our perspective here.

Definition 10.1. In the sequel we will frequently have to refer to pull-backs of maps that are mot uniquely specified from the setting. In this case we write $X \times_Z^{f,g} Y$ for the pull-back of $f\colon X \to Z$ and $g\colon Y \to Z$ (in case that one map is a submersion). In order to simplify notation we will from now on assume that the pull-back is always given by the submanifold

$$X \times_Z^{f,g} Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subseteq X \times Y,$$

rather than a abstractly defined manifold up to diffeomorphism.

A *Lie groupoid* $X = (X_1 \rightrightarrows X_0)$ consists of a smooth manifold X_1 (elements of X_1 are called *morphisms*) and X_0 (elements of X_0 are called *objects*), two submersions $s, t\colon X_1 \to X_0$ (called *source* and *target*), and smooth maps $i\colon X_0 \to X_1$, $c\colon X_1 \times_{X_0}^{s,t} X_1 \to X_1$, $\iota\colon X_1 \to X_1$ (called *identity*, *composition* and *inversion*) satisfying the axioms of a small groupoid:

$$\begin{aligned}
s(c(x,y)) = s(y) \text{ and } t(c(x,y)) = t(x) &\text{ for all } (x,y) \in\colon X_1 \times_{X_0}^{s,t} X_1\\
c(x,c(y,z)) = c(c(x,y),z) &\text{ for all } (x,y,z) \in X_1 \times_{X_0}^{s,t} X_1 \times_{X_0}^{s,t} X_1,\\
s(i(x)) = t(i(x)) = x &\text{ for all } x \in X_0,\\
c(x, i(s(x))) = x = c(i(t(x)), x) &\text{ for all } x \in X_1,\\
s(\iota(x)) = t(x) \text{ and } t(\iota(x)) = s(x) &\text{ for all } x \in X_1,\\
c(x, \iota(x)) = i(t(x)) \text{ and } c(\iota(x), x) = i(s(x)) &\text{ for all } x \in X_1.
\end{aligned}$$

We will often denote $c(x,y)$ by $x \cdot y$ and $\iota(x)$ by x^{-1}. ■

The discussion of morphisms of Lie groupoids will be a bit longer and will not be finished before the end of Section 11. So let us first consider some examples.

Example 10.2. a) Each smooth manifold M gives a Lie groupoid $\underline{M}$ with $\underline{M}_0 = \underline{M}_1 = M$ and all structure maps the identity on M (note that $M \times_M M \cong M$).

b) Each Lie group G gives a Lie groupoid BG with $BG_1 = G$, $BG_0 = *$ and composition and inversion induces my multiplication and inversion in G.

c) If $\mathcal{U} = (U_i)_{i \ni I}$ is an open cover of a manifold M, then the Čech groupoid $\check{C}\mathcal{U}$ is a Lie groupoid (cf. Example c)) with respect to the smooth structure on $\coprod_{i \in I} U_i$ coming from declaring $\coprod_{i \in I} U_i \to M$ to be a local diffeomorphism (and similarly for $\coprod_{i,j \in J} U_{ij}$).

d) If G is a Lie group acting smoothly from the right on some manifold M, then we get a action groupoid $M \rtimes G$ with $(M \rtimes G)_0 = M$, $(M \rtimes G)_1 = M \times G$ and structure maps given by

$$s(m,g) = m \cdot g, \quad t(m,g) = m, \quad (m,g) \cdot (m.g, h) = (m, gh), \quad \iota(m,g) = (m.g, g^{-1}).$$

Note that in the special case that $M = *$ we get $BG = * \rtimes G$.

e) If M is a smooth manifold, then we obtain a Lie groupoid $\mathrm{Pair}(M)$ with $\mathrm{Pair}(M)_1 = M \times M$, $\mathrm{Pair}(M)_0 = M$, $s(m,n) = n$, $t(m,n) = m$, $(m,n) \cdot (n,l) = (m,l)$ and $\iota((m,n)) = (n,m)$. This is called the *pair groupoid* of M.

f) Let $\pi\colon P \to M$ be a principal G-bundle. Then we obtain a modified version of the pair groupoid of P as follows. Consider the action of G on $P \times P$ from the right via $(p,q).g := (p.g, q.g)$. Since the local trivialisations $\Phi\colon \pi^{-1}(U) \to U \times G$ of P commute with the G-action, we have that

$$(P|_U \times P|_U)/G \cong (U \times G \times U \times G)/G \cong U \times U \times G,$$

which we take to endow $(P \times P)/G$ with a smooth manifold structure. The maps $s, t\colon (P \times P)/G \to M$, $s([p,q]) = \pi(q)$, $t([p,q]) = \pi(p)$ are well-defined, as well as the composition map

$$([p,q]) \cdot ([v,w]) := ([p, w.\delta(q,v)]),$$

where $\delta\colon P \times_M P \to G$ is the map uniquely determined by $(p,q) = (p, p \cdot \delta(p,q))$ (see Exercise D.16 and note that if $s([p,q]) = t([v,w])$, then $(q,v) \in P \times_M P$). Finally, the identity is given by $m \mapsto [\Phi^{-1}(m,e), \Phi^{-1}(m,e)]$ if $m \in U$ (note that this is also well-defined on M) and the inversion is given by $[p,q] \mapsto [q,p]$. The resulting Lie groupoid is also called the *gauge groupoid* of P and is denoted by $\mathrm{Gauge}(P)$. ■

Remark 10.3. There is an obvious source for morphisms and 2-morphism of Lie groupoids: A smooth functor $f\colon X \to Y$ of Lie groupoids consists of smooth maps $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to Y_1$ such that it is a functor of the underlying groupoids. A smooth natural transformation $\alpha\colon f \Rightarrow g$ between the smooth functors f, g is a smooth map $\alpha\colon X_0 \to Y_1$ constituting a natural transformation from f to g. We will denote the category of smooth functors from X to Y and smooth natural transformations between them by $\mathbf{Fun}^{\mathbf{sm}}(X,Y)$. ■

Lemma 10.4. *The composition of functors and horizontal composition of natural transformation induces a composition functor*

$$c\colon \mathbf{Fun^{sm}}(X,Y)\times \mathbf{Fun^{sm}}(Y,Z)\to \mathbf{Fun^{sm}}(X,Z).$$

Together with the identity functor this constitutes a 2-category $\mathbf{Lie\text{-}Grpd^{Fun}}$.

Example 10.5. a) If $f\colon M\to N$ is a smooth map, then we obtain a smooth functor $\underline{f}\colon \underline{M}\to\underline{N}$, uniquely determined on objects by $m\mapsto f(m)$. Obviously, we have $\underline{g\circ f}=\underline{g}\circ\underline{f}$. Note that each smooth functor is of this form and that there are no non-trivial smooth natural transformations between two such functors.

b) If $\varphi\colon G\to H$ is a morphism of Lie groups, then we obtain a smooth functor $B\varphi\colon BG\to BH$, uniquely determined by $g\mapsto\phi(g)$ on morphisms. Note again that each smooth functor $BG\to BH$ is of this form, but now a smooth natural transformation $\alpha\colon\varphi\to\psi$ is a element $h\in H$ such that $h\cdot\varphi(g)\cdot h^{-1}=\psi(g)$ for all $g\in G$.

c) If $\varphi\colon G\to H$ is a morphism of Lie groups and $f\colon M\to N$ is a smooth function such that $f(m.g)=f(m).\varphi(g)$, then there is an induced smooth functor $f\rtimes\varphi\colon M\rtimes G\to N\rtimes H$, given on morphisms by $(m,g)\mapsto(f(m),\varphi(g))$. A smooth natural transformation $\alpha\colon f\rtimes\varphi\Rightarrow g\rtimes\psi$ between two such smooth functors is then given by a smooth map $\alpha\colon M\to H$ such that $\alpha(m)\cdot\varphi(g)\cdot\alpha(m)^{-1}=\psi(m)$ for all $m\in M$, $g\in G$. ■

Proof. We only have to check that the usual compositions of functors and natural transformations preserve the property of being smooth. But this is obviously true since these compositions are given by compositions of smooth functions. ■

However, smooth functors and smooth natural transformations do not give the most general notion of morphisms and 2-morphisms between Lie groupoids. This is in particular due to the following failure.

Remark 10.6. Suppose Z is a Lie groupoid. Then we have the weak presheaf

$$\mathbf{Fun^{sm}}(\,\cdot\,,Z)\colon \mathbf{Man}^{\mathrm{op}}\to\mathbf{Grpd}$$

of groupoids, given by

$$M\mapsto\mathbf{Fun^{sm}}(\underline{M},Z)\quad (f\colon N\to M)\mapsto\mathbf{Fun^{sm}}(f,Z):=(f^{*}\colon\mathbf{Fun^{sm}}(\underline{M},Z)\to\mathbf{Fun^{sm}}(\underline{N},Z))\,,$$

where f^{*} denotes the pull-back of functors and natural transformations along the map f. Since $(f\circ g)^{*}=g^{*}\circ f^{*}$, we may take the identity as transformation $\mathbf{Fun^{sm}}(g,Z)\circ\mathbf{Fun^{sm}}(f,Z)\Rightarrow\mathbf{Fun^{sm}}(f\circ g,Z)$ to complete the definition of $\mathbf{Fun^{sm}}(\,\cdot\,,Z)$ as a weak presheaf in groupoids.

However, note that if $Z=BG$ for a Lie group G, then $\mathbf{Fun^{sm}}(M,BG)=B(C^{\infty}(M,G))$, since there is only one smooth functor $\underline{M}\to BG$. Thus $\mathbf{B}G_{triv}=\mathbf{Fun^{sm}}(\,\cdot\,,BG)$ (cf. Example 9.4 c)), and thus $\mathbf{Fun^{sm}}(\,\cdot\,,Z)$ is in general not a stack. ■

11 Smooth Stacks

In this section we construct the bicategory of smooth (infinite-dimensional) stacks. The construction is motivated by the observation of the last section that smooth functors between Lie groupoids are not general enough for describing certain phenomena. The section comprises

- a discussion of actions of Lie groupoids on manifolds,
- the definition of bibundles between Lie groupoids
- a discussion of why bibundles generalise smooth functors
- the construction of the tensor product of bibundles
- the construction of the bicategory **Bun** of Lie groupoids and bibundles and bibundle morphisms (compare to the case of algebras, bimodules and bimodules morphisms).

We have already seen how to remedy the failure of $\mathbf{B}G_{triv}$ to being a stack by considering $\mathbf{B}G$ instead (cf. Proposition 9.10).

For this we need to *interpret principal bundles as (generalised) morphisms from* $\underline{M}$ *to* BG.

Those who feel uncomfortable with the preceding statement should recall that something very similar happens when considering bimodules of algebras as generalisations of morphisms between them. This leads to a more flexible notion of morphism between rings, that helps for instance in characterising rings that have isomorphic module categories (cf. [Wei94, Section 9.5] or [Lam99, Section 18]).

Technically, the aforementioned passage will be done by considering bibundles between Lie groupoids.

Definition 11.1. If P is a manifold and X is a Lie groupoid, then a *smooth right action* of X on P is given by a smooth map $\sigma\colon P \to X_0$ and an a map

$$\rho\colon P \times_{X_0}^{\sigma,t} X_1 \to P,$$

frequently denoted by $p.x := \rho(p,x)$ such that

$$\begin{aligned} \sigma(p.x) = s(x) &\text{ for all } (p,x) \in P \times_{X_0}^{\sigma,t} X_1, \\ p.(x \cdot y) = (p.x).y &\text{ for all } (p,x,y) \in P \times_{X_0} X_1 \times_{X_0} X_1, \\ p.i(x) = p &\text{ for all } p \in P, x \in X_0 \end{aligned}$$

The map σ is also called *moment map* or *anchor map* and the map ρ is also called *action map* (note that it plays a rôle similar to the source map, whence the symbol σ). We then also say that P is a *right X-space*. One similarly defines a smooth left action of X on P and a left X-space (by substituting t by s in the above definition).

A morphism of (right) X-spaces is a smooth map $f\colon P \to Q$ such that $\sigma_Q \circ f = \sigma_Q$ and that $f(m.x) = f(m).x$ for all $(m,x) \in P \times_{X_0} X_1$ (note that then $(f(m),x) \in Q \times_{X_0} X_1$). ■

Example 11.2. a) Each Lie groupoid acts on its own manifold of morphisms with source map as anchor map.

b) A right action of $\underline{N}$ on M is the same thing as a smooth map $f\colon M \to N$, since the action map is entirely determined by requiring $n.i(x) = n$.

c) A right action of BG on M is the same thing as a right action of the Lie group G on M (in the sense of Definition B.6). ■

Remark 11.3. Note that the preceding example tells us that a principal G-bundle over M can be viewed simultaneously as right BG-space and a left $\underline{M}$-space. In order to generalise this to arbitrary Lie groupoids we will need a slightly different perspective to principal bundles in comparison to the definition in Section D. Let M be a smooth manifold and G be a Lie group. We define the category $\mathbf{G\text{-}Sp}_M^{\mathbf{pr}}$ of *principal G-spaces over M* to have as objects right G-spaces P and a submersion $\pi\colon P \to M$ such that $\pi(p.g) = \pi(p)$ for all $p \in P$, $g \in G$ and $P \times G \to P \times_M P$, $(p, g) \mapsto (p, p.g)$ is a diffeomorphism. A morphism in $\mathbf{G\text{-}Sp}_M^{\mathbf{pr}}$ is a morphism $\varphi\colon P \to Q$ of G-spaces such that $\pi^Q \circ \varphi = \pi^P$. Then $\mathbf{G\text{-}Sp}_M^{\mathbf{pr}}$ is canonically isomorphic to $\mathbf{Bun}(M, G)$ (cf. Exercise 11.17). ■

Definition 11.4. Suppose Y is a Lie groupoid and M is a smooth manifold. Then a *principal Y-bundle over M* is a manifold P, together with a smooth right action (σ, ρ) of Y on P and a surjective submersion $\pi\colon P \to M$ such that $\pi(p.y) = \pi(p)$ for all $(p, y) \in P \times_{Y_0}^{\sigma,t} Y_1$ and such that the map

$$P \times_{Y_0}^{\sigma,t} Y_1 \to P \times_M P, \quad (p, y) \mapsto (p, p.y)$$

is a diffeomorphism. A *morphism* of principal Y-bundles P, Q over M is a morphism of right Y-spaces $f\colon P \to Q$ such that $\pi^Q \circ f = \pi^P$.

If, moreover, X is another Lie groupoid, then a *right principal Y-bundle over X* is a manifold P, together with a smooth right action (σ, ρ) of Y and a smooth left action (τ, λ) of X such that that actions commute:

$$\begin{aligned} \sigma(x.p) = \sigma(p) \quad & \text{for all} \quad (x, p) \in X_1 \times_{X_0}^{s,\tau} P \\ \tau(p.y) = \tau(p) \quad & \text{for all} \quad (p, y) \in P \times_{Y_0}^{\sigma,t} Y_1 \\ (x.p).y = x.(p.y) \quad & \text{for all} \quad (x, p, y) \in X_1 \times_{X_0}^{s,\tau} P \times_{Y_0}^{\sigma,t} Y_1 \end{aligned}$$

and the right action of Y and $\tau\colon P \to X_0$ turn P into a principal Y-bundle over X_0 (note that the first two conditions make σ and τ behave even more like a source and target map). We denote such a right principal Y-bundle over X shortly by ${}_X P_Y$, a more exhaustive illustration of the relations of the different maps is

$$\begin{tikzcd} X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \arrow[r, phantom, "\overset{\lambda}{\curvearrowright}"] & P \arrow[dl, "\tau"] \arrow[dr, "\sigma"'] & Y_1 \arrow[l, phantom, "\overset{\rho}{\curvearrowleft}"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ X_0 & & Y_0 \end{tikzcd}$$

A morphism $\varphi\colon {}_X P_Y \to {}_X Q_Y$ of principal right Y-bundles over X is a morphism $\varphi\colon P \to Q$ of right Y-spaces and left X-spaces. The obvious composition and identity morphism then give a category $\mathbf{Bun}(X, Y)$ of right principal Y-bundles over X. ■

Lemma 11.5. *For each two Lie groupoids X, Y the category* $\mathbf{Bun}(Y, X)$ *is a gropuoid.*

Proof. The proof is completely analogous to the one for principal bundles (cf. Exercise D.14) and thus omitted. ∎

Example 11.6. Let X, Y be Lie groupoids and let $f\colon X \to Y$ be a smooth functor. Then we obtain a right principal Y-bundle $P(f)$ over X by setting

$$P(f) := X_0 \times^{f_0,t}_{Y_0} Y_1, \quad \tau(x, y) = x, \quad \sigma(x, y) = s(y)$$

and

$$x'.(x, y) := (t(x'), f_1(x') \cdot y), \quad (x, y) \cdot y' = (x, y \cdot y').$$

This defines the left and right actions of X and Y and they commute since we clearly have

$$\tau((x, y).y') = x = \tau((x, y)), \quad \sigma(x'.(x, y)) = s(y) = \sigma((x, y))$$

and

$$(x'.(x, y)).y' = (t(x'), (f_1(x') \cdot y) \cdot y') = (t(x'), f_1(x') \cdot (y \cdot y')) = x'.(x, y \cdot y') = x'.((x, y).y').$$

Moreover, $\tau\colon P(f) \to X_0$ is a submersion since $t\colon Y_1 \to Y_0$ is so (cf. Proposition C.8). Eventually, the induced map

$$P(f) \times^{\sigma,t}_{Y_0} Y_1 \to P(f) \times_{X_0} P(f), \quad ((x, y), y') \mapsto ((x, y), (x, y \cdot y'))$$

is a diffeomorphism, since $((x, y), (x, y')) \mapsto ((x, y), y^{-1} \cdot y')$ provides an inverse for it.

We sometimes call $P(f)$ the *bundlisation* of the smooth functor f. If $\alpha\colon f \Rightarrow g$ is a smooth natural transformation, then we obtain a morphism $P(\alpha)\colon P(f) \to P(g)$ $(x, y) \mapsto (x, \alpha(x) \cdot y)$. We obviously have $P(\alpha \circ \beta) = P(\alpha) \circ P(\beta)$ and $P(\mathrm{id}_f) = \mathrm{id}_{P(f)}$. Thus P can be interpreted as a functor

$$P\colon \mathbf{Fun}^{\mathbf{sm}}(X, Y) \to \mathbf{Bun}(Y, X).$$ ∎

If we want to consider right principal Y-bundles over X as generalisations of smooth morphisms between Lie groupoids, then we need a way to compose them. This composition should be thought of as somehow analog to the composition of bimodules, if interpreted as generalised morphisms of rings.

Remark 11.7. Suppose X, Y, Z are Lie groupoids and ${}_X P_Y$ is a right prinicpal Y-bundle over X and ${}_Y Q_Z$ is a right principal Z-bundle over Y. We denote by $\tau\colon P \to X_0$, $\sigma\colon P \to Y_0$, $\tau'\colon Q \to Y_0$ and $\sigma'\colon Q \to Z_0$ the respective anchor maps. The depiction of this situation is

$$\begin{tikzcd}
X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \arrow[r, phantom, "\overset{\lambda}{\curvearrowright}"] & P \arrow[dl, "\tau"] \arrow[dr, "\sigma"'] \arrow[r, phantom, "\overset{\rho}{\curvearrowleft}"] & Y_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \arrow[r, phantom, "\overset{\lambda'}{\curvearrowright}"] & Q \arrow[dl, "\tau'"] \arrow[dr, "\sigma'"'] \arrow[r, phantom, "\overset{\rho'}{\curvearrowleft}"] & Z_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
X_0 & & Y_0 & & Z_0
\end{tikzcd} .$$

We construct a new principal Z-bundle ${}_X P \otimes_Y Q_Z$ over X as follows.

In order to construct the underlying manifold we first consider the pull-back

$$P \times_{Y_0}^{\sigma,\tau'} Q.$$

Note that the pull-back $P\times_{Y_0}^{\sigma,\tau'}Q$ exists since τ' is a submersion. Now there is a map $\pi\colon P\times_{Y_0}^{\sigma,\tau'}Q \to Y_0$, $(p,q)\mapsto \sigma(p)=\tau'(q)$, and we obtain a right action of Y on $P\times_{Y_0}^{\sigma,\tau'}Q$ with π as anchor map by setting

$$(p,q).y := (p.y, y^{-1}.q).$$

We endow the quotient $(P\times_{Y_0}^{\sigma,\tau'}Q)/Y$ of this action, i.e., the quotient of the equivalence relation

$$(p,q)\sim(p',q') :\Leftrightarrow (p,q)=(p'.y,y^{-1}.q') \text{ for some } y\in Y_1,$$

with the quotient topology. This turns the quotient map $\mu\colon (P\times_{Y_0}^{\sigma,\tau'}Q)\to(P\times_{Y_0}^{\sigma,\tau'}Q)/Y$ into a continuous map. Since for each $y\in Y$ the map $(p,q)\mapsto(p.y,y^{-1}.q)$ is a homeomorphism of $P\times_{Y_0}^{\sigma,\tau'}Q$ we have that

$$\Big(O\Subset P\times_{Y_0}^{\sigma,\tau'}Q\Big)\Rightarrow\Big(O.Y=\cup_{y\in Y}O.y\Subset P\times_{Y_0}^{\sigma,\tau'}Q\Big)$$

and since $O.Y=\mu^{-1}(\mu(O))$ we have that μ is an open map. ■

Lemma 11.8. *In the setting of the previous remark, there exists a unique manifold structure on*

$$P\otimes_Y Q := (P\times_{Y_0}^{\sigma,\tau'}Q)/Y$$

turning the quotient map $\mu\colon (P\times_{Y_0}^{\sigma,\tau'}Q)\to P\otimes_Y Q$ into a surjective submersion. Moreover, τ induces a well-defined map $P\otimes_Y Q\to X_0$, $[(p,q)]\mapsto\tau(p)$ (which we also denote by τ), that is again a surjective submersion.

Proof. We first observe that $[(p,q)]\mapsto\tau(p)$ is indeed well-defined since we have $\tau(p.y)=\tau(p)$ by assumption. Since $\tau\colon P\to X_0$ is a surjective submersion there exists an open cover $(U_i)_{i\in I}$ of X_0 such that there exist $S_i\colon U_i\to P$ smooth with $\tau\circ S_i=\mathrm{id}_{U_i}$ for each $i\in I$. From the open cover $(\tau^{-1}(U_i)=:P_i)_{i\in I}$ of P we obtain the open cover $(P_i\otimes_Y Q)_{i\in I}$ of $P\otimes_Y Q$ (where we set $P_i\otimes_Y Q:=(P_i\times_{Y_0}^{\sigma,\tau'}Q)/Y$, which is open since μ is so).

We will have to make extensive use of the diffeomorphism $P\times_{X_0}P\to P\times_{Y_0}^{\sigma,t}Y_1$, $(p,y)\mapsto(p,p.y)$, whose inverse we denote by $(\mathrm{pr}_1\times\delta)\colon P\times_{Y_0}^{\sigma,t}Y_1\to P\times_{X_0}P$. Note that then we have $t(\delta(p,p'))=\sigma(p))$, $s(\delta(p,p'))=\sigma(p')$, $\delta(p,p'.y)=\delta(p,p')\cdot y$, $p.\delta(p,p')^{-1}=p'$ and $\delta(p,p)=i(\sigma(p))$. From this we can construct a homeomorphism

$$\Phi_i\colon P_i\otimes_Y Q\to U_i\times_{Y_0}^{\sigma\circ S_i,\tau'}Q,\quad [(p,q)]\mapsto(\tau(p),\delta(S_i(\tau(p)),p).q).$$

This is well-defined, since

$$s(\delta(S_i(\tau(p)),p))=\tau'(q),\quad \sigma(S_i(\tau(p)))=\tau'(\delta(S_i(\tau(p)),p).q)$$

and

$$(\tau(p.y),\delta(S_i(\tau(p.y)),p.y).y^{-1}.q)=(\tau(p),\delta(S_i(\tau(p)),p).q)$$

(check this!). The inverse of Φ_i is given by $(x,q) \mapsto [(S_i(x),q)]$.

We endow $P \otimes_Y Q$ with the smooth structure turning all Φ_i into diffeomorphisms. This yields a well-defined manifold structure on $P \otimes_Y Q$ since the "chart changes"

$$\Phi_i \circ \Phi_j^{-1} \colon U_{ij} \times_{Y_0}^{\sigma\circ S_j,\tau'} Q \to U_{ij} \times_{Y_0}^{\sigma\circ S_i,\tau'}, (x,q) \mapsto (x, \delta(S_i(x)), S_j(x)).q)$$

are smooth (and thus diffeomorphisms) for all $i,j \in I$.

It remains to check that μ and τ are submersion. If we apply the preceding construction to the right principal bundle $Q = Y_1$ over Y, then we obtain a diffeomorphism $P_i \cong U_i \times_{Y_0} Y_1$, with respect to which the quotient map $P_i \times Q \to P_i \otimes_Y Q$ is given by $\mathrm{pr}_{\widehat{2}} \colon U_i \times_{Y_0} Y_1 \times_{Y_0} Q \to U_i \times_{Y_0} Q$. Since this is a submersion, so is μ. Since surjective submersions are effective epimorphisms (cf. Definition 12.1 and Exercise 12.9), the target manifold is uniquely determined up to diffeomorphism. Eventually, we also have that $[(p,q)] \mapsto \tau(p)$, which is locally given by $\mathrm{pr}_1 \colon U_i \times_{Y_0}^{\sigma\circ S_i,\tau'} Q \to U_i$, is a submersion.■

Remark 11.9. Note that the previous statement is in the literature often proved by employing the fact that in finite dimensions the quotient of a manifold by a proper action of a Lie groupoid is again a manifold and the quotient map is a submersion. This statement is not true true any more in infinite dimensions: if we take a non-complemented subspace Y of a Banach space X, then the quotient map $X \to X/Y$ is not a submersion (the differential of any local smooth right inverse of the quotient map would have a continuous linear right inverse of the projection as differential). However, $X \times Y \to X \times X$, $(x,y) \mapsto (x, x+y)$ is a homeomorphism and thus in particular proper.

The situation becomes ways more subtle if one requires that the tangent spaces of the source and target fibres are complemented. In this case one can in principle try to invoke implicit function theorems to obtain a a similar statement also in infinite dimensions. However, the most simple proof in the greatest generality of the construction of P know to the author is the one from the preceding lemma. ■

Proposition 11.10. *In the setting of the previous remark, the maps $P \otimes_Y Q \to X_0$, $[(p,q)] \mapsto \tau(p)$ and $P \otimes_Y Q \to Z_0$, $[(p,q)] \mapsto \sigma'(q)$, along with*

$$x.[(p,q)] := [(x.p,q)] \quad \textit{and} \quad [(p,q)].z := [(p,q.z)] \tag{36}$$

are all well-defined and turn $P \otimes_Y Q$ into a right principal Z-bundle over X, which we denote by ${}_X P \otimes_Y Q_Z$.

Proof. We have already seen that τ is well-defined and $[(p,q)] \mapsto \sigma'(q)$ is well-defined since $\sigma'(y.q) = \sigma'(q)$ by assumption. This implies that (36) is well-defined, and obviously the two actions commute. Since τ is a submersion by the preceding lemma, it remains to check that the induced map

$$(P \otimes_Y Q) \times_{Z_0} Z_1 \to (P \otimes_Y Q) \times_{X_0} (P \otimes_Y Q), \quad [(p,q)], z \mapsto ([(p,q)], [(p,q.z)]) \tag{37}$$

is a diffeomorphism. The inverse to this map is given by

$$([(p,q)], [(p',q')]) \mapsto ([(p,q)], \delta(q, \delta(p,p')^{-1}.q'))$$

(check that this is well-defined!), which is smooth since in local coordinates given by the smooth map

$$(U_i \times_{Y_0}^{\sigma\circ S_i,\tau'} Q) \times (U_i \times_{Y_0}^{\sigma\circ S_i,\tau'} Q) \to U_i \times_{Y_0}^{\sigma\circ S_i,\tau'} Q, \quad (x,q),(x,q') \mapsto (x, \delta(q,q')).$$

Thus (37) is a diffeomorphism. ■

The following is the "consistency check" for the above construction of "composition bundles".

Lemma 11.11. *If $f\colon Y \to X$ and $g\colon Z \to Y$ are smooth functors of Lie groupoids, then there is a unique isomorphism $\gamma(f,g)\colon {}_Z P(g) \otimes_Y P(f)_X \to {}_Z P(f \circ g)_X$ that makes the diagram*

$$\begin{tikzcd} & P(g) \otimes_Y P(f) & \\ Z_0 \arrow[ur, hook] \arrow[dr, hook] & & X_1 \arrow[ul, hook'] \arrow[dl, hook'] \\ & P(f \circ g) & \end{tikzcd}$$

commute.

Proof. This is left as Exercise 11.22. ∎

Remark 11.12. If we have morphisms $\alpha\colon {}_X P_Y \to {}_X P'_Y$ and $\beta\colon {}_Y Q_Z \to {}_Y Q'_Z$ of right principal bundles, then we obtain an induced morphism

$$\alpha * \beta\colon {}_X P \otimes_Y Q_Z \to {}_X P' \otimes_Y Q'_Z, \quad [(p,q)] \mapsto [(\alpha(p), \beta(q))]$$

of principal bundles (the smoothness is a straight-forward check in local coordinates). Since this assignment is obviously compatible with composition and identities, we obtain a functor

$$\otimes_Y\colon \mathbf{Bun}(X,Y) \times \mathbf{Bun}(Y,Z) \to \mathbf{Bun}(X,Z), \quad ({}_X P_Y, {}_Y Q_Z) \mapsto {}_X P \otimes_Y Q_Z, \quad (\alpha,\beta) \mapsto \alpha * \beta.$$

If X, Y, Z and W are Lie groupoids and ${}_X P_Y$ (respectively ${}_Y Q_Z$ and ${}_Z R_W$) is a right principal Y-bundle (respectively Z-bundle and Q-bundle) over X (respectively Y and Z), then we have a canonical isomorphism

$$C_{X,Y,Z,W}(P,Q,R)\colon {}_X((P \otimes_Y Q) \otimes_Z R)_W \to {}_X(P \otimes_Y (Q \otimes_Z R))_W,$$

induced from the diffeomorphism

$$(P \times_{Y_0}^{\sigma,\tau'} Q) \times_{Z_0}^{\sigma',\tau''} R \to P \times_{Y_0}^{\sigma,\tau'} (Q \times_{Z_0}^{\sigma',\tau''} R)$$

given by restricting the identity of $P \times Q \times R$ to the pull-back. Note that $C_{X,Y,Z,W}(P,Q,R)$ is *not* the identity, since source and target are not the same manifolds (they do even not have the same underlying set). However, $C_{X,Y,Z,W}$ constitutes a natural isomorphism $C_{X,Y,Z,W}\colon \otimes_Z \circ (\otimes_Y \times \mathrm{id}) \Rightarrow \otimes_Y \circ (\mathrm{id} \times \otimes_Z)$, where $C_{X,Y,Z,W}$ evaluates on morphisms to the unique morphism induced on the corresponding pull-backs (check the naturality condition as an exercise!).

Moreover, for each Lie groupoid X there is a canonical principal X-bundle $i(X)$ over X, simply given by $i(X) = X_1$ with anchor maps $\tau = t$ and $\sigma = s$ and left and right action by composition in X. If ${}_X P_Y$ is a principal Y-bundle over X, then we have isomorphisms $L_{X,Y}(P)\colon {}_X P_Y \to {}_X i(X) \otimes_X P_Y$ and $R_{X,Y}(P)\colon {}_X P_Y \to {}_X P \otimes_Y i(Y)_Y$, given by

*Comme
isn't this simp
$P(\mathrm{id}_X)$?

$$L_{X,Y}(P)(p) = [(p, i(\tau(p)))] \quad \text{and} \quad R_{X,Y}(P)(p) = [(i(\sigma(p)), p)].$$

As above, we leave it as an exercise to complete the definition of natural isomorphisms $L_{X,Y}\colon \mathrm{id} \Rightarrow \otimes_X \circ (i \times \mathrm{id})$ and $R_{X,Y}\colon \mathrm{id} \Rightarrow C \otimes_Y (\mathrm{id} \times i)$. ∎

Lemma 11.13. *If $f\colon N \to M$ is a smooth map and P is a principal Z-bundle over M, then the principal Z-bundle $P(f) \otimes_M P$ is canonically isomorphic to $N \times_M P$, together with $\mathrm{pr}_1\colon N \times P \to N$ as surjective submersion and the right action of X on the second factor.*

Proof. This is left as Exercise 11.23. ∎

Theorem 11.14. *The assignment $(X, Y) \mapsto \mathbf{Bun}(X, Y)$,*

$$\otimes_Y \colon \mathbf{Bun}(X, Y) \times \mathbf{Bun}(Y, Z) \to \mathbf{Bun}(X, Z)$$

and $X \mapsto P(\mathrm{id}_X)$, together with $C_{X,Y,Z,W}$, $L_{X,Y}$ and $R_{X,Y}$ constitute a bicategory $\mathbf{Bun_{pr}}$.

Proof. It only remains to check that $C_{X,Y,Z,W}$, $L_{X,Y}$ and $R_{X,Y}$ make the corresponding diagrams commute, but this follows readily from the uniqueness assertion in the definition of the corresponding pull-backs. ∎

We now return to the motivation for considering principal bundles as generalised morphisms between Lie groupoids. The original problem was to the failure of

$$\mathbf{Fun^{sm}}(\,\cdot\,, Z)\colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grpd}$$

to being a stack (cf. Remark 10.6). We now know that we should instead consider

$$\mathbf{Bun}(\,\cdot\,, Z)\colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grpd}, \quad M \mapsto \mathbf{Bun}(\underline{M}, Z).$$

We will now complete this assignment to a weak presheaf in groupoids and show that it is indeed a stack.

Remark 11.15. If L, M are smooth manifolds, Z is a Lie groupoid and $f\colon M \to L$ is smooth, then we obtain a smooth functor $\underline{f}\colon \underline{M} \to \underline{L}$ and thus a principal $\underline{L}$-bundle ${}_{\underline{M}}P(\underline{f})_{\underline{L}}$ over $\underline{M}$. Plugging this as first argument into (11.14) yields a functor

$$f^*\colon \mathbf{Bun}(\underline{L}, Z) \to \mathbf{Bun}(\underline{M}, Z), \quad {}_{\underline{L}}P_Z \mapsto {}_{\underline{M}}P(\underline{f}) \otimes_{\underline{L}} P_Z\,, \quad \varphi \mapsto \mathrm{id} * \varphi.$$

If, moreover, $g\colon N \to M$ is smooth, then we have isomorphisms

$$C_{N,M,L,Z}(P(\underline{g}), P(\underline{f}), P)^{-1}\colon {}_N P(\underline{g}) \otimes_M (P(\underline{f}) \otimes_L P)_Z \to {}_N (P(\underline{g}) \otimes_M P(\underline{f})) \otimes_L P_Z$$

and

$$\gamma(\underline{f}, \underline{g}) \otimes_L \mathrm{id}_P \colon {}_N (P(\underline{g}) \otimes_M P(\underline{f})) \otimes_L P_Z \to {}_N P(\underline{f \circ g}) \otimes_L P_Z$$

of principal Z-bundles over $\underline{N}$. We set

$$\varphi(f, g) := \gamma(\underline{f}, \underline{g}) \otimes_L \mathrm{id}_P \circ C_{N,M,L,Z}(P(\underline{g}), P(\underline{f}), P)^{-1}.$$

Clearly, $\varphi(f, g)$ is a natural transformation. ∎

We now can finally prove the main result of this section.

Theorem 11.16. *Let X be a Lie groupoid. Then the assignment $\mathbf{B}X\colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grpd}$*

$$M \mapsto \mathbf{Bun}(\underline{M}, X), \quad (f\colon M \to L) \mapsto f^* \quad \text{and} \quad (f\colon M \to L, g\colon N \to M) \mapsto \varphi(f,g)$$

is a weak presheaf in groupoids. Moreover, it is a stack for the open cover and the surjective submersion coverage on **Man**.

Proof. The proof is in both cases almost entirely analogous to the one of Proposition 9.10. We repeat here the proof for the surjective submersion coverage for convenience. Recall that $f\colon N \to M$ is a smooth map and P is a principal X-bundle over M, then $P(f) \otimes_M P$ is canonically isomorphic to $N \times_M P$. We will throughout identify $P(f) \otimes_M P$ with $N \times_M P$ via this isomorphism.

Let $R = \{f\colon U \to M\}$ be a cover of some manifold M. Then we have the canonical maps $\mathrm{pr}_1, \mathrm{pr}_2\colon U \times_M U \to M$ and also $\mathrm{pr}_{\hat{1}}, \mathrm{pr}_{\hat{2}}, \mathrm{pr}_{\hat{3}}\colon U \times_M U \times_U \to U \times_M U$. We first show that the functor $F(M) \to \mathbf{Match}(R, \mathbf{B}X)$ is essentially surjective. An object in $\mathbf{Match}(R, \mathbf{B}X)$ is given by a principal X-bundles over U and an isomorphism $\varphi\colon\ \mathrm{pr}_1^*(P) \to \mathrm{pr}_2^*(P)$ such that $\mathrm{pr}_{\hat{3}}^*(\varphi) \circ \mathrm{pr}_{\hat{1}}^*(\varphi) = \mathrm{pr}_{\hat{2}}^*(\varphi)$. Since $f\colon U \to M$ is a surjective submersion, there exists an open cover $(U_i)_{i\in I}$ of M and $S_i\colon U_i \to U$ such that $f \circ S_i = \mathrm{id}_{U_i}$. From this we obtain smooth maps

$$\coprod_i S_i\colon \coprod_i U_i \to U \quad \text{and} \quad \coprod_{i,j} S_i \times S_j\colon \coprod_{i,j} U_{ij} \to U \times_M U.$$

We now set $P_i := S_i^*(P) = U_i \times_M^{S_i,\tau} P$, i.e., we have principal X-bundles

$$\begin{tikzcd}
U_i \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] \arrow[r, bend left, "\lambda_i=\mathrm{id}"] & P_i \arrow[dl, "\tau_i"] \arrow[dr, "\sigma_i"'] & X_1 \arrow[l, bend right, "\rho_i"'] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
U_i & & X_0
\end{tikzcd},$$

with $\tau_i(u_i, p) = u_i$, $\sigma_i(u_i, p) = \sigma(p)$ and $(u_i, p).x = (u_i, p.x)$. Moreover, the isomorphism $\mathrm{pr}_1^*(P) \to \mathrm{pr}_2^*(P)$ is given by $(u, v, p) \mapsto (u, v, \varphi(u, v, p))$ and induces isomorphisms

$$\varphi_{ij} := (S_i \times S_j)^*(\varphi)\colon U_{ij} \times_M P \to U_{ij} \times_M P, \quad (u_{ij}, p) \mapsto (u_{ij}, \varphi(S_i(u_{ij}), S_j(u_{ij}), p)).$$

From this we now construct the manifold $\coprod_i P_i$. We abbreviate the element $(u_i, p) \in P_i$ also by p_i and define an equivalence relation on $\coprod_i P_i$ by setting

$$p_i \sim q_j :\Leftrightarrow \tau_i(p_i) = \tau_j(q_j) \quad \text{and} \quad \varphi_{ii}(p_i) = \varphi_{ij}(q_j).$$

Then we set

$$Q := \coprod_{i\in I} P_i / \sim,$$

endowed with the quotient topology. The quotient map $\coprod_{i\in I} P_i \to Q$ is clearly open since it is a local homeomorphism. Since $\tau_i \circ \varphi_{ii}^{-1} = \tau_j \circ \varphi_{ij}^{-1}$ we have a well-defined map $\tau\colon Q \to M$, $[p_i] \mapsto \tau_i(p_i)$. We have for each $i \in I$ a bijection

$$\tau^{-1}(U_i) \to P_i, \quad [q_j] \mapsto \varphi_{ij}^{-1}(q_j)$$

that we declare to be a diffeomorphism. Since the coordinate changes are then given by

$$P_i \ni p_i \mapsto \varphi_{ji}^{-1}(\varphi_{ii}(p_i)) \in P_j$$

and since these are diffeomorphisms, this gives rise to a well-defined manifold structure on Q. Then τ is a smooth submersion since the restriction of τ to each $\tau^{-1}(U_i)$ is so. Moreover, we get a well-defined smooth map $\sigma\colon Q \to X_1$, $[p_i] \mapsto \sigma_i(p_i)$ and an induced right action of X by $[p_i].x := [p_i.x]$.

In order to check that Q is a principal X-bundle over M we have to check that

$$Q \times_{X_0}^{\sigma,t} X_1 \to Q \times_M^{\tau,\tau} Q, \quad ([p_i], x) \mapsto ([p_i].[p_i].x) \tag{38}$$

is a diffeomorphism. But we already know that

$$P \times_{X_0}^{\sigma,t} X_1 \to P \times_U^{\tau,\tau} P, \quad (p, x) \mapsto (p, p.x)$$

is a diffeomorphism, whose inverse we denote by $(\mathrm{pr}_1 \times \delta)$. An element of $Q \times_M^{\tau,\tau} Q$ is a pair $[(u_i, p), (v_j, q)]$ with $u = v$ (as elements of M), $\sigma_i(u_i) = \tau(p)$ and $\sigma_j(v_j) = \tau(q)$. This implies $\sigma_j(u_i) = \tau(\varphi(p))$ and thus $(\varphi(p), q) \in P \times_U^{\tau,\tau} P$. With this we obtain the inverse $([(u_{ij}, p)], [(u_{ij}, q)]) \mapsto ([(u_{ij}, p)], \delta(\varphi(p), q))$ to (38).

By construction we have $f^*Q = \{(u, [(u_i, p)] \mid S_i(u_i) = \tau(p) \text{ and } f(u) = u_i\}$. Thus

$$(S_i(u_i), u, p) \in (U \times_M^{f,f} U) \times_U^{\mathrm{pr}_1,\tau} P$$

and we obtain an isomorphism

$$f^*Q \to P, \quad (u, [(u_i, p)]) \mapsto \varphi(S_i(u_i), u, p)$$

of principal X-bundles over U with inverse $p \mapsto (\tau(p), [f(\tau(p))_i, \varphi_{ii}^{-1}(f(\tau(p))_{ii}, p)])$. This provides an isomorphism in $\mathbf{Match}(R, \mathbf{B}X)$ from $f^*(P)$ to Q and thus $F(M) \to \mathbf{Match}(R, \mathbf{B}X)$ is essentially surjective.

To check that $F(M) \to \mathbf{Match}(R, \mathbf{B}G)$ is fully faithful, let $P, Q \to M$ be a principal X-bundles over M and let

$$\alpha\colon f^*(P) \to f^*(Q), \quad (u, p) \mapsto (u, \alpha(u, p))$$

be an isomorphism such that $\mathrm{pr}_1^*\alpha = \mathrm{pr}_2^*\alpha$ (i.e., $\alpha(u, p) = \alpha(v, p)$ if $f(u) = f(v)$). Then there exists a unique $\beta\colon P \to Q$ such that $\alpha = f^*(\beta)$. Indeed, if $S\colon U_{\tau(p)} \to U$ is a smooth and local right inverse of f, then $\beta(p) = \alpha(S(\tau(p)), p)$ is smooth and independent of the choice of S. Thus $F(M) \to \mathbf{Match}(R, \mathbf{B}G)$ is fully faithful. ■

Exercises for Section 10

Exercise 11.17. Show that each principal G-space over M is in fact a principal G-bundle over M and that the functor $\mathbf{G\text{-}Sp}_M^{\mathbf{pr}} \to \mathbf{Bun}(M, G)$ yields an isomorphism of categories $\mathbf{G\text{-}Sp}_M^{\mathbf{pr}} \cong \mathbf{Bun}(M, G)$.■

Exercise 11.18. Let G be a Lie group, acting on itself by right translation. Show that the action groupoid $G \rtimes G$ is actually equal to the pair groupoid $\mathrm{Pair}(G)$. ■

Exercise 11.19. Fill in the details of the construction of the gauge groupoid $\mathrm{Gauge}(P)$ of a principal G-bundle $\pi\colon P \to M$ over G. In particular, show that the identity map $M \to (P \times P)/G$, $m \mapsto \Phi^{-1}(m, e), \Phi^{-1}(M, e)$ (for Φ a local trivialisation at m) is well-defined, that the composition map $[(p, q)], [(v, w)] \mapsto [(p, w.\delta(q, v))]$ is well-defined and smooth and that the inversion map is given by $[(p, q)] \mapsto [(q, p)]$ and is smooth. ■

Exercise 11.20. Let G be a Lie group and M be a manifold. Show that a principal BG-bundle over $\underline{M}$ is the same thing as a principal G-bundle over M. Then show that the bundlisation $P(f)$ of a smooth functor $f\colon \underline{M} \to BG$ always gives the trivial principal G-bundle over M. ■

Exercise 11.21. Show that the bundlisation functor

$$P\colon \mathbf{Fun}^{\mathbf{sm}}(X, Y) \to \mathbf{Bun}(Y, X).$$

is fully faithful, but in general not essentially surjective. ■

Exercise 11.22. Show that composition of principal bundles is compatible with composition of smooth functors (under bundlisation), i.e., that if $f\colon Y \to X$ and $g\colon Z \to Y$ are smooth functors of Lie groupoids, then there is a unique isomorphism $\gamma(f,g)\colon {}_Z P(g) \otimes_Y P(f)_X \to {}_Z P(f \circ g)_X$ that makes the diagram

$$\begin{tikzcd}
& P(g) \otimes_Y P(f) & \\
Z_0 \arrow[ur, hook] \arrow[dr, hook] & & X_1 \arrow[ul, hook'] \arrow[dl, hook'] \\
& P(f \circ g) &
\end{tikzcd}$$

commute. ■

Exercise 11.23. Show that $f\colon N \to M$ is a smooth map and P is a principal Z-bundle over M, then the principal Z-bundle $P(f) \otimes_M P$ is canonically isomorphic to $N \times_M P$, together with $\mathrm{pr}_1\colon N \times P \to N$ as surjective submersion and the right action of X on the second factor. ■

Exercise 11.24. Let G be a Lie group and $\pi\colon P \to M$ be a principal G-bundle over M. Suppose that G acts smoothly on a lcs V from the left by linear continuous automorphisms (i.e, we have a homomorphism $\lambda\colon G \to \mathrm{Gl}(V)$ such that the action map $G \times V \to V$ is smooth). Then we may view V as a principal $\underline{\mathrm{pt}}$-bundle over BG (pt is the point, so a principal $\underline{\mathrm{pt}}$ no additional structure). Then we can build the bundle

$$P \times_{BG} V,$$

which is a principal $\underline{\mathrm{pt}}$-bundle over M. Show that

$$P \times_{BG} V = (P \times V)/G,$$

where G acts on $P \times V$ via $(p, v).g := (p.g, g^{-1}.v)$. Moreover, show that $P \times_{BG} V$ is in fact a vector bundle over M (which is usually called the *vector bundle associated to P via λ*). ■

Exercise 11.25. (The Hopf fibration) Let S^3 be the three sphere, given by $S^3 = \{(z, w) \in \mathbb{C}^2 \mid z\overline{z} + w\overline{w} = 1\}$. Then $S^1 = \{x \in \mathbb{C} \mid x\overline{x} = 1\}$ acts on S^3 by $(z, w).x = (z \cdot x, w \cdot x)$. Show that $S^3/S^1 \cong S^2$ and that the quotient map turns S^3 into a principal S^1-bundle over S^2 (with respect to the above identification). ■

12 Subcanonical Sites

This short section collects the concepts from topos theory on subcanonical sites that we need in the following sections.

The fact that in the representable functors in Example 2.29 were sheaves was not a coincidence. We usually want that they are, and now consider under which condition on the site this is the case.

Definition 12.1. Let $(\mathcal{C}, K)$ be a site and $R := \{f_i \colon D_i \to C \mid i \in I\}$ be a cover in $K(C)$. Define J_R to be the category with set of objects $\{D_i \mid i \in I\}$ and

$$J_R(D_i, D_j) = \left\{ f \in \mathcal{C}(D_i, D_j) \;\middle|\; \begin{tikzcd} D_i \arrow[rr, "f"] \arrow[dr, "f_i"'] & & D_j \arrow[dl, "f_j"] \\ & C & \end{tikzcd} \text{ commutes} \right\}.$$

Then J_R is a subcategory of $\mathcal{C}$, and the inclusion $J_R \to \mathcal{C}$ defines a diagram A of type J_R in $\mathcal{C}$. By definition, $(f_i \colon D_i \to C)_{i \in I}$ is a cocone over A, since for each morphism $f \colon D_i \to D_j$ in J_R the diagram

$$\begin{tikzcd} D_i \arrow[rr, "f"] \arrow[dr, "f_i"'] & & D_j \arrow[dl, "f_j"] \\ & C & \end{tikzcd}$$

commutes. If $(f_i \colon D_i \to C)_{i \in I}$ is actually universal (a colimit), then R is called an *effective-epimorphic* cover. If each $R \in K(C)$ is effective-epimorphic for each object C of $\mathcal{C}$, then the site $(\mathcal{C}, K)$ (or sometimes also only the coverage K) is called *subcanonical*. ■

The importance of this definition is the following

Proposition 12.2. *A site $(\mathcal{C}, K)$ is subcanonical if and only if each representable presheaf is a sheaf.*

Proof. We only show the if part, the only if part is left as Exercise 12.7. Let h^Z, $X \mapsto \mathcal{C}(X, Z)$ be representable, $R := \{f_i \colon D_i \to C \mid i \in I\}$ be a cover of C and $(g_i) \in \prod_i \mathcal{C}(D_i, Z)$ be a matching family. We want to check that $(g_i \colon D_i \to Z)_{i \in I}$ is a cocone over J_R. To this end, we claim that if φ satisfies $f_j \circ \varphi = f_i$ (makes the inner square of the following diagram commute), then it also satisfies $g_j \circ \varphi = g_i$ (makes the diagram without f_i, f_j and C commute):

$$\begin{tikzcd} D_i \times_C D_j \arrow[r, "\rho_{ij}"] \arrow[d, "\pi_{ij}"'] & D_j \arrow[d, "f_j"] \arrow[ddr, bend left, "g_j"] & \\ D_i \arrow[r, "f_i"] \arrow[ur, dotted, "\varphi"] \arrow[drr, bend right, "g_i"'] & C & \\ & & Z \end{tikzcd}$$

Indeed, from the universal property of the pull-back

$$\begin{tikzcd} D_i \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, equal, bend right] & & \\ & D_i \times_C D_j \arrow[r, "\rho_{ij}"] \arrow[d, "\pi_{ij}"'] & D_j \arrow[d, "f_j"] \\ & D_i \arrow[r, "f_i"] & C \end{tikzcd}$$

we obtain a morphism $h\colon D_i \to D_i \times_C D_j$ such that $\varphi = \rho_{ij} \circ h$ and $\pi_{ij} \circ h = \mathrm{id}_{D_i}$. Thus

$$g_j \circ \varphi = g_j \circ \rho_{ij} \circ h = g_i \circ \pi_{ij} \circ h = g_i.$$

This implies that $(g_i\colon D_i \to Z)_{i\in I}$ is a cocone over J_R and thus there exists a unique morphism $k\colon C \to Z$ such that $g_i = k \circ f_i = h^Z(f_i)(k)$ for all i. This is precisely the condition on h^Z for being a sheaf. ∎

With a similar idea as in the previous proof we also show the following

Lemma 12.3. *Le $(\mathcal{C}, K)$ be a site such that each cover in $K(C)$ consists of a single morphism $\{f\colon D \to C\}$. Then $(\mathcal{C}, K)$ is subcanonical if and only if for each cover the morphism $f\colon D \to C$ is a colimit (or coequaliser) for the two canonical morphism $D \times_C D \rightrightarrows D$.*

Proof. We only show the if part, the only if part is left as Exercise 12.8. Let $R = \{f\colon D \to C\}$ be a cover of C. We have to show that whenever we have $g\colon D \to E$ with the property

$$\big(\varphi \in \mathcal{C}(D, D) \text{ satisfies } f \circ \varphi = f\big) \Rightarrow g = g \circ \varphi,$$

then there exists a unique $h\colon C \to E$ such that $g = h \circ f$:

$$\begin{tikzcd}
& D \arrow[d, "f"] \arrow[ddr, bend left, "g"] & \\
D \arrow[ur, "\varphi"] \arrow[r, "f"] \arrow[drr, bend right, "g"'] & C \arrow[dr, dashed, "\exists! h"] & \\
& & E
\end{tikzcd}$$

Note that φ is variable, so that $D \xrightarrow{f} C$ is then a colimit for the diagram

$$\left\{ \varphi \circlearrowright D \;\middle|\; \varphi \in \mathcal{C}(D, D) \text{ satisfies } f \circ \varphi = f \right\}.$$

If one such morphism $g\colon D \to E$ is given, then it also make

$$\begin{tikzcd}
D \arrow[dr] \arrow[drr, bend left, "\varphi"] \arrow[ddr, equals] & & & \\
& D \times_C D \arrow[r] \arrow[d] & D \arrow[d, "f"] \arrow[ddr, bend left, "g"] & \\
& D \arrow[r, "f"] \arrow[drr, bend right, "g"'] & C & \\
& & & E
\end{tikzcd}$$

commute. Since $f\colon D \to C$ is assumed to be the coequaliser there now exists a unique $h\colon C \to E$ satisfying $g = h \circ f$. ∎

Definition 12.4. Let $f\colon C \to D$ be a morphism in a category such that the pull-back $D \times_C D$ exists. Then f is called an *effective epimorphism* if $f\colon D \to C$ is a colimit for the two canonical morphisms $D \times_C D \rightrightarrows D$. ∎

Remark 12.5. One calls a square

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "f_2"'] & E_1 \arrow[d, "g_1"] \\ E_2 \arrow[r, "g_2"] & D \end{tikzcd}$$

of morphisms in a category $\mathcal{C}$ *cartesian* if $(C \xrightarrow{f_i} E_i)_{i=1,2}$ is a limit (pullback) of $E_2 \xrightarrow{g_2} D \xleftarrow{g_1} E_1$ and *cocartesian* if $(E_i \xrightarrow{g_i} D)_{i=1,2}$ is a colimit of $E_2 \xleftarrow{f_2} C \xrightarrow{f_1} E_1$. Thus in the previous lemma, $f\colon D \to C$ is a colimit for $D \times_C D \rightrightarrows D$ if and only if the square

$$\begin{tikzcd} D \times_C D \arrow[r] \arrow[d] & C \arrow[d] \\ C \arrow[r] & C \end{tikzcd}$$

is cartesian and cocartesian (a limit and a colimit). ■

Example 12.6. a) The local section coverage on **Top** is subcanonical: if $f\colon D \to C$ is surjective and admits local sections, then

$$\begin{tikzcd} D \times_C D \arrow[r] \arrow[d] & D \arrow[d] \\ D \arrow[r] & C \end{tikzcd}$$

is cartesian and cocartesian. Indeed, if we use the realisation $D \times_C D = \{(d, d') \in D \times D \mid f(d) = f(d')\}$, then a continuous map $g\colon D \to E$ makes both maps $D \times_C D \to E$ coincide iff $f(d) = f(d')$ implies $g(d) = g(d')$. Thus there is a well-defined and unique map $h\colon C \to E$ with $g = h \circ f$. Since f admits local sections, h is locally expressible as $g \circ \sigma$ for some section $\sigma\colon V \subseteq C \to D$ and thus h is continuous. ■

Exercises for Section 12

Exercise 12.7. Show the only if part of Proposition 12.2: If each representable functor on a site $(\mathcal{C}, K)$ is a sheaf, then the site is subcanonical. ■

Exercise 12.8. Show the only if part of Lemma 12.3: If the covers of a coverage consist only of singletons $\{f\colon D \to C\}$, then the morphism $f\colon D \to C$ is a colimit (or coequaliser) for the two canonical morphism $D \times_C D \rightrightarrows D$ if the site $(\mathcal{C}, K)$ is subcanonical. ■

Exercise 12.9. Show that **Man**, together with the surjective submersion coverage is a subcanonical site. ■

A Differential Calculus in Infinite Dimensions

In this appendix we give the background on differential calculus in locally convex spaces.

If we want to understand function spaces like $C^\infty(S^1, \mathbb{R})$ as geometric objects (smooth manifolds), then we first have to understand the differentiability properties of maps between infinite-dimensional vector spaces. The approach to infinite-dimensional differential calculus that we take here goes back to Hamilton [Ham82] and Milnor [Mil84]. More recently, Glckner and Neeb used this approach in a quite general account on infinite-dimensional Lie groups [GN14]. Note that this approach does note require the modelling spaces to be complete (see [Glö02]), which makes the notion quite easy accessible.

Definition A.1. If X is a real vector space, then a *half-norm* on X is a map $p\colon X \to \mathbb{R}_{\geq 0}$ such that $p(t \cdot x) = |t| \cdot p(x)$ and $p(x+y) \leq p(x) + p(y)$ for each $t \in \mathbb{R}$, $x, y \in X$. If, in addition, $p(x) = 0$ implies $x = 0$, then p is called a *norm* on X. The norm of a vector is also often denoted by $\|x\|$. A family $(p_i)_{i \in I}$ of semi-norms on X is said to *separate the points* of X if

$$p_i(x) \text{ for all } i \in I \Rightarrow x = 0$$

holds.

A topology τ on X is called *vector topology* if it is Hausdorff and the addition and scalar multiplication

$$X \times X \to X, \quad (x, y) \mapsto x + y \qquad \mathbb{R} \times X \to X, \quad (t, x) \mapsto t \cdot x$$

are continuos (with respect to the product topology). The pair (X, τ) (often simply abbreviated by X) is then called *topological vector space* (shortly tvs). A morphism of topological vector spaces is a linear map that is continuous. With respect to composition of maps this defines the category **TopVect** of topological vector spaces. The morphisms from X to $\mathbb{R}$ will throughout be denoted by X'. ■

Example A.2. a) If p is a norm on X, then the sets

$$B_{\varepsilon,x} := \{y \in X \mid p(x - y) < \varepsilon\} \tag{39}$$

form the basis for a vector topology on X.

b) If $(p_i)_{i \in I}$ is a family of semi-norms on X that separate the points of X, then the sets

$$B_{J,\varepsilon_1,\ldots,\varepsilon_{|J|},x} := \{y \in X \mid p_{i_k}(x - y) < \varepsilon_k \text{ for } 1 \leq k \leq |J|\} \tag{40}$$

(where $J \subseteq I$ runs through all finite subsets of I) form a basis for a vector topology on X. If $\ker(p_n) := \{x \in X \mid p_n(x) = 0\}$ is the kernel of p_n, then $X/\ker(p_n)$ is a normed space and the topology induced by (40) is precisely the initial topology for the projections $\pi_n\colon X \to X/\ker(p_n)$.

If X is a topological vector space and there exists a family of semi-norms such that the induced topology is the originally given one, then we say that X is a *locally convex space* (shortly lcs). Note that the semi-norms are *not* part of the data of a lcs, but that local convexity is (by definition) a property of the topology. If fact, a vector topology on X is locally convex if and only if it possesses a basis of zero neighbourhoods that are convex. Supplements on this notion can be found in [Rud91, Ch. I.1] and [Wer00, Ch. VIII]. If we restrict to locally convex spaces, then we obtain the category **Lcs** as a full subcategory of **TopVect**.

c) If $(p_1, \ldots, p_n)$ is a *finite* family of semi-norms on X that separates the points of X, then

$$p(x) := \max\{p_1(x), \ldots, p_n(x)\}$$

defines a norm on X (note that the ∞-norm on $\mathbb{R}^n$ is of this form). In this case, the topologies on X that are induced by (39) and (40) coincide. In this case we call X *normable*. If X is complete is complete with respect to the metric $d(f,g) := p(f-g)$, then X is a *Banach space*.

d) If X is a vector space and $d\colon X \times X \to \mathbb{R}_{\geq 0}$ is a translation invariant metric on X (i.e., if we have $d(x-z, y-z) = d(x,y)$ for all $x,y,z \in X$), then the pair (X,d) is called *metric* vector space. The topology induced from d then turns X into a topological vector space. This need not be locally convex, as the example

$$d((x_1,x_2),(y_1,y_2)) := (\sqrt{|x_1-y_1|} + \sqrt{|x_2-y_2|})^2$$

shows. Moreover, the metric will be (in general) not compatible with the scalar multiplication in the sense that $d(t\cdot x, t\cdot y) = |t| d(x,y)$, since then $x \mapsto d(0,x)$ defines a norm on X inducing the same topology.

e) A natural source for metric vector spaces that are not normable is the following. If $(p_n)_{n\in\mathbb{N}}$ is a *countable* family of semi-norms on X, then the assignment

$$d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n} \min\{p_n(x-y), 1\} \tag{41}$$

defines a translation invariant metric on X. If X is complete with respect to d, then X is called *Fréchet space*. Observe that the topology on X that is induced by d coincides with the topology induced by (40) (cf. Exercise A.15).

For instance, on $\mathbb{R}^{\mathbb{N}} = \prod_{n\in\mathbb{N}} \mathbb{R}$ we have the semi-norms $p_n((x_i)) = |x_n|$ that clearly separate the points. The induced topology is the product topology. This is even a Fréchet space (why?).

f) The first geometrically motivated example of a lcs is the vector space $C(X,\mathbb{R})$ of all continuous real valued functions on a topological space X. If X is compact, then we have on $C(X,\mathbb{R})$ the norm

$$p_\infty(f) := \sup\{|f(x)| : x \in X\},$$

which is finite sine $f(X) \subseteq \mathbb{R}$ is compact and thus bounded. Since the limit of a uniformly convergent sequence of continuous functions is again continuous (see [Wer00, Ex. I.1.c] if you have not seen the argument in this generality before) it also follows that $C(X,\mathbb{R})$ is a Banach space.

If X is not compact, then a similar construction works if we assume that there exists a sequence $K_1, K_2, \ldots$ of compact subsets wiht $\cup_{i\in\mathbb{N}} K_i = X$ (this applies for instance to $X = \mathbb{R}^n$ and thus to each submanifold of $\mathbb{R}^n$). By replacing K_n with $K_1 \cup \ldots \cup K_n$ we may assume without loss of generality that $K_1 \subseteq K_2 \subseteq \ldots$. From this we obtain the family of semi-norms

$$p_n(f) := \sup\{|f(x)| : x \in K_n\},$$

which is finite since $f(K_n) \subseteq \mathbb{R}$ is bounded. Then $(p_n)_{n\in\mathbb{N}}$ separates the points of $C(X,\mathbb{R})$, since

$$p_n(f) = 0 \,\forall n \in \mathbb{N} \Leftrightarrow f|_{K_n} \equiv 0 \,\forall n \in \mathbb{N} \Leftrightarrow f \equiv 0$$

follows from $\cup_{i\in\mathbb{N}} K_i = X$. The resulting topology turns $C(X,\mathbb{R})$ into a Fréchet space. In order to check that $C(X,\mathbb{R})$ is complete, take a Cauchy sequence $(f_k)_{k\in\mathbb{N}}$ with respect to the metric (41). Then $(f_k|_{K_n})$ is a Cauchy sequence in $C(K_n,\mathbb{R})$ (check this!) and thus has a limit $g_n \in C(K_n,\mathbb{R})$. Since for $m < n$ we have that $g_n|_{K_m}$ is a limit of $f|_{K_n}|_{K_m} = f|_{K_m}$ (check this!) it follows that $g_n|_{K_m} = g_m$ and thus

$$x \mapsto g_n(x) \text{ if } x \in K_n$$

defines an element $f \in C(X,\mathbb{R})$. Since $(p_n(f_k - f))_{k\in\mathbb{N}}$ converges to 0 for each $n \in \mathbb{N}$, it follows that $f_n \to f$ in the metric (41) (check this!).

g) Now suppose X is compact, Y is an lcs and $(p_i)_{i\in I}$ is a family of point-separating semi-norms on Y. Then

$$P_i(f) := \sup\{p_i(f(x)) : x \in X\}$$

is a point-separating family of semi-norms on $C(X,Y)$ which endows $C(X,Y)$ with the structure of a lcs. If, moreover, I is countable and Y is a Fréchet space, then $C(X,Y)$ then is so. In fact, if $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C(X,Y)$, then $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence for each $x \in X$. Since Y is complete we may define

$$f(x) := \lim_{k\to\infty} f_k(x).$$

In order to check that f is continuous it suffices to check that $\pi_n \circ f$ is continuous for each $n \in \mathbb{N}$, where $\pi_n \colon Y \to Y/\ker(p_n)$ is the canonical projection. Since $Y/\ker(p_n)$ is complete [Rud91, Theorem 1.41], it is a Banach space. We thus have that the point-wise limit of $(\pi_n(f_k(x)))_{k\in\mathbb{N}}$ is continuous and coincides with $\pi_n \circ f$. Thus f is continuous. Since the topology on $C(X,Y)$ may also be constructed as the initial topology for the maps $C(X,Y) \to C(X,Y/\ker(p_n))$, $f \mapsto \pi_n \circ f$ (check this!) we may deduce $\lim_{k\to\infty} f_k = f$ from $\lim_{k\to\infty} \pi_n \circ f_k = \pi_n \circ f$ for each $n \in \mathbb{N}$.

If X is not compact, but if we assume that there exists a sequence $K_1, K_2, \ldots$ of compact subsets with $\cup_{i\in\mathbb{N}} K_i = X$, then we have the point-separating family of semi-norms

$$P_{n,i} := \sup\{p_i(f(x)) : x \in K_n\} \qquad \blacksquare$$

which endow $C(X,Y)$ with the structure of a lcs. As above we may conclude that $C(X,Y)$ is a Fréchet space if Y is so.

Lemma A.3. *If X, Y are two topological vector spaces, then a linear map $f \colon X \to Y$ is continuous if and only if it is continuous in 0.*

Proof. We have to show that if f is continuous in 0, then it is continuous in each $x_0 \in X$. Since the translations $x \mapsto x - x_0$ and $y \mapsto y + f(x_0)$ are continuous, we have that the composition of

$$x \mapsto x - x_0 \mapsto f(x - x_0) = f(x) - f(x_0) \mapsto (f(x) - f(x_0)) + f(x_0) = f(x)$$

is continuous in x_0. Since this maps equals f, this shows the claim. $\blacksquare$

The following two theorems are of fundamental importance for us.

Theorem A.4. *If X is a topological vector space that possesses a compact zero neighbourhood, then X is finite-dimensional.*

Proof. This is [Rud91, Thm. I.1.22]. ■

Theorem A.5 (Hahn-Banach). *If X is a lcs, then the continuous linear functionals separate the points of X, i.e., for each two $x, y \in X$ with $x \neq y$ there exists some $\lambda \in X'$ such that $\lambda(x) \neq \lambda(y)$.*

Proof. This is [Rud91, Sect. I.3] or [Wer00, Cor. VIII.2.13]. ■

The following notion is the reason for dealing with locally convex spaces, instead of even larger classes of spaces.

Definition A.6. Let X be a lcs and $\gamma\colon [a, b] \to X$ be continuous. Then we say that $x \in X$ is the *weak integral of* γ and write $x = \int_a^b \gamma(t)dt$ if and only if $\lambda(x) = \int_a^b \lambda(\gamma(t))dt$ for each $\lambda \in X'$. In the latter expression, the integral is the usual integral of continuous real-valued functions on $[0, 1]$. ■

Lemma A.7. *The weak integral of γ is unique if it exists.*

Proof. If x and y are both weak integrals of γ, then we have by definition

$$\lambda(x) = \int_a^b \lambda(\gamma(t))dt = \lambda(y)$$

for each $\lambda \in X'$, and thus by Hahn-Banach that $x = y$. ■

The following is an important source for continuous maps:

Proposition A.8. *If X is lcs, P is a topological space and $f\colon [a, b] \times P \to X$ is continuous such that for each $p \in P$ the integral $\int_a^b f(t, p)dt$ exists, then $p \mapsto \int_a^b f(t, p)dt$ is continuous.*

Proof. This is a more or less direct consequence of the Hahn-Banach Theorem, cf. [GN14]. ■

With these results in shape, we can now turn to differentiability questions on locally convex spaces.

Definition A.9. Let X, Y be lcs, $U \subseteq X$ be open and $f\colon U \to Y$ be a map. We will abbreviate this by $f\colon U \Subset X \to Y$.

a) We say that f is *differentiable* if for each $x \in U$ and $v \in X$ the *differential*

$$df(x)(v) := \lim_{s \to 0} \frac{f(x + sv) - f(x)}{s} = \left.\frac{\partial}{\partial s} f(x + sv)\right|_{s=0}$$

exists in X. In order to simplify notation, we sometimes also write $df(x, v)$ for $df(x)(v)$. If, moreover, the map $U \times X \to Y$, $(x, v) \mapsto df(x)(v)$ is continuous with respect to the product topology, then f is said to be *continuously differentiable* or shortly a *C^1-map*. We denote the set of all C^1-maps $f\colon U \Subset X \to Y$ by $C^1(U, Y)$.

b) We define the higher differentials recursively by setting $d^1 f(x)(v) := df(x)(v)$ and

$$d^k f(x)(v_1, ..., v_k) := d(d^{k-1} f)(x, v_1, ..., v_{k-1})(v_k, 0, ..., 0) = \left.\frac{\partial}{\partial s_k} \cdots \frac{\partial}{\partial s_1} f(x + s_1 v_1 + ..., s_k v_k)\right|_{s_1 = ... = s_k = 0}$$

for $k \geq 2$. We say that f is k-fold differentiable if all $d^j(x, v_1, ..., v_j)$ exists for $1 \leq j \leq k$. If, moreover, the maps

$$U \times X^k \to Y, \quad (x, v_1, ..., v_k) \mapsto d^k(x, v_1, ..., v_k)$$

are continuous with respect to the product topology, then we say that f is k-fold continuously differentiable or shortly a C^k*-map*. We say that f is *smooth* if it is C^k for each $k \in \mathbb{N}_0$ (defining a C^0-map to be a continuous map). We denote the set of all C^k-maps (respectively smooth maps) $f\colon U \Subset X \to Y$ by $C^k(U, Y)$ (respectively $C^\infty(U, Y)$). ■

Remark A.10. There are several other natural definitions of C^1 and C^k maps on locally convex spaces.

a) A more convenient definition for f to be a C^k-map would be to demand f to be a C^1-map and df to be a C^{k-1}-map (the latter defined recursively). This is the definition in [Nee06].

b) One could also define f to be a C^k-map if it is a C^{k-1}-map (defined recursively) and $D^{k-1}f$ (also defined recursively with $Df := df$) is a C^1-map. Noting that $D^k f = d(D^{k-1}f) = D^{k-1}(df)$, an easy induction argument shows that this notion is equivalent to the one given in a).

c) A more general notion of C^1-map is to define $f\colon U \Subset X \to Y$ to be C^1 if there exists a continuous map

$$f^{[1]}\colon U^{[1]} := \{(x, v, s) \in U \times X \times \mathbb{R} \mid x + sv \in U\} \to Y$$

such that

$$f(x + sv) - f(x) = t \cdot f^{[1]}(x, v, t) \text{ for all } (x, v, t) \in U^{[1]}.$$

Note that this definition works if X, Y are merely topological modules over a topological ring (cf. [BGN04]). If X, Y are lcs, then this notion of C^1 map it is equivalent to the above one (cf. Exercise A.18). As above, this gives rise to two definitions of C^k maps that are both equivalent.

Eventually, all these notions of C^k-maps are (for lcs X, Y) equivalent to the one given in Definition A.9. In fact, it is trivial that C^k in any of a)-c) implies C^k in the sense of Definition A.9, since d^k only evaluates $D^k f$ in directions with multiple 0's. Moreover, C^k in the sense of Definition A.9 implies C^k in the sense of c) by [BGN04, 7.4]. Be aware that this is due to the fact that we are working with locally convex spaces. For non-locally convex spaces, the above notion of smoothness is not the appropriate one (cf. [Glö04a]). ■

Example A.11. If X, Y are lcs and $f\colon X \to Y$ is continuous and linear, then f is C^1, since $df(x, v) = f(v)$. This also shows that $d^k f = 0$ for each $k \geq 2$ and thus that f is in particular smooth. If, moreover, Z is a lcs, $U \subseteq Z$ is open and $g\colon U \to X$ is C^k, then from the definition it follows immediately that $f \circ g$ is C^k and that $d^k(f \circ g) = f \circ d^k g$.

One also sees that $df(x, v) = f(v)$ holds for each linear map, so that differentiability does not imply continuity in general. ■

Remark A.12. It is in general not easy to prove, that a given arbitrarily defined map $f\colon U \to Y$ is (continuously) differentiable or even smooth. Note that this is also the case in finite dimensions, where smooth maps often come from power series (and thus actually from analytic functions).

It will be important to have a similarly rich source for smooth maps between infinite-dimensional manifolds as well. However, power series will *not* serve this purpose (or rather in a quite limited way), since that spaces that we consider will not be normable. ■

Proposition A.13. *Let X, Y be lcs and $f\colon U \Subset X \to Y$ be C^1.*

a) *For any $x \in U$ the map $v \mapsto df(x, v)$ is real linear and continuous.*

b) *We have the **Fundamental Theorem of Calculus:** If $x \in U$, $v \in X$ such that $x+[0,1]v \subseteq U$, then*
$$f(x+v) = f(x) + \int_0^1 df(x+tv)(v)dt.$$
In particular, f is locally constant if and only if $df = 0$.

c) *f is continuous.*

d) *If f is C^k for $k \geq 2$, then for each $x \in U$ the functions $d^k(x)(\,\cdot\,)$ are symmetric and n-linear.*

e) *We have the **Chain Rule:** If, moreover, Z is a lcs and $g\colon V \Subset Z \to U$ is C^1, then $f \circ g$ is C^1 and*
$$d(f \circ g)(z, w) = df(g(z), dg(z, w)) \text{ for all } z \in V, w \in Z.$$

Proof. a) We only proof the additivity in order to illustrate the idea. For each $\lambda \in X'$ and $v, w \in X$ we define $F(s,t) := \lambda(f(x+sv+tw))$. This is a continuous map defined on some open neighbourhood of 0 in $\mathbb{R}^2$. It has the continuous partial derivatives
$$\frac{\partial}{\partial s}F(s,t) = \lambda(df(x+sv,tw)(v)) \quad \text{and} \quad \frac{\partial}{\partial t}F(s,t) = \lambda(df(x+sv,tw)(w)),$$
which are continuous. Thus from finite-dimensional calculus we know that F is continuously differentiable and $dF(0,0)$ is linear. This implies that
$$\begin{aligned}\lambda(df(x, v+w)) = d(\lambda \circ f)(x, v+w) = dF((0,0)(1,1)) &= dF((0,0)(1,0)) + dF((0,0)(0,1)) \\ &= d(\lambda \circ f)(x,v) + d(\lambda \circ f)(x,w) = \lambda(df(x,v) + df(x,w)).\end{aligned}$$
Since λ was arbitrary, this shows that $df(x, v+w)) = df(x,v) + df(x,w)$ by Hahn-Banach.

b) It follows from the Fundamental Theorem for continuous cunctions from $\mathbb{R}$ to $\mathbb{R}$ that
$$\lambda(f(x+v)-f(x)) = \lambda(f(x+v))-\lambda(f(x)) = \int_0^1 d(\lambda\circ f)(x+tv)(v)dt = \lambda\left(\int_0^1 df(x+tv)(v)dt\right)$$
for each $\lambda \in X'$.

c) Is omitted, cf. [Nee06, Lem. II.2.3]. The case that X is metrisable is also treated in Section 4.

d) Follows from the corresponding finite-dimensional statement as in part a).

e) Is left as Exercise A.18. ■

Remark A.14. Observe that we have not used any completeness assertion on the locally convex spaces in order to prove the chain rule. This makes the approach via Definition A.9 and the derived characterisation from Remark A.10 c) particularly handy. ■

Exercises for Appendix A

Exercise A.15. Let $(p_n)_{n\in\mathbb{N}}$ be a countable point separating family of semi-norms on the vector space X. Then show that the topologies induced by (40) and the metric (41) coincide. ■

Exercise A.16. Fill in the details of Example A.2 f). For this it could help to first show/realise the following fact: If d is a metric on X, then

$$d'(x,y) := \min\{d(x,y), 1\}$$

is an equivalent metric on X (i.e., id_X is continuous with respect to d and d'). ■

Exercise A.17. Let $X_1, ..., X_n, Y$ be lcs and $f\colon X_1 \times ... \times X_n \to Y$ be continuous and multi-linear. Show that f is smooth. ■

Exercise A.18. Let X, Y be lcs, $f\colon U \Subset X \to Y$ be a map and set $U^{[1]} := \{(x,v,s) \in U \times X \times \mathbb{R} \mid x + sv \in U\}$. Show that f is a C^1-map if and only there exists a continuous map $f^{[1]}\colon U^{[1]} \to Y$ such that

$$f^{[1]}(x,v,s) = \frac{1}{s}\left(f(x+sv) - f(x)\right) \qquad \text{if } s \neq 0. \tag{42}$$

Hint: Try

$$f^{[1]}(x,v,s) := \begin{cases} \int_0^1 df(x+stv)(v)dt & \text{if } x + [0,1]s \subseteq U \\ \frac{1}{s}\left(f(x+sv) - f(x)\right) & \text{else} \end{cases}$$

and use the fact that integrals depend continuously on their integrands.

Observe that (42) implies in particular that $df(x,v) = f^{[1]}(x,v,0)$ and thus that df is continuous. Use this to show the **Chain Rule:** If $f\colon U \Subset X \to Y$ and $g\colon V \Subset Z \to U$ are C^1-maps, then so is $f \circ g$ and

$$d(f\circ g)(z)(w) = df(g(z))(dg(z)(w)). \qquad ■$$

B Manifolds and Lie Groups

In this appendix we define the category **Man** of (possible infinite-dimensional) manifolds (with boundary). Moreover, we also treat Lie groups and some examples of them.

We now turn to the definition of (abstract) manifolds. As probably known from the undergraduate courses, the idea of a manifold should be a "somehow curved" object that locally looks like an open subset of a (locally convex) vector space.

Definition B.1. Let M be a Hausdorff topological space. A *chart* on M is a homeomorphism

$$\varphi\colon U \to \varphi(U) \Subset X_\varphi$$

for X_φ a lcs and $U \Subset M$ and $\varphi(U) \Subset X_\varphi$ open. Two charts $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ are *compatible* if the map

$$\varphi(U\cap V) \to \psi(U \cap V), \quad x \mapsto \psi(\varphi^{-1}(x))$$

and its inverse are smooth in the sense of Definition A.9[14]. An *atlas* on M is a family $(\varphi_i\colon U_i \to \varphi_i(U_i))_{i\in I}$ of pairwise compatible charts with $\cup_{i\in I} U_i = M$. The set of all atlases is ordered with respect to

$$(\varphi_i\colon U_i \to \varphi_i(U_i))_{i\in I} \leq (\psi_j\colon U_j \to \psi_j(U_j))_{j\in J} :\Leftrightarrow I \subseteq J \text{ and } \varphi_i = \psi_i \,\forall i \in I$$

and a smooth manifold is the pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a maximal atlas on M. We will often simply say that M is a smooth manifold, assuming that $\mathcal{A}$ is understood. If all the lcs X_φ is some atlas are isomorphic (as lcs), then we find an equivalent atlas in which all X_φ are equal to some lcs X. In this case M is also said to be *modelled* on X[15].

If $(M, \mathcal{A})$ is a smooth manifold and $(N, \mathcal{B})$ is a smooth manifold, then a map $f\colon M \to N$ is *smooth* if for each chart $\varphi\colon U \to \varphi(U)$ occurring in $\mathcal{A}$ and chart $\psi\colon V \to \psi(V)$ occurring in $\mathcal{B}$, the *coordinate representation*

$$\varphi(U \cap f^{-1}(V)) \to \psi(V), \quad x \mapsto \psi(f(\varphi^{-1}(x))) \tag{43}$$

is smooth in the sense of Definition A.9. We then also call f a *morphism* of manifolds. We denote the set of morphisms between smooth manifolds by $C^\infty(M, N)$ and call them also *smooth maps*. If there is no risk of confusion with the notation for diffeological spaces, then we use $\mathrm{Diff}(M)$ to refer to the *diffeomorphisms* of M, i.e., the invertible elements in $C^\infty(M, M)$.

In case that (43) is only a C^k-map for some $k \in \mathbb{N}_0$, then we call f is a C^k-map. Clearly, f is smooth if and only if it is C^k for each $k \in \mathbb{N}_0$. ■

Remark B.2. a) From the Chain Rule it follows that compatibility of charts is an equivalence relation. From this it follows, in turn, that a maximal atlas is uniquely determined by the choice of a single (not necessarily maximal) atlas. Likewise, it also follows from the Chain Rule that for a map $f\colon M \to N$ to be smooth it suffices to check (43) only for charts from two atlases on M and N, determining the respective maximal atlases.

b) It follows from the chain rule that the composition of smooth maps between manifolds is again smooth. Thus we obtain the category **Man** of *manifolds modelled on locally convex spaces* if we set

$$\mathbf{Man}(M, N) := \{f\colon M \to N \mid f \text{ is smooth}\}$$

with respect to the composition of smooth maps. Note that the restriction to locally convex spaces was important for obtaining the Chain Rule in Proposition A.13 e). As always, re will put a cardinality bound on the manifolds that we consider and thus turn **Man** into a small category (cf. Remark 2.6).

c) We call a manifold in which all X_φ are Fréchet spaces, (respectively Banach or finite-dimensional spaces) a Fréchet (respectively Banach or finite-dimensional) manifold. In contrast to this, a *metrisable* manifold is one whose underlying topological space is metrisable and a manifold in which all X_φ are metrisable is a *locally metrisable* manifold.

d) Each manifold will give rise to a diffeological space (see Section 3). In this context we will use the term *smooth map* to refer to a morphism of manifolds (a morphism in **Man**), rather than a morphism of diffeological spaces. However, we will see in Section 4 that these concepts agree on locally metrisable manifolds. ■

[14] From this it follows in particular, that if $U \cap V \neq \emptyset$, then the lcs that $\varphi(U)$ and $\psi(V)$ are open subsets of are in fact isomorphic.

[15] By the previous footnote and an easy compactness argument this is for instance the case if M is connected

Example B.3. a) Each set X is a manifold modelled on $\mathbb{R}^0 = \{0\}$ if we endow it with the discrete topology. This provides a functor $\delta\colon \mathbf{Set} \to \mathbf{Man}$, which is left adjoint to the forgetful functor $F\colon \mathbf{Man} \to \mathbf{Set}$. Since we have that each map $f\colon \delta(X) \to M$ is automatically smooth we have

$$\mathbf{Man}(\delta(X), M) \cong \mathbf{Set}(X, M)$$

and thus $\delta \dashv F$.

b) Each lcs X is a manifold with respect to the identity as (global) chart. Likewise, each $U \subseteq X$ is a manifold with the inclusion $U \hookrightarrow X$ as (global) chart.

c) Each submanifold of $\mathbb{R}^n$ is a smooth manifold.

d) If $M_1, ..., M_n$ are smooth manifolds, each modelled on some X_i, then their product $M_1 \times ..., \times M_n$ is a smooth manifold, modelled on $X_1 \times ... \times X_n$. Note that this is possible since charts $\varphi_i\colon U_i \to \varphi_i(U_i)$ give rise to a chart

$$\varphi_1 \times ... \varphi_n\colon U_1 \times ... \times U_n \to \varphi_1(U_1) \times ... \times \varphi_n(U_n).$$

This is not possible for infinite products any more, i.e., the category **Man** do not possess arbitrary products. ■

Constructing more exciting examples of smooth manifolds is not so easy, since smooth maps between lcs are not easily constructed in an ad-hoc way. Natural candidates for such gadgets would be mapping spaces, i.e., $C^\infty(M, N)$ for M and N say finite-dimensional manifolds.

On the other hand, the tool that we will get to know in Section 4 will provide us with a rich source of such maps. For the moment we will develop the theory mostly with finite-dimensional examples.

Note also that by Whitney's Embedding Theorem, each finite-dimensional manifold which is paracompact is a submanifold of some $\mathbb{R}^n$ [Pra07, Section 3.1].

Definition B.4. Let G be a group which is also a smooth manifold modelled on some lcs X. If the group operations

$$G \times G \to G, \quad (g, h) \mapsto g \cdot h, \quad \text{and} \quad G \to G, \quad g \mapsto g^{-1}$$

are smooth maps, then we call G a Lie group modelled on X. A morphism $f\colon G \to H$ between Lie groups G, H is a group homomorphism which is also a smooth map. This defines the category **LieGp** of Lie groups. It is a (non-full) subcategory of **Man** and of the category **Gp** of groups. ■

Example B.5. a) Finite-dimensional Lie groups are for instance matrix Lie groups like $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{R})$. Clearly, $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\}) \subseteq \mathbb{R}^{n^2}$ is open and thus naturally a manifold. The matrix multiplication is a polynomial in the matrix entries and thus smooth (cf. Exercise A.17). The inversion is by Cramer's Rule given by a rational function on $\mathrm{GL}_n(\mathbb{R})$ and thus also smooth. Moreover, $\mathrm{SL}_n(\mathbb{R}) = \det^{-1}(1)$ is a submanifold by the Regular Value Theorem. Thus the restrictions of the multiplication and inversion maps are also smooth on $\mathrm{SL}_n(\mathbb{R})$.

b) Let A be a Banach space and $m\colon A \times A \to A$, $(x, y) \mapsto x \cdot y$ be an associative, bilinear multiplication with identity $1 \in A$ that satisfies

$$\|x \cdot y\| \leq \|x\| \cdot \|y\|.$$

Then A is also called *Banach algebra*. In a Banach algebra the multiplication m is continuous: if $(x_n, y_n) \to (x, y)$, then $(x_n \cdot y_n) \to x \cdot y$, since

$$\begin{aligned}\|x_n y_n - xy\| = \|x_n y_n - xy_n + xy_n - xy\| &\leq \|x_n y_n - xy_n\| + \|xy_n - xy\| \\ &\leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \to 0.\end{aligned}$$

In particular, m is smooth (cf. Exercise A.17). Moreover, we have that if $\|x\| < 1$, then $1 - x$ is a unit since

$$(1 - x) \cdot \sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} 1 - x^n = 1.$$

If $a \in A$ is a unit, and $\|x\| < \|a^{-1}\|^{-1}$, then $a - x = a(1 - a^{-1}x)$ and $\|a^{-1}x\| < 1$ implies that $1 - a^{-1}x$ is a unit and thus is $a - x$. This implies that the group $A^{\times}$ of units in A is open and thus naturally a smooth manifold. We have already seen that the group multiplication is smooth and it will we shown in Exercise B.10 that $a \mapsto a^{-1}$ is also smooth. Thus $A^{\times}$ is a Lie group.

This applies in particular to the Banach algebra $A = \mathrm{B}(X)$ of bounded linear operators on the Banach space X with respect to the operator norm. Thus $\mathrm{Aut}(X) := \mathrm{B}(X)^{\times}$ is a Lie group, modelled on $B(X)$. Another example of a Banach algebra is $C(Y, A)$ for Y a compact topological space and A another Banach algebra with respect to $(f \cdot g)(y) := f(y) \cdot g(y)$ and the supremum norm (cf. Exercise B.11). Clearly, $f\colon Y \to A$ is invertible if and only if $f(y) \in A^{\times}$ for each $y \in Y$. Thus

$$C(Y, A)^{\times} = C(Y, A^{\times})$$

is a Lie group modelled on $C(Y, A)$. Examples thereof include for instance $C(Y, \mathbb{R}^{\times})$, $C(Y, \mathbb{C}^{\times})$ or $C(Y, \mathrm{GL}_n(\mathbb{R}))$. ■

One of the most important things that Lie groups can do is that they can act on other manifolds.

Definition B.6. If M is a manifold and G is a Lie group, then a *smooth left action* of G on M is an abstract left action $G \to \mathrm{Diff}(M)$ such that the map $G \times M \ni (g, m) \mapsto g.m \in M$ is smooth. This is also referred to as a *smooth left G-space*. Smooth right actions are defined similarly. An action is called faithful, free and transitive if the underlying abstract action is so. A morphism of smooth G-spaces M, N is a smooth equivariant map $f\colon M \to N$, i.e., we have $f(g.m) = g.f(m)$ for all $g \in G, m \in M, n \in N$. We denote the set of morphisms between G-spaces by $C^{\infty}(M, N)^G$. ■

The following actions will be of particular importance for us.

Definition B.7. A *torsor* over a Lie group G is a smooth right G-space M such that the map $M \times G \to M \times M$, $(m, g) \mapsto (m, m.g)$ is a diffeomorphism. ■

Example B.8. Each Lie group G is naturally a left and a right G-torsor with respect to the group multiplication. We clearly have $C^{\infty}(G, G)^G \cong G$ for both, the left and the left G-action. ■

Lemma B.9. *If M is a G-torsor, then $G \cong M$ as G-space and $G \cong C^{\infty}(M, M)^G$ as groups (non-canonically).*

Proof. If we fix some $m_0 \in M$, then each m may be written uniquely as $m = m_0.g$ and we denote suggestively $m_0^{-1} \cdot m := g$. Then $m \mapsto m_0^{-1} \cdot m$ is smooth and has as inverse the smooth map $g \mapsto m_0.g$. If we pre-and postcompose with the identification $M \cong G$, then we obtain a bijective group homomorphism $G \to C^{\infty}(M, M)^G$. ■

Exercises for Appendix B

Exercise B.10. Let A be a Banach algebra and denote by $A^\times$ the group of units in A. Show that the inversion $\iota\colon A^\times \to A^\times$, $a \mapsto a^{-1}$ is smooth by the following steps

a) Verify that $a^{-1} - b^{-1} = a^{-1}(a-b)b^{-1}$ for each $a, b \in A^\times$.

b) Calculate $d\iota(a)(v)$ and conclude that ι is of class C^1.

c) Show inductively that if ι is of class C^k, then $d\iota$ is of class C^k and conclude that ι is of class C^k for each $k \in \mathbb{N}_0$. ■

Exercise B.11. If A is a Banach algebra and X is a compact topological space, show that $C(X, A)$ is a Banach algebra with respect to $(f \cdot g)(x) := f(x) \cdot g(x)$ and the supremum norm

$$\|f\|_\infty := \sup\{\|f(x)\| : x \in X\}.$$ ■

C Fibreproducts

This section contains a self-contained treatment of submersions and fibreproducts in infinite dimensions (the necessary background for Lie groupoids and the construction of the surjective submersion coverage on **Man**).

In this section we will introduce fibreproducts, the perhaps most important tool for the construction and understanding of higher geometric objects. For the moment this is a purely technical notion, but it will later play a foundational rle for the understanding of smooth stacks. Supplements on this sections can be found in [Lan99, Section II.2].

Remark C.1. Recall from functional analysis that a subspace $X_1 \leq X$ of a tvs is called *complemented* if there exists a tvs X_2 such that $X \cong X_1 \times X_2$. Then X_1 is automatically closed in X.

The condition on being complemented is equivalent to the existence of a continuous projection $\pi\colon X \to X$ with $\pi(X) = X_1$. In fact, if π is given, then $\ker(\pi) \leq X$ is closed and $X_1 \times \ker(\pi) \to X$, $(x, y) \mapsto x+y$ is continuous with inverse given by $x \mapsto (\pi(x), x-\pi(x))$. Conversely, if $\varphi\colon X \to X_1 \times X_2$ is an isomorphism of tvs, then $\varphi^{-1} \circ \mathrm{pr}_1 \circ \varphi$ provides a continuous projection onto X_1.

Another condition for being complemented is that the projection $X \to X/X_1$ has a continuous linear right inverse $\sigma\colon X/X_1 \to X$. Indeed, if σ is given, then $x \mapsto x - \sigma(x + X_1)$ provides a projection onto X_1 and if $X \cong X_1 \times X_2$, then $X \to X/X_1 \cong X_2$ admits a linear an continuous right inverse, provided by $X_2 \hookrightarrow X_1 \times X_2$. ■

Definition C.2. Let M be a smooth manifold and $N \subseteq M$ a subset. Then a submanifold chart (for N) is a chart $\varphi\colon U \to \varphi(U) \Subset X$ such that there exists $Y \subseteq X$ closed and $\varphi(U \cap N) = \varphi(U) \cap Y$. If N is covered by domains of submanifold charts, then N is called *submanifold* of M. If, moreover, N is closed in M, then it is called *closed* submanifold. If G is a Lie group, then a *Lie subgroup* is a subgroup $H \subseteq G$ which is also a submanifold. If H is also closed, then it is called *closed* Lie subgroup.

If $N \subseteq M$ is a submanifold and in the above definition, each Y can be chosen to be complemented, then N is called *split submanifold*. Clearly, a split Lie subgroup is a Lie subgroup that is a split submanifold. ■

Remark C.3. a) Each submanifold is in fact a manifold, where an atlas is determined by all the charts on M which are also submanifold charts for N (as defined above). This is called the induced manifold structure on N. Note that this determines the manifold N uniquely.

b) If $f\colon M \to P$ is smooth, then $f|_N$ is smooth for the induced manifold structure, since the corresponding coordinate representations are smooth for the atlas of submanifold charts (restrictions of smooth maps to closed subspaces remain smooth in the induced topology). Likewise, if $g\colon Q \to M$ is smooth and $g(Q) \subseteq N$, then g is smooth as a map to N with the induced manifold structure.

c) Each open subset $U \Subset M$ is a submanifold, which is automatically split. We then call U also an *open submanifold*. Note also that submanifolds of finite-dimensional manifolds are automatically split, we thus omit the term *split* from the notation.

d) If $N \subseteq M$ is a split submanifold, then the inclusion $N \hookrightarrow M$ can locally be written as the injection $X_1 \hookrightarrow X_1 \times \{0\} \subseteq X$ of a complemented subspace. More generally, if P is an arbitrary manifold, then a map $f\colon P \to M$ is called an immersion if it is locally given by the inclusion of a complemented subspace, i.e., for each $p \in P$ there exist charts $\varphi\colon U \to \varphi(U) \Subset X_1$ of P and $\psi\colon V \to \psi(V) \Subset X$ of M such that $X_1 \leq X$ is complemented, $p \in U$, $f(U) \subseteq V$ and the coordinate representation $\psi \circ f \circ \varphi^{-1}$ is the restriction of the inclusion $X_1 \hookrightarrow X$ to $\varphi(U)$. ■

Example C.4. a) If G is a finite-dimensional Lie group and $H \leq G$ is a closed subgroup, then H is a submanifold and thus a closed Lie subgroup, see [Hel01, Theorem II.2.3] or [Mic08, Theorem 5.5].

b) Let $f\colon M \to P$ be smooth and $g = \mathrm{pr}_2\colon N \times P \to P$ be the projection. Then

$$M \times_P (N \times P) := \{(m,n,p) \mid f(m) = \mathrm{pr}_2(n,p) = p\}$$

is a split submanifold of $M \times N \times P$. In fact, let $\varphi\colon U \to \varphi(U)$ be a chart of M and $\psi \times \xi\colon V \times W \to \psi(V) \times \xi(W)$ be a product chart of $N \times P$. Then

$$(m,n,p) \mapsto \big(\varphi(m), \psi(n), \xi(p) - \xi(f(m))\big) \tag{44}$$

defines a chart of $M \times N \times P$ with inverse

$$(x,y,z) \mapsto (\varphi^{-1}(x), \psi^{-1}(y), \xi^{-1}(z + \xi(f(\varphi^{-1}(x))))).$$

Moreover, $(m,n,p) \in \big(U \cap f^{-1}(W)\big) \times V \times W$ maps under (44) to $(x,y,0)$ if and only if $f(m) = p$, i.e., if $(m,n,p) \in M \times_P (N \times P)$. Thus (44) is a submanifold chart for $M \times_P (N \times P)$. ■

Remark C.5. Note that already for Banach-Lie groups it is not true any more that closed subgroups are Lie subgroups. Indeed, if G is the additive group of the Hilbert space $\ell^2(\mathbb{N})$, then for each $n \in \mathbb{N}$ we consider the group $H := \{(x_n) \in \ell^2(\mathbb{N}) \mid x_n \in \frac{1}{n}\mathbb{Z}\}$. Since $\lambda_i(x_n) = x_i$ is continuous for each i we have that $H = \bigcap_{i \in \mathbb{N}} \lambda_i^{-1}(\frac{1}{i}\mathbb{Z})$ is closed. It is non-discrete (in the induced topology), since each open ball $B_{\frac{1}{m}}(0)$ contains the sequence which is zero except at $n = 2m$, where it is $\frac{1}{2m}$. But no neighbourhood of 0 in H can contain a continuous path γ connecting 0 with $(x_n) \neq 0$, since then $\lambda_i \circ \gamma$ would take all values between 0 and x_i. Thus no neighbourhood of 0 in H can be homeomorphic to an open subset of a lcs, since the latter would admit continuous paths joining sufficiently close distinct points. ■

The dual version of an immersion is that of a submersion, i.e., a smooth map that can locally be written as the projection onto a complemented subspace.

Definition C.6. Let M,N be manifolds and $f\colon M\to N$ be smooth. Then f is a *submersion* if for each $m\in M$ there exist charts $\varphi\colon U\to\varphi(U)\Subset X$ of M and $\psi\colon V\to\psi(V)\Subset X_1$ of N with $m\in U$, $f(U)\subseteq V$, $X_1\leq X$ complemented and a continuous projection $\pi\colon X\to X_1$ with $\pi(X)=X_1$ such that the coordinate representation $\psi\circ f\circ\varphi^{-1}$ is the restriction of π to $\varphi(U)$. ∎

The most important source of submersions will be bundle projections, that we introduce in Appendix D. We here already give some basic examples of those.

Example C.7. a) If G is a finite-dimensional Lie group and $H\leq G$ is a closed subgroup, then G/H carries a unique manifold structure such that $G\to G/H$ is a submersion, see [DK00, Corollary 1.11.5] or [Mic08, Theorem 5.11].

b) If M is a manifold and $\widetilde{M}\to M$ is a covering of M, then $\widetilde{M}$ has a unique manifold structure such that $\widetilde{M}\to M$ is a local diffeomorphism. Indeed, $\widetilde{M}\to M$ is a local homeomorphism and we may thus pull charts that are defined on sufficiently small open subsets in M via $\widetilde{M}\to M$ back to charts around each point of $\widetilde{M}$. Thus the coordinate representation of $\widetilde{M}\to M$ is the identity, and thus in particular a projection. ∎

Proposition C.8. *Let $f\colon M\to P$, $g\colon N\to P$ be smooth maps and g be a submersion. Then $M\times_P N=\{(m,n)\mid f(m)=g(n)\}$ is a split submanifold of $M\times N$ and the projection $\mathrm{pr}_1\colon M\times_p N\to M$ is a submersion.*

Proof. Let $(m,n)\in M\times_P N$ and let $\psi\colon V\to\psi(V)\Subset Y$ be a chart on N, $\xi\colon W\to\psi(W)\Subset Y_1$ be a chart on P such that $n\in V$, $g(V)\subseteq W$ and $\xi\circ g\circ\psi^{-1}=\pi$ for some continuous projection $\pi\colon Y\to Y_1$ with $\pi(Y)=Y_1$. After shrinking $\varphi(V)$ and $\xi(W)$ if necessary, we may assume w.l.o.g. that $Y=Y_1\times Y_2$ for $Y_2\leq Y$ closed, that $\psi(V)=V_1\times V_2$ with $V_i\Subset Y_i$, that $\xi(W)\subseteq V_1$ and that $\pi=\mathrm{pr}_1$:

$$\begin{tikzcd}
V \arrow[rr, "\psi"] \arrow[d, "g"'] & & \psi(V) \arrow[r, "\cong"] \arrow[d, "\pi"] & V_1\times V_2 \arrow[d, "\mathrm{pr}_1"] \\
g(V)\subseteq W \arrow[rr, "\xi"] & & \xi(W) \arrow[r, hook] & V_1
\end{tikzcd}$$

As in (44) we obtain from this a chart of $M\times N$

$$\big(U\cap f^{-1}(W)\times V\big)\ni(m,n)\mapsto(\varphi(m),\psi_1(n),\psi_2(n)-\xi(f(m)))\in\varphi(U)\times V_1\times V_2 \tag{45}$$

with inverse

$$(x,y,z)\mapsto(\varphi^{-1}(x),\psi^{-1}(y,z+\xi(f(\varphi^{-1}(x)))))$$

such that $(m,n)\in\big(U\cap f^{-1}(W)\times V\big)\cap(M\times_P N)$ if and only if (m,n) maps under (45) to $(x,y,0)$. Thus (45) is a submanifold chart. Since (m,n) was arbitrary, this shows that $M\times_P N$ is a submanifold of $M\times N$ and since $\mathrm{pr}_1\colon M\times N\to M$ is smooth, so is the restriction to $M\times_P N$. Moreover, its coordinate representation in with respect to the charts (45) and φ is given by pr_1, and thus $\mathrm{pr}_1\colon M\times N\to M$ is a submersion. ∎

Corollary C.9. *If $f\colon M \to P$ and $g\colon N \to P$ are smooth an either f or g is a submersion, then the pull-back $M \times_P N$ exists in* **Man**. *Moreover, the forgetful functor* **Man** $\to$ **Top** *maps $M \times_P N$ to the pullback of $f\colon M \to P$ and $g\colon N \to P$ in* **Top**.

Note that the last assertion is non-trivial, since in general the forgetful functor **Man** $\to$ **Top** does not preserve pull-backs.

Proof. If $k\colon Q \to M$ and $l\colon Q \to N$ satisfy $f \circ k = g \circ l$, then $h\colon Q \to M \times N$, $q \mapsto (k(q), l(q))$ is smooth and takes values in $M \times_P N$, and is uniquely determined by requiring $\mathrm{pr}_1 \circ h = k$ and $\mathrm{pr}_2 \circ h = l$. ■

In practice it is good to have criterions for a map to be a submersion that are easier to check that the definition.

Proposition C.10. *a) A smooth map $f\colon M \to N$ between finite-dimensional manifolds is a submersion if and only if for each $m \in M$ the tangent map $T_m f\colon T_m M \to T_{f(m)} N$ (see Appendix D) is surjective.*

b) A smooth map $f\colon M \to N$ between Banach-manifolds is a submersion if and only if for each $m \in M$ the tangent map $T_m f\colon T_m M \to T_{f(m)} N$ (see Appendix D) is surjective and $\ker(T_m f) \leq T_m M$ is complemented.

c) A smooth map $f\colon M \to N$ between Banach-manifolds is a submersion if and only if it admits smooth sections through each point in M, i.e., for each $m \in M$ there is $V \Subset N$ with $f(m) \in V$ and $\sigma\colon V \to M$ smooth such that $\sigma(f(m)) = m$ and $f \circ \sigma = \mathrm{id}_V$.

Proof. b) is [Lan99, Proposition II.2.3] and this clearly implies a).

If $f\colon M \to N$ admits sections through each point, then $T_m f$ is surjective for each $m \in M$ since $T_{f(m)}\sigma$ provides a continuous and linear right inverse to $T_m f\colon T_m M \to T_{f(m)} N$. Since $T_{f(m)} N \cong T_m/\ker(T_m f)$ this shows also that $\ker(T_m f)$ is complemented. ■

We end this section by a discussion of the following fundamental notion for differential topology.

Definition C.11. Let M, N be manifolds and $f\colon M \to N$ be smooth and $Q \subseteq N$ be a split submanifold. Then f is *transversal* over Q if for each $m \in f^{-1}(Q)$ and submanifold chart $\psi\colon V \to V_1 \times V_2$ with $\psi(f(m)) = (0,0)$ there exists $U \Subset M$ such that $m \in U$, $f(U) \subseteq V$ and that

$$U \xrightarrow{f} V \xrightarrow{\psi} V_1 \times V_2 \xrightarrow{\mathrm{pr}_2} V_2 \tag{46}$$

is a submersion. ■

The following proposition is crucial.

Proposition C.12. *If $f\colon M \to N$ is transversal over a split submanifold Q of N, then $f^{-1}(Q)$ is a split submanifold of M.*

Proof. Since (46) is a submersion we may assume, after possibly shrinking U and V_2, that we have charts $\varphi\colon U \to U_1 \times U_2$ of U (where $U_i \Subset X_i$ are open zero neighbourhoods in some lcs X_1 and X_2) and $\xi\colon V_2 \to U_2$ (w.l.o.g. $\varphi(m) = (0,0)$ and $\xi(0) = 0$) of V_2 such that

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[r, "\psi"] & V_1 \times V_2 \arrow[r, "\mathrm{pr}_2"] & V_2 \arrow[d, "\xi"] \\ U_1 \times U_2 \arrow[rrr, "\mathrm{pr}_2"] & & & U_2 \end{tikzcd}$$

commutes. We claim that φ is a submanifold chart for $f^{-1}(Q)$, i.e., that

$$\varphi(U \cap f^{-1}(Q)) = \varphi(U) \cap (U_1 \times \{0\}).$$

Indeed, we have for $x \in U$ that

$$f(x) \in Q \Leftrightarrow \mathrm{pr}_2(\psi(f(x))) = 0 \Leftrightarrow \mathrm{pr}_2(\varphi(x)) = 0 \Leftrightarrow \varphi(x) \in \mathrm{pr}_2^{-1}(0) = U_1 \times \{0\},$$

where the first equivalence follows from the fact that ψ is a submanifold chart. This shows the claim.■

Corollary C.13. *If $f\colon M \to N$ is a submersion, then the fibre $f^{-1}(n)$ is a split submanifold for each $n \in N$.*

D Locally Trivial and Principal Bundles

This section contains a quick treatment of different types of bundles (in particular infinite-dimensional ones), in particular vector bundles, principal bundles and Lie group bundles.

We now come to a very important concept encoding most of the structure in differential geometry. There is a close similarity to the concept of a submersion: a submersion is a morphism of manifolds that is locally a projection to a factor in a product space, where a bundle is a *space* that is locally a product of spaces.

Definition D.1. Let X, Y, Z be manifolds. Then the trivial bundle (over Z with fibre X) is the projection $\mathrm{pr}_1\colon Z \times X \to Z$. A locally trivial bundle with fibre X is a smooth map $\pi\colon Y \to Z$ such that there exists an open cover $(U_i)_{i\in I}$ of Z and diffeomorphisms $\Phi_i\colon \pi^{-1}(U_i) \to U_i \times X$, called *local trivialisations*, such that the diagram

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\Phi_i"] \arrow[dr, "\pi"'] & & U_i \times X \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd} \tag{47}$$

commutes. We will frequently use the notation $Y|_U := \pi^{-1}(U)$ for $U \Subset Z$ (note that this is then also locally trivial bundle over U). We shortly refer to $\pi\colon Y \to Z$ (or simply Y) as *bundle* (over Z). We refer to X as the *fibre*, to Y as the *total space* and to Z as the *base* of the bundle. A smooth map $\sigma\colon Z \to Y$ such that $\pi \circ \sigma = \mathrm{id}_Z$ is called a *section* of the bundle. The set of all sections is denoted by $\Gamma(\pi\colon Y \to Z)$ (shortly $\Gamma^\pi(Z)$ and $\Gamma^\pi(U) := \Gamma^\pi(\pi|_{Y|_U}\colon\ Y|_U \to U)$).

If $\pi\colon Y \to Z$ and $\pi'\colon Y' \to Z$ are bundles over Z with fibre X, then a morphism from $Y \to Y'$ of bundles is a smooth map $f\colon Y \to Y'$ such that

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y' \arrow[dl, "\pi'"] \\ & Z & \end{tikzcd}$$

commutes. A morphism from Y to the trivial bundle is called a *trivialisation* and in case that it exists Y is called *trivialisable*. Obviously, compositions of bundle morphisms are bundle morphisms.■

Example D.2. a) If G is a finite-dimensional Lie group and $H \subseteq G$ is a closed subgroup, then H is a submanifold and on G/H there exists a unique manifold structure such that $\pi\colon G \to G/H$ is a submersion (cf. Examples C.4 a) and C.7 a)). In particular, there exists an open cover $(U_i)_{i\in I}$ and $\sigma_i\colon U_i \to G$ such that $\pi \circ \sigma_i = \mathrm{id}_{U_i}$ (one may always define σ_i in local coordinates that render π a projection). Then

$$\pi^{-1}(U_i) \to U_i \times H, \quad g \mapsto (\pi(g), \sigma_i(\pi(g))^{-1} \cdot g) \tag{48}$$

is a local trivialisation, i.e., it is smooth, has

$$U_i \times H \to \pi^{-1}(U_i), \quad (y,h) \mapsto \sigma_i(y) \cdot h \tag{49}$$

as smooth inverse and makes (47) commute. Thus $\pi\colon G \to G/H$ is a bundle with fibre H.

The same construction applies to infinite-dimensional G if we assume that H is a closed Lie subgroup and that G/H carries a manifold structure turning $G \to G/H$ into a submersion.

b) Each covering $\pi\colon M \to N$ is a bundle. By definition, there exists an open cover $(U_i)_{i\in I}$ of N such that $\pi^{-1}(U_i) \cong F \times U_i$ for F a discrete set.

c) Let M be a manifold modelled on the lcs X and $(\varphi_i\colon U_i \to \varphi_i(U_i) \Subset X)_{i\in I}$ be an atlas for M. Then we define the *tangent bundle* to M to be the set

$$TM := \left(\coprod_{i\in I} \varphi_i(U_i) \times X \right) / \sim,$$

where we define $(x_i, v) \in \varphi_i(U_i)\times X$ to be equivalent to $(y_j, w) \in \varphi_j(U_j)\times X$ if $y_j = \varphi_j(\varphi_i^{-1}(x_i))$ and

$$w = d(\varphi_j \circ \varphi_i^{-1})(x_i)(v).$$

This defines an equivalence relation due to the chain rule. If we endow TM with the quotient topology of the disjoint union topology, then this becomes a manifold, since we have the charts

$$T\varphi_i\colon \varphi_i(U_i) \times X \to \varphi_i(U_i) \times X \Subset X \times X, \quad [(x_i, v)] \mapsto (x_i, v)$$

(where we have identified $\varphi_i(U_i)\times X$ with the subset of TM that have an (automatically unique) representative in $\varphi_i(U_i) \times X$). We get a canonical map $\pi\colon TM \to M$, $[(x_i,v)] \mapsto \varphi_i^{-1}(x_i)$ (note that this is well-defined). This is clearly smooth since the coordinate representation is the projection $\mathrm{pr}_1\colon \varphi_i(U_i) \times X \to \varphi_i(U_i)$. Moreover, $\pi^{-1}(U_i) = \varphi_i(U_i) \times X$, so that the identity provides a local trivialisation. Thus $\pi\colon TM \to M$ is a bundle with fibre X. ■

In many cases, bundles have actually more structure than just being bundles. We now turn to those structured bundles and their additional properties.

Remark D.3. Let $\pi\colon Y \to Z$ be a bundle with fibre X. For each two local trivialisations $\Phi_{i,j}\colon \pi^{-1}(U_{i,j}) \to U_{i,j}\times X$ we consider the diffeomorphism

$$\Phi_i \circ \Phi_j^{-1}\colon U_{ij} \times X \to U_{ij} \times X, \tag{50}$$

which we also call *trivialisation change*. Since Φ_i and Φ_j make (47) commute it follows that $\Phi_i(\Phi_j^{-1}(y,x)) = (y, \varphi_{ij}(y,x))$, where $\varphi_{ij}\colon U_{ij}\times X \to X$ is smooth. Moreover, we have $\varphi_{ij}(x, \varphi_{ji}(x,y)) =$

y and thus that for all $i, j \in I$ and each $x \in U_{ij}$ the map $y \mapsto \varphi_{ij}(x, y)$ is a diffeomorphism of X, which we denote by $\varphi_{ij}(x)$. Depending on which additional structure X has and whether $\varphi_{ij}(x)$ preserves this structure we call $\pi\colon Y \to Z$ a

$$\left.\begin{array}{r}\text{vector bundle}\\ \text{Lie group bundle}\\ \text{principal } G\text{-bundle}\end{array}\right\}\text{ if }\left\{\begin{array}{c}X \text{ is a lcs}\\ X \text{ is a Lie group}\\ X=G \text{ is a Lie group}\end{array}\right\}\begin{array}{l}\text{and there exist trivialisations}\\ (\Phi_i)_{i\in I} \text{ with } \cup_{i\in I}U_i = Z \text{ such that}\\ \text{each } \varphi_{ij}(x) \text{ is an isomorphism of}\end{array}\left\{\begin{array}{l}\text{lcs}\\ \text{Lie groups}\\ G\text{-torsors}\end{array}\right. .$$

This scheme also applies to define Lie algebra bundles, (matrix) algebra bundles and other types of bundles whose fibres have additional structure. ■

Example D.4. a) The bundle $\pi\colon G \to G/H$ for $H \subseteq G$ a closed Lie subgroup and π a submersion is a principal H-bundle. In fact, the trivialisation changes are given by

$$U_{ij} \times H \to U_{ij} \times H, \quad (y, h) \mapsto \sigma_i(y) \cdot h \mapsto (\underbrace{\pi(\sigma_i(y) \cdot h)}_{=y}, \sigma_j(y)^{-1} \cdot \sigma_i(y) \cdot h)$$

and thus each $\varphi_{ij}(x)$ (defined as above) is in fact the torsor isomorphism $h \mapsto \sigma_j(y)^{-1} \cdot \sigma_i(y) \cdot h$.

b) Coverings have no additional structure, they are simply bundles with discrete fibres. However, if the covering $\pi\colon M \to N$ is regular, then M is a G-principal bundle, where G is the group of deck transformations of the covering.

c) The tangent bundle $\pi\colon TM \to M$ is a vector bundle. Indeed, the trivialisation changes are given by

$$U_{ij} \times X \ni (x_i, v) \mapsto (x_i, d(\varphi_j \circ \varphi_i^{-1})(x_i)(v)) \in U_{ij} \times X$$

(why?), and each $d(\varphi_j \circ \varphi_i^{-1})(x_i)$ is an isomorphism of lcs. The fibres $T_mM := \pi^{-1}(m)$ are called the *tangent spaces* in m. They are lcs that are (non-canonically) isomorphic to X. ■

In case that bundles have additional structure, we surely also want the morphisms to preserve this additional structure.

Remark D.5. Let $\pi\colon Y \to Z$ and $\pi'\colon Y' \to Z$ be bundles with fibre X and $f\colon Y \to Y'$ be a bundle morphism. If $\Phi_i\colon \pi^{-1}(U_i) \to U_i \times X$ and $\Phi_i'\colon \pi'^{-1}(U_i) \to U_i \times X$ are local trivialisations, then we have

$$\Phi_i' \circ f \circ \Phi_i^{-1}\colon U_i \times X \to U_i \times X, \quad (y, x) \mapsto (y, \xi_i(y, x)) \tag{51}$$

for $\xi_i\colon U_i \times X \to X$ smooth. If the maps $x \mapsto \xi_i(y, x)$ (denoted by $\xi_i(y)$ in the sequel) preserve the additional structure on X that we might have, then it is a morphism of structured bundles, i.e., f is a morphism of

$$\left.\begin{array}{r}\text{vector bundles}\\ \text{Lie group bundles}\\ \text{principal } G\text{-bundles}\end{array}\right\}\text{ if each } \xi_i(x) \text{ is a morphism of }\left\{\begin{array}{l}\text{lcs}\\ \text{Lie groups}\\ G\text{-torsors}\end{array}\right. .$$ ■

Lemma D.6. *For each manifold M the category* $\mathbf{Bun}(M, G)$ *of principal G-bundles over M is a groupoid, i.e., each morphism is automatically invertible.*

Proof. The proof is left as Exercise D.14. ■

One nice feature of structured bundles is that they allow for additional structures on the total space and on the set of sections.

Remark D.7. Let $\pi\colon Y \to Z$ be a structured bundle with fibre X, i.e., there exists a family $(\Phi_i)_{i\in I}$ of local trivialisations such that each $\varphi_{ij}(x)$ is an isomorphism of the structure in question.

a) If $\pi\colon Y \to Z$ is a vector bundle, then we can define for each $y \in Y$ and $\lambda \in \mathbb{R}$ the element

$$\lambda \cdot y := \Phi_i^{-1}(\lambda \cdot \Phi_i(y)),$$

if $z := \pi(y) \in U_i$ each $\mathbb{R}$ acts on $U_i \times X$ by $\lambda \cdot (z,x) := (z, \lambda \cdot x)$. This is well-defined, since if $z \in U_{ij}$, then

$$\Phi_j^{-1}(\lambda \cdot \Phi_j(y)) = \Phi_j^{-1}(\lambda \cdot \Phi_j(\Phi_i^{-1}(\Phi_i(y)))) = \Phi_j^{-1}(\Phi_j(\Phi_i^{-1}(\lambda \cdot \Phi_i(y)))) = \Phi_i^{-1}(\lambda \cdot \Phi_i(y)),$$

where the equality in the middle follows from the linearity of $\varphi_{ij}(z)$, i.e.,

$$\lambda \cdot \Phi_j(\Phi_i^{-1}(z,x)) = (z, \lambda \cdot \varphi_{ij}(z)(x)) = (z, \varphi_{ij}(z)(\lambda \cdot y)) = \Phi_j(\Phi_i^{-1}(z, \lambda \cdot y)).$$

In a similar fashion, we may add two elements in the same fibre of π : if $y, y' \in \pi^{-1}(z)$, then

$$y + y' := \Phi_i^{-1}(\Phi_i(y) + \Phi_i(y'))$$

if $z \in U_i$ and $(z,x) + (z,x') := (z, x + x')$. This is well-defined since $\varphi_{ij}(z)$ is additive and thus

$$\Phi_j^{-1}(\Phi_j(y) + \Phi_i(y')) = \Phi_j^{-1}(\Phi_j(\Phi_i^{-1}(\Phi_i(y))) + \Phi_i(\Phi_i^{-1}(\Phi_i(y')))) = \\ \Phi_j^{-1}(\Phi_j(\Phi_i^{-1}(\Phi_i(y)) + \Phi_i^{-1}(\Phi_i(y')))) = \Phi_i^{-1}(\Phi_i(y) + \Phi_i(y')).$$

This defines a smooth and partially defined addition map $+\colon Y \times_X Y \to Y$ that is associative, commutative and distributive over the above $\mathbb{R}$-action (these assertions can easily be checked in local coordinates). In particular, the fibres of π are lcs isomorphic to X. Moreover, this leads to a vector space structure on the sections $\Gamma^\pi(Z)$ by point-wise operations, since for $\sigma, \tau \in \Gamma^\pi(Z)$ the map $z \mapsto (\sigma(z), \tau(z))$ has values in $Y \times_X Y$.

b) If $\pi\colon Y \to Z$ is a Lie group bundle, then we obtain a partially defined multiplication map $m\colon Y \times_X Y \to Y$ which is associative (wherever defined) and admits inverses. In particular, the fibres of π are Lie groups isomorphic to X and the sections carry a group structure, defined by point-wise operations.

c) If $\pi\colon Y \to Z$ is a principal G-bundle, i.e., the fibres are right G-spaces and the trivialisation changes $\Phi_j \circ \Phi_i^{-1}$ are equivariant with respect to the right action $(z,g) \cdot h := (z, g \cdot h)$ of G on $U_i \times G$, then we obtain a smooth right action of G on Y by setting

$$y \cdot g := \Phi_i^{-1}(\Phi_i(y) \cdot g)$$

if $z := \pi(y) \in U_i$. This is well-defined since if $z \in U_{ij}$, then

$$\Phi_j^{-1}(\Phi_j(y) \cdot g) = \Phi_j^{-1}(\Phi_j(\Phi_i^{-1}(\Phi_i(y))) \cdot g) = \Phi_j^{-1}(\Phi_j(\Phi_i^{-1}(\Phi_i(y) \cdot g))) = \Phi_j^{-1}(\Phi_j(y) \cdot g).$$

Moreover, each fibre $Y_z := \pi^{-1}(z)$ is a G-torsor since G is so and

$$Y_z \ni y \mapsto \Phi(z) \in \{z\} \times G \cong G$$

is an isomorphism of G-spaces by definition. Thus Y might be thought of as a "parametrised" family of G-torsors. ∎

Example D.8. If $H \subseteq G$ is a closed Lie subgroup and $\pi\colon G \to G/H$ is a submersion, then we obtain a right H-action of H on G from the previous description. From (48) and (49) it follows that this is given by

$$g \cdot h := \sigma_i(\pi(g)) \cdot \sigma_i(\pi(g))^{-1} \cdot g \cdot h = g \cdot h.$$

More generally, one can show that if $\pi\colon Y \to Z$ is a locally trivial bundle with fibre some Lie group G and if there exist trivialisations Φ_i that are morphisms of right G-spaces (for the action of G on $U_i \times G$ by right multiplication in the second component), then $\pi\colon Y \to Z$ is a principal G-bundle and the induced action of G on Y coincides with the original one.

Note that this also tells one how to distinguish between Lie group bundles with fibre G and principal G-bundles. Both are locally trivial bundles, but for principal bundles the trivialisations have to preserve the right G-action, while for a Lie group bundle the trivialisations have to be compatible with the Lie group structure on the fibres. ∎

Lemma D.9. *If $\pi\colon Y \to Z$ is a locally trivial bundle, then π is a submersion and for each smooth map $f\colon M \to Z$ the projection $\mathrm{pr}_1\colon M \times_Z Y \to M$ from of the pull-back $M \times_Z Y = \{(m,y) \in M \times Y \mid f(m) = \pi(y)\}$ is a locally trivial bundle. Moreover, if $\pi\colon Y \to Z$ is a vector bundle (respectively, a Lie group bundle or a principal G-bundle), then so is $\mathrm{pr}_1\colon M \times_Z Y \to M$.*

Proof. Since $\mathrm{pr}_1\colon U_i \times X \to U_i$ is a submersion it is clear that $\pi\colon Y \to Z$ is so. Let $\Phi_i\colon \pi^{-1}(U_i) \to U_i \times X$ be a local trivialisation of Y. Then a local trivialisation of $M \times_Z Y$ over $\mathrm{pr}_1^{-1}(f^{-1}(U_i)) = f^{-1}(U_i) \times_Z Y$ is given by

$$f^*(\Phi_i)\colon f^{-1}(U_i) \times_Z Y \to f^{-1}(U_i) \times X, \quad (m,y) \mapsto (m, \mathrm{pr}_2(\Phi_i(y)))$$

with inverse

$$f^*(\Phi_i^{-1})\colon f^{-1}(U_i) \times X \to f^{-1}(U_i) \times_Z Y, \quad (m,x) \mapsto (m, \Phi_i^{-1}(f(m),x)).$$

From this it follows that the trivialisation changes are given by

$$f^{-1}(U_{ij}) \times X \ni (m,x) \mapsto (m, \varphi_{ij}(f(m))(x)) \in f^{-1}(U_{ij}) \times X,$$

where $\varphi_{ij}(x)$ are the trivialisation changes of Y. If some additional structure on X in preserved by each $\varphi_{ij}(x)$, then it is also preserved by each $\varphi_{ij}(f(m))$ and thus $M \times_Z Y$ is a vector bundle (respectively, a Lie group bundle or a principal G-bundle) if Y is so. ∎

Remark D.10. The bundle $M \times_Z Y$ in the previous proposition if called *pull-back* bundle of $\pi\colon Y \to Z$ along $f\colon M \to Z$ and is denoted by $f^*(Y)$. If $\varphi\colon Y \to Y'$ is a bundle morphism, then

$$f^*(Y) \to f^*(Y'), \quad (m,y) \mapsto (m, \varphi(y))$$

is a bundle morphism, also called the pull-back $f^*(\varphi)$ of φ along f. Clearly, $f^*(\varphi)$ is a morphism of vector bundles (respectively Lie group bundles or principal bundles) if φ is so. ■

Sometimes it is important to talk about morphisms of bundles over different base spaces.

Definition D.11. If $\pi\colon Y \to Z$ and $\psi\colon M \to N$ are a locally trivial bundles (respectively vector bundles, Lie group bundles or principal bundles) and $f\colon N \to Z$ it smooth, then a morphism of locally trivial bundles (respectively vector bundles, Lie group bundles or principal bundles) *covering* f is a respective morphism $\varphi\colon M \to f^*(Y)$. As a map to $f^*(Y) = N \times_Z Y$ it is completely determined by its second component $M \to Y$, since the first component has to agree with f. We will throughout identify a bundle morphism $\varphi\colon M \to f^*(Y)$ with its second component $\mathrm{pr}_2 \circ \varphi\colon M \to Y$. We denote a bundle morphism also shortly by the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\psi"'] & Y \arrow[d, "\pi"] \\ N \arrow[r, "f"] & Z \end{tikzcd} .$$
■

Example D.12. If M, N are smooth manifolds (modelled on fixed lcs X and Y) and $f\colon M \to N$ is smooth, then there is a bundle morphism $Tf\colon TM \to TN$ covering f, given by

$$[(x_i, v)] \mapsto [(\psi_i(f(\varphi_i^{-1}(x_i))), d(\psi_j \circ f \circ \varphi^{-1})(x_i)(v))],$$

where $\psi_j\colon V_j \to \psi_j(V_j)$ is some chart with $f(x_i) \in V$. Note that this is smooth and does not depend on the choices of the charts. The bundle morphism Tf is sometimes also called the *differential* of f. Its restriction to the tangent space T_mM is a continuous linear map denoted T_mf. We clearly have $T(f \circ g) = Tf \circ Tg$, so that T can be understood as a functor **Man** $\to$ **Man**. Not that if $f\colon U \Subset X \to Y$ is smooth, then $TU = U \times X$ and Tf is given by $(x, v) \mapsto (f(x), df(x)(v))$. In this case we thus have $df = \mathrm{pr}_2 \circ Tf$.

If G is a Lie group, then G is modelled on some fixed lcs X. In fact, if $\varphi\colon U \to \varphi(U) \Subset X$ is an arbitrary chart, then $(\varphi \circ \lambda_g^{-1}\colon g \cdot U \to \varphi(U) \Subset X)_{g \in G}$ is an equivalent atlas of G (where λ_g denotes the left multiplication with g). If $\mu\colon G \times G \to G$ denotes the group multiplication, then it follows that

$$TG \to T_eG \times G, \quad v_g \mapsto (T\mu_{(g^{-1},g)}(0, v_g), g)$$

is a bundle morphism covering id_G with inverse

$$T_eg \times G \to TG, (v_e, g) \mapsto T\mu_{(g,e)}(0, v_e).$$

Thus the tangent bundle of a Lie group is always trivialisable. ■

Note that in the last example we implicitly mad use of the following easy lemma.

Lemma D.13. *If $f\colon O \Subset X \to Y$ is smooth, then so is $d^k\colon U \times X^k \to Y$.*

Proof. The proof is left as Exercise D.17. ■

Exercises for Appendix D

Exercise D.14. Show that morphisms of principal G-bundles are in local trivialisations (51) always given by $(y,g) \mapsto \xi_i(y) \cdot g$, where $\xi_i \colon U_{ij} \to G$ is smooth. Deduce from this that morphisms of principal bundles are automatically isomorphisms. ■

Exercise D.15. Let $\pi \colon Y \to Z$ be a bundle. Show that

$$(U \Subset Z) \mapsto \Gamma^\pi(U) := \{\gamma \in C^\infty(U,Y) \mid \pi \circ \gamma = \mathrm{id}_U\}$$

is a sheaf on $\mathbf{Open}_X$. This is also called the *sheaf of sections* of $\pi \colon Y \to Z$. ■

Exercise D.16. Let $\pi \colon Y \to Z$ be a principal G-bundle. Show that $Y \times_Z Y$ is isomorphic to the trivial principal G-bundle over Y. **Hint:** $(y,y') \in Y \times_Z Y$ gives rise to a unique $g \in G$ with $y' = y \cdot g$.■

Exercise D.17. Show that if $f \colon O \Subset X \to Y$ is smooth, then so is $d^k \colon U \times X^k \to Y$. ■

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