Exercises for Higher Structures in Differential Geometry
SS 2013

Sheet 10

Exercise 49

If $(\mathcal{C}, K)$ is a site and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Grpd}$ is a weak presheaf in groupoids, show that $\mathbf{Match}(R, F)$ is for each $R = \{f_i\colon D_i \to C \mid i \in I\} \in K(C)$ a category with respect to $\mathrm{id}_{(X_i,\varphi_{ij})} = (\mathrm{id}_{X_i})$ and $(\alpha_i) \circ (\beta_i) = (\alpha_i \circ \beta_i)$. Moreover, show that

$$F(C) \to \mathbf{Match}(R, F), \quad (X \mapsto ((X|_{D_i}), (\varphi_{ij}(F, X)))),\ (\alpha \mapsto (\alpha|_{D_i})), \tag{1}$$

where $\varphi_{ij}(F, X) := F(f_i, \pi_{ij})(X)^{-1} \circ F(f_j, \rho_{ij})(X)$, is a functor.

Exercise 50

Give other examples of manifolds M and Lie groups G (besides the one given in the proof of Proposition III.11) such that the functor

$$\mathbf{B}G_{triv}(M) \to \mathbf{Match}(R, \mathbf{B}G_{triv})$$

is not essentially surjective.

Exercise 51

Let $(\mathcal{C}, K)$ be a site such that each $K(C) = \{f\colon D \to C\}$ is a singleton. Recall that then for each morphism $f\colon X \to C$ the pull-back $X \times_C D$ exists. For objects X, Y of $\mathcal{C}$ we the category $\mathbf{Span}(X, Y)$ to be given by objects

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

with $f \in K(X)$ and a morphism from $(X \xleftarrow{f} Z \xrightarrow{g} Y)$ to $(X \xleftarrow{f'} Z' \xrightarrow{g'} Y)$ to be given by a morphism $\alpha\colon Z \to Z'$ such that $f' \circ \alpha = f$ and $g' \circ \alpha = g$:

$$\begin{tikzcd}
& Z \arrow[ld, "f"'] \arrow[rd, "g"] \arrow[dd, "\alpha"] & \\
X & & Y . \\
& Z' \arrow[lu, "f'"] \arrow[ru, "g'"'] &
\end{tikzcd}$$

(the composition and identity in $\mathbf{Span}(X, Y)$ is induced from the one in $\mathcal{C}$). We define a "composition functor"

$$\mathbf{Span}(X, Y) \times \mathbf{Span}(Y, Z) \to \mathbf{Span}(X, Z).$$

as follows. If $(X \xleftarrow{f} Z \xrightarrow{g} Y, Y \xleftarrow{h} W \xrightarrow{k} Z)$ is an object in $\mathbf{Span}(X, Y) \times \mathbf{Span}(Y, Z)$, then we define the composite object to be

$$\begin{tikzcd}
& & Z \times_Y W \arrow[ld] \arrow[rd] & & \\
& Z \arrow[ld, "f"'] \arrow[rd, "g"] & & W \arrow[ld, "h"'] \arrow[rd, "k"] & . \\
X & & Y & & Z
\end{tikzcd}$$

If (α, α') is a morphism in $\mathbf{Span}(X, Y) \times \mathbf{Span}(Y, Z)$, then this induces a unique morphism on the pull-backs, which we take to be the composite morphism.
Show that this way we obtain a bicategory **Span** and nail down why we do (in general) *not* obtain a 2-category (or under which conditions we obtain a 2-category rather than a bicategory).

Exercise 52

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, f, g, h, k be functors, α, β be natural transformations such that

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} .
\end{tikzcd}$$

Then show that

$$\beta(G(X)) \circ h(\alpha(X)) = k(\alpha(X)) \circ \beta(f(X)) \tag{2}$$

for all objects X of $\mathcal{C}$ and that (2) defines a natural transformation $\beta * \alpha \colon h \circ f \Rightarrow k \circ g$. Moreover, if

$$\begin{tikzcd}[row sep=large]
\mathcal{C} \arrow[rr, bend left=50, "f"] \arrow[rr, "g" description] \arrow[rr, bend right=50, "p"'] & \overset{\Downarrow\alpha}{\underset{\Downarrow\gamma}{}} & \mathcal{D} \arrow[rr, bend left=50, "h"] \arrow[rr, "k" description] \arrow[rr, bend right=50, "q"'] & \overset{\Downarrow\beta}{\underset{\Downarrow\delta}{}} & \mathcal{E} ,
\end{tikzcd}$$

then show that $(\delta * \gamma) \circ (\beta * \alpha) = (\gamma \circ \alpha) * (\delta \circ \beta)$.

Exercise 53

Show that if $(\mathcal{C}, K)$ is a site and $A \colon J \to \mathbf{PSh}_{\mathcal{C}}$ is a diagram in $\mathcal{C}$ such that for each object j of J the presheaf A_j is a sheaf, then the presheaf $\varprojlim_J A$ is actually a sheaf.
Hint: You may use without further proof that if you have a diagram of diagrams, then it does not matter in which order you take the limit (provided all limits exist), cf. [Mac98, Sect. V.4].

References

[Mac98] MacLane, S. *Categories for the working mathematician*, *Graduate Texts in Mathematics*, vol. 5 (Springer-Verlag, New York, 1998), second edn.