Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 08

Exercise 39

Show that if X is a compact space and $(U_i)_{i\in I}$ is an open cover of X, then there exists an open cover $(V_i)_{i\in I}$ such that $\overline{V_i} \subseteq U_i$. Does this also hold if X is not assumed to be compact?

Exercise 40

Let $(X_n)_{n\in\mathbb{N}}$ be a countable family of metrisable spaces. Then show that the product $\prod_{n\in\mathbb{N}} X_n$ is also metrisable. **Hint:** We have already encountered the phenomenon that metrisability is implied by countability assumptions.

Exercise 41

Let $(X_n)_{n\in\mathbb{N}}$ be a countable family of complete metrisable spaces. Then show that the product $\prod_{n\in\mathbb{N}} X_n$ is also complete.

Exercise 42

Let X be a Fréchet space and M be a second countable manifold.

a) Show that if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^\infty(V,X)$, (for $V \Subset \mathbb{R}^n$) then also $(d^k f)_n$ is a Cauchy sequence for each $k \in \mathbb{N}_0$.

b) In the same setting as in a), show that $d^k(\lim_{n\to\infty} f_n) = \lim_{n\to\infty} d^k f_n$ for each $k \in \mathbb{N}_0$.

c) Show that $C^\infty(M,X)$ is a Fréchet space.

Exercise 42

Fill in the details of the proof of Proposition II.45, i.e., show that if $E \to M$ is a vector bundle with compact M and metrisable fibre X, then

$$\Gamma(E \to M) = \{\sigma \in C^\infty(M,E) \mid \pi \circ \sigma = \mathrm{id}_M\}$$

is closed in $C^\infty(M,E)$, that the fibre-wise addition and scalar multiplication turns $\Gamma(E \to M)$ into a tvs and that Lemma F.3 may be used to identify $\Gamma(E \to M)$ with a space that we already know to be a (metrisable) lcs.

Exercise 43

The following exercise should assure you that the arguments that you know from the exponential function for matrices carry over to Banach algebras in the following sense: Let A be a Banach algebra and $A^\times$ be the Lie group of units in A. Define the *exponential function* of A to be

$$\exp\colon A \to A, \quad x \mapsto \sum_{n=0}^{\infty} (\frac{x^n}{n!}).$$

a) Show that $\exp(A) \subseteq A^\times$ and that $\exp(0) = 1$.

b) Show that exp restricts to a diffeomorphism from some open neighbourhood of 0 onto some open neighbourhood of 1.

c) Show that for $g \in A^\times$ we have $\exp(g \cdot x \cdot g^{-1}) = g \cdot \exp(x) \cdot g^{-1}$.

d) For $x \in A$ we set $\xi_x\colon \mathbb{R} \to A^\times$, $t \mapsto \exp(t \cdot x)$. The show that $\frac{\partial}{\partial t}\xi_x(t) = \xi_x(t) \cdot x$.