Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 06 – Solution Sketch

Exercise 29

Let $\pi\colon Y \to Z$ be a principal G-bundle. Show that $Y \times_Z Y$ is isomorphic to the trivial principal G-bundle over Y. **Hint:** $(y, y') \in Y \times_Z Y$ gives rise to a unique $g \in G$ with $y' = y \cdot g$.

We first observe that $y \cdot g = y \cdot h$ implies $g = h$. In fact, if Φ is a trivialisation around $\pi(y)$ and $\Phi(y) = (\pi(y), k)$, then $y \cdot g = y \cdot h$ implies $\Phi(y) \cdot g = (\pi(y), kg) = \Phi(y) \cdot h = (\pi(y), kh)$ and thus $g = h$. If $(y, y') \in Y \times_G Y$, then there exists a $g(y, y') \in G$ such that $y' = y \cdot g(y, y')$, which is unique by the previous argument. Thus $(y, y') \mapsto (y, g(y, y'))$ is a map, which has $(y, g) \mapsto (y, y \cdot g)$ as inverse. That the first map is smooth and G-equivariant can be seen in local trivialisations. Thus it is an isomorphism of principal G-bundles.

Exercise 30

Let X be a lcs such that the topology of X is induced by a countable family of semi-norms $(p_n)_{n\in\mathbb{N}}$.

a) Show that if we set $p'_n := \sum_{i\leq n} p_i$, we obtain another family of semi-norms inducing the same topology on X. We may thus w.l.o.g. assume that the family satisfies $p_n \leq p_{n+1}$.

b) Show that the following statements are equivalent conditions for a sequence (x_k) in X and $p \in X$:

   i) $(x_k) \xrightarrow{k\to\infty} p$ in the topology of X.

   ii) $(d(x_k, p)) \xrightarrow{k\to\infty} 0$, where d is the metric $d(x, y) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}$.

   iii) $(p_n(x_k - p)) \xrightarrow{k\to\infty} 0$ for each n.

   Moreover, if the family satisfies $p_n \leq p_{n+1}$, then show that any of these conditions is implied by

   iv) $(p_k(x_k - p)) \xrightarrow{k\to\infty} 0$.

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a) It is clear that p'_n is a point-separating family of semi-norms and since $p_n \leq p'_n$ we have that the topology induced from the p'_n is finer that the topology induced from the p_n. Conversely, we have $B_{J,\varepsilon_1,\ldots,\varepsilon_{|J|},x} \subseteq B'_{J,\frac{\varepsilon_1}{M},\ldots,\frac{\varepsilon_{|J|}}{M},x}$, where $M = \max J$, and thus he topology induced from the p_n is finer that the topology induced from the p'_n.

b) i) and iii) are equivalent be the definition of the topology induced by (p_n).

If $(d(x_k,p)) \xrightarrow{k\to\infty} 0$, then we have for each n that $\frac{p_n(x_k-p)}{1+p_n(x_k-p)} \leq 2_n d(x_k,p) \to 0$ and thus $p_n(x_k-p) \to 0$. Hence ii) implies iii).

If $(p_n(x_k-p)) \xrightarrow{k\to\infty} 0$ for each n and $\varepsilon > 0$ is given, then there exists M such that $2 \cdot 2^{-M} < \varepsilon$. If $N \in \mathbb{N}$ is such that $p_n(x_k-p) < 2^{-M}$ for all $n < M$ and all $k > N$, then $d(x_k,p) < 2^{-M} + 2^{-M} < \varepsilon$. Since ε was arbitrary this shows that iii) implies ii).

Since for each n we have $p_n \leq p_k$ if $k > n$ we have that iv) implies iii).

Exercise 31

Show that for a Hausdorff space X and a metrisable space Y the topology of compact convergence equals the compact open topology. **Hint:** This involves various typical compactness arguments.

Let d be a metric on Y inducing the topology. Recall that $C(X,Y)_{c.o.}$ has as a subbasis consisting of the sets

$$\lfloor K,U \rfloor := \{\gamma \in C(X,Y) \mid \gamma(K) \subseteq U\},$$

where K runs through the compacts of X and U through the opens of Y. To see that each $\lfloor K,U \rfloor$ is also open in $C(X,Y)_c$ take $f \in \lfloor K,U \rfloor$. For each $x \in K$ we have $f(x) \in U$ and this there exists $\varepsilon_x > 0$ such that $B_{f(x)}(\varepsilon_x) \subseteq U$ and $x \in V_x \Subset X$ such that $f(V_x) \subseteq B_{f(x)}(\frac{\varepsilon_x}{2})$. Since K is compact it is covered by $V_{x_1},...,V_{x_n}$ and we set $\varepsilon := \frac{1}{2}\min\{\varepsilon_{x_1},...,\varepsilon_{x_n}\}$. Then $O_f(K,\varepsilon) \subseteq \lfloor K,U \rfloor$: each $x \in K$ is contained in $x \in V_{x_i}$ for some i and thus $\gamma \in O_f(K,\varepsilon)$ implies

$$d(\gamma(x), f(x_i)) \leq d(\gamma(x), f(x)) + d(f(x), f(x_i)) < \frac{\varepsilon}{2} + \frac{\varepsilon_{x_i}}{2} \leq \varepsilon_i$$

and thus $\gamma(x) \in U$. Hence $O_f(K,\varepsilon) \subseteq \lfloor K,U \rfloor$ and $\lfloor K,U \rfloor$ is open in $C(X,Y)_c$.
To see that each $O_f(K,\varepsilon)$ is also open in $C(X,Y)_{c.o.}$, choose for each $x \in K$ some $x \in V_x \Subset X$ such that $f(V_x) \subseteq B_{f(x)}(\frac{\varepsilon}{4})$. This implies

$$f(\overline{V_x}) \subseteq \overline{f(V_x)} \subseteq \overline{B_{f(x)}(\frac{\varepsilon}{4})} \subseteq B_{f(x)}(\frac{\varepsilon}{2}).$$

Now K is covered by $V_{x_1},...,V_{x_n}$ and we set $K_i := K \cap \overline{V_{x_i}}$ and $U_i = B_{f(x_i)}(\frac{\varepsilon}{2})$. If

$$\gamma \in \lfloor K_1,U_1 \rfloor \cap ... \cap \lfloor K_n,U_n \rfloor,$$

then each $x \in K$ is in some $\overline{V_{x_i}}$ and this implies

$$d(f(x),\gamma(x)) \leq d(f(x), f(x_i)) + d(f(x_i),\gamma(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\gamma \in O_f(K,\varepsilon)$. This implies that $\lfloor K_1,U_1 \rfloor \cap ... \cap \lfloor K_n,U_n \rfloor \subseteq O_f(K,2\varepsilon)$ and since $f \in \lfloor K_1,U_1 \rfloor \cap ... \cap \lfloor K_n,U_n \rfloor$ we see that $O_f(K,2\varepsilon)$ is open in $C(X,Y)_{c.o.}$.

Exercise 32

Show that a morphism of diffeological spaces is continuous for the d-topology.

If $f\colon X \to Y$ is a morphism in **Diff**, then for each plot $\varphi\colon U \to X$ we have that $f \circ \varphi\colon U \to X$ is a plot. Thus if $O \subseteq Y$ is open, then $\varphi^{-1}(f^{-1}(O)) = (f \circ \varphi)^{-1}(O)$ is open, and thus $f^{-1}(O)$ is open in X.

Exercise 33

Show that if M, N are manifolds, M is compact, N is locally metrisable and $O \subseteq N$ is open, then $C^\infty(M, O)$ is open in the d-topology on $C^\infty(M, N)$.

Since the d-topology on $C^\infty(M, N)$ is the final topology for all plots we have to show that for each $U \Subset \mathbb{R}^n$ and for each plot $\varphi\colon U \to C^\infty(M, N)$ the inverse image $\varphi^{-1}(C^\infty(M, O))$ is open. Note that

$$x \in \varphi^{-1}(C^\infty(M, O)) \Leftrightarrow \varphi(x)(y) \in O \text{ for all } y \in M.$$

thus if $x \in \varphi^{-1}(C^\infty(M, O))$, then there exists for each $y \in M$ open neighbourhoods $x \in U_y \Subset U$ and $y \in V_y \Subset M$ such that $\varphi(U_y)(V_y) \subseteq O$. Since M is compact it is covered by $V_{y_1}, ..., V_{y_n}$ and $\widetilde{U} := U_{y_1} \cap U_{y_1} \cap U_{y_n}$ is an open neighbourhood of x. If $\widetilde{x} \in \widetilde{U}$ and $\widetilde{y} \in M$, then $\widetilde{y} \in V_{y_k}$ for some k and since $\widetilde{x} \in U_{y_k}$ we thus have $\varphi(\widetilde{x}, \widetilde{y}) \in O$. Since $\widetilde{y}$ was arbitrary this shows $\widetilde{x} \in \varphi^{-1}(C^\infty(M, O))$ and thus the latter is open.