Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 10

Exercise 49

If (\mathcal{C}, K) is a site and $F: \mathcal{C}^{\text{op}} \to \mathbf{Grpd}$ is a weak presheaf in groupoids, show that $\mathbf{Match}(R, F)$ is for each $R = \{f_i: D_i \to C \mid i \in I\} \in K(C)$ a category with respect to $\mathrm{id}_{(X_i,\varphi_i)} = (\mathrm{id}_{X_i})$ and $(\alpha_i) \circ (\beta_i) = (\alpha_i \circ \beta_i)$. Moreover, show that

$$F(C) \to \operatorname{Match}(R, F), \quad (X \mapsto ((X|_{D_i}), (\varphi_{ij}(F, X)))), \ (\alpha \mapsto (\alpha|_{D_i})), \tag{1}$$

where $\varphi_{ij}(F, X) := F(f_i, \pi_{ij})(X)^{-1} \circ F(f_j, \rho_{ij})(X)$, is a functor.

Exercise 50

Give other examples of manifolds M and Lie groups G (besides the one given in the proof of Proposition III.11) such that the functor

$$\mathbf{B}G_{triv}(M) \to \mathbf{Match}(R, \mathbf{B}G_{triv})$$

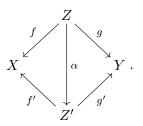
is not essentially surjective.

Exercise 51

Let (\mathcal{C}, K) be a site such that each $K(C) = \{f : D \to C\}$ is a singleton. Recall that then for each morphism $f : X \to C$ the pull-back $X \times_C D$ exists. For objects X, Y of \mathcal{C} we the category $\mathbf{Span}(X, Y)$ to be given by objects

 $X \xleftarrow{f} Z \xrightarrow{g} Y$

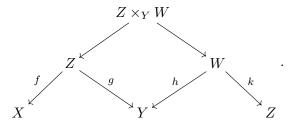
with $f \in K(X)$ and a morphism from $(X \xleftarrow{f} Z \xrightarrow{g} Y)$ to $(X \xleftarrow{f'} Z' \xrightarrow{g'} Y)$ to be given by a morphism $\alpha \colon Z \to Z'$ such that $f' \circ \alpha = f$ and $g' \circ \alpha = g$:



(the composition and identity in $\mathbf{Span}(X, Y)$ is induced from the one in \mathcal{C}). We define a "composition functor"

$$\mathbf{Span}(X, Y) \times \mathbf{Span}(Y, Z) \to \mathbf{Span}(X, Z).$$

as follows. If $(X \xleftarrow{f} Z \xrightarrow{g} Y, Y \xleftarrow{h} W \xrightarrow{k} Z)$ is an object in $\mathbf{Span}(X, Y) \times \mathbf{Span}(Y, Z)$, then we define the composite object to be

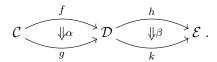


If (α, α') is a morphism in **Span** $(X, Y) \times$ **Span**(Y, Z), then this induces a unique morphism on the pull-backs, which we take to be the composite morphism.

Show that this way we obtain a bicategory **Span** and nail down why we do (in general) *not* obtain a 2-category (or under which conditions we obtain a 2-category rather than a bicategory).

Exercise 52

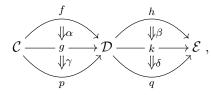
Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, f, g, h, k be functors, α, β be natural transformations such that



Then show that

$$\beta(G(X)) \circ h(\alpha(X)) = k(\alpha(X)) \circ \beta(f(X))$$
(2)

for all objects X of C and that (2) defines a natural transformation $\beta * \alpha \colon h \circ f \Rightarrow k \circ g$. Moreover, if



then show that $(\delta * \gamma) \circ (\beta * \alpha) = (\gamma \circ \alpha) * (\delta \circ \beta).$

Exercise 53

Show that if (\mathcal{C}, K) is a site and $A: J \to \mathbf{PSh}_{\mathcal{C}}$ is a diagram in \mathcal{C} such that for each object j of J the presheaf A_j is a sheaf, then the presheaf $\lim_{J} A$ is actually a sheaf. **Hint:** You may use without further proof that if you have a diagram of diagrams, then it does not matter in which order you take the limit (provided all limits exist), cf. [Mac98, Sect. V.4].

References

[Mac98] MacLane, S. Categories for the working mathematician, Graduate Texts in Mathematics, vol. 5 (Springer-Verlag, New York, 1998), second edn.